Potential theory of infinite dimensional Lévy processes

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Abstract

We study the potential theory of a large class of infinite dimensional Lévy processes, including Brownian motion on abstract Wiener spaces. The key result is the construction of compact Lyapunov functions, i.e. excessive functions with compact level sets. Then many techniques from classical potential theory carry over to this infinite dimensional setting. Thus a number of potential theoretic properties and principles can be proved, answering long standing open problems even for the Brownian motion on abstract Wiener space, as e.g. formulated by R. Carmona in 1980. In particular, we prove the analog of the known result, that the Cameron-Martin space is polar, in the Lévy case and apply the technique of controlled convergence to solve the Dirichlet problem with general (not necessarily continuous) boundary data.

Keywords: abstract Wiener space, infinite dimensional Brownian motion, Lévy process on Hilbert space, capacity, polar set, Lyapunov function, Dirichlet problem, controlled convergence
2000 MSC: 60J45, 60J40, 60J35, 47D07, 31C15.

1. Introduction

The purpose of this paper is to study the potential theory of infinite dimensional Lévy processes. Such processes, in particular, the special case of infinite dimensional Brownian motion, are of fundamental importance as driving (i.e. noise) processes for stochastic partial differential equations. In addition, there had been interest in solving Dirichlet problems for infinite dimensional Ornstein-Uhlenbeck processes (see [15]). Nevertheless, there are very few papers in the last 30 years analyzing these fundamental processes in infinte dimensions from a potential theoretic point of view, as e.g. in the nice papers [36] and [37] on Liouville properties for the Ornstein-Uhlenbeck process with Lévy noise. Therefore, many questions about the validity of fundamental potential theoretic properties and principles even in the case of Brownian motion on abstract Wiener space remained open problems, since they were posed e.g. in [12], and the more so for infinite dimensional Lévy processes.

In this paper we shall establish a number of such properties and principles answering positively a substantial number of R. Carmona's questions in [12]. Naturally, in the meantime the "technology" and methodology in potential theory, in particular, in its analytic component, has been developed much further (see e.g. [3]).

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The main tool, however, to make this modern analytic potential theory work in our situation, is the construction of explicit compact Lyapunov functions, i.e. (β-) excessive functions with compact level sets, which is done in a very explicit way for the first time in this paper. Through such functions the usual local compactness assumption on the topology can be avoided.

The structure and main results of this paper are the following:

In Section 2 we start with the case of Brownian motion on abstract Wiener space. The compact Lyapunov functions are constructed in Proposition 2.4 and Theorem 2.7. First consequences are presented in Theorem 2.9 and Remark 2.10. The crucial integrability of the norm q_x (cf. (2.6)) with respect to the Gaussian measure follows from an application of Fernique's Theorem (see Proposition 2.4 (iv)).

Section 3 is devoted to infinite dimensional Lévy processes. The explicit compact Lyapunov functions are constructed in Proposition 3.3 and Theorem 3.4. Because of lack of an analog of Fernique's Theorem in this case, we can only consider Hilbert state spaces and require the existence of weak second moments (see assumption (H)(i) in Section 3 below). Examples include perturbations of nondegenerate Gaussian cases and the Poisson case (see Examples 3.2 and 3.6).

In Section 4 we present the potential theoretic consequences. We here mention the most important ones only: (a) we prove that Meyer's Hypothesis (L) (i.e. existence of a reference measure for the resolvent) does not hold; (b) we derive a natural condition ensuring that points are polar; (c) we prove that the "Cameron-Martin space" H is polar (including the Lévy case); (d) we introduce natural Choquet capacities (replacing the Newton capacity in finite dimensions) and show their tightness; (e) we prove quasi continuity properties for the excessive functions; (f) we prove the existence of bounded functions invariant under the semigroup; (g) we prove that the state space E can be decomposed into an uncountable union of disjoint affine spaces each being invariant under the Lévy process (Brownian motion respectively) and that the restriction of the process to any of such affine subspace is càdlàg; (h) we prove that the so-called "balayage principle" holds.

Results (d) and (h) above are even new in the infinite dimensional Brownian motion case.

Section 5 is devoted to the so-called "controlled convergence" for the solution to the Dirichlet problem for strongly regular open subsets of E. This type of convergence provides a way to describe the boundary behavior of the solution to the Dirichlet problem for general (not necessarily continuous) boundary data. Our main result here is Theorem 5.3.

Finally, we would like to point out that many of the above potential theoretic results extend to infinite dimensional α-stable or more general processes obtained by the above ones by standard subordination. In particular, if one considers processes subordinate to infinite dimensional Brownian motion, such as α-stable processes, one can cover jump processes without any conditions on their weak moments. We thank Masha Gordina and Sergio Albeverio for pointing this out to us. More details on this will be the subject of forthcoming work.

In the Appendix we prove a type of analogue to the necessity-part of L. Gross famous result on measurable norms (see [24]) in the non-Gaussian case.

2. Brownian motion on abstract Wiener space

Let (E, H, μ) be an *abstract Wiener space*, i.e. $\big(H, \langle\ ,\ \rangle\big)$ is a separable real Hilbert space with corresponding norm $|\cdot|$, which is continuously and densely embedded into a Banach space $\big(E, \|\cdot\|\big)$, which is hence also separable; μ is a Gaussian measure on $\mathcal{B}$ (= the Borel σ-algebra of E), that is, each $l \in E'$, the dual space of E, is normally distributed with mean zero and variance $|l|^2$. Here we use the standard continuous and dense embeddings

$$E' \subset (H' \equiv) H \subset E\ .$$

Clearly, we then have that

$$_{E'}\langle l, h\rangle_E = \langle l, h\rangle \text{ for all } l \in E' \text{ and } h \in H. \tag{2.1}$$

We recall that the embedding $H \subset E$ is automatically compact (see Ch.III, Section 2 in [10]) and that μ is H-quasi-invariant, that is for $T_h(z) := z + h$, $z, h \in E$, we have

$$\mu \circ T_h^{-1} \ll \mu \quad \text{for all } h \in H.$$

By the famous Dudley-Feldman-Le Cam Theorem (see [16] and also Theorem 4.1 in [41] for a concise presentation) we know that the norm $\|\cdot\|$ is *μ-measurable* in the sense of L. Gross (cf. [25], see also [29]). Hence also the centered Gaussian measures μ_t, $t > 0$, exist on $\mathcal{B}$, whose variance are given by $t|l|^2$, $l \in E'$, $t > 0$. So,

$$\mu_1 = \mu \ .$$

Clearly, μ_t is the image measure of μ under the map $z \longmapsto \sqrt{t}z$, $z \in E$.

For $x \in E$, the probability measure $p_t(x, \cdot\,)$ is defined by

$$p_t(x, A) := \mu_t(A - x) \quad \text{ for all } A \in \mathcal{B}.$$

Let $(P_t)_{t>0}$ be the associated family of Markovian kernels:

$$P_t f(x) := \int_E f(y)\, p_t(x, \mathrm{d}y) = \int_E f(x+y)\, \mu_t(\mathrm{d}y)\ , \quad f \in p\mathcal{B},\ x \in E;$$

we have denoted by $p\mathcal{B}$ the set of all positive, numerical, $\mathcal{B}$-measurable functions on E. By Proposition 6 in [25] it follows that $(P_t)_{t\geq 0}$ (where $P_0 := Id_E$) induces a strongly continuous semigroup of contractions on the space $\mathcal{C}_u(E)$ of all bounded uniformly continuous real-valued functions on E.

Let $\mathcal{U} = (U_\alpha)_{\alpha>0}$ be the associated Markovian resolvent of kernels on $(E, \mathcal{B})$ given by $U_\alpha := \int_0^\infty e^{-\alpha t} P_t \,\mathrm{d}t$, $\alpha > 0$. Recall that $\mathcal{U} = (U_\alpha)_{\alpha>0}$ induces a strongly continuous resolvent of contractions on $\mathcal{C}_u(E)$. By $\mathcal{E}(\mathcal{U})$ we denote the set of all $\mathcal{B}$-measurable *$\mathcal{U}$-excessive functions:* $u \in \mathcal{E}(\mathcal{U})$ if and only if u is a positive numerical $\mathcal{B}$-measurable function, $\alpha U_\alpha u \leq u$ for all $\alpha > 0$ and $\lim_{\alpha\to\infty} \alpha U_\alpha u(x) = u(x)$ for all $x \in E$. By Remark 3.5 in [25] it follows that the potential kernel U defined by

$$Uf = \int_0^\infty P_t f \;\mathrm{d}t$$

is proper, that is, there exists a bounded strictly positive $\mathcal{B}$-measurable function f such that Uf is finite.

If $\beta > 0$ we denote by $\mathcal{U}_\beta$ the sub-Markovian resolvent of kernels $(U_{\beta+\alpha})_{\alpha>0}$. Our first aim is to construct a $\mathcal{U}_\beta$-excessive function v such that: *the set $[v \leq \alpha]$ is relatively compact for all $\alpha > 0$* (and having some further useful properties). Such a function will be called **compact Lyapunov function** further on.

Consider an orthonormal basis $\{e_n : n \in \mathbb{N}\}$ of H in E' which separates the points of E. For each $n \in \mathbb{N}$ define $\widetilde{P}_n : E \longrightarrow H_n := span\{e_1, \ldots, e_n\} \subset E'$ by

$$\widetilde{P}_n z = \sum_{k=1}^{n} {}_{E'}\langle e_k, z\rangle_E \, e_k,\ z \in E, \tag{2.2}$$

and $P_n := \widetilde{P}_n \restriction_H$, so

$$P_n h = \sum_{k=1}^{n} \langle e_k, h\rangle\, e_k, \quad h \in H,$$

and $P_n \longrightarrow Id_H$ strongly as $n \to \infty$.

Lemma 2.1. *(i) Let $y, z \in E$. Then*

$${}_{E'}\langle \widetilde{P}_n z, y\rangle_E = \sum_{k=1}^{n} {}_{E'}\langle e_k, z\rangle_E \, {}_{E'}\langle e_k, y\rangle_E = {}_{E'}\langle \widetilde{P}_n y, z\rangle_E.$$

(ii) Let $y \in E'$, $z \in E$. Then

$${}_{E'}\langle \widetilde{P}_n y, z\rangle_E = \langle y, \widetilde{P}_n z\rangle = {}_{E'}\langle y, \widetilde{P}_n z\rangle_E.$$

(iii) For $n \geq m$ we have $\widetilde{P}_n\widetilde{P}_m = \widetilde{P}_m\widetilde{P}_n = \widetilde{P}_m$.

Proof. The proof of (i) is elementary and that of (ii) follows from (i) and (2.1). (iii) in turn is a consequence of (ii). $\square$

Proposition 2.2. *We have*

$$\lim_{n\to\infty} ||\widetilde{P}_n z - z|| = 0 \text{ in } \mu\text{-measure.}$$

Proof. Let ν be the cylinder measure on H corresponding to μ. Let $i : H \longrightarrow E$ denote the above embedding. Then again by the Dudley-Feldman-LeCam Theorem ([41], Theorem 4.1, in particular (iv)) for all $\varepsilon > 0$

(2.3) $\quad \lim_{m,n\to\infty} \nu(\{h \in H : ||P_n h - P_m h|| > \varepsilon\}) = 0.$

But $\mu(\{z \in E : ||\widetilde{P}_n z - \widetilde{P}_m z|| > \varepsilon\}) = \nu(\{h \in H : ||P_n h - P_m h|| > \varepsilon\})$. Hence by (2.3) there exists a $\mathcal{B}(E)/\mathcal{B}(E)$-measurable function $F : E \longrightarrow E$ such that

$$\lim_{n\to\infty} ||F - \widetilde{P_n}||_E = 0 \text{ in } \mu\text{-measure,}$$

and therefore μ-a.e. for a subsequence $(n_k)_{k\in\mathbb{N}}$. Thus for all $m \in \mathbb{N}$ and μ-a.e. $z \in E$

$${}_{E'}\langle e_m, z\rangle_E = \lim_{k\to\infty} {}_{E'}\langle e_m, \widetilde{P}_{n_k} z\rangle_E = {}_{E'}\langle e_m, F(z)\rangle_E \,,$$

and we conclude that $F(z) = z$ for μ-a.e. $z \in E$. $\square$

Passing to a subsequence if necessary, which we denote by Q_n, $\widetilde{Q}_n$, $n \in \mathbb{N}$, respectively, we may assume that

(2.4) $\quad ||Id_H - Q_n||_{\mathcal{L}(H,E)} \le \frac{1}{2^n}$

and

(2.5) $\quad \mu(\{z \in E : ||z - \widetilde{Q}_n z|| > \frac{1}{2^n}\}) \le \frac{1}{2^n},$

where we used the compactness of the embedding $H \subset E$ for (2.4) and Proposition 2.2 for (2.5).

Let $x \in E \setminus H$. We note that assuming the existence of such a point implies that $\dim H = \infty$ and a standard argument shows that $\mu(H) = 0$ (see [10]).

The following lemma is due to R. Carmona.

Lemma 2.3. *Let $x \in E \setminus H$. There exists an orthonormal basis $\{e_n^x : n \in \mathbb{N}\}$ of H such that $e_n^x \in E'$ for all $n \in \mathbb{N}$, $\{e_n^x : n \in \mathbb{N}\}$ separates the points of E and*

$${}_{E'}\langle e_n^x, x\rangle_E \ge 2^{\frac{n}{2}} \text{ for all } n.$$

Proof. This follows from the proof of Lemma 1 in [12]. Concerning the claim that $\{e_n^x : n \in \mathbb{N}\}$ separates the points of E, one just realizes that $\{x_n^* : n \in \mathbb{N}\}$ in the proof of [12], Proposition 1, separates the points of E and it follows by the construction there, that so does $\{e_n^x : n \in \mathbb{N}\}$. $\square$

Define the function $q_x : E \longrightarrow \overline{\mathbb{R}}_+$ by

(2.6) $$q_x(z) := \left[\sum_{n\ge 0} 2^n ||\widetilde{Q}_{n+1} z - \widetilde{Q}_n z||^2 + \left(\sum_{n\ge 1} 2^{-\frac{n}{2}} |{}_{E'}\langle e_n^x, z\rangle_E|\right)^2\right]^{\frac{1}{2}}, \quad z \in E,$$

where $\widetilde{Q}_0 := 0$ and e_n^x, $n \in \mathbb{N}$, is as defined in Lemma 2.3. Also $\widetilde{Q_n}$, $n \in \mathbb{N}$, is defined as above with this particular ONB. Let

$$E_x := \{z \in E : q_x(z) < \infty\}.$$

Note that by Lemma 2.3 we have

$$x \in E \setminus E_x.$$

Recall that if $l \in E'$ then for all $z \in E$ we have

$$\int_E l^2(y)\, p_t(z, dy) = t|l|^2 + l^2(z), \tag{2.7}$$

where $|l|$ denotes the H-norm of l ($\in E' \subset H' \equiv H$).

Modifying the arguments in [30] we can now prove:

Proposition 2.4. *Let $x \in E \setminus H$. The following assertions hold.*
(i) $\mu(E_x) = 1$.
(ii) For all $h \in H$ we have $q_x(h) \leq \sqrt{3}|h|$. In particular, $H \subset E_x$ continuously.
(iii) For all $z \in E$ we have

$$||z|| \leq \sqrt{2}\, q_x(z).$$

In particular, (E_x, q_x) is complete. Furthermore, (E_x, q_x) is compactly embedded into $(E, ||\cdot||)$.
(iv) (E_x, H, μ) is an abstract Wiener space. In particular, $q_x \in L^2(E, \mu)$.

Proof. (i) Let us set

$$g(z) := \sum_{n\geq 1} 2^{-\frac{n}{2}} |{}_{E'}\langle e_n^x, z\rangle_E|, \quad z \in E.$$

We show that

$$g \in L^2(E, \mu). \tag{2.8}$$

Indeed, by (2.7) and Minkowski's inequality we have

$$\int_E g^2(z)\mu(dz) \leq \left(\sum_{n\geq 1} 2^{-\frac{n}{2}} \sqrt{\int_E {}_{E'}\langle e_n^x, z\rangle_E^2\, \mu(dz)}\right)^2 = \left(\sum_{n\geq 1} 2^{-\frac{n}{2}} |e_n^x|\right)^2 < \infty.$$

Consequently g is finite μ-a.s. and assertion (i) is now a direct consequence of (2.5) and the Borel-Cantelli Lemma.

(ii) For all $h \in H$, by (2.4), we have

$$||\widetilde{Q}_{n+1}h - \widetilde{Q}_n h|| \leq 2^{-n}|Q_{n+1}h| \leq 2^{-n}|h|$$

and therefore

$$q_x(h)^2 \leq \sum_{n\geq 0} 2^{-n}|h|^2 + (\sum_{n=1}^{\infty} 2^{-n}) \sum_{n=1}^{\infty} \langle e_n^x, h\rangle^2,$$

which implies the assertions of (ii).

(iii) We have for all $n \in \mathbb{N}$ and $z \in E$

$$\sup_{m\geq n} ||\widetilde{Q}_m z - \widetilde{Q}_n z|| \leq \sup_{m\geq n} \sum_{k=n}^{m-1} ||\widetilde{Q}_{k+1}z - \widetilde{Q}_k z|| 2^{\frac{k}{2}} 2^{-\frac{k}{2}} \leq$$

$$(\sum_{k=n}^{\infty} 2^k ||\widetilde{Q}_{k+1}z - \widetilde{Q}_k z||^2)^{\frac{1}{2}} (\sum_{k=n}^{\infty} 2^{-k})^{\frac{1}{2}} \leq q_x(z)(\sum_{k=n}^{\infty} 2^{-k})^{\frac{1}{2}}.$$

In particular (restricting the above to $z \in E_x$), $(\widetilde{Q}_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{L}(E_x, E)$ with respect to the operator norm. Hence by completeness there exists $T \in \mathcal{L}(E_x, E)$ such that $\widetilde{Q}_n \to T$ as $n \to \infty$ in operator norm and T is compact since each $\widetilde{Q}_n$ is of finite rank. By Lemma 2.1 (ii) it follows that for each e_n^x

$$_{E'}\langle e_n^x, Tz\rangle_E = \lim_{m\to\infty} {_{E'}\langle e_n^x, \widetilde{Q}_m z\rangle_E} = \lim_{m\to\infty} {_{E'}\langle Q_m e_n^x, z\rangle_E} = {_{E'}\langle e_n^x, z\rangle_E}.$$

Therefore, for all $z \in E_x$, $Tz = z$ and thus $E_x \subset E$ compactly and furthermore

$$||z|| = ||Tz|| = \lim_m ||\widetilde{Q}_m z|| = \lim_m ||\widetilde{Q}_m z - \widetilde{Q}_0 z|| \leq \sup_{m\geq 0} ||\widetilde{Q}_m z - \widetilde{Q}_0 z|| \leq q_x(z)(\sum_{k=0}^{\infty} 2^{-k})^{\frac{1}{2}}.$$

The completeness of (E_x, q_x) then easily follows by Fatou's lemma.

(iv) *Claim 1.* Let $z \in E_x$. Then $\lim_{n\to\infty} q_x(z - \widetilde{Q}_n z) = 0$. In particular, $H \subset E_x$ densely.

Proof of Claim 1. For all $n \in \mathbb{N}$ by Lemma 2.1 (ii) and (iii)

$$q_x^2(z - \widetilde{Q}_n z) = \sum_{k=0}^{\infty} 2^k ||\widetilde{Q}_{k+1} z - \widetilde{Q}_{(k+1)\wedge n} z - \widetilde{Q}_k z + \widetilde{Q}_{k\wedge n} z||^2 + \sum_{k=1}^{\infty} 2^{-\frac{k}{2}} |_{E'}\langle (Id_H - Q_n) e_k^x, z\rangle_E|$$

$$= \sum_{k\geq n} 2^k ||\widetilde{Q}_{k+1} z - \widetilde{Q}_k z||^2 + \sum_{k\geq N_n} 2^{-\frac{k}{2}} |_{E'}\langle e_k^x, z\rangle_E|$$

for some $N_n \nearrow \infty$ when $n \to \infty$. Now the first part of the assertion follows, since $z \in E_x$. The second part is then a consequence thereof, since $\widetilde{Q}_n z \in H$ for all $n \in \mathbb{N}$.

Claim 2. Let $l \in E'_x$ and $l_n := l \circ \widetilde{Q}_n$, $n \in \mathbb{N}$. Then $l_n \in E'$ and $\lim_{n\to\infty} l_n(z) = l(z)$ for all $z \in E_x$.

Proof of Claim 2. Since each $\widetilde{Q}_n : E \to H$ is continuous and $H \subset E_x$ continuously, we have that $l_n \in E'$ for all $n \in \mathbb{N}$. The last part of the assertion follows from Claim 1.

We shall now see that Claim 1 and Claim 2 imply assertion (iv). Indeed, since $H \subset E_x$ continuously by (ii) and densely by Claim 1, it remains to show that μ is centered Gaussian as a measure on the Banach space (E_x, q_x), with Cameron-Martin space H, i.e. every $l \in E'_x$ has a mean zero normal distribution with variance $|l|^2$. (Recall that $E'_x \subset (H' \equiv) H \subset E_x$ continuously and densely.) So, let $l \in E'_x$ and let l_n, $n \in \mathbb{N}$, be as in Claim 2. Then l_n, $n \in \mathbb{N}$, are jointly Gaussian with mean zero and $l_n \longrightarrow l$ μ-a.e. as $n \longrightarrow \infty$ by (i), hence $l_n \longrightarrow l$ in $L^2(E, \mu)$ as $n \longrightarrow \infty$. Since then $l_n \longrightarrow h$ in H as $n \longrightarrow \infty$ for some $h \in H$, considering the Fourier transforms we see that l under μ has a mean zero normal distribution with variance $|h|^2$. But obviously $l_n \longrightarrow l$ weakly in H, hence $l = h$. The last part of assertion (iii) then follows by Fernique's Theorem (see e.g. [10, Theorem 2.8.5]). □

Corollary 2.5. (cf. [12], Proposition 1) *Let $x \in E \setminus H$. Then there exists a Borel linear subspace E_x of E such that $H \subset E_x$, $\mu(E_x) = 1$, and $x \notin E_x$. In particular, $\mu(H + x) = 0$*

Proof. The first part is just Proposition 2.4 (i). Since $(x+H) \cap E_x = \emptyset$, also the second part of the assertion follows. □

Lemma 2.6. *Let $L \in \mathcal{B}$ be a linear subspace of E such that $\mu(L) = 1$. Then for all $z \in E$ the set $L + z$ is invariant with respect to $(P_t)_{t\geq 0}$, i.e. $P_t(1_{L+z}) = 1_{L+z}$ for all $t > 0$. In particular, the measure $p_t(x, \cdot)$ is carried by $L + z$ for every $x \in L + z$.*

Proof. We have $\mu_t(L) = \mu_1(t^{-\frac{1}{2}} L) = \mu(L) = 1$. Let $z \in E$. If $x \in L + z$ then $p_t(x, L+z) = \mu_t(L + z - x) = \mu_t(L) = 1$. If $x \notin L + z$ then $(L + z - x) \cap L = \emptyset$ and thus $p_t(x, L + z) = \mu_t(L + z - x) \leq \mu_t(E \setminus L) = 0$. □

Theorem 2.7. *Let $x \in E \setminus H$. Define $v_0^x := U_1 q_x^2$ and for every $z \in E$, $v_z^x := v_0^x \circ T_z^{-1}$. Then v_z^x is a compact Lyapunov function such that $E_x + z = [v_z^x < \infty]$ and each $E_x + z$ is invariant with respect to $(P_t)_{t\geq 0}$.*

Proof. By Proposition 2.4 and Lemma 2.6 it follows that $E_x + z$ is absorbing and invariant with respect to $(P_t)_{t\geq 0}$.

We show that v_0^x is a compact Lyapunov function on E such that $E_x = [v_0^x < \infty]$. By Proposition 2.4 (iv) and by (2.8) we have $q_x \in L^2(E, \mu)$. Let $M := \int_E q_x^2(y)\mu(dy)$. Then for all $t > 0$, $z \in E$, $\int_E q_x^2(y)\mu_t(dy) = Mt$, and by the sublinearity of q_x

$$P_t(q_x^2)(z) = \int_E q_x^2(z+y)\mu_t(dy) \leq 2\int_E (q_x^2(z) + q_x^2(y))\mu_t(dy) \leq 2(q_x^2(z) + Mt).$$

We conclude that

$$v_0^x(z) = U_1(q_x^2)(z) = \int_0^\infty e^{-t}P_t(q_x^2)(z)dt \leq 2q_x^2(z) + 2M\int_0^\infty e^{-t}tdt.$$

Hence $E_x \subset [v_0^x < \infty]$.

We claim that v_0 has compact level sets in E. Obviously, q_x is lower semicontinuous on E. Therefore, because U_1 maps bounded continuous functions to bounded continuous functions, v_0^x is also lower semicontinuous on E. Then by Proposition 2.4 the sets $[q_x \leq \beta]$ are compact in E, hence it will be sufficient to prove that

$$v_0^x \geq q_x^2.$$

Let $f_n(z) := ||\widetilde{Q}_{n+1}z - \widetilde{Q}_n z||^2$ and $(l_k)_k \subset E'$, $||l_k|| = 1$, be such that for all $z \in E$

$$||z|| = \sup_k l_k(z).$$

The functionals $l_{k,n} := l_k \circ (\widetilde{Q}_{n+1} - \widetilde{Q}_n)$ belong to E' and using (2.7) we get for all $z \in E$, $t > 0$ and natural number n:

$$P_t f_n(z) = \int_E f_n(y)p_t(z, dy) = \int_E \sup_k l_{k,n}^2(y)p_t(z, dy) \geq$$
$$\sup_k \int_E l_{k,n}^2(y)p_t(z, dy) \geq \sup_k l_{k,n}^2(z) = f_n(z).$$

Hence $P_t f_n \geq f_n$. Recall that g denotes the second sum occurring in the definition of q_x. We have

$$P_t(g^2)(z) \geq (P_t g(z))^2 = \left(\sum_{n\geq 1} \frac{1}{2^{\frac{n}{2}}} \int_E |_{E'}\langle e_n^x, y\rangle_E|\, p_t(z, dy)\right)^2 \geq$$
$$\left(\sum_{n\geq 1} \frac{1}{2^{\frac{n}{2}}} |\int_E {}_{E'}\langle e_n^x, z+y\rangle_E \mu_t(dy)|\right)^2 = \left(\sum_{n\geq 1} \frac{1}{2^{\frac{n}{2}}} |_{E'}\langle e_n^x, z\rangle_E|\right)^2 = g^2(z).$$

Hence we also have $P_t(g^2) \geq g^2$. Since $q_x^2 = \sum_{n\geq 0} 2^n f_n + g^2$ we obtain

$$P_t(q_x^2) \geq q_x^2 \text{ for all } t > 0$$

and thus

$$v_0^x = \int_0^\infty e^{-t}P_t(q_x^2)\,dt \geq q_x^2 \int_0^\infty e^{-t}dt = q_x^2.$$

Since $P_t(f \circ T_z) = P_t f \circ T_z$ for all $f \in p\mathcal{B}$ and $z \in E$, we deduce that if $u \in \mathcal{E}(\mathcal{U}_\beta)$ then $u \circ T_z \in \mathcal{E}(\mathcal{U}_\beta)$. Consequently, by the first part of the proof, the function $v_z^x = v_0^x \circ T_{-z}$ is a compact Lyapunov function for every $z \in E$ and $E_x + z = [v_z^x < \infty]$. $\square$

Remark 2.8. *Fix $x \in E$ and for $y, z \in E$ define the equivalence relation $y \sim z$ if and only if $y - z \in E_x$, and let τ be defined as a set in E containing exactly one representative of each equivalence class. Note that since $\alpha x + E_x$, $\alpha \in \mathbb{R}$, are pairwise disjoint, τ is uncountable, and*

$$E = \dot{\bigcup_{z\in\tau}} (E_z + x).$$

Hence E is an uncountable union of disjoint Borel sets which are invariant for the Brownian motion.

As one consequence of Theorem 2.7, we can reprove Gross's famous result on the existence of the infinite dimensional Brownian motion (cf. [25]; see also [34] and [35] for constructions of diffusion processes on abstract Wiener spaces) and give some additional information, based on a general technique we developed in [9]; the proof will be sketched.

Recall that a *Ray cone* associated with $\mathcal{U}_\beta$, $\beta > 0$, is a cone $\mathcal{R}$ of bounded $\mathcal{U}_\beta$-excessive functions such that: $U_\alpha(\mathcal{R}) \subset \mathcal{R}$ for all $\alpha > 0$, $U_\beta\big((\mathcal{R} - \mathcal{R})_+\big) \subset \mathcal{R}$, $\sigma(\mathcal{R}) = \mathcal{B}$, it is min-stable, separable in the supremum norm and $1 \in \mathcal{R}$. The topology on E generated by a Ray cone is called *Ray topology.*

Theorem 2.9. (i) *There exists a diffusion process $\mathcal{W} = (\Omega, \mathcal{F}, \mathcal{F}_t, W_t, \theta_t, P^x)$ with state space E (*the Brownian motion on E*), having $(P_t)_{t\geq 0}$ as transition function.*

(ii) *The topology of E is a Ray one. For every finite measure λ on $(E, \mathcal{B})$ there exists a natural capacity associated with the Brownian motion on an abstract Wiener space, which in particular is tight. More precisely, the functional $M \longmapsto c_\lambda(M)$, $M \subset E$, defined by*

$$c_\lambda(M) := \inf\big\{\lambda(P_{T_G} p) : M \subset G \text{ open}\big\}$$

is a Choquet capacity on E, where P_{T_G} denotes the hitting kernel of the set G (see, e.g., Section 5 below for further details) and p is a bounded $\mathcal{U}$-excessive function of the form $p = Uf_0$ with $f_0 \in bp\mathcal{B}$ strictly positive; $bp\mathcal{B}$ denotes the bounded elements of $p\mathcal{B}$.

(iii) *Every $\mathcal{U}$-excessive function u of the form $u = Uf$, $f \in p\mathcal{B}$, is c_λ-quasi continuous, provided it is finite λ-a.e. More generally, every potential of a continuous additive functional (cf. [39] or [3]) is c_λ-quasi continuous if it is finite λ-a.e. In particular, every $\mathcal{U}$-excessive function is c_λ-quasi lower semicontinuous.*

Sketch of the proof. (i) We show first that $\mathcal{U}$ satisfies condition $(*)$ from [9], Corollary 5.4, namely for some $\beta > 0$ and every $z \in E$ we have:

$(*)$ if $\xi \in \mathsf{Exc}(\mathcal{U}_\beta)$ and $\xi \leq U_\beta(z, \cdot)$ then $\xi \in \mathsf{Pot}(\mathcal{U}_\beta)$;

we have denoted by $\mathsf{Exc}(\mathcal{U}_\beta)$ (resp. $\mathsf{Pot}(\mathcal{U}_\beta)$) the set of all $\mathcal{U}_\beta$-excessive measures (resp. of all potential $\mathcal{U}_\beta$-excessive measures). Let $x, z \in E$. Theorem 2.7 and assertion (ii) of Corollary 5.4 from [9] imply that the restriction of $\mathcal{U}$ to $E_x + z$ is the resolvent of a right process with state space $E_x + z$. Therefore it verifies in particular $(*)$ for $z \in E_x + z$; cf. assertion $(ii.1)$ of Corollary 5.4 from [9]. Hence $(*)$ holds for all $z \in E$ and so, by assertion (i) of Corollary 5.4 in [9], we conclude now that $(P_t)_{t\geq 0}$ is the transition function of a Borel right process with state space E.

The argument in [25], page 134, ensures (using a criterion of E. Nelson, [32]) that the process has continuous paths.

(ii) Since the semigroup $(P_t)_{t\geq 0}$ is strongly continuous on $\mathcal{C}_u(E)$, we deduce from Proposition 2.2 in [9] that the topology of E is a Ray one. By the above considerations and Proposition 4.1 in [9] we get the desired capacity and its tightness property.

Assertion (iii) is a consequence of Proposition 3.2.6 from [3], using essentially the property of the topology to be a Ray one, proved above.

Remark 2.10. (i) *The existence of the compact Lyapunov function v_z^x was crucial in our approach. To underline this, we present here the main arguments from the proof of Theorem 5.2 from [9], on which (the above crucially used) Corollary 5.4 is based: The resolvent $\mathcal{U}$ is always associated to a Borel right process, but on a bigger set E_1, the so called "entry space". However, if there exists a nest of Ray compact sets,* *then the set, $E_1 \setminus E$ is polar and consequently $\mathcal{U}$ is the resolvent of the process restricted to E (see, e.g., Lemma 3.5 in [5]). The level sets $[v_z^x \leq n]$, $n \in \mathbb{N}$, offer precisely the required nest of Ray compact subsets of $E_x + z$ and therefore the restriction of $\mathcal{U}$ to $E_x + z$ is the resolvent of a Borel right process with state space $E_x + z$, for all $x, z \in E$.*

(ii) In [12], page 41, R. Carmona asked whether there is a relevant notion of Newtonian capacity in the setting of the infinite dimensional Brownian motion. The second assertion of (ii) in Theorem 2.9 answers this question; see also Section 4 below. The quasi continuity properties stated by assertion (iii) of Theorem 2.9 are exactly analogous to those which hold in the classical case with respect to the Newtonian capacity.

3. Lévy processes on Hilbert space

The purpose of this section is to show that a slight modification of the construction in the previous section gives rise to explicit compact Lyapunov functions for Lévy processes in infinite dimensions provided they have finite (weak) second moments. For simplicity we restrict ourselves to the case of Hilbert state spaces. As in Section 2 we start with a separable real Hilbert space $(H, \langle , \rangle)$ with corresponding norm $|\cdot|$ and Borel σ-algebra $\mathcal{B}(H)$.

Let $\lambda : H \longrightarrow \mathbb{C}$ be a continuous negative definite function such that $\lambda(0) = 0$. Then by Bochner's Theorem there exists a finitely additive measure ν_t, $t > 0$, on $(H, \mathcal{B}(H))$ such that for its Fourier transform we have

$$\widehat{\nu_t}(\xi) := \int_H e^{i\langle \xi, h\rangle} \nu_t(dh) = e^{-t\lambda(\xi)}, \quad \xi \in H.$$

Let E be a Hilbert space such that $H \subset E$ continuously and densely, with inner product $\langle , \rangle_E$ and norm $\|\cdot\|$. Then, identifying H with its dual H' we have

$$E' \subset H \subset E \tag{3.1}$$

continuously and densely, and ${}_{E'}\langle \xi, h\rangle_E = \langle \xi, h\rangle$, for all $\xi \in E'$, $h \in H$.

In addition, we assume that the following assumption holds

(HS) $\quad H \subset E$ is Hilbert-Schmidt.

(Such a space E always exists.) Then, since $\widehat{\nu_t}$ is continuous on H, by the Bochner-Minlos Theorem (see, e.g., [Ya89]) each ν_t extends to a measure on $(E, \mathcal{B}(E))$, which we denote again by ν_t, such that

$$\widehat{\nu_t}(\xi) = \int_E e^{i\,{}_{E'}\langle \xi, z\rangle_E} \nu_t(dz) \text{ for all } \xi \in E'. \tag{3.2}$$

Clearly, λ restricted to E' is Sazonov continuous, i.e., continuous with respect to the topology generated by all Hilbert-Schmidt operators on E'. Hence by Lévy's continuity theorem on Hilbert spaces (see [35, Theorem IV.3.1 and Proposition VI.1.1]), $\nu_t \to \delta_0$ weakly as $t \to 0$. Here δ_0 denotes Dirac measure on $(E, \mathcal{B}(E))$ concentrated at $0 \in E$. Furthermore, by the Lévy-Khintchine Theorem on Hilbert space (see, e.g., Theorem VI.4.10 in [33])

$$\lambda(\xi) = -i\,{}_{E'}\langle \xi, b\rangle_E + \tfrac{1}{2}\,{}_{E'}\langle \xi, R\xi\rangle_E - \int_E \left(e^{i\,{}_{E'}\langle \xi, z\rangle_E} - 1 - \frac{i\,{}_{E'}\langle \xi, z\rangle_E}{1 + \|z\|^2} \right) M(dz), \xi \in E', \tag{3.3}$$

where $b \in E$, $R : E' \longrightarrow E$ is linear such that its composition $R \circ i_R$ with the Riesz isomorphism $i_E : E \to E'$ is a non-negative symmetric trace class operator on E, and M is a Lévy measure on $(E, \mathcal{B}(E))$, i.e. a positive measure on $(E, \mathcal{B}(E))$ such that

$$M(\{0\}) = 0, \quad \int_E (1 \wedge \|z\|^2) M(dz) < \infty.$$

Defining the probability measures

(3.4) $\qquad p_t(x,A) := \nu_t(A-x),\ t>0,\ x \in E, A \in \mathcal{B}(E),$

we obtain a semigroup of Markovian kernels $(P_t)_{t\geq 0}$ on $(E, \mathcal{B}(E))$ just like for the Gaussian case in the previous section. It has been proved in [21], that there exists a conservative Markov process $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, P^x)$ with transition function $(P_t)_{t\geq 0}$ which has càdlàg paths (see Theorem 5.1 in [21]). X is just an infinite dimensional version of a classical Lévy process. Obviously, each P_t maps $C_b(E)$ into $C_b(E)$, hence so does its associated resolvent $U_\beta = \int_0^\infty e^{-t\beta} P_t\, dt$, $\beta > 0$. In addition, $P_t f(z) \to f(z)$ as $t \to 0$, hence $\beta U_\beta f(z) \to f(z)$ as $\beta \to \infty$ for all $f \in C_b(E)$, $z \in E$. Hence X is also quasi-left continuous, and thus a standard process.

By (HS) there exists an orthonormal basis $\{e_n : \ n \in \mathbb{N}\}$ of H contained in E' having the following properties:
There exist $\lambda_n \in (0,\infty)$, $n \in \mathbb{N}$, such that

$$\sum_{n=1}^{\infty} \lambda_n < \infty$$

and $\overline{e}_n := \frac{e_n}{\sqrt{\lambda_n}}$, $n \in \mathbb{N}$, form an orthonormal basis of E. Furthermore,

(3.5) $\qquad \lambda_n {}_{E'}\langle e_n, z\rangle_E = \langle e_n, z\rangle_E \quad$ for all $n \in \mathbb{N}, z \in E$.

In particular, $\{e_n : n \in \mathbb{N}\}$ separates the points of E. The construction of $\{e_n : n \in \mathbb{N}\}$ is standard. We refer, e.g., to Proposition 3.5 from [1]. For $n \in \mathbb{N}$ define $\widetilde{P}_n : E \longrightarrow E'$ by

$$\widetilde{P}_n z := \sum_{k=1}^{n} {}_{E'}\langle e_k, z\rangle_E\, e_k,\ z \in E,$$

and $P_n := \widetilde{P}_n \upharpoonright_H$. Since by (3.5) for all $n \in \mathbb{N}$ and $z \in E$

$$\widetilde{P}_n z = \sum_{k=1}^{n} \langle \overline{e}_k, z\rangle_E\, \overline{e}_k,$$

we have

(3.6) $\qquad \lim_{n\to\infty} ||\widetilde{P}_n z - z|| = 0$ for all $z \in E$.

Remark 3.1. *Let $t > 0$ and consider the (non-Gaussian) triple (E, H, ν_t). As mentioned at the beginning of Section 2, in the Gaussian case the Dudley-Feldman-Le Cam Theorem says that $||\cdot||$ is a μ-measurable norm in the sense of Gross, which, however, is not known to be true for our not necessarily Gaussian measure ν_t. Recall that in [16] only a weaker notion of "μ-measurability" was shown and this notion was proved to be equivalent with Gross's μ-measurability only in the Gaussian case (see [16, Theorem 3]).* (3.6) *above, however, provides a suitable substitute for the special sequence $(P_n)_{n\in\mathbb{N}}$ of projections considered above, whose existence follows from assumption (HS). It is an interesting question whether this depends on this special sequence $(P_n)_{n\in\mathbb{N}}$ or, whether* (3.6) *is true at least ν_t-a.s. for any sequence of projections $(P_n)_{n\in\mathbb{N}}$ of the type considered in Section 2, i.e., whether Proposition 2.2 is true for ν_t or even more general measures. This question (of independent interest) is answered in the Appendix. The corresponding Proposition A.2 can be considered as a kind of generalization of the Dudley-Feldman-Le Cam Theorem to non-Gaussian measures under assumption (HS).*

Now we want to extend the construction of compact Lyapunov functions from Section 2 to this case. To this end we have to make the following further assumption (H) below, which as we shall see (cf. Example 3.2 below), is always fulfilled if λ is sufficiently regular.

(H)(i) There exists $C > 0$ such that for all $\xi \in E'$

$$\int_E {}_{E'}\langle \xi, z\rangle_E^2 \, \nu_t(dz) \le C(1+t^2)|\xi|^2, \quad t > 0.$$

(ii) $\nu_t(H) = 0$ for all $t > 0$.

Example 3.2. (i) If λ is sufficiently regular, by a straightforward computation one deduces from the representation in (3.3) that for every $\xi \in E'$

$$\begin{aligned}\int_E {}_{E'}\langle \xi, z\rangle_E^2 \nu_t(dz) &= -\frac{d^2}{d\varepsilon^2} e^{-t\lambda(\varepsilon\xi)}\big|_{\varepsilon=0} \\ &= t^2 \left({}_{E'}\langle \xi, b\rangle_E + \int_E {}_{E'}\langle \xi, z\rangle_E \frac{\|z\|^2}{1+\|z\|^2} \, M(dz)\right)^2 \\ &\quad + t\left({}_{E'}\langle \xi, R\xi\rangle_E + \int_E {}_{E'}\langle \xi, z\rangle_E^2 \, M(dz)\right)\end{aligned}$$

where we assume that ξ is such that $\int_E {}_{E'}\langle \xi, z\rangle_E^2 \, M(dz) < \infty$. Hence assuming that $b \in H$, $R(E') \subset H$ and $R : E' \longrightarrow H$ is continuous with respect to the norm $|\cdot|$ on E', we have that $(H)(i)$ holds provided $\int_E {}_{E'}\langle \xi, z\rangle_E^2 \, M(dz) < \infty$ for all ξ in E', because then by the uniform boundedness principle $\sup\{\int_E {}_{E'}\langle \xi, z\rangle_E^2 \, M(dz) : |\xi| \le 1\} < \infty$.

(ii) Assume that λ is such that in (3.3) $R = i_H \circ i_H^* \circ i_E^{-1}$, where i_H denotes the embedding $H \subset E$ and $i_H^* : E \longrightarrow H$ its adjoint. Fix $t > 0$. Then there exist probability measures μ_t, ν_t^0 on $(E, \mathcal{B}(E))$ and $b \in E$ such that

$$\nu_t = \delta_{tb} * \mu_t * \nu_t^0,$$

where μ_t is Gaussian such that (E, H, μ_t) is an abstract Wiener space, i.e., μ_t is exactly the Gaussian measure from Section 2. Therefore, if $\dim H = \infty$, by Corollary 2.5

$$\mu_t(H + x) = 0 \text{ for all } x \in E,$$

hence for all $t > 0$

$$\nu_t(H) = \int\int 1_H(tb + z + y) \, \mu_t(dy) \, \nu_t^0(dz) = 0.$$

So, $(H)(ii)$ holds in this case.

Let $\alpha_n \in (0, \infty)$, $n \in \mathbb{N}$, such that $\alpha_n \nearrow \infty$ as $n \to \infty$ and

$$\sum_{n=1}^\infty \alpha_n \lambda_n < \infty. \tag{3.7}$$

Let us fix $x \in E \setminus H$, and e_n^x, $n \in \mathbb{N}$, be as in Lemma 2.3. Define $q_x : E \to \overline{\mathbb{R}}_+$ by

$$q_x(z) := \left[\sum_{n=1}^\infty \alpha_n \lambda_n \, {}_{E'}\langle e_n, z\rangle_E^2 + \left(\sum_{n=1}^\infty 2^{-\frac{n}{2}} |{}_{E'}\langle e_n^x, z\rangle_E| \right)^2 \right]^{\frac{1}{2}}, \tag{3.8}$$

where $\{e_n : n \in \mathbb{N}\}$ is the special orthonormal basis of H from above. Then clearly q_x has compact level sets in E. Define again

$$E_x := \{z \in E : q_x(z) < \infty\}.$$

Then obviously $x \notin E_x$. Furthermore, we have an analog of Proposition 2.4.

Proposition 3.3. *Let $t > 0$. Then the following assertions hold.*
(i) $q_x \in L^2(E, \nu_t)$, in particular $\nu_t(E_x) = 1$ and $\nu_t(H + x) = 0$.
(ii) $H \subset E_x$ continuously.
(iii) For all $z \in E$ we have

$$||z|| \leq q_x(z).$$

In particular, (E_x, q_x) is complete. Furthermore, (E_x, q_x) is compactly embedded into $(E, ||\cdot||)$.

Proof. (i) By $(H)(i)$ we have

$$\int_E q_x^2(z)\nu_t(dz) \leq C(1+t^2)\sum_{n=1}^{\infty}\alpha_n\lambda_n + \left(\sum_{n=1}^{\infty} 2^{-\frac{n}{2}}\sqrt{\int_E {}_{E'}\langle e_n^x, z\rangle_E^2\,\nu_t(dz)}\right)^2 \leq \tag{3.9}$$

$$C(1+t^2)\left(\sum_{n=1}^{\infty}\alpha_n\lambda_n + (\sum_{n=1}^{\infty} 2^{-\frac{n}{2}})^2\right) < \infty.$$

(ii) This is obvious by (2.1) and (3.7).

(iii) By (3.5) we have for all $z \in E$

$$\sum_{n=1}^{\infty}\alpha_n\lambda_n\, {}_{E'}\langle e_n, z\rangle_E^2 = \sum_{n=1}^{\infty}\alpha_n\lambda_n^{-1}\langle e_n, z\rangle_E^2 = \sum_{n=1}^{\infty}\alpha_n\langle \overline{e}_n, z\rangle_E^2. \tag{3.10}$$

Hence since $\alpha_n \nearrow \infty$ as $n \to \infty$, we have

$$q_x^2(z) \geq \alpha_1||z||_E^2,$$

and, therefore, (E_x, q_x) is complete by Fatou's Lemma and (E_x, q_x) is compactly embedded into $(E, ||\cdot||)$. □

The following result is an analog to Theorem 2.7 for infinite dimensional Lévy processes.

Theorem 3.4. *Assume that (HS) and (H) hold. Let $v_0^x := U_1 q_x^2$ and for every $z \in E$, $v_z^x := v_0^x \circ T_z^{-1}$. Then v_z^x is a compact Lyapunov function such that $E_x + z = [v_z^x < \infty]$ and each $E_x + z$ is invariant with respect to $(P_t)_{t\geq 0}$. In particular, $E_x + z$ is left invariant by the infinite dimensional Lévy process $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, P^x)$. Furthermore, the restriction of X to $E_x + z$ is càdlàg in the trace topology.*

Proof. For $y \in E$, using the sublinearity of q_x, by (3.9) we obtain that for some constant $\widetilde{C} > 0$

$$P_t q_x^2(y) \leq 2q_x^2(y) + 2\int_E q_x^2(z)\nu_t(dz) \leq 2q_x^2(y) + 2\widetilde{C}(1+t^2).$$

Hence

$$v_0^x(y) = U_1 q_x^2(y) = \int_0^{\infty} e^{-t} P_t q_x^2(y)dt \leq 2q_x^2(y) + 2\widetilde{C}\int_0^{\infty}(1+t^2)e^{-t}dt. \tag{3.11}$$

On the other hand, since q_x is a norm, for all $y, z \in E$ by the triangle inequality we have that

$$q_x^2(y+z) \geq (q_x(y) - q_x(z))^2 \geq \frac{1}{2}q_x^2(y) - q_x^2(z).$$

Hence by (3.9)

$$P_t q_x^2(y) \geq \frac{1}{2}q_x^2(y) - \int_E q_x^2(z)\nu_t(dz) \geq \frac{1}{2}q_x^2(y) - \widetilde{C}(1+t^2)$$

and therefore

$$(3.12) \qquad v_0^x(y) = U_1 q_x^2(y) = \int_0^\infty e^{-t} P_t\, q_x^2(y)\, dt \geq \frac{1}{2} q_x^2(y) - \widetilde{C} \int_0^\infty e^{-t}(1+t^2)\, dt.$$

Finally, by (3.11) and (3.12) it follows that

$$E_x = [v_0^x < \infty].$$

v_0^x is a Lyapunov function for $(P_t)_{t\geq 0}$, which is compact by (3.12).
Since the measure ν_t is carried by E_x, it follows by the same argument as in the proof of Lemma 2.6 that each $E_x + z$ is an invariant set for $(P_t)_{t\geq 0}$.

To prove the next part of the assertions let us more generally consider any set $L \in \mathcal{B}(E)$ instead of $E_x + z$ just with the property that $P_t 1_L = 1_L$ for all $t > 0$. Then $1_L \in \mathcal{E}(\mathcal{U})$, hence it is finely continuous and therefore $[1_L = 0] = [1_L < \frac{1}{2}]$ is finely closed and finely open. Consequently, for all $x \in L$, $t > 0$,

$$P^x([1_L(X_t) > \frac{1}{2}]) = P^x([X_t \in L]) = E^x[1_L(X_t)] = P_t 1_L(x) = 1$$

and thus, since $t \longmapsto 1_L(X_t)$ is continuous because $1_L \in \mathcal{E}(\mathcal{U})$, we obtain

$$P^x(X_t \in L \quad \forall t \geq 0) = P^x(\bigcap_{t\geq 0} [1_L(X_t) > \frac{1}{2}]) = P^x(\bigcap_{t\in\mathbb{Q}^+} [1_L(X_t) > \frac{1}{2}]) = 1.$$

To prove the final assertion let X' be the restriction of X to L, $\mathcal{U}' = (U'_\alpha)_{\alpha>0}$ be its resolvent, and recall that $U_\alpha(C_b(E)) \subset C_b(E)$ for all $\alpha > 0$, where $\mathcal{U} = (U_\alpha)_{\alpha>0}$ is the resolvent of X. Consequently, U'_α maps $C_b(E)|_L$ into $C_b(E)|_L$ for all $\alpha > 0$. From the first part of the proof there exists on L a real valued compact Lyapunov function with respect to $\mathcal{U}'$. The claimed càdlàg property of X' follows now by Theorem 5.2 from [9]. □

Remark 3.5. (*i*) *The analog of Remark 2.8 holds, i.e. E is an uncountable disjoint union of Borel sets which are invariant for the Lévy process on E.*

(*ii*) *Subsection* 3.2 *from [8] presents an informal description of constructing compact Lyapunov functions for the infinite dimensional Lévy processes.*

Example 3.6. Let $(S, \mathcal{B}, \sigma)$ be a finite measure space and $H := L^2(S, \mathcal{B}, \sigma)$. Define $\lambda : H \to \mathbb{C}$ by

$$\lambda(h) := \int_S (1 - e^{ih}) d\sigma, \ h \in H.$$

Then $\lambda(0) = 0$, λ is negative definite and continuous on H. Choosing a Hilbert-Schmidt extension E of H as above there exist probability measures ν_t, $t > 0$, on $(E, \mathcal{B}(E))$ such that

$$\widehat{\nu_t}(\xi) = \int_E e^{i_{E'}\langle \xi, z\rangle_E}\, \nu_t(dz) = e^{-t\int_S (1-e^{i\xi})d\sigma}, \ \xi \in E'.$$

ν_t is just the Poisson measure with intensity t on E. Hence for all $\xi \in E'$

$$\int \langle \xi, z\rangle^2 \nu_t(dz) = t \int_S \xi^2 d\sigma + t^2 \left(\int_S \xi d\sigma \right)^2 \leq \sup(2\sigma(S)^2)(1+t^2)|\xi|_H^2.$$

In particular, (H)(i) holds.

Now take $S = (0,1)$, $\mathcal{B}$=Borel σ-algebra on $(0,1)$ and σ=Lebesgue measure ds. Let H_0^1 be the Sobolev space of order 1 in $L^2((0,1), ds)$ with Dirichlet boundary conditions. Let

$$E := (H_0^1)' (= H^{-1}).$$

Then we have the Hilbert-Schmidt embeddings

$$E' = H_0^1 \subset L^2((0,1), ds) := H \subset E.$$

So, each ν_t extends to a probability measure on $(E, \mathcal{B}(E))$. Since H_0^1 continuously embeds into the bounded continuous on $(0,1)$ equipped with the sup-norm, E contains all measure of finite total variation. It is, however, well-known (see, e.g., [28]) that each ν_t is supported by positive measures of type $\sum_{i=1}^{N} \varepsilon_{x_i}$, where ε_{x_i} is a Dirac measure with mass in $x_i \in [0,1]$, $1 \leq i \leq N_x \in \mathbb{N}$, and x_i are pairwise distinct. In particular, $\nu_t(H) = 0$ for all $t > 0$. So, also $(H)(ii)$ holds in this case.

Similar arguments can be used in the case where S is replaced by an open bounded set in $\mathbb{R}^d$. Then one has to take E as the dual of a Sobolev space of sufficiently (with respect to d) high order. Likewise one can treat the case $S = \mathbb{R}^d$, but then one has to use weighted Sobolev spaces.

4. Potential theory

4.1. Preliminaries

In this section we consider the Banach space E and the Hilbert space H as in Section 2. Let $(\nu_t)_{t\geq 0}$ be a convolution semigroup of probability measures on $(E, \mathcal{B})$ and $(P_t)_{t\geq 0}$ the associated family of Markovian kernels:

$$P_t f(x) = \int_E f(y) p_t(x, dy) = \int_E f(x+y)\nu_t(dy), \quad f \in p\mathcal{B},\ x \in E,$$

where $p_t(x, \cdot)$ is the probability measure on $(E, \mathcal{B})$ such that

$$p_t(x, A) := \nu_t(A - x) \text{ for all } A \in \mathcal{B}.$$

Let further $\mathcal{U} = (U_\alpha)_{\alpha>0}$ be the Markovian resolvent of kernels on $(E, \mathcal{B})$ associated with $(P_t)_{t\geq 0}$, i.e., $U_\alpha := \displaystyle\int_0^\infty e^{-\alpha t} P_t \, dt$, $\alpha > 0$, and set $U := \int_0^\infty P_t dt$. U is called potential kernel of $\mathcal{U}$. Clearly, for $\mathcal{U}_\beta := (U_{\beta+\alpha})_{\alpha>0}$ the corresponding potential kernel is U_β.

We consider an orthonormal basis $\{e_n : n \in \mathbb{N}^*\}$ of H formed by $e_n \in E'$, $n \in \mathbb{N}^*$. For each n define

$$\widetilde{P_n} : E \to H_n := span\{e_1, e_2, \cdots, e_n\} \subset E' \subset H$$

by

$$\widetilde{P_n} z := \sum_{k=1}^{n} {}_{E'}\langle e_k, z\rangle_E \; e_k, \quad z \in E.$$

Whenever necessary, H_n is identified with $\mathbb{R}^n$. For each $t > 0$ and $n \in \mathbb{N}^*$ we consider the probability measure $\nu_t^{\{n\}}$ on $\mathbb{R}^n$ defined by

$$\nu_t^{\{n\}} := \nu_t \circ \widetilde{P_n}^{-1}.$$

Analogously, we consider the kernel $P_t^{\{n\}}$ on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ induced by $\nu_t^{\{n\}}$:

$$P_t^{\{n\}}\varphi(x) = \int_{\mathbb{R}^n} \varphi(x+z)\nu_t^{\{n\}}(dz), \quad \varphi \in p\mathcal{B}(\mathbb{R}^n),\ x \in \mathbb{R}^n.$$

We obtain a Markovian semigroup of kernels $(P_t^{\{n\}})_{t\geq 0}$ on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ and let $\mathcal{U}^n = (U_\alpha^{\{n\}})_{\alpha>0}$ be the associated resolvent of kernels.

Let $n \in \mathbb{N}^*$, $t > 0$, and f be a positive cylinder function on E based on H_n, i.e., there exists a function $\varphi \in p\mathcal{B}(\mathbb{R}^n)$ such that $f = \varphi \circ \widetilde{P_n}$. Then for all $x \in E$ we have

$$P_t f(x) = \int_E f(x+y)\nu_t(dy) = \int_{\mathbb{R}^n} \varphi(\widetilde{P_n}x + z)\nu_t^{\{n\}}(dz) = P_t^{\{n\}}\varphi(\widetilde{P_n}x).$$

Consequently, for all $\alpha > 0$ we have

$$U_\alpha f = (U_\alpha^{\{n\}}\varphi) \circ \widetilde{P_n}. \tag{4.1}$$

Proposition 4.1. *Let $v \in p\mathcal{B}(\mathbb{R}^n)$ and $\beta > 0$. Then v is $\mathcal{U}_\beta^{\{n\}}$-excessive (resp. $\mathcal{U}_\beta^{\{n\}}$-supermedian, i.e., $\alpha U_{\beta+\alpha}^{\{n\}} v \leq v$ for all $\alpha > 0$) if and only if $v \circ \widetilde{P_n}$ is $\mathcal{U}_\beta$-excessive (resp. $v \circ \widetilde{P_n}$ is $\mathcal{U}_\beta$-supermedian).*

Proof. The assertion follows from the equality (4.1):

$$U_\alpha(v \circ \widetilde{P_n}) = (U_\alpha^{\{n\}} v) \circ \widetilde{P_n}.$$

□

We assume further that $(P_t)_{t\geq 0}$ (resp. $(P_t^{\{n\}})_{t\geq 0}$) is the transition function of a right process $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, P^x)$ with state space E (resp. $X^{\{n\}} = (\Omega^{\{n\}}, \mathcal{F}^{\{n\}}, \mathcal{F}_t^{\{n\}}, X_t^{\{n\}}, \theta_t^{\{n\}}, P^x)$ with state space $\mathbb{R}^n$), i.e.,

$$P_t f(x) = E_x(f \circ X_t), \quad x \in E,\ f \in p\mathcal{B}(E).$$

Remark 4.2. (*i*) *The Gaussian measures in an abstract Wiener space (presented in Section 2) and the convolution semigroup of a Lévy process on a Hilbert space (studied in Section 3) are examples for which the results from this section apply.*

(*ii*) *If $\nu_t = \mu_t$, a Gaussian measure with parameter t in an abstract Wiener space, then $\nu_t^{\{n\}}$ is the n-dimensional Gaussian measure with parameter t. Consequently, Proposition 4.1 has the following interpretation:* every superharmonic function in an n-dimensional Euclidean space is "superharmonic" with respect to the Gross-Laplace operator, i.e., it is an excessive function for the infinite dimensional Brownian motion, when it is canonically transported on the abstract Wiener space.

Corollary 4.3. *Suppose that $(\nu_t)_{t\geq 0}$ is the convolution semigroup of a Lévy process on an Hilbert space as in Section 3. If for some $n \in \mathbb{N}^*$ the process $X^{\{n\}}$ is transient then X is also transient. If X is not transient then $X^{\{n\}}$ is recurrent for all n.*

Proof. If the process $X^{\{n\}}$ is transient, or equivalently the potential kernel $U^{\{n\}} = \int_0^\infty P_t^{\{n\}} dt$ of $X^{\{n\}}$ is proper, then by (4.1) we get that the potential kernel U of X is also proper. The second assertion follows from the first one and by the transience–recurrence dichotomy which holds for Lévy processes (cf., e.g., Theorem 35.4 in [38]). □

4.2. Excessive measures and the energy functional

Let $Exc(\mathcal{U})$ be the set of all $\mathcal{U}$-*excessive measures* on E: $\xi \in Exc(\mathcal{U})$ if and only if it is a σ-finite measure on $(E, \mathcal{B})$ such that $\xi \circ \alpha U_\alpha \leq \xi$ for all $\alpha > 0$.

By $Pot(\mathcal{U})$ we denote the set of all *potential* $\mathcal{U}$-excessive measures, i.e. all σ-finite measures ξ of the form $\xi = \mu \circ U$, where μ is a measure on $(E, \mathcal{B})$. Clearly, by the resolvent equation we have that $Pot(\mathcal{U}) \subset Exc(\mathcal{U})$. Note that the **mass uniqueness principle** holds for the Gaussian measures in an abstract Wiener space and the convolution semigroup of a Lévy process on a Hilbert space:

If $\beta > 0$ and μ, ν are two positive measures on $(E, \mathcal{B})$ such that $\mu \circ U_\beta$, $\nu \circ U_\beta$ are σ-finite and $\mu \circ U_\beta = \nu \circ U_\beta$, then $\mu = \nu$.

The assertion follows from (10.40) in [39]; see Proposition 5 in [12] for the Gaussian case.

If $\beta > 0$ then the *energy functional* $L_\beta : Exc(\mathcal{U}_\beta) \times \mathcal{E}(\mathcal{U}_\beta) \longrightarrow \overline{\mathbb{R}}_+$ is defined by

$$L_\beta(\xi, v) := \sup\{\mu(v) : Pot(\mathcal{U}_\beta) \ni \mu \circ U_\beta \leq \xi\}.$$

The following result is a consequence of (4.1) and Proposition 4.1.

Corollary 4.4. *The following assertions hold.*

(i) If $\xi \in Exc(\mathcal{U}_\beta)$ *then* $\xi \circ \widetilde{P_n}^{-1} \in Exc(\mathcal{U}_\beta^{\{n\}})$ *provided it is a* σ*-finite measure on* $\mathbb{R}^n$*. If in addition* $\xi \in Pot(\mathcal{U}_\beta)$ *then* $\xi \circ \widetilde{P_n}^{-1} \in Pot(\mathcal{U}_\beta^{\{n\}})$*.*

(ii) Let $\xi \in Exc(\mathcal{U}_\beta)$ *such that* $\xi \circ \widetilde{P_n}^{-1}$ *is* σ*-finite,* $v \in \mathcal{E}(\mathcal{U}_\beta^{\{n\}})$*, and let* $L_\beta^{\{n\}}$ *be the energy functional with respect to* $\mathcal{U}_\beta^{\{n\}}$*. Then*

$$L_\beta^{\{n\}}(\xi \circ \widetilde{P_n}^{-1}, v) = L_\beta(\xi, v \circ \widetilde{P_n}).$$

4.3. Absence of a reference measure

Recall that a right Markov process satisfies the *hypothesis (L)* of P.A. Meyer provided that there exists a finite measure on $(E, \mathcal{B})$ with respect to which all the measures $U_\alpha(x, \cdot)$, $x \in E$, are absolutely continuous, where $\mathcal{U} = (U_\alpha)_{\alpha>0}$ is the resolvent family of the process. Such a measure is called *reference measure* for $\mathcal{U}$. Recall that the *fine topology* is the topology on E generated by $\mathcal{E}(\mathcal{U}_\beta)$.

Proposition 4.5. *The hypothesis* (L) *of P.A. Meyer does not hold for the Lévy processes on an infinite dimensional Hilbert space.*

Proof. The main argument in the proof is the same as in the Gaussian case (cf. Proposition 8 in [12]; see also [17] for another proof), namely, the existence of an uncountable family of mutually disjoint finely open sets. More precisely, assume that there exists a reference measure λ for $\mathcal{U}$. Note that λ charges every non-empty finely open set. Indeed, if $G \in \mathcal{B}$ is finely open and we suppose that $\lambda(G) = 0$ then $U_\beta(1_G) \equiv 0$, which contradicts the fact that $U_\beta(1_G)(x) > 0$ for all $x \in G$. (cf., e.g., Proposition 1.3.2 from [3]). Since $\dim H = \infty$, there exist $x \in E \setminus H$ and the space E_x defined in Section 3. By Theorem 3.4 the sets $E_x + z$, $z \in E$, are invariant with respect to $(P_t)_{t\geq 0}$. In particular, $E_x + z$ is finely open for every $z \in E$. Because $x \notin E_x$, it follows that $(E_x + \alpha x)_{\alpha \in \mathbb{R}_+}$ is an uncountable family of mutually disjoint sets and from the above considerations we get $\lambda(E_x + \alpha x) > 0$ for all $\alpha \in \mathbb{R}_+$, which leads to a contradiction. $\square$

4.4. Reduced functions and polar sets

If $M \subset E$ and $v \in \mathcal{E}(\mathcal{U}_\beta)$, then the *reduced function* (with respect to $\mathcal{U}_\beta$) of v on M is the function $R_\beta^M v$ defined by:

$$R_\beta^M v := \inf\{u \in \mathcal{E}(\mathcal{U}_\beta) : u \geq v \text{ on } M\}.$$

If M is a Souslin subset of E then the reduced function $R_\beta^M v$ is universally $\mathcal{B}$-measurable. The maps $v \longmapsto R_\beta^M v$ and $v \longmapsto \widehat{R_\beta^M} v$ extend to kernels on E and by Hunt's Theorem we have

$$R_\beta^M v(x) = E^x(e^{-\beta D_M} v \circ X_{D_M}; D_M < \infty),$$

$$\widehat{R_\beta^M} v(x) = E^x(e^{-\beta T_M} v \circ X_{T_M}; T_M < \infty),$$

where $D_M(\omega) := \inf\{t \geq 0 \mid X_t(\omega) \in M\}$, $T_M(\omega) := \inf\{t > 0 \mid X_t(\omega) \in M\}$, $\omega \in \Omega$, and for a $\mathcal{U}_\beta$-supermedian function u, $\widehat{u}$ denotes its $\mathcal{U}_\beta$-excessive regularization, $\widehat{u}(x) = \sup_{\alpha>0} \alpha U_{\beta+\alpha} u(x)$ for all $x \in E$.

The set $M \in \mathcal{B}$ is called *polar* (resp. ν*-polar*; where ν is a σ-finite measure on $(E, \mathcal{B})$) if $\widehat{R_\beta^M} 1 = 0$ (resp. $\widehat{R_\beta^M} 1 = 0$ ν-a.e.). By the above mentioned Hunt's Theorem a set $M \in \mathcal{B}$ will be polar (resp. ν-polar) if and only if $T_M = \infty$ P^x-a.s. for all $x \in E$ (resp. $T_M = \infty$ P^ν-a.e.).

Corollary 4.6. *If* $M \in \mathcal{B}$*,* $n \in \mathbb{N}^*$*, and* $v \in \mathcal{E}(\mathcal{U}_\beta^{\{n\}})$ *then*

$$R_\beta^M(v \circ \widetilde{P_n}) \leq ({}^{\{n\}}R_\beta^{\widetilde{P_n}(M)} v) \circ \widetilde{P_n},$$

where for a set $F \subset \mathbb{R}^n$ *we have denoted by* ${}^{\{n\}}R_\beta^F v$ *the reduced function (with respect to* $\mathcal{U}_\beta^{\{n\}}$*) of* v *on* F*. In particular, if* $\widetilde{P_n}(M)$ *is a polar subset of* $\mathbb{R}^n$ *then* M *is a polar subset of* E*.*

Proof. Let $u \in \mathcal{E}(\mathcal{U}_\beta^{\{n\}})$, $u \geq v$ on $\widetilde{P_n}(M)$. Then $u \circ \widetilde{P_n} \geq v \circ \widetilde{P_n}$ on M and by Proposition 4.1 we have $u \circ \widetilde{P_n} \in \mathcal{E}(\mathcal{U}_\beta)$. Consequently, we get that $u \circ \widetilde{P_n} \geq R_\beta^M(v \circ \widetilde{P_n})$ on E and thus for all $x \in E$ we have

$$ {}^{\{n\}}\!R_\beta^{\widetilde{P_n}(M)} v(\widetilde{P_n}x) = \inf\{u(\widetilde{P_n}x) \,:\, u \in \mathcal{E}(\mathcal{U}_\beta^{\{n\}}),\ u \geq v \text{ on } \widetilde{P_n}(M)\} \geq R_\beta^M(v \circ \widetilde{P_n})(x). $$

Assume now that $\widetilde{P_n}(M)$ is a polar subset of $\mathbb{R}^n$. Using (4.1) we get for all $x \in E$

$$ U_\alpha^{\{n\}}({}^{\{n\}}\!R_\beta^{\widetilde{P_n}(M)} v)(\widetilde{P_n}x) = U_\alpha({}^{\{n\}}\!R_\beta^{\widetilde{P_n}(M)} v \circ \widetilde{P_n})(x) \geq U_\alpha(R_\beta^M(v \circ \widetilde{P_n}))(x) $$

and therefore, taking $v = 1$ we have

$$ 0 = \widehat{{}^{\{n\}}\!R^{\widetilde{P_n}(M)}} 1(\widetilde{P_n}x) \geq \widehat{R_\beta^M} 1(x), $$

hence M is a polar subset of E. □

Proposition 4.7. *Assume that $(\nu_t)_{t \geq 0}$ is the convolution semigroup of a Lévy process on an Hilbert space as in Section 3 and suppose that for all $t > 0$ ν_t charges no proper closed linear subspace of E. Then the points of E are polar sets.*

Proof. By Corollary 4.6 it is sufficient to show that the points are polar for one finite dimensional projection $(\nu_t^{\{n\}})_{t \geq 0}$ of $(\nu_t)_{t \geq 0}$. By Theorem 4 in [11] it follows that the points are polar for a Lévy process in $\mathbb{R}^n$, $n \geq 2$, provided that the points are not finely open sets for all 1-dimensional projections. Suppose that $\{0\} \subset \mathbb{R}$ is a finely open set for $(\nu_t^{\{1\}})_{t \geq 0}$. Proposition 4.1 implies that $\widetilde{P_n}^{-1}(G)$ is a finely open subset of E for every $G \subset \mathbb{R}^n$ which is finely open with respect to $\mathcal{U}_\beta^{\{n\}}$. Consequently, the set $F := \widetilde{P_1}^{-1}(\{0\})$ will be a closed proper subspace of E which is finely open, hence $U_\beta(1_F) > 0$ on F. This contradicts the hypothesis on ν_t which implies $\nu_t(F) = 0$. Therefore $\{0\} \subset \mathbb{R}$ is not finely open and we conclude that the set $\{0\} \subseteq E$ is polar. □

Proposition 4.8. *Let $(\nu_t)_{t \geq 0}$ be either the Gaussian semigroup in an abstract Wiener space or the convolution semigroup of a Lévy process on an Hilbert space as in Section 3, satisfying hypotheses (HS) and (H). Then the "Cameron-Martin" space H is a polar set.*

Proof. Let $x \in E \setminus H$. By Corollary 2.5 and Lemma 2.6 (in the Gaussian case) and by Proposition 3.3 (in the Lévy process case) there exists $E_x \in \mathcal{B}$, a linear subspace of E, such that $H \subset E_x$, $\nu_t(E_x) = 1$ and $x \notin E_x$. Using again Lemma 2.6 (in the Gaussian case) and Theorem 3.4 in the Lévy process case) we get that E_x is invariant with respect to $(P_t)_{t \geq 0}$, hence $1_{E_x} \in \mathcal{E}(\mathcal{U}_\beta)$. Consequently, we get $R_\beta^H 1(x) \leq 1_{E_x}(x) = 0$ and thus $R_\beta^H 1 = 0$ on $E \setminus H$. Since $p_t(y, H) = 0$ for all $y \in E$ and $t > 0$, we get $U_\alpha(1_H) = 0$ and so

$$ \widehat{R_\beta^H} 1(x) = \lim_{\alpha \to \infty} \alpha U_\alpha(R_\beta^H 1)(x) = 0 \text{ for all } x \in E. $$

□

Remark 4.9. (i) *The result of Proposition 4.8 was proved in the Gaussian case in [12], Proposition 4. Note that the main probabilistic argument used in that proof (see Remark 7 in [12]) remains valid here: The property of $E_x + x$ to be invariant with respect to $(P_t)_{t \geq 0}$ implies that the process starting from x never leaves the set $E_x + x$. Since $H \subset E \setminus (E_x + x)$, it follows that the process starting from x never hits H.*

(ii) *If H is polar, then clearly all the points are polar sets. So, the conclusion of Proposition 4.8 is stronger than that of Proposition 4.7.*

4.5. Choquet capacities and quasi continuity

In this subsection we assume again that $(\nu_t)_{t\geq 0}$ is the convolution semigroup of a Lévy process on an Hilbert space as in Section 3; see Theorem 2.9 and [7] for the Gaussian case.

In Remark 2.10 (ii) we recalled Carmona's question on the existence of a relevant capacity for the infinite dimensional Brownian motion. We can present now the corresponding capacity for the Lévy processes. Note that in this case, since these processes are not necessarily transient, we have to consider the "β-level" capacity, $\beta > 0$.

Let $p := U_\beta f_0$, with $0 < f_0 \leq 1$, $f_0 \in p\mathcal{B}$, and let λ be a finite measure on $(E, \mathcal{B})$. Then the functional $M \longmapsto c_\lambda(M)$, $M \subset E$, defined by

$$c_\lambda(M) := \inf\bigl\{\lambda(R_\beta^G p) \,:\, M \subset G \text{ open}\bigr\}$$

is a Choquet capacity on E (see e.g. [3]).

We complete this subsection with an analog of Theorem 2.9 for Lévy processes.

Theorem 4.10. (i) *The topology of E is a Ray one and the capacity c_λ is tight, i.e., there exists an increasing sequence $(K_n)_n$ of compact sets such that* $\inf_n c_\lambda(E \setminus K_n) = 0$.

(ii) *Let $M \in \mathcal{B}$. Then*

$$c_\lambda(M) = \lambda(R_\beta^M p) = \sup\bigl\{\nu(p \cdot 1_M) \,:\, \nu \circ U_\beta \leq \lambda \circ U_\beta\bigr\}\ .$$

The set M will be λ-polar and λ-zero if and only if $c_\lambda(M) = 0$.

(iii) *Every $\mathcal{U}_\beta$-excessive function of the form $U_\beta f$, $f \in p\mathcal{B}$, is c_λ-quasi continuous, provided it is finite λ-a.e. More generally, every (β)-level potential of a continuous additive functional (cf. [39] or [3] in the transient case) is c_λ-quasi continuous if it is finite λ-a.e. In particular, every $\mathcal{U}_\beta$-excessive function is c_λ-quasi lower semicontinuous.*

Proof. (i) Let $C_{bl}(E)$ be the set of all bounded Lipschitz continuous functions on E. Using (3.4) one can check that $(U_\alpha)_{\alpha>0}$ induces a strongly continuous resolvent of contractions on $C_{bl}(E)$ and then one can construct an appropriate Ray cone (see Proposition 2.2 from [9] for details). The tightness property follows by [31] (see also [4]) since we already remarked in Section 3 that an infinite dimensional Lévy process has càdlàg paths.

Assertion (ii) is a consequence of Proposition 1.6.3 and Proposition 1.6.4 from [3], because by (i) the topology of E is a Ray one.

As in the proof of Theorem 2.9, assertion (iii) follows by Proposition 3.2.6 from [3], using again the property of the topology to be a Ray one. $\square$

4.6. Existence of bounded invariant functions

Remark 4.11. (i) *Suppose that $(\nu_t)_{t\geq 0}$ is the convolution semigroup of a Lévy process on an infinite dimensional Hilbert space as in Section 3 and $x \notin H$. By Theorem 3.4 the function 1_{E_x} is invariant with respect to $(P_t)_{t\geq 0}$, it is identically equal to one on H and zero at x. This shows that the answer given by R. Carmona (see Remark 6 in [12]) to a conjecture of V. Goodman (cf. [23], page 219) for the infinite dimensional Brownian motion, remains valid for the Lévy processes on an Hilbert space.*

(ii) *Unbounded invariant functions may be further constructed as in [12], the proof of Proposition 3, namely, consider the function f defined as*

$$f = \sum_{n=1}^{\infty} r_n 1_{\frac{1}{n}x + E_x},$$

where $(r_n)_n$ is a sequence of real numbers with $\lim_{n\to\infty} r_n = \infty$. Then clearly f is invariant and it is unbounded in every neighborhood of each point.

(iii) *Let $v \in bp\mathcal{B}$ be invariant with respect to $(P_t)_{t\geq 0}$, assume that the Lévy process has continuous paths (i.e., M in* (3.3) *is the zero measure), and consider an open set $V \subset E$ which is transient, i.e., we have a.s.*

$\sup\{t > 0 : X_t \in V\} < \infty$. *Then the function v is harmonic on V in the sense considered in the Gaussian case (see Section 5 below): v is finely continuous and there exists $\rho > 0$ such that*

$$v(x) = P_{T_{E \setminus B_r(x)}} v(x)$$

for all $r < \rho$ whenever $\bar{B}_r(x) \subset V$; $\bar{B}_r(x)$ denotes the closed ball or radius r centered at x. Indeed, since V is transient we get that a.s. $T_{E\setminus B_r(x)} < \infty$. The assertion follows from a straightforward consequence of Dynkin's formula (cf., e.g., (12.18) *in [39]): if v is a bounded $\mathcal{U}$-invariant function and T is a terminal time with $T < \infty$ a.s., then $v = P_T v$.*

4.7. Domination principle

Proposition 4.12. *Let μ, ν be two σ-finite measures on $(E, \mathcal{B})$, $G \in \mathcal{B}$ a finely open set such that $\mu(E\setminus G) = 0$. Assume that $\mu \circ U_\beta$, $\nu \circ U_\beta$ are σ-finite measures and $\mu \circ U_\beta \le \nu \circ U_\beta$ on G for some $\beta > 0$. Then $\mu \circ U_\beta \le \nu \circ U_\beta$ on E.*

Proof. For $\xi \in Exc(\mathcal{U}_\beta)$ and $M \in \mathcal{B}$ define ${}^*R^M \xi := \bigwedge\{\eta \in Exc(\mathcal{U}_\beta) : \eta \ge \xi \text{ on } M\}$, where $\bigwedge$ denotes the infimum in $Exc(\mathcal{U}_\beta)$. If $u \in \mathcal{E}(\mathcal{U}_\beta)$, then by Theorem 1.4.12 in [3]

$$L_\beta({}^*R^G\xi, u) = L_\beta(\xi, R^G_\beta u). \tag{4.2}$$

Since $R^G_\beta U_\beta f = U_\beta f$ on G, $f \in bp\mathcal{B}$, and using (4.2) we have

$$\mu \circ U_\beta(f) = \mu(R^G_\beta U_\beta f) = L_\beta({}^*R^G_\beta(\mu \circ U_\beta), U_\beta f) = {}^*R^G_\beta(\mu \circ U_\beta)(f) \le \nu \circ U_\beta(f).$$

□

Remark 4.13. (i) *Proposition 4.12 is a version of the domination principle stated for the Gaussian case in Proposition 6 from [12]. However, our statement is valid for general right processes, it holds also for $\beta = 0$ in the transient case (i.e., if the kernel $U = \int_0^\infty P_t \, dt$ is proper), and it is closer to the original assertion from [27]. The use of the "duality formula"* (4.2) *enabled us to avoid the assumption on the strong duality from [27].*

(ii) *In [19], Theorem 2.13 and Corollary 2.15, P.J. Fitzsimmons proved an analogous result, even for a general Borel set G, for right Markov processes. His proof uses more specialized techniques from probabilistic potential theory, while our proof is more analytic in nature (at least to our taste).*

4.8. Balayage principle

The next proposition points out that the balayage principle holds for the infinite dimensional Lévy processes; see Proposition 7 in [12] for the Gaussian case.

Proposition 4.14. *Let $\beta > 0$, $M \in \mathcal{B}$, ν a σ-finite measure on $(E, \mathcal{B})$, and consider the measure ν_M defined by*

$$\nu_M := \nu \circ \widehat{R^M_\beta}.$$

Then ν_M is carried by the fine closure of M, $\nu_M \circ U_\beta \le \nu \circ U_\beta$, and

$$\nu_M \circ U_\beta = \nu \circ U_\beta \text{ on } M.$$

Proof. By Proposition 1.7.11 from [3] the measure ν_M is carried by the fine closure of M. Since $\widehat{R^M_\beta} u \le u$ for every $u \in \mathcal{E}(\mathcal{U}_\beta)$, it follows that $\nu_M \circ U_\beta \le \nu \circ U_\beta$. If $\mathcal{B} \ni F \subset M$ then $\widehat{R^M_\beta} U_\beta(1_F) = U_\beta(1_F)$ and so $\nu_M \circ U_\beta(F) = \int_E \widehat{R^M_\beta} U_\beta(1_F) \, d\nu = \nu \circ U_\beta(F)$. □

Remark 4.15. *(i) The assertion of Proposition 4.14 holds also for $\beta = 0$ in the transient case.*

(ii) Recall that the fine closure of M is precisely the union of M with the set of all its regular points; a point $x \in E$ is called regular *for M if $P^x(T_M = 0) = 1$ (see, e.g., [39] or [3]).*

(iii) The measure ν_M is called the balayage *of ν on M. Proposition 4.14 offers an analytic construction of the balayage of a measure, and therefore, in the particular case of the Brownian motion on an abstract Wiener space, this gives the answer to a question of R. Carmona (cf. Remark 8 in [12]).*

Open problem: It is still an open question (formulated in [12], page 38) whether the axiom of polarity holds for the infinite dimensional Brownian motion. At this point we would like to thank an anonymous referee for reminding us of the connection between the axiom of polarity and the so called Choquet dichotomy property of the capacity (cf. [18], [26], and [20]). However, this interesting approach does not seem easy to be implemented here, but it will be the subject of future study.

5. Dirichlet problem and controlled convergence

Let $\mathcal{W} = (\Omega, \mathcal{F}, \mathcal{F}_t, W_t, \theta_t, P^x)$ be the path continuous Borel right process with state space E, having $(P_t)_{t\geq 0}$ as transition function, given by Theorem 2.9; recall that $\mathcal{W}$ is called the *Brownian motion* on E.

We already noted in Section 2 that the process $\mathcal{W}$ is *transient*, i.e., there exists a bounded strictly positive $\mathcal{B}$-measurable function f such that $Uf = \int_0^\infty P_t f dt$ is finite. Therefore in this case we may use the "0-level" excessive functions and potential theoretical tools. Let $M \in \mathcal{B}$ and P_{T_M} be the associated *hitting kernel*,

$$P_{T_M} f(x) = E^x(f \circ W_{T_M}\, ;\, T_M < \infty)\ , \quad x \in E,\ f \in p\mathcal{B},$$

where $T_M(\omega) := \inf\{t > 0 : W_t(\omega) \in M\}$, $\omega \in \Omega$. If $u \in \mathcal{E}(\mathcal{U})$, then $P_{T_M} u = \widehat{R^M} u$.

Remark 5.1. *If V is an open set and $x \in V$ then the hitting distribution $P_{T_{E\setminus V}}(\cdot, x)$ (i.e., the measure $f \longmapsto P_{T_{E\setminus V}} f(x)$) is concentrated on the boundary ∂V of V. Indeed, by (10.6) from [39] $W_{T_{E\setminus V}}$ belongs to $E \setminus V$ a.s. on $[T_{E\setminus V} < \infty]$. On the other hand we have $T_{E\setminus V} > 0$ P^x-a.s. and clearly $W_t(\omega) \in V$ provided that $t < T_{E\setminus V}(\omega)$. By the path continuity of $\mathcal{W}$ we conclude that $W_{T_{E\setminus V}} \in \partial V$ P^x-a.s.*

Following [22], a real-valued function f defined on an open set $V \subset E$ is called *harmonic* on V, if it is locally bounded, Borel measurable, finely continuous and there exists $\rho > 0$ such that

$$f(x) = P_{T_{E\setminus B_r(x)}} f(x)$$

for all $r < \rho$ whenever $\bar{B}_r(x) \subset V$; $\bar{B}_r(x)$ denotes the closed ball or radius r centered at x.

We shall denote by $H^V : p\mathcal{B}(\partial V) \longrightarrow p\mathcal{B}(V)$ the kernel defined by

$$H^V f := P_{T_{E\setminus V}} \bar{f}|_V\ , \quad f \in p\mathcal{B}(\partial V),$$

where $\bar{f}$ is a Borel measurable extension of f to E; $H^V f$ is well defined by Remark 5.1. Hence

$$H^V f(x) = E^x(f \circ W_{T_{E\setminus V}}\, ;\, T_{E\setminus V} < \infty)\ , \quad x \in V.$$

$H^V f$ is called the *stochastic solution of the Dirichlet problem* for f (cf. [22]).

Recall that (cf. [25]) an open set V is called *strongly regular* provided that for each $y \in \partial V$ there exists a cone K in E with vertex y such that $V \cap K = \emptyset$; a cone in E with vertex y is the closed convex hull of the set $\{y\} \cup \bar{B}_r(z)$ and $y \notin \bar{B}_r(z)$.

By Corollary 1.2 and Remark 3.4 in [25] it follows that:

(5.1) if V is strongly regular and $f \in \mathcal{C}(\partial V)$ is bounded, then $H^V f$ is harmonic on V and $\lim\limits_{V \ni x \to y} H^V f(x) = f(y)$ for all $y \in \partial V$.

(5.2) If $f \in \mathcal{B}(\partial V)$ is bounded, then $H^V f$ is harmonic on V (see also Remark 3.4 in [25] and page 453 in [22]). Consequently, for every $f \in p\mathcal{B}(\partial V)$, $H^V f$ is the sum of a series of positive harmonic functions on V.

Proof of (5.2)**.** We may assume that $f \geq 0$. By Theorem 3.6.4 in [3] it follows that $H^V f$ is an excessive function with respect to the process on V obtained by killing $\mathcal{W}$ at the boundary of V. Therefore $H^V f$ is finely continuous on V and $H^B H^V f \leq H^V f$ for all $B := B_r(x)$, $\overline{B}_r(x) \subset V$. Since $H^B H^V 1(x) = H^V 1(x)$ we conclude that $H^B H^V f(x) = H^V f(x)$, hence $H^V f$ is harmonic on V. If $f \in p\mathcal{B}(\partial V)$ then $H^V f = \sum_n H^V f_n$, where $(f_n)_n \subset bp\mathcal{B}(\partial V)$ is such that $f = \sum_n f_n$.

Controlled convergence

Let $f : \partial V \to \overline{\mathbb{R}}$, $V_0 \subset V$, and $h, k : V \to \overline{\mathbb{R}}$ be such that $k \geq 0$ and $h|_{V_0}, k|_{V_0}$ are real valued. We say that h *converges to* f *controlled by* k *on* V_0, if the following conditions hold: For every set $A \subset V_0$ and $y \in \partial V \cap \bar{A}$ we have

(c1) If $\limsup\limits_{A \ni x \to y} k(x) < \infty$, then $f(y) \in \mathbb{R}$ and $f(y) = \lim\limits_{A \ni x \to y} h(x)$.

(c2) If $\lim\limits_{A \ni x \to y} k(x) = \infty$, then $\lim\limits_{A \ni x \to y} \frac{h(x)}{1+k(x)} = 0$.

Remark 5.2. (i) *Following [13] and [14], the controlled convergence intends to offer a new method for setting and solving the Dirichlet problem for general open sets and general boundary data. In the above definition the function f should be interpreted as being the boundary data of the harmonic function h. The function k is called* control function*, it is controlling the convergence of the solution h to the given boundary data f. If $\alpha > 0$ then αk and any majorant of k are also control functions.*

(ii) *The case $k = 0$, $V_0 = V$, corresponds to the classical solution: $\lim\limits_{V \ni x \to y} h(x) = f(y)$ for any boundary point y.*

(iii) *In [13] it was considered only the case $V_0 = V$ for the controlled convergence. It turns out that for the application we present here (see Theorem 5.3 below) we need to take into account an exceptional set $V \setminus V_0$.*

(5.3) If h_n converges to f_n controlled by k on V_0 for each n and $(\alpha_n)_n \subset \mathbb{R}$, $\alpha_n \nearrow +\infty$, is such that $l := \sum_n \alpha_n |h_n| < \infty$, and $\sum_n h_n < \infty$ on V_0, then $\sum_n h_n$ converges to $\sum_n f_n$ controlled by $k + l$ on V_0 (cf. Proposition 1.7 in [14]).

Theorem 5.3. *Let $V \subset E$ be a strongly regular open set, λ be a finite measure on V, $\widehat{\lambda}$ be the measure on ∂V defined by $\widehat{\lambda} := \lambda \circ H^V$, and let $f \in \mathcal{L}^1_+(\widehat{\lambda})$. Then there exist $g \in p\mathcal{B}(\partial V)$ and a λ-zero set $M \subset V$ which is finely closed and λ-polar with respect to the Brownian motion on V (killed at the hitting time of ∂V), such that $k := H^V g \in \mathcal{L}^1_+(\lambda)$ and $H^V f$ converges to f controlled by k on $V \setminus M$.*

Proof. Let $\mathcal{M} = \{f \in \mathcal{L}^1_+(\widehat{\lambda}) : \exists\, g \in p\mathcal{B}(\partial V)$ such that $H^V f$ converges to f controlled by $k = H^V g \in \mathcal{L}^1(\lambda)$ on $[k < \infty]\}$. Note that by (5.1) the set of all positive bounded continuous functions on ∂V is a subset of $\mathcal{M}$ (taking $k = 0$). Note also that the λ-zero set $[k = \infty]$ is finely closed λ-polar because k is a 0-excessive function with respect to the Brownian motion on V. The proof will be complete if we show that $\mathcal{M}$ is a monotone class in $\mathcal{M}$.

Let $(f_n)_{n \geq 1} \subset \mathcal{M}$ be increasing to $f \in \mathcal{L}^1_+(\widehat{\lambda})$. We show that $f \in \mathcal{M}$. Let $h_n = H^V f_n$ and $h = H^V f$. Then $(h_n)_n$ increases to $h \in \mathcal{L}^1_+(\lambda)$ and by hypothesis h_n converges to f_n controlled by k_n on $[k_n < \infty]$ for all $n \geq 1$. We may assume $\lambda(k_n) = 1$ for all n. If

$$k_0 := \sum_n \frac{1}{2^n} k_n.$$

then h_n converges to f_n controlled by k_0 on $[k_0 < \infty]$ for all n. Let

$$l := \sum_{n \geq 1} n(h_{n+1} - h_n) = \sum_{n \geq 1} (h - h_n).$$

Since $\lambda(h_n) \nearrow \lambda(h) < \infty$, passing to a subsequence, we may assume that $\sum_n(\lambda(h) - \lambda(h_n)) < \infty$ and consequently $l = \mathcal{L}^1_+(\lambda)$, $l = H^V g$ with $g \in p\mathcal{B}(\partial V)$. By (5.3) it follows that h converges to f controlled by $k_0 + l$ on $[k_0 + l < \infty]$, hence $f \in \mathcal{M}$. $\square$

Remark 5.4. *(i) By* (5.2) *the "solution" $H^V f$ of the Dirichlet problem with boundary data $f \in \mathcal{L}^1_+(\widehat{\lambda})$ from Theorem 5.3 is a sum of a series of positive harmonic functions on V.*

(ii) The result from Theorem 5.3 holds in a more general setting, e.g., for a path continuous Borel right process, if (5.1) *holds.*

Appendix

Let $(H, \langle , \rangle)$ be a separable real Hilbert space with norm $|\cdot|$. Let $(E, \langle , \rangle_E)$ be another Hilbert space with norm $||\cdot||$ such that $H \subset E$ continuously and densely by a Hilbert-Schmidt map. Identifying H with its dual we have

$$E' \subset H \subset E$$

continuously and densely. Let μ be a finitely additive measure on H such that its Fourier transform $\widehat{\mu} : H \longrightarrow \mathbb{C}$, defined by

$$\widehat{\mu}(\xi) := \int_H e^{i\langle \xi, h\rangle} \mu(dh),\ \xi \in H,$$

is continuous on H and $\widehat{\mu}(0) = 1$. Then by the Bochner-Minlos Theorem (see, e.g., [41]) μ extends to a probability measure on $(E, \mathcal{B}(E))$ again denoted by μ.

Lemma A.1. *Assume that apart from the Hilbert-Schmidt embedding $E' \subset H \subset E$ we have another such embedding*

$$E_1^{'} \subset H \subset E_1,$$

i.e., $(E_1, \langle\cdot,\cdot\rangle_{E_1})$ is a Hilbert space with norm $||\cdot||_1 := \langle\cdot,\cdot\rangle_{E_1}^{\frac{1}{2}}$ such that $H \subset E_1$ continuously and densely by a Hilbert-Schmidt embedding. Suppose that there exists a linear subspace $K \subset E' \cap E_1'$ such that K separates the points both of E_1 and E (i.e., for each $x \in E \cup E_1$ such that $l(x) = 0$, for all $l \in K$, it follows that $x = 0$). Then there exists a Hilbert space $(E_0, \langle , \rangle_{E_0})$ such that $H \subset E_0$ continuously and densely by a Hilbert-Schmidt map and both $E_0 \subset E$ and $E_0 \subset E_1$ continuously. (Note that by Kuratowski's theorem $E_0 \in \mathcal{B}(E) \cap \mathcal{B}(E_1)$.)

Proof. Set $\langle h_1, h_2\rangle_{E_0} := \langle h_1, h_2\rangle_E + \langle h_1, h_2\rangle_{E_1}$, for all $h_1, h_2 \in H$ with corresponding norm $||\cdot||_{E_0} := \langle , \rangle_{E_0}^{\frac{1}{2}}$. Let E_0:=completion of H with respect to $||\cdot||_{E_0}$. Then clearly, $H \subset E_0$ continuously and densely by a Hilbert-Schmidt map.

Claim 1. $E_0 \subset E$ continuously.

To prove the claim we have to show that if $u_n \in H$, $n \in \mathbb{N}$, is an $||\cdot||_{E_0}$-Cauchy sequence and at the same time an $||\cdot||_E$-zero sequence, then it is also an $||\cdot||_{E_0}$-zero sequence. But u_n, $n \in \mathbb{N}$, is also an $||\cdot||_{E_1}$-Cauchy sequence, hence there exists $u \in E_1$ such that $\lim_{n\to\infty} ||u_n - u||_{E_1} = 0$. It suffices to show that $u = 0$. To this end let $k \in K$. Then ${}_{E_1'}\langle k, u\rangle_{E_1} = \lim_{n\to\infty} \langle k, u_n\rangle_H = \lim_{n\to\infty} {}_{E'}\langle k, u_n\rangle_E = 0$.

By assumption on K, it follows that $u = 0$, and Claim 1 follows.

Likewise one proves:

Claim 2. $E_0 \subset E_1$ continuously.

$\square$

Proposition A.2. *Let $\{e_n : n \in \mathbb{N}\} \subset E'$ be any orthonormal basis in H separating the points of E. For $n \in \mathbb{N}$ let $\widetilde{P}_n$ be defined by* (2.2) *and $P_n := \widetilde{P}_n \upharpoonright_H$. Let μ be a probability measure on E coming from a cylinder measure on H, i.e., μ is the image of a cylinder measure ν on H under the Hilbert-Schmidt embedding $H \subset E$, and the Fourier transform $\widehat{\nu}$ of ν is continuous on H. Then*

$$\lim_{n\to\infty} ||z - P_n z|| = 0 \text{ for } \mu\text{-a.e. } z \in E.$$

Proof. Let $\lambda_n \in (0,\infty)$ such that $\sum_{n=1}^{\infty} \lambda_n < \infty$ and for $h_1, h_2 \in H$ define

$$\langle h_1, h_2 \rangle_{E_1} := \sum_{n=1}^{\infty} \lambda_n \langle e_n, h_1 \rangle_H \ \langle e_n, h_2 \rangle_H$$

with corresponding norm $||\cdot||_{E_1} := \langle \cdot, \cdot \rangle_{E_1}^{\frac{1}{2}}$. Let E_1 be the completion of H with respect to $||\cdot||_{E_1}$. Then $H \subset E_1$ continuously and densely by a Hilbert-Schmidt map and hence we have the Hilbert-Schmidt embeddings $E_1' \subset H \subset E_1$. Furthermore $\overline{e}_n := \lambda_n^{-\frac{1}{2}} e_n$, $n \in \mathbb{N}$, form an orthonormal basis of E_1 and for all $n \in \mathbb{N}$, $h \in H$

$$(A.1) \qquad \lambda_n \langle e_n, h \rangle_H = \langle e_n, h \rangle_{E_1} ,$$

hence

$$(A.2) \qquad {}_{E'}\langle e_n, h \rangle_E \, e_n = \langle e_n, h \rangle_H \, e_n = \langle \overline{e}_n, h \rangle_{E_1} \overline{e}_n.$$

Furthermore, for all $n \in \mathbb{N}$ by $(A.1)$

$$h \longmapsto \langle e_n, h \rangle_H$$

extends to a linear functional in E_1' again denoted by e_n. Hence $(A.1)$ implies by continuity that

$$(A.3) \qquad \lambda_n \, {}_{E_1'}\langle e_n, z \rangle_{E_1} = \langle e_n, z \rangle_{E_1} \ \text{ for all } n \in \mathbb{N}, z \in E_1,$$

in particular (since $\{\lambda_n^{-\frac{1}{2}} e_n \ : n \in \mathbb{N}\}$ forms an ONB of E_1), $\{e_n : n \in \mathbb{N}\}$ also separates the points of E_1. Hence we can apply Lemma A.1 with $K := linspan\{e_n \ : \ n \in \mathbb{N}\} \subset E'$ (since K also separates the points of E) to get the Hilbert space $E_0 \subset E \cap E_1$. Then the assertion of the proposition follows from the following two claims.

Claim 1. $\mu(E_0) = 1$.

Claim 2. $\lim_{n\to\infty} ||P_n z - z||_E = 1$, for all $z \in E_0$.

To prove Claim 1 we note that the cylinder measure on H generating μ, mapped under the Hilbert-Schmidt embedding $H \subset E_0$ on E_0, extends to a σ-additive probability measure on $(E_0, \mathcal{B}(E_0))$. Clearly, because $H \subset E_0 \subset E$ continuously, we have $\mathcal{B}(E) \cap E_0 = \mathcal{B}(E_0)$, $E_0 \in \mathcal{B}(E)$, by Kuratowski's theorem. Hence it follows that this image measure coincides with μ, because the Fourier transforms coincide on E' and $E' \subset E_0' \subset H \subset E_0 \subset E$ continuously and densely. So, $\mu(E_0) = 1$.

Now let us prove Claim 2. By $(A.2)$ for all $h \in H$

$$(A.4) \qquad P_n h = \textstyle\sum_{k=1}^{n} \langle \overline{e}_k, h \rangle_{E_1} \overline{e}_k.$$

Let $z \in E_0$. Then there exists $h_l \in H$, $l \in \mathbb{N}$, such that $\lim_{l\to\infty} ||z - h_l||_{E_0} = 0$. Hence, since both $E_0 \subset E$ and $E_0 \subset E_1$ continuously,

$$\lim_{l\to\infty} ||z - h_l||_E = 0 = \lim_{l\to\infty} ||z - h_l||_{E_1}.$$

Therefore, by $(A.4)$

$$(A.5) \qquad \widetilde{P}_n z = \lim_{l\to\infty} P_n h_l = \sum_{k=1}^{n} \langle \overline{e}_k, z \rangle_{E_1} \overline{e}_k, \quad \text{for all } n \in \mathbb{N}.$$

But the right hand side of $(A.5)$ converges to z, since $\{\overline{e}_n : n \in \mathbb{N}\}$ is an orthonormal basis of $(E_1, \langle \cdot, \cdot \rangle_{E_1})$, and Claim 2 is proved. $\square$

Acknowledgement. Financial support by the Deutsche Forschungsgemeinschaft, through Project GZ:436 RUM 113/23/0-1 and the CRC 701 is gratefully acknowledged. The first named author acknowledges support from the Romanian Ministry of Education, Research, Youth and Sport (PN II Program, CNCSIS code ID 209/2007). The third named author would also like to thank the I. Newton Institute in Cambridge for a very pleasant stay during which part of this work was done.

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