# ON UNIQUENESS OF MILD SOLUTIONS FOR DISSIPATIVE STOCHASTIC EVOLUTION EQUATIONS

CARLO MARINELLI AND MICHAEL RÖCKNER

Abstract. In the semigroup approach to stochastic evolution equations, the fundamental issue of uniqueness of mild solutions is often "reduced" to the much easier problem of proving uniqueness for strong solutions. This reduction is usually carried out in a formal way, without really justifying why and how one can do that. We provide sufficient conditions for uniqueness of mild solutions to a broad class of semilinear stochastic evolution equations with coefficients satisfying a monotonicity assumption.

## 1. Introduction

The purpose of this work is to prove uniqueness of mild and weak (in the PDE sense) solutions to dissipative stochastic evolution equations of the type

$$du(t) + Au(t)\,dt + Fu(t)\,dt = B(t,u(t))\,dW(t) + \int_Z G(t,u(t-),z)\,\bar{\mu}(dt,dz) \tag{1}$$

on a real Hilbert space $H$, where $W$ is a Wiener process and $\bar{\mu}$ a compensated Poisson measure (precise definitions and assumptions will be given below).

The motivation for this work is that we have not been able to understand the arguments, often very concise, used in the literature. It is probably worth elaborating more on this observation, as most readers will be surprised that we return to such a basic issue as uniqueness for solutions of equations that are indeed expected to be well-posed thanks to their dissipative nature. Essentially all proofs of uniqueness that we have been able to find go as follows: suppose that (1) has two solutions, and further assume they are strong. Then an application of Itô's formula for the square of the norm, monotonicity, and Gronwall's lemma quickly yield that the two strong solutions must coincide. Now comes the trouble: if the solutions are not strong, consider a "suitable" regularization. The problem is that mild solutions, except in the non-interesting cases of Lipschitz nonlinearities, are constructed using regularizations, usually of $A$ and $F$. Therefore, without fully elaborating the argument (something that is not done in the literature we know of), one could at most prove uniqueness of mild solutions constructed by regularization, and thus it seems like one is trapped in a vicious circle. In fact, there is no guarantee that other mild solutions could be constructed without resorting to regularized equations and limit passages. We should also clarify that we are not claiming that the literature contains errors, but it is probably fair to say that the usual arguments are a bit mysterious and perhaps not fully convincing.

After we realized we could not easily understand how to prove uniqueness by the "simple" regularization procedure alluded to in the literature, each of us obtained an independent proof, by different arguments. The two proofs are collected in the present paper.

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## 2. Preliminaries and notation

Let $T > 0$ and $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$, $\mathbf{F} = \{\mathcal{F}_t\}_{0\leq t\leq T}$ a fixed filtered probability space (satisfying the "usual" assumptions), on which all random elements will be defined. The predictable sigma-field on this stochatic basis will be denoted by $\mathcal{P}$. Let $K$, $H$ be real separable Hilbert spaces, $\mathcal{L}_2(K \to H)$ the set of Hilbert-Schmidt operators from $K$ to $H$. By $W$ we shall denote a $K$-valued Wiener process with covariance operator $Q$, while $\mu$ will denote a Poisson measure on $[0, T] \times Z$ with compensator $\mathrm{Leb} \otimes m$, where $(Z, \mathcal{Z}, m)$ is a measure space, and $\bar{\mu} := \mu - \mathrm{Leb} \otimes m$ stands for the compensated measure associated to $\mu$. Here and in the following we shall denote Lebesgue measure by Leb.

Let us recall a few facts about stochastic integration with respect to Wiener processes and compensated Poisson measures. For all unexplained (but classical) results and notations we refer to [7] (cf. also [8]). Let $\mathfrak{I}_Q$ denote the set of all progressively measurable processes $\Phi : [0, T] \to \mathcal{L}_2^Q(K \to H)$ such that

$$\mathbb{P}\Big(\int_0^T |\Phi(t)|_Q^2\, dt < \infty\Big) = 1,$$

where $\mathcal{L}_2^Q(K \to H)$ denotes the space of linear (possibly unbounded) operators from $K$ to $H$ that belong to $\mathcal{L}_2(Q^{1/2}K \to H)$, endowed with the norm

$$|\cdot|_Q := |\cdot|_{\mathcal{L}_2(Q^{1/2}K\to H)}.$$

Then for any $F \in \mathfrak{I}_Q$ the stochastic integral $(F \cdot W)_t := \int_0^t F(s)\, dW(s)$ is well-defined for all $t \leq T$ and $F \cdot W$ is a local martingale. Similarly, denoting the set of all (random) functions $\phi : [0, T] \times Z \to H$ that are $\mathcal{P} \otimes \mathcal{Z}$-measurable and satisfy

$$\mathbb{P}\Big(\int_0^T \int_Z |\phi(s, z)|^2\, m(dz)\, ds < \infty\Big) = 1$$

by $\mathfrak{I}_m$, we have that, for any $g \in \mathfrak{I}_m$, the stochastic integral

$$(g \star \bar{\mu})_t := \int_{]0,t]} \int_Z g(s, z)\, \bar{\mu}(ds, dz)$$

is well-defined for all $t \leq T$ and $g \star \bar{\mu}$ is a local martingale. Moreover, if $F_n(s) \to F(s)$ in probability for a.a. $s$ and there exists $\Phi \in \mathfrak{I}_Q$ such that $|F_n(s)|_Q \leq |\Phi(s)|_Q$ $\mathbb{P}$-a.s. for a.a. $s$, then $(F_n \cdot W)_t \to (F \cdot W)_t$ in probability for all $t$. Similarly, if $g_n(s, z) \to g(s, z)$ in probability for $\mathrm{Leb} \otimes m$-a.a. $(s, z)$ and there exists $\phi \in \mathfrak{I}_m$ such that $|g_n(s, z)| \leq \phi(s, z)$ $\mathbb{P}$-a.s. for $\mathrm{Leb} \otimes m$-a.a. $(s, z)$, then $(g_n \star \bar{\mu})_t \to (g \star \bar{\mu})_t$ in probability for all $t$. For simplicity of notation we shall write $\int_0^t$ instead of $\int_{]0,t]}$ when integrating against random measures, and we shall denote the norm in $L^2(Z, m)$ by $|\cdot|_m$. Finally, we recall that $u$ is a mild solution to (1) with initial condition $u(0) = u_0$ if one has

$$\begin{aligned} u(t) &+ \int_0^t e^{-(t-s)A} F u(s)\, ds \\ &= e^{-tA}u_0 + \int_0^t e^{-(t-s)A} B(s, u(s))\, dW(s) + \int_0^t \int_Z e^{-(t-s)A} G(s, u(s-), z)\, \bar{\mu}(ds, dz) \end{aligned}$$

$\mathbb{P}$-a.s. for all $t \leq T$ (and all integrals are well-defined).

## 3. Uniqueness of mild solutions by a bootstrap argument

In this section we give a proof of uniqueness that is somewhat reminiscent of bootstrap arguments used in deterministic PDE. In the first subsection we consider the simpler case of equations with additive noise (i.e. with $B$ and $G$ in (1) independent of $u$), and in the second subsection we consider the general case. The first subsection is included both because the argument is relatively short, and also because it will be used later to prove uniqueness for a class of equations which is not covered by the results of this section.

3.1. **Additive noise.** Assuming that the coefficients in front of the Wiener process and the Poisson measure do not depend explicitly on the unknown, we obtain uniqueness by a pathwise argument, thus using essentially a deterministic argument. More precisely, we have the following

**Theorem 1.** *Consider the stochastic evolution equation on $H$*

$$du(t) + Au(t)\,dt + Fu(t)\,dt = B(t)\,dW(t) + \int_Z G(t,z)\,\bar{\mu}(dt,dz), \tag{2}$$

*where $A$ is a linear monotone operator on $H$, $F$ a (nonlinear) operator on $H$ such that $x \mapsto Fx+\eta x$ is monotone for some $\eta \in \mathbb{R}$, $B$ is progressively measurable, $G$ is $\mathcal{P}\otimes\mathcal{Z}$-measurable, and*

$$\int_0^t \Big(\big|e^{(t-s)A}B(s)\big|_Q^2 + \big|e^{(t-s)A}G(s,\cdot)\big|_m^2\Big)\,ds < \infty$$

*$\mathbb{P}$-a.s. for all $t \leq T$. Then (2) admits at most one mild solution $u$ such that $|Fu|_{L^1([0,T]\to H)} < \infty$ $\mathbb{P}$-a.s..*

*Proof.* Let $u$, $v$ be two mild solutions of (2), and define

$$y(s) := u(s) - v(s), \qquad g(s) := Fv(s) - Fu(s).$$

We shall keep $g$ fixed from now on, and we will work "$\omega$-by-$\omega$". In particular, the hypotheses imply the existence of $\Omega' \subset \Omega$ such that $\mathbb{P}(\Omega') = 1$ and $g \in L^1([0,T] \to H)$ for all $\omega \in \Omega'$. Let us fix $\omega \in \Omega'$ from now on. Then $y$ is a mild solution of the equation

$$dy(t) + Ay(t)\,dt = g(t)\,dt$$

with initial condition $y(0) = 0$ and $\sup_{t\leq T}|y(t)| < \infty$. Let $A_\varepsilon := A(I+\varepsilon A)^{-1}$ denote the Yosida approximation of $A$, and let $y_\varepsilon$ be the (unique) strong solution of the equation obtained by replacing $A$ with $A_\varepsilon$ in the previous one. Trotter-Kato's approximation theorem (see e.g. [1, p. 241]) then yields

$$\lim_{\varepsilon\to 0}\sup_{t\leq T}|y_\varepsilon(t) - y(t)| = 0.$$

We also have

$$\frac{1}{2}\frac{d}{dt}|y_\varepsilon(t)|^2 + \langle A_\varepsilon y_\varepsilon(t), y_\varepsilon(t)\rangle = \langle g(t), y_\varepsilon(t)\rangle = \langle g(t), y(t)\rangle + \langle g(t), y_\varepsilon(t) - y(t)\rangle,$$

therefore, by the monotonicity of $A_\varepsilon$ and

$$-\langle g(t), y(t)\rangle = \langle Fu(t) - Fv(t), u(t) - v(t)\rangle \geq -\eta|u(t) - v(t)|^2,$$

we get

$$\begin{aligned}|y_\varepsilon(t)|^2 &\leq \eta\int_0^t |y(s)|^2\,ds + \int_0^t \langle g(s), y_\varepsilon(s) - y(s)\rangle\,ds \\ &\leq \eta\int_0^t |y(s)|^2\,ds + \sup_{s\leq t}|y_\varepsilon(s) - y(s)|\int_0^t |g(s)|\,ds.\end{aligned}$$

Letting $\varepsilon \to 0$, we conclude, recalling that $g \in L^1([0,T] \to H)$, that $y(t) = u(t) - v(t) = 0$ for all $t \leq T$ by an application of Gronwall's inequality. Since $\omega \in \Omega'$ was arbitrary, we have $u(t) = v(t)$ $\mathbb{P}$-a.s.. □

*Remark* 2. Note that the above proof does not use anywhere the linearity of $A$. The same observation applies to the method used in the next subsection.

3.2. **General case.** In this subsection we consider the general case of equations with multiplicative noise under a monotonicity assumption on $A$ and on the triplet $(F, B, G)$.

Throughout this subsection we shall always assume, without further mention, that $B : [0,T] \times H \to \mathcal{L}_2^Q$ and $G : [0,T] \times H \times Z \to H$ satisfy the usual measurability conditions needed to ensure that the corresponding stochastic integrals with respect to $W$ and $\bar{\mu}$ are meaningful.

**Theorem 3.** *Assume that $A$ is a linear monotone operator on $H$, $F$ is a (nonlinear) operator on $H$, and $B$, $G$ satisfy the monotonicity property*

$$2\langle Fu - Fv, u - v\rangle - \big|B(s,u) - B(s,v)\big|_Q^2 - \big|G(s,u,\cdot) - G(s,v,\cdot)\big|_m^2 \geq \alpha|u-v|^2$$

*for all $u$, $v \in \mathrm{dom}(F)$ and all $s \leq T$, for some $\alpha \in \mathbb{R}$ independent of $s$. Then there is at most one càdlàg mild solution of the stochastic evolution equation (1) such that*

$$\int_0^T \big(|Fu(s)| + |B(s,u(s))|_Q^2 + |G(s,u(s),\cdot)|_m^2\big)\, ds < \infty \tag{3}$$

$\mathbb{P}$-*a.s.*.

*Proof.* Let $u$, $v$ be two càdlàg mild solutions of (1) satisfying condition (3), and define

$$y(s) := u(s) - v(s), \qquad g(s) := Fu(s) - Fv(s),$$
$$C(s) := B(s,u(s)) - B(s,v(s)), \qquad D(s,z) := G(s,u(s-),z) - G(s,v(s-),z).$$

We shall keep $g$, $C$, $D$ fixed from now on. Then $y$ is a mild solution of the equation

$$dy(t) + Ay(t)\,dt + g(t)\,dt = C(t)\,dW(t) + \int_Z D(t,z)\,\bar{\mu}(dt,dz)$$

with initial condition $y(0) = 0$, and $\sup_{t\leq T}|y(t)| < \infty$ $\mathbb{P}$-a.s. (because $y$ has càdlàg paths). Let us define

$$g_\varepsilon(t) := (I + \varepsilon A)^{-1} g(t),$$
$$C_\varepsilon(t) := (I + \varepsilon A)^{-1} C(t),$$
$$D_\varepsilon(t,z) := (I + \varepsilon A)^{-1} D(t,z).$$

Then the equation

$$dy(t) + Ay(t)\,dt + g_\varepsilon(t)\,dt = C_\varepsilon(t)\,dW(t) + \int_Z D_\varepsilon(t,z)\,\bar{\mu}(dt,dz),$$

with initial condition $y(0) = 0$, admits a unique mild solution $y_\varepsilon$, which is also a strong solution. One can actually immediately verify that $y_\varepsilon(t) = (I + \varepsilon A)^{-1} y(t)$, so that, in particular,

$$\lim_{\varepsilon\to 0} \sup_{t\leq T} \big|y_\varepsilon(t) - y(t)\big| = 0 \qquad \mathbb{P}\text{-a.s.} \tag{4}$$

(we shall omit the indication that statements are meant to hold $\mathbb{P}$-a.s. in the rest of the proof, if no confusion may arise). Itô's formula for the square of the norm yields

$$\begin{aligned}
|y_\varepsilon(t)|^2 &+ 2\int_0^t \langle Ay_\varepsilon(s), y_\varepsilon(s)\rangle\, ds + 2\int_0^t \langle g_\varepsilon(s), y_\varepsilon(s)\rangle\, ds \\
&= 2\int_0^t \langle y_\varepsilon(s), C_\varepsilon(s)\,dW(s)\rangle + 2\int_0^t\int_Z \langle y_\varepsilon(s-), D_\varepsilon(s,z)\rangle\,\bar{\mu}(ds,dz) \\
&+ \int_0^t |C_\varepsilon(s)|_Q^2\, ds + \int_0^t\int_Z |D_\varepsilon(s,z)|^2\,\mu(ds,dz).
\end{aligned} \tag{5}$$

Clearly we have $g_\varepsilon(t) \to g(t)$ for all $t \leq T$ as $\varepsilon \to 0$, and

$$\langle g_\varepsilon(s), y_\varepsilon(s)\rangle \leq |g(s)| \sup_{s\leq t} |y(s)|,$$

$$\int_0^t |g(s)| \sup_{s\leq t} |y(s)|\, ds = \sup_{s\leq t} |y(s)| \int_0^t |g(s)|\, ds < \infty,$$

hence, by the dominated convergence theorem,

$$\lim_{\varepsilon\to 0} \int_0^t \langle g_\varepsilon(s), y_\varepsilon(s)\rangle\, ds = \int_0^t \langle g(s), y(s)\rangle\, ds.$$

For any $s \leq T$, setting $\tilde{C}(s) : K \to \mathbb{R}$, $\tilde{C}(s) : \zeta \mapsto \langle y(s), C(s)\zeta\rangle$, and defining $\tilde{C}_\varepsilon$ replacing $y$ and $C$ with $y_\varepsilon$ and $C_\varepsilon$, respectively, we have $|\tilde{C}_\varepsilon(s) - \tilde{C}(s)|_Q \to 0$ in probability. By the inequality

$$|\tilde{C}_\varepsilon(s)|_Q \leq |y_\varepsilon(s)|\, |C_\varepsilon(s)|_Q \leq |y(s)|\, |C(s)|_Q$$

we infer

$$\int_0^T |\tilde{C}_\varepsilon(s)|_Q^2\, ds \leq \sup_{s\leq T} |y(s)|^2 \int_0^T |C(s)|_Q^2 < \infty.$$

Since the above bounds are uniform with respect to $\varepsilon$, we immediately deduce that we can apply the convergence results for stochastic integrals mentioned in Section 2, obtaining

$$\int_0^t \langle y_\varepsilon(s), C_\varepsilon(s)\, dW(s)\rangle = \int_0^t \tilde{C}_\varepsilon(s)\, dW(s) \to \int_0^t \tilde{C}(s)\, dW(s) = \int_0^t \langle y(s), C(s)\, dW(s)\rangle$$

in probability for all $t$. An analogous argument proves that we have

$$\int_0^t \int_Z \langle y_\varepsilon(s-), D_\varepsilon(s,z)\rangle\, \bar{\mu}(ds,dz) \to \int_0^t \int_Z \langle y(s-), D(s,z)\rangle\, \bar{\mu}(ds,dz)$$

in probability for all $t$.

Passing to the limit as $\varepsilon \to 0$ in (5) we are left with

$$|y(t)|^2 + 2\int_0^t \langle g(s), y(s)\rangle\, ds \leq M(t) + \int_0^t |C(s)|_Q^2\, ds + \int_0^t \int_Z |D(s,z)|^2\, \mu(ds,dz),$$

where $M$ is the local martingale defined by

$$M(t) = 2\int_0^t \langle y(s), C(s)\, dW(s)\rangle + 2\int_0^t \int_Z \langle y(s-), D(s,z)\rangle\, \bar{\mu}(ds,dz).$$

Let us define the sequences of stopping times

$$\tau_n^1 := \inf\Big\{t \geq 0 :\ \int_0^t |C(s)|_Q^2\, ds \geq n\Big\},$$
$$\tau_n^2 := \inf\Big\{t \geq 0 :\ \int_0^t |D(s,\cdot)|_m^2\, ds \geq n\Big\},$$

and note that both are predictable (as hitting times of continuous adapted processes). Then $\tau_n := \tau_n^1 \wedge \tau_n^2$ is easily seen to be a localizing sequence of stopping times for $M$, so that we get

$$\mathbb{E}|y(\tau_n \wedge t)|^2 + 2\mathbb{E}\int_0^{\tau_n\wedge t} \langle g(s), y(s)\rangle\, ds \leq \mathbb{E}\int_0^{\tau_n\wedge t} |C(s)|_Q^2\, ds + \mathbb{E}\int_0^{\tau_n\wedge t}\int_Z |D(s,z)|^2\, \mu(ds,dz).$$

We also have, recalling that $\mathrm{Leb} \otimes m$ is the compensator of $\mu$,

$$\begin{aligned}\mathbb{E}\int_0^{\tau_n\wedge t}\int_Z |D(s,z)|^2\,\mu(ds,dz) &= \mathbb{E}\int_0^t\int_Z |D(s,z)|^2\mathbf{1}_{s\leq\tau_n}\,\mu(ds,dz)\\ &= \mathbb{E}\int_0^t\int_Z |D(s,z)|^2\mathbf{1}_{s\leq\tau_n}\,m(dz)\,ds\\ &= \mathbb{E}\int_0^{\tau_n\wedge t}\int_Z |D(s,z)|^2\,m(dz)\,ds,\end{aligned}$$

hence

$$\mathbb{E}|y(\tau_n\wedge t)|^2 + \mathbb{E}\int_0^{\tau_n\wedge t}\big(2\langle g(s),y(s)\rangle - |C(s)|_Q^2 - |D(s,\cdot)|_m^2\big)\,ds \leq 0,$$

and, by the monotonicity assumption on the coefficients,

$$0 \geq \mathbb{E}|y(\tau_n\wedge t)|^2 + 2\alpha\mathbb{E}\int_0^{\tau_n\wedge t}|y(s)|^2\,ds = \mathbb{E}|y(\tau_n\wedge t)|^2 + 2\alpha\mathbb{E}\int_0^t \mathbb{E}|y(\tau_n\wedge s)|^2\,ds$$

Appealing to Gronwall's inequality and recalling that $\tau_n \to \infty$ as $n \to \infty$, we obtain $y(s) = u(s) - v(s) = 0$ $\mathbb{P}$-a.s. for all $s \leq T$, thus completing the proof. □

*Remark* 4. Note that uniqueness in the previous theorem is obtained in the class of solutions satisfying (3), which is stronger than necessary for existence. On the other hand, since stochastic convolutions of pseudo-contraction semigroups with respect to general (locally square integrable) martingales are càdlàg (see e.g. [4]), it follows that solutions to (1) will also be càdlàg.

## 4. Uniqueness of weak solutions

In this section we prove that (1) admits a unique weak solution. By weak solution we shall always mean weak solution in the analytic sense, not in the probabilistic sense. In particular, we shall say that $u$ is a weak solution of (1) if

$$\begin{aligned}&\langle u(t),\phi\rangle + \int_0^t\langle u(s),A^*\phi\rangle\,ds + \int_0^t\langle Fu(s),\phi\rangle\,ds\\ &\qquad = \Big\langle\int_0^t B(s,u(s))\,dW(s),\phi\Big\rangle + \int_0^t\int_Z\langle G(s,u(s-),z),\phi\rangle\,\bar\mu(ds,dz)\end{aligned}$$

for all $\phi \in D(A^*)$. Here and in the following $A^*$ stands for the adjoint of $A$.

The assumptions of Theorem 3 will be in force throughout this section.

**Theorem 5.** *The stochastic equation (1) admits at most one càdlàg weak solution satisfying the integrability condition (3).*

*Proof.* Let $u$ and $v$ be two weak solutions of (1), and set $\tilde e_k := (I+\varepsilon A^*)^{-1}e_k$, where $\{e_k\}_{k\in\mathbb{N}}$ is a complete orthonormal basis of $H$. We can then write (all claims will meant to hold $\mathbb{P}$-a.s., if not otherwise stated)

$$\begin{aligned}\langle\tilde e_k,u(t)-v(t)\rangle &+ \int_0^t\langle A^*\tilde e_k,u(s)-v(s)\rangle\,ds + \int_0^t\langle\tilde e_k,Fu(s)-Fv(s)\rangle\,ds\\ &= \int_0^t\big\langle\tilde e_k,\big(B(s,u(s))-B(s,v(s))\big)\,dW(s)\big\rangle\\ &\quad + \int_0^t\int_Z\langle\tilde e_k,G(s,u(s-),z)-G(s,v(s-),z)\rangle\,\bar\mu(ds,dz)\\ &=: \psi_k(t)+\xi_k(t).\end{aligned}$$

Setting $\phi_k(t) := \langle \tilde{e}_k, u(t) - v(t) \rangle$, Itô's formula yields

$$\phi_k(t)^2 = 2 \int_0^t \phi_k(s-)\, d\phi_k(s) + [\phi_k](t), \tag{6}$$

where

$$\begin{aligned}
\int_0^t \phi_k(s-)\, d\phi_k(s) = &- \int_0^t \langle A^* \tilde{e}_k, u(s) - v(s) \rangle \langle \tilde{e}_k, u(s) - v(s) \rangle \, ds \\
&- \int_0^t \langle \tilde{e}_k, Fu(s) - Fv(s) \rangle \langle \tilde{e}_k, u(s) - v(s) \rangle \, ds \\
&+ \int_0^t \phi_k(s)\, d\psi_k(s) + \int_0^t \phi_k(s-)\, d\xi_k(s),
\end{aligned}$$

and

$$[\phi_k](t) = I_k^1(t) + I_k^2(t),$$

$$\begin{aligned}
I_k^1(t) &= \Big[ \int_0^\cdot \big\langle \tilde{e}_k, \big(B(s,u(s)) - B(s,v(s))\big)\, dW(s) \big\rangle \Big](t) \\
&= \Big[ \int_0^\cdot \big\langle \big(B(s,u(s))^* - B(s,v(s))^*\big) \tilde{e}_k, dW(s) \big\rangle \Big](t) \\
&= \int_0^t \big| (I + \varepsilon A)^{-1} \big(B(s,u(s)) - B(s,v(s))\big) e_k \big|^2 \, ds, \\
I_k^2(t) &= \Big[ \int_0^\cdot \int_Z \big\langle \tilde{e}_k, \big(G(s,u(s-),z) - G(s,v(s-),z)\big) \big\rangle \, \bar{\mu}(ds,dz) \Big](t) \\
&= \int_0^t \int_Z \big| (I + \varepsilon A)^{-1} \big(G(s,u(s-),z) - G(s,v(s-),z)\big) e_k \big|^2 \, \mu(ds,dz).
\end{aligned}$$

Note that since $A^*$ and $(I + \varepsilon A^*)^{-1}$ commute, and $(I + \varepsilon A^*)^{-1} = ((I + \varepsilon A)^{-1})^*$, the dominated convergence theorem yields the following relations:

$$\begin{aligned}
&\sum_{k \le N} \int_0^t \langle A^* \tilde{e}_k, u(s) - v(s) \rangle \langle \tilde{e}_k, u(s) - v(s) \rangle \, ds \\
&\qquad = \int_0^t \sum_{k \le N} \big\langle e_k, A(I + \varepsilon A)^{-1}(u(s) - v(s)) \big\rangle \big\langle e_k, (I + \varepsilon A)^{-1}(u(s) - v(s)) \big\rangle \, ds \\
&\qquad \xrightarrow{N \to \infty} \int_0^t \big\langle A(I + \varepsilon A)^{-1}(u(s) - v(s)), (I + \varepsilon A)^{-1}(u(s) - v(s)) \big\rangle \, ds,
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{k \le N} \int_0^t \langle \tilde{e}_k, Fu(s) - Fv(s) \rangle \langle \tilde{e}_k, u(s) - v(s) \rangle \, ds \\
&\qquad = \int_0^t \sum_{k \le N} \big\langle e_k, (I + \varepsilon A)^{-1}(Fu(s) - Fv(s)) \big\rangle \big\langle e_k, (I + \varepsilon A)^{-1} u(s) - v(s) \big\rangle \, ds \\
&\qquad \xrightarrow{N \to \infty} \int_0^t \big\langle (I + \varepsilon A)^{-1}(Fu(s) - Fv(s)), (I + \varepsilon A)^{-1}(u(s) - v(s)) \big\rangle \, ds.
\end{aligned}$$

In fact, one has

$$\big\langle A(I + \varepsilon A)^{-1}(u(s) - v(s)), (I + \varepsilon A)^{-1}(u(s) - v(s)) \big\rangle \lesssim_\varepsilon |u(s) - v(s)|^2$$

and

$$\big\langle (I + \varepsilon A)^{-1}(Fu(s) - Fv(s)), (I + \varepsilon A)^{-1}(u(s) - v(s)) \big\rangle \le |Fu(s) - Fv(s)|\, |u(s) - v(s)|,$$

which imply the claim recalling that $\sup_{t\leq T}|u(t)-v(t)|<\infty$ and $Fu$, $Fv \in L^1([0,T]\to H)$ $\mathbb{P}$-a.s.. Furthermore, let us define, for each $s\leq T$, the operators $C(s):K\to\mathbb{R}$,

$$C(s):\zeta\mapsto\langle (I+\varepsilon A)^{-1}u(s)-v(s),(I+\varepsilon A)^{-1}(B(s,u(s))-B(s,v(s)))\zeta\rangle,$$

and $C_N(s):K\to\mathbb{R}$,

$$C_N(s):\zeta\mapsto\sum_{k\leq N}\langle \tilde{e}_k,u(s)-v(s)\rangle\langle\tilde{e}_k,(B(s,u(s))-B(s,v(s))\zeta\rangle.$$

Then it is clear that $C_N(s)\to C(s)$ in probability as $N\to\infty$ for all $s\leq T$, and

$$|C_N(s)|_Q\leq |u(s)-v(s)|\,\big|B(u(s)-B(s,v(s))\big|_Q,$$

$$\begin{aligned}\int_0^T &|u(s)-v(s)|^2\,\big|B(u(s)-B(s,v(s))\big|_Q^2\,ds\\ &\leq\sup_{s\leq T}|u(s)-v(s)|^2\int_0^T\big|B(u(s)-B(s,v(s))\big|_Q^2\,ds<\infty,\end{aligned}$$

which implies that $(C_N\cdot W)_t\to(C\cdot W)_t$ in probability as $N\to\infty$ for all $t\leq T$, or equivalently

$$\begin{aligned}&\sum_{k\leq N}\int_0^t\phi_k(s)\,d\psi_k(s)\\ &\xrightarrow{N\to\infty}M_\varepsilon^1(t):=\int_0^t\big\langle(I+\varepsilon A)^{-1}(u(s)-v(s)),(I+\varepsilon A)^{-1}\big(B(s,u(s))-B(s,v(s))\big)\,dW(s)\big\rangle\end{aligned}$$

in probability for all $t\leq T$. An analogous reasoning yields

$$\sum_{k\leq N}\int_0^t\phi_k(s)\,d\xi_k(s)\xrightarrow{N\to\infty}M_\varepsilon^2(t),$$

$$M_\varepsilon^2(t):=\int_0^t\int_Z\big\langle(I+\varepsilon A)^{-1}(u(s)-v(s)),(I+\varepsilon A)^{-1}\big(G(s,u(s-),z)-G(s,v(s-),z)\big)\big\rangle\,\bar{\mu}(ds,dz)$$

in probability for all $t\leq T$. Finally, the following obvious inequalities hold:

$$\begin{aligned}\sum_{k\leq N}I_k^1(t)&\leq\int_0^t\big|(I+\varepsilon A)^{-1}\big(B(s,u(s))-B(s,v(s))\big)\big|_Q^2\,ds,\\ \sum_{k\leq N}I_k^2(t)&\leq\int_0^t\int_Z\big|(I+\varepsilon A)^{-1}\big(G(s,u(s-),z)-G(s,v(s-),z)\big)\big|^2\,\mu(ds,dz)\end{aligned}$$

for all $N$.

Summing up over $k\leq N$ in (6) and letting $N\to\infty$ yields

$$\begin{aligned}&\big|(I+\varepsilon A)^{-1}(u(t)-v(t))\big|^2\\ &\qquad+2\int_0^t\big\langle A(I+\varepsilon A)^{-1}(u(s)-v(s)),(I+\varepsilon A)^{-1}(u(s)-v(s))\big\rangle\,ds\\ &\qquad+2\int_0^t\big\langle(I+\varepsilon A)^{-1}(Fu(s)-Fv(s)),(I+\varepsilon A)^{-1}(u(s)-v(s))\big\rangle\,ds\\ &\quad\leq M_\varepsilon(t)+\int_0^t\big|(I+\varepsilon A)^{-1}\big(B(s,u(s))-B(s,v(s))\big)\big|_Q^2\,ds\\ &\qquad+\int_0^t\int_Z\big|(I+\varepsilon A)^{-1}\big(G(s,u(s-),z)-G(s,v(s-),z)\big)\big|^2\,\mu(ds,dz)\end{aligned}$$

where $M_\varepsilon := M_\varepsilon^1 + M_\varepsilon^2$ is a local martingale. By the monotonicity of $A$, the previous inequality yields

$$\begin{aligned}
&\big|(I+\varepsilon A)^{-1}(u(t)-v(t))\big|^2 \\
&\qquad + 2\int_0^t \big\langle (I+\varepsilon A)^{-1}(Fu(s)-Fv(s)), (I+\varepsilon A)^{-1}(u(s)-v(s))\big\rangle\,ds \\
&\quad \le M_\varepsilon(t) + \int_0^t \big|\big(B(s,u(s))-B(s,v(s))\big)\big|_Q^2\,ds \\
&\qquad + \int_0^t\int_Z \big|\big(G(s,u(s-),z)-G(s,v(s-),z)\big)\big|^2\,\mu(ds,dz).
\end{aligned}$$

We are now going to pass to the limit as $\varepsilon \to 0$ in the inequality just obtained. Trivially, the first-term on the right hand side converges to $|u(t)-v(t)|^2$, while the second term on the left-hand side converges to

$$2\int_0^t \langle Fu(s)-Fv(s), u(s)-v(s)\rangle\,ds$$

by the dominated convergence, in analogy to a situation already entountered. The contractivity of $(I+\varepsilon A)^{-1}$ also implies $M_\varepsilon(t) \to M(t)$ as $\varepsilon \to 0$ in probability for all $t$, where $M$ is the same local martingale defined in the proof of Theorem 3. We are thus left with

$$\begin{aligned}
&|u(t)-v(t)|^2 + 2\int_0^t \langle Fu(s)-Fv(s), u(s)-v(s)\rangle\,ds \\
&\quad \le M(t) + \int_0^t \big|\big(B(s,u(s))-B(s,v(s))\big)\big|_Q^2\,ds + \int_0^t\int_Z \big|\big(G(z,u(s))-G(z,v(s))\big)\big|^2\,\mu(ds,dz),
\end{aligned}$$

and the proof is completed exactly as in the previous section, i.e. taking a sequence of localizing stopping times for $M$, etc. □

*Remark* 6. Using a stochastic Fubini theorem in infinite dimensions (see e.g. [5]), it is not difficult to see that weak and mild solutions of (1) coincide, provided the integrability condition (3) is satisfied (cf. [2]).

## 5. Uniqueness of generalized solutions

The purpose of this section is to show that, in certain cases, one can still prove uniqueness for equations whose solutions $u$ do not satisfy the integrability condition $Fu \in L^1([0,T] \to H)$ $\mathbb{P}$-a.s.. In fact, in general it is difficult (and we are not aware of any general results or techniques) to prove well-posedness in the mild sense without imposing rather restrictive conditions on the initial condition and on the coefficients of the equations. A possible way out is to define "generalized" mild solutions as limits of solutions of equations with more regular $u_0$, $B$, and $G$. Let us make this notion precise. In the following we shall say that $\zeta \in \mathcal{H}_2(T)$ if $\zeta : [0,T] \to H$ is an adapted process such that $\sup_{t\le T} \mathbb{E}|\zeta(t)|^2 < \infty$.

**Definition 7.** *Let*

$$\mathbb{E}|u_{0n}-u_0|^2 + \mathbb{E}\int_0^T \big(|B_n(t)-B(t)|_Q^2 + |G_n(t,\cdot)-G(t,\cdot)|_m^2\big)\,dt \to 0$$

*as $n \to \infty$, and assume that the equation*

$$du(t) + Au(t)\,dt + Fu(t)\,dt = B_n(t)\,dW(t) + G_n(t,z)\,\bar{\mu}(dt,dz)$$

*with initial condition $u(0) = u_{0n}$ admits a unique mild solution $u_n \in \mathcal{H}_2(T)$ for all $n \in \mathbb{N}$, such that $|u_n - u|_{\mathcal{H}_2(T)} \to 0$ as $n \to \infty$. Then $u$ is called a* generalized mild *solution of (2).*

Unfortunately we cannot give general sufficient conditions ensuring well-posedness of (2), but we limit ourselves to giving one criterion which can be verified, for instance, for reaction-diffusion equations with polynomial nonlinearity $F$, as considered in e.g. [3] in the case of Wiener noise, and in [6] in the case of Poisson noise. In the latter reference one may also find a fixed-point argument leading to existence and uniqueness of generalized mild solutions for equations with multiplicative noise.

Uniqueness of generalized mild solutions can be obtained by a priori estimates for mild solutions. For instance, let $u^1$, $u^2$ be solutions of (2) with initial conditions $u_0^1$, $u_0^2$, and coefficients $B^1$, $B^2$ and $G^1$, $G^2$, respectively. Assume that the following estimate holds

$$\mathbb{E}|u^1(t) - u^2(t)|^2 \le N\Big(\mathbb{E}|u_0^1 - u_0^2|^2 + \mathbb{E}\int_0^t \big(|B^1(s) - B^2(s)|_Q^2 + |G^1(s,\cdot) - G^2(s,\cdot)|_m^2\big)\,ds\Big),$$

where the constant $N$ depends continuously on $t$. Since the inequality is stable with respect to the limit passages of the previous definition, it is immediate to see that the same estimate holds also for generalized mild solution. This in turn implies that the generalized mild solution, if it exists, is unique, simply by taking $u_0^1 = u_0^2$ and $G^1 = G^2$.

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(C. Marinelli) Facoltà di Economia, Università di Bolzano, Piazza Università 1, I-39100 Bolzano, Italy, and Dipartimento di Matematica, Università di Trento, I-38123 Trento, Italy.

*URL*: `http://www.uni-bonn.de/~cm788`

(M. Röckner) Fakultät für Mathematik, Universität Bielefeld, Postfach 100 131, D-33501 Bielefeld, Germany.

*E-mail address*: `roeckner@math.uni-bielefeld.de`