WEAK UNIQUENESS OF FOKKER-PLANCK EQUATIONS WITH DEGENERATE AND BOUNDED COEFFICIENTS

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Abstract. In this note, by using the theory of stochastic differential equations (SDE), we prove uniqueness of measure-valued solutions and L^p-solutions to *degenerate* second order Fokker-Planck equations under weak conditions on the coefficients. Our uniqueness results are based on the natural connection between Fokker-Planck equations and SDEs.

Résumé. Dans cette Note, en utilisant la théorie des équations différentielles stochastiques (EDS), nous démontrons l'unicité de solutions L^p et à valeurs mesures pour des équations de Fokker-Planck du second ordre dégénérées, sous des conditions faibles sur les coefficients. Nos résultats d'unicité sont fondés sur le lien naturel existant entre les équations de Fokker-Planck et les EDS.

1. Introduction and Main Results

Let $\mathbb{W}^d := C([0,1];\mathbb{R}^d)$ be the space of all continuous functions from $[0,1]$ to $\mathbb{R}^d$. Let $\mathscr{W}_t$ be the canonical filtration generated by the coordinate process $W_t(\omega) = \omega(t)$, where $\omega \in \mathbb{W}^d$. Write $\mathscr{W} := \mathscr{W}_1$. Let ν be the standard Wiener measure on $(\mathbb{W}^d, \mathscr{W})$ so that $(t,\omega) \to W_t(\omega)$ is a standard d-dimensional Brownian motion.

Let $(X_t)_{t\in[0,1]}$ be a continuous $\mathscr{W}_t$-adapted process and solve the following SDE in $\mathbb{R}^d$:

$$\mathrm{d}X_t = \sigma_t(X_t)\mathrm{d}W_t + b_t(X_t)\mathrm{d}t, \tag{1.1}$$

where $\sigma : [0,1]\times\mathbb{R}^d \to \mathbb{R}^d\times\mathbb{R}^d$ and $b : [0,1]\times\mathbb{R}^d \to \mathbb{R}^d$ are two bounded measurable functions. Denote by μ_t the law of X_t in $\mathbb{R}^d$, i.e.: for any $\varphi \in C_0^\infty(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d} \varphi(x)\mu_t(\mathrm{d}x) = \mathbb{E}\varphi(X_t). \tag{1.2}$$

Write

$$L_t(x) = b_t^i(x)\partial_i + \frac{1}{2}[\sigma_t^{ik}\sigma_t^{jk}](x)\partial_{ij}^2.$$

By Itô's formula, μ_t solves the following Fokker-Planck equation in the sense of distribution:

$$\partial_t\mu_t = L_t^*\mu_t, \tag{1.3}$$

where L_t^* is the adjoint operator of L_t. More precisely, for any $\varphi \in C_0^\infty(\mathbb{R}^d)$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} \varphi(x)\mu_t(\mathrm{d}x) = \int_{\mathbb{R}^d} L_t\varphi(x)\mu_t(\mathrm{d}x),$$

where the initial condition means that μ_t weakly * converges to μ_0 as $t \to 0$. In particular, if the law of X_t is absolutely continuous with respect to Lebesgue measure, i.e., $\mu_t(\mathrm{d}x) = u_t(x)\mathrm{d}x$, then $u_t(x)$ solves the following PDE in the weak sense:

$$\partial_t u_t = L_t^* u_t. \tag{1.4}$$

Let $\mathcal{P}(\mathbb{R}^d)$ be the set of all probability measures on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$. It is well known from [1] that if σ^{ik} is uniformly *non-degenerate* and Lipschitz continuous, b is locally integrable and coercive, then the uniqueness for (1.3) holds in $\mathcal{P}(\mathbb{R}^d)$, at least if the initial measure has finite entropy. In the degenerate case, in order to prove the uniqueness for (1.4), one usually needs to impose some regularity on the solution. For example, Le Bris and Lions [7] proved the uniqueness of solutions to (1.4) for a given initial condition in the following class:

$$\{u \in L^\infty(0,1;(L^1 \cap L^\infty)(\mathbb{R}^d)), \sigma^{\mathrm{t}}\nabla u \in L^2(0,1;L^2(\mathbb{R}^d))\},$$

where σ^{t} denotes the transpose of σ. However, to the best of our knowledge, if σ is degenerate, there are only a few results about the uniqueness of measure-valued solutions to (1.3) and L^p-solutions to (1.4).

For $p \geqslant 1$, define

$$\mathcal{M}^p(\mathbb{R}^d) := \left\{ u \in L^p(0,1;L^p_{loc}(\mathbb{R}^d)); u \geqslant 0 \text{ and } \int_{\mathbb{R}^d} u_t(x)\mathrm{d}x = 1, \forall t \in [0,1] \right\}.$$

Below, by $B_R := \{x \in \mathbb{R}^d : |x| \leqslant R\}$ we denote the ball around zero in $\mathbb{R}^d$. Our main results are:

Theorem 1.1. *Assume that σ and b are bounded measurable functions and for some $q \in [1,\infty]$ and any $R > 0$, there exists a real function $f_R \in L^q([0,1] \times B_R)$ such that for almost all $(t,x,y) \in [0,1] \times B_R \times B_R$*

$$2\langle x-y, b_t(x)-b_t(y)\rangle + \|\sigma_t(x)-\sigma_t(y)\|^2 \leqslant (f_{R,t}(x)+f_{R,t}(y)) \cdot |x-y|^2. \tag{1.5}$$

Then for any given probability distribution density ρ, there is at most one weak solution u_t to PDE (1.4) in the class $\mathcal{M}^p(\mathbb{R}^d)$, where $p = \frac{q}{q-1}$, with $u_0 = \rho$.

Remark 1.2. *Condition (1.5) is satisfied if for some $q \in (1,\infty]$,*

$$b \in L^q(0,1;W^{q,1}_{loc}(\mathbb{R}^d)), \quad \sigma \in L^{q\vee 2}(0,1;W^{q\vee 2,1}_{loc}(\mathbb{R}^d)).$$

Indeed, in this case, there exists a constant $C_d > 0$ such that for almost all $(t,x,y) \in [0,1] \times B_R \times B_R$ (cf. [2, Lemma A.2, A.3] *or* [9, Lemma 3.7]*)*

$$|b_t(x)-b_t(y)| \leqslant C_d \cdot (M_R|\nabla b_t|(x) + M_R|\nabla b_t|(y)) \cdot |x-y|$$

and

$$\|\sigma_t(x)-\sigma_t(y)\| \leqslant C_d \cdot (M_R|\nabla \sigma_t|(x) + M_R|\nabla \sigma_t|(y)) \cdot |x-y|,$$

where $M_R g(x) := \sup_{0<r<R} \fint_{B_r} g(x+y)\mathrm{d}y$ denotes the maximal function of g. By the boundedness of the maximal operator in L^q (cf. [8]*), one knows that $M_R|\nabla b_\cdot| \in L^q([0,1] \times B_R)$ and $M_R|\nabla \sigma_\cdot| \in L^{q\vee 2}([0,1] \times B_R)$.*

Theorem 1.3. *Assume that σ and b are bounded measurable functions and for some $q \in [1,\infty]$ and any $R > 0$, there exists a real function $f_R \in L^q([0,1] \times B_R)$ such that for almost all $(t,x) \in [0,1] \times B_R$ and all $y \in B_R$*

$$2\langle x-y, b_t(x)-b_t(y)\rangle + \|\sigma_t(x)-\sigma_t(y)\|^2 \leqslant f_{R,t}(x) \cdot |x-y|^2. \tag{1.6}$$

Suppose that there exists a solution $u_t(x)$ to (1.4) in the class $\mathcal{M}^p(\mathbb{R}^d)$, where $p = \frac{q}{q-1}$. Then for any measure-valued solution μ_t to (1.3) in $\mathcal{P}(\mathbb{R}^d)$ with initial value $\mu_0(\mathrm{d}x) = u_0(x)\mathrm{d}x$,

$$\mu_t(\mathrm{d}x) = u_t(x)\mathrm{d}x, \quad \forall t \in [0,1].$$

Remark 1.4. *Condition (1.6) is satisfied if for some $q \in (d,\infty]$*

$$b \in L^q(0,1;W^{q,1}_{loc}(\mathbb{R}^d)), \quad \sigma \in L^{q\vee 2}(0,1;W^{q\vee 2,1}_{loc}(\mathbb{R}^d)).$$

Indeed, in this case, there exists a constant $C_{d,q} > 0$ such that for almost all $(t,x,y) \in [0,1] \times B_R \times B_R$ (cf. [5, p.143, Theorem 3]*)*

$$|b_t(x)-b_t(y)| \leqslant C_{d,q} \cdot (M_R|\nabla b_t|^q(x))^{1/q} \cdot |x-y|$$

and

$$\|\sigma_t(x)-\sigma_t(y)\| \leqslant C_{d,q} \cdot (M_R|\nabla \sigma_t|^q(x))^{1/q} \cdot |x-y|.$$

Since b and σ are continuous by Sobolev's embedding theorem, the above two inequalities hold for all $y \in B_R$.

Theorems 1.1 and 1.3 will be proven in the next section. Our argument is based on the representation (1.2) (see Theorem 2.5 below) and Yamada-Watanbe's theorem (cf. [6]).

2. Proofs of Main Results

For proving our main results, we first recall some facts from the theories of SDEs and PDEs.

Definition 2.1. *(Martingale solutions) Given $\mu_0 \in \mathcal{P}(\mathbb{R}^d)$, a probability measure P_{μ_0} on $(\mathbb{W}^d, \mathscr{W})$ is called a martingale solution of SDE (1.1) with initial distribution μ_0 if $P_{\mu_0} \circ \omega_0^{-1} = \mu_0$ and for any $\varphi \in C_0^\infty(\mathbb{R}^d)$, $\varphi(\omega_t) - \varphi(\omega_0) - \int_0^t L_s\varphi(\omega_s)\mathrm{d}s$ is an $\mathscr{W}_t$-martingale under P_{μ_0}.*

Definition 2.2. *(Weak solutions) Let $\mu_0 \in \mathcal{P}(\mathbb{R}^d)$. We say that Eq.(1.1) has a weak solution with initial law μ_0 if there exist a stochastic basis $(\Omega, \mathscr{F}, P; (\mathscr{F}_t)_{t\in[0,1]})$, a $\mathbb{R}^d$-valued continuous $(\mathscr{F}_t)$-adapted stochastic process X and a d-dimensional standard $(\mathscr{F}_t)$-Brownian motions $(W_t)_{t\in[0,1]}$ such that X_0 has law μ_0 and $X_t = X_0 + \int_0^t b_s(X_s)\mathrm{d}s + \int_0^t \sigma_s(X_s)\mathrm{d}W_s, \forall t \geqslant 0, a.s.$ This solution is denoted by $(\Omega, \mathscr{F}, P; (\mathscr{F}_t)_{t\in[0,1]}; W, X)$.*

The following two propositions are well known (cf. [6, Chapter IV, Theorem 1.1 and Proposition 2.1]).

Proposition 2.3. *(Equivalence between martingale solutions and weak solutions) Given $\mu_0 \in \mathcal{P}(\mathbb{R}^d)$ and let P_{μ_0} be a martingale solution of SDE (1.1). Then there exists a weak solution $(\Omega, \mathscr{F}, P; (\mathscr{F}_t)_{t\in[0,1]}; W, X)$ to SDE (1.1) such that $P \circ X^{-1} = P_{\mu_0}$.*

Proposition 2.4. *Given two weak solutions to SDE (1.1)*

$$(\Omega^{(i)}, \mathscr{F}^{(i)}, P^{(i)}; (\mathscr{F}_t^{(i)})_{t\in[0,1]}; W^{(i)}, X^{(i)}), \quad i = 1, 2,$$

with the same initial law $\mu_0 \in \mathcal{P}(\mathbb{R}^d)$, there exists a stochastic basis $(\Omega, \mathscr{F}, P; (\mathscr{F}_t)_{t\in[0,1]})$, a standard d-dimensional $(\mathscr{F}_t)$-Brownian motion W and two continuous $(\mathscr{F}_t)$-adapted processes $Y^{(i)}, i = 1, 2$ such that $P(Y_0^{(1)} = Y_0^{(2)}) = 1$ and $(\Omega, \mathscr{F}, P; (\mathscr{F}_t)_{t\in[0,1]}; W, Y^{(i)}),\ \ i = 1, 2$ are two weak solutions of (1.1), and $X^{(i)}$ and $Y^{(i)}$ have the same laws in $\mathbb{W}^d$ for $i = 1, 2$.

The following result is due to Figalli [4, Theorem 2.6].

Theorem 2.5. *Assume that σ and b are bounded measurable functions. Given $\mu_0 \in \mathcal{P}(\mathbb{R}^d)$, let $\mu_t \in \mathcal{P}(\mathbb{R}^d)$ be a measure-valued solution of PDE (1.3) with initial value μ_0. Then there exists a martingale solution P_{μ_0} to SDE (1.1) with initial distribution μ_0 such that for any $\varphi \in C_0^\infty(\mathbb{R}^d)$*

$$\int_{\mathbb{R}^d} \varphi(x)\mu_t(\mathrm{d}x) = \int_{\mathbb{W}^d} \varphi(\omega_t)P_{\mu_0}(\mathrm{d}\omega), \quad \forall t \in [0, 1].$$

We are now in a position to give the proofs of our main results.

Proof of Theorem 1.1. Let $u_t^{(i)}, i = 1, 2$ be two weak solutions of (1.4) in the class $\mathcal{M}^p(\mathbb{R}^d)$ with the same initial value u_0. By Theorem 2.5, there exists two martingale solutions $P_{u_0}^{(i)}, i = 1, 2$ to SDE (1.1) with the same initial law $u_0(x)\mathrm{d}x$ such that for any $\varphi \in C_0^\infty(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d} \varphi(x)u_t^{(i)}(x)\mathrm{d}x = \int_{\mathbb{W}^d} \varphi(\omega_t)P_{u_0}^{(i)}(\mathrm{d}\omega), \quad i = 1, 2. \tag{2.1}$$

By Propositions 2.3 and 2.4, there is a common stochastic basis $(\Omega, \mathscr{F}, P; (\mathscr{F}_t)_{t\in[0,1]})$, a standard d-dimensional $(\mathscr{F}_t)$-Brownian motion W and two continuous $(\mathscr{F}_t)$-adapted processes $Y^{(i)}, i = 1, 2$ such that $P(Y_0^{(1)} = Y_0^{(2)}) = 1$ and for $i = 1, 2$

$$Y_t^{(i)} = Y_0^{(i)} + \int_0^t b_s(Y_s^{(i)})\mathrm{d}s + \int_0^t \sigma_s(Y_s^{(i)})\mathrm{d}W_s, \tag{2.2}$$

and $Y^{(i)}$ has law $P_{u_0}^{(i)}$ in $(\mathbb{W}^d, \mathscr{W})$.

Set now $Z_t := Y_t^{(1)} - Y_t^{(2)}$ and for $R > 0$, $\tau_R := \inf\{t \in [0,1] : |Y_t^{(1)}| \vee |Y_t^{(2)}| \geqslant R\}$. By Itô's formula, for any $\delta > 0$, we have

$$\begin{aligned}
\log\left(\frac{|Z_{t\wedge\tau_R}|^2}{\delta^2}+1\right) &= \int_0^{t\wedge\tau_R} \frac{2\langle Z_s, b_s(Y_s^{(1)}) - b_s(Y_s^{(2)})\rangle + \|\sigma_s(Y_s^{(1)})) - \sigma_s(Y_s^{(2)})\|^2}{|Z_s|^2+\delta^2}\mathrm{d}s \\
&\quad +2\int_0^{t\wedge\tau_R} \frac{\langle Z_s, (\sigma_s(Y_s^{(1)}) - \sigma_s(Y_s^{(2)}))\mathrm{d}W_s\rangle}{|Z_s|^2+\delta^2} \\
&\quad -2\int_0^{t\wedge\tau_R} \frac{|(\sigma_s(Y_s^{(1)}) - \sigma_s(Y_s^{(2)}))^{\mathrm{t}} \cdot Z_s|^2}{(|Z_s|^2+\delta^2)^2}\mathrm{d}s. \qquad (2.3)
\end{aligned}$$

Let ρ be a nonnegative smooth function on $\mathbb{R}^d$ with support in $\{x \in \mathbb{R}^d : |x| < 1\}$ and $\int_{\mathbb{R}^d}\rho(x)\mathrm{d}x = 1$. For $\varepsilon \in (0,1)$, let $\rho_\varepsilon(x) := \varepsilon^{-d}\rho(x/\varepsilon)$ be a mollifier and define $b_s^\varepsilon := b_s * \rho_\varepsilon, \sigma_s^\varepsilon := \sigma_s * \rho_\varepsilon$, where $*$ denotes the convolution. By the property of mollifier, we have

$$\lim_{\varepsilon\downarrow 0}\int_0^t\int_{B_R}(|b_s^\varepsilon(x) - b_s(x)|^p + \|\sigma_s^\varepsilon(x) - \sigma_s(x)\|^p)\mathrm{d}x\mathrm{d}s = 0, \quad p \in [1,\infty),$$

and by (1.5) and the property of convolution, for almost all t and all $x, y \in B_R$

$$2\langle x-y, b_t^\varepsilon(x) - b_t^\varepsilon(y)\rangle + \|\sigma_t^\varepsilon(x) - \sigma_t^\varepsilon(y)\|^2 \leqslant (f_{R+1,t}^\varepsilon(x) + f_{R+1,t}^\varepsilon(y)) \cdot |x-y|^2.$$

Thus, by taking expectations for (2.3), we obtain

$$\begin{aligned}
\mathbb{E}\log\left(\frac{|Z_{t\wedge\tau_R}|^2}{\delta^2}+1\right) &\leqslant \mathbb{E}\int_0^{t\wedge\tau_R} \frac{2\langle Z_s, b_s(Y_s^{(1)}) - b_s(Y_s^{(2)})\rangle + \|\sigma_s(Y_s^{(1)}) - \sigma_s(Y_s^{(2)})\|^2}{|Z_s|^2+\delta^2}\mathrm{d}s \\
&\leqslant \mathbb{E}\int_0^{t\wedge\tau_R} \frac{2\langle Z_s, b_s^\varepsilon(Y_s^{(1)}) - b_s^\varepsilon(Y_s^{(2)})\rangle + \|\sigma_s^\varepsilon(Y_s^{(1)}) - \sigma_s^\varepsilon(Y_s^{(2)})\|^2}{|Z_s|^2+\delta^2}\mathrm{d}s \\
&\quad +\frac{2}{\delta}\mathbb{E}\int_0^{t\wedge\tau_R}(|b_s^\varepsilon(Y_s^{(1)}) - b_s(Y_s^{(1)})| + |b_s^\varepsilon(Y_s^{(2)}) - b_s(Y_s^{(2)})|)\mathrm{d}s \\
&\quad +\frac{3}{\delta^2}\mathbb{E}\int_0^{t\wedge\tau_R}(\|\sigma_s^\varepsilon(Y_s^{(1)}) - \sigma_s(Y_s^{(1)})\|^2 + \|\sigma_s^\varepsilon(Y_s^{(2)}) - \sigma_s(Y_s^{(2)})\|^2)\mathrm{d}s \\
&=: I_1^\varepsilon + I_2^\varepsilon + I_3^\varepsilon.
\end{aligned}$$

For I_1^ε, we have

$$\begin{aligned}
I_1^\varepsilon &\leqslant \mathbb{E}\int_0^{t\wedge\tau_R}(f_{R+1,s}^\varepsilon(Y_s^{(1)}) + f_{R+1,s}^\varepsilon(Y_s^{(2)}))\mathrm{d}s \\
&\leqslant \mathbb{E}\int_0^t\Big(1_{|Y_s^{(1)}|\leqslant R}\cdot f_{R+1,s}^\varepsilon(Y_s^{(1)}) + 1_{|Y_s^{(2)}|\leqslant R}\cdot f_{R+1,s}^\varepsilon(Y_s^{(2)})\Big)\mathrm{d}s \\
&= \int_0^t\int_{B_R} f_{R+1,s}^\varepsilon(x)u_s^{(1)}(x)\mathrm{d}x\mathrm{d}s + \int_0^t\int_{B_R} f_{R+1,s}^\varepsilon(x)u_s^{(2)}(x)\mathrm{d}x\mathrm{d}s \\
&\leqslant \|f_{R+1}^\varepsilon\|_{L^q([0,1]\times B_R)}\|u^{(1)}\|_{L^p([0,1]\times B_R)} + \|f_{R+1}^\varepsilon\|_{L^q([0,1]\times B_R)}\|u^{(2)}\|_{L^p([0,1]\times B_R)}.
\end{aligned}$$

Similarly, we have

$$I_2^\varepsilon \leqslant C\left(\int_0^t\int_{B_R}|b_s^\varepsilon(x) - b_s(x)|^q\mathrm{d}x\mathrm{d}s\right)^{1/q}$$

and

$$I_3^\varepsilon \leqslant C\left(\int_0^t\int_{B_R}|\sigma_s^\varepsilon(x) - \sigma_s(x)|^q\mathrm{d}x\mathrm{d}s\right)^{1/q},$$

where the constant C depends on $\|u^{(i)}\|_{L^p([0,1]\times B_R)}$, but is independent of ε.

Combining the above calculations and letting ε go to zero, we get

$$\mathbb{E}\log\left(\frac{|Z_{t\wedge\tau_R}|^2}{\delta^2}+1\right)\leqslant \|f_{R+1}\|_{L^q([0,1]\times B_R)}\cdot\Big(\|u^{(1)}\|_{L^p([0,1]\times B_R)}+\|u^{(2)}\|_{L^p([0,1]\times B_R)}\Big).$$

Now, letting $\delta\to 0$, we obtain that for any $R>0$ and $t\in[0,1]$

$$Z_{t\wedge\tau_R}=0,\quad a.s. \tag{2.4}$$

Since b and σ are bounded, from (2.2), it is now standard to prove that

$$\mathbb{E}\left(\sup_{t\in[0,1]}|Y^{(i)}_t|\right)<+\infty,\quad i=1,2.$$

Hence,

$$P\left\{\omega:\lim_{R\to\infty}\tau_R(\omega)=1\right\}=1$$

and letting $R\to\infty$ in (2.4), we further have

$$Z_t=0,\quad a.s.,\quad \forall t\in[0,1].$$

So, $P^{(1)}_{u_0}=P^{(2)}_{u_0}$. Now, the uniqueness follows by (2.1). □

Proof of Theorem 1.3. Following the proof of Theorem 1.1, let $Y^{(1)}_t$ (resp. $Y^{(2)}_t$) be the weak solution corresponding to $u_t(x)\mathrm{d}x$ (resp. $\mu_t(\mathrm{d}x)$). By (1.6) and (2.3), we have

$$\mathbb{E}\log\left(\frac{|Z_{t\wedge\tau_R}|^2}{\delta^2}+1\right)\leqslant \mathbb{E}\int_0^{t\wedge\tau_R} f_{R,s}(Y^{(1)}_s)\mathrm{d}s\leqslant \|f_R\|_{L^q([0,1]\times B_R)}\cdot\|u\|_{L^p([0,1]\times B_R)}.$$

From this, as above we obtain the uniqueness. □

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