Probabilistic representation for solutions of an irregular porous media type equation: the degenerate case.

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Summary: We consider a possibly degenerate porous media type equation over all of $\mathbb{R}^d$ with $d = 1$, with monotone discontinuous coefficients with linear growth and prove a probabilistic representation of its solution in terms of an associated microscopic diffusion. This equation is motivated by some singular behaviour arising in complex self-organized critical systems. The main idea consists in approximating the equation by equations with monotone non-degenerate coefficients and deriving some new analytical properties of the solution.

Key words: singular degenerate porous media type equation, probabilistic representation.

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1 Introduction

We are interested in the probabilistic representation of the solution to a porous media type equation given by

$$\begin{cases} \partial_t u &= \frac{1}{2}\partial_{xx}^2(\beta(u)), \ t \in [0, \infty[\\ u(0, x) &= u_0(x), \ x \in \mathbb{R}, \end{cases} \tag{1.1}$$

in the sense of distributions, where u_0 is an initial bounded probability density. We look for a solution of (1.1) with time evolution in $L^1(\mathbb{R})$.

We make the following assumption.

Assumption 1.1

- *$\beta : \mathbb{R} \to \mathbb{R}$ is monotone increasing.*
- $|\beta(u)| \leq \mathrm{const}|u|, \ u \geq 0$.
  In particular, β is right-continuous at zero and $\beta(0) = 0$.
- *There is $\lambda > 0$ such that $(\beta + \lambda id)(x) \to \mp\infty$ when $x \to \mp\infty$.*

Remark 1.2 *(i) By one of the consequences of our main result, see Remark 1.6 below, the solution to (1.1) is non-negative, since $u_0 \geq 0$. Therefore, it is enough to assume that only the restriction of β to $\mathbb{R}_+$ is increasing such that $|\beta(u)| \leq \mathrm{const}|u|$ for $u \geq 0$, and $(\beta + \lambda id)(x) \to \infty$ when $x \to +\infty$. Otherwise, we can just replace β by an extension of the restriction of β to $\mathbb{R}_+$ which satisfies Assumption 1.1, e.g. take its odd symmetric extension.*

(ii) In the main body of the paper, we shall in fact replace β with the "filled" associated graph, see remarks after Definition 2.2 for details; in this way, we consider β as a multivalued function and Assumption 1.1 will be replaced by Hypothesis 3.1.

Since β is monotone, (1.1) implies $\beta(u) = \Phi^2(u)u, \ u \geq 0$, Φ being a non-negative bounded Borel function. We recall that when $\beta(u) = |u|^{m-1}u$, $m > 1$, (1.1) is nothing else but the classical *porous media equation.*

One of our targets is to consider Φ as continuous except for a possible jump at one positive point, say $e_c > 0$. A typical example is

$$\Phi(u) = H(u - e_c), \tag{1.2}$$

H being the Heaviside function.

The analysis of (1.1) and its probabilistic representation can be done in the framework of monotone partial differential equations (PDE) allowing multi-valued coefficients and will be discussed in detail in the main body of the paper. In this introduction, for simplicity, we restrict our presentation to the single-valued case.

Definition 1.3
- *We will say that equation (1.1) or β is* **non-degenerate** *if on each compact, there is a constant $c_0 > 0$ such that $\Phi \geq c_0$.*
- *We will say that equation (1.1) or β is* **degenerate** *if* $\lim_{u \to 0_+} \Phi(u) = 0$ *in the sense that for any sequence of non-negative reals (x_n) converging to zero, and $y_n \in \Phi(x_n)$ we have* $\lim_{n \to \infty} y_n = 0$.

Remark 1.4
1. *β may be in fact neither non-degenerate nor degenerate. If β is odd, which according to Remark 1.2 (ii), we may always assume, then β is non-degenerate if and only if* $\liminf_{u \to 0+} \Phi(u) > 0$.
2. *Of course, Φ in (1.2) is degenerate. In order to have Φ non-degenerate, one could add a positive constant to it.*

There are several contributions to the analytical study of (1.1), starting from [11] for existence, [13] for uniqueness in the case of bounded solutions and [12] for continuous dependence on the coefficients. The authors consider the case where β is continuous, even if their arguments allow some extensions for the discontinuous case.

As mentioned in the abstract, the first motivation of this paper was to discuss continuous time models of self-organized criticality (SOC), which are described by equations of type (1.1) with $\beta(u) = u\Phi^2(u)$ and Φ as in (1.2), see e.g. [3] for a significant monography on the subject and the interesting physical papers [4] and [14]. For other comments related to SOC, one can read the introduction of [9]. The recent papers, [8, 7], discuss (1.1) in the case (1.2), perturbed by a multiplicative noise.

The singular non-linear diffusion equation (1.1) models the *macroscopic* phenomenon for which we try to give a *microscopic* probabilistic representation, via a non-linear stochastic differential equation (NLSDE) modelling the evolution of a single point.

The most important contribution of [9] was to establish a probabilistic representation of (1.1) in the non-degenerate case. For the latter we established both existence and uniqueness. In the degenerate case, even if the irregular diffusion equation (1.1) is well-posed, at that time, we could not prove existence of solutions to the corresponding NLSDE. This is now done in the present paper.

To the best of our knowledge the first author who considered a probabilistic representation (of the type studied in this paper) for the solutions of a non-linear deterministic PDE was McKean [24], particularly in relation with the so called propagation of chaos. In his case, however, the coefficients were smooth. From then on the literature has steadily grown and nowadays there is a vast amount of contributions to the subject, especially when the non-linearity is in the first order part, as e.g. in Burgers equation. We refer the reader to the excellent survey papers [29] and [21].

A probabilistic interpretation of (1.1) when $\beta(u) = |u|u^{m-1}, m > 1$, was provided for instance in [10] and the propagation of chaos was investigated by [20]. When β is Lipschitz, [22] considered probabilisitic representation and propagation of chaos.

Let us now describe the principle of the mentioned probabilistic representation. The stochastic differential equation (in the weak sense) rendering the probabilistic representation is given by the following (random) non-linear diffusion:

$$\left\{ \begin{array}{rcl} Y_t & = & Y_0 + \int_0^t \Phi(u(s,Y_s))dW_s \\ \text{Law} \quad \text{density}(Y_t) & = & u(t,\cdot), \end{array} \right. \tag{1.3}$$

where W is a classical Brownian motion. The solution of that equation may be visualised as a continuous process Y on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ equipped with a Brownian motion W. By looking at a properly chosen version, we can and shall assume that $Y : [0,T] \times \Omega \to \mathbb{R}_+$ is $\mathcal{B}([0,T]) \otimes \mathcal{F}$-measurable. Of course, we can only have (weak) uniqueness for (1.3) fixing the initial distribution, i.e. we have to fix the distribution (density) u_0 of Y_0.

The connection with (1.1) is then given by the following result, see also [9].

Theorem 1.5 *Let us assume the existence of a solution Y for (1.3). Then $u : [0,T] \times \mathbb{R} \to \mathbb{R}_+$ provides a solution in the sense of distributions of (1.1) with $u_0 := u(0,\cdot)$.*

Remark 1.6 *An immediate consequence for the associated solution of (1.1) is its positivity at any time if it starts with an initial value u_0 which is positive. Also the mass 1 of the initial condition is conserved in this case. However this property follows already by approximation from Corollary 4.5 of [9], which in turn is based on the probabilistic representation in the non-degenerate case, see Corollary 4.2 below for details.*

The main purpose of this paper is to show existence of the probabilistic representation equation (1.3), in the case where β is degenerate and not necessarily continuous. The uniqueness is only known if β is non-degenerate and in some very special cases in the degenerate case.

Let us now briefly and consecutively explain the points that we are able to treat and the difficulties which naturally appear in the probabilistic representation.

For simplicity we do this for β being single-valued (and) continuous. However, with some technical complications this generalizes to the multi-valued case, as spelt out in the subsequent sections.

1. Monotonicity methods allow us to show existence and uniqueness of solutions to (1.1) in the sense of distributions under the assumption that β is monotone, that there exists $\lambda > 0$ with $(\beta + \lambda id)(\mathbb{R}) = \mathbb{R}$ and that β is continuous at zero, see Proposition 3.2 of [9] and the references therein.

2. If β is non-degenerate, Theorem 4.3 of [9], allows to construct a unique (weak) solution Y to the non-linear SDE in the first line of (1.3), for any intial bounded probability density u_0 on $\mathbb{R}$.

3. Suppose β to be degenerate. We fix a bounded probability density u_0. We set $\beta_\varepsilon(u) = \beta(u) + \varepsilon u, \quad \Phi_\varepsilon = \sqrt{\Phi^2 + \varepsilon}$ and consider the weak

solution Y^ε of

$$Y_t^\varepsilon = Y_0^\varepsilon + \int_0^t \Phi_\varepsilon(u^\varepsilon(s, Y_s^\varepsilon))dW_s, \tag{1.4}$$

where $u^\varepsilon(t, \cdot)$ is the law of $Y_t^\varepsilon, t \geq 0$ and Y_0^ε is distributed according to $u_0(x)dx$. The sequence of laws of the processes (Y^ε) are tight, but the limiting process of a convergent subsequence a priori may not necessarily solve the SDE

$$Y_t = Y_0 + \int_0^t \Phi(u(s, Y_s))dW_s. \tag{1.5}$$

However, this will be shown to be the case in the following two general situations.

(a) The case when the initial condition u_0 is locally of bounded variation, without any further restriction on the coefficient β.

(b) The case when β is strictly increasing after some zero, see Definition 4.20, and without any further restriction on the initial condition.

In this paper, we proceed as follows. Section 2 is devoted to preliminaries and notations. In Section 3, we analyze an elliptic non-linear equation with monotone coefficients which constitutes the basis for the existence of a solution to (1.1). We recall some basic properties and we establish some other which will be useful later. In Section 4, we recall the notion of C^0- solution to (1.1) coming from an implicit scheme of non-linear elliptic equations presented in Section 3. Moreover, we prove three significant properties. The first is that $\beta(u(t, \cdot))$ is in H^1, therefore continuous, for almost all $t \in [0, T]$. The second is that the solution $u(t, \cdot)$ is locally of bounded variation if u_0 is. The third is that if β is strictly increasing after some zero, then $\Phi(u(t, \cdot))$ is continuous for almost all t. Section 5 is devoted to the study of the probabilistic representation of (1.1).

Finally, we would like to mention that, in order to keep this paper self-contained and make it accessible to a larger audience, we include the analytic background material and necessary (through standard) definitions. Likewise, we tried to explain all details on the analytic delicate and quite technical parts of the paper which form the back bone of the proofs for our main result.

2 Preliminaries

We start with some basic analytical framework.

If $f : \mathbb{R} \to \mathbb{R}$ is a bounded function we will set $\|f\|_\infty = \sup_{x\in\mathbb{R}} |f(x)|$. By $C_b(\mathbb{R})$ we denote the space of bounded continuous real functions and by $C_\infty(\mathbb{R})$ the space of all continuous functions on $\mathbb{R}$ vanishing at infinity. $\mathcal{D}(\mathbb{R})$ will be the space of all infinitely differentiable functions with compact support $\varphi : \mathbb{R} \to \mathbb{R}$, and $\mathcal{D}'(\mathbb{R})$ will be its dual (the space of Schwartz distributions). $\mathcal{S}(\mathbb{R})$ is the space of all rapidly decreasing infinitely differentiable functions $\varphi : \mathbb{R} \to \mathbb{R}$, and $\mathcal{S}'(\mathbb{R})$ will be its dual (the space of tempered distributions).

If $p \geq 1$ by $L^p(\mathbb{R})$ (resp. $L^p_{\mathrm{loc}}(\mathbb{R})$), we denote the space of all real Borel functions f such that $|f|^p$ is integrable (resp. integrable on each compact interval). We denote the space of all Borel essentialy bounded real functions by $L^\infty(\mathbb{R})$. In several situations we will even omit $\mathbb{R}$.

We will use the classical notation $W^{s,p}(\mathbb{R})$ for Sobolev spaces, see e.g. [1]. $\|\cdot\|_{s,p}$ denotes the corresponding norm. We will use the notation $H^s(\mathbb{R})$ instead of $W^{s,2}(\mathbb{R})$. If $s \geq 1$, this space is a subspace of the space $C(\mathbb{R})$ of real continuous functions. We recall that, by Sobolev embedding, $W^{1,1}(\mathbb{R}) \subset C_\infty(\mathbb{R})$ and that each $u \in W^{1,1}(\mathbb{R})$ has an absolutely continuous version. Let $\delta > 0$. We will denote by $< \cdot, \cdot >_{-1,\delta}$ the inner product

$$< u, v >_{-1,\delta} = < (\delta - \frac{1}{2}\Delta)^{-1/2} u, (\delta - \frac{1}{2}\Delta)^{-1/2} v >_{L^2(\mathbb{R})},$$

and by $\|\cdot\|_{-1,\delta}$ the corresponding norm. For details about $(\delta - \frac{1}{2}\Delta)^{-s}$, see [27, 30] and also [9], section 2. In particular, given $s \in \mathbb{R}$, $(\delta - \frac{1}{2}\Delta)^s$ maps $\mathcal{S}'(\mathbb{R})$ (resp. $\mathcal{S}(\mathbb{R})$) onto itself. If $u \in L^2(\mathbb{R})$.

$$(\delta - \frac{1}{2}\Delta)^{-1} u(x) = \int_{\mathbb{R}} K_\delta(x - y) u(y) dy,$$

with

$$K_\delta(x) = \frac{1}{\sqrt{2\delta}} e^{-\sqrt{2\delta}|x|}. \tag{2.6}$$

Moreover the map $(\delta - \frac{1}{2}\Delta)^{-1}$ continuously maps H^{-1} onto H^1 and a tempered distribution u belongs to H^{-1} if and only if $(\delta - \frac{1}{2}\Delta)^{-1/2}u \in L^2$.

Remark 2.1 *$L^1 \subset H^{-1}$ continuously. Moreover for $u \in L^1$,*

$$\|u\|_{-1,\delta} \leq \|K_\delta\|_\infty^{\frac{1}{2}} \|u\|_{L^1} = (2\delta)^{-\frac{1}{4}} \|u\|_{L^1}.$$

Let $T > 0$ be fixed. For functions $(t, x) \to u(t, x)$, the notation u' (resp. u'') will denote the first (resp. second) derivative with respect to x.

Let E be a Banach space. One of the most basic notions of this paper is the one of a multivalued function (graph). A **multivalued function** (graph) β on E will be a subset of $E \times E$. It can be seen, either as a family of couples $(e, f), e, f \in E$ and we will write $f \in \beta(e)$ or as a function $\beta : E \to \mathcal{P}(E)$.

We start with the definition in the case $E = \mathbb{R}$.

Definition 2.2 *A multivalued function β defined on $\mathbb{R}$ with values in subsets of $\mathbb{R}$ is said to be* **monotone** *if given $x_1, x_2 \in \mathbb{R}$, $(x_1 - x_2)(\beta(x_1) - \beta(x_2)) \geq 0$.*

We say that β is **maximal monotone** *(or a* **maximal monotone graph***) if it is monotone and if for one (hence all) $\lambda > 0$, $\beta + \lambda id$ is surjective, i.e.*

$$\mathcal{R}(\beta + \lambda id) := \bigcup_{x \in \mathbb{R}} (\beta(x) + \lambda x) = \mathbb{R}.$$

For a maximal monotone graph $\beta : \mathbb{R} \to 2^{\mathbb{R}}$, we define a function $j : \mathbb{R} \to \mathbb{R}$ by

$$j(u) = \int_0^u \beta^\circ(y)dy, \ u \in \mathbb{R}, \tag{2.7}$$

where β° is the minimal section of β. It fullfills the property that $\partial j = \beta$ in the sense of convex analysis see e.g. [6]. In other words β is the subdifferential of j. j is convex, continuous and if $0 \in \beta(0)$, then $j \geq 0$.

We recall that one motivation of this paper is the case where $\beta(u) = H(u - e_c)u$. It can be considered as a multivalued map *by filling the gap*. More generally, let us consider a monotone function ψ. Then all the discontinuities are of jump type. At every discontinuity point x of ψ, it is possible to *complete* ψ by setting $\psi(x) = [\psi(x-), \psi(x+)]$. Since ψ is a monotone function, the corresponding multivalued function will be, of course, also monotone.

Now we come back to the case of our general Banach space E with norm $\|\cdot\|$. An operator $T : E \to E$ is said to be a **contraction** if it is Lipschitz of norm less or equal to 1 and $T(0) = 0$.

Definition 2.3 *A map $A : E \to E$, or more generally a multivalued map $A : E \to \mathcal{P}(E)$ is said to be* **accretive** *if for any $f_1, f_2, g_1, g_2 \in E$ such that $g_i \in Af_i, i = 1, 2$, we have*

$$\|f_1 - f_2\| \leq \|f_1 - f_2 + \lambda(g_1 - g_2)\|,$$

for any $\lambda > 0$.

This is equivalent to saying the following: for any $\lambda > 0$, $(I + \lambda A)^{-1}$ is a contraction on $Rg(I + \lambda A)$. We remark that a contraction is necessarily single-valued.

Proposition 2.4 *Suppose that E is a Hilbert space equipped with the scalar product $(\ ,\ )_H$. Then A is accretive if and only if A is* **monotone** *i.e. $(f_1 - f_2, g_1 - g_2)_H \geq 0$ for any $f_1, f_2, g_1, g_2 \in E$ such that $g_i \in Af_i, i = 1, 2$, see Corollary 1.3 of [26].*

Definition 2.5 *An accretive map $A : E \to E$ (possibly multivalued) is said to be* **m-accretive** *if for some $\lambda > 0$, $I + \lambda A$ is surjective (as a graph in $E \times E$).*

Remark 2.6 *An accretive map $A : E \to E$ is m-accretive if and only if $I + \lambda A$ is surjective for any $\lambda > 0$.*

So, A is m-accretive, if and only if for all λ strictly positive, $(I + \lambda A)^{-1}$ is a contraction on E.

If E is a Hilbert space, by the celebrated Minty's theorem, see e.g. [5], a mapping $A : E \to E$ is m-accretive if it is maximal monotone, i.e. it is monotone and has no proper monotone extension.

Now, let us consider the case $E = L^1(\mathbb{R})$, so $E^* = L^\infty(\mathbb{R})$. The following is taken from [12], Section 1.

Theorem 2.7 *Let $\beta : \mathbb{R} \to \mathbb{R}$ be a monotone (possibly multi-valued) function such that the corresponding graph is maximal monotone. Suppose that $0 \in \beta(0)$. Let $f \in E = L^1(\mathbb{R})$.*

1. *There is a unique $u \in L^1(\mathbb{R})$ for which there is $w \in L^1_{\mathrm{loc}}(\mathbb{R})$ such that*

   $$u - \Delta w = f \quad \text{in} \quad \mathcal{D}'(\mathbb{R}), \quad w(x) \in \beta(u(x)), \quad \text{for a.e.} \quad x \in \mathbb{R}, \tag{2.8}$$

   see Proposition 2 of [12].

2. *Then, a (possibly multivalued) operator $A := A_\beta : D(A) \subset E \to E$ is defined with $D(A)$ being the set of $u \in L^1(\mathbb{R})$ for which there is $w \in L^1_{\mathrm{loc}}(\mathbb{R})$ such that $w(x) \in \beta(u(x))$ for a.e. $x \in \mathbb{R}$ and $\Delta w \in L^1(\mathbb{R})$ and for $u \in D(A)$*

   $$Au = \{-\frac{1}{2}\Delta w | w \text{ as in definition of } D(A)\}.$$

   This is a consequence of the remarks following Theorem 1 in [12].

   In particular, if β is single-valued, then $Au = -\frac{1}{2}\Delta\beta(u)$. (We will adopt this notation also if β is multi-valued).

3. *The operator A defined in 2. above is m-accretive on $E = L^1(\mathbb{R})$, see Proposition 2 of [12]. Moreover $\overline{\mathcal{D}(A)} = E$.*

4. *We set $J_\lambda = (I + \lambda A)^{-1}$, which is a single-valued operator. If $f \in L^\infty(\mathbb{R})$, then $\|J_\lambda f\|_\infty \leq \|f\|_\infty$, see Proposition 2 (iii) of [12]. In particular, for every positive integer n, $\|J^n_\lambda f\|_\infty \leq \|f\|_\infty$.*

Let us summarize some important results of the theory of non-linear semigroups, see for instance [18, 5, 6, 11] or the more recent monograph [26], which we shall use below. Let $A : E \to E$ be a (possibly multivalued) accretive operator. We consider the equation

$$0 \in u'(t) + A(u(t)), \quad 0 \leq t \leq T. \tag{2.9}$$

A function $u : [0, T] \to E$ which is absolutely continuous such that for a.e. t, $u(t, \cdot) \in D(A)$ and fulfills (2.9) in the following sense is called **strong solution**.

There exists $\eta : [0,T] \to E$, Bochner integrable, such that $\eta(t) \in A(u(t))$ for a.e. $t \in [0,T]$ and

$$u(t) = u_0 - \int_0^t \eta(s)ds, \quad 0 < t \leq T.$$

A weaker notion for (2.9) is the so-called C^0**- solution**, see chapter IV.8 of [26], or **mild solution**, see [6]. In order to introduce it, one first defines the notion of ε-solution related to (2.9).

An ε**-solution** is a discretization

$$\mathcal{D} = \{0 = t_0 < t_1 < \ldots < t_N = T\}$$

and an E-valued step function

$$u^\varepsilon(t) = \begin{cases} u_0 & : \quad t = t_0 \\ u_j \in D(A) & : \quad t \in]t_{j-1}, t_j], \end{cases}$$

for which $t_j - t_{j-1} \leq \varepsilon$ for $1 \leq j \leq N$, and

$$0 \in \frac{u_j - u_{j-1}}{t_j - t_{j-1}} + Au_j, 1 \leq j \leq N.$$

We remark that, since A is maximal monotone, u^ε is determined by $\mathcal{D}$ and u_0, see Theorem 2.7 3.

Definition 2.8 *A* C^0**- solution** *of (2.9) is an* $u \in C([0,T];E)$ *such that for every* $\varepsilon > 0$*, there is an* ε*-solution* u^ε *of (2.9) with*

$$\|u(t) - u^\varepsilon(t)\| \leq \varepsilon, \quad 0 \leq t \leq T.$$

Proposition 2.9 *Let A be a maximal monotone (multivalued) operator on a Banach space E. We set again* $J_\lambda := (I + \lambda A)^{-1}, \lambda > 0$*. Suppose* $u_0 \in \overline{D(A)}$*. Then:*

1. *There is a unique* C^0*- solution* $u : [0,T] \to E$ *of (2.9)*
2. $u(t) = \lim_{n\to\infty} J^n_{\frac{t}{n}} u_0$ *uniformly in* $t \in [0,T]$*.*

Proof.

1) is stated in Corollary IV.8.4. of [26] and 2) is contained in Theorem IV 8.2 of [26]. ∎

The complications coming from the definition of C^0-solution arise because the dual E^* of $E = L^1(\mathbb{R})$ is not uniformly convex. In general a C^0-solution is not absolutely continuous and not a.e. differentiable, so it is not a strong solution. For uniformly convex Banach spaces, the situation is much easier. Indeed, according to Theorem IV 7.1 of [26], for a given $u_0 \in D(A)$, there would exist a (strong) solution $u : [0,T] \to E$ to (2.9). Moreover, Theorem 1.2 of [16] says the following. Given $u_0 \in \overline{D(A)}$ and given a sequence (u_0^n) in $D(A)$ converging to u_0, then the sequence of the corresponding strong solutions (u_n) would converge to the unique C^0-solution of the same equation.

3 Elliptic equations with monotone coefficients

Let us fix our assumptions on β which we assume to be in force in this entire section.

Hypothesis 3.1 *Let $\beta : \mathbb{R} \to 2^{\mathbb{R}}$ be a maximal monotone graph with the property that there exists $c > 0$ such that*

$$w \in \beta(u) \Rightarrow |w| \leq c|u|. \tag{3.1}$$

We note that (3.1) implies that $\beta(0) = 0$, hence $j(u) \geq 0$, for any $u \in \mathbb{R}$, where j is defined in (2.7). Furthermore, by Hypothesis 3.1,

$$j(u) \leq \int_0^{|u|} |\beta^\circ(y)| dy \leq c|u|^2. \tag{3.2}$$

We recall from [12] that the first ingredient to study well-posedness of equation (1.1) is the following elliptic equation

$$u - \lambda \Delta \beta(u) \ni f \tag{3.3}$$

where $f \in L^1(\mathbb{R})$ and u is the unknown function in $L^1(\mathbb{R})$.

Definition 3.2 *Let $f \in L^1(\mathbb{R})$. Then $u \in L^1(\mathbb{R})$ is called a solution of* (3.3) *if there is $w \in L^1_{loc}$ with $w \in \beta(u)$ a.e. and*

$$u - \lambda \Delta w = f \tag{3.4}$$

in the sense of distributions.

According to Theorem 4.1 of [11], and Theorem 1, Ch.1, of [12], equation (3.3) admits a unique solution. Moreover, w is also uniquely determined by u. Sometimes, we will also call the couple (u, w) the solution to (3.4).

We recall some basic properties of the couple (u, w).

Lemma 3.3 *Let (u, w) be the unique solution of* (3.3)*. Let $J_\beta^\lambda : L^1(\mathbb{R}) \to L^1(\mathbb{R})$ be the map which associates the solution u of* (3.3) *to $f \in L^1(\mathbb{R})$. We have the following:*

1. *$J_\beta^\lambda 0 = 0$.*
2. *J_β^λ is a contraction in the sense that*
$$\left\| J_\beta^\lambda(f_1) - J_\beta^\lambda(f_2) \right\|_{L^1} \leq \|f_1 - f_2\|_{L^1}$$
for every $f_1, f_2 \in L^1$.
3. *If $f \in L^1 \bigcap L^\infty$, then $\|u\|_\infty \leq \|f\|_\infty$*
4. *If $f \in L^1 \bigcap L^2$, then $u \in L^2$ and*
$$\int_{\mathbb{R}} u^2(x)dx \leq \int_{\mathbb{R}} f^2(x)dx.$$
and $\int_{\mathbb{R}} j(u)(x)dx \leq \int_{\mathbb{R}} j(f)(x)dx \leq \mathrm{const}\|f\|_{L^2}$.
5. *Let $f \in L^1$. Then $w, w' \in W^{1,1} \subset C_\infty(\mathbb{R})$. Hence, in particular $w \in W^{1,p}$ for any $p \in [1, \infty]$.*

Proof.

1. is obvious and comes from uniqueness of (3.3).
2. See Proposition 2.i) of [12].
3. See Proposition 2.iii) of [12].
4. This follows from [11], Point III, Ch. 1 and (3.2).

5. We define $g := \frac{1}{2\lambda}(w+f-u)$. Since $f \in L^1$, also $u \in L^1$, hence $w \in L^1$ by (3.1). Altogether it follows that $g \in L^1$. (3.4) and (2.6) imply that $w = K_\delta \star g$ with $\delta = \frac{1}{2\lambda}$ and hence

$$w' = K'_\delta \star g = \int \operatorname{sign}(y-x) e^{-\lambda^{-\frac{1}{2}}|x-y|} g(y)dy,$$

where

$$\operatorname{sign} x = \begin{cases} -1 & : \quad x < 0 \\ 0 & : \quad x = 0 \\ 1 & : \quad x > 0. \end{cases}$$

This implies $w, w' \in L^1 \bigcap L^\infty$. By (3.4) we know that also $w'' \in L^1$, hence $w, w' \in W^{1,1} (\subset C_\infty)$.

■

Remark 3.4 *Let $\delta > 0$. The same results included in Lemma 3.3 are valid for the equation*

$$u + \lambda\delta\beta(u) - \lambda\Delta(\beta(u)) \ni f. \tag{3.5}$$

In fact, [11] treats the equation $\Delta v + \gamma(v) \ni f$, with $\gamma : \mathbb{R} \to 2^{\mathbb{R}}$ a maximal monotone graph. We reduce equation (3.3) and (3.5) to this equation, by setting $v = \lambda\beta(u), \gamma(v) = -\beta^{-1}(\frac{v}{\lambda})$, where β^{-1} is the inverse graph of β, and setting $v = \lambda\beta(u), \quad \gamma(v) = -\beta^{-1}(\frac{v}{\lambda}) - \delta v$, respectively. In both cases γ is a maximal monotone graph such that $0 \in \gamma(0)$.

Since $w'' \in L^1$, (3.4) can be written as

$$\int_{\mathbb{R}} u(x)\varphi(x)dx - \lambda \int_{\mathbb{R}} w''(x)\varphi(x)dx = \int_{\mathbb{R}} f(x)\varphi(x)dx \quad \forall \varphi \in L^\infty(\mathbb{R}). \tag{3.6}$$

Since $w \in L^\infty$, we may replace φ by w in (3.6). In addition, $w' \in L^2$, so by a simple approximation argument, it follows that

$$\int_{\mathbb{R}} (u(x) - f(x))w(x)dx + \lambda \int_{\mathbb{R}} w'^2(x)dx = 0. \tag{3.7}$$

Now, we are ready to prove the following.

Lemma 3.5 *Let $f \in L^1 \bigcap L^2$ and (u, w) be a solution to (3.3). Then* $\int_{\mathbb{R}} (j(u) - j(f))(x)dx \leq -\lambda \int_{\mathbb{R}} w'^2(x)dx.$

Proof. By definition of the subdifferential and since $w(x) \in \beta(x)$ for a.e. $x \in \mathbb{R}$, we have

$$(j(u) - j(f))(x) \leq w(x)(u - f)(x) \quad \text{a.e. } x \in \mathbb{R}. \tag{3.8}$$

Again (3.7) implies the result after integrating (3.8). ■

We go on analysing the local bounded variation character of the solution u of (3.3).

If $f : \mathbb{R} \to \mathbb{R}$, for $h \in \mathbb{R}$, we define

$$f^h(x) = f(x + h) - f(x). \tag{3.9}$$

Writing $w^{h''} := (w^h)''$ we observe that

$$u^h - \lambda w^{h''} = f^h, \tag{3.10}$$

where $w(x) \in \beta(u(x))$, and $w(x + h) \in \beta(u(x + h))$ a.e.

Let $\zeta \geq 0$ be a smooth function with compact support.

Lemma 3.6 *Assume β is strictly monotone, i.e.*

$$\beta(x) \bigcap \beta(y) = \emptyset \text{ if } x \neq y. \tag{3.11}$$

Let u be a solution of (3.3)*. Then, for each $h \in \mathbb{R}$*

$$\int_{\mathbb{R}} \zeta(x)|u^h(x)|dx \leq \int_{\mathbb{R}} \zeta(x) f^h(x) sign(w^h(x))dx + c\|\zeta'''\|_\infty \lambda |h| \, \|u\|_{L^1}. \tag{3.12}$$

where c is the constant from (3.1)*.*

Proof. (3.10) gives

$$\int_{\mathbb{R}} u^h(x)\varphi(x)dx = \int_{\mathbb{R}} (\lambda w^{h''}(x) + f^h(x))\varphi(x)dx, \quad \forall \varphi \in L^\infty(\mathbb{R}). \tag{3.13}$$

We set $\varphi(x) = \text{sign}(u^h(x))\zeta(x)$. By (3.11) we have $w^h \neq 0$ on $\{u^h \neq 0\}$, dx a.e. Hence, by strict monotonicity we have $\varphi(x) = \text{sign}(w^h(x))\zeta(x)$ a.e.

on $\{u^h \neq 0\}$. By (3.10), up to a Lebesgue null set, we have $\{u_h = 0\} = \{\lambda w^{h''} + f^h = 0\}$. Hence (3.13) implies

$$\begin{aligned}\int_{\mathbb{R}} \zeta(x)|u^h(x)|dx &= \int_{\{u^h\neq 0\}} (\lambda w^{h''}(x) + f^h(x))\mathrm{sign}(w^h(x))\zeta(x)dx \\ &= \lambda \int_{\mathbb{R}} w^{h''}(x)\mathrm{sign}(w^h)(x)\zeta(x)dx \\ &+ \int_{\mathbb{R}} f^h(x)\mathrm{sign}(w^h(x))\zeta(x)dx.\end{aligned}$$

It remains to control

$$\lambda \int_{\mathbb{R}} w^{h''}(x)\mathrm{sign}(w^h(x))\zeta(x)dx. \tag{3.14}$$

Let $\varrho = \varrho_L : \mathbb{R} \to \mathbb{R}$, be an odd smooth function such that $\varrho \leq 1$ and $\varrho(x) = 1$ on $[\frac{1}{L}, \infty[$. (3.14) is the limit when L goes to infinity of

$$\begin{aligned}\lambda \int_{\mathbb{R}} w^{h''}(x)\varrho(w^h(x))\zeta(x)dx = & -\lambda \int_{\mathbb{R}} w^{h'}(x)^2\varrho'(w^h(x))\zeta(x)dx \\ - & \lambda \int_{\mathbb{R}} w^{h'}(x)\varrho(w^h(x))\zeta'(x)dx.\end{aligned}$$

Since the first integral of the right-hand side of the previous expression is positive, (3.14) is upper bounded by the lim sup when L goes to infinity of

$$\begin{aligned}-\lambda \int_{\mathbb{R}} w^{h'}(x)\varrho(w^h(x))\zeta'(x)dx &= -\lambda \int_{\mathbb{R}} (\tilde{\varrho}(w^h(x)))'\zeta'(x)dx \\ &= \lambda \int_{\mathbb{R}} \tilde{\varrho}(w^h(x))\zeta''(x)dx,\end{aligned}$$

where $\tilde{\varrho}(x) = \int_0^x \varrho(y)dy$. Since $\lim_{L\to\infty} \varrho(w^h(x)) = w^h(x)$ and $\tilde{\varrho}(x) \leq |x|$, pointwise, by Lebesgue's dominated convergece theorem, the mentioned $\limsup_{L\to\infty}$ is equal to

$$\lambda \int_{\mathbb{R}} w^h(x)\zeta''(x)dx = \lambda \int_{\mathbb{R}} \tilde{w}(x)(\zeta''(x-h) - \zeta''(x))dx.$$

Since $w \in \beta(u)$ a.e., $u \in L^1$, the previous integral is bounded by

$$\lambda \left\|\zeta'''\right\|_\infty |h| \int_{\mathbb{R}} |w(x)|dx \leq c\lambda \left\|\zeta'''\right\|_\infty |h| \int_{\mathbb{R}} |u(x)|dx, \tag{3.15}$$

with c coming from (3.1). ■

Remark 3.7 *Using similar arguments as in Section 4 below, we can show that u is locally of bounded variation whenever f is. We have not emphasized this result since we will not directly use it.*

4 Some properties of the porous media equation

Let $\beta : \mathbb{R} \to 2^{\mathbb{R}}$. Throughout this section, we assume that β satisfies Hypothesis 3.1. Our first aim is to prove Theorem 4.15 below, for which we need some preparations. Let $u_0 \in (L^1 \cap L^\infty)(\mathbb{R})$. The proposition below collects some basic results, stated and used in [9].

Proposition 4.1 *1. Let $u_0 \in (L^1 \bigcap L^\infty)(\mathbb{R})$. Then, there is a unique solution to* (1.1) *in the sense of distributions. This means that there exists a unique couple $(u, \eta_u) \in \left(L^1 \cap L^\infty\right)([0,T] \times \mathbb{R})^2$ with*

$$\int_{\mathbb{R}} u(t,x)\varphi(x)dx = \int_{\mathbb{R}} u_0(x)\varphi(x)dx + \frac{1}{2}\int_0^t \int_{\mathbb{R}} \eta_u(s,x)\varphi''(x)dx\, ds, \tag{4.1}$$

where $\varphi \in C_\circ^\infty(\mathbb{R})$. Furthermore, $t \to u(t,\cdot)$ is in $C([0,T], L^1)$ and $\eta_u(t,x) \in \beta(u(t,x))$ for $dt \otimes dx$-a.e. $(t,x) \in [0,T] \times \mathbb{R}$.

2. *We define the multivalued map $A = A_\beta : D(A) \subset E \to E$, $E = L^1(\mathbb{R})$, where $D(A)$ is the set of all $u \in L^1$ for which there is $w \in L^1_{\mathrm{loc}}(\mathbb{R})$ such that $w(x) \in \beta(u(x))$ a.e. $x \in \mathbb{R}$ and $\Delta w \in L^1(\mathbb{R})$. For $u \in D(A)$ we set*

$$Au = \{-\frac{1}{2}\Delta w | w \text{ as in the definition of } D(A)\}.$$

Then A is m-accretive on $L^1(\mathbb{R})$. Therefore there is a unique C^0-solution of the evolution problem

$$\begin{cases} 0 \in u'(t) + Au(t), \\ u(0) = u_0. \end{cases}$$

3. *The C^0-solution under 2. coincides with the solution in the sense of distributions under 1.*

4. $\|u\|_\infty \le \|u_0\|_\infty$.

5. *Let $\beta^\varepsilon(u) = \beta(u) + \varepsilon u$, $\varepsilon > 0$ and consider the solution $u^{(\varepsilon)}$ to*

$$\begin{cases} \partial_t u^{(\varepsilon)} = \frac{1}{2}\beta^\varepsilon(u^{(\varepsilon)})'' \\ u^{(\varepsilon)}(0,\cdot) = u_0. \end{cases}$$

Then $u^{(\varepsilon)} \to u$ in $C([0,T], L^1(\mathbb{R}))$ when $\varepsilon \to 0$, see [12].

Proof. Points 1. to 4. were objects of Propositions 2.8 and 3.4 in [9]. Point 5. is essentially a consequence of results [13]. In fact, given a monotone graph associated with a real non-decreasing function φ such that $0 \in \varphi(0)$, its minimal section $\varphi^\circ(x)$ coincides with the left-limit (resp. right-limit) $\varphi(x-)$ (resp. $\varphi(x+)$) for $x \geq 0$ (resp. $x \leq 0$). For the general notion of minimal section, the reader may consult [26] ch. IV.1, Definition before Theorem 1.1. Since $(\beta^\varepsilon)^\circ \to \beta^\circ$, Theorem 3 and considerations above that theorem in [12] together with points 1. and 3. in this proposition allow to conclude. ∎

Corollary 4.2 *We have $u(t,\cdot) \geq 0$ a.e. for any $t \in [0,T]$. Moreover $\int_{\mathbb{R}} u(t,x)dx = \int_{\mathbb{R}} u_0(x)dx = 1$, for any $t \geq 0$.*

Proof. In fact the functions $u^{(\varepsilon)}$ introduced in point 5. of Proposition 4.1 have the desired property. Taking the limit when ε goes to zero, the assertion follows.

∎

Remark 4.3 *Uniqueness to* (4.1) *holds even only with the assumptions β monotone, continuous at zero and $\beta(0) = 0$.*

Below we fix on an initial condition $u_0 \in L^1 \bigcap L^\infty$.

Lemma 4.4 *Let $\varepsilon > 0$. We consider an ε-solution given by*

$$\mathcal{D} = \{0 = t_0 < t_1 < \ldots < t_N = T\}$$

and

$$u^\varepsilon(t) = \begin{cases} u_0, & t = 0 \\ u_j, & t \in]t_{j-1}, t_j] \end{cases},$$

for which for $1 \leq j \leq N$

$$\begin{cases} u_j - \frac{t_j - t_{j-1}}{2} w_j'' = u_{j-1}, \\ w_j \in \beta(u_j) \quad a.e. \end{cases}$$

We set

$$\eta^\varepsilon(t,\cdot) = \begin{cases} \beta(u_0) & : t = 0, \\ w_j & : t \in]t_{j-1}, t_j]. \end{cases}$$

When $\varepsilon \to 0$, η^ε converges weakly in $L^1([0,T]\times\mathbb{R})$ to η_u, where (u,η_u) solves equation (1.1). Furthermore, for $p = 1$ or $p = \infty$,

$$\begin{aligned} \sup_{t\le T} \|u^\varepsilon(t,\cdot)\|_{L^p} &\le \|u_0\|_{L^p} \\ &\text{and} \\ \sup_{t\le T} \|\eta^\varepsilon(t,\cdot)\|_{L^p} &\le c\|u_0\|_{L^p}, \end{aligned} \tag{4.2}$$

where c is as in Hypothesis 3.1. Hence,

$$\sup_{t\le T} \|u(t,\cdot)\|_{L^p} \le \|u_0\|_{L^p}. \tag{4.3}$$

and

$$\|\eta\|_{L^r([0,T]\times\mathbb{R})} \le cT^{\frac{1}{r}} \|u_0\|_{L^\infty}^{\frac{r-!}{r}} \|u_0\|_{L^1}^{\frac{1}{r}} \tag{4.4}$$

for all $r \in [1,\infty[$.

Proof. See point 3. in the proof of the Proposition 3.2 in [9]. (4.2) follows by Lemma 3.3, 1-3. and Hypothesis 3.1 by induction. (4.3) is an immediate consequence of the first part of (4.2) and the fact that u is a C^0 solution. (4.4) follows by an elementary interpolation argument. Indeed, for $r \in [1,\infty[$, $r' := \frac{r}{r-1}$, $\varepsilon_n \to 0$ and $\varphi \in (L^r \bigcap L^\infty)([0,T] \bigcap \mathbb{R})$ we have

$$\begin{aligned} \left|\int_0^T \int_{\mathbb{R}} \varphi(t,x)\eta_u(t,x)dxdt\right| &= \lim_{n\to\infty} \left|\int_0^T \int_{\mathbb{R}} \varphi(t,x)\eta^{\varepsilon_n}(t,x)dxdt\right| \\ &\le \|\varphi\|_{L^{r'}([0,T]\times\mathbb{R})} \liminf_{n\to\infty} \left(\int_0^T \int_{\mathbb{R}} |\eta^{\varepsilon_n}(t,x)|^r dxdt\right)^{\frac{1}{r}} \\ &\le \|\varphi\|_{L^{r'}([0,T]\times\mathbb{R})} \liminf_{n\to\infty} \sup_{t\le T} \|\eta^{\varepsilon_n}(t,\cdot)\|_{L^\infty}^{\frac{r-1}{r}} T^{\frac{1}{r}} \sup_{t\le T} \|\eta^{\varepsilon_n}(t,\cdot)\|_{L^1}^{\frac{1}{r}} \\ &\le \|\varphi\|_{L^{r'}([0,T]\times\mathbb{R})} T^{\frac{1}{r}} c\|u_0\|_{L^\infty}^{\frac{r-1}{r}} \|u\|_{L^1}^{\frac{1}{r}}, \end{aligned}$$

where we used the second part of (4.2) in the last step. ■

If not mentioned otherwise, in the sequel for $N > 0$ and $\varepsilon = \frac{T}{N}$, we will consider the subdivision

$$\mathcal{D} = \{t_i = \varepsilon i,\ 0 \le i \le N\}. \tag{4.5}$$

We now discuss some properties of the solution exploiting the fact that the initial condition is square integrable.

Proposition 4.5 *Let $u_0 \in (L^1 \bigcap L^\infty)(\mathbb{R})$. Then the solution $(u, \eta_u) \in (L^1 \cap L^\infty)([0,T] \times \mathbb{R})^2$ of (1.1) has the following properties.*

a) $t \mapsto \int_{\mathbb{R}} j(u(t,x))dx$ is continuous on $[0,T]$.

b) $\eta_u(t,\cdot)$ is absolutely continuous for a.e. $t \in [0,T]$ and $\eta_u \in L^2([0,T]; H^1(\mathbb{R}))$.

c)

$$\int_{\mathbb{R}} j(u(t,x))dx + \frac{1}{2}\int_r^t \int_{\mathbb{R}} (\eta_u')^2(s,x)dxds \leq \int_{\mathbb{R}} j(u(r,x))dx \quad \forall\, 0 \leq r \leq t \leq T. \tag{4.6}$$

In particular $t \mapsto \int_{\mathbb{R}} j(u(t,x))dx$ is decreasing and

$$\int_{[0,T)} \int_{\mathbb{R}} (\eta_u')^2(s,x)dxds \leq 2\int_{\mathbb{R}} j(u_0(x))dx := C \leq \left(\mathrm{const.} \|u_0\|_{L^2}^2 < \infty\right). \tag{4.7}$$

Proof. To prove a), by (4.3) and Lebesgue dominated convergence theorem, using again that $u_0 \in (L^1 \bigcap L^\infty)(\mathbb{R}) \subset L^2$, we deduce that $t \mapsto u(t,\cdot)$ is in $C([0,T], L^2)$ since it is in $C([0,T], L^1)$ by Proposition 4.1, 1. Now let $t_n \to t$ in $[0,T]$ as $n \to \infty$, then $u(t_n,\cdot) \to u(t,\cdot)$ in L^2 as $n \to \infty$, in particular $\{u^2(t_n,\cdot) | n \in \mathbb{N}\}$ is equiintegrable, hence by (3.2) $\{j(u(t_n,\cdot)) | n \in \mathbb{N}\}$ is equiintegrable. Since j is continuous, assertion a) follows.
For $N \in N^*$, we consider the dyadic subdivision $\mathcal{D} = \mathcal{D}_N = \{t_j = jT2^{-N}, j = 0, \cdots, 2^N\}$ and we refer to the scheme considered in Lemma 4.4 corresponding to $\varepsilon = \varepsilon_N = T2^{-N}$. In the sequel of the proof, taking the limit ε to zero, of course means letting the limit when $N \to \infty$.
By Lemma 3.3 5., we have $w_i \in H^1(\mathbb{R})$, $1 \leq i \leq N$, and by Lemma 3.5

$$\int_{\mathbb{R}} j(u_i(x))dx + \frac{\varepsilon}{2}\int_{\mathbb{R}} w_i'(x)^2 dx \leq \int_{\mathbb{R}} j(u_{i-1}(x))dx \quad \forall i = 1, \ldots, N. \tag{4.8}$$

Hence for any $0 \leq l \leq m \leq N$

$$\int_{\mathbb{R}} j(u_m(x))dx + \frac{\varepsilon}{2}\sum_{i=l+1}^{m} \int_{\mathbb{R}} (w_i')(x)^2 dx \leq \int_{\mathbb{R}} j(u_l(x))dx. \tag{4.9}$$

Using the notation introduced in Lemma 4.4, for all $0 \leq r \leq t \leq T$ such that $r, t \in \mathcal{D}_N$, we obtain

$$\int_{\mathbb{R}} j(u^\varepsilon(t,x))dx + \frac{1}{2}\int_r^t \int_{\mathbb{R}} (\eta^{\varepsilon\prime})^2(s,x)dsdx \leq \int_{\mathbb{R}} j(u^\varepsilon(r,x))dx. \tag{4.10}$$

On the other hand, Lemma 4.4 and Lemma 3.3 4. imply that

$$\|u^\varepsilon(t)\|_{L^2} \leq \|u_0\|_{L^2} \quad \forall t \in [0,T]. \tag{4.11}$$

Therefore,

$$\int_0^T ds \int_{\mathbb{R}} u^\varepsilon(s,x)^2 dx \leq T \|u_0\|_{L^2}^2 . \tag{4.12}$$

Since $|\beta(u)| \leq c|u|$, (4.12) implies that

$$\sup_{\varepsilon>0} \int_0^T ds \int_{\mathbb{R}} \eta^\varepsilon(s,x)^2 dx < \infty. \tag{4.13}$$

(4.13) and (4.10) with $r = 0$ and $t = T$ say that $\eta^\varepsilon, \varepsilon > 0$, are bounded in $L^2([0,T]; H^1(\mathbb{R}))$. There is then a subsequence (ε_n) with η^{ε_n} converging weakly in $L^2([0,T]; H^1(\mathbb{R}))$ and therefore also weakly in $L^2([0,T] \times \mathbb{R})$ to some ξ. According to Lemma 4.4 and the uniqueness of the limit, it follows $\eta_u = \xi$ and so $\eta_u \in L^2([0,T]; H^1(\mathbb{R}))$, which implies b).
We go on with the proof of c). We remark that for any $0 \leq r \leq t \leq T$,

$$\int_r^t ds \int_{\mathbb{R}} (\eta_u')^2(s,x)dx \leq \liminf_{\varepsilon \to 0} \int_r^t ds \int_{\mathbb{R}} (\eta^{\varepsilon\prime})^2(s,x)dx. \tag{4.14}$$

In fact the sequence $(\eta^{\varepsilon\prime})$ is weakly relatively compact in $L^2([r,t] \times \mathbb{R})$. It follows by (3.2) and (4.11) that $j(u^\varepsilon(t)), \varepsilon > 0$, are uniformly integrable for each $t \in [0,T]$. Since j is continuous, and $u^\varepsilon(t,\cdot) \to u(t,\cdot)$ in $L^1(\mathbb{R})$ for each $t \in [0,T]$, it follows that $j(u^\varepsilon(t,\cdot)) \to j(u(t,\cdot))$ as $\varepsilon \to 0$ in $L^1(\mathbb{R})$ for each $t \in [0,T]$. Since the dyadic set $\bigcup_N \mathcal{D}_N$ is dense in $[0,T]$ and the three integral members of (4.6) are continuous in t because of previous points a), b), it will be enough to show (4.6) for $r, t \in \mathcal{D}_{N_0}$ for every positive integer N_0. We observe that (4.10) holds for every $N > N_0$ since $\mathcal{D}_{N_0} \subset \mathcal{D}_N$. Taking the lim inf when $\varepsilon \to 0$ (i.e. $N \to \infty$), (4.14) and (4.10) imply that

$$\int_{\mathbb{R}} j(u(t,x))dx + \frac{1}{2}\int_r^t \int_{\mathbb{R}} (\eta_u')^2(s,x)dxds \leq \int_{\mathbb{R}} j(u(r,x))dx, \tag{4.15}$$

for every $0 \leq r \leq t \leq T$, which is inequality b).

∎

Corollary 4.6

$$\int_{\mathbb{R}} u^2(t,x)dx \le \|u_0\|_{L^2}^2 \,, \forall t \in [0,T]. \qquad (4.16)$$

Proof. The result follows by Fatou's lemma, from (4.11). ■

Inequality (4.15) will be shown in Theorem 4.15 below to be indeed an equality.

Remark 4.7 *According to Proposition 4.1 1., for every $\varphi \in C_0^\infty(\mathbb{R})$, we have*

$$\int_{\mathbb{R}} u(t,x)\varphi(x)dx - \int_{\mathbb{R}} u_0(x)\varphi(x)dx = \frac{1}{2}\int_0^t \int_{\mathbb{R}} \eta_u(s,x)\varphi''(x)dxds. \qquad (4.17)$$

Since $s \mapsto \eta_u(s,\cdot)$ belongs to $L^2([0,T];H^1(\mathbb{R}))$, by Proposition 4.5 b), we have $s \mapsto \eta_u''(s,\cdot) \in L^2([0,T];H^{-1}(\mathbb{R}))$. This, together with (4.17), imply that $t \mapsto u(t,\cdot)$ is absolutely continuous from $[0,T] \to H^{-1}(\mathbb{R})$. So, in $H^{-1}(\mathbb{R})$ we have

$$\frac{d}{dt}u(t,\cdot) = \frac{1}{2}\eta_u''(t,\cdot) \quad t \in [0,T] \quad a.e.. \qquad (4.18)$$

Before proving that (4.15) is in fact an equality, we need to improve the upper bound established in (4.7).

Proposition 4.8 *In addition to Hypothesis 3.1, we suppose that*

$$\beta(\mathbb{R}) = \mathbb{R}. \qquad (4.19)$$

Then there is a constant $C > 0$ such that

$$\int_{\mathbb{R}} \eta_u'(t,x)^2 dx \le C \quad \text{for a.e.} t \in [0,T]. \qquad (4.20)$$

This proposition will be important to prove that the real function $t \mapsto \int_{\mathbb{R}} j(u(t,x))dx$ is absolutely continuous.

Proof. We equip $H = H^{-1}(\mathbb{R})$ with the inner product $\langle\cdot,\cdot\rangle_{-1,\delta}$ where $\delta \in]0,1]$ and

$$\langle u,v\rangle_{-1,\delta} = \langle(\delta - \frac{1}{2}\Delta)^{-\frac{1}{2}}u, (\delta - \frac{1}{2}\Delta)^{-\frac{1}{2}}v\rangle_{L^2(\mathbb{R})},$$

and corresponding norm $\|\cdot\|_{-1,\delta}$. We define $\Gamma : H \to [0, \infty]$ by

$$\Gamma(u) = \begin{cases} \int_{\mathbb{R}} j(u(x))dx, & \text{if } u \in L^1_{\text{loc}} \\ +\infty, & \text{otherwise.} \end{cases}$$

and $\mathcal{D}(\Gamma) = \{u \in H | \, \Gamma(u) < \infty\}$. We also consider

$$D(A_\delta) = \{u \in \mathcal{D}(\Gamma) | \, \exists \, \eta_u \in H^1, \eta_u \in \beta(u) \text{ a.e.}\}.$$

For $u \in D(A_\delta)$, we set $A_\delta u = \{(\delta - \frac{1}{2}\Delta)\eta_u | \eta_u$ as in the definition of $D(A_\delta)\}$. Obviously, Γ is convex since j is convex, and Γ is proper since $\mathcal{D}(\Gamma)$ is non-empty and even dense in H^{-1}, because $L^2(\mathbb{R}) \subset \mathcal{D}(\Gamma)$. The rest of the proof will be done in a series of lemmas.

Lemma 4.9 *The function Γ is lower semicontinuous.*

Proof. First of all we observe that Γ is lower semicontinuous on $L^1_{\text{loc}}(\mathbb{R})$. In fact, defining $\Gamma_N, N \in \mathbb{N}$, analogously to Γ, with $j \wedge N$ replacing j, by the continuity of j and Lebesgue's dominated converegence theorem, Γ_N is continuous in L^1_{loc}. Since $\Gamma = \sup_{N \in \mathbb{N}} \Gamma_N$, it follows that Γ is lower continuous on L^1_{loc}. Let us suppose now that $u_n \to u$ in $H^{-1}(\mathbb{R})$. We have to prove that

$$\int_{\mathbb{R}} j(u(x))dx \leq \liminf_{n \to \infty} \int_{\mathbb{R}} j(u_n(x))dx. \tag{4.21}$$

Let us consider a subsequence such that $\int j(u_n(x))dx$ converges to the right-hand side of (4.21) denoted by C. We may suppose $C < \infty$. According to (4.19), we have

$$\lim_{R \to \infty} \frac{j(R)}{R} = \infty,$$

which implies that the sequence $(u_n)_{n \in \mathbb{N}}$ is uniformly integrable on $[-K, K]$ for each $K > 0$. Hence, by Dunford-Pettis theorem, the sequence (u_n) is weakly relatively compact in L^1_{loc}. Therefore, there is a subsequence (n_l) such that (u_{n_l}) converges weakly in L^1_{loc}, necessarily to u, since $u_n \to u$ strongly, hence also weakly in $H^{-1}(\mathbb{R})$. Since Γ is convex and lower semicontinuous on L^1_{loc}, it is also weakly lower semicontinuous on L^1_{loc}, see [15] p.62, 22.1. This implies that

$$\int j(u(x))dx \leq \liminf_{l \to \infty} \int j(u_{n_l}(x))dx = C.$$

Finally, (4.21) and thus the assertion of Lemma 4.9 is proved. ∎

An important intermediate step is the following.

Lemma 4.10 *$\mathcal{D}(\Gamma) = D(A_\delta)$ and $\partial_H\Gamma(u) = A_\delta u$, $\forall u \in \mathcal{D}(\Gamma)$. In particular $D(A_\delta)$ is dense in H^{-1}.*
We observe, that ∂_H depends in fact on δ since the inner product on H^{-1} depends on δ.

Proof. Let $u \in \mathcal{D}(\Gamma)$, $h \in L^2(\subset \mathcal{D}(\Gamma))$. For $z \in \partial_H\Gamma(u)$ we have

$$\Gamma(u+h) - \Gamma(u) \geq \langle z, h\rangle_{-1,\delta} = \int_{\mathbb{R}} v(x)h(x)dx, \tag{4.22}$$

where $v = (\delta - \frac{1}{2}\Delta)^{-1}z$. Clearly $v \in H^1$. By (4.22) it follows that $v \in \partial_{L^2}\tilde{\Gamma}(u)$ where $\tilde{\Gamma}$ is the restriction of Γ to $L^2(\mathbb{R})$. By Example 2B of Chapter IV.2 in [26], this yields that $v \in \beta(u)$ a.e. Consequently, $\mathcal{D}(\Gamma) = D(A_\delta)$ and $\partial_H\Gamma(u) \subset A_\delta u, \forall u \in \mathcal{D}(\Gamma)$.

It remains to prove that $A_\delta(u) \subset \partial_H\Gamma(u), \forall u \in \mathcal{D}(\Gamma)$. Let $u \in \mathcal{D}(\Gamma)$, $h \in L^2$, $\eta_u \in \beta(u)$ a.e. with $\eta_u \in H^1$. Since

$$j(u+h) - j(u) \geq \eta_u h \quad \text{a.e.},$$

it follows

$$\Gamma(u+h) - \Gamma(u) \geq \int_{\mathbb{R}} \eta_u(x)h(x)dx = \langle(\delta - \frac{1}{2}\Delta)\eta_u, h\rangle_{-1,\delta}. \tag{4.23}$$

It remains to show that (4.23) holds for any $h \in H^{-1}$ such that $u+h \in \mathcal{D}(\Gamma)$. Then we have $u+h$, $u \in L^1_{\text{loc}}$ and $j(u), j(u+h) \in L^1$. We first prove that (4.23) holds if $h \in L^1(\subset H^{-1})$. We truncate h setting

$$h_n = 1_{\{|h|\leq n\}}h, n \in \mathbb{N},$$

so that $h_n \in L^2(\mathbb{R})$. Now

$$j(u+h_n)(x) = \begin{cases} j(u(x)+h(x)) & \text{if } |h(x)| \leq n, \\ j(u(x)) & \text{if } |h(x)| > n, \end{cases}$$

and it is dominated by

$$j(u+h) + j(u) \in L^1.$$

We have

$$\int_{\mathbb{R}} j(u+h_n)(x)dx \geq \int_{\mathbb{R}} j(u(x))dx + \langle(\delta - \frac{1}{2}\Delta)\eta_u, h_n\rangle_{-1,\delta}. \tag{4.24}$$

Since $h_n \to h$ in L^1 (and so in H^{-1}), using Lebesgue's dominated convergence theorem, (4.23) follows for $h \in L^1$.

Let $M > 0$ and consider a smooth function $\chi : \mathbb{R} \to [0,1]$ such that $\chi(r) = 1$ for $0 \leq |r| \leq 1$, $\chi(r) = 0$ for $2 \leq |r| < \infty$. We define

$$\chi_M(x) = \chi\left(\frac{x}{M}\right), \quad x \in \mathbb{R}.$$

Then

$$\chi_M(x) = \begin{cases} 1 : & |x| \leq M \\ 0 : & |x| \geq 2M. \end{cases}$$

Since $h\chi_M \in L^1$, we have

$$\int_{\mathbb{R}} (j(u+h\chi_M)(x) - j(u)(x))dx \geq \langle(\delta - \frac{1}{2}\Delta)\eta_u, h\chi_M\rangle_{-1,\delta}. \tag{4.25}$$

Since j is convex and non-negative, we have

$$\begin{aligned} j(u+h\chi_M) &= j((1-\chi_M)u + \chi_M(u+h)) \\ &\leq (1-\chi_M)j(u) + \chi_M j(u+h) \leq j(u) + j(u+h). \end{aligned}$$

Hence Lebesgue's domintated convergence theorem allows to take the limit in the left-hand side, when $M \to \infty$ of (4.25) to obtain

$$\int_{\mathbb{R}} (j(u+h)(x)) - j(u(x))dx.$$

The right-hand side of (4.25) converges to $\langle(\delta - \frac{1}{2}\Delta)\eta_u, h\rangle_{H^{-1}}$ because of the next lemma. Hence, the assertion of Lemma 4.10 follows. ■

Lemma 4.11 *Define $h_M := \chi_M h$ in $H^{-1}, M > 0$. Then*

$$\lim_{M\to\infty} h_M = h \quad \text{weakly } in\ H^{-1}.$$

Proof (of Lemma 4.11). Let us first show that the sequence (h_M) is bounded in H^{-1}.

In fact, given $\varphi \in H^1$,

$$\begin{aligned}\left|\int_{\mathbb{R}} h_M(x)\varphi(x)dx\right| &= \left|\int_{\mathbb{R}} h(x)\chi_M(x)\varphi(x)dx\right| \\ &\leq \ \|h\|_{H^{-1}} \left(\|\chi_M\varphi\|_{L^2} + \left\|\chi'_M\varphi + \varphi'\chi_M\right\|_{L^2}\right) \\ &\leq \|h\|_{H^{-1}} \left(\|\varphi\|_{L^2} + \left\|\varphi'\right\|_{L^2} + \frac{\|\chi'\|_\infty}{M}\|\varphi\|_{L^2}\right) \\ &\leq \text{const } \ \|h\|_{H^{-1}}\|\varphi\|_{H^1}\end{aligned}$$

for some positive constant independent of M.

Hence there is a subsequence weakly converging to some $k \in H^{-1}$. Since

$$\int_{\mathbb{R}} h_M(x)\varphi(x)dx \to_{M\to\infty} \int_{\mathbb{R}} h(x)\varphi(x)dx$$

for any $\varphi \in C_0^\infty(\mathbb{R})$, k must be equal to h. Now the assertion of Lemma 4.11 follows. ■

By Corollary IV 1.2 in [26], we know that A_δ is maximal monotone on H^{-1} and therefore m-accretive with domain $D(A_\delta) = \mathcal{D}(\Gamma)$.

We go on with the proof of Proposition 4.8. Since our initial condition u_0 belongs to $L^1 \cap L^\infty$ and $L^2 \subset \mathcal{D}(\Gamma)$, clearly $u_0 \in \mathcal{D}(A_\delta)$. According to Komura-Kato theorem, see [26, Proposition IV.3.1], there exists a (strong) solution $u = u_\delta : [0,T] \to E = H^{-1}$ of

$$\begin{cases} \frac{du}{dt} + A_\delta u \ni 0, \quad t \in [0,T] \\ u(0,\cdot) = u_0, \end{cases} \tag{4.26}$$

which is Lipschitz. In particular, for almost all $t \in [0,T]$, $u_\delta(t,\cdot) \in \mathcal{D}(A_\delta)$ and there is $\xi_\delta(t,\cdot) \in H^1$ such that $\xi_\delta(t,\cdot) \in \beta(u_\delta(t,\cdot))$ a.e., $t \mapsto (\delta\xi_\delta - \frac{1}{2}\Delta\xi_\delta)(t,\cdot) \in H^{-1}$ is measurable and

$$u_\delta(t,\cdot) = u_0 - \int_0^t \left(\delta\xi_\delta - \frac{1}{2}\Delta\xi_\delta\right)(s,\cdot)ds \tag{4.27}$$

in H^{-1}.

Furthermore, for the right-derivative $D^+u_\delta(t)$, we have

$$D^+u_\delta(t,\cdot)) + (A_\delta)^\circ u_\delta(t,\cdot) = 0 \quad \text{in } H^{-1}, \forall t \in [0,T], \tag{4.28}$$

where $(A_\delta)^\circ$ denotes the minimal section of A_δ and the map $t \mapsto \|(A_\delta)^\circ u_\delta(t,\cdot)\|_{-1,\delta}$ is decreasing. On the other hand (4.27) implies that

$$\frac{du_\delta}{dt}(t,\cdot) + \delta\xi_\delta(t,\cdot) - \frac{1}{2}(\xi_\delta)''(t,\cdot) = 0 \quad \text{for a.e. } t \in [0,T]. \tag{4.29}$$

Consequently, for almost all $t \in [0,T]$

$$\left\|\delta\xi_\delta(t,\cdot) - \frac{1}{2}(\xi_\delta)''(t,\cdot)\right\|_{-1,\delta} = \|(A_\delta)^\circ u_\delta(t,\cdot)\|_{-1,\delta} \leq \|(A_\delta)^\circ u_0\|_{-1,\delta}\,, \tag{4.30}$$

i.e. setting $\xi_0 = (\delta - \frac{1}{2}\Delta)^{-1}(A_\delta)^\circ u_0$, we observe that it belongs to H^1 and that

$$_{H^{-1}}\langle(\delta - \frac{1}{2}\Delta)\xi_\delta(t,\cdot), \xi_\delta(t,\cdot)\rangle_{H^1} \leq \; _{H^{-1}}\langle(\delta - \frac{1}{2}\Delta)\xi_0, \xi_0\rangle_{H^1},$$

for a.e. $t \in [0,T]$.

Consequently, for a.e. $t \in [0,T]$,

$$\begin{aligned}\int_{\mathbb{R}}(\delta\xi_\delta(t,x)^2 &+ \frac{1}{2}\xi_\delta'(t,x)^2)dx \\ &\leq \delta\int_{\mathbb{R}}\xi_0^2(x)dx + \int_{\mathbb{R}}{\xi_0'}^2(x)dx \\ &\leq \|\xi_0\|_{H^1} =: C\end{aligned} \tag{4.31}$$

since $\delta \leq 1$.

We now consider equation (4.26) from an L^1 perspective, similarly as for equation (1.1), see Proposition 4.1 2. Since our initial condition u_0 belongs to $(L^1 \cap L^\infty)(\mathbb{R})$, equation (4.26) can also be considered as an evolution problem on the Banach space $E = L^1(\mathbb{R})$. More precisely define

$$D(\tilde{A}_\delta) := \{u \in L^1(\mathbb{R}) | \exists w \in L^1_{\text{loc}} : w \in \beta(u) \text{ a.e. and } (\delta - \frac{1}{2}\Delta)w \in L^1(\mathbb{R})\}$$

and for $u \in D(\tilde{A}_\delta)$,

$$\tilde{A}_\delta u := \{(\delta - \frac{1}{2}\Delta)w | w \text{ as in } D(\tilde{A}_\delta)\}.$$

Note that for w as in the definition of $D\tilde{(A_\delta)}$, we have $(\delta - \frac{1}{2}\Delta)w \in H^{-1}$, since $L^1(\mathbb{R}) \subset H^{-1}$. Therefore, $w \in H^1$, hence

$$D(\tilde{A}_\delta) \subset D(A_\delta) \text{ and } \tilde{A}_\delta = A_\delta \text{ on } D(\tilde{A}_\delta). \tag{4.32}$$

Furthermore, as indicated in Section 3, it is possible to show that $\tilde{A}_\delta$ is an m-accretive operator on L^1.

For $\lambda > 0$, the following four points are then a consequence of Remark 3.4 and Lemma 3.3.

1. For each $f \in L^1(\mathbb{R})$ there is $u \in L^1$, $w \in L^1$ with $w \in \beta(u)$ a.e. and
$$u + \lambda(\delta w - \frac{1}{2}w'') = f. \tag{4.33}$$

2. The map
$$f \mapsto u := (I + \lambda\tilde{A}_\delta)^{-1}(f) \text{ is a contraction on } L^1. \tag{4.34}$$

3. $\overline{D(\tilde{A}_\delta)} = L^1$.

4. We recall that whenever $f \in L^\infty$, then $u \in L^\infty$ and
$$\|u\|_\infty \leq \|f\|_\infty . \tag{4.35}$$

Therefore, there is a C^0-solution $\tilde{u} : [0,T] \times \mathbb{R} \to \mathbb{R}$ of (4.26). Since by (4.32), every ε-solution of (4.26) in $L^1(\mathbb{R})$ is also an ε-solution of (4.26) in H^{-1} and $L^1 \subset H^{-1}$ continuously, $\tilde{u}$ is also a C^0-solution of (4.26) in H^{-1}. Since, by Proposition IV 8.2 and 8.7 of [26], the solution above is the unique C^0-solution of (4.26) in H^{-1}, we have proved the first part of the following lemma.

Lemma 4.12 *The solution $\tilde{u}$ coincides with the H^{-1}-valued solution u_δ. Moreover, for $p = 1$ or $p = \infty$ and c as in Hypothesis 3.1*

$$\sup_{t\leq T} \|u_\delta(t,\cdot)\|_{L^p} \leq \|u_0\|_{L^p} \text{ and } \operatorname{ess\,sup}_{t\leq T} \|\xi_\delta(t,\cdot)\|_{L^p} \leq c\,\|u_0\|_{L^p} \tag{4.36}$$

Proof.

It remains to show (4.36). As in the proof of (4.2) by (4.34), (4.35) and induction, we easily obtain that for any ε-solution in L^1 and $p = 1$ or $p = \infty$,

$$\sup_{t\leq T} \|u^\varepsilon(t,\cdot)\|_{L^p} \leq \|u_0\|_{L^p} .$$

The conclusion follows because for every $t \in [0,T]$, there is a sequence (ε_n) such that $u^{\varepsilon_n}(t,\cdot) \to \tilde{u}(t,\cdot) = u_\delta(t,\cdot)$ a.e. as $n \to \infty$. The second part of (4.36) then obviously follows by Hypothesis 3.1, since $\xi_\delta(t,\cdot) \in \beta(u_\delta(t,\cdot))$ a.e. for a.e. $t \in [0,T]$.

■

Lemma 4.13 *We have $u_\delta \to u$ in $C([0,T]; L^1(\mathbb{R}))$ as $\delta \to 0$, where u is the solution to* (1.1).

Proof. It will be enough to prove that for δ small enough, we have

$$\sup_{t \le T} \int_{\mathbb{R}} |u_\delta(t,x) - u(t,x)| dx \le cT \|u_0\|_{L^1} \delta. \tag{4.37}$$

Using point 5. of Proposition 4.1 in a slightly modified form, and approximating β by $\beta^\varepsilon(u) = \beta(u) + \varepsilon u$, it is enough to suppose that β is strictly monotone, i.e. (3.11) holds. In the lines below the parameter ε will play however a different role.
We need to go back to the L^1-ε-solutions related to u_δ and u.

For $\varepsilon > 0$ we consider a subdivision $0 = t_0^\varepsilon < \ldots < t_j^\varepsilon < \ldots < t_N^\varepsilon = T$ such that $t_j^\varepsilon - t_{j-1}^\varepsilon < \varepsilon$, $j = 1, \ldots, N$. Similarly as in Lemma 4.4

$$\begin{aligned} u_\delta^\varepsilon(t_j,\cdot) &= u_\delta^\varepsilon(t_{j-1},\cdot) \\ &+ (t_j - t_{j-1})\frac{(\eta_\delta^\varepsilon)''}{2}(t_j,\cdot) \\ &- (t_j - t_{j-1})\delta\eta_\delta^\varepsilon(t_j,\cdot) \end{aligned} \tag{4.38}$$

and

$$u^\varepsilon(t_j,\cdot) = u^\varepsilon(t_{j-1},\cdot) + (t_j - t_{j-1})\frac{1}{2}(\eta^\varepsilon)''(t_j,\cdot) \tag{4.39}$$

with $\eta_\delta^\varepsilon \in \beta(u_\delta^\varepsilon)$, $\eta^\varepsilon \in \beta(u^\varepsilon)$ a.e.. Taking the difference of the previous two equations we obtain

$$\begin{aligned} u_\delta^\varepsilon(t_j,\cdot) - u^\varepsilon(t_j,\cdot) &= u_\delta^\varepsilon(t_{j-1},\cdot) - u^\varepsilon(t_{j-1},\cdot) \\ &+ \left(\frac{t_j - t_{j-1}}{2}\right)(\eta_\delta^\varepsilon - \eta^\varepsilon)''(t_j,\cdot) - \delta(t_j - t_{j-1})\eta_\delta^\varepsilon(t_j,\cdot). \end{aligned} \tag{4.40}$$

Let $\Psi_\kappa : \mathbb{R} \to [-1,1]$ be an odd smooth increasing function such that $\Psi_\kappa(x) \to \text{sign}\, x$ as $\kappa \to 0$ pointwise, We integrate (4.40) against $\Psi_\kappa(\eta^\varepsilon_\delta(t_j,\cdot) - \eta^\varepsilon(t_j,\cdot))$ and we get

$$\begin{aligned}
&\int_{\mathbb{R}} (u^\varepsilon_\delta(t_j,x) - u^\varepsilon(t_j,x))\Psi_\kappa(\eta^\varepsilon_\delta(t_j,x) - \eta^\varepsilon(t_j,x))dx \\
= &\int_{\mathbb{R}} (u^\varepsilon_\delta(t_{j-1},x) - u^\varepsilon(t_{j-1},x))\Psi_\kappa(\eta^\varepsilon_\delta(t_j,x) - \eta^\varepsilon(t_j,x))dx \\
&- \frac{(t_j - t_{j-1})}{2} \int_{\mathbb{R}} (\eta^\varepsilon_\delta - \eta^\varepsilon)'(t_j,x)^2 \Psi'_\kappa(\eta^\varepsilon_\delta(t_j,x) - \eta^\varepsilon(t_j,x))dx \\
&- \delta(t_j - t_{j-1}) \int_{\mathbb{R}} (\eta^\varepsilon_\delta(t_j,x)\Psi_\kappa(\eta^\varepsilon_\delta(t_j,x) - \eta^\varepsilon(t_j,x))dx.
\end{aligned}$$

Using the fact that $\Psi'_\kappa \geq 0$, $|\Psi_\kappa| \leq 1$, that, by strict monotonicity of β

$$\text{sign}(\eta^\varepsilon_\delta(t_j,\cdot) - \eta^\varepsilon(t_j,\cdot)) = \text{sign}(u^\varepsilon_\delta(t_j,\cdot) - u^\varepsilon(t_j,\cdot)),$$

a.e. on $\{u^\varepsilon_\delta(t_j,\cdot) \neq u^\varepsilon(t_j,\cdot)\}$, and letting $\kappa \to 0$, by (4.2), we obtain

$$\begin{aligned}
&\int_{\mathbb{R}} |u^\varepsilon_\delta(t_j,x) - u^\varepsilon(t_j,x)|dx \\
\leq &\int_{\mathbb{R}} |u^\varepsilon_\delta(t_{j-1},x) - u^\varepsilon(t_{j-1},x)|dx + c\delta(t_j - t_{j-1}) \|u_0\|_{L^1}.
\end{aligned} \tag{4.41}$$

Since $\int_{\mathbb{R}} |u^\varepsilon_\delta(0,x) - u^\varepsilon(0,x)|dx = 0$, an induction argument implies that

$$\int_{\mathbb{R}} |u^\varepsilon_\delta(t_j,x) - u^\varepsilon(t_j,x)|dx \leq cT \|u_0\|_{L^1} \cdot \delta$$

for every $j \in \{0,\dots,N\}$. Consequently, for any $t \in [0,T]$

$$\int_{\mathbb{R}} |u^\varepsilon_\delta(t,x) - u^\varepsilon(t,x)|dx \leq cT \|u_0\|_{L^1} \delta.$$

Letting $\varepsilon \to 0$, (4.37) follows and Lemma 4.13 is proved. ■

By (4.27), for every $\alpha \in C_0^\infty(\mathbb{R})$ and all $t \in [0,T]$,

$$\begin{aligned}
\int_{\mathbb{R}} u_\delta(t,x)\alpha(x)dx = &\int_{\mathbb{R}} u_0(x)\alpha(x)dx \\
&- \delta \int_0^t ds \int_{\mathbb{R}} \xi_\delta(s,x)\alpha(x)dx + \frac{1}{2}\int_0^t ds \int_{\mathbb{R}} dx \xi_\delta(s,x)\alpha''(x),
\end{aligned}$$

$\xi_\delta \in \beta(u_\delta)$ a.e. Letting $\delta \to 0$, by (4.36) and Lemma 4.13, we obtain that

$$\int_{\mathbb{R}} u(t,x)\alpha(x)dx = \int_{\mathbb{R}} u_0(x)\alpha(x)dx + \frac{1}{2}\lim_{\delta\to 0}\int_0^t ds \int_{\mathbb{R}} \xi_\delta(s,x)\alpha''(x)dx. \tag{4.42}$$

By (4.36) it follows that for each $K > 0$, $u_\delta \to u$ in $L^2([0,T] \times [-K,K])$ and that (ξ_δ), is bounded in $L^2([0,T] \times [-K,K])$. Since, by [26] Example IV.2C, the map $u \mapsto \beta(u)$ is m-accretive on $L^2([0,T] \times [-K,K])$, it is weakly-strongly closed, see [6], p.37 Proposition 1.1 (i) and (ii). So, there is a sequence (δ_n) such that $\xi_{\delta_n} \to \xi$ weakly in $L^2([0,T] \times [-K,K])$ for some $\xi \in \beta(u)$ a.e. Hence, (4.42) implies

$$\int_{\mathbb{R}} u(t,x)\alpha(x)dx = \int_{\mathbb{R}} u_0(x)\alpha(x)dx + \frac{1}{2}\int_0^t ds \int_{\mathbb{R}} \xi(s,x)\alpha''(x)dx. \quad (4.43)$$

By the uniqueness part of Proposition 4.1 1., we conclude that $\xi \equiv \eta_u$.

By Proposition 4.5, we already knew that $\eta_u(t,\cdot) \in H^1(\mathbb{R})$ for almost any t. By (4.31) for a.e. fixed t, there is a sequence (δ_n) such that $(\xi_{\delta_n})(t,\cdot)$ weakly converges to some $\tilde{\xi}(t,\cdot)$ in $H^1(\mathbb{R})$ hence in $L^2(\mathbb{R})$.

Consequently $\tilde{\xi}(t,\cdot) = \eta_u(t,\cdot)$ for almost all $t \in [0,T]$.

Recalling (4.31) for a.e. $t \in [0,T]$ we get

$$\int_{\mathbb{R}} dx\ \eta_u'(t,x)^2 = \int_{\mathbb{R}} dx\ \tilde{\xi}'(t,x)^2 \leq \liminf_{\delta \to 0} \int_{\mathbb{R}} dx\ (\xi_\delta)'(t,x)^2 \leq C.$$

This finally completes the proof of Proposition 4.8. ■

At this point, we can state and prove the following important theorem.

Theorem 4.14 *Assume that Hypothesis 3.1 and condition* (4.19) *hold. Let u be the solution of* (1.1) *(or equivalently of* (4.1)*, from Proposition 4.1). Then the function $t \mapsto \int_{\mathbb{R}} j(u(t,x))dx$ is absolutely continuous.*

Proof. Let $0 \leq s < t \leq T$. Let Γ and $\mathcal{D}(\Gamma)$ be as defined in the proof of Proposition 4.8. Since $u(t,\cdot) \in \mathcal{D}(\Gamma)$ for a.e. $t \in [0,T]$, Lemma 4.10 applies and thus for a.e. $t, s \in [0,T]$ we have that $(\delta - \frac{1}{2}\Delta)\eta_u(t,\cdot) \in A_\delta u(t,\cdot)$, and

$$\begin{aligned}
|\Gamma(u(t,\cdot)) - \Gamma(u(s,\cdot))| &\leq \max_{r \in \{t,s\}} |< (\delta - \frac{1}{2}\Delta)\eta_u(r,\cdot), u(t,\cdot) - u(s,\cdot) >_{-1,\delta}| \\
&\leq \max_{r \in \{t,s\}} \|(\delta - \frac{1}{2}\Delta)\eta_u(r,\cdot)\|_{-1,\delta} \|u(t,\cdot) - u(s,\cdot)\|_{-1,\delta} \\
&\leq \left(\text{ess sup}_{r \in [0,T]} \sqrt{\delta \int_{\mathbb{R}} \eta_u(r,x)^2 dx + \frac{1}{2}\int_{\mathbb{R}} \eta_u'(r,x)^2 dx} \right) \\
&\quad \|u(t,\cdot) - u(s,\cdot)\|_{-1,\delta}.
\end{aligned}$$

By (4.20) and (3.1), this is bounded by

$$\max(c, C)\sqrt{\delta\|u_0\|_{L^2}^2 + 1}\|u(t,\cdot) - u(s,\cdot)\|_{-1,\delta},$$

where we recall that by Remark 4.7 the map $t \mapsto u(t,\cdot)$ is absolutely continuous in H^{-1}. Since by Proposition 4.5 a), $t \mapsto \Gamma(u(t,\cdot))$ is continuous, we have

$$|\Gamma(u(t,\cdot)) - \Gamma(u(s,\cdot))| \leq \text{const}\|u(t,\cdot) - u(s,\cdot)\|_{-1,\delta}, \ \forall t, s \in [0,T],$$

and the assertion follows. ■

We are now prepared to prove the first main result of this section, which will be used in the next section in a crucial way.

Theorem 4.15 *Under Assumption* (4.19)*, the unique solution to* (1.1) *verifies*

$$\int_{\mathbb{R}} j(u(t,x))dx = \int_{\mathbb{R}} j(u(r,x))dx - \frac{1}{2}\int_r^t ds \int_{\mathbb{R}} {\eta_u'}^2(s,x)dx \tag{4.44}$$

for every $0 \leq r \leq t \leq T$.

Proof. For a.e. $t \in [0,T]$, (4.18) gives

$$\langle \frac{d}{dt}u(t,\cdot), \varphi\rangle = \frac{1}{2}\langle \eta_u(t,\cdot), \varphi''\rangle, \quad \forall \varphi \in C_0^\infty(\mathbb{R}).$$

By density arguments,

$${}_{H^{-1}}\langle \frac{d}{dt}u(t,\cdot), \psi\rangle_{H^1} = -\frac{1}{2}\int_{\mathbb{R}} \eta_u'(t,x)\psi'(x)dx$$

for every $\psi \in H^1(\mathbb{R})$. For $\psi = \eta_u(t,\cdot)$, we get

$${}_{H^{-1}}\langle \frac{d}{dt}u(t,\cdot), \eta_u(t,\cdot)\rangle_{H^1} = -\frac{1}{2}\int_{\mathbb{R}} {\eta_u'}^2(t,x)dx. \tag{4.45}$$

Since $u \in (L^1 \bigcap L^\infty)([0,T] \times \mathbb{R})$ and $|j(u)| \leq c|u|^2$, then, in particular, it belongs to $L^2([0,T], L^2(\mathbb{R}))$. We need the following lemma.

Lemma 4.16 *For a.e.* $t \in [0,T]$

$${}_{H^{-1}}\langle \frac{d}{dt}u(t,\cdot), \eta_u(t,\cdot)\rangle_{H^1} = \frac{d}{dt}\int_{\mathbb{R}} j(u(t,x))dx. \tag{4.46}$$

Proof. Let $t \in]0,T]$ such that

$$\frac{u(t+h,\cdot)-u(t,\cdot)}{h} \overset{h\to 0}{\longrightarrow} \frac{d}{dt}u(t,\cdot) \quad \text{in } H^{-1}(\mathbb{R}).$$

Let $h>0$ such $t-h, t+h$ are both positive. We have by (4.23)

$$\int_{\mathbb{R}} \frac{j(u(t,x))-j(u(t-h,x))}{h}dx \le \langle \frac{u(t,\cdot)-u(t-h,\cdot)}{h}, \eta_u(t,\cdot)\rangle_{L^2}.$$

Taking $\limsup$ for $h\to 0$, we get

$$\limsup_{h\to 0} \int_{\mathbb{R}} \frac{j(u(t,x))-j(u(t-h,x))}{h}dx \le_{H^{-1}} \langle \frac{d}{dt}u(t,\cdot), \eta_u(t,\cdot)\rangle_{H^1}. \tag{4.47}$$

On the other hand

$$\langle \frac{u(t+h,\cdot)-u(t,\cdot)}{h}, \eta_u(t,\cdot)\rangle_{L^2} \le \int_{\mathbb{R}} \frac{j(u(t+h,x))-j(u(t,x))}{h}dx.$$

So

$$_{H^{-1}}\langle \frac{d}{dt}u(t,\cdot), \eta_u(t,\cdot)\rangle_{H^1} \le \liminf_{h\to 0} \int_{\mathbb{R}} \frac{j(u(t+h,x))-j(u(t,x))}{h}dx.$$

Consequently for a.e. $t\in[0,T]$,

$$\begin{aligned}&\limsup_{h\to 0} \int_{\mathbb{R}} \frac{j(u(t,x))-j(u(t-h,x))}{h}dx \le_{H^{-1}} \langle \frac{du}{dt}(t,\cdot), \eta_u(t,\cdot)\rangle_{H^1}\\ &\le \liminf_{h\to 0} \int_{\mathbb{R}} \frac{j(u(t+h,x))-j(u(t,x))}{h}dx.\end{aligned} \tag{4.48}$$

On the other hand we know already by Theorem 4.14 that for a.e. $t\in[0,T]$, the $\limsup$ and $\liminf$-terms in (4.48) coincide. Hence the assertion follows.

■

At this point (4.45) and Lemma 4.16 imply that for a.e. $t\in[0,T]$,

$$\frac{d}{dt}\int_{\mathbb{R}} j(u(t,x))dx = -\frac{1}{2}\int_{\mathbb{R}} {\eta_u'}^2(t,x)dx. \tag{4.49}$$

Theorem 4.14 says that $t\mapsto \int_{\mathbb{R}} j(u(t,x))dx$ is absolutely continuous. So, after integrating in time, we get

$$\int_{\mathbb{R}} j(u(t,x))dx = \int_{\mathbb{R}} j(u(r,x))dx - \frac{1}{2}\int_r^t ds \int_{\mathbb{R}} \eta_u'(s,x)^2 dx. \tag{4.50}$$

This completes the proof of Theorem 4.15. ■

The second main result of this section, also crucially used in Section 5 below, is the following.

Proposition 4.17 *Let Hypothesis 3.1 hold and let u be the unique solution to* (1.1) *with initial condition $u_0 \in L^1 \bigcap L^\infty$ being locally of bounded variation. Then, for each $t \in [0,T]$, $u(t,\cdot)$ also has locally bounded variation.*

Remark 4.18 1. *We note that* (4.19) *is not needed for the above proposition.*

2. *Since $u(t,\cdot)$ has locally bounded variation, it has at most a countable number of discontinuities. We will see that in the degenerate case, i.e. if $\Phi(0)=0$, a suitable section of $\Phi(u(t,\cdot))$, also has at most countably many discontinuities, see Lemma 4.19 below.*

Proof (of Proposition 4.17). For h small real fixed, we set

$$u^h(t,x) = u(t,x+h) - u(t,x).$$

Let ζ be a smooth nonnegative function with support on some compact interval. We aim at establishing the following intermediate result:

$$\int_{\mathbb{R}} \zeta(x)|u^h(t,x)|dx \le \int_{\mathbb{R}} \zeta(x)|(u_0)^h(x)|dx + c\|\zeta'''\|_\infty |h| \int_{[0,T]\times\mathbb{R}} |u(s,x)|dsdx. \tag{4.51}$$

Approximating β with β^ε as in Proposition 4.1 5., we may suppose that β satisfies (3.11). In the rest of this proof ε will however be the discretization mesh related to an ε-solution. We recall that u is the unique C^0-solution to (1.1). So for fixed $t \in]0,T]$

$$u(t,\cdot) = \lim_{\varepsilon\to 0} u^\varepsilon(t,\cdot) \quad \text{in } L^1, \tag{4.52}$$

where $u^\varepsilon(t,\cdot)$ is given in Lemma 4.4.

According to Lemma 3.6 we have, for $i = 1,\ldots,N$,

$$\begin{aligned}\int_{\mathbb{R}} \zeta(x)|u_i^h(x)|dx &\le \int_{\mathbb{R}} \zeta(x)u_{i-1}^h(x)\mathrm{sign}(w_i^h(x))dx + c\|\zeta'''\|_\infty|h|\varepsilon \int_{\mathbb{R}} |u_i(x)|dx \\ &\le \int_{\mathbb{R}} \zeta(x)|u_{i-1}^h(x)|dx + c\|\zeta'''\|_\infty|h|\varepsilon \int_{\mathbb{R}} |u_i(x)|dx,\end{aligned}$$

where $u_i^h = (u_i)^h, w_i^h = (w_i)^h$, $i \in \{0, \dots, N\}$, and u_i is defined as in Lemma 4.4 with partition as in (4.5).

Let $t \in]0, T]$ and m be an integer such that $t \in]\frac{(m-1)T}{N}, \frac{mT}{N}]$. Summing on $i = 0, \cdots, m$, we get

$$\int_{\mathbb{R}} \zeta(x)|u_m^h(x)|dx \leq \int_{\mathbb{R}} \zeta(x)|u_0^h(x)|dx + c\|\zeta'''\|_\infty |h|\varepsilon \sum_{i=1}^m \int_{\mathbb{R}} |u_i(x)|dx.$$

Setting $u^{\varepsilon,h} := (u^\varepsilon)^h$ we obtain

$$\int_{\mathbb{R}} \zeta(x)|u^{\varepsilon,h}(t,x)|dx \leq \int_{\mathbb{R}} |u_0^h(x)|\zeta(x)dx + c\|\zeta'''\|_\infty |h| \int_0^T ds \int_{\mathbb{R}} |u^\varepsilon(s,x)|dx.$$

So, letting $\varepsilon \to 0$ and using (4.52) we get

$$\int_{\mathbb{R}} \zeta(x)|u^h(t,x)|dx \leq \int_{\mathbb{R}} |(u_0)^h(x)|\zeta(x)dx + c\|\zeta'''\|_\infty |h| \int_0^T ds \int_{\mathbb{R}} |u(s,x)|dx$$

and so (4.51). Let $M > 0$ such that the support of ζ is included in $[-2M, 2M]$. Then,

$$\limsup_{h\to 0} \frac{1}{|h|} \int_{\mathbb{R}} \zeta(x)|u^h(t,x)|dx \leq \|\zeta\|_\infty \left\|u_{0[-2M,2M+1]}\right\|_{\mathrm{var}} + c\|\zeta'''\|_\infty \int_{[0,T]\times\mathbb{R}} |u(s,x)|dsdx, \tag{4.53}$$

where $\|\cdot\|_{\mathrm{var}}$ denotes the total variation.

We denote the right hand-side of (4.53) by $\mathcal{C}(\zeta)$. Let $\varphi \in C_0^\infty(\mathbb{R})$ such that $\mathrm{supp}\varphi \subset]-M, M[$, and $t \in [0, T]$. Taking $\zeta \equiv 1$ on $]-M, M[$, we can replace φ with $\varphi\zeta$. Then

$$\begin{aligned}\left|\int_{\mathbb{R}} u(t,x)\frac{\varphi(x) - \varphi(x-h)}{h}dx\right| &= \left|\int_{\mathbb{R}} \frac{u^h(t,x)}{h}\zeta(x)\varphi(x)dx\right| \\ &\leq \frac{1}{|h|}\|\varphi\|_\infty \int_{\mathbb{R}} \zeta(x)|u^h(t,x)|dx.\end{aligned}$$

So taking the $\limsup$ and using (4.53) we obtain

$$\left|\int_{\mathbb{R}} u(t,x)\varphi'(x)dx\right| \leq \|\varphi\|_\infty \mathcal{C}(\zeta)$$

Hence $u(t,\cdot)$ has locally bounded variation on $]-M, M[$ and the assertion follows. ∎

We now show that, without particular assumptions on the initial conditions, in the degenerate case, a suitable "section" of $\Phi(u(t,\cdot))$ has at most countably many discontinuities if so has $u(t,\cdot)$. We again consider equation (1.1) in the sense of distributions

$$\begin{cases} \partial_t u = \frac{1}{2}\eta_u'', & \eta_u \in \beta(u) \\ u(0,\cdot) = u_0 \in L^1 \cap L^\infty. \end{cases}$$

We recall that by Proposition 4.5 b), $\eta_u(t,\cdot) \in H^1(\mathbb{R})$ for a.e. $t \in]0,T]$, hence has an absolutely continuous version, which will be still denoted by $\eta_u(t,\cdot)$. Likewise, since $u(t,\cdot) \geq 0$ a.e., for $\forall t \in [0,T]$, we shall take a version which is nonnegative everywhere, which will be still denoted by $u(t,\cdot)$ below.

Define

$$\chi_u = \sqrt{\frac{\eta_u}{u}} 1_{\{|u|>0\}}. \tag{4.54}$$

Here we recall that $u\eta_u \geq 0$, hence $\frac{\eta_u}{u} \geq 0$ on $\{|u| > 0\}$, and that χ_u is bounded by Hypothesis 3.1.

Lemma 4.19 *Suppose β is degenerate, let $t \in [0,T[$ such that $\eta_u(t,\cdot) \in H^1(\mathbb{R})$ and $x \in \mathbb{R}$. If $u(t,\cdot)$ is continuous in x, then so is $\chi_u(t,\cdot)$. In particular, $\chi_u(t,\cdot)$ has at most countably many discontinuities if so has $u(t,\cdot)$.*

Proof. It is enough to show that $\chi_u^2(t,\cdot)$ is continuous in x. Let $x_n \in \mathbb{R}, n \in \mathbb{N}$, converge to x. We have

$$\chi_u^2(t,x_n) = \begin{cases} \frac{\eta_u(t,x_n)}{u(t,x_n)}, & \text{if} \quad u(t,x_n) > 0 \\ 0, & \text{if} \quad u(t,x_n) = 0. \end{cases}$$

- If $u(t,x) > 0$, then

$$\chi_u^2(t,x_n) \to \frac{\eta_u(t,x)}{u(t,x)} = \chi_u^2(t,x).$$

- If $u(t,x) = 0$ then, since β is degenerate,

$$\chi_u^2(t,x_n) \overset{n\to\infty}{\longrightarrow} 0 = \chi_u^2(t,x).$$

■

We have observed that for a relatively general coefficient β, but with a restriction on the initial condition, $u(t,\cdot)$ (and therefore a suitable section of $\Phi(u(t,\cdot))$) is a.e. continuous, for a.e. $t \in [0,T]$, see Proposition 4.12. We now provide some conditions on β (degenerate) for which a suitable section of $\Phi(u(t,\cdot))$ is continuous for any initial condition in $L^2(\mathbb{R})$. This will prepare the third main result of this section, crucially to be used in the next section.

Let (u,η_u) be as usual the solution to (1.1) and χ_u as in (4.54).

Definition 4.20 *We say that β is* **strictly increasing after some zero** *if there is $e_c \geq 0$ such that*

i) $\beta|_{[0,e_c[} = 0$.

ii) β *is strictly increasing on* $[e_c,\infty[$.

iii) If $e_c = 0$, *then* $\lim_{u\to 0_+} \Phi(u) = 0$.

Remark 4.21 *1. Condition iii) guarantees that β is degenerate.*

2. A typical example of a function that is strictly increasing after some zero is given by

$$\beta(u) = uH(u - e_c),$$

where $e_c > 0$ and H is the Heaviside function, i.e.

$$H(u-e_c) = \begin{cases} 0 & : \quad u < e_c \\ [0,1] & : \quad u = e_c \\ 1 & : \quad u > e_c \end{cases}$$

3. We recall that for almost all $t \in]0,T]$, $\eta_u(t,\cdot)$ is continuous. This will constitute the main ingredient in the proof of the proposition below.

4. Suppose that β is as in Definition 4.20. Then β^{-1} is single-valued and continuous on $]0,\infty[$.

Proposition 4.22 *Suppose β strictly increasing after some zero. Then for almost all $t \in]0,T[$, $\chi_u(t,\cdot)$ is continuous.*

Proof. We first recall that by Corollary 4.2, $u(t,\cdot) \geq 0$ a.e. for all $t \in [0,T]$. Let e_c be as in Definition 4.20. Let $t \in]0,T]$ for which $\eta_u(t,\cdot)$ is continuous. Let (x_n) be a sequence converging to some $x_0 \in \mathbb{R}$. The principle is to find a subsequence (n_k) such that $\chi_u^2(t,x_{n_k}) \to \chi_u^2(t,x_0)$. In the sequel of the proof, we will omit t and denote the functions $u(t,x)$ (resp. $\eta_u(t,x), \chi_u(t,x)$) by $u(x)$ (resp. $\eta_u(x), \chi_u(x)$).

We distinguish several cases

1. $\underline{u(x_0) \in [0,e_c[}$. Then $e_c > 0$ and $\eta_u(x_0) \in \beta(u(x_0)) = 0$.
   Hence $\chi_u(x_0) = 0$.

   - If $u(x_{n_k}) < e_c$ for some subsequence (n_k), then
   $$\chi_u^2(x_{n_k}) \equiv 0 \overset{k\to\infty}{\longrightarrow} 0.$$
   - If there is a subsequence (n_k) such that $u(x_{n_k}) \geq e_c$, then
   $$\chi_u^2(x_{n_k}) = \frac{\eta_u(x_{n_k})}{u(x_{n_k})} \leq \frac{\eta_u(x_{n_k})}{e_c} \overset{k\to\infty}{\longrightarrow} \frac{\eta_u(x_0)}{e_c} = 0.$$

2. We suppose now $\underline{u(x_0) \in]e_c,\infty[}$.
   Since β^{-1} is single-valued, continuous on $]0,\infty[$ and $\eta_u(x_0) \in \beta(u(x_0))$, so $\eta_u(x_0) > 0$, we have
   $$\begin{aligned} u(x_0) = \beta^{-1}(\eta_u(x_0)) &= \beta^{-1}(\lim_{n\to\infty} \eta_u(x_n)) \\ &= \lim_{n\to\infty} \beta^{-1}(\eta_u(x_n)) = \lim_{n\to\infty} u(x_n). \end{aligned}$$
   Consequently
   $$\chi_u^2(x_n) \to \chi_u^2(x_0).$$

3. $\underline{u(x_0) = e_c}$.
   Clearly there are three possibilities.

   (a) there is a subsequence (n_k) with $u(x_{n_k}) \in]e_c,\infty[$,

   (b) there is a subsequence (n_k) with $u(x_{n_k}) \in [0,e_c[$,

   (c) there is a subsequence (n_k) with $u(x_{n_k}) = e_c \; \forall k \in \mathbb{N}$.

Case (a). First we suppose $e_c > 0$. We have $\eta_u(x_{n_k}) \to \eta_u(x_0)$. If $\eta_u(x_0) = 0$ then

$$\chi_u^2(x_{n_k}) = \frac{\eta_u(x_{n_k})}{u(x_{n_k})} \to 0 = \frac{\eta_u(x_0)}{u(x_0)} = \chi_u^2(x_0).$$

If $\eta_u(x_0) \neq 0$ then the continuity of β^{-1} implies

$$u(x_{n_k}) = \beta^{-1}(\eta_u(x_{n_k})) \to \beta^{-1}(\eta_u(x_0)) = u(x_0) = e_c,$$

so $\chi_u^2(x_{n_k}) \to \chi_u^2(x_0)$.
If $e_c = 0$, the result follows since β is degenerate.

Case (b). In this case e_c is again strictly positive. Since $\eta_u(x_{n_k}) \in \beta(u(x_{n_k}) = 0$ we have $\chi_u(x_{n_k}) = 0$, hence $\chi_u(x_{n_k}) \overset{k\to\infty}{\longrightarrow} 0$. But $0 = \eta_u(x_{n_k}) \overset{k\to\infty}{\longrightarrow} \eta_u(x_0)$. This implies that $\eta_u(x_0) = 0$, so $\chi_u^2(x_0) = 0$.

Case (c). We have $u(x_{n_k}) = e_c$. If $e_c = 0$ the result follows trivially by definition of χ_u. Therefore we can suppose again that $e_c > 0$. Then $u(x_{n_k}) = e_c \overset{k\to\infty}{\longrightarrow} e_c$, so

$$\chi_u^2(x_{n_k}) = \frac{\eta_u(x_{n_k})}{e_c} \to \frac{\eta_u(x_0)}{e_c} = \frac{\eta_u(x_0)}{u(x_0)} = \chi_u^2(x_0).$$

This completes the proof.

■

5 The probabilistic representation of the deterministic equation

We again consider the $\beta : \mathbb{R} \to 2^{\mathbb{R}}$ satisfying Hypothesis 3.1. We aim at providing a probabilistic representation for solutions to equation (1.1). Let $u_0 \geq 0$ such that $\int_{\mathbb{R}} u_0(x)dx = 1$ and $u_0 \in L^\infty(\mathbb{R})$.

We consider a multi-valued map $\Phi : \mathbb{R} \to 2^{\mathbb{R}_+}$ such that

$$\beta(u) = \Phi^2(u)u, \quad u \in \mathbb{R},$$

which is bounded, i.e.

$$\sup_{u\in\mathbb{R}} \sup \Phi(u) < \infty.$$

The degenerate case is much more difficult than the non-degenerate case which was solved in [9].

Definition 5.1 *Let* (u, η_u) *be the solutions in the sense of Proposition 4.1 to equation* (1.1). *i.e.*

$$\begin{cases} \partial_t u &= \frac{1}{2}\partial^2_{xx}(\eta_u), \quad \text{on } L^1(\mathbb{R}) \\ u(0,x) &= u_0(x). \end{cases} \tag{5.1}$$

We say that (1.1) has a **probabilistic representation**, if there is a filtered probability space $(\Omega, \mathcal{F}, P, (\mathcal{F}_t))$, an $(\mathcal{F}_t)$)-Wiener process W and, at least one process Y, such there exists $\chi_u \in (L^1 \bigcap L^\infty)([0,T] \times \mathbb{R})$ with

$$\begin{cases} Y_t &= Y_0 + \int_0^t \chi_u(s, Y_s))dW_s \text{ in law} \\ \chi_u(t,x) &\in \Phi(u(t,x)) \text{ for } dt \otimes dx \text{ a.e.}(t,x) \in [0,T] \times \mathbb{R}, \\ \text{Law} \quad \text{density}(Y_t) &= u(t,\cdot). \\ u(0,\cdot) &= u_0. \end{cases} \tag{5.2}$$

We recall the main result of [9], Theorem 4.3.

Theorem 5.2 *When* β *is non-degenerate then* (1.1) *has a probabilistic representation, with*

$$\chi_u = \sqrt{\frac{\eta_u}{u}} 1_{\{|u|>0\}}.$$

Remark 5.3 *In the non-degenerate case the representation is unique.*

We will show that, even in the degenerate case, (1.1) has a probabilistic representation.

Theorem 5.4 *Suppose that* β *is degenerate. Then equation* (1.1) *admits a probabilistic representation if one of the following conditions are verified.*

1. β *is strictly increasing after some (non-negative) zero.*
2. u_0 *has locally bounded variation.*

Proof. We will make use of Theorem 5.2. Let $\varepsilon \in]0,1]$ and set

$$\Phi_\varepsilon(u) = \sqrt{\Phi^2(u) + \varepsilon},\ \beta^\varepsilon(u) = \beta(u) + \varepsilon u.$$

Let $(u^{(\varepsilon)}, \eta_{u^{(\varepsilon)}})$ the solution to the deterministic PDE (1.1), with β^ε replacing β. Define

$$\chi^\varepsilon = \sqrt{\frac{\eta_{u^{(\varepsilon)}}}{u^{(\varepsilon)}}} 1_{\{|u^{(\varepsilon)}|>0\}}. \tag{5.3}$$

We note that since $\Phi_\varepsilon, \varepsilon \in]0,1]$ are uniformly bounded, so are $\chi^\varepsilon, \varepsilon \in]0,1]$.

By Theorem 5.2, there exists a unique solution $Y = Y^\varepsilon$ in law of

$$\begin{cases} Y_t & = & Y_0 + \int_0^t \chi^\varepsilon(s, Y_s))dW_s \\ \chi^\varepsilon(t,x) & \in & \Phi_\varepsilon(u^{(\varepsilon)}(t,x)) \text{ for } dt \otimes dx \text{ a.e.}(t,x) \in [0,T] \times \mathbb{R}. \\ \text{Law density}(Y_t) & = & u^{(\varepsilon)}(t,\cdot) \\ u^{(\varepsilon)}(0,\cdot) & = & u_0. \end{cases} \tag{5.4}$$

Since Φ is bounded, using the Burkholder-Davies-Gundy inequality one obtains

$$\mathbb{E}|Y_t^\varepsilon - Y_s^\varepsilon|^4 \leq \text{const.}(t-s)^2. \tag{5.5}$$

This implies (see for instance [23] Problem 4.11 of Section 2.4) that the laws of $Y^\varepsilon, \varepsilon > 0$ are tight. Consequently, there is a subsequence $Y^n := Y^{\varepsilon_n}$ converging in law (as $C[0,T]$-valued random elements) to some process Y. We set $u^n := u^{(\varepsilon_n)}$, where we recall that $u^n(t,\cdot)$ is the law of Y_t^n, and $\chi^n := \chi^{\varepsilon_n}$.

Since

$$[Y^n]_t = \int_0^t (\chi^n)^2(s, Y_s^n) ds,$$

and $E([Y^n]_T)$ is finite, Φ being bounded, the continuous local martingales Y^n are indeed martingales.

By Skorokhod's theorem there is a new probability space $(\Omega, \mathcal{F}, P)$ and processes $\tilde{Y}^n$, with the same distribution as Y^n so that $\tilde{Y}^n$ converge to some process $\tilde{Y}$, distributed as Y, as $C([0,T])$- random elements P-a.s. In particular, those processes $\tilde{Y}^n$ remain martingales with respect to the filtrations generated by them. We denote the sequence $\tilde{Y}^n$ (resp. $\tilde{Y}$), again by Y^n (resp. Y).

Remark 5.5 *We observe that, for each* $t \in [0,T]$, $u(t,\cdot)$ *is the law density of* Y_t. *In fact, for any* $t \in [0,T]$, Y_t^n *converges in probability to* Y_t*; on the other hand* $u^n(t,\cdot)$, *which is the law of* Y_t^n *converges to* $u(t,\cdot)$ *in* $L^1(\mathbb{R})$, *by Proposition 4.1 5.*

Remark 5.6 *Let* $\mathcal{Y}^n$ *(resp.* $\mathcal{Y}$*) be the canonical filtration associated with* Y^n *(resp.* Y*).*

We set

$$W_t^n = \int_0^t \frac{1}{\chi^n}(s, Y_s^n) dY_s^n.$$

Those processes W^n *are standard* $(\mathcal{Y}_t^n)$ *-Wiener processes since* $[W^n]_t = t$ *and because of Lévy's characterization theorem of Brownian motion. Then one has*

$$Y_t^n = Y_0^n + \int_0^t \chi^n(s, Y_s^n) dW_s^n.$$

■

We aim to prove first that

$$Y_t = Y_0 + \int_0^t \chi_u(s, Y_s) dW_s. \tag{5.6}$$

where χ_u is defined as in (4.54). Once this equation is established for the given u, the statement of Theorem 5.4 would be completely proven because of Remark 5.5. In fact, that remark shows in particular the third line of (5.2).

Taking into account, Theorem 4.2 of Ch. 3 of [23], to establish (5.6), it will be enough to prove that Y is a $\mathcal{Y}$- martingale with quadratic variation $[Y]_t = \int_0^t \chi_u^2(s, Y_s) ds$.

Let $s, t \in [0,T]$ with $t > s$ and Θ a bounded continuous function from $C([0,s])$ to $\mathbb{R}$.

In order to prove the martingale property for Y, we need to show that

$$E\left((Y_t - Y_s)\Theta(Y_r, r \le s)\right) = 0.$$

This follows by (5.5) because $Y^n \to Y$ a.s. as $C([0,T])$-valued process and

$$E\left((Y_t^n - Y_s^n)\Theta(Y_r^n, r \le s)\right) = 0.$$

It remains to show that $Y_t^2 - \int_0^t \chi_u^2(s, Y_s)ds, t \in [0,T]$, defines a $\mathcal{Y}$-martingale, which in turn follows, if for $t > s$ we can verify

$$E\left((Y_t^2 - Y_s^2 - \int_s^t \chi_u^2(r, Y_r)dr)\Theta(Y_r, r \le s) \right) = 0.$$

The left-hand side decomposes into $I^1(n) + I^2(n) + I^3(n)$ where

$$\begin{aligned} I^1(n) &= E\left((Y_t^2 - Y_s^2 - \int_s^t \chi_u^2(r, Y_r)dr)\Theta(Y_r, r \le s) \right) \\ &- E\left(\left((Y_t^n)^2 - (Y_s^n)^2 - \int_s^t \chi_u^2(r, Y_r^n)dr \right) \Theta(Y_r^n, r \le s) \right), \\ I^2(n) &= E\left(\left((Y_t^n)^2 - (Y_s^n)^2 - \int_s^t \chi^n(r, Y_r^n)^2 dr \right) \Theta(Y_r^n, r \le s) \right), \end{aligned}$$

and

$$I^3(n) = E\left(\int_s^t \left(\chi^n(r, Y_r^n)^2 - \chi_u^2(r, Y_r^n) \right) dr \Theta(Y_r^n, r \le s) \right).$$

We start showing the convergence of $I^3(n)$. Now $\Theta(Y_r^n, r \le s)$ converges a.s. to $\Theta(Y_r, r \le s)$ and it is dominated by a constant. so that it suffices to consider the expectation of

$$\int_s^t \left| \chi^n(r, Y_r^n)^2 - \chi_u(r, Y_r^n)^2 \right| dr$$

which is equal to

$$I(n) = \int_s^t dr \int_{\mathbb{R}} \left| \eta_{u(\varepsilon_n)}(r, y) - \chi_u(r, y)^2 u^n(r, y) \right| dy.$$

By Proposition 5.7 below $\eta_{u(\varepsilon)} \to \eta_u$ in $L^1([0,T] \times \mathbb{R})$ as $\varepsilon \to 0$. Furthermore, Proposition 4.1 5), see also the theorem in the introduction of [12], implies that $u^\varepsilon(t, \cdot)$ converges to $u(t, \cdot)$ in $L^1(\mathbb{R})$, as $\varepsilon \to 0$, uniformly in $t \in [0,T]$. Hence Lebesgue's dominated convergence theorem implies that $I(n) \to 0$, since χ_u is bounded.

We go on with the analysis of $I^2(n)$ and $I^1(n)$. $I^2(n)$ equals to zero because Y^n is a martingale with quadratic variation given by

$$[Y^n]_t = \int_0^t \chi^n(r, Y_r^n)^2 dr.$$

We finally treat $I^1(n)$. We recall that $Y^n \to Y$ a. s. as random elements in $C([0,T])$ and that the sequence $E\left((Y^n_t)^4\right)$, is bounded, so $(Y^n_t)^2$ are uniformly integrable. Therefore, for $t > s$ we have

$$E\left((Y^n_t)^2 - (Y^n_s)^2)\Theta(Y^n_r, r \le s)\right) - E\left((Y^2_t - Y^2_s)\Theta(Y_r, r \le s)\right) \to 0,$$

when $n \to \infty$. It remains to prove that

$$E\left(\int_s^t \chi_u^2(r, Y_r) dr \Theta(Y_r, r \le s) - \int_s^t \chi_u^2(r, Y^n_r) dr \Theta(Y^n_r, r \le s)\right) \to 0. \quad (5.7)$$

Under the assumptions of the theorem, for fixed $r \in [0,T]$, by the second and third main results of Section 4 (see Propositions 4.17, 4.22 and Remark 4.18), $\chi_u(r,\cdot)$ has at most a countable number of discontinuities. Moreover, the law of Y_r has a density and it is therefore non atomic. So, let $N(r)$ be the null event of $\omega \in \Omega$ such that $Y_r(\omega)$ is a point of discontinuity of $\chi_u(r,\cdot)$. For $\omega \notin N(r)$ we have

$$\lim_{n\to\infty} \chi_u^2(r, Y^n_r(\omega)) = \chi_u^2(r, Y_r(\omega)).$$

Now, Lebesgue's dominated convergence and Fubini's theorem imply (5.7). So equation (5.6) is shown.

It remains to prove the following result which is based on our first main result of Section 4, see Theorem 4.15.

Proposition 5.7 *Let* $\eta_u^\varepsilon := \eta_{u^{(\varepsilon)}}, \varepsilon > 0$. *Then* $\eta_u^\varepsilon \to \eta_u$ *in* $L^1([0,T] \times \mathbb{R})$ *as* $\varepsilon \to 0$.

Proof. We set

$$j_\varepsilon(x) := \int_0^x \beta_\varepsilon^\circ(y) dy = j(x) + \varepsilon \frac{x^2}{2}.$$

where β_ε° is the minimal section of β^ε and clearly $\beta_\varepsilon^\circ(x) = \beta^\circ(x) + \varepsilon x, x \in \mathbb{R}$. According to Theorem 4.15 we have

$$\int_{\mathbb{R}} j_\varepsilon(u^{(\varepsilon)}(T,x)) dx + \frac{1}{2}\int_0^T ds \int_{\mathbb{R}} (\eta_u^\varepsilon)'^2(s,x) dx = \int_{\mathbb{R}} j(u_0(x)) dx. \quad (5.8)$$

In particular,

$$\sup_{\varepsilon>0} \int_0^T ds \int_{\mathbb{R}} (\eta_u^\varepsilon)'^2(s,x) dx \le \int_{\mathbb{R}} j(u_0(x)) dx < \infty. \quad (5.9)$$

So, by (4.4), the family $\{\eta_u^\varepsilon, \varepsilon \in]0,1]\}$ is weakly relatively compact in $L^2([0,T]; H^1(\mathbb{R}))$, hence also in $L^2([0,T]; L^2(\mathbb{R})) = L^2([0,T] \times \mathbb{R})$. We recall that $u^{(\varepsilon)}(t,\cdot) \to u(t,\cdot)$ in $L^1(\mathbb{R})$ uniformly in $t \in [0,T]$.

Let (ε_n) be a sequence converging to zero. There is a subsequence (n_k) such that $\eta_u^k := \eta_u^{\varepsilon_{n_k}}$ converges weakly in $L^2([0,T]\times\mathbb{R})$ to some $\xi \in L^2([0,T]\times\mathbb{R})$. For any $\alpha \in C_0^\infty(\mathbb{R})$,

$$\int_{\mathbb{R}} u^{(\varepsilon)}(t,x)\alpha(x)dx = \int_{\mathbb{R}} u_0(x)\alpha(x)dx + \frac{1}{2}\int_0^t ds \int_{\mathbb{R}} \eta_u^\varepsilon(s,x)\alpha''(x)dx.$$

Taking the limit when $k \to \infty$, we get

$$\int_{\mathbb{R}} u(t,x)\alpha(x)dx = \int_{\mathbb{R}} u_0(x)\alpha(x)dx + \frac{1}{2}\int_0^t ds \int_{\mathbb{R}} \xi(s,x)\alpha''(x)dx. \tag{5.10}$$

Let $K > 0$. Since β is maximal monotone, $v \mapsto \beta(v)$ is a maximal monotone map from $L^2(\mathbb{R}\times[-K,K])$ to $L^2(\mathbb{R}\times[-K,K])$. Therefore, [6], p.37, Proposition 1.1 (i) and (ii), imply that this map is weakly-strongly closed. Since, by (4.3), $u^{(\varepsilon)}$ converges to u in $L^2([0,T]\times[-K,K])$, it follows that $\xi \in \beta(u)$ a.e. on $[0,T]\times[-K,K]$ for all $K > 0$, so, $\xi \in \beta(u)$ a.e. By the uniqueness of (1.1) we get $\xi = \eta_u$ a.e.

Let $\varepsilon_n \to 0$. The rest of the paper will be devoted to the proof of the existence of a subsequence $(\eta_u^k) := \eta_{u^{(\varepsilon_{n_k})}}$ converging (strongly) to η_u in $L^2_{\text{loc}}([0,T]) \times \mathbb{R})$. Since $\eta_u^k \in \beta(u^{(\varepsilon_{n_k})})$, we have

$$|\eta_u^k| \le (c + \varepsilon_{n_k})|u^{(\varepsilon_{n_k})}|.$$

Hence $\{\eta_u^k\}$ is equintegrable on $[0,T]\times\mathbb{R}$. Therefore, the existence of such a subsequence completes the proof.

We will need the following well-known lemma.

Lemma 5.8 *Let H be a Hilbert space, (f_n) be a sequence in H converging weakly to some $f \in H$. Suppose*

$$\limsup_{n\to\infty} \|f_n\|^2 \le \|f\|^2.$$

Then $f_n \to f$ strongly in H.

We apply previous Lemma to establish the existence of a subsequence still denoted by (η_u^k) such that $(\eta_u^k)'$ converges strongly to η_u' in $L^2([0,T]\times\mathbb{R})$. For this, we will prove that

$$\limsup_{k\to\infty}\int_0^T ds\int_{\mathbb{R}} dx\ (\eta_u^k)'^2(s,x)\leq \int_0^T ds\int_{\mathbb{R}} dx\ {\eta_u'}^2(s,x).$$

The idea consists in letting k go to infinity in (5.8) for $\varepsilon=\varepsilon_{n_k}$. First, for $t\in[0,T]$ we have

$$\int_{\mathbb{R}} j_\varepsilon(u^{(\varepsilon)}(t,x))dx=\int_{\mathbb{R}} j(u^{(\varepsilon)}(t,x))dx+\frac{\varepsilon}{2}\int_{\mathbb{R}}(u^{(\varepsilon)})^2(t,x)dx. \tag{5.11}$$

Since j is continuous,

$$\lim_{k\to\infty} j(u^{(\varepsilon_{n_k})}(t,x))=j(u(t,x))\quad \text{a.e..}$$

By Fatou's lemma

$$\int_{\mathbb{R}} j(u(t,x))dx\leq\liminf_{k\to\infty}\int_{\mathbb{R}} j(u^{(\varepsilon_{n_k})}(t,x))dx. \tag{5.12}$$

Again Theorem 4.15 implies

$$\int_{\mathbb{R}} j(u(T,x))dx+\frac{1}{2}\int_0^T ds\int_{\mathbb{R}}(\eta_u)'^2(s,x)dx=\int_{\mathbb{R}} j(u_0(x))dx.$$

This together with (5.8) gives

$$\begin{aligned}\frac{1}{2}\int_0^T ds\int_{\mathbb{R}}(\eta_u^k)'^2(s,x)dx &= \frac{1}{2}\int_0^T ds\int_{\mathbb{R}}{\eta_u'}^2(s,x)dx+\int_{\mathbb{R}} j(u(T,x))dx \\ &- \int_{\mathbb{R}} j(u^{(\varepsilon_{n_k})}(T,x))dx-\frac{\varepsilon_{n_k}}{2}\int_{\mathbb{R}} u^{(\varepsilon_{n_k})}(T,x)^2dx.\end{aligned} \tag{5.13}$$

Since by Corollary 4.6

$$\int_{\mathbb{R}} u^{(\varepsilon)}(T,x)^2dx\leq\int_{\mathbb{R}} u_0(x)^2dx,$$

the last term in (5.13) converges to zero when $k\to\infty$. Taking the lim sup when $k\to\infty$ in (5.13) and using (5.12) we get

$$\limsup_{k\to\infty}\int_0^T ds\int_{\mathbb{R}}(\eta_u^k)'^2(s,x)dx\leq\int_0^T ds\int_{\mathbb{R}}{\eta_u'}^2(s,x)dx.$$

Consequently, by Lemma 5.8

$$\lim_{k\to\infty}\int_0^T ds\int_{\mathbb{R}} dx((\eta_u^k)'(s,x)-\eta_u'(s,x))^2=0. \tag{5.14}$$

Now let us finally prove that $\eta_u^k \to \eta_u$ (strongly) in $L^2_{\rm loc}([0.T], \times\mathbb{R})$. Let $x\in\mathbb{R}$. We recall that $\eta_u, \eta_u^k(t,\cdot)$ vanish at infinity since they belong to $H^1(\mathbb{R})=H^1_0(\mathbb{R})$. So we can write, for $x\in\mathbb{R}$,

$$\begin{aligned}(\eta_u^k(t,x)-\eta_u(t,x))^2 &= 2\int_{-\infty}^x (\eta_u^k)'(t,y)-\eta_u'(t,y)(\eta_u^k(t,y)-\eta_u(t,y))dy\\ &\le 2\left\{\int_{-\infty}^x ((\eta_u^k)'-\eta_u')^2(t,y)dy\int_{-\infty}^x(\eta_u^k-\eta_u)^2(t,y)dy\right\}^{\frac12}.\end{aligned}$$

Integrating from 0 to T, by the Cauchy-Schwarz inequality, the quantity

$$\int_0^T dt({\eta_u}^k-\eta_u)^2(t,x)$$

is bounded by

$$2\sqrt{\int_0^T dt\int_{\mathbb{R}}(\eta_u^{k\prime}-\eta_u')^2(t,y)dy}\sqrt{\int_0^T dt\int_{\mathbb{R}}((\eta_u^k)-\eta_u)^2(t,y)dy}.$$

On the other hand, using Corollary 4.6 and (3.1), we have

$$\int_0^T ds\int_{\mathbb{R}} dy\;\eta_u^k(t,y)^2\le \mathrm{const}\int_0^T ds\int_{\mathbb{R}} dy\;u^{(\varepsilon_{n_k})}(t,y)^2\le \mathrm{const}\,T\left\|u_0\right\|^2_{L^2}$$

and likewise

$$\int_0^T ds\int_{\mathbb{R}} dy\;\eta_u^2(t,y)\le \mathrm{const}\,.T\left\|u_0\right\|^2_{L^2}.$$

Consequently, maybe with another const,

$$\sup_{x\in\mathbb{R}}\int_0^T dt(\eta_u^k-\eta_u)^2(t,x)\le\sqrt{T}\,\mathrm{const}\left\|u_0\right\|_{L^2}\left\|\eta_u^{k\prime}-\eta_u'\right\|_{L^2([0,T]\times\mathbb{R})},$$

which by (5.14), converges to zero. ■

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