DIMENSION-INDEPENDENT HARNACK INEQUALITIES FOR SUBORDINATED SEMIGROUPS

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Abstract. Dimension-independent Harnack inequalities are derived for a class of subordinate semigroups. In particular, for a diffusion satisfying the Bakry-Emery curvature condition, the subordinate semigroup with power α satisfies a dimension-free Harnack inequality provided $\alpha \in \left(\frac{1}{2}, 1\right)$, and it satisfies the log-Harnack inequality for all $\alpha \in (0,1)$. Some infinite-dimensional examples are also presented.

Contents

1. Introduction

By using the gradient estimate for diffusion semigroups, the following dimension-free Harnack inequality was established in [19] for the diffusion semigroup P_t generated by $L = \Delta + Z$ on a complete Riemannian manifold M with curvature $\mathrm{Ric} - \nabla Z$ bounded below by $-K \in \mathbb{R}$

(1.1)
$$(P_t f(x))^p \leqslant \exp\left(\frac{pK\rho(x,y)^2}{2(p-1)(\mathrm{e}^{2Kt}-1)}\right) P_t f^p(y), \quad t>0, x,y \in M, f \in \mathcal{B}_b^+(M),$$

where $p > 1$, ρ is the Riemannian distance, and $\mathcal{B}_b^+(M)$ is the class of all bounded positive measurable functions on M. This inequality has been extended and applied in the study of contractivity properties, heat kernel bounds, strong Feller properties

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and cost-entropy properties for finite- and infinite-dimensional diffusions. In particular, using the coupling method and Girsanov transformations developed in [4], this inequality has been derived for diffusions without using curvature conditions, see e.g. [5, 6, 9, 13–15, 17, 20] and references therein. See also [1–3] for applications to the short time behavior of transition probabilities. On the other hand, however, due to absence of a chain rule for the "gradient estimate" argument and an explicit Girsanov theorem, this technique of proving a dimension independent Harnack inequalities is not applicable to pure jump processes. The main purpose of this paper is to establish such inequalities for a class of α-stable like jump processes by using subordination.

Let (E, ρ) be a Polish space with the Borel3 σ-algebra $\mathcal{B}(E)$, and P_t the semigroup for a time-homogenous Markov process on E. Let $\{\mu_t\}_{t\geq 0}$ be a convolution semigroup of probability measures on $[0, \infty)$, i.e. one has $\mu_{t+s} = \mu_t * \mu_s$ for $s, t \geq 0$ and $\mu_t \to \mu_0 := \delta_0$ weakly as $t \to 0$. Thus, the Laplace transform for μ_t has the form

$$\int_0^\infty \mathrm{e}^{-xs}\,\mu_t(\mathrm{d}s) = \mathrm{e}^{-tB(x)}, \text{ for any } x \geqslant 0, t \geqslant 0 \tag{1.2}$$

for some Bernstein function B, see e.g. [12]. We shall study the Harnack inequality for the subordinated semigroup

$$P_t^B := \int_0^\infty P_s \mu_t(\mathrm{d}s), \quad t \geqslant 0. \tag{1.3}$$

Obviously, if P_t is generated by a negatively definite self-adjoint operator $(L, \mathcal{D}(L))$ on $L^2(\nu)$ for some σ-finite measure ν on E, then P_t^B is generated by $-B(-L)$. In particular, if $B(x) = x^\alpha$ for $\alpha \in (0, 1]$, we shall denote the corresponding μ_t by μ_t^α, and P_t^B by P_t^α respectively.

We shall use (1.3) and a known dimension independent Harnack inequality for P_t to establish the corresponding Harnack inequality for P_t^B. For instance, suppose we know that

$$(P_t f(x))^p \leqslant \exp\left(\Phi(p, t, x, y)\right) P_t f^p(y), x, y \in E, t > 0, p > 1, f \in \mathcal{B}_b^+(E)$$

for some $\Phi : (1, \infty) \times (0, \infty) \times E^2 \to [0, \infty)$. Then (1.3) implies

$$\begin{aligned}
(P_t^B f(x))^p &= \left(\int_0^\infty P_s f(x) \mu_t(\mathrm{d}s)\right)^p \\
&\leqslant \left(\int_0^\infty (P_s f^p(y))^{1/p} \exp\left(\frac{\Phi(p, s, x, y)}{p}\right) \mu_t(\mathrm{d}s)\right)^p \\
&\leqslant (P_t^B f^p(y)) \left(\int_0^\infty \exp\left(\frac{\Phi(p, s, x, y)}{p-1}\right) \mu_t(\mathrm{d}s)\right)^{p-1}.
\end{aligned} \tag{1.4}$$

In general, $\Phi(p, s, x, y) \to \infty$ as $s \to 0$, so we have to verify that $\exp[\Phi(p, s, x, y)/(p-1)]$ is integrable w.r.t. $\mu_t(\mathrm{d}s)$. Similarly to (1.1), for many specific models the singularity of $\Phi(p, s, x, y)$ at $s = 0$ behaves like $\mathrm{e}^{\delta/s^\kappa}$ for some $\delta = \delta\,(p, x, y) > 0, \kappa \geqslant 1$ (see Section 3 below for specific examples). In this case, the following results say that the Harnack inequality provided by (1.4) is valid for P_t^α with $\alpha > \kappa/(\kappa + 1)$.

Theorem 1.1. *Let $p>1,\kappa>0$ and $\alpha\in\left(\frac{\kappa}{\kappa+1},1\right)$ be fixed. Suppose that P_t satisfies the Harnack inequality*

$$(P_tf(x))^p\leqslant \exp\left(H(x,y)(\varepsilon+t^{-\kappa})\right)P_tf^p(y),\quad x,y\in E, f\in\mathcal{B}_b^+(E), t>0, \tag{1.5}$$

for some positive measurable function H on $E\times E$ and a constant $\varepsilon\geqslant 0$. Then there exists a constant $c>0$ depending on α and κ such that

$$\begin{aligned}
&(P_t^\alpha f(x))^p\\
&\leqslant \mathrm{e}^{\varepsilon H(x,y)}\left(1+\left[\exp\left(\left(\frac{cH(x,y)}{(p-1)t^{\kappa/\alpha}}\right)^{1/(1-(\alpha^{-1}-1)\kappa)}\right)-1\right]^{(1-(\alpha^{-1}-1)\kappa)}\right)^{p-1}P_t^\alpha f^p(y)\\
&\leqslant 2^{p-1}\exp\left(\varepsilon H(x,y)+C_{p,\kappa,\alpha}\left(\frac{H(x,y)}{t^{\kappa/\alpha}}\right)^{1/(1-(\alpha^{-1}-1)\kappa)}\right)P_t^\alpha f^p(y),\quad t>0, x,y\in E
\end{aligned}$$

holds for all $f\in\mathcal{B}_b^+(E)$, where

$$C_{p,\kappa,\alpha}=\frac{(1-(\alpha^{-1}-1)\kappa)c^{1/(1-(\alpha^{-1})\kappa)}}{(p-1)^{(\alpha^{-1}-1)\kappa/(1-(\alpha^{-1}-1)\kappa)}}.$$

Consequently, if P_t has an invariant probability measure μ, we have that

(i) for any $p,q>1$,

$$\frac{\|P_t^\alpha\|_{p\to q}}{2^{(p-1)/p}}\leqslant\left(\int_E\frac{\mu(\mathrm{d}x)}{\left(\int_E\exp\left[-\varepsilon H(x,y)-C_{p,\kappa,\alpha}\left(\frac{H(x,y)}{t^{\kappa/\alpha}}\right)^{1/(1-(\alpha^{-1}-1)\kappa)}\right]\mu(\mathrm{d}y)\right)^{q/p}}\right)^{1/q};$$

(ii) if P_t^α has a transition density $p_t^\alpha(x,y)$ w.r.t. μ such that for any $x\in\operatorname{supp}(\mu)$

$$\begin{aligned}
&\int_E p_t^\alpha(x,y)^2\mu(\mathrm{d}y)\\
&\leqslant 2\left(\int_E\exp\left(-\varepsilon H(x,y)-C_{p,\kappa,\alpha}\left(\frac{H(x,y)}{t^{\kappa/\alpha}}\right)^{1/(1-(\alpha^{-1}-1)\kappa)}\right)\mu(\mathrm{d}y)\right)^{-1}.
\end{aligned}$$

As an application of Theorem 1.1 (*ii*), we have the following explicit heat kernel upper bounds for stable like processes.

Example 1.2. *Let P_t be generated by $L=\Delta+Z$ on a complete Riemannian manifold such that $\mathrm{Ric}-\nabla Z\geq -K$. By (1.1), (1.5) holds for $H(x,y)=\rho(x,y)^2$ and $\kappa=1$. So, for $\alpha\in(1/2,1]$, Theorem 1.1 (ii) implies*

$$p_{2t}^\alpha(x,x)\leqslant\frac{c}{\mu(\{y:\rho(x,y)\leqslant t^{1/2\alpha}\})},\quad x\in M,\ t>0$$

for some constant $c>0$. In particular, for $L=\Delta$ on $\mathbb{R}^d$, $\mu(\mathrm{d}x)=\mathrm{d}x$ and $K=0$, we have

$$\sup_{x,y\in\mathbb{R}^d}p_t^\alpha(x,y)=\sup_{x\in\mathbb{R}^d}p_t^\alpha(x,x)\leqslant ct^{-d/2\alpha}, t>0,$$

for some constant $c > 0$. This is sharp due to the well known explicit bounds of heat kernels for the classical stable processes on $\mathbb{R}^d$.

Theorem 1.1 does not apply to $\alpha \in (0, \frac{\kappa}{\kappa+1}]$, since in this case $\int_0^\infty \mathrm{e}^{\delta/s^\kappa}\, \mu_t^\alpha(\mathrm{d}s) = \infty$ for large $\delta > 0$. A more careful analysis allows us to treat the case $\alpha = \frac{\kappa}{\kappa+1}$ under certain restrictions on x, y, t. Thus results of this type apply also to the Cauchy process.

Proposition 1.3 (The case $\alpha = \frac{\kappa}{\kappa+1}$)**.** *Suppose that P_t satisfies the Harnack inequality* (1.5) *for some positive measurable function H on $E \times E$ and a constant $\varepsilon \geqslant 0$. Then there exists a constant $C > 0$ depending on κ such that*

$$\begin{aligned}&(P_t^{\frac{\kappa}{\kappa+1}} f(x))^p\\ &\leqslant \mathrm{e}^{\varepsilon H(x,y)} \left(1 + \frac{C}{\frac{e(p-1)}{H(x,y)\kappa}\left(\frac{\kappa t}{\kappa+1}\right)^{\kappa+1} - 1}\right)^{p-1} P_t^{\frac{\kappa}{\kappa+1}} f^p(y), \quad f \in \mathcal{B}_b^+(E)\end{aligned}$$

holds for all $t > 0, x, y \in \mathbb{E}$ such that

$$e(p-1)(t\kappa)^{\kappa+1} > \kappa(\kappa+1)^{\kappa+1} H(x,y).$$

In other cases we can still prove the log-Harnack inequality. For diffusion semigroups, the known log-Harnack inequality looks like

$$P_t \log f(x) \leqslant \log P_t f(y) + H(x,y)(\varepsilon + t^{-\kappa}), x, y \in E, t > 0, f \geq 1, \tag{1.6}$$

for some positive measurable function H on $E \times E$ and some constants $\varepsilon \geq 0, \kappa \geq 1$. In many cases, one has $H(x,y) = c\rho(x,y)^2$ for a constant $c > 0$ and the intrinsic distance ρ induced by the diffusion (see e.g. [18]).

Theorem 1.4. *If* (1.6) *holds, then for any $\alpha \in (0,1]$,*

$$P_t^\alpha \log f(x) \leqslant \log P_t^\alpha f(y) + H(x,y)\left(\varepsilon + \log P_t^\alpha f(y) + H(x,y)\left(\varepsilon + \frac{\Gamma\left(\frac{\kappa}{\alpha}\right)}{\alpha t^{\frac{\kappa}{\alpha}} \Gamma(\kappa)}\right)\right),$$

$t > 0, x, y \in E, f \geq 1$.

As observed in [6] and [18] , the log-Harnack inequality implies an entropy-cost inequality for the semigroup and an entropy inequality for the corresponding transition density. Let W_H be the Wasserstein distance induced by H, i.e.

$$W_H(\mu_1, \mu_2) = \inf_{\pi \in \mathcal{C}(\mu_1,\mu_2)} \int_{E\times E} H(x,y)\pi(\mathrm{d}x, \mathrm{d}y),$$

where μ_1, μ_2 are probability measures on E and $\mathcal{C}(\mu_1, \mu_2)$ is the set of all couplings for μ_1 and μ_2.

Corollary 1.5. *Assume that* (1.6) *holds and let P_t have an invariant probability measure μ. Then for any $\alpha \in (0,1]$:*

(1) *The entropy-cost inequality*

$$\mu(((P_t^\alpha)^* f) \log (P_t^\alpha)^* f) \leqslant W_H(f\mu, \mu)\left(\varepsilon + \log P_t^\alpha f(y) + H(x,y)\left(\varepsilon + \frac{\Gamma\left(\frac{\kappa}{\alpha}\right)}{\alpha t^{\frac{\kappa}{\alpha}} \Gamma(\kappa)}\right)\right),$$

$t > 0, f \geqslant 0, \mu(f) = 1$

holds for all $\alpha \in (0,1]$, where $(P_t^\alpha)^*$ is the adjoint of P_t^α in $L^2(E;\mu)$.

(2) *If $H(x,y) \to 0$ as $y \to x$ holds for any $x \in E$, then P_t^α is strong Feller and thus has a transition density $p_t(x,y)$ w.r.t. μ on $\mathrm{supp}\mu$, which satisfies the entropy inequality*

$$\int_E p_t(x,z)\log\frac{p_t(x,z)}{p_t(y,z)}\,\mu(\mathrm{d}z) \leqslant H(x,y)\left(\varepsilon + \frac{\Gamma\left(\frac{\kappa}{\alpha}\right)}{\alpha t^{\frac{\kappa}{\alpha}}\Gamma(\kappa)}\right), \quad t>0, x,y \in \mathrm{supp}\,\mu.$$

2. Proofs

Proof of Theorem 1.1. The consequences of the desired Harnack inequality are straightforward. Indeed, (i) follows by noting that the claimed Harnack inequality implies

$$\begin{aligned}&(P_t^\alpha f(x))^p \int_E \exp\Big[-\varepsilon H(x,y) - C_{p,\kappa,\alpha}\Big(\frac{H(x,y)}{t^{\kappa/\alpha}}\Big)^{1/(1-(\alpha^{-1}-1)\kappa)}\Big]\mu(\mathrm{d}y)\\ &\leqslant \mu(P_t^\alpha f^p) = \mu^\alpha(f^p),\end{aligned}$$

which also implies (ii) by taking $p=2$ and $f(z)=p_t^\alpha(x,z), z\in E$. Indeed, with $f=1_A$ for a μ-null set A, this inequality implies that the associated transition probability $P_t^\alpha(x,\cdot)$ is absolutely continuous w.r.t. μ and hence, has a density $p_t^\alpha(x,\cdot)$ for every $x\in E$. Then the desired upper bound for $\int_E p_t^\alpha(x,y)^2\mu(\mathrm{d}y)$ follows by first applying the above inequality with $p=2$ and $f(z)=p_t^\alpha(x,z)\wedge n$ then letting $n\to\infty$. So, it remains to prove the first assertion.

By (1.5), (1.4) holds for $\Phi(p,s,x,y)=H(x,y)(\varepsilon+s^{-\kappa})$, i.e.

$$(P_t^\alpha f(x))^p \leqslant \mathrm{e}^{\varepsilon H(x,y)}(P_t^\alpha f^p(y))\left(\int_0^\infty \exp\left[\frac{H(x,y)}{(p-1)s^\kappa}\right]\mu_t(\mathrm{d}s)\right)^{p-1}. \tag{2.1}$$

So it suffices to estimate the integral $\int_0^\infty \mathrm{e}^{\delta/s^\kappa}\,\mu_t(\mathrm{d}s)$ for $\delta := \frac{H(x,y)}{(p-1)} > 0$.

We use the formula

$$s^{-r} = \frac{1}{\Gamma(r)}\int_0^\infty x^{r-1}\,\mathrm{e}^{-xs}\,\mathrm{d}x,\ \ r>0.$$

to obtain

$$\begin{aligned}&\int_0^\infty \frac{\mu_t^\alpha(ds)}{s^r} = \int_0^\infty \frac{1}{\Gamma(r)}\int_0^\infty x^{r-1}e^{-xs}dx\mu_t(ds) =\\ &\frac{1}{\Gamma(r)}\int_0^\infty x^{r-1}\int_0^\infty e^{-xs}\mu_t(ds)\,dx = \frac{1}{\Gamma(r)}\int_0^\infty x^{r-1}e^{-tB(x)}dx.\end{aligned}$$

In particular, for $B(x)=x^\alpha$ we have

$$\int_0^\infty \frac{\mu_t^\alpha(ds)}{s^r} = \frac{1}{\Gamma(r)}\int_0^\infty x^{r-1}e^{-tx^\alpha}dx = \frac{1}{\alpha\Gamma(r)}\int_0^\infty y^{\frac{r}{\alpha}-1}e^{-ty}dy = \frac{\Gamma\left(\frac{r}{\alpha}\right)}{\alpha\Gamma(r)}t^{-\frac{r}{\alpha}}. \tag{2.2}$$

We can use the generalization of Stirling's formula giving the asymptotic behavior of the Gamma function for large r

$$\Gamma(r) = \sqrt{2\pi}r^{r-\frac{1}{2}}e^{-r+\eta(r)},$$

where

$$\eta(r)=\sum_{n=0}^{\infty}\left(r+n+\frac{1}{2}\right)\ln\left(1+\frac{1}{r+n}\right)-1=\frac{\theta}{12r},0<\theta<1.$$

We apply this estimate to $\Gamma(\kappa n)$, $\Gamma\left(\frac{\kappa n}{\alpha}\right)$ and $n!$. Thus

$$\begin{aligned}
&\int_0^{\infty} e^{\frac{\delta}{s^{\kappa}}}\mu_t^{\alpha}(\mathrm{d}s)=1+\sum_{n=1}^{\infty}\frac{\delta^n}{n!}\frac{\Gamma\left(\frac{\kappa n}{\alpha}\right)}{\alpha\Gamma(\kappa n)}t^{-\frac{\kappa n}{\alpha}}=\\
&1+\frac{1}{\alpha}\sum_{n=1}^{\infty}\frac{\delta^n}{n!}(\kappa n)^{\kappa n\left(\frac{1}{\alpha}-1\right)}e^{-\kappa n\left(\frac{1}{\alpha}-1\right)}\alpha^{\frac{1}{2}-\frac{\kappa n}{\alpha}}e^{\frac{\theta_1\alpha-\theta_2}{12\kappa n}}t^{-\frac{\kappa n}{\alpha}}\leqslant\\
&1+\frac{1}{\sqrt{\alpha}}\sum_{n=1}^{\infty}\frac{\delta^n}{n!}(\kappa n)^{\kappa n\left(\frac{1}{\alpha}-1\right)}e^{-\kappa n\left(\frac{1}{\alpha}-1\right)}\alpha^{-\frac{\kappa n}{\alpha}}e^{\frac{\alpha}{12\kappa n}}t^{-\frac{\kappa n}{\alpha}}=\\
&1+\frac{1}{\sqrt{\alpha}}\sum_{n=1}^{\infty}\frac{n^{\kappa n\left(\frac{1}{\alpha}-1\right)}}{n!}\left(\delta\left(\frac{\kappa}{e}\right)^{\kappa\left(\frac{1}{\alpha}-1\right)}\alpha^{-\frac{\kappa}{\alpha}}t^{-\frac{\kappa}{\alpha}}\right)^n e^{\frac{\alpha}{12\kappa n}}\leqslant\\
&1+\frac{1}{\sqrt{2\pi\alpha}}\sum_{n=1}^{\infty}n^{\kappa n\left(\frac{1}{\alpha}-1\right)-n-\frac{1}{2}}\left(\delta\left(\frac{\kappa}{e}\right)^{\kappa\left(\frac{1}{\alpha}-1\right)}\alpha^{-\frac{\kappa}{\alpha}}t^{-\frac{\kappa}{\alpha}}\right)^n e^{\frac{\alpha}{12\kappa n}}.
\end{aligned}\tag{2.3}$$

This series converges for $\alpha>\frac{\kappa}{\kappa+1}$, moreover, there is a constant c depending only on κ such that

$$\frac{1}{\sqrt{2\pi\alpha n}}\left(\left(\frac{\kappa}{e}\right)^{\kappa\left(\frac{1}{\alpha}-1\right)}\alpha^{-\frac{\kappa}{\alpha}}t^{-\frac{\kappa}{\alpha}}\right)^n e^{\frac{\alpha}{12\kappa n}}\leqslant c^n.$$

Denote

$$c(\delta,\alpha,\kappa):=1+\sum_{n=1}^{\infty}n^{n\left(\kappa\left(\frac{1}{\alpha}-1\right)-1\right)}\left(c\delta t^{-\frac{\kappa}{\alpha}}\right)^n,$$

then

$$\left(P_t^{\alpha}f(x)\right)^p\leqslant \mathrm{e}^{\varepsilon H(x,y)}\left(c\left(\frac{H(x,y)}{p-1},\alpha,\kappa\right)\right)^{p-1}P_t^{\alpha}f^p(y).$$

Note that for $a>0, 1\geqslant b>0$ we have the following estimate

$$\begin{aligned}
&\sum_{n=1}^{\infty}\frac{a^n}{n^{bn}}=\sum_{n=1}^{\infty}\frac{(2a)^n}{n^{bn}}\frac{1}{2^n}\leqslant\left(\sum_{n=1}^{\infty}\frac{(2a)^{\frac{n}{b}}}{n^n}\frac{1}{2^n}\right)^b\leqslant\\
&\left(\sum_{n=1}^{\infty}\frac{(2a)^{\frac{n}{b}}}{n!}\frac{1}{2^n}\right)^b=\left(e^{\frac{(2a)^{1/b}}{2}}-1\right)^b,
\end{aligned}$$

where we used Jensen's inequality. Thus for any $\alpha\in\left(\frac{\kappa}{\kappa+1},1\right)$ we use the above estimate with $b:=\kappa\left(1-\frac{1}{\alpha}\right)+1\leqslant 1$ to see that

$$c\left(\delta,\alpha,\kappa\right)=1+\sum_{n=1}^{\infty}n^{n\left(\kappa\left(\frac{1}{\alpha}-1\right)-1\right)}\left(c\delta t^{-\frac{\kappa}{\alpha}}\right)^{n}\leqslant$$

$$1+\left(\exp\left(\frac{\left(2c\delta t^{-\frac{\kappa}{\alpha}}\right)^{\frac{1}{\kappa\left(1-\frac{1}{\alpha}\right)+1}}}{2}\right)-1\right)^{\kappa\left(1-\frac{1}{\alpha}\right)+1}.$$

Thus we can say that there is $c>0$ depending on α and κ such that

$$\int_0^\infty e^{\frac{H(x,y)}{(p-1)s^\kappa}}\mu_t^\alpha(\mathrm{d}s)\leqslant 1+\left(\exp\left(\left(\frac{cH\left(x,y\right)}{\left(p-1\right)t^{\frac{\kappa}{\alpha}}}\right)^{\frac{1}{\kappa\left(1-\frac{1}{\alpha}\right)+1}}\right)-1\right)^{\kappa\left(1-\frac{1}{\alpha}\right)+1}$$

Using the inequality

$$1+(x-1)^a\leqslant 2x^a$$

for any $x\geqslant 1$ and $0\leqslant a\leqslant 1$ we see that

$$\int_0^\infty e^{\frac{\delta}{s^\kappa}}\mu_t^\alpha(\mathrm{d}s)\leqslant 2\exp\left(\left(\kappa\left(1-\frac{1}{\alpha}\right)+1\right)\left(\frac{cH\left(x,y\right)}{\left(p-1\right)t^{\frac{\kappa}{\alpha}}}\right)^{\frac{1}{\kappa\left(1-\frac{1}{\alpha}\right)+1}}\right)$$

which completes the proof.

□

Proof of Proposition 1.3. In the case $\alpha=\frac{\kappa}{\kappa+1}$ the series in (2.3) converges for $t>0$ and $x,y\in E$ such that

$$e(p-1)(t\kappa)^{\kappa+1}>\kappa(\kappa+1)^{\kappa+1}H(x,y). \tag{2.4}$$

Note that for $\delta:=\frac{H(x,y)}{p-1}$ the last line of (2.3) reduces to

$$\begin{aligned}&1+\sqrt{\frac{\kappa+1}{2\pi\kappa}}\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}\left(\frac{\delta\kappa}{e}\left(\frac{\kappa+1}{\kappa t}\right)^{\kappa+1}\right)^{n}e^{\frac{1}{12(\kappa+1)n}}\\&\leqslant 1+C\sum_{n=1}^{\infty}\left(\frac{\delta\kappa}{e}\left(\frac{\kappa+1}{\kappa t}\right)^{\kappa+1}\right)^{n}\\&=1+\frac{C}{\frac{e}{\delta\kappa}\left(\frac{\kappa t}{\kappa+1}\right)^{\kappa+1}-1}.\end{aligned}$$

This completes the proof. □

Proof of Theorem 1.4. By (2.2) with $r=\kappa$, we have

$$\int_0^\infty\frac{\mu_t^\alpha\left(ds\right)}{s^\kappa}=\frac{\Gamma\left(\frac{\kappa}{\alpha}\right)}{\alpha t^{\frac{\kappa}{\alpha}}\Gamma\left(\kappa\right)}.$$

Using (1.2), (1.6) we obtain

$$P_t^\alpha \log f(x) = \int_0^\infty P_s \log f(x) \mu_t^\alpha(\mathrm{d}s) \leqslant \int_0^\infty \big(\log P_s f(y) + H(x,y)(\varepsilon + s^{-\kappa})\big) \mu_t^\alpha(\mathrm{d}s)$$
$$= \log P_t^\alpha f(y) + H(x,y) \left(\varepsilon + \frac{\Gamma\left(\frac{\kappa}{\alpha}\right)}{\alpha t^{\frac{\kappa}{\alpha}} \Gamma(\kappa)} \right).$$

This completes the proof.

□

Proof of Corollary 1.5. (1) It suffices to prove for $f \in \mathcal{B}_b^+(E)$ such that $\inf f > 0$ and $\mu(f) = 1$. In this case, there exists a constant $c > 0$ such that $cf \geqslant 1$. By Theorem 1.4 for $cP_t^\alpha f$ in place of f, we obtain

$$P_t^\alpha \log (P_t^\alpha)^* f(x) \leqslant \log P_t^\alpha (P_t^\alpha)^* f(y) + H(x,y) \left(\varepsilon + \frac{\Gamma\left(\frac{\kappa}{\alpha}\right)}{\alpha t^{\frac{\kappa}{\alpha}} \Gamma(\kappa)} \right).$$

Since μ is invariant for P_t^α and $(P_t^\alpha)^*$, taking the integral for both sides w.r.t. $\pi \in (f\mu, \mu)$ and minimizing in π, we prove the first assertion.

(2) The strong Feller property follows from Theorem 1.4 according to [18, Proposition 2.3], while by [18, Proposition 2.4] the desired entropy inequality for the transition density is equivalent to the log-Harnack inequality for P_t^α provided by Theorem 1.4.

□

3. Some infinite-dimensional examples

As explained in Section 1, Theorems 1.1 and 1.4 hold for $\kappa = 1$ if P_t is a diffusion semigroup on a Riemannian manifold with the Ricci curvature bounded below. In this section we present some infinite dimensional examples where these theorems can be used.

3.1. **Stochastic porous medium equation.** Let Δ be the Dirichlet Laplace operator on a bounded interval (a,b) and W_t the cylindrical Brownian motion on $L^2((a,b); \mathrm{d}x)$. Since the eigenvalues $\{\lambda_i\}$ of $-\Delta$ satisfies $\sum_{i=1}^\infty \lambda_i^{-1} < \infty$, W_t is a continuous process on $\mathbb{H}$, the completion of $L^2((a,b); \mathrm{d}x)$ under the inner product

$$\langle x, y \rangle := \sum_{i=1}^\infty \frac{1}{\lambda_i} \langle x, e_i \rangle \langle y, e_i \rangle,$$

where e_i is the unit eigenfunction corresponding to λ_i for each $i \geq 1$. Let $\|\cdot\|$ denote the norm on $\mathbb{H}$, and suppose $r > 1$. Then the following stochastic porous medium equation has a unique strong solution on $\mathbb{H}$ for any $X_0 \in \mathbb{H}$ (see e.g. [7]):

$$\mathrm{d}X_t = \Delta X_t^r \mathrm{d}t + \mathrm{d}W_t.$$

Let P_t be the corresponding Markov semigroup. According to [20, Remark 1.1 and Theorem 1.2], Theorem 1.1 in [20] holds for $\theta = r - 1$ and some constant $\gamma, \delta, \xi > 0$. Thus, there exist two constants $c_1, c_2 > 0$ depending on r such that

$$(P_t f)^p(x) \leqslant (P_t f^p(y)) \exp\left[\frac{c_1 p \|x - y\|^{4/(1+r)}}{(p-1)(1 - \mathrm{e}^{-c_2 t})^{(3+r)/(1+r)}} \right], \quad p > 1, t > 0, x, y \in \mathbb{H}$$

holds for all $f \in \mathcal{B}_b^+(\mathbb{H})$. By [18, Proposition 2.2] for $\rho(x,y)^2 = \|x-y\|^{2/(1+r)}$, this implies the log-Harnack inequality

$$P_t \log f(x) \leqslant \log P_t f(x) + \frac{c_1 \|x-y\|^{4/(1+r)}}{(1-\mathrm{e}^{-c_2 t})^{(3+r)/(1+r)}}, \quad x, y \in \mathbb{H}, f \geq 1.$$

Therefore, Theorems 1.1 and 1.4 apply to P_t^α for

$$\kappa = \frac{r+r}{1+r}$$

and some constant ε depending on r.

3.2. **Singular stochastic semi-linear equations.** Let $\mathbb{H}$ be a separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$, and W_t the cylindrical Brownian motion on $\mathbb{H}$. Consider the stochastic equation

$$\mathrm{d}X_t = (AX_t + F(X_t))\mathrm{d}t + \sigma \mathrm{d}W_t, \quad X_0 \in H. \tag{3.1}$$

Let A, F and σ satisfy the following hypotheses:

(H1) $(A, \mathcal{D}(A))$ is the generator of a C_0-semigroup, $T_t = e^{tA}$, $t \geq 0$, on $\mathbb{H}$ and for some $\omega \in \mathbb{R}$

$$\langle Ax, x \rangle \leq \omega \|x\|^2, \quad \forall\, x \in \mathcal{D}(A). \tag{3.2}$$

(H2) σ is a bounded positively definite, self-adjoint operator on $\mathbb{H}$ such that σ^{-1} is bounded and $\int_0^\infty \|T_t \sigma\|_{HS}^2 dt < \infty$, where $\|\cdot\|_{HS}$ denotes the norm on the space of all Hilbert–Schmidt operators on $\mathbb{H}$.
(H3) $F : \mathcal{D}(F) \subset \mathbb{H} \to \mathbb{H}$ is an m-dissipative map, i.e.,

$$\langle F(x) - F(y), x - y \rangle \leqslant 0, \qquad x, y \in \mathcal{D}(F),\ u \in F(x),\ v \in F(y),$$

("dissipativity") and

$$\text{Range } (I - F) := \bigcup_{x \in \mathcal{D}(F)} (x - F(x)) = \mathbb{H}.$$

Furthermore, $F_0(x) \in F(x),\ x \in \mathcal{D}(F)$, is such that

$$\|F_0(x)\| = \min_{y \in F(x)} \|y\|.$$

Here we recall that for F as in (H3) we have that $F(x)$ is closed, non empty and convex.

The corresponding Kolmogorov operator is then given as follows: Let $\mathcal{E}_A(H)$ denote the linear span of all real parts of functions of the form $\varphi = e^{i\langle h, \cdot \rangle}$, $h \in D(A^*)$, where A^* denotes the adjoint operator of A, and define for any $x \in \mathcal{D}(F)$,

$$L_0\varphi(x) = \frac{1}{2} \text{ Tr } (\sigma^2 D^2 \varphi(x)) + \langle x, A^* D\varphi(x) \rangle + \langle F_0(x), D\varphi(x) \rangle, \quad \varphi \in \mathcal{E}_A(H).$$

Additionally, we assume:

(H4) There exists a probability measure μ on H (equipped with its Borel σ-algebra $\mathcal{B}(H)$) such that
  (i) $\mu(\mathcal{D}(F)) = 1$,
  (ii) $\int_H (1 + \|x\|^2)(1 + \|F_0(x)\|)\mu(dx) < \infty$,
  (iii) $\int_H L_0 \varphi d\mu = 0$ for all $\varphi \in \mathcal{E}_A(H)$.

By [8], the closure of $(L_0, \mathcal{E}_A(\mathbb{H}))$ in $L^1(\mathbb{H};\mu)$ generates a Markov semigroup P_t with μ as an invariant probability measure, which is point-wisely determined on $\mathbb{H}_0 := \mathrm{supp}\mu$. If moreover the following hypotheses holds:

(H5) (i) $(1+\omega-A, \mathcal{D}(A))$ satisfies the weak sector condition: there exists a constant $K > 0$ such that

$$\langle (1+\omega-A)x, y\rangle \leqslant K\langle (1+\omega-A)x, x\rangle^{1/2}\langle (1+\omega-A)y, y\rangle^{1/2}, \quad \forall\, x, y \in \mathcal{D}(A). \tag{3.3}$$

(ii) There exists a sequence of A-invariant finite dimensional subspaces $\mathbb{H}_n \subset \mathcal{D}(A)$ such that $\bigcup_{n=1}^{\infty} \mathbb{H}_n$ is dense in $\mathbb{H}$.

Then (see [9, Theorem 1.6])

$$(P_t f(x))^p \leqslant P_t f^p(y) \exp\left[\|\sigma^{-1}\|^2 \, \frac{p\omega\|x-y\|^2}{(p-1)(1-\mathrm{e}^{-2\omega t})}\right], \quad t > 0,\ x, y \in \mathbb{H}_0.$$

As mentioned above, according to [18, Proposition 2.2] this implies the corresponding log-Harnack inequality. Therefore, our Theorems 1.1 and 1.4 apply to P_t^p for $\kappa = 1$.

3.3. **The Ornstein–Uhlenbeck type semigroups with jumps.** Consider the following stochastic differential equation driven by a Lévy process

$$\mathrm{d}X_t = AX_t \mathrm{d}t + \mathrm{d}Z_t, \quad X_0 = x \in \mathbb{H}, \tag{3.4}$$

where A is the infinitesimal generator of a strongly continuous semigroup $(T_t)_{t\geq 0}$ on $\mathbb{H}$, $Z_t := \{Z_t^u,\ u \in \mathbb{H}\}$ is a cylindrical Lévy process with characteristic triplet (a, R, M) on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, that is, for every $u \in \mathbb{H}$ and $t \geq 0$

$$\mathbb{E}\exp(\mathrm{i}\langle Z_t, u\rangle) = \exp\big(\mathrm{i}t\langle a, u\rangle - \frac{t}{2}\langle Ru, u\rangle - \int_{\mathbb{H}} \left[1 - \exp(\mathrm{i}\langle x, u\rangle) + \mathrm{i}\langle x, u\rangle 1_{\{\|x\|\leqslant 1\}}(x)\right], M(\mathrm{d}x)\big),$$

where $a \in \mathbb{H}$, R is a symmetric linear operator on $\mathbb{H}$ such that

$$R_t := \int_0^t T_s R T_s^* \,\mathrm{d}s$$

is a trace class operator for each $t > 0$, and M is a Lévy measure on $\mathbb{H}$. (For simplicity, we shall write $Z_t^u = \langle Z_t, u\rangle$ for every $u \in \mathbb{H}$.) In this case, (3.4) has a unique mild solution

$$X_t = T_t x + \int_0^t T_{t-s}\mathrm{d}Z_s, t \geqslant 0.$$

Let

$$P_t f(x) = \mathbb{E}f(X_t), \quad x \in \mathbb{H},\ f \in \mathbb{B}_b(\mathbb{H}).$$

If

$$\|R^{-1/2}T_t R x\| \leqslant \sqrt{h(t)}\,\|R^{1/2}x\|, \quad x \in \mathbb{H},\ t \geq 0$$

holds for some positive function $h \in C([0,\infty))$. Then by [16, Theorem 1.2] (see also [17] for the diffusion case),

$$(P_t f)^\alpha(x) \leqslant \exp\left[\frac{\alpha\|R^{-1/2}(x-y)\|^2}{2(\alpha-1)\int_0^t h(s)^{-1}\mathrm{d}s}\right] P_t f^\alpha(y), \quad t>0, x-y\in R^{1/2}\mathbb{H}$$

holds for all $f \in \mathcal{B}_b^+(\mathbb{H})$. By this and [18, Proposition 2.2] which implies the corresponding log-Harnack inequality, Theorems 1.1 and 1.4 apply to some $\varepsilon \geq 0$ and $\kappa \geq 1$ if

$$\limsup_{t\to 0} \frac{1}{t^\kappa}\int_0^t \frac{\mathrm{d}s}{h(s)} > 0.$$

3.4. **Infinite-dimensional Heisenberg groups.** In [10] an integrated Harnack inequality similar to (1.1) has been established for a Brownian motion on infinite-dimensional Heisenberg groups modeled on an abstract Wiener space. The inequality is the consequence of the Ricci curvature bounds for both finite-dimensional approximations to these groups and the group itself, and the results established for inductive limits of finite-dimensional Lie groups in [11]. Even though the methods described in that paper are applicable to inductive and projective limits of finite-dimensional Lie groups, the infinite-dimensional Heisenberg groups provide a very concrete setting. We follow the exposition in [10].

Let (W, H, μ) be an abstract Wiener space over $\mathbb{R}(\mathbb{C})$, $\mathbf{C}$ be a real(complex) finite dimensional inner product space, and $\omega : W \times W \to \mathbf{C}$ be a continuous skew symmetric bilinear quadratic form on W. Further, let

$$\|\omega\|_0 := \sup\{\|\omega(w_1, w_2)\|_{\mathbf{C}} : w_1, w_2 \in W \text{ with } \|w_1\|_W = \|w_2\|_W = 1\} \tag{3.5}$$

be the uniform norm on ω which is finite since ω is assumed to be continuous. We will need the Hilbert-Schmidt norm of ω which is defined as

$$\|\omega\|_2^2 = \|\omega\|_{H^*\otimes H^*\otimes\mathbf{C}} := \sum_{i,j=1}^{\infty} \|\omega(e_i, e_j)\|_{\mathbf{C}}^2,$$

which is finite by Proposition 3.14 in [10].

Definition 3.1. Let $\mathfrak{g}$ denote $W \times \mathbf{C}$ when thought of as a Lie algebra with the Lie bracket operation given by

$$[(A,a),(B,b)] := (0, \omega(A,B)). \tag{3.6}$$

Let $G := G(\omega)$ denote $W \times \mathbf{C}$ when thought of as a group with the multiplication law given by

$$g_1 g_2 = g_1 + g_2 + \frac{1}{2}[g_1, g_2] \text{ for any } g_1, g_2 \in G. \tag{3.7}$$

It is easily verified that $\mathfrak{g}$ is a Lie algebra and G is a group. The identity of G is the zero element, $\mathbf{e} := (0,0)$.

Notation 3.2. *Let $\mathfrak{g}_{CM}$ denote $H \times \mathbf{C}$ when viewed as a Lie subalgebra of $\mathfrak{g}$ and G_{CM} denote $H \times \mathbf{C}$ when viewed as a subgroup of $G = G(\omega)$. We will refer to $\mathfrak{g}_{CM}$ (G_{CM}) as the* ***Cameron–Martin subalgebra (subgroup)*** *of $\mathfrak{g}$ (G). (For explicit examples of such $(W, H, \mathbf{C}, \omega)$, see [10].)*

We equip $G=\mathfrak{g}=W\times\mathbf{C}$ with the Banach space norm

$$\|(w,c)\|_{\mathfrak{g}} := \|w\|_W + \|c\|_{\mathbf{C}} \tag{3.8}$$

and $G_{CM}=\mathfrak{g}_{CM}=H\times\mathbf{C}$ with the Hilbert space inner product,

$$\langle (A,a),(B,b)\rangle_{\mathfrak{g}_{CM}} := \langle A,B\rangle_H + \langle a,b\rangle_{\mathbf{C}}. \tag{3.9}$$

The associate Hilbertian norm is given by

$$\|(A,\delta)\|_{\mathfrak{g}_{CM}} := \sqrt{\|A\|_H^2+\|\delta\|_{\mathbf{C}}^2}. \tag{3.10}$$

As was shown in [10, Lemma 3.3], these Banach space topologies on $W\times\mathbf{C}$ and $H\times\mathbf{C}$ make G and G_{CM} into topological groups.

Then we can define a Brownian motion on G starting at $\mathbf{e}=(0,0)\in G$ to be the process

$$g(t)=\left(B(t),B_0(t)+\frac{1}{2}\int_0^t \omega\left(B(\tau),dB(\tau)\right)\right). \tag{3.11}$$

We denote by ν_t the corresponding heat kernel measure on G. The following estimate was used in the proof of Theorem 8.1 in [10]. For any $h\in G_{CM}$, $1<p<\infty$

$$\int_G |f(xh)|\,d\nu_t(x) \leqslant \|f\|_{L^p(G,\nu_t)} \exp\left(\frac{c(-k(\omega)t)(p-1)}{2t} d_{G_{CM}}^2(e,h)\right). \tag{3.12}$$

where

$$c(t)=\frac{t}{e^t-1} \quad \text{for all} \quad t\in\mathbb{R}$$

with the convention that $c(0)=1$ and

$$k(\omega):=\frac{1}{2}\sup_{\|A\|_H=1}\|\omega(\cdot,A)\|_{H^*\otimes\mathbf{C}}^2 \leqslant \frac{1}{2}\|\omega\|_2^2<\infty.$$

Equation (3.12) implies the corresponding L^p-estimates of Radon-Nikodym derivatives of ν_t relative to the left and right multiplication by elements in G_{CM}. This in turn is equivalent to the Harnack inequality (1.1) following an argument similar to Lemma D.1 in [11]

$$[(P_tf)(x)]^p \le C^p(P_tf^p)(y) \text{ for all } f\geqslant 0.$$

Thus we are in position to apply our results to the heat kernel measure ν_t subordinated as described in Section 1.

References

1. Shigeki Aida, *Uniform positivity improving property, Sobolev inequalities, and spectral gaps*, J. Funct. Anal. **158** (1998), no. 1, 152–185. MR MR1641566 (2000d:60125)
2. Shigeki Aida and Hiroshi Kawabi, *Short time asymptotics of a certain infinite dimensional diffusion process*, Stochastic analysis and related topics, VII (Kusadasi, 1998), Progr. Probab., vol. 48, Birkhäuser Boston, Boston, MA, 2001, pp. 77–124. MR MR1915450 (2003m:60219)
3. Shigeki Aida and Tusheng Zhang, *On the small time asymptotics of diffusion processes on path groups*, Potential Anal. **16** (2002), no. 1, 67–78. MR MR1880348 (2003e:58052)
4. Marc Arnaudon, Anton Thalmaier, and Feng-Yu Wang, *Harnack inequality and heat kernel estimates on manifolds with curvature unbounded below*, Bull. Sci. Math. **130** (2006), no. 3, 223–233. MR MR2215664 (2007i:58032)
5. ———, *Gradient estimates and harnack inequalities on non-compact riemannian manifolds*, to appear **x** (2009), x.

6. Sergey G. Bobkov, Ivan Gentil, and Michel Ledoux, *Hypercontractivity of Hamilton-Jacobi equations*, J. Math. Pures Appl. (9) **80** (2001), no. 7, 669–696. MR MR1846020 (2003b:47073)
7. G. Da Prato, Michael Röckner, B. L. Rozovskii, and Feng-Yu Wang, *Strong solutions of stochastic generalized porous media equations: existence, uniqueness, and ergodicity*, Comm. Partial Differential Equations **31** (2006), no. 1-3, 277–291. MR MR2209754 (2007b:60153)
8. Giuseppe Da Prato and Michael Röckner, *Singular dissipative stochastic equations in Hilbert spaces*, Probab. Theory Related Fields **124** (2002), no. 2, 261–303. MR MR1936019 (2003k:60151)
9. Giuseppe Da Prato, Michael Röckner, and Feng-Yu Wang, *Singular stochastic equations on Hilbert spaces: Harnack inequalities for their transition semigroups*, J. Funct. Anal. **257** (2009), no. 4, 992–1017. MR MR2535460
10. Bruce K. Driver and Maria Gordina, *Heat kernel analysis on infinite-dimensional Heisenberg groups*, J. Funct. Anal. **255** (2008), no. 9, 2395–2461. MR MR2473262
11. ———, *Integrated Harnack inequalities on Lie groups*, Journal of Differential Geometry **3** (2009), 501–550.
12. N. Jacob, *Pseudo differential operators and Markov processes. Vol. I*, Imperial College Press, London, 2001, Fourier analysis and semigroups. MR MR1873235 (2003a:47104)
13. Wei Liu, *Fine properties of stochastic evolution equations and their applications*, Ph.D. thesis, Doctor-Thesis, Bielefeld University, 2009.
14. Wei Liu and Feng-Yu Wang, *Harnack inequality and strong Feller property for stochastic fast-diffusion equations*, J. Math. Anal. Appl. **342** (2008), no. 1, 651–662. MR MR2440828 (2009k:60137)
15. Shun-Xiang Ouyang, *Harnack inequalities and applications for stochastic equations*, Ph.D. thesis, Ph.D. thesis, Bielefeld University, 2009.
16. Shun-Xiang Ouyang, Michael Röckner, and Feng-Yu Wang, *Harnack inequalities and applications for Ornstein-Uhlenbeck semigroups with jumps*, preprint.
17. Michael Röckner and Feng-Yu Wang, *Harnack and functional inequalities for generalized Mehler semigroups*, J. Funct. Anal. **203** (2003), no. 1, 237–261. MR MR1996872 (2005d:47077)
18. Feng-Yu Wang, *Heat kernel inequalities for curvature and second fundamental form*, preprint.
19. ———, *Logarithmic Sobolev inequalities on noncompact Riemannian manifolds*, Probab. Theory Related Fields **109** (1997), no. 3, 417–424. MR MR1481127 (98i:58253)
20. ———, *Harnack inequality and applications for stochastic generalized porous media equations*, Ann. Probab. **35** (2007), no. 4, 1333–1350. MR MR2330974 (2008e:60192)

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