q-EXCHANGEABILITY VIA QUASI-INVARIANCE

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Abstract. For positive q, the q-exchangeability is introduced as quasi-invariance under permutations, with a special cocycle. This allows us to extend the q-analogue of de Finetti's theorem for binary sequences [GO2] to the general real-valued sequences. In contrast to the classical case with $q = 1$, the order on $\mathbb{R}$ plays for the q-analogues a significant role. An explicit construction of ergodic q-exchangeable measures involves a random shuffling of $\mathbb{N} = \{1, 2, \dots\}$ by iteration of the geometric choice. For q distinct from 1, the shuffling yields a probability measure Q that is supported by the group of bijections of $\mathbb{N}$, and has the property of quasi-invariance under both left and right multiplications by finite permutations. We establish connections of the q-exchangeability to certain transient Markov chains on the q-Pascal pyramids and to invariant random flags over the Galois fields.

Contents

1. Introduction

Let $\mathbb{A}$ be a standard Borel space thought of as 'alphabet'. Consider the infinite product space $\mathbb{A}^\infty$, whose elements are written as infinite words $w = w_1 w_2 \dots$ with letters $w_i \in \mathbb{A}$. Let $\mathbb{N} = \{1, 2, \dots\}$ and $\mathfrak{S}_\infty$ be the infinite symmetric group of bijections $\sigma : \mathbb{N} \to \mathbb{N}$ which move only finitely many integers. The group $\mathfrak{S}_\infty$ acts on $\mathbb{A}^\infty$ by operators $w \mapsto T_\sigma w$ which change the succession of letters in a word:

$$(T_\sigma w)_i = w_{\sigma^{-1}(i)}, \qquad i = 1, 2, \dots, \quad w \in \mathbb{A}^\infty, \quad \sigma \in \mathfrak{S}_\infty.$$

A probability measure P on $\mathbb{A}^\infty$ is called *exchangeable* if P is $\mathfrak{S}_\infty$-invariant. By de Finetti's theorem (see, e.g., [F, Section 27.4]), all extreme (=ergodic) exchangeable measures on $\mathbb{A}^\infty$ are the homogeneous product measures $\nu^{\otimes\infty}$, and every exchangeable probability measure is a unique mixture of the extremes. See [Al], [K] for survey of ideas around exchangeability and multiple generalizations of this fundamental kind of stochastic symmetry.

The infinite symmetric group has the structure of inductive limit, that is $\mathfrak{S}_\infty = \cup_{n\geq 1}\mathfrak{S}_n$, where $\mathfrak{S}_n$ is the subgroup of bijections satisfying $g(i) = i$ for $i > n$, so $\mathfrak{S}_n$ is in essence the group of permutations of $\mathbb{N}_n := \{1, \dots, n\}$. Moreover, the action on $\mathbb{A}^\infty$ is the system of consistent actions of $\mathfrak{S}_n$ on $\mathbb{A}^n$ for $n = 1, 2, \dots$. From a combinatorial perspective, de Finetti's theorem appears as a consequence of large-n properties of the binomial coefficients that enter the multivariate hypergeometric distribution in the model of random sampling without replacement, inherent to the action of $\mathfrak{S}_n$. See [DF] for quantitative aspects of the relation between finite and infinite exchangeability.

Alternative sampling schemes associated with other arrays of combinatorial numbers can be considered. In particular, it is natural to wonder what is the analogue of de Finetti's theorem if the binomial coefficients are replaced by their Gaussian q-analogues, and what kind of symmetry corresponds to this framework. One obvious (for algebraist!) direction is to consider the group $GL(\infty, \mathbb{F}_q)$ of invertible matrices over the Galois field $\mathbb{F}_q$ as a q-analogue of $\mathfrak{S}_\infty$; this makes sense, however, only for q a power of a prime integer.

In our recent paper [GO2] we observed that $GL(\infty, \mathbb{F}_q)$-invariant measures on the Grassmannian in $(\mathbb{F}_q)^\infty$ correspond to certain *quasi*-invariant measures for the action of $\mathfrak{S}_\infty$ on $\mathbb{A}^\infty$, for the two-element base space $\mathbb{A} = \{0, 1\}$. The approach based on the quasi-invariance is valid for arbitrary $q > 0$, thus it suggests an attractive way to understand the 'q-exchangeability'.

In this paper we continue the line initiated in [GO2] by introducing the q-exchangeability of random infinite words on arbitrary Borel alphabet $\mathbb{A} \subseteq \mathbb{R}$. The order on reals is essential, and reversing the order changes the type of symmetry by transforming q-exchangeability into q^{-1}-exchangeability. For finite $\mathbb{A}$ the q-exchangeable processes correspond to a class of lattice random walks. We show that for each extreme quasi-invariant probability measure on $\mathbb{A}^\infty$ the generic random word w only involves letters from a fixed finite or countable sub-alphabet (depending on the measure); in the first case only the maximal letter appears infinitely often, and in the second case none of the letters. We also give the explicit construction of the extremes by means of a *q-shuffle*[1] which iterates a single choice by geometric variable. In particular, the q-shuffle of the infinite word $1\,2\,\dots$ is a remarkable probability measure on the group $\mathfrak{S}$ of *all* permutations of $\mathbb{N}$, analogous to the familiar Mallows measures on the finite symmetric groups. In Section 9 we turn to the algebraic setting and connect the q-exchangeability with random flags in $(\mathbb{F}_q)^\infty$ invariant under the natural action of $GL(\infty, \mathbb{F}_q)$.

[1]Not to be confused with the notion of a-shuffle with integer parameter a, see [BD], [St], [GO1].

2. The q-exchangeability

We recall first a general framework and basic facts from [GS]. Let W be a standard Borel space, and let G be a countable group acting on W on the left by Borel isomorphisms $T_g : W \to W$, $g \in G$. Then G also acts on the space of all Borel probability measures on W: namely, T_g transforms such a measure P to $T_g P := P \circ T_g^{-1}$. We prefer to write this relation as $T_g^{-1} P = P \circ T_g$, which means that $(T_g^{-1} P)(X) = P(T_g(X))$ for every Borel set $X \subseteq W$.

A probability measure P on W is said to be *quasi-invariant* if $T_g^{-1} P$ is equivalent to P for all $g \in G$, that is, $T_g^{-1} P$ and P have the same null sets. Then there exists a function $\rho(g, w)$ on $G \times W$ such that $w \mapsto \rho(g, w)$ is Borel and $T_g^{-1} P = \rho(g, \cdot\,) P$ for each $g \in G$. That is to say, $\rho(g, \cdot\,)$ is the Radon–Nikodým derivative $dT_g^{-1} P/dP$. The function ρ is unique modulo P-null sets and satisfies the relation

$$\rho(gh, w) = \rho(g, T_h w)\rho(h, w), \qquad g, h \in G, \quad w \in W$$

(again modulo null sets). A function ρ with this property is called a *multiplicative cocycle*.

Conversely, given a multiplicative cocycle ρ, let $\mathfrak{M}(\rho)$ stand for the set of all quasi-invariant probability measures on W satisfying the relation $dT_g^{-1} P/dP = \rho(g, \cdot\,)$, $g \in G$. The set $\mathfrak{M}(\rho)$ has itself the structure of a standard Borel space, and if $\mathfrak{M}(\rho)$ is nonempty then it is convex and has a nonempty subset $\operatorname{Ex}\mathfrak{M}(\rho)$ of extreme points. The set of extremes $\operatorname{Ex}\mathfrak{M}(\rho)$ is also Borel. Moreover, every measure $M \in \mathfrak{M}(\rho)$ is uniquely representable as a mixture of the extreme measures, meaning that there exists a unique probability measure $\varkappa$ on $\operatorname{Ex}\mathfrak{M}(\rho)$ such that

$$M(X) = \int_{\operatorname{Ex}\mathfrak{M}(\rho)} P(X)\varkappa(dP)$$

for every Borel subset $X \subseteq W$.

Since the generic element of $\mathfrak{M}(\rho)$ is a unique mixture of extremes, it is important to describe as explicitly as possible the set of extremes $\operatorname{Ex}\mathfrak{M}(\rho)$. A useful criterion is that the extreme measures can be characterized as ergodic measures from $\mathfrak{M}(\rho)$. Recall that a G-quasi-invariant probability measure P on W is *ergodic* if every G-invariant Borel subset of W has P-measure 0 or 1. Since the group G is countable, the ergodicity is equivalent to the formally stronger condition that every invariant mod 0 subset has measure 0 or 1.

After these general preliminaries we focus on a concrete instance. We shall consider the action of the group $G = \mathfrak{S}_\infty$ on the infinite product space $W = \mathbb{A}^\infty$, where $\mathbb{A}$ is a Borel subset of the ordered space $(\mathbb{R}, <)$. Although we assume $\mathbb{A} \subseteq \mathbb{R}$ many considerations of the present paper remain valid for arbitrary standard Borel space endowed with a Borel-measurable linear order (for instance, $\mathbb{R}^k$ with the lexicographic order).

Given a finite word $w = w_1 w_2 \ldots w_n \in \mathbb{A}^n$, let

$$\operatorname{inv}(w_1 \ldots w_n) := \#\{(i, j) \mid 1 \le i < j \le n,\ w_i > w_j\}$$

denote the number of inversions in w. For an infinite word $w = w_1 w_2 \ldots \in \mathbb{A}^\infty$, let

$$\operatorname{inv}_n(w) = \operatorname{inv}(w_1 \ldots w_n)$$

be the number of inversions in the n-truncated word $w_1 \ldots w_n$.

For $w \in \mathbb{A}^\infty$ and $\sigma \in \mathfrak{S}_\infty$, the difference $\operatorname{inv}_n(T_\sigma w) - \operatorname{inv}_n(w)$ stabilizes as n becomes so large that $\sigma(i) = i$ for all $i \geq n$. We set

$$c(\sigma, w) = \text{stable value of the difference } \operatorname{inv}_n(T_\sigma w) - \operatorname{inv}_n(w). \tag{2.1}$$

For instance, if σ is the elementary transposition of i and $i+1$ then $T_\sigma w$ differs from w by transposition of the adjacent letters w_i and w_{i+1} only, and then $c(\sigma, w)$ equals 1, -1 or 0 depending on whether $w_i < w_{i+1}$, $w_i > w_{i+1}$ or $w_i = w_{i+1}$, respectively.

The function $c(\sigma, w)$ is an *additive cocycle* in the sense that

$$c(\sigma\tau, w) = c(\sigma, T_\tau w) + c(\tau, w), \qquad \sigma,\, \tau \in \mathfrak{S}_\infty.$$

Equivalently, for $q > 0$,

$$\rho_q(\sigma, w) := q^{c(\sigma, w)} \tag{2.2}$$

is a multiplicative cocycle. In accord with the terminology of ergodic theory, the additive cocycle $c = \log_q \rho_q$ may be also called the 'modular function'.

Our considerations are based on the following definition.

Definition 2.1. For fixed $q > 0$, a Borel probability measure P on $\mathbb{A}^\infty$ is called q-*exchangeable* if P is quasi-invariant with respect to the action of the group $\mathfrak{S}_\infty$, with the multiplicative cocycle given by (2.2).

Note that it is enough to require (2.2) to hold for the elementary transpositions, because these permutations generate the group $\mathfrak{S}_\infty$.

In the special case $q = 1$ the order on $\mathbb{A}$ plays no role, as the cocycle ρ_q is identically equal to 1, hence our definition turns then into the conventional exchangeability.

It is important to understand how the q-exchangeability behaves under transformations. For $f : \mathbb{A} \to \mathbb{B}$ let f^∞ denote the induced mapping $\mathbb{A}^\infty \to \mathbb{B}^\infty$ which replaces each letter w_i in a word by $f(w_i)$. Consider first the identity mapping from $(\mathbb{A}, <)$ to $(\mathbb{A}, >)$.

Proposition 2.2. *If P is a q-exchangeable measure on words over $(\mathbb{A}, <)$ then P is q^{-1}-exchangeable with respect to $(\mathbb{A}, >)$, that is when the order on the basic space is reversed.*

Proof. Indeed, the claim is easily checked for the elementary transpositions which swap i and $i+1$. ☐

It is obvious that if f is an injective morphism of ordered Borel spaces then f^∞ sends one q-exchangeable measure to another q-exchangeable measure. This applies, in particular, to $\mathbb{A} \subseteq \mathbb{R}$ and strictly increasing function $f : \mathbb{A} \to \mathbb{R}$. It is less obvious that q-exchangeability is preserved by arbitrary monotone transformations:

Proposition 2.3. *Let $\mathbb{A}$ and $\mathbb{B}$ be Borel subsets of $\mathbb{R}$. Suppose $f : \mathbb{A} \to \mathbb{B}$ is weakly increasing, that is, $a < b$ implies $f(a) \leq f(b)$. Then the induced Borel map $f^\infty : \mathbb{A}^\infty \to \mathbb{B}^\infty$ preserves the q-exchangeability.*

This proposition will be reduced to its restricted version involving finite random words and finite alphabet $\mathbb{A}$ (see Proposition 2.5 to follow). In the case $q = 1$ the assertion turns into a familiar property of exchangeability, which holds for arbitrary Borel f.

Definition 2.1 has a straightforward counterpart for *finite* random words $w \in \mathbb{A}^n$. We say that a probability measure P_n on $\mathbb{A}^n$ is *finitely q-exchangeable* if for each $\sigma \in \mathfrak{S}_n$ the measure $T_\sigma^{-1} P_n$ is equivalent to P_n and the Radon–Nikodým derivative $dT_\sigma^{-1} P_n / dP_n$ is given by the function $q^{\operatorname{inv}(T_\sigma w) - \operatorname{inv}(w)}$. If $\mathbb{A}$ is finite or countable, then P_n is purely atomic and this condition means that for $w = w_1 \dots w_n \in \mathbb{A}^n$

$$P_n(T_\sigma w) = q^{\operatorname{inv}(T_\sigma w) - \operatorname{inv}(w)} P_n(w), \qquad \sigma \in \mathfrak{S}_n . \tag{2.3}$$

Consider the canonical projection $\mathbb{A}^\infty \to \mathbb{A}^n$ assigning to an infinite word $w = w_1 w_2 \dots$ its n-truncation $w_1 \dots w_n$, $n = 1, 2, \dots$. Given a probability measure P on $\mathbb{A}^\infty$, let P_n stand for the push-forward of P under the projection. Easily from the definitions we have:

Lemma 2.4. *A probability measure P on $\mathbb{A}^\infty$ is q-exchangeable if and only if P_n is finitely q-exchangeable for every $n = 1, 2, \dots$.*

In principle, the structure of the set of finitely q-exchangeable measures on $\mathbb{A}^n$ is clear: by finiteness of the group $\mathfrak{S}_n$ every such measure is a unique mixture of the extreme measures, and every extreme (=ergodic) measure is supported by a single $\mathfrak{S}_n$-orbit in $\mathbb{A}^n$. Moreover, every $\mathfrak{S}_n$-orbit carries a unique q-exchangeable probability measure, hence the extreme measures are in the bijective correspondence with the set of $\mathfrak{S}_n$-orbits in $\mathbb{A}^n$. Each $\mathfrak{S}_n$-orbit in $\mathbb{A}^n$ contains exactly one *inversion-free* word $v_1 \dots v_n \in \mathbb{A}^n$, that is satisfying $v_1 \le \dots \le v_n$. Thus the collection of inversion-free words of length n parameterizes the orbits of $\mathfrak{S}_n$ and all finitely q-exchangeable measures on $\mathbb{A}^n$.

Now we can state a simplified version of Proposition 2.3:

Proposition 2.5. *Let $\mathbb{A}$ and $\mathbb{B}$ be finite ordered alphabets and let $f : \mathbb{A} \to \mathbb{B}$ be a weakly increasing map. Then the induced map $f^n : \mathbb{A}^n \to \mathbb{B}^n$ preserves the finite q-exchangeability of measures.*

We show first how to deduce Proposition 2.3 from Proposition 2.5. To this end, let $\mathbb{A}$, $\mathbb{B}$ and f be as required in Proposition 2.3. Furthermore, let P be a q-exchangeable probability measure on $\mathbb{A}^\infty$ and $f^\infty(P)$ be its push-forward under f^∞. Observe that $(f^\infty(P))_n = f^n(P_n)$ for all $n = 1, 2, \dots$. By the virtue of Lemma 2.4, it suffices to prove that if a measure P_n on $\mathbb{A}^n$ is finitely q-exchangeable then so is its push-forward $f^n(P_n)$. This in turn shows that it suffices to inspect the particular case of extreme P_n. As pointed out above, every extreme measure P_n is concentrated on a single $\mathfrak{S}_n$-orbit, so that P_n actually lives on words from a finite alphabet. This provides the desired reduction to Proposition 2.5.

Proof of Proposition 2.5. Let P_n be a finitely q-exchangeable measure on $\mathbb{A}^n$ and $\widetilde{P}_n = f^n(P_n)$ be its push-forward on $\mathbb{B}^n$. Since the alphabets are finite, the measures are purely atomic, supported by finite sets, so that we may deal with probabilities of individual words.

It suffices to prove that for every word $u \in \mathbb{B}^n$ and every elementary transposition $\sigma = (i, i+1)$, one has

$$\widetilde{P}_n(u^*) = q^{\mathrm{inv}(u^*)-\mathrm{inv}(u)} \widetilde{P}_n(u), \qquad u^* := T_\sigma u.$$

Let us fix u and i. There are three possible cases: $u_i = u_{i+1}$, $u_i < u_{i+1}$, and $u_i > u_{i+1}$. In the first case, $u^* = u$, and the desired relation is trivial. By symmetry between the second and third cases, it suffices to examine one of them, say, the second case. Then $\mathrm{inv}(u^*) - \mathrm{inv}(u) = 1$. Consider the inverse images $X = (f^n)^{-1}(u)$ and $X^* = (f^n)^{-1}(u^*)$. Then we have $\widetilde{P}_n(u) = P_n(X)$ and $\widetilde{P}_n(u^*) = P_n(X^*)$. Thus, we are reduced to showing that

$$P_n(X^*) = q P_n(X).$$

Since f is weakly increasing, $u_i < u_{i+1}$ implies that $w_i < w_{i+1}$ for every $w \in X$, hence $P(T_\sigma w) = qP(w)$. It remains to note that the transformation $T_\sigma : \mathbb{A}^n \to \mathbb{A}^n$ maps X bijectively onto X^*. This concludes the proof. $\square$

Another proof will be given in the end of Section 3.

Proposition 2.6. *Let $f : \mathbb{A} \to \mathbb{B}$ be as in Proposition 2.3. If a probability measure P on $\mathbb{A}^\infty$ is q-exchangeable and extreme then so is its push-forward $f^\infty(P)$.*

Proof. By Proposition 2.3, $f^\infty(P)$ is q-exchangeable, hence quasi-invariant under the action of $\mathfrak{S}_\infty$. Obviously, the map f^∞ commutes with that action. Recall that extremality of quasi-invariant measures is equivalent to their ergodicity, so that it suffices to show that $f^\infty(P)$ is ergodic if P is such, but this follows straightforwardly from the definitions. $\square$

3. The finite q-shuffle

We fix a positive parameter q (later on we will assume $0 < q < 1$). For a finite permutation $\sigma \in \mathfrak{S}_n$ we denote by $\mathrm{inv}(\sigma)$ the number of inversions, meaning the number of inversions in the permutation word $\sigma(1) \dots \sigma(n)$. It is well known that

$$\sum_{\sigma \in \mathfrak{S}_n} q^{\mathrm{inv}(\sigma)} = [n]_q!\,,$$

where

$$[n]_q! := [1]_q [2]_q \dots [n]_q, \qquad [n]_q := \sum_{i=0}^{n-1} q^i$$

(this is a particular case of formula (5.4) below).

Definition 3.1. For $n = 1, 2, \dots$, the *Mallows measure* $\mathcal{Q}_n$ is the probability measure on $\mathfrak{S}_n$ defined by

$$\mathcal{Q}_n(\sigma) = \frac{q^{\mathrm{inv}(\sigma)}}{[n]_q!}\,, \qquad \sigma \in \mathfrak{S}_n.$$

The Mallows measure and its relatives, introduced in [M], have been studied in statistics in the context of ranking problems. See recent work [DR], [BBHM] for connections with card shuffling and exclusion processes, and [Sta] for a scaling limit of $\mathcal{Q}_n$.

If $q = 1$ then $\mathcal{Q}_n$ is just the uniform measure on $\mathfrak{S}_n$. Thus, for general $q > 0$, $\mathcal{Q}_n$ may be viewed as a deformation of the uniform measure.

The Mallows measure is the unique finitely q-exchangeable measure supported by the set of permutation words of length n, i.e. corresponding to the inversion-free word $1\,2\,\dots\,n$.

The measure $\mathcal{Q}_n$ can be characterized by means of important independence property partly mentioned in [M] [2]. To prepare, we need more notation. For $n = 1, 2, \dots$ we denote by $G_{q,n}$ the n-truncated geometric distribution on $\mathbb{N}_n = \{1, \dots, n\}$ with parameter q:

$$G_{q,n}(i) = \frac{q^{i-1}}{[n]_q}\,, \qquad i \in \mathbb{N}_n.$$

For a permutation $\sigma \in \mathfrak{S}_n$ (or the corresponding permutation word $\sigma(1)\dots\sigma(n)$) define *backward ranks*

$$\beta_j = \beta_j(\sigma) := \#\{i \le j \mid \sigma(i) \le \sigma(j)\}, \qquad j = 1, \dots, n. \tag{3.1}$$

For instance, the permutation word 1324 has $\beta_1 = 1$, $\beta_2 = 2$, $\beta_3 = 2$, $\beta_4 = 4$. The correspondence $\sigma \mapsto (\beta_1(\sigma), \dots, \beta_n(\sigma))$ is a well-known bijection between $\mathfrak{S}_n$ and the Cartesian product $\mathbb{N}_1 \times \dots \times \mathbb{N}_n$.

Proposition 3.2. *Mallows measure $\mathcal{Q}_n$ is the unique measure on $\mathfrak{S}_n$ under which the backward ranks are independent, with each variable $j - \beta_j + 1$ distributed according to $G_{q,j}$.*

Proof. Decompose the number of inversions as $\mathrm{inv}(\sigma) = \sum_{j=1}^n (j - \beta_j)$, and multiply probabilities of the truncated geometric distribution to see that $\mathcal{Q}_n$ coincides with the product measure. □

The following shuffling algorithm is central for our construction of finitely q-exchangeable measures. The procedure is a variation of 'absorption sampling' which was studied under various guises in [B], [Ke1], [R].

Definition 3.3. Given an arbitrary finite word $v_1 \dots v_n$, its *q-shuffle* is the random word $w_1 \dots w_n$ obtained by a random permutation of the letters $v_1, \dots, v_n$, determined by the following n-step algorithm. Let $\xi_1, \dots, \xi_n$ be independent random variables, with ξ_j having distribution $G_{q,n-j+1}$.

At step 1 take for w_1 the ξ_1th letter from the word $v^{(1)} := v_1 \dots v_n$. Then remove the letter v_{ξ_1} from $v^{(1)}$ and denote by $v^{(2)}$ the resulting word of length $n-1$. Iterate. So at each following step $m = 2, \dots, n$ there is a word $v^{(m)}$ which was derived from the initial word by deleting some $m-1$ letters, a new letter $w_m = v^{(m)}_{\xi_m}$ is then chosen, and if $m < n$ the word $v^{(m+1)}$ is obtained by removing this letter from $v^{(m)}$.

[2] On top of page 125 in [M] substitute $q^{-1/2}$ for Mallows' ϕ.

Proposition 3.4. *Let $v = v_1 \dots v_n$ be an inversion-free word on the ordered alphabet $\mathbb{A}$, so $v_1 \le \dots \le v_n$. Let w be the random word obtained from v by the q-shuffle algorithm, and let P_n be the distribution of w, which is a probability measure concentrated on the $\mathfrak{S}_n$-orbit of v. Then P_n is finitely q-exchangeable.*

Proof. First of all, observe that the probability $P_n(w)$ of any word w from the $\mathfrak{S}_n$-orbit of v is strictly positive. By the very definition of the finite q-exchangeability, it suffices to prove that if σ is an elementary transposition $(i, i+1)$, $i = 1, \dots, n-1$, then the ratio $P_n(T_\sigma(w))/P_n(w)$ equals q, q^{-1} or 1 depending on whether $w_i < w_{i+1}$, $w_i > w_{i+1}$ or $w_i = w_{i+1}$. The latter case being trivial, we may assume by symmetry that $w_i < w_{i+1}$.

For $w_1 < w_2$ suppose a word starts with $w_1 w_2$ and examine the transposition $\sigma = (1, 2)$, which swaps w_1 and w_2. Let I and J denote the sets of indices i and j for which $v_i = w_1$ and $v_j = w_2$, respectively. If the q-shuffle algorithm results in the word w, then the first chosen letter is v_i for some $i \in I$ and the second chosen letter is v_j for some $j \in J$. Likewise, if the resulting word starts with $w_2 w_1$, then we have to choose first v_j with some $j \in J$ and afterwards v_i with some $i \in I$. Let $P_{v_i v_j}$ and $P_{v_j v_i}$ stand for the corresponding probabilities.

If we fix $i \in I$ and $j \in J$ then the word $v^{(3)}$ obtained from the initial word v at the third step of the algorithm does not depend on the order in which v_i and v_j were chosen. Thus, it suffices to prove that $P_{v_i v_j}/P_{v_j v_i} = 1/q$.

The probabilities in question are easily computed. Note that $i < j$, because $v_i < v_j$. It follows that

$$P_{v_i v_j} = G_{q,n}(i) G_{q,n-1}(j-1) = \frac{q^{i+j-1}}{[n]_q [n-1]_q},$$

because after the first step the letter v_j acquires number $j-1$. On the other hand,

$$P_{v_j v_i} = G_{q,n}(j) G_{q,n-1}(i) = \frac{q^{j+i}}{[n]_q [n-1]_q},$$

because now the position of the second letter does not change after the first step. Therefore, the ratio in question is indeed equal to $1/q$.

Finally, transpositions $\sigma = (i, i+1)$ with $i = 2, 3, \dots$ are handled in the same way: the key point being that each of the words $v^{(2)}$, $v^{(3)} \dots$ is inversion-free. □

Remark 3.5. Note that the claim of Proposition 3.4 fails if one drops the assumption that v is inversion-free. For instance, if $v_1 \ge \dots \ge v_n$ then the resulting probability measure on the orbit will be q^{-1}-exchangeable and hence not q-exchangeable, except the trivial cases when $v_1 = \dots = v_n$ or $q = 1$.

The connection between Definitions 3.1 and 3.3 is established by the following

Corollary 3.6. *The q-shuffle, as introduced in Definition 3.3, coincides with the action of the random permutation $\sigma \in \mathfrak{S}_n$ distributed according to the Mallows measure $\mathcal{Q}_n$.*

Proof. As is seen from the description of the q-shuffle, it actually acts on positions of the letters rather than on letters themselves. Thus, it is given by the action of the random permutation $\sigma \in \mathfrak{S}_n$ distributed according to some probability measure $\mathcal{Q}'_n$ on $\mathfrak{S}_n$, which does not depend on the word to be q-shuffled. Let us identify permutations $\sigma \in \mathfrak{S}_n$ with the corresponding permutation words $\sigma(1) \dots \sigma(n)$. Then $\mathcal{Q}'_n$ can be characterized as the outcome of q-shuffling of the inversion-free word $v = 12 \dots n$. By Proposition 3.4, $\mathcal{Q}'_n$ is a finitely q-exchangeable probability measure concentrated on the $\mathfrak{S}_n$-orbit of v. Such a measure is unique, and the orbit can identified with the group $\mathfrak{S}_n$ itself. On the other hand, $\mathcal{Q}_n$ is q-exchangeable, thus $\mathcal{Q}'_n = \mathcal{Q}_n$. □

As yet another application of Proposition 3.4 we obtain an alternative proof of Proposition 2.5.

Second proof of Proposition 2.5. We will show that if P_n is an extreme q-exchangeable measure on $\mathbb{A}^n$ then so is $f^n(P_n)$. This will imply the claim of the proposition.

By Proposition 3.4, P_n is obtained by the q-shuffle applied to an inversion-free word $v \in \mathbb{A}^n$. Therefore, the same holds for the measure $f^n(P_n)$ and the word $f(v) := f(v_1) \dots f(v_n)$, because the q-shuffle commutes with the map f^n. Since f is weakly increasing, the word $f(v)$ is inversion-free. Applying again Proposition 3.4 we get the desired result. □

4. The infinite q-shuffle and statement of the main result

The above discussion of the finite q-exchangeability can be summarized as follows: the extreme finitely q-exchangeable probability measures are parameterized by finite inversion-free words and can be obtained by application of the q-shuffle procedure to these words. Now our aim is to find a counterpart of this result for measures on infinite words. As in Section 2, we are dealing with an ordered alphabet $(\mathbb{A}, <)$, where $\mathbb{A}$ is a Borel subset of $\mathbb{R}$. So far the parameter q was an arbitrary positive number, and

- throughout the rest of the paper we assume $0 < q < 1$.

By Proposition 2.2, this restriction does not lead to loss of generality, because the case $q > 1$ is reduced to the case $q < 1$ by inverting the order on the alphabet.

Let $\mathbb{N} = \{1, 2, \dots\}$ and G_q be the geometric distribution on $\mathbb{N}$ with parameter q:

$$G_q(i) = (1-q)q^{i-1}, \qquad i \in \mathbb{N}.$$

Definition 4.1. Let $v = v_1 v_2 \dots \in \mathbb{A}^\infty$ be an arbitrary infinite word. The *infinite q-shuffle* of v is the infinite random word $w = w_1 w_2 \dots$ produced by the algorithm similar to that in Definition 3.3. The only changes are that (i) the independent variables with varying truncated geometric distributions should be replaced by the independent variables $\xi_1, \xi_2, \dots$ with the same geometric distribution G_q, and (ii) the number of steps becomes infinite.

Although the infinite q-shuffle involves countably many steps, the first n letters in the output word w are specified after n steps of the algorithm. This shows, in particular, that the law of the random word w is well defined as a Borel probability measure on $\mathbb{A}^\infty$.

Lemma 4.2. *The output random word w is a random permutation of the letters of the input word v. That is to say, all letters of v appear in w, with probability* 1.

Proof. The probability that the first letter v_1 will not be chosen at the first m steps of the algorithm is equal to q^m. As $m \to \infty$, this quantity goes to 0, so that v_1 will appear in w with probability 1. Iterating this argument we arrive to the same conclusion for all other letters. □

As above, we say that an infinite word $v \in \mathbb{A}^\infty$ is *inversion-free* if it has no inversions, that is, $v_1 \le v_2 \le \dots$.

Proposition 4.3. *If $v \in \mathbb{A}^\infty$ is an inversion-free word then its q-shuffle produces a q-exchangeable Borel probability measure on $\mathbb{A}^\infty$.*

Proof. Let $P^{(v)}$ denote the measure in question. For any $n = 1, 2, \dots$, let $P_n^{(v)}$ be the nth marginal measure of P, as in Lemma 2.4. The same argument as in the proof of Proposition 3.4 shows that each of the measures $P_n^{(v)}$ is q-exchangeable. Consequently, by virtue of Lemma 2.4, $P^{(v)}$ is q-exchangeable, too. □

Let $\mathfrak{S}$ stand for the set of *all* permutations (i.e., bijections) of the set $\mathbb{N}$. We will often identify permutations $\sigma \in \mathfrak{S}$ with the corresponding infinite words $\sigma(1)\sigma(2)\dots \in \mathbb{N}^\infty$. In this way we get an embedding $\mathfrak{S} \hookrightarrow \mathbb{N}^\infty$. It is easy to check that $\mathfrak{S}$ is a Borel subset of $\mathbb{N}^\infty$, so that one can speak about Borel measures on $\mathfrak{S}$.

On the other hand, $\mathfrak{S}$ is a group containing $\mathfrak{S}_\infty$ as a proper subgroup. The group $\mathfrak{S}$ acts on $\mathbb{A}^\infty$ in the same way as $\mathfrak{S}_\infty$ does. Namely, if $\sigma \in \mathfrak{S}$ and $w \in \mathbb{A}^\infty$ then $(T_\sigma w)_i = w_{\sigma^{-1}(i)}$.

Definition 4.4. By virtue of Proposition 4.3 and Lemma 4.2, an application of the infinite q-shuffle to the inversion-free word $v = 1\,2\,\dots \in \mathbb{N}^\infty$ produces a q-exchangeable Borel probability measure on $\mathbb{N}^\infty$, which is concentrated on the group $\mathfrak{S}$. We call this measure *the Mallows measure on* $\mathfrak{S}$, and denote it $\mathcal{Q}$.

Remark 4.5. In accordance with our definition of the action of permutations on words, the permutation word $\sigma(1)\sigma(2)\dots$ corresponding to an element $\sigma \in \mathfrak{S}$ coincides with $T_{\sigma^{-1}}(12\dots)$, and not with $T_\sigma(12\dots)$. It follows that the infinite q-shuffle of any infinite word coincides with the action on it by the random permutation T_σ with $\sigma \in \mathfrak{S}$ distributed according to the push-forward of $\mathcal{Q}$ under the inversion map $\sigma \mapsto \sigma^{-1}$. However, as will be shown in Section 10, $\mathcal{Q}$ is actually preserved by this map, so that we may simply choose random σ distributed itself according to the Mallows measure $\mathcal{Q}$.

Given a word $v \in \mathbb{A}^\infty$, its *support*, denoted $\mathrm{supp}(v)$, is the subset of $\mathbb{A}$ comprised of all distinct letters that appear in v, without regard to their multiplicities. If no assumption on v is made, $\mathrm{supp}(v)$ may be any finite or countable subset of $\mathbb{R}$ and the letters from $\mathrm{supp}(v)$ may enter v with arbitrary multiplicities, finite or infinite. This is not the case, however, if v is inversion-free, as is demonstrated by the following evident proposition.

Proposition 4.6. *The inversion-free words $v \in \mathbb{R}^\infty$ belong to one of the following two types, depending on whether the support* $\mathrm{supp}(v)$ *is finite or infinite:*

(I) *The finite type:* $\operatorname{supp}(v)$ *is a finite set* $a_1 < \dots < a_d$. *Then for each* $i = 1, \dots, d-1$, *the letter* a_i *enters* v *with a finite nonzero multiplicity* l_{a_i}, *while the last letter* a_d *has infinite multiplicity, and*

$$v = \underbrace{a_1 \dots a_1}_{l_{a_1}} \dots \underbrace{a_{d-1} \dots a_{d-1}}_{l_{a_{d-1}}} \underbrace{a_d a_d \dots}_{l_{a_d}=\infty}.$$

(II) *The infinite type:* $\operatorname{supp}(v)$ *is a countable set* $a_1 < a_2 < \dots$. *Then for each* $i = 1, 2, \dots$, *the letter* a_i *enters* v *with a finite nonzero multiplicity* l_{a_i}, *and*

$$v = \underbrace{a_1 \dots a_1}_{l_{a_1}} \underbrace{a_2 \dots a_2}_{l_{a_2}} \dots$$

For both types, the finite multiplicities l_{a_i} *may take arbitrary positive integer values.*

For an inversion-free word $v \in \mathbb{R}^\infty$, let $\Omega^{(v)}$ denote its $\mathfrak{S}$-orbit, $\Omega^{(v)} := \{T_\sigma v \mid \sigma \in \mathfrak{S}\}$, which is a Borel subset in $\mathbb{R}^\infty$. By the very definition, the measure $P^{(v)}$ is concentrated on $\Omega^{(v)}$.

Remark 4.7. If $\operatorname{supp}(v)$ is finite then $\Omega^{(v)}$ coincides with the $\mathfrak{S}_\infty$-orbit of v and hence is countable (except when $\operatorname{supp}(v)$ is a singleton). Therefore, in this case the measure $P^{(v)}$ is purely atomic: for $w \in \Omega^{(v)}$, $P^{(v)}(w)$ is proportional to $q^{\operatorname{inv}(w)}$. Note that here $\operatorname{inv}(w)$, the total number of inversions in w, is finite. Moreover, the number

$$\mathcal{I}^{(v)}(k) := \#\{w \in \Omega^{(v)} \mid \operatorname{inv}(w) = k\}$$

has polynomial growth in k as $k \to \infty$, so that the series $\sum_k \mathcal{I}^{(v)}(k) q^k$ converges, which explains why the measure does exist. (Note that in the situation of the conventional de Finetti's theorem there are no finite invariant measures supported by a nontrivial $\mathfrak{S}_\infty$-orbit.) In contrast to that, if $\operatorname{supp}(v)$ is infinite then $\Omega^{(v)}$ has cardinality continuum and the measure $P^{(v)}$ is diffuse.

Now we are in a position to state the main result of the paper.

Theorem 4.8. *Let* $\mathbb{A}$ *be an arbitrary Borel subset of* $\mathbb{R}$ *with the order inherited from* $\mathbb{R}$. *The extreme* q*-exchangeable Borel probability measures on* $\mathbb{A}^\infty$ *are parameterized by the infinite inversion-free words* v *with support contained in* $\mathbb{A}$. *The measure* $P^{(v)}$ *corresponding to such a word* v *is obtained by application of the infinite* q*-shuffle to* v, *as described in Proposition 4.3.*

Observe that the orbits $\Omega^{(v)}$ with different v's are pairwise disjoint. It follows that Theorem 4.8 is reduced to the following seemingly weaker claim.

Proposition 4.9. *For* $\mathbb{A}$ *as in Theorem 4.8, the extreme* q*-exchangeable measures on* $\mathbb{A}^\infty$ *belong to the family of measures* $\{P^{(v)}\}$, *where* v *ranges over the set of inversion-free words in* $\mathbb{A}^\infty$.

Indeed, combining this proposition with the above observation, we see that none of the measures in the family $\{P^{(v)}\}$ can be written as a nontrivial mixture of other measures, which implies that each $P^{(v)}$ is extreme. A proof of Proposition 4.9 will be given in the next sections.

Remark 4.10. Given an element $\tau \in \mathfrak{S}$, let $\widetilde{\tau} \in \mathbb{N}^\infty$ denote the corresponding permutation word, $\widetilde{\tau} = \tau(1)\tau(2)\dots$. The Mallows measure $\mathcal{Q}$ (Definition 4.4) can be characterized as the only probability measure on the group $\mathfrak{S}$, which is quasi-invariant under the right shifts $\tau \mapsto \tau\sigma^{-1}$ by elements σ of the subgroup $\mathfrak{S}_\infty$, with the cocycle $\rho_q(\sigma, \widetilde{\tau})$. This follows from Theorem 4.8 and the very definition of $\mathcal{Q}$.

We shall inspect next the nature of random word $w \in \mathbb{A}^\infty$ under $P^{(v)}$. The sequence of truncations $\varnothing, w_1, w_1w_2, \dots$ has transition probabilities described in the following proposition. To explain the notation: letters a, b range over $\mathbb{A}$; l_a is the multiplicity of a in v, as above; $u = w_1 \dots w_{n-1}$ is a finite word; and $\mu_a(u)$ is the multiplicity of a in u.

Proposition 4.11. *Let w be the infinite random word distributed according to $P^{(v)}$. The transition probabilities have the form*

$$P^{(v)}(u \to ua) = q^{\sum_{b<a}(l_b-\mu_b(u))}(1 - q^{l_a-\mu_a(u)}) = q^{\sum_{b<a}(l_b-\mu_b(u))} - q^{\sum_{b\le a}(l_b-\mu_b(u))}. \tag{4.1}$$

Proof. Assume first $n = 1$, that is $u = \varnothing$. Then the left-hand side of (4.1) is the probability of $w_1 = a$, as in the first step of the q-shuffling algorithm. The string of a's in v starts from position $i := 1 + \sum_{b<a} l_b$ and ends at position $j := \sum_{b\le a} l_b$. Therefore, the probability in question equals

$$(1 - q)(q^{i-1} + \dots + q^{j-1}) = q^{i-1}(1 - q^{j-i+1}).$$

The same quantity appears in the right-hand side of (4.1) when $u = \varnothing$, because then $\mu_b(u) = 0$ for all $b \in \mathbb{A}$.

For $n = 2, 3, \dots$, the argument is exactly the same, taking into account that we are dealing with the nth step of the algorithm and the word $v^{(n)}$ is inversion-free, with letter multiplicities $l'_b = l_b - \mu_b(u)$. $\square$

Remark 4.12. Next are some comments to formula (4.1).

1. If $\mu_a(u) = l_a$ then (4.1) shows that the transition $u \to ua$ has probability zero. This agrees with the fact that if $l_a < \infty$ then the letter a cannot enter the random word more than l_a times. In particular, if $l_a = 0$ (which means $a \notin \operatorname{supp}(v)$) then a never appears.

2. The transition probability $P^{(v)}(u \to ua)$ depends on u only through the collection of multiplicities $\{\mu_a(u)\}_{a\in\mathbb{A}}$. That is, it depends only on the $\mathfrak{S}_n$-orbit of u.

3. Recall that the support of v is either of the form $a_1 < \dots < a_d$ or $a_1 < a_2 < \dots$. Let us set

$$x_0(u) = 1, \qquad x_i(u) = q^{\sum_{j\le i}(l_{a_j}-\mu_{a_j}(u))},$$

where $j = 1, \dots, d$ or $j = 1, 2, \dots$ for finite or infinite support, respectively. In this notation, (4.1) can be rewritten as

$$P^{(v)}(u \to ua_i) = x_{i-1}(u) - x_i(u), \qquad a_i \in \operatorname{supp}(v). \tag{4.2}$$

Now observe that

$$1 = x_0(u) \geq x_1(u) \geq \cdots \geq x_d(u) = 0$$

or

$$1 = x_0(u) \geq x_1(u) \geq x_2(u) \geq \cdots \geq 0 \quad \text{with} \quad \lim_{i\to\infty} x_i(u) = 0$$

for finite or infinite support, respectively. This makes evident the fact that the transition probabilities given by (4.2) indeed sum to 1.

4. We have deduced the formula (4.1) from the q-shuffling algorithm. Conversely, starting from (4.1), one can easily recover the algorithm itself.

Proposition 4.11 describes the measures $P^{(v)}$ via transition probabilities. The next proposition characterizes $P^{(v)}$ in terms of the marginal measures $P_n^{(v)}$, which are the joint distributions of the first n letters. Note that $P_n^{(v)}$ is a purely atomic measure, because it is supported by the words $u = u_1 \dots u_n$ with letters u_i from the finite or countable set $\operatorname{supp}(v)$, and the set of all such words is finite or countable. Thus, we may speak about probabilities $P_n^{(v)}(u)$ of individual words.

We recall some standard q-notation. Denote

$$(x;q)_0 = 1, \quad (x;q)_k := \prod_{i=0}^{k-1}(1 - xq^i), \quad k = 1, 2, \dots .$$

Likewise, we define $(x;q^{-1})_k$. Below we use the same notation as in Proposition 4.11.

Proposition 4.13. *Let $v \in \mathbb{R}^\infty$ be an inversion-free word, and let u be a word of length n with letters belonging to the support of v. We have*

$$P_n^{(v)}(u) = q^{\operatorname{inv}(u)} q^{-\sum_{b<a} \mu_b(u)\mu_a(u)} \prod_a (q^{l_a}; q^{-1})_{\mu_a(u)} q^{\mu_a(u)\sum_{b<a} l_b}, \tag{4.3}$$

where a and b assume values in $\operatorname{supp}(v)$.

Note that the product over $a \in \operatorname{supp}(v)$ is actually finite even if $\operatorname{supp}(v)$ is infinite. This follows from the fact that $\mu_a(u) = 0$ implies that the corresponding factor equals 1, and there are only finitely many a's with $\mu_a(u) \neq 0$.

Proof. Computing the ratio $P_{n+1}^{(v)}(ua)/P_n^{(v)}(u)$ from (4.3) one sees that the formula agrees with transition probabilities (4.1). □

5. The case of finite alphabet

In this section we prove Proposition 4.9 (and hence Theorem 4.8) for finite alphabet $\mathbb{A}$ with cardinality $d = \#\mathbb{A} \geq 2$. The simplest case $d = 2$ was examined in [GO2], and we will apply here the same method. To be definite, we take $\mathbb{A} = \mathbb{N}_d$. Following the formalism due to Kerov and Vershik [VK] it is insightful to interpret the q-exchangeability as a property of measures on the path space of a graded graph (Bratteli diagram) which captures the branching of orbits of $\mathfrak{S}_n$ on $\mathbb{A}^n$ as n varies.

Let $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$ and consider the d-dimensional lattice $\mathbb{Z}_+^d$. The lattice points will be denoted λ or μ. We write lattice points as vectors $\lambda = (\lambda_1, \dots, \lambda_d)$ in the canonical basis $e_1, \dots, e_d$, and we call $|\lambda| = \lambda_1 + \dots + \lambda_d$ the *degree* of λ. We write $\mu \prec \lambda$ if $\mu \neq \lambda$ and $\lambda - \mu \in \mathbb{Z}_+^d$; in this case there is a nondecreasing lattice path connecting μ with λ.

Each λ of degree n corresponds to an inversion-free word

$$v(\lambda) = v_1 \dots v_n = \underbrace{1 \dots 1}_{\lambda_1} \underbrace{2 \dots 2}_{\lambda_2} \dots \underbrace{d \dots d}_{\lambda_d}, \tag{5.1}$$

where letter a does not enter if $\lambda_a = 0$. This correspondence yields a bijection between $\mathfrak{S}_n$-orbits in $\mathbb{A}^n$ and vectors $\lambda \in \mathbb{Z}_+^d$ of degree n.

Definition 5.1. The *q-Pascal pyramid of dimension d*, denoted $\Gamma(q, d)$, is the oriented graph with the vertex set $\mathbb{Z}_+^d$ and directed edges $(\lambda, \lambda + e_a)$ endowed with weights

$$\mathrm{weight}(\lambda, \lambda + e_a) := q^{\lambda_{a+1} + \dots + \lambda_d}, \qquad a \in \mathbb{N}_d. \tag{5.2}$$

Note that $\mathrm{weight}(\lambda, \lambda + e_d) = 1$ for any λ. The nth level of the graph consists of the vertices $\lambda \in \mathbb{Z}_+^d$ with $|\lambda| = n$. Level 0 has a sole *root* vertex $\bar{0} := (0, \dots, 0)$. A *standard path* terminating at λ is a lattice path which connects $\bar{0}$ to λ and is nondecreasing in each coordinate. Similarly, we define *infinite standard path* in $\Gamma(q, d)$ as an infinite coordinate-wise nondecreasing path, with the initial vertex $\bar{0}$.

Observe that there is a natural bijection between $\mathbb{A}^n$ and standard paths in $\Gamma(q, d)$ of length n. By this bijection a word $w_1 \dots w_n$ is mapped to the path

$$\mu(\varnothing) = \bar{0},\ \mu(w_1) = e_{w_1},\ \mu(w_1 w_2) = e_{w_1} + e_{w_2}, \dots, \mu(w_1 \dots w_n) = e_{w_1} + \dots + e_{w_n},$$

where the ath coordinate of the terminal vertex is equal to the multiplicity of letter a in $w_1 \dots w_n$. For $n = 1, 2, \dots$ the bijections are consistent, hence define a bijection between $\mathbb{A}^\infty$ and the set of infinite standard paths in $\Gamma(q, d)$: under this bijection $w_n = a$ means that the nth edge of the path connects a vertex $\mu(w_1 \dots w_{n-1})$ of degree $n-1$ with $\mu + e_a$. Fixing the first n vertices of a standard path corresponds to a cylinder $[w_1 \dots w_n] \subset \mathbb{A}^\infty$. A measure P on $\mathbb{A}^\infty$ translates as a measure on the space of infinite standard paths, with $P([w_1 \dots w_n])$ being the probability of the corresponding initial path of length n.

Definition 5.2. The *weight* of a standard path with endpoint λ is defined as the product of weights of the edges comprising the path. Let us say that a probability measure on the path space of $\Gamma(q, d)$ is a *Gibbs measure* if for every λ the conditional measure of a standard path terminating at λ is proportional to the weight of this path[3].

Proposition 5.3. *For $\mathbb{A} = \mathbb{N}_d$, the q-exchangeable measures on $\mathbb{A}^\infty$ correspond uniquely to the Gibbs measures on the space of infinite standard paths in the q-Pascal pyramid $\Gamma(q, d)$.*

[3]In the terminology of Kerov and Vershik [VK] such a measure is called 'central'.

Proof. Let $w \in \mathbb{A}^\infty$. Under the correspondence between words and paths, $q^{\mathrm{inv}_n(w)}$ is equal to the weight of the standard path encoded in $w_1 \dots w_n$, as is seen by induction. Indeed, if the finite word $w_1 \dots w_{n-1}$ corresponds to λ and $w_n = a$ is appended, then the number of inversions increases by $\mathrm{inv}_n(w) - \mathrm{inv}_{n-1}(w) = \lambda_{a+1} + \dots + \lambda_d$, which is the same quantity that appears in (5.2); then we use the telescoping representation

$$\mathrm{inv}_n(w) = [\mathrm{inv}_n(w) - \mathrm{inv}_{n-1}(w)] + [\mathrm{inv}_{n-1}(w) - \mathrm{inv}_{n-2}(w)] + \dots + [\mathrm{inv}_1(w) - 0].$$

On the other hand, the words in $\mathbb{A}^n$ that correspond to standard paths with a given endpoint make up a $\mathfrak{S}_n$-orbit. Thus we see that the Gibbs condition for fixed n is equivalent to finite q-exchangeability. Since this holds for every n, Proposition 2.4 allows to translate the finite q-exchangeability for $n = 1, 2, \dots$ as the Gibbs property, and conversely. □

Now we shall proceed along the lines in [KOO]. Denote by $\mathrm{Path}(d)$ the space of all infinite standard paths in $\Gamma(q, d)$. With each $\lambda \in \mathbb{Z}_+^d$ we associate a unique *elementary* probability measure supported by the finite set of standard paths with endpoint λ: this measure corresponds to an orbital finitely q-exchangeable probability measure on $\mathbb{A}^n$. We can understand this measure as a function which assigns to λ value 1 and to each $\mu \prec \lambda$ assigns the probability that a path passes through μ. The *Martin boundary* of $\Gamma(q, d)$ consists of probability measures on $\mathrm{Path}(d)$ representable as weak limits of these elementary measures along a sequence of lattice points with $|\lambda| \to \infty$. We will prove that under the correspondence of Proposition 5.3, the Martin boundary is exactly the images of the measures $P^{(v)}$, with v ranging over the set of inversion-free words in $\mathbb{A}^\infty$. By the general theory (see [KOO]), the Martin boundary contains all extreme Gibbs measures, so that this will imply Proposition 4.9.

To determine the boundary we need to identify all asymptotic regimes for λ which guarantee convergence of the ratios

$$\frac{\dim(\mu, \lambda)}{\dim(\lambda)}, \tag{5.3}$$

where $\dim(\lambda) = \dim(\bar{0}, \lambda)$, and $\dim(\mu, \lambda)$ is equal to the sum of weights of all nondecreasing lattice paths connecting μ and λ (the weight of each such path is defined as the product of the weights of its edges). We set $\dim(\mu, \lambda) = 0$ if $\lambda - \mu \notin \mathbb{Z}_+^d$. The ratio (5.3) is the Martin kernel for a certain Markov chain and, by analogy with the Gibbs formalism in statistical physics, $\dim \lambda$ may be called 'partition function'.

Recall the notation

$$[0]_q! = 1, \qquad [n]_q! = [1]_q [2]_q \dots [n]_q = \frac{(q;q)_n}{(1-q)^n}, \quad n = 1, 2, \dots .$$

For nonnegative integers $n_1, \dots, n_d$ with $n_1 + \dots + n_d = n$ the number

$$\begin{bmatrix} n \\ n_1, \dots, n_d \end{bmatrix}_q := \frac{[n]_q!}{[n_1]_q! \dots [n_d]_q!} = \frac{(q;q)_n}{(q;q)_{n_1} \dots (q;q)_{n_d}}$$

is known as the *Gaussian multinomial coefficient*.

Lemma 5.4. *We have for* $\lambda = (\lambda_1, \dots, \lambda_d)$ *and* $\mu \prec \lambda$

$$\dim(\lambda) = \begin{bmatrix} |\lambda| \\ \lambda_1, \dots, \lambda_d \end{bmatrix}_q, \qquad \dim(\mu, \lambda) = q^{N(\mu,\lambda)} \dim(\lambda - \mu)$$

where

$$N(\mu, \lambda) = \sum_a \mu_a \sum_{b\colon b<a} \lambda_b - \sum_{b<a} \mu_b \mu_a.$$

Proof. Recall that the set of finite standard paths ending at λ is encoded by the words w belonging to the $\mathfrak{S}_{|\lambda|}$-orbit of the inversion-free word $v(\lambda)$ as defined in (5.1). Let $\{w\}$ stand for the set of these paths. MacMahon's formula for the generating function for the number of inversions in permutations of a multiset (see [An, Theorem 3.6]) says in our notation that

$$\sum_{\{w\}} q^{\mathrm{inv}(w)} = \begin{bmatrix} |\lambda| \\ \lambda_1, \dots, \lambda_d \end{bmatrix}_q. \tag{5.4}$$

This yields the formula for $\dim(\lambda)$. The formula for $\dim(\mu, \lambda)$ with

$$N(\mu, \lambda) = (\lambda_1 - \mu_1)(\mu_2 + \dots + \mu_d) + (\lambda_2 - \mu_2)(\mu_3 + \dots + \mu_d) \\ + \dots + (\lambda_{d-1} - \mu_{d-1})\mu_d$$

follows by counting inversions in the corresponding words, which in turn is done by comparing the oriented subgraph rooted at μ with the whole graph $\Gamma(q, d)$. □

A weakly increasing function $h : \mathbb{N}_d \to \{0, 1, \dots, \infty\}$ with $h(d) = \infty$ will be called a *height function* on $\mathbb{A} = \mathbb{N}_d$. We also set $h(0) := 0$ where appropriate. There is a natural bijection $h \leftrightarrow v$ between the height functions on $\mathbb{N}_d$ and the inversion-free words in $\mathbb{N}_d^\infty$:

$$v = \underbrace{1 \dots 1}_{h(1)} \ \underbrace{2 \dots 2}_{h(2)-h(1)} \ \underbrace{3 \dots 3}_{h(3)-h(2)} \ \cdots \ \underbrace{r \dots r}_{h(r)-h(r-1)} \ \underbrace{r+1\, r+1 \dots}_{h(r+1)=\infty}, \tag{5.5}$$

where for some $0 \le r < d$ each letter $1 \le a \le r$ appears $h(a) - h(a-1) < \infty$ times (if any), and infinitely many times for $a = r + 1$.

Proposition 5.5. *The Martin boundary of the graph* $\Gamma(q, d)$ *can be parameterized, in a natural way, by the height functions on* $\mathbb{N}_d$.

Proof. Using the identity

$$\frac{(q; q)_n}{(q; q)_{n-m}} = (q^n; q^{-1})_m, \qquad n \ge m \ge 0,$$

we derive from Lemma 5.4 for $\mu \prec \lambda,\ m = |\mu|$ and $n = |\lambda|$ that

$$\frac{\dim(\mu, \lambda)}{\dim \lambda} = q^{-\sum_{b<a} \mu_b \mu_a} \frac{(q; q)_{n-m}}{(q; q)_n} \prod_{a=1}^d (q^{\lambda_a}; q^{-1})_{\mu_a} q^{\mu_a \sum_{b\colon b<a} \lambda_b}. \tag{5.6}$$

Observe that the constraint $\mu \prec \lambda$ can be removed; indeed, if it is not satisfied then $\dim(\mu,\lambda) = 0$ and the right-hand side of (5.6) also vanishes, because $(q^{\lambda_a}; q^{-1})_{\mu_a} = 0$ for $\lambda_a < \mu_a$.

Let us rewrite (5.6) using the notation

$$h_\lambda(a) := \lambda_1 + \cdots + \lambda_a, \qquad a = 1, \dots, d, \quad h_\lambda(0) := 0,$$

in the form

$$\frac{\dim(\mu,\lambda)}{\dim\lambda} = q^{-\sum_{b<a}\mu_b\mu_a} \frac{(q;q)_{n-m}}{(q;q)_n} \prod_{a=1}^{d} (q^{h_\lambda(a)-h_\lambda(a-1)}; q^{-1})_{\mu_a} q^{\mu_a h_\lambda(a-1)}. \tag{5.7}$$

Now it is easy to analyze the asymptotics of this expression assuming that μ remains fixed while λ varies so that $n = |\lambda| \to \infty$. First of all, note that

$$\lim_{n\to\infty} \frac{(q;q)_{n-m}}{(q;q)_n} = \frac{(q;q)_\infty}{(q;q)_\infty} = 1.$$

Next, observe that

$$0 \le h_\lambda(1) \le \cdots \le h_\lambda(d-1) \le h_\lambda(d) = n.$$

Passing to a subsequence, we may assume that there exist finite or infinite limits

$$\lim_{n\to\infty} h_\lambda(a) = h(a) \in \mathbb{Z}_+ \cup \{+\infty\}, \qquad a = 1, \dots, d\,.$$

This means that there exists $0 \le r < d$ such that the numbers $h_\lambda(1), \dots h_\lambda(r)$ stabilize for n large enough, $h_\lambda(a) = h(a) < \infty$ for $1 \le a \le r$, while $h_\lambda(a) \to h(a) = +\infty$ for $a > r$. Note that $h_\lambda(d) = n$ always goes to infinity, so that $h(d) = \infty$ in any case.

Clearly, the product in (5.7) up to $a = r$ stabilizes. Next, we have

$$(q^{h_\lambda(r+1)-h_\lambda(r)}; q^{-1})_{\mu_{r+1}} q^{\mu_{r+1} h_\lambda(r)} \to q^{\mu_{r+1} h_\lambda(r)},$$

because $q^{h_\lambda(r+1)-h_\lambda(r)} \to 0$. As for the factors with $a > r+1$, we have

$$(q^{h_\lambda(a)-h_\lambda(a-1)}; q^{-1})_{\mu_a} q^{\mu_a h_\lambda(a-1)} \to \delta_{\mu_a,0},$$

with the Kronecker delta in the right-hand side, because $h_\lambda(a-1) \to \infty$.

We conclude that convergence $h_\lambda \to h$ implies

$$\frac{\dim(\mu,\lambda)}{\dim\lambda} \to q^{-\sum_{b<a}\mu_b\mu_a} \prod_{a=1}^{d} (q^{h(a)-h(a-1)}; q^{-1})_{\mu_a} q^{\mu_a h(a-1)}, \tag{5.8}$$

with the convention that $h(0) = 0$ and $h(a) - h(a-1) = 0$ if $h(a) = h(a-1) = +\infty$. Since for distinct h the limits in (5.8) are all distinct, the Martin boundary can indeed be parameterized by the height functions. ☐

Observe that if $h(a) = h(a-1)$ then the limit value (5.8) vanishes unless $\mu_a = 0$. Returning to random words $w = w_1 w_2 \cdots \in \mathbb{A}^\infty$, this means that if $h(a) = h(a-1)$, then the letter a does not occur in w, with probability 1.

Proposition 5.6. *Under the correspondence $h \leftrightarrow v$, the measures on* Path(d) *afforded by Proposition 5.5 correspond exactly to the measures $P^{(v)}$, where v ranges over the set of inversion-free words on the alphabet $\mathbb{N}_d$.*

Proof. Fix a height function h and let $\mathcal{P}$ be the corresponding Gibbs measure on Path(d). Next, let P be the measure on $\mathbb{N}_d^\infty$, which corresponds to $\mathcal{P}$ via the bijection of Proposition 5.3. Finally, let $v \in \mathbb{N}_d^\infty$ be the inversion-free word associated with h. We have to prove that $P = P^{(v)}$. To do this it suffices to check that $P_n = P_n^{(v)}$ for all n. Let $u \in \mathbb{N}_d^n$. Then $P_n(u)$ equals $q^{\mathrm{inv}(v)}$ times the right-hand side of (5.8), where we set $\mu_a = \mu_a(u)$. Comparing with (4.3) we see that this coincides with $P_n^{(v)}(u)$. $\square$

This concludes the proof of Proposition 4.9 in the case of finite alphabet $\mathbb{A}$.

6. The case $\mathbb{A} = \mathbb{N}$

In this section we assume that $\mathbb{A}$ is the countable ordered set $(\mathbb{N}, <)$ of positive integers. Our aim is to prove, for this case, Proposition 4.9 and hence Theorem 4.8.

Definition 6.1. By a *height function* on $\mathbb{N}$ we shall mean a map $h : \mathbb{N} \to \mathbb{Z}_+ \cup \{+\infty\}$ which is weakly increasing (that is, $h(a) \le h(b)$ for $a < b$) and satisfies $\lim_{a\to\infty} h(a) = +\infty$. The set of all height functions on $\mathbb{N}$ will be denoted $H(\mathbb{N})$.

Obviously, setting

$$l_a = h(a) - h(a-1), \qquad a \in \mathbb{N},$$

with the understanding that $h(0) = 0$ and $l_a = 0$ if $h(a) = h(a-1) = +\infty$, we get a bijection $h \leftrightarrow v$ between $H(\mathbb{N})$ and the set of all inversion-free words $v \in \mathbb{N}^\infty$.

Proof of Proposition 4.9 for $\mathbb{A} = \mathbb{N}$. Let P be an extreme q-exchangeable measure on $\mathbb{N}^\infty$. We have to show that $P = P^{(v)}$ for some v. The idea is to reduce this claim to the case $\mathbb{A} = \mathbb{N}_d$, which has been examined in Section 5, by using Propositions 2.3 and 2.6.

For $d = 1, 2, \dots$ and $a \in \mathbb{N}$, set $f_d(a) = a \wedge d = \min(a, d)$. Clearly, this gives us a weakly increasing map $f_d : \mathbb{N} \to \mathbb{N}_d$. By Proposition 2.6, $f_d^\infty(P)$ is an extreme q-exchangeable measure on $\mathbb{N}_d^\infty$. By the result of Section 5, it coincides with some measure $P^{(v(d))}$ where $v(d) \in \mathbb{N}_d^\infty$ is an inversion-free word. Denote by h_d the corresponding height function on $\mathbb{N}_d$.

Let $w \in \mathbb{N}^\infty$ be the random word with law P. For each $a = 1, \dots, d-1$, the letter a enters the random word $f_d(w)$ exactly $h_d(a) - h_d(a-1)$ times, with probability 1. Since the map f_d does not change the letters $a = 1, \dots, d-1$, the same holds for the initial random word w. This implies that $h_d(a) = h_{d+1}(a)$ for all $a = 1, \dots, d-1$. Therefore, for every $a \in \mathbb{N}$, the value $h_d(a)$ stabilizes as $d \to \infty$, starting from $d = a+1$; denote by $h(a)$ this stable value. We claim that h is a height function on $\mathbb{N}$. Indeed, it is obvious that h weakly increases, so that we only have to check that $h(a) \to \infty$ as $a \to \infty$. If this were not the case then $h(a)$ would assume the same (finite) value for all a large enough. But this would mean that w contained only finitely many letters, each with a prescribed finite multiplicity $l_a = h(a) - h(a-1)$, which is clearly impossible. Thus, h should be a height function.

Now, let $v \in \mathbb{N}^\infty$ be the inversion-free word corresponding to h. By the very definition of h, we have $f_d^\infty(P) = f_d^\infty(P^{(v)})$ for all d. Clearly, this implies $P_n = P_n^{(v)}$ for all n, so that $P = P^{(v)}$, as desired. □

Remark 6.2. An alternative proof can be based on the notion of the *q-Pascal pyramid of dimension* ∞, denoted $\Gamma(q,\infty)$, which is the graph with the vertex set

$$\{\lambda \in \mathbb{Z}_+^\infty \mid \lambda_1 + \lambda_2 + \cdots < +\infty\},$$

the edges $(\lambda, \lambda + e_a)$, where

$$e_a = (\underbrace{0,\ldots,0}_{a-1}, 1, 0, 0, \ldots), \qquad a \in \mathbb{N},$$

and the weight $q^{\sum_{b>a}\lambda_b}$ assigned to the edge $(\lambda, \lambda+e_a)$. Note that the sum in the exponent is finite because $|\lambda| := \sum_a \lambda_a$ is finite by the definition of $\Gamma(q,\infty)$. The nth level of $\Gamma(q,\infty)$ consists of vertices with $|\lambda| = n$.

The graph $\Gamma(q,d)$ is embedded in $\Gamma(q,\infty)$ as the set of vertices with $\lambda_b = 0$ for $b > d$. Obviously, $\Gamma(q,\infty) = \cup_{d\geq 1}\Gamma(q,d)$. The definition of Gibbs measures on the space of standard paths in $\Gamma(q,\infty)$, and the correspondence with q-exchangeable measures on $\mathbb{N}^\infty$ extend straightforwardly the definitions from Section 5. Then one can repeat the arguments in Proposition 5.5 to show that the Martin boundary of $\Gamma(q,\infty)$ consists precisely of the Gibbs measures corresponding to measures $P^{(v)}$.

7. The case $\mathbb{A} = \mathbb{R}$

Here we prove Proposition 4.9 and hence Theorem 4.8 for $\mathbb{A} = \mathbb{R}$. This will also cover the seemingly more general case with $\mathbb{A}$ an arbitrary Borel subset of $(\mathbb{R}, <)$.

Assume measure P on $\mathbb{R}^\infty$ is q-exchangeable and extreme. Our aim is to show that there exists a finite or countable subset $A \subset \mathbb{R}$, of the form $a_1 < \cdots < a_d$ or $a_1 < a_2 < \ldots$, such that P is supported by A^∞. Then the results of Sections 5 and 6 will imply that $P = P^{(v)}$ for some inversion-free word v.

For an arbitrary word $w \in \mathbb{R}^\infty$, set $h_w(x) := \#\{j : w_j \leq x\}$. The function $h_w : \mathbb{R} \to \mathbb{Z}_+ \cup \{+\infty\}$ is weakly increasing and right-continuous, hence it is completely determined by its restriction on the set $\mathbb{Q}$ of rational numbers.

For $x \in \mathbb{R}$ let $\phi_x : \mathbb{R}^\infty \to \{1,2\}^\infty$ be the mapping which replaces each $w_j \in (-\infty, x]$ by 1 and each $w_j \in (x, +\infty)$ by 2. The measure $\phi_x^\infty(P)$ on $\{1,2\}^\infty$ is q-exchangeable and extreme, by the virtue of Proposition 2.6. Since $h_w(x)$ is the number of 1's in $\phi_x(w)$, the ergodicity implies that the value $h_w(x)$ is the same for P-almost all words w. Letting x to run over $\mathbb{Q}$ we see that, outside a P-null set of words, the value $h_w(x)$ does not depend on w for each $x \in \mathbb{R}$; we denote $h(x)$ this common value. The function $h(x)$ is again weakly increasing and right-continuous, and assumes values in $\mathbb{Z}_+ \cup \{+\infty\}$.

Recall that in the $d = 2$ case q-exchangeability implies the dichotomy: either 1 appears finitely many times and 2 appears infinitely often, or 2 does not appear at all. From this, $h(x) \equiv \infty$ would imply $w_j \le x$ for all j, which is impossible. It follows that $h(x)$ cannot be identically equal to $+\infty$.

By a similar argument, $h(x)$ cannot be identically equal to a finite constant as well.

Defining A to be the set of the jump points of h, we see that A is either a nonempty finite set $a_1 < \cdots < a_d$ or a countably infinite set of the form $a_1 < a_2 < \ldots$. In the latter case we set $a^* = \sup\{a_i\} = \lim a_i \in \mathbb{R} \cup \{+\infty\}$. By the very definition of $h(x)$, the function is constant on every interval of the form

$$(-\infty, a_1), \quad [a_{i-1}, a_i), \quad [a^*, +\infty).$$

Finally, observe that if one ignores a P-null set of words mentioned above, then any word w does not contain letters from the open intervals

$$(-\infty, a_1), \quad (a_{i-1}, a_i), \quad (a^*, +\infty).$$

We conclude that P is concentrated on A^∞.

Remark 7.1. We note in passing that this argument fails for more general ordered spaces. For instance, it cannot be applied to $\mathbb{R}^k$ $(k > 1)$ with lexicographic order, because the order is not separable and h cannot be determined by its restriction to a countable set.

8. Quantization

A motivation to study the q-exchangeability is that this property can be viewed as a quantization of the conventional exchangeability. We comment briefly on this connection.

In the classical setting, each extreme exchangeable P on $\mathbb{R}^\infty$ is of the form $\nu^{\otimes\infty}$, where ν is the limit of empirical measures, meaning that for every Borel $B \subset \mathbb{R}$, as $n \to \infty$, the random word satisfies the strong law of large numbers

$$\#\{j \le n \,|\, w_j \in B\} \sim n\,\nu(B) \qquad P\text{–a.s.} \tag{8.1}$$

Trivially, $0 < P(w_1 \in B) < 1$ if and only if $0 < \nu(B) < 1$, in which case letters from A appear in w infinitely many times for both $A = B$ and $A = B^c$.

In the framework of q-exchangeability (with $q < 1$), the analogue of (8.1) is

$$\#\{j \le n \,|\, w_j \in B\} \to \nu_q(B) \qquad P\text{–a.s.} \tag{8.2}$$

where ν_q is a *counting* measure associated with some height function h, so that the letters from B are represented in w exactly $\nu_q(B)$ times. Similarly to the above, one sees from the formula

$$P(w_1 \in B) = \sum_{\{x \in B \,|\, \nu\{x\} > 0\}} q^{\nu(-\infty, x)}(1 - q^{\nu\{x\}})$$

that $0 < P(w_1 \in B) < 1$ if and only if $0 < \nu_q(B) < \infty$.

There are many ways to approach the exchangeability through q-exchangeability, that is to obtain independent sampling in the classical limit $q \to 1$. One possible explicit realization of such limit is the following quantization of homogeneous product measures.

Let ν be a probability measure on $\mathbb{R}$, with distribution function $F(x) := \nu(-\infty, x]$. Let $F^{-1}(p) := \inf\{x \in \mathbb{R} : F(x) \geq p\}$ be the corresponding quantile function, and consider the countable collection of quantiles $\alpha_k := F^{-1}(1-q^k),\ k \in \mathbb{N}$, as letters of the inversion-free word $v := \alpha_1\alpha_2 \ldots$. The idea is to bridge between independent sampling from ν and the q-shuffle for the counting measure $\nu_q = \sum_{j\in\mathbb{N}} \delta_{\alpha_j}$ by means of independent sampling from the measures

$$\widetilde{\nu}_q = \sum_{k\in\mathbb{N}} G_q(k)\, \delta_{\alpha_k}.$$

Proposition 8.1. *As $q \to 1$, for $v = \alpha_1\alpha_2 \ldots$ the q-shuffle measures $P^{(v)}$ converge, in the sense of weak convergence of the finite-dimensional marginal measures $P_n^{(v)},\ n \in \mathbb{N}$, to the product measure $\nu^{\otimes\infty}$.*

Proof. For ξ a random variable with geometric distribution G_q, the distribution of randomized quantile α_ξ is $\widetilde{\nu}_q$. It is convenient to introduce two more random variables: ζ with uniform distribution on $[0, 1]$, and ζ_q with the discrete distribution

$$\sum_{k\in\mathbb{N}} G_q(k)\, \delta_{1-q^k}. \tag{8.3}$$

From standard properties of the quantile function, the distribution of $F^{-1}(\zeta)$ is ν, and the distribution of $F^{-1}(\zeta_q)$ is $\widetilde{\nu}_q$, so that we can identify $\alpha_\xi = F^{-1}(\zeta_q)$.

Now, measure (8.3) was designed so, that the mass of each interval $[0, 1-q^k]$ is $1-q^k$, and the largest atom has mass $1-q$, which approaches 0 as $q \to 1$. Therefore, ζ_q converges in distribution to ζ. On the other hand, the set of discontinuities of the quantile function is at most countable and so has Lebesgue measure zero, hence F^{-1} preserves the convergence relation (see e.g. [Bi, Theorem 5.1]), meaning that $F^{-1}(\zeta_q) \to_d F^{-1}(\zeta)$. The latter is the same as

$$\mathbb{P}(\alpha_\xi \leq x) \to F(x) \quad \text{as} \quad q \to 1,$$

where x is arbitrary continuity point of F. For any nonnegative integer m, the total variation distance between ξ and the shift $\xi + m$ equals $1 - q^m$, from which the above can be strengthened as

$$\mathbb{P}(\alpha_{\xi+m} \leq x) \to F(x) \quad \text{as} \quad q \to 1.$$

In the like way, if $\xi_1, \xi_2, \ldots$ are independent copies of ξ and $m_1, \ldots, m_n$ are arbitrary fixed nonnegative integers, then we have

$$\mathbb{P}(\alpha_{\xi_1+m_1} \leq x_1, \ldots, \alpha_{\xi_n+m_n} \leq x_n) \to F(x_1) \cdots F(x_n) \quad \text{as} \quad q \to 1,$$

where $x_1, \ldots, x_n$ are arbitrary continuity points of F.

Let $w_1w_2 \ldots$ be the q-shuffle of $1\,2\ldots$, constructed from the independent geometric $\xi_1, \xi_2, \ldots$, as in Definition 4.1. Easily from the definition, $\xi_j \leq w_j < \xi_j + j$, whence the above implies

$$\mathbb{P}(\alpha_{w_1} \leq x_1, \ldots, \alpha_{w_n} \leq x_n) \to F(x_1) \cdots F(x_n) \quad \text{as} \quad q \to 1$$

for continuity points $x_1, \dots x_n$, which is precisely the property of weak convergence of $P_n^{(v)}$ we wanted to prove. $\square$

The construction offers quantization of homogeneous product measures on $\mathbb{R}^\infty$. Extension to the general exchangeable case is straightforward in the light of de Finetti's theorem: just randomize ν.

9. Random flags over a Galois field

Fix $q \in (0,1)$ and set $\tilde{q} = q^{-1}$, so that $\tilde{q} > 1$. In this section we assume that $\tilde{q}$ is a power of a prime number.

Let $\mathbb{F}_{\tilde{q}}$ be the Galois field with $\tilde{q}$ elements and let V_∞ be an infinitely-dimensional vector space over $\mathbb{F}_{\tilde{q}}$ with a countable basis $\{v_1, v_2, \dots\}$. Defining V_n to be the linear span of vectors $v_1, \dots, v_n$, we have $\cup_{n\ge 0} V_n = V_\infty$, so that each element of V_∞ can be uniquely written in the basis as an infinite vector with finitely many nonzero components.

For $d \in \mathbb{N}$, by *decreasing d-flag in V_∞* we shall mean a $(d+1)$-tuple $X = (X(i))$ of linear subspaces in V_∞ such that

$$V_\infty = X(0) \supseteq X(1) \supseteq \cdots \supseteq X(d-1) \supseteq X(d) = \{0\}$$

Keep in mind that our definition disagrees with the conventional notion of a flag in that the inclusions are not necessarily strict. In the same way we define decreasing d-flags in each space V_n. Let $\mathbf{X}_d(V_\infty)$ and $\mathbf{X}_d(V_n)$ denote the sets of the decreasing d-flags in V_∞ and V_n, respectively.

Lemma 9.1. *One can identify $\mathbf{X}_d(V_\infty)$ with the projective limit space $\varprojlim \mathbf{X}_d(V_n)$, where the projection $\mathbf{X}_d(V_{n+1}) \to \mathbf{X}_d(V_n)$ is determined by taking intersection with V_n.*

Proof. Indeed, the map $\mathbf{X}_d(V_\infty) \to \varprojlim \mathbf{X}_d(V_n)$ is defined by assigning to a flag $X = (X(i))$ in V_∞ the sequence $\{X_n \in \mathbf{X}_d(V_n)\}$ of flags with $X_n(i) = X(i) \cap V_n$. Clearly, the flags X_n are consistent with the projections $\mathbf{X}_d(V_{n+1}) \to \mathbf{X}_d(V_n)$ and hence determine an element of the projective limit space. The inverse map assigns to any such sequence $\{X_n\}$ the flag $X \in \mathbf{X}_d(V_\infty)$ with $X(i) = \cup X_n(i)$. $\square$

Using the lemma we endow $\mathbf{X}_d(V_\infty)$ with the topology of projective limit. In other words, a small neighborhood of a flag $X = (X(i))$ is formed by the flags $Y = (Y(i))$ such that $X(i) \cap V_n = Y(i) \cap V_n$ for all i and some fixed large n. We will consider the sigma-algebra of Borel sets in $\mathbf{X}_d(V_\infty)$ relative to this topology.

Let $\mathscr{G}_n$ be the group of all invertible linear transformations of the space V_∞ that leave V_n invariant and fix the basis vectors $v_{n+1}, v_{n+2}, \dots$. We have then $\{e\} = \mathscr{G}_0 \subset \mathscr{G}_1 \subset \mathscr{G}_2 \subset \dots$ and we define $\mathscr{G}_\infty := \cup_{n\ge 1} \mathscr{G}_n$. The group $\mathscr{G}_n$ is finite and isomorphic to the group $GL(n, \mathbb{F}_{\tilde{q}})$ of invertible $n \times n$ matrices over $\mathbb{F}_{\tilde{q}}$. The countable group $\mathscr{G}_\infty$ is isomorphic to the group $GL(\infty, \mathbb{F}_{\tilde{q}})$ of infinite invertible matrices (g_{ij}), such that $g_{ij} = \delta_{ij}$ for large enough $i + j$.

The group $\mathscr{G}_n$ acts, in a natural way, on $\mathbf{X}_d(V_n)$, and the group $\mathscr{G}_\infty$ acts on $\mathbf{X}_d(V_\infty)$ by continuous transformations. The next proposition is an extension of [GO2, Lemma 5.2].

Proposition 9.2. *There exists a natural bijection $P \leftrightarrow \mathcal{P}$ between q-exchangeable Borel probability measures on $\mathbb{N}_d^\infty$ and $\mathscr{G}_\infty$-invariant Borel probability measures on $\mathbf{X}_d(V_\infty)$.*

Proof. The desired bijection is constructed by understanding P as a Gibbs measure on the path space $\mathrm{Path}(d)$ of the q-Pascal pyramid $\Gamma(q,d)$, as defined in Section 5.

We assign to P a function $\varphi(\lambda)$ on the vertices in the following way. Given a vertex $\lambda \in \Gamma(q,d)$, the probability of a finite path ending at λ equals the weight of the path times a quantity that (for given P) depends on λ only; let us denote this quantity $\varphi(\lambda)$.

The Gibbs measure is uniquely determined by this function φ, which must satisfy the rule of addition of probabilities along the path:

$$\varphi(\lambda) = \sum_{a=1}^{d} \mathrm{weight}(\lambda, \lambda + e_a)\varphi(\lambda + e_a) \tag{9.1}$$

for all $\lambda \in \mathbb{Z}_+^d$, where the weight of the edge $(\lambda, \lambda + e_w)$ is specified in (5.2) as

$$\mathrm{weight}(\lambda, \lambda + e_a) = q^k \quad \text{for} \quad k = \lambda_{a+1} + \cdots + \lambda_d. \tag{9.2}$$

One has also to add the normalization condition $\varphi(\bar{0}) = 1$, which implies that

$$\sum_{\lambda \in \mathbb{Z}_+^d \colon |\lambda| = n} \dim(\lambda)\varphi(\lambda) = 1, \qquad n = 1, 2, \dots, \tag{9.3}$$

so that $\dim(\lambda)\varphi(\lambda)$ is the probability that a random walk on $\Gamma(q,d)$ driven by P ever visits λ.

Conversely, if a nonnegative function φ satisfies (9.1) and the normalization condition then it defines a Gibbs measure[4].

Now we wish to show that precisely the same functions are associated with the $\mathscr{G}_\infty$-invariant measures. Indeed, there is a one-to-one correspondence between $\mathscr{G}_\infty$-invariant probability measures $\mathcal{P}$ on $\mathbf{X}_d(V_\infty)$ and sequences $\{\mathcal{P}_n\}$ of probability measures such that each $\mathcal{P}_n$ is a measure on $\mathbf{X}_d(V_n)$ invariant under $\mathscr{G}_n$, and various $\mathcal{P}_n$'s are consistent with respect to the projections $\mathbf{X}_d(V_{n+1}) \to \mathbf{X}_d(V_n)$. Specifically, the correspondence is established by letting $\mathcal{P}_n$ to be the push-forward of $\mathcal{P}$ under the projection $\mathbf{X}_d(V_\infty) \to \mathbf{X}_d(V_n)$.

Observe that the $\mathscr{G}_n$-orbit of a d-flag $X_n = (X_n(i)) \in \mathbf{X}_d(V_n)$ is uniquely determined by the d-tuple of nonnegative integers [5]

$$\lambda_i = \dim V_n(i-1) - \dim V_n(i), \qquad i = 1, \dots, d,$$

which determine a vector $\lambda \in \mathbb{Z}_+^d$ with $|\lambda| = n$. We will say that the vertex λ is the *type* of the flag. Conversely, every such λ corresponds to an orbit. Let $\psi(\lambda)$ be the mass that

[4]Such functions φ play a central role in the work of Kerov and Vershik (see, e.g., [VK]), who called them 'harmonic'. This is unfortunate terminology which disagrees with the conventional concept of harmonic function in the literature on Markov processes.

[5]Warning: the dimension of a linear space over $\mathbb{F}_q$ in this formula and below should not be confused with the combinatorial dimension function in the Pascal pyramid, like e.g. in (9.3).

$\mathcal{P}_n$ gives to each of the flags of type λ. The consistency of the measures $\mathcal{P}_n$ with respect to the projections means that

$$\psi(\lambda)=\sum_{a=1}^{d}\operatorname{weight}'(\lambda,\lambda+e_a)\psi(\lambda+e_a),\qquad \lambda\in\mathbb{Z}_+^d, \tag{9.4}$$

where $\operatorname{weight}'(\lambda,\lambda+e_a)$ stands for the number of flags $X_{n+1}\in\mathbf{X}_d(V_{n+1})$ of type $\lambda+e_a$ projecting onto any fixed flag $X_n\in\mathbf{X}_d(V_n)$ of type λ. Conversely, each function $\psi(\lambda)\ge 0$ satisfying (9.4) and the normalization condition $\psi(\bar{0})=1$ determines a consistent sequence $\{\mathcal{P}_n\}$ and hence a $\mathscr{G}_\infty$-invariant probability measure $\mathcal{P}$ on $\mathbf{X}_d(V_\infty)$.

We claim that

$$\operatorname{weight}'(\lambda,\lambda+e_a)=\tilde{q}^{\,n-k}=q^{k-n},$$

where k is the same as in (9.2), that is, $k=\dim X_n(a)$. Indeed, if a flag X_{n+1} is projected onto X_n, then it has type $\lambda+e_a$ if and only if

$$\dim X_{n+1}(i)=\dim X_n(i)+1\quad\text{for}\quad 0\le i\le a-1,$$

and

$$\dim X_{n+1}(j)=\dim X_n(j)\quad\text{for}\quad a\le j\le d.$$

This means that there exists a nonzero vector $v\in V_{n+1}\setminus V_n$ such that, for every $i=0,\dots,a-1$, the subspace $X_{n+1}(i)$ is spanned by $X_n(i)$ and v. Such a vector is defined uniquely up to a scalar multiple and addition of an arbitrary vector from $X_n(a)$. Therefore, the number of options is equal to the number of lines in $V_{n+1}/X_n(a)$ not contained in $V_n/X_n(a)$, which equals

$$\frac{\tilde{q}^{\,n+1-k}-1}{\tilde{q}-1}-\frac{\tilde{q}^{\,n-k}-1}{\tilde{q}-1}=\tilde{q}^{\,n-k}.$$

Viewing equations (9.1) and (9.4) as recursions on φ, respectively ψ, we see that they are similar, with the coefficients related as

$$\operatorname{weight}'(\lambda,\lambda+e_a)=\operatorname{weight}(\lambda,\lambda+e_a)q^{-n},\qquad n=|\lambda|.$$

Setting

$$\varphi(\lambda)=q^{n(n-1)/2}\psi(\lambda)$$

yields an isomorphism $\{\varphi\}\leftrightarrow\{\psi\}$ between the convex compact sets of nonnegative solutions to (9.1) and (9.4), respectively. Note also that the above relation does not affect the normalization condition. This completes the proof. □

Remark 9.3. By the virtue of isomorphism in Proposition 9.2, the extreme measures P correspond bijectively to extreme measures $\mathcal{P}$.

Remark 9.4. Define a *decreasing* $\mathbb{N}$-*flag* in V_∞ as an infinite collection $X=(X(i))$ of subspaces such that

$$V_\infty=X(0)\supseteq X(1)\supseteq\dots,\qquad \bigcap_{i\in\mathbb{N}}X(i)=\{0\}.$$

The result of Proposition 9.2 remains true when $\mathbb{N}_d$ is replaced by $\mathbb{N}$. That is, q-exchangeable probability measures on $\mathbb{N}^\infty$ correspond bijectively to $\mathscr{G}_\infty$-invariant probability measures on the space of decreasing $\mathbb{N}$-flags. The proof is literally the same, with $\Gamma(q,d)$ replaced by $\Gamma(q,\infty)$.

Remark 9.5. Let V^∞ be the dual vector space to V_∞. We endow V^∞ with the topology of simple convergence of linear functionals; then it becomes a compact topological space. As an additive group, V^∞ is also the Pontryagin dual to V_∞ viewed as a discrete additive group. Passing to the orthogonal complement establishes a bijection between arbitrary linear subspaces in V_∞ and *closed* linear subspaces in V^∞. Define an *increasing* d-flag in V^∞ as a collection of closed subspaces

$$\{0\} = Y(0) \subseteq Y(1) \subseteq \cdots \subseteq Y(d) = V^\infty$$

and an *increasing* $\mathbb{N}$-flag in V^∞ as an infinite collection of closed subspaces

$$\{0\} = Y(0) \subseteq Y(1) \subseteq \dots, \qquad \overline{\bigcup_{i\in\mathbb{N}} Y(i)} = V^\infty,$$

where the horizontal line means closure. By duality, the increasing d-flags in V^∞ are in a one-to-one correspondence with the decreasing d-flags in V_∞. Moreover, this correspondence is consistent with the natural action of the group $\mathscr{G}_\infty$ on V^∞. The same holds for the $\mathbb{N}$-flags as well. Thus, instead of considering invariant measures on decreasing flags in V_∞ one can equally well deal with invariant measures on the set of increasing flags in V^∞.

10. Appendix: Mallows' measure

In this Section we sketch some properties of the Mallows measures $\mathcal{Q}_n$ and $\mathcal{Q}$. To state the results we need some preparation. It is convenient to represent a generic permutation $\sigma \in \mathfrak{S}_n$ as an $n \times n$ permutation matrix $\sigma(i,j)$, where the entry $\sigma(i,j)$ equals 1 or 0 depending on whether $\sigma(j) = i$ or not. Such permutation matrices are *strictly monomial*, in the sense that they have one and only one non-zero element per row and per column. Note that this realization of permutations by strictly monomial matrices takes the group multiplication into the conventional matrix multiplication, and the inversion map $\sigma \mapsto \sigma^{-1}$ corresponds to the matrix transposition. Likewise, the group $\mathfrak{S}$ can be realized as the group of strictly monomial matrices of infinite size.

More generally, a 0-1 matrix of finite or infinite size is *weakly monomial* if each row and each column contains *at most* one 1, the other entries being 0's. Let $M(n)$ and M denote the sets of weakly monomial 0-1 matrices of size $n \times n$ and $\infty \times \infty$, respectively. Both $M(n)$ and M are semigroups under the matrix multiplication, and $\mathfrak{S}_n \subset M(n)$ and $\mathfrak{S} \subset M$ are respective subgroups of invertible elements. An additional operation in $M(n)$ and M is the matrix transposition, which is an involutive antiautomorphism.

For $k = 1, 2, \dots$, the *truncation operation* θ_k assigns to a matrix of size $\infty \times \infty$ or $l \times l$ with $l \geq k$ the $k \times k$ submatrix comprised of the entries (i,j) with $i,j \leq k$. Obviously, θ_k

projects $M(n)$ onto $M(k)$ for any $n > k$. Likewise, θ_k projects M onto $M(k)$. Using these projections we may identify M with the projective limit space $\varprojlim M(k)$. We endow M with the corresponding projective limit topology; then M becomes a compact topological space. By the very definition, a fundamental system of neighborhoods of a matrix $m \in M$ is formed by the subsets $\{m' \in M \mid \theta_k(m') = \theta_k(m)\}$, $k = 1, 2, \dots$.

It is readily checked that the restriction of $\theta_k : M \to M(k)$ to the subset $\mathfrak{S} \subset M$ is surjective for every k. It follows that $\mathfrak{S}$ is dense in M (and even $\mathfrak{S}_\infty$ is dense). Recall that we endowed $\mathfrak{S}$ with the sigma-algebra of Borel sets inherited via the embedding $\mathfrak{S} \subset \mathbb{N}^\infty$. Clearly, this Borel structure coincides with that induced by the embedding $\mathfrak{S} \subset M$. Thus, any Borel probability measure on $\mathfrak{S}$ or on $\mathfrak{S}_n \subset \mathfrak{S}$ can be viewed as a measure on M. In particular, we may view the Mallows measures $\mathcal{Q}_n$ and $\mathcal{Q}$ as probability measures on the compact space M. This makes sense of the following assertion:

Proposition 10.1. *As $n \to \infty$, $\mathcal{Q}_n$ weakly converge to $\mathcal{Q}$.*

Proof. Let $\theta_k(\mathcal{Q}_n)$ and $\theta_k(\mathcal{Q})$ denote the push-forwards of $\mathcal{Q}_n$ and $\mathcal{Q}$ under θ_k. By the definition of topology in M and finiteness of $M(k)$, it suffices to prove that for any k and any fixed matrix $m \in M(k)$, $\theta_k(\mathcal{Q}_n)(\{m\})$ converges to $\theta_k(\mathcal{Q})(\{m\})$.

Taking in account Remark 4.5, it is convenient to replace $\mathcal{Q}_n$ and $\mathcal{Q}$ by their push-forwards under the matrix transposition; let us denote them as $\mathcal{Q}'_n$ and $\mathcal{Q}'$, respectively. Thus, we will prove the equivalent assertion that $\theta_k(\mathcal{Q}'_n)(\{m\})$ converge to $\theta_k(\mathcal{Q})(\{m\})$.

Let $w = w_1 w_2 \dots$ be the output of the q-shuffling algorithm applied to the infinite word $12\dots$. As usual, we identify w with the random permutation $\sigma \in \mathfrak{S}$ by writing $w = \sigma(1)\sigma(2)\dots$. From this, one sees that the quantity $\theta_k(\mathcal{Q}')(\{m\})$ is equal to the probability of the event that for each $j = 1, \dots, k$, the letter w_j either equals some $i \in \{1, \dots, k\}$ if the matrix m has 1 in the jth column in position (i, j), or $w_j > k$ if the jth column of m consists entirely of 0's.

For instance, if $m = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in M(2)$ then the event in question is that the first step of the algorithm yields $w_1 > 2$ and the second step yields $w_2 = 1$.

The quantity $\theta_k(\mathcal{Q}'_n)(\{m\})$ admits exactly the same interpretation in terms of the finite q-shuffle applied to the finite word $1 \dots n$.

Now, the desired convergence of the probabilities follows from the fact that, as $n \to \infty$, the truncated geometric distributions directing the finite q-shuffle (Definition 3.3) converge to the infinite geometric distribution directing the infinite q-shuffle (Definition 4.1). $\square$

Corollary 10.2. *The Mallows measures $\mathcal{Q}_n$ and $\mathcal{Q}$ are invariant under the group inversion map $\sigma \mapsto \sigma^{-1}$.*

Proof. Given a matrix $m \in M(n)$, let us say that two distinct positions $\{(i_1, j_1), (i_2, j_2)\}$ occupied by 1's are *in inversion* if the two differences $i_1 - i_2$ and $j_1 - j_2$ have opposite signs (note that these differences cannot vanish), and denote by $\mathrm{inv}(m)$ the total number of unordered pairs of positions in inversion. Clearly, $\mathrm{inv}(m) = \mathrm{inv}(m')$ where m' stands for the transposed matrix.

On the other hand, if $\sigma \in \mathfrak{S}_n$ and $m := [\sigma(i,j)]$ is the corresponding permutation matrix then we obviously have $\mathrm{inv}(\sigma) = \mathrm{inv}(m)$. If σ is replaced by σ^{-1} then m is replaced by m'. Therefore, $\mathrm{inv}(\sigma) = \mathrm{inv}(\sigma^{-1})$, which implies the desired symmetry property of $\mathcal{Q}_n$. Now, the similar property of $\mathcal{Q}$ follows from Proposition 10.1. $\square$

Remark 10.3. The 'absorption sampling' mentioned above (see [Ke2] for history and references) seems to have not been identified with the Mallows measure on M. This connection along with the invariance of $\mathcal{Q}$ under the matrix transposition make obvious the unexplained symmetry in formulas like [Ke1, Equation (10)] and [B, Equation (2.12)].

Likewise, the number of inversions is also invariant under reflection with respect to the secondary matrix diagonal, which swaps (i, j) and $(n + 1 - j, n + 1 - i)$, hence $\mathcal{Q}_n$ is preserved by this transformation as well. However, this operation has no analogue for the infinite group $\mathfrak{S}$.

Remark 10.4. Observe that the group $\mathfrak{S}_\infty$ acts on $\mathfrak{S}$ both by left and right shifts: an element $\sigma \in \mathfrak{S}_\infty$ maps an element $\tau \in \mathfrak{S}$ to $\sigma\tau$ or $\tau\sigma^{-1}$, respectively. Under the right action, the elementary transposition $\sigma_i := (i, i + 1) \in \mathfrak{S}_\infty$ swaps the letters of a permutation word $\widetilde{\tau}$ in the ith and $(i + 1)$th positions, while under the left action, the same element σ_i swaps the letters i and $(i+1)$ in $\widetilde{\tau}$. That is to say, under the right action on permutation words we look at positions, while under the left action we look at the letters themselves. The inversion map intertwines the both actions.

We know that $\mathcal{Q}$ is a unique probability measure on $\mathfrak{S}$ that is quasi-invariant under the right action, with a special cocycle, (2.2). The symmetry property of the measure $\mathcal{Q}$ implies that it is quasi-invariant under the left action as well. To compute the corresponding cocycle, return to the definition (2.1) of the additive cocycle and observe that instead of taking the n-truncated word with large n we can equally well deal with arbitrary finite subwords, provided that they are large enough. Using this re-formulation, one sees that the additive cocycle is preserved under the group inversion on $\mathfrak{S}$, and so is the corresponding multiplicative cocycle.

It follows that the cocycle corresponding to the left action remains the same. Consequently, $\mathcal{Q}$ can be also characterized as a unique probability measure on $\mathfrak{S}$, which is quasi-invariant under the left action of $\mathfrak{S}_\infty$, with the same cocycle as before.

The next proposition describes the finite-dimensional distributions of the Mallows measure $\mathcal{Q}$ viewed as a measure on $M = \varprojlim M(k)$. We use the following notation: m is an arbitrary matrix from $M(k)$; $I \subset \{1, \ldots, k\}$ is the set of indices of the rows in m containing 1's; $J \subset \{1, \ldots, k\}$ is the set of indices of the columns in m containing 1's; $r = |I| = |J|$ is the rank of m; and $\mathrm{inv}(m)$ has the same meaning as in the proof of Corollary 10.2.

Proposition 10.5. *In the above notation,*

$$\theta_k(\mathcal{Q})(\{m\}) = (1 - q)^r q^{k^2 - 2kr - r + \mathrm{inv}(m) + \sum_{i \in I} i + \sum_{j \in J} j}. \tag{10.1}$$

Proof. We apply the same method as in Section 6, i.e. reduce the alphabet $\mathbb{N}$ to the finite alphabet $\mathbb{N}_{k+1}$ using the monotone map $f_{k+1}(a) = a \wedge (k + 1)$. The key idea is that if

$w = w_1 w_2 \dots = \sigma(1)\sigma(2)\dots$ is the random output of the infinite q-shuffle of the word $v = 1\,2\dots$ then, as is seen from the proof of Proposition 10.1, the truncated matrix $\theta_k(\sigma)$ depends only of the first k letters of the word $f^\infty_{k+1}(w)$ (that is to say, all the letters $\ge k+1$ become indistinguishable).

On the other hand, by virtue of Proposition 2.3, the random word $f^\infty_{k+1}(w)$ is the output of the infinite q-shuffle applied to the inversion-free word

$$v' := 1 \dots k \underbrace{(k+1)(k+1)\dots}_{\infty} \in (\mathbb{N}_{k+1})^\infty.$$

In the notation of Section 4, the law of the random word $f^\infty_{k+1}(w)$ is given by the measure $P^{(v')}$, and the distribution of the first k letters is given by the marginal $P^{(v')}_k$, for which we have an explicit expression, see (4.3). In this formula, we need to take

$$l_1 = \dots = l_k = 1, \quad l_{k+1} = \infty, \quad \mu_{k+1} = k - r, \quad \mu_a = \begin{cases} 1, & a \in I \\ 0, & a \in \{1, \dots, k\} \setminus I, \end{cases}$$

and then the direct computation gives (10.1). □

There is another way of approximating $\mathcal{Q}$ by the $\mathcal{Q}_n$'s. Namely, we will see that $\mathcal{Q}$ can be represented as the projective limit of the $\mathcal{Q}_n$'s. Incidentally, we will realize $\mathcal{Q}$ as a product measure.

As usual, we will identify permutations with the corresponding permutation words. For any $n \ge 2$, we define the projection $\mathfrak{S}_n \to \mathfrak{S}_{n-1}$ as deletion of n from a permutation word. Using these projections we construct the projective limit space $\varprojlim \mathfrak{S}_n$, which is a compact topological space in the standard topology. We have a natural embedding

$$\mathfrak{S} \hookrightarrow \varprojlim \mathfrak{S}_n\,, \tag{10.2}$$

which is specified by the projection $\mathfrak{S} \to \mathfrak{S}_n$ which removes from an infinite permutation word all letters larger than n.

Note that $\mathfrak{S}$ is a *proper* subset of $\varprojlim \mathfrak{S}_n$. Indeed, there is a natural one-to-one correspondence between elements of $\varprojlim \mathfrak{S}_n$ and all possible linear orders on the set $\mathbb{N}$, of which the orders induced by permutation words $\sigma(1)\sigma(2)\dots$ comprise a relatively small part. Still, $\mathfrak{S}$ is dense in $\varprojlim \mathfrak{S}_n$.

Proposition 10.6. *The measures $\mathcal{Q}_n$ are consistent with the projections $\mathfrak{S}_n \to \mathfrak{S}_{n-1}$, so that one can define the projective limit $\mathcal{Q}_\infty := \varprojlim \mathcal{Q}_n$, which is a probability measure on $\varprojlim \mathfrak{S}_n$. The image of $\mathfrak{S}$ under the embedding (10.2) has full $\mathcal{Q}_\infty$-measure, and the restriction of $\mathcal{Q}_\infty$ to $\mathfrak{S}$ coincides with the Mallows measure $\mathcal{Q}$.*

Proof. For a permutation $\sigma \in \mathfrak{S}_n$ (which we identify with the corresponding permutation word) set

$$\widetilde{\beta}_j = \widetilde{\beta}_j(\sigma) = \#\{i < j \mid i \text{ precedes } j\} + 1, \qquad j = 1, \dots, n,$$

cf. (3.1). The link with (3.1) is the following: $\widetilde{\beta}_j(\sigma) = \beta_j(\sigma^{-1})$.

The correspondence $\sigma \mapsto (\widetilde{\beta}_1(\sigma), \dots, \widetilde{\beta}_n(\sigma))$ is a bijection

$$\mathfrak{S}_n \to \mathbb{N}_1 \times \cdots \times \mathbb{N}_n, \tag{10.3}$$

and we have a counterpart of Proposition 3.2: under $\mathcal{Q}_n$, the coordinates $\widetilde{\beta}_j$ are independent and $j+1-\widetilde{\beta}_j$ is distributed according to $G_{q,j}$. This can be deduced from Proposition 3.2 taken together with the symmetry property of $\mathcal{Q}_n$ (Proposition 10.2) or can be easily checked directly.

Under the bijection (10.3), the projection $\mathfrak{S}_n \to \mathfrak{S}_{n-1}$ is simply the deletion of the last letter. This enables us to identify $\varprojlim \mathfrak{S}_n$ with the infinite product space $\prod_{n=1}^{\infty} \mathbb{N}_n$. Under this identification, the measure $\varprojlim \mathcal{Q}_n$ turns into the product of truncated geometric distributions. The image of $\mathfrak{S}$ in $\prod_{n=1}^{\infty} \mathbb{N}_n$ consists of those sequences $(i_1, i_2, \dots)$ for which $i_n \to \infty$. From this it is readily checked that $\mathfrak{S}$ has full measure.

It remains to check that the measure $\varprojlim \mathcal{Q}_n$ coincides on $\mathfrak{S}$ with the measure $\mathcal{Q}$. To this end we use the characterization of $\mathcal{Q}$ in terms of the left action of $\mathfrak{S}_\infty$ as described in Remark 10.4. It is easy to see that the measure $\varprojlim \mathcal{Q}_n$ has the same transformation property with respect to the left action of elementary transpositions σ_i. Consequently, $\varprojlim \mathcal{Q}_n = \mathcal{Q}$. $\square$

Alternatively, one can use another chain of projections, such that the projection $\mathfrak{S}_n \to \mathfrak{S}_{n-1}$ first cuts the last letter in $\sigma(1) \dots \sigma(n)$ then re-labels the letters $\sigma(1) \dots \sigma(n-1)$ by the increasing bijection with $\mathbb{N}_{n-1}$. A random element of $\mathfrak{S}$ under $\mathcal{Q}$ is representable by an infinite sequence of backward ranks $(\beta(1), \beta(2), \dots)$, which are independent and have distribution as in Proposition 3.2.

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