Existence and approximation of Hunt processes associated with generalized Dirichlet forms

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Abstract. We show that any strictly quasi-regular generalized Dirichlet form that satisfies the mild structural condition **D3** is associated to a Hunt process, and that the associated Hunt process can be approximated by a sequence of multivariate Poisson processes. This also gives a new proof for the existence of a Hunt process associated to a strictly quasi-regular generalized Dirichlet form that satisfies **SD3** and extends all previous results.

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1 Introduction

The theory of Dirichlet forms is a powerful tool in the study of Markov processes, since it combines different areas of mathematics such as probability, potential, and semigroup theory, as well as the theory of partial differential equations (see monographs [2], [3] and references therein). For instance, the classical energy calculus in combination with the potential theory of additive functionals allows to obtain an extension of Itô's formula for only weakly differentiable functions, i.e. functions in the domain of the form. This celebrated extension of Itô's formula where the martingale and the possibly unbounded variation drift part are controlled through the energy is well-known as *Fukushima's decomposition* of additive functionals (see e.g. [2, Theorem 5.2.2]).

Until recent years the applicability of Dirichlet form theory was limited to symmetric Markov processes (see [2]) or, more generally, to Markov processes satisfying a sector condition (cf. [3]). Within the theory of *generalized* (non-sectorial) *Dirichlet forms* (see [10], and [11] for the associated stochastic calculus including the extension of [2, Theorem 5.2.2] to the non-sectorial case) this limitation has been overcome since in this generalized framework only the existence of a positive measure μ is required for which the transition semigroup of the Markov process operates as a C_0-semigroup of contractions on $L^2(\mu)$. In particular, as no sector condition has to be verified, the theory of generalized Dirichlet forms is robust and well-suited for far-reaching perturbation methods.

In this note we are concerned with several questions related to probabilistic and analytic potential theory of generalized Dirichlet forms. A particular aim is to find general analytic conditions for non-sectorial Dirichlet forms that ensure the existence of an associated Hunt process. The question whether the associated process is a Hunt process is crucial for localizing purposes (see e.g. introduction of [12]).

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A fundamental consequence of [12, Theorem 2] and [8, Theorem 3.2(ii)] is that any *transient* Hunt process $\mathbb{M}$ on a metrizable and separable state space is strictly properly associated in the resolvent sense with a strictly quasi-regular generalized Dirichlet form. This is relevant because we can then apply all the fine results from the potential theory of generalized Dirichlet forms w.r.t. the strict capacity (see [12] for some strict potential theory, and [8, Remark 3.3(iv)] which applies also to strictly quasi-regular generalized Dirichlet forms and Hunt processes). Moreover, if the state space is only slightly less general, namely (for tightness reasons) a metrizable Lusin space, then by [5, Theorem 2.1] the Hunt process can be approximated by multivariate Poisson processes and the approximation works for all P_x, i.e. for all x in the state space. The canonical approximation of the Hunt process by Markov chains is useful as it provides an additional tool for its analysis and for the analysis of the underlying generalized Dirichlet form. Note that the just mentioned line of arguments is not valid for sectorial Dirichlet forms, which underlines a strength of generalized Dirichlet form theory. In fact, for a given arbitrary Hunt process we first do not know how to check whether it is associated to a sectorial Dirichlet form, and second this is clearly is not true in general.
Here, we establish the “quasi converse” of the above with nearly no restriction on the state space. We consider two problems, which, due to the method, are in fact solved simultaneously. The first problem is to establish the existence of an associated Hunt process to a strictly quasi-regular generalized Dirichlet form on a general state space, and the second is the approximation of this Hunt process in a canonical way through Markov chains. The second problem goes back to an original idea of S. Ethier and T. Kurtz. In fact, it is shown in [1, Chapter 4.2] that for nice state spaces such as locally compact and separable state spaces and nice transition semigroups like Feller ones, the Yoshida approximation via multivariate Poisson processes converges for all starting points to a Markov process with the given semigroup. This was generalized in [6] where it is shown that the Yoshida approximation of the generator together with some tightness arguments that result from the strict quasi-regularity leads to the approximation via multivariate Poisson processes of any Hunt process that is associated with a strictly quasi-regular sectorial Dirichlet form. This also led to a new proof for the existence of an associated Hunt process. However, the price for the increased generality is that the approximation only works for strictly quasi-every starting point x of the state space. We have to pay the same price, and even more we have to assume the additional structural condition **D3** that is however trivially satisfied for any sectorial Dirichlet form (see Proposition 2.3). Nonetheless, since the class of generalized Dirichlet forms is much larger than the class of sectorial Dirichlet forms our results represent a considerable generalization. In particular time-dependent processes and processes corresponding to large perturbations of symmetric forms, are covered.
Besides the canonical approximation scheme through Markov chains we want to emphazise that our main result Theorem 4.11 is an improvement of [12, Theorem 3]. We were able to relax condition **SD3** to the weaker one **D3** of W. Stannat which was used to show the existence of an associated standard process (see [10, IV. Theorem 2.2]). Therefore our general analytic conditions for non-sectorial Dirichlet forms to ensure the existence of an associated Hunt process are just **D3** and the strict quasi-regularity. The state space is only assumed to be a Hausdorff topological space such that its Borel σ-algebra is generated by the set of continuous functions on the state space. It should also be noted that our conditions, when applied to the symmetric case on a locally compact and separable metric state space, are weaker than the regularity assumed in [2, Chapter 7] (see [12, Corollary 1], [3, Corollary 2.16]).
Finally let us very briefly summarize the main contents of this paper. Section 2 contains the preliminaries and the fundamental technical results. In particular our way of defining the strict

capacity (cf. Definition 2.4 and [12, Definition 1]) is simpler than in [3, V.2], [6, Section 2], but still equivalent (see Remark 2.5). The strict capacity is defined w.r.t. some reference function φ, but it turns out to be independent of that function (see Remark 2.6). One of the most important technical results is the construction of the modified functions $\widehat{e}_n$ in Lemma 2.13 in comparison to the functions e_n of [6, Lemma 3.5]. This makes the difference and allows to handle the non-sectorial case (see also Remark 2.14 for some related explanations). Lemma 2.13 allows to get the crucial tightness result of Lemma 4.4. Note that we also correct an inaccuracy that appears in the proof of the statements corresponding to Lemma 4.4 in both papers [6] and [5] (see Remark 4.3). Having developed the fundamental results of potential theory in Section 2, the results of Sections 3 and 4 follow similarly to the line of arguments in [6].

2 Strict quasi-regularity, strict capacity, and the construction of R_α

In this section we recall basic notions and consequences related to strict notions and generalized Dirichlet forms. For notations that might not be defined here we refer to [12].

2.1 Quasi-regular generalized Dirichlet forms and the conditions D3 and SD3

Let E be a Hausdorff space such that its Borel σ-algebra $\mathcal{B}(E)$ is generated by the set $\mathcal{C}(E)$ of all continuous functions on E. Let m be a σ-finite measure on $(E, \mathcal{B}(E))$ such that $\mathcal{H} = L^2(E, m)$ is a separable (real) Hilbert space with inner product $(\cdot,\cdot)_{\mathcal{H}}$. Let $(\mathcal{A}, \mathcal{V})$ be a real valued coercive closed form on $\mathcal{H}$, i.e. $\mathcal{V}$ is a dense linear subspace of $\mathcal{H}$, $\mathcal{A} : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ is a positive definite bilinear map, $\mathcal{V}$ is a Hilbert space with inner product $\widetilde{\mathcal{A}}_1(u, v) := \frac{1}{2}(\mathcal{A}(u, v) + \mathcal{A}(v, u)) + (u, v)_{\mathcal{H}}$, and $\mathcal{A}$ satisfies the *weak sector condition*

$$|\mathcal{A}_1(u, v)| \leq K \mathcal{A}_1(u, u)^{1/2} \mathcal{A}_1(v, v)^{1/2},$$

$u, v \in \mathcal{V}$, with *sector constant* K. Identifying $\mathcal{H}$ with its dual $\mathcal{H}'$ we have that $\mathcal{V} \subset \mathcal{H} \subset \mathcal{V}'$ densely and continuously. Since $\mathcal{V}$ is a dense linear subspace of $\mathcal{H}$, $(\mathcal{V}, \widetilde{\mathcal{A}}_1(\cdot,\cdot)^{1/2})$ is again a separable real Hilbert space. Let $\|\cdot\|_{\mathcal{V}}$ be the corresponding norm.

For a linear operator Λ defined on a linear subspace D of one of the Hilbert spaces $\mathcal{V}$, $\mathcal{H}$ or $\mathcal{V}'$ we will use from now on the notation (Λ, D). Let $(\Lambda, D(\Lambda, \mathcal{H}))$ be a linear operator on $\mathcal{H}$ satisfying the following conditions:

D1 (i) $(\Lambda, D(\Lambda, \mathcal{H}))$ generates a C_0-semigroup of contractions $(U_t)_{t\geq 0}$.
(ii) $(U_t)_{t\geq 0}$ can be restricted to a C_0-semigroup on $\mathcal{V}$.

Denote by $(\Lambda, D(\Lambda, \mathcal{V}))$ the generator corresponding to the restricted semigroup. From [10, Lemma I. 2.3, p.12] we know that $\Lambda : D(\Lambda, \mathcal{H}) \cap \mathcal{V} \to \mathcal{V}'$ is closable as an operator from $\mathcal{V}$ into $\mathcal{V}'$ if $(\Lambda, D(\Lambda, \mathcal{H}))$ satisfies (i) and (ii). Let $(\Lambda, \mathcal{F})$ denote its closure, then $\mathcal{F}$ is a real Hilbert space with corresponding norm

$$\|u\|_{\mathcal{F}}^2 := \|u\|_{\mathcal{V}}^2 + \|\Lambda u\|_{\mathcal{V}'}^2.$$

By [10, Lemma I. 2.4, p.13] the adjoint semigroup $(\widehat{U}_t)_{t\geq 0}$ of $(U_t)_{t\geq 0}$ can be extended to a C_0-semigroup on $\mathcal{V}'$ and the corresponding generator $(\widehat{\Lambda}, D(\widehat{\Lambda}, \mathcal{V}'))$ is the dual operator of $(\Lambda, D(\Lambda, \mathcal{V}))$. Let $\widehat{\mathcal{F}} := D(\widehat{\Lambda}, \mathcal{V}') \cap \mathcal{V}$. Then $\widehat{\mathcal{F}}$ is a real Hilbert space with corresponding norm

$$\|u\|_{\widehat{\mathcal{F}}}^2 := \|u\|_{\mathcal{V}}^2 + \|\widehat{\Lambda} u\|_{\mathcal{V}'}^2.$$

Let the form $\mathcal{E}$ be given by

$$\mathcal{E}(u,v) := \begin{cases} \mathcal{A}(u,v) - \langle \Lambda u, v\rangle & \text{for } u \in \mathcal{F},\ v \in \mathcal{V} \\ \mathcal{A}(u,v) - \langle \widehat{\Lambda} v, u\rangle & \text{for } u \in \mathcal{V},\ v \in \widehat{\mathcal{F}} \end{cases}$$

and $\mathcal{E}_\alpha(u,v) := \mathcal{E}(u,v) + \alpha(u,v)_{\mathcal{H}}$ for $\alpha > 0$. $\mathcal{E}$ is called the *bilinear form associated with* $(\mathcal{A}, \mathcal{V})$ and $(\Lambda, D(\Lambda, \mathcal{H}))$ (that satisfies **D1**). Here, $\langle \cdot, \cdot \rangle$ denotes the dualization between $\mathcal{V}'$ and $\mathcal{V}$. Note that $\langle \cdot, \cdot \rangle$ restricted to $\mathcal{H} \times \mathcal{V}$ coincides with $(\cdot,\cdot)_{\mathcal{H}}$ and that $\mathcal{E}$ is well-defined.

It follows, from [10, Proposition I. 3.4, p.19], that for all $\alpha > 0$ there exist continuous, linear bijections $W_\alpha : \mathcal{V}' \to \mathcal{F}$ and $\widehat{W}_\alpha : \mathcal{V}' \to \widehat{\mathcal{F}}$ such that $\mathcal{E}_\alpha(W_\alpha f, u) = \langle f, u\rangle = \mathcal{E}_\alpha(u, \widehat{W}_\alpha f)$, $\forall f \in \mathcal{V}'$, $u \in \mathcal{V}$. Furthermore $(W_\alpha)_{\alpha>0}$ and $(\widehat{W}_\alpha)_{\alpha>0}$ satisfy the resolvent equation

$$W_\alpha - W_\beta = (\beta - \alpha) W_\alpha W_\beta \ \ \textit{and} \ \ \widehat{W}_\alpha - \widehat{W}_\beta = (\beta-\alpha)\widehat{W}_\alpha \widehat{W}_\beta.$$

Restricting W_α to $\mathcal{H}$ we get a strongly continuous contraction resolvent $(G_\alpha)_{\alpha>0}$ on $\mathcal{H}$ satisfying $\lim_{\alpha\to\infty} \alpha G_\alpha f = f$ in $\mathcal{V}$ for all $f \in \mathcal{V}$. The resolvent $(G_\alpha)_{\alpha>0}$ is called the (L^2-)*resolvent associated with* $\mathcal{E}$. Let $(\widehat{G}_\alpha)_{\alpha>0}$ be the adjoint of $(G_\alpha)_{\alpha>0}$ in $\mathcal{H}$. $(\widehat{G}_\alpha)_{\alpha>0}$ is called the (L^2-)*coresolvent associated with* $\mathcal{E}$.

A bounded linear operator $G : \mathcal{H} \to \mathcal{H}$ is called *sub-Markovian* if $0 \leq Gf \leq 1$ for all $f \in \mathcal{H}$ with $0 \leq f \leq 1$. Consider the following condition

D2 αG_α is submarkovian for any $\alpha > 0$.

The bilinear form $\mathcal{E}$ associated with $(\mathcal{A}, \mathcal{V})$ and $(\Lambda, D(\Lambda, \mathcal{H}))$ is called a **generalized Dirichlet form** if the resolvent associated with $\mathcal{E}$ is submarkovian, i.e. if **D2** holds.

The class of generalized Dirichlet forms contains in particular symmetric and coercive Dirichlet forms (choose $\Lambda = 0$) (cf. [2], [3], and [4]) and also time dependent Dirichlet forms (choose $\Lambda = \frac{\partial}{\partial t}$) as in [7], [13]. But generalized Dirichlet forms contain also the following important example:

Example 2.1 *Let $\mathcal{A} = 0$ on $\mathcal{V} := \mathcal{H}$ and $(\Lambda, D(\Lambda))$ be a Dirichlet operator (cf. e.g. [3]) generating a C_0-semigroup of contractions on $\mathcal{H}$. In this case $\mathcal{F} = D(\Lambda)$, $\widehat{\mathcal{F}} = D(\widehat{\Lambda})$ and the corresponding bilinear form $\mathcal{E}(u,v) = (-\Lambda u, v)_{\mathcal{H}}$ if $u \in D(\Lambda)$, $v \in \mathcal{H}$, and $\mathcal{E}(u,v) = (u, -\widehat{\Lambda} v)_{\mathcal{H}}$ if $u \in \mathcal{H}$, $v \in D(\widehat{\Lambda})$, is a generalized Dirichlet form.*

An element u of $\mathcal{H}$ is called 1-excessive (resp. 1-coexcessive) if $\beta G_{\beta+1} u \leq u$ (resp. $\beta \widehat{G}_{\beta+1} u \leq u$) for all $\beta \geq 0$. Let $\mathcal{P}$ (resp. $\widehat{\mathcal{P}}$) denote the 1-excessive (resp. 1-coexcessive) elements of $\mathcal{V}$. Let $\mathcal{C},\mathcal{D} \subset \mathcal{H}$. We define $\mathcal{D}_{\mathcal{C}} := \{u \in \mathcal{D} \mid \exists f \in \mathcal{C},\ u \leq f\}$. For an arbitrary Borel set $B \in \mathcal{B}(E)$ and an element $u \in \mathcal{H}$ such that $\{v \in \mathcal{H} \mid v \geq u \cdot 1_B\} \cap \mathcal{F} \neq \emptyset$ (resp. $\hat{u} \in \widehat{\mathcal{P}}_{\widehat{\mathcal{F}}}$) let $u_B := e_{u \cdot 1_B}$

be the 1-*reduced function* (resp. $\hat{u}_B := \hat{e}_{\hat{u}\cdot 1_B}$ be the 1-*coreduced function*) of $u \cdot 1_B$ (resp. $\hat{u} \cdot 1_B$) as defined in [10, Definition III. 1.8, p.65]. Here we use the notation 1_B for the characteristic function of B. If $B = E$ we rather use the notation e_u instead of u_E.
The following is useful (cf. [10, Proposition III. 1.6 and proof of Proposition III. 1.7] for some intermediate steps in the sequel): for $\hat{u} \in \widehat{\mathcal{P}}_{\widehat{\mathcal{F}}}$, $B \in \mathcal{B}(E)$ there exists $\hat{u}_B^\alpha \in \widehat{\mathcal{F}} \cap \widehat{\mathcal{P}}$ such that $\hat{u}_B^\alpha \le \hat{u}_B^\beta$, $0 < \alpha \le \beta$, $\hat{u}_B^\alpha \to \hat{u}_B$, $\alpha \to \infty$, strongly in $\mathcal{H}$ and weakly in $\mathcal{V}$, and

$$\mathcal{E}_1(v, \hat{u}_B^\alpha) = \alpha((\hat{u}_B^\alpha - \hat{u} \cdot 1_B)^-, v)_{\mathcal{H}} \text{ for any } v \in \mathcal{V}$$

where f^- denotes the negative part of f. Similarly for $u \in \mathcal{P}_{\mathcal{F}}$ there exists $u_B^\alpha \in \mathcal{F} \cap \mathcal{P}$ such that $u_B^\alpha \le u_B^\beta$, $0 < \alpha \le \beta$, $u_B^\alpha \to u_B$, $\alpha \to \infty$, strongly in $\mathcal{H}$ and weakly in $\mathcal{V}$ and

$$\mathcal{E}_1(u_B^\alpha, v) = \alpha((u_B^\alpha - u \cdot 1_B)^-, v)_{\mathcal{H}} \text{ for any } v \in \mathcal{V}.$$

Since by [10, Proposition III. 1.7(ii)] $\hat{u}_B \cdot 1_B = \hat{u} \cdot 1_B$, $u_B \cdot 1_B = u \cdot 1_B$ we then have

$$\lim_{\alpha\to\infty} \mathcal{E}_1(u_B^\alpha, \hat{u}) = \lim_{\alpha\to\infty} \mathcal{E}_1(u, \hat{u}_B^\alpha). \tag{1}$$

Both sides of (1) exist as increasing and bounded limits. In particular (by our definition of reduced functions for not necessarily open sets) [10, Lemma III. 2.9] extends to general Borel sets, i.e. $\mathcal{E}_1(f_B, \hat{f}) = \mathcal{E}_1(f, \hat{f}_B)$ for any $(f, \hat{f}) \in \mathcal{F} \cap \mathcal{P} \times \widehat{\mathcal{F}} \cap \widehat{\mathcal{P}}$, $B \in \mathcal{B}(E)$.
Let $A \subset E$. We set $A^c := E \setminus A$, i.e. the complement of A in E. An increasing sequence of closed subsets $(F_k)_{k\ge1}$ is called an $\mathcal{E}$-nest, if for every function $u \in \mathcal{P} \cap \mathcal{F}$ it follows that $u_{F_k^c} \to 0$ in $\mathcal{H}$ and weakly in $\mathcal{V}$. Equivalently, a sequence of closed subsets $(F_k)_{k\ge1}$ is an $\mathcal{E}$-nest, if

$$\mathrm{Cap}_\varphi(F_k^c) = \int_E (G_1\varphi)_{F_k^c} \varphi \, dm \longrightarrow 0 \quad \text{as } k \to \infty$$

for some (and hence all) $\varphi \in L^2(E, m)$, $\varphi > 0$ (see [10, IV. Proposition 2.10]).
A subset $N \subset E$ is $\mathcal{E}$-exceptional if there is an $\mathcal{E}$-nest $(F_k)_{k\ge1}$ such that $N \subset \cap_{k\ge1} E \setminus F_k$. A property of points in E holds $\mathcal{E}$-quasi-everywhere ($\mathcal{E}$-q.e.) if the property holds outside some $\mathcal{E}$-exceptional set. A function f defined up to some $\mathcal{E}$-exceptional set $N \subset E$ is called $\mathcal{E}$-quasi-continuous ($\mathcal{E}$-q.c.)(resp. $\mathcal{E}$-quasi-lower-semicontinuous ($\mathcal{E}$-q.l.s.c.)) if there exists an $\mathcal{E}$-nest $(F_k)_{k\in\mathbb{N}}$, such that $\bigcup_{k\ge1} F_k \subset E \setminus N$ and $f_{|F_k}$ is continuous (resp. lower-semicontinuous) for all k. We denote by $\widetilde{f}$ an $\mathcal{E}$-q.c. m-version of f, conversely f denotes the m-class represented by an $\mathcal{E}$-q.c. m-version $\widetilde{f}$ of f.
The quasi-regularity of a generalized Dirichlet form is defined similarly to [3] as follows

Definition 2.2 *The generalized Dirichlet form $\mathcal{E}$ is called quasi-regular if:*

(i) There exists an $\mathcal{E}$-nest $(E_k)_{k\ge1}$ consisting of compact sets.

(ii) There exists a dense subset of $\mathcal{F}$ whose elements have $\mathcal{E}$-q.c. m-versions.

(iii) There exist $u_n \in \mathcal{F}$, $n \in \mathbb{N}$, having $\mathcal{E}$-q.c. m-versions $\widetilde{u}_n$, $n \in \mathbb{N}$, and an $\mathcal{E}$-exceptional set $N \subset E$ such that $\{\widetilde{u}_n \mid n \in \mathbb{N}\}$ separates the points of $E \setminus N$.

In contrast to the theory of sectorial Dirichlet forms in [3] and [4] it is not known whether quasi-regularity alone implies the existence of an associated standard process in case of a generalized Dirichlet form. Therefore the following condition

D3 There exists a linear subspace $\mathcal{Y} \subset \mathcal{H} \cap L^\infty(E,m)$ such that $\mathcal{Y} \cap \mathcal{F}$ is dense in $\mathcal{F}$, $\lim_{\alpha\to\infty} e_{\alpha G_\alpha u - u} = 0$ in $\mathcal{H}$ for all $u \in \mathcal{Y}$ and for the closure $\overline{\mathcal{Y}}$ of $\mathcal{Y}$ in $L^\infty(E;m)$ it follows that $u \wedge \alpha \in \overline{\mathcal{Y}}$ for $u \in \overline{\mathcal{Y}}$ and $\alpha \geq 0$.

is introduced in [10, IV. 2, D3] and it is shown in [10, IV. Theorem 2.2] that a quasi-regular generalized Dirichlet form satisfying **D3** is associated with an m-tight special standard process. By an algebra of functions we understand a *linear* space that is closed under multiplication. The following condition

SD3 There exists an algebra of functions $\mathcal{G} \subset \mathcal{H} \cap L^\infty(E,m)$ such that $\mathcal{G} \cap \mathcal{F}$ is dense in $\mathcal{F}$ and $\lim_{\alpha\to\infty} e_{\alpha G_\alpha u - u}$ in $\mathcal{H}$ for every $u \in \mathcal{G}$.

was introduced in [12].

Proposition 2.3 *It holds:*

(i) **SD3** *implies* **D3**.

(ii) **SD3** *holds for any (sectorial semi-)Dirichlet form.*

Proof (i) The proof is the same as in [10, IV. Proposition 2.1].
(ii) If $(\mathcal{E}, D(\mathcal{E}))$ be a (sectorial semi-)Dirichlet form we can choose $\mathcal{G} = D(\mathcal{E}) \cap L^\infty(E,m)$.

□

Up to the end of this subsection we fix a quasi-regular generalized Dirichlet form and an m-tight special standard process $\mathbb{M} = (\Omega, (\mathcal{F}_t)_{t\geq 0}, (Y_t)_{t\geq 0}, (P_z)_{z\in E\cup\{\Delta\}})$ with lifetime ζ such that the resolvent $V_\alpha f$ of $\mathbb{M}$ is an $\mathcal{E}$-q.c. m-version of $G_\alpha f$ for all $\alpha > 0$, $f \in \mathcal{H} \cap L^\infty(E,m)$. $\mathbb{M}$ is then said to be properly associated in the resolvent sense with $\mathcal{E}$.
Note that $u \in \mathcal{P}$ not necessarily admits an $\mathcal{E}$-q.c. m-version. However, under the quasi-regularity we know that there exists an $\mathcal{E}$-q.c. m-version $\widetilde{\alpha G_{\alpha+1} u}$ of $\alpha G_{\alpha+1} u$. Since $\alpha G_{\alpha+1} u$ increases m-a.s. if α increases we know from [10, Corollary III. 3.3] that $\widetilde{\alpha G_{\alpha+1} u}$ increases $\mathcal{E}$-q.e. if α increases. Hence we may define an $\mathcal{E}$-q.l.s.c. m-version of u by

$$\overline{u} := \sup_{\alpha>0} \widetilde{\alpha G_{\alpha+1} u}$$

$\overline{u}$ is called an $\mathcal{E}$-q.l.s.c. regularization of $u \in \mathcal{P}$. Since any two $\mathcal{E}$-q.l.s.c. regularizations of $u \in \mathcal{P}$ coincide $\mathcal{E}$-q.e. it follows that any $\mathcal{E}$-q.l.s.c. regularization of $u \in \mathcal{P}$ coincides $\mathcal{E}$-q.e. with the "canonical" regularization $\overline{u} = \sup_{\alpha>0} \alpha V_{\alpha+1} u$.

2.2 Strict capacities and strictly quasi-regular generalized Dirichlet forms

In this subsection we introduce strict notions corresponding to a generalized Dirichlet form, as well as simple consequences of our definitions and we revise results from [12]. In case of Dirichlet forms similar (slightly more general) strict notions were introduced in [3, Chapter V] (cf. Remark 2.5 and the definition of the strict capacity in [3, Chapter V.2]).

Let $(\mathcal{E}, \mathcal{F})$ be a generalized Dirichlet form as defined in the previous subsection. We fix $\varphi \in L^1(E,m) \cap \mathcal{B}$ with $0 < \varphi(z) \leq 1$ for every $z \in E$.

Definition 2.4 *For $U \subset E$, U open, set*

$$Cap_{1,\widehat{G}_1\varphi}(U) := \int_E e_U \varphi \, dm$$

where $e_U := \lim_{k\to\infty}(G_1(k\varphi)\wedge 1)_U$ exists as a bounded and increasing limit in $L^\infty(E,m)$. If $A \subset E$ arbitrary then $Cap_{1,\widehat{G}_1\varphi}(A) := \inf\{Cap_{1,\widehat{G}_1\varphi}(U) | \, U \supset A, U \, open\}$.

Remark 2.5 *From [12, Theorem 1] we know that $Cap_{1,\widehat{G}_1\varphi}$ is a finite Choquet capacity. Moreover, we have $Cap_\varphi \leq Cap_{1,\widehat{G}_1\varphi}$ (see [12, Remark 1]). Thus the strict notions defined below are indeed "stricter". Moreover, if $(\mathcal{E},\mathcal{F})$ is quasi-regular then*

$$Cap_{1,\widehat{G}_1\varphi}(U) = \mu_{(\widehat{G}_1\varphi)_U}(E) \tag{2}$$

where $\mu_{(\widehat{G}_1\varphi)_U}$ is the smooth measure associated to the 1-coexcessive function $(\widehat{G}_1\varphi)_U$ (see [11, Theorem 2.3]). If $(\mathcal{E},\mathcal{F})$ is a quasi-regular Dirichlet form and $Cap_{1,\widehat{G}_1\varphi}$ is the strict capacity as defined in [3, Chapter V.2], then (2) clearly also holds. Therefore, if $(\mathcal{E},\mathcal{F})$ is a Dirichlet form and at least quasi-regular, then our notion of strict capacity coincides with the strict capacity defined in [3, Chapter V.2].

Adjoining the cemetery Δ to E we let $E_\Delta := E \cup \{\Delta\}$ and $\mathcal{B}(E_\Delta) = \mathcal{B}(E) \cup \{B \cup \{\Delta\} | B \in \mathcal{B}(E)\}$. We will consider different topologies on E_Δ. If E is a locally compact separable metric space but not compact, E_Δ will be the one point compactification of E, i.e. the open sets of E_Δ are the open sets of E together with the sets of the form $E_\Delta \setminus K$, $K \subset E$, K compact in E. Otherwise we adjoin the cemetery Δ to E as an isolated point. We extend m to $(E_\Delta, \mathcal{B}(E_\Delta))$ by setting $m(\{\Delta\}) = 0$. Any real-valued function u on E is extended to E_Δ by setting $u(\Delta) = 0$.
Given an increasing sequence $(F_k)_{k\in\mathbb{N}}$ of closed subsets of E, we define

$$C(\{F_k\}) = \{f : A \to \mathbb{R} \mid \bigcup_{k\geq 1} F_k \subset A \subset E, \, f_{|F_k} \text{ is continuous } \forall k\},$$

$$C_\infty(\{F_k\}) = \{f : A \to \mathbb{R} \mid \bigcup_{k\geq 1} F_k \subset A \subset E, \, f_{|F_k \cup \{\Delta\}} \text{ is continuous } \forall k\},$$

$$C_{l,\infty}(\{F_k\}) = \{f : A \to \mathbb{R} \mid \bigcup_{k\geq 1} F_k \subset A \subset E, \, f_{|F_k \cup \{\Delta\}} \text{ is lower semicontinuous } \forall k\}.$$

Obviously $C_\infty(\{F_k\}) \subset C(\{F_k\})$ and $C_\infty(\{F_k\})$ only differs from $C(\{F_k\})$ if E_Δ is the one point compactification since otherwise Δ is an isolated point of E.

A subset $N \subset E$ is called strictly $\mathcal{E}$-exceptional if $\text{Cap}_{1,\widehat{G}_1\varphi}(N) = 0$. An increasing sequence $(F_k)_{k\in\mathbb{N}}$ of closed subsets of E is called a strict $\mathcal{E}$-nest if $\text{Cap}_{1,\widehat{G}_1\varphi}(F_k^c) \downarrow 0$ as $k \to \infty$. A property of points in E holds strictly $\mathcal{E}$-quasi-everywhere (s.$\mathcal{E}$-q.e.) if the property holds outside some strictly $\mathcal{E}$-exceptional set. A function f defined up to some strictly $\mathcal{E}$-exceptional set $N \subset E$ is called strictly $\mathcal{E}$-quasi-continuous (s.$\mathcal{E}$-q.c.) if there exists a strict $\mathcal{E}$-nest $(F_k)_{k\in\mathbb{N}}$, such that $f \in C_\infty(\{F_k\})$.

Remark 2.6 *It follows by [12, Lemma 1] that the strict notions do not depend on the special choice of φ if $(\mathcal{E},\mathcal{F})$ is associated with a m-tight m-special standard process. More precisely, in that case the strict capacity and hence the strict $\mathcal{E}$-nests do not depend on the special choice of φ.*

Definition 2.7 *The generalized Dirichlet form $\mathcal{E}$ is called strictly quasi-regular if:*

(i) There exists a strict $\mathcal{E}$-nest $(E_k)_{k\geq 1}$ such that $E_k \cup \{\Delta\}$, $k \geq 1$, is compact in E_Δ.

(ii) There exists a dense subset of $\mathcal{F}$ whose elements have strictly $\mathcal{E}$-q.c. m-versions.

(iii) There exist $u_n \in \mathcal{F}$, $n \in \mathbb{N}$, having strictly $\mathcal{E}$-q.c. m-versions $\widetilde{u}_n$, $n \in \mathbb{N}$, and a strictly $\mathcal{E}$-exceptional set $N \subset E$ such that $\{\widetilde{u}_n \mid n \in \mathbb{N}\}$ separates the points of $E_\Delta \setminus N$.

From now on we fix a *generalized Dirichlet form $(\mathcal{E}, \mathcal{F})$ that is strictly quasi-regular.* For some consequences of this we refer to [12]. For instance, by [12, Proposition 2(i)] every $f \in \mathcal{F}$ admits a s.$\mathcal{E}$-q.c. m-version $\widetilde{f}$.

Most of the related statements in [10] remain true if we replace Cap_φ by $\text{Cap}_{1,\widehat{G}_1\varphi}$, $C(\{F_k\})$ by $C_\infty(\{F_k\})$, and add strict or strictly to quasi-regular, $\mathcal{E}$-nest, $\mathcal{E}$-quasi-uniformly, $\mathcal{E}$-quasi-continuous, etc. We shall refer to these statements as "strict versions". However, some statements are not easily seen to have strict versions (cf e.g. Lemma 2.10(ii) below). These shall be proved here in detail. First we give some complements to results in [12]. Using (1) as main ingredient [12, Proposition 2(i)] can easily be modified (with proof remaining nearly the same) as follows.

Proposition 2.8 *Let $u \in \mathcal{H}$ with s.$\mathcal{E}$-q.c. m-version $\widetilde{u}$ and suppose further that e_u exist. Then for any $\varepsilon > 0$*

$$Cap_{1,\widehat{G}_1\varphi}(\{\widetilde{u} > \varepsilon\}) \leq \varepsilon^{-1} \int_E e_u \, \varphi \, dm.$$

Next we have the following strict versions:

Lemma 2.9 *(i) Let S be a countable family of s.$\mathcal{E}$-q.c. functions (resp. s.$\mathcal{E}$-q.l.s.c. functions). Then there exists a s.$\mathcal{E}$-nest $(F_k)_{k\geq 1}$ such that $S \subset C_\infty(\{F_k\})$ (resp. $S \subset C_{l,\infty}(\{F_k\})$).*

(ii) If f is s.$\mathcal{E}$-q.s.l.c. and $f \leq 0$ m-a.e. on an open set $U \subset E$, then $f \leq 0$ s.$\mathcal{E}$-q.e. on U. If f, g are s.$\mathcal{E}$-q.c. and $f = g$ m-a.e. on an open set $U \subset E$, then $f = g$ s.$\mathcal{E}$-q.e. on U.

(iii) Let $u_n \in \mathcal{H}$ with s.$\mathcal{E}$-q.c. m-version $\widetilde{u}_n$, $n \geq 1$, such that $e_{u_n - u} + e_{u-u_n} \to 0$ in $\mathcal{H}$ as $n \to \infty$ for some $u \in \mathcal{H}$. Then there is a subsequence $(\widetilde{u}_{n_k})_{k\geq 1}$ and a s.$\mathcal{E}$-q.c. m-version $\widetilde{u}$ of u such that $\lim_{k\geq 1} \widetilde{u}_{n_k} = \widetilde{u}$ s.$\mathcal{E}$-quasi-uniformly.

(iv) Let $u_n \in \mathcal{F}$ with s.$\mathcal{E}$-q.c. m-version $\widetilde{u}_n$, $n \geq 1$, and $u_n \to u$ in $\mathcal{F}$. Then there is a subsequence $(\widetilde{u}_{n_k})_{k\geq 1}$ and a s.$\mathcal{E}$-q.c. m-version $\widetilde{u}$ of u such that $\lim_{k\geq 1} \widetilde{u}_{n_k} = \widetilde{u}$ s.$\mathcal{E}$-quasi-uniformly.

Proof (i) The proof is similar to the corresponding one in [10].
(ii) By strict quasi-regularity we obtain the existence of a strict $\mathcal{E}$-nest of compact metrizable sets as in [10, IV. Lemma 1.10]. Then, we may apply the strict versions of [10, III. Lemma 2.1, Corolarry 3.4, and Corollary 3.4] in order to conclude.
(iii) For any s.$\mathcal{E}$-q.c. m-version $\widetilde{u}$ of u (which exists) we have

$$\{|\widetilde{u}_n - \widetilde{u}_m| > \varepsilon\} \subset \left\{\widetilde{u}_n - \widetilde{u} > \frac{\varepsilon}{2}\right\} \cup \left\{\widetilde{u} - \widetilde{u}_n > \frac{\varepsilon}{2}\right\} \cup \left\{\widetilde{u} - \widetilde{u}_m > \frac{\varepsilon}{2}\right\} \cup \left\{\widetilde{u}_m - \widetilde{u} > \frac{\varepsilon}{2}\right\}.$$

Using Proposition 2.8 the proof now follows as in [10, III. Proposition 3.7].
(iv) Follows immediately from (iii) and [10, III. Lemma 2.2(i)].

$\square$

Let

$$Y_1 := \bigcup_{k\in\mathbb{N}} E_k$$

where $(E_k)_{k\geq 1}$ is a strict $\mathcal{E}$-nest of compact metrizable sets as in the proof of Lemma 2.9(ii). Then Y_1 is a Lusin space. Since $E\setminus Y_1$ is strictly $\mathcal{E}$-exceptional it is $\mathcal{E}$-exceptional, hence $m(E\setminus Y_1)=0$ and we may identify $L^2(E;m)$ with $L^2(Y_1,m)$ canonically.
By [12, Lemma 2] we know that for any $\alpha>0$ there exists a kernel $\widetilde{R}_\alpha$ from $(E,\mathcal{B}(E))$ to $(Y_1,\mathcal{B}(Y_1))$ such that

(R1) $\alpha\widetilde{R}_\alpha(z,Y_1)\leq 1$ for all $z\in E$.

(R2) $\widetilde{R}_\alpha f$ is a s.$\mathcal{E}$-q.c. m-version of $G_\alpha f$ for all (measurable) $f\in\mathcal{H}$.

Moreover, the kernel $\widetilde{R}_\alpha$ is unique in the sense that, if K is another kernel from $(E,\mathcal{B}(E))$ to $(Y_1,\mathcal{B}(Y_1))$ satisfying $(R1)$ and $(R2)$, it follows that $K(z,\cdot)=\widetilde{R}_\alpha(z,\cdot)$ s.$\mathcal{E}$-q.e.

Proposition 2.10 *(i) Let $(u_n)_{n\geq 1}\subset\mathcal{H}$ densely. Then $\{\widetilde{R}_1u_n^+,\widetilde{R}_1u_n^-;n\geq 1\}$ separates the points of $E_\Delta\setminus N$, where N is some s.$\mathcal{E}$-exceptional set.*

(ii) There is some $\varphi\in L^1(E,m)\cap\mathcal{B}$ such that $0<\varphi(z)\leq 1$ for every $z\in E$, and such that $\widetilde{R}_1\varphi>0$ s.$\mathcal{E}$-q.e.

Proof (i) Using Lemma 2.9(iv) the proof is the same as in [10, IV. Proposition 1.9].
(ii) Choose $(u_n)_{n\geq 1}\subset L^1(E,m)\cap\mathcal{B}_b$ such that $(u_n)_{n\geq 1}\subset\mathcal{H}$ densely. Then by (i) $\{\widetilde{R}_1u_n^+,\widetilde{R}_1u_n^-;n\geq 1\}$ separates the points of $E_\Delta\setminus N$, where N is some s.$\mathcal{E}$-exceptional set. Define

$$h(x):=\sum_{n\geq 1}c_n\widetilde{R}_1(u_n^++u_n^-)(x),\quad\text{with}\ \ c_n:=2^{-n}(1+\|u_n\|_{L^1(E,m)}+\|u_n\|_{L^\infty(E,m)}).$$

Since $\{\widetilde{R}_1u_n^+,\widetilde{R}_1u_n^-;n\geq 1\}$ separates the points of $E_\Delta\setminus N$ we have $h(x)>0$ for all $x\in E\setminus N$. Since $g_k:=\sum_{n=1}^k c_n(u_n^++u_n^-)$ converges in $L^1(E,m)$ to some g with $0\leq g\leq 1$ and $\widetilde{R}_1$ is a kernel we obtain $h=\widetilde{R}_1g$. Now choose $\rho\in L^1(E,m)$ with $0<\rho\leq 1$. Then $\varphi:=\rho\vee g$ is the desired function.

$\square$

From now on we assume that the strictly quasi-regular generalized Dirichlet form $\mathcal{E}$ satisfies **D3**.
Using Lemma 2.9, Proposition 2.10, and the strict version of [10, IV. Proposition 2.8] we obtain the following:

Lemma 2.11 *There exists a countable family J_0 of bounded strictly $\mathcal{E}$-quasi-continuous 1-excessive functions and a Borel set $Y\subset Y_1$ satisfying:*

(i) If $u,v\in J_0,\ \alpha,c_1,c_2\in\mathbb{Q}_+^*$, then $\widetilde{R}_\alpha u,\ u\wedge v,\ u\wedge 1,\ (u+1)\wedge v,\ c_1u+c_2v$ are all in J_0.

(ii) $N:=E\setminus Y$ is strictly $\mathcal{E}$-exceptional and $\widetilde{R}_\alpha(x,N)=0$, for all $x\in Y,\ \alpha\in\mathbb{Q}_+^*$.

(iii) *J_0 separates the points of Y_Δ.*

(iv) *If $u \in J_0,\ x \in Y$, then $\beta\widetilde{R}_{\beta+1}u(x) \le u(x)$ for all $\beta \in \mathbb{Q}_+^*$,*
$\widetilde{R}_\alpha u(x) - \widetilde{R}_\beta u(x) = (\beta-\alpha)\widetilde{R}_\alpha\widetilde{R}_\beta u(x)$ for all $\alpha,\beta \in \mathbb{Q}_+^*$,
$\lim_{\mathbb{Q}_+^* \ni \alpha\to\infty} \alpha\widetilde{R}_{\alpha+1}u(x) = u(x)$.

Next, we extend the kernel $\widetilde{R}_\alpha$ to the point Δ. Define for $\alpha \in \mathbb{Q}_+^*,\ A \in \mathcal{B}(Y_\Delta) := \mathcal{B}(E_\Delta) \cap Y_\Delta$

$$R_\alpha(x,A) := \begin{cases} \widetilde{R}_\alpha(x, A\cap Y) + \left(\frac{1}{\alpha} - \widetilde{R}_\alpha(x,Y)\right) 1_A(\Delta), & if\ x \in Y \\ \frac{1}{\alpha} 1_A(\Delta), & if x = \Delta \end{cases} \tag{3}$$

and set

$$J := \{u + c1_{Y_\Delta} \mid u \in J_0, c \in \mathbb{Q}_+\}. \tag{4}$$

Since J_0 separates the points of Y_Δ, so does J. The following lemma is also clear.

Lemma 2.12 *Let $(R_\alpha)_{\alpha\in\mathbb{Q}_+^*}$ and J be as in (3), (4). Then the statements of Lemma 2.11 remain true with J_0, Y and $\widetilde{R}_\alpha$ replaced by J, Y_Δ and R_α respectively.*

2.3 The construction of nice excessive functions

Since strict quasi-regularity implies quasi-regularity by [12, Proposition 2 (ii)] we obtain by [10, IV. Theorem 2.2] that $(\mathcal{E},\mathcal{F})$ is associated with some m-tight m-special standard process. We denote the process resolvent by

$$V_\alpha f(z) = E_z\left[\int_0^\infty e^{-\alpha t} f(Y_t)dt\right], \qquad \alpha > 0,\ f \in \mathcal{H} \cap L^\infty(E,m),$$

just as at the end of subsection 2.1.
By Remark 2.6 the strict capacity does not depend on the special choice of φ. We may and will hence **from now on** assume that φ is as in Proposition 2.10(ii).
The following two lemmas are crucial for the later study of weak limits.

Lemma 2.13 *Let $U_n \subset E,\ n \ge 1$ be a decreasing sequence of open sets such that $Cap_{1,\widehat{G}_1\varphi}(U_n) \to 0$, as $n \to \infty$. Then we can find m-versions e_n of e_{U_n} such that:*

(i) *$e_n \ge 1$ $\mathcal{E}$-q.e. on U_n, $n \ge 1$. In particular, there are $\mathcal{E}$-exceptional sets $N_n \in \mathcal{B}(E)$, $N_n \subset U_n$, such that*

$$\widehat{e}_n(x) := e_n(x) + 1_{N_n}(x) \ge 1\ \forall x \in U_n,\ n \ge 1.$$

(ii) *$\alpha\widetilde{R}_{\alpha+1}e_n \le e_n$ and $\alpha\widetilde{R}_{\alpha+1}\widehat{e}_n \le \widehat{e}_n$ s.$\mathcal{E}$-q.e. for any $\alpha \in \mathbb{Q}_+^*,\ n \ge 1$.*

(iii) *$e_n \searrow 0$ and $\widehat{e}_n \to 0$ s.$\mathcal{E}$-q.e. as $n \to \infty$.*

Proof Define for $n \geq 1$

$$e_n := \sup_{\alpha \geq 1} \sup_{l \geq 1} \alpha \widetilde{R}_{\alpha+1} \left(V_1(l\varphi) \wedge 1\right)_{U_n}. \tag{5}$$

where $\left(V_1(l\varphi) \wedge 1\right)_{U_n}$ is some everywhere bounded measurable m-version of $\left(G_1(l\varphi) \wedge 1\right)_{U_n}$. Clearly e_n is an m-version of e_{U_n}. Since $\left(G_1(l\varphi) \wedge 1\right)_{U_n}$ is 1-excessive, and $\widetilde{R}_{\alpha+1} f$ is s.$\mathcal{E}$-q.c. for any (measurable) $f \in \mathcal{H}$ by (R2), by Lemma 2.9(ii) it is clear that the first part of (ii) holds. The second part of (ii) similarly also holds once we have shown that N_n is $\mathcal{E}$-exceptional, hence in particular m-negligible. This is done at the end of the proof.
Obviously e_n is s.$\mathcal{E}$-q.l.s.c, s.$\mathcal{E}$-q.e. decreasing in n, $\lim_{n\to\infty} e_n$ exists s.$\mathcal{E}$-q.e. and $\lim_{n\to\infty} e_n \geq 0$ s.$\mathcal{E}$-q.e. We have (for the intermediate steps (1) is the main ingredient, cf. e.g. [12])

$$\mathrm{Cap}_{1,\widehat{G}_1\varphi}\left(\{\lim_{n\to\infty} e_n > 0\}\right) \quad \leq \quad \sum_{k\geq 1} \mathrm{Cap}_{1,\widehat{G}_1\varphi}\left(\cap_{n\geq 1}\left\{e_n > k^{-1}\right\}\right),$$

and

$$\mathrm{Cap}_{1,\widehat{G}_1\varphi}\left(\cap_{n\geq 1}\left\{e_n > k^{-1}\right\}\right) \quad \leq \quad k \inf_{n\geq 1} \int_E e_n \varphi \, dm = 0.$$

Thus the first part of (iii) holds. The second part of (iii) is clear since $\limsup_{n\geq 1} 1_{N_n} \leq 1_{\cap_{n\geq 1} U_n} = 0$ s.$\mathcal{E}$-q.e.
Since $\widetilde{R}_{\alpha+1} f$, $V_{\alpha+1} f$, $f \in \mathcal{H} \cap L^\infty(E,m)$, are $\mathcal{E}$-q.c. and coincide m-a.e. by (R2), it follows by [10, III. Corollary 3.4] that $V_{\alpha+1} f = \widetilde{R}_{\alpha+1} f$ $\mathcal{E}$-q.e. Using $\left(V_1(l\varphi) \wedge 1\right)_{U_n} = V_1(l\varphi) \wedge 1$ m-a.e. on U_n, and $V_1(l\varphi) \wedge 1 \nearrow \mathbb{I}_E$ as $l \to \infty$, it follows $\mathcal{E}$-q.e.

$$e_n \geq \sup_{\alpha \geq 1} \alpha V_{\alpha+1} \mathbb{I}_{U_n}. \tag{6}$$

By right-continuity and normality of the process Y we obtain for all $z \in U_n$

$$\lim_{\alpha\to\infty} \alpha V_{\alpha+1} \mathbb{I}_{U_n}(z) = \lim_{\alpha\to\infty} \frac{\alpha}{\alpha+1} E_z \left[\int_0^\infty e^{-t} \mathbb{I}_{U_n}(Y_{\frac{t}{\alpha+1}}) dt\right] = 1.$$

Hence the first part of (i) holds. For the second part of (i) we can find $\mathcal{E}$-exceptional sets $N_n \in \mathcal{B}(E)$, $N_n \subset U_n$, with $e_n \cdot 1_{U_n \setminus N_n} + 1_{N_n} \geq 1$ pointwise on U_n. But then $e_n \geq 0$ everywhere since $\widetilde{R}_{\alpha+1}$ is a kernel and so we obtain $\widehat{e}_n \geq 1$ on U_n as desired.

□

Remark 2.14 *(i) In Lemma 2.13(i) we were not able to show directly*

$$e_n \geq 1 \text{ s.}\mathcal{E}\text{-q.e. on } U_n, \; n \geq 1. \tag{7}$$

(Unfortunately, $\left(V_1(l\varphi) \wedge 1\right)_{U_n}$ has only a s.$\mathcal{E}$-q.l.s.c. m-version in general and the inequality in Lemma 2.9(ii) is just the wrong way around.) (7) is used in [6, Lemma 3.5, Lemma 3.6, proof of Theorem 3.3] in an essential way. Instead, we will use the functions $\widehat{e}_n$ defined in Lemma 2.13(i) which is sufficient (cf. Lemmas 2.15 and 4.2, and Theorem 4.5). We remark that it is even sufficient to only know that $e_n \geq 1$ m-a.e. on U_n, so that the sets N_n in Lemma 2.13(i) are only m-negligible.
It will turn out a posteriori that (7) actually holds. In fact, by our main result Theorem 4.11 below it follows that the process resolvent $V_{\alpha+1} f$, $f \in \mathcal{H} \cap L^\infty(E,m)$, is s.$\mathcal{E}$-q.c. Thus applying Lemma 2.9(ii) $V_{\alpha+1} f = \widetilde{R}_{\alpha+1} f$ s.$\mathcal{E}$-q.e. Therefore (6) holds s.$\mathcal{E}$-q.e. and (7) follows.

(ii) There are some other, however strong assumptions that imply (directly) the existence of m-versions e_n of e_{U_n} such that (7) (and of course also Lemma 2.13(ii) and (iii)) holds. For instance:

(a) Any $u \in \mathcal{P}_{\mathcal{F}}$ admits a s.$\mathcal{E}$-q.c. m-version $\widetilde{u}$.

(b) $\widetilde{R}_\alpha f \geq V_\alpha f$ s.$\mathcal{E}$-q.e. for $\alpha > 0$ and $f \in \mathcal{H} \cap L^\infty(E, m)$ with $f \geq 0$.

(c) $\limsup_{\alpha\to\infty} \alpha \widetilde{R}_{\alpha+1} \mathbb{I}_U \geq \mathbb{I}_U$ s.$\mathcal{E}$-q.e.

In case of (a), we may define

$$e_n := \sup_{l\geq 1} (\widetilde{G_1(l\varphi)} \wedge 1)_{U_n}, \quad n \geq 1.$$

Then by Lemma 2.9(ii) $(\widetilde{G_1(l\varphi)} \wedge 1)_{U_n} = \widetilde{R}_1(l\varphi) \wedge 1$ s.$\mathcal{E}$-q.e. on U_n for any $n \geq 1$. Letting $l \nearrow \infty$ and noting that both limits are s.$\mathcal{E}$-q.e. increasing we can see by Proposition 2.10(ii) that (7) holds. (a) holds for instance if $(\mathcal{E}, \mathcal{F})$ is a semi-Dirichlet form. In case of (b) or (c), we may define e_n by (5). Then (7) follows similarly to the proof of Lemma 2.13(i).

Lemma 2.15 *In the situation of Lemma 2.13 there exists $S \in \mathcal{B}(E)$, $S \subset Y$ such that $E \setminus S$ is strictly $\mathcal{E}$-exceptional and the following holds:*

(i) $\widetilde{R}_\alpha(x, Y \setminus S) = 0 \ \forall x \in S, \ \alpha \in \mathbb{Q}_+^*$.

(ii) $\widehat{e}_n(x) \geq 1$ for $x \in U_n, \ n \geq 1$, and $\widetilde{R}_\alpha 1_{N_n}(x) = 0 \ \forall x \in S, \ \alpha \in \mathbb{Q}_+^*, n \geq 1$.

(iii) $\alpha \widetilde{R}_{\alpha+1} \widehat{e}_n(x) \leq \widehat{e}_n(x), \ \forall x \in S, \ \alpha \in \mathbb{Q}_+^*, \ n \geq 1$.

(iv) $\widehat{e}_n \to 0, \ \forall x \in S$.

Proof In view of Lemma 2.13 the proof is similar to [6, Lemma 3.6].

□

3 The approximating forms $\mathcal{E}^\beta$ and the approximating processes X^β

Let J, Y_Δ and $(R_\alpha)_{\alpha\in\mathbb{Q}_+^*}$ be as in Lemma 2.12.
First we collect some results of [1, Chapter 4 section 2]. For a fixed $\beta \in \mathbb{Q}_+^*$, let $\{Y^\beta(k), \ k = 0, 1, \ldots\}$ be a Markov chain in Y_Δ with initial distribution ν and transition function βR_β. Let further $(\Pi_t^\beta)_{t\geq 0}$ be a Poisson process with parameter β and independent of $\{Y^\beta(k), \ k = 0, 1, \ldots\}$. Then it is known that

$$X_t^\beta := Y^\beta(\Pi_t^\beta)$$

is a strong Markov process in Y_Δ with transition semigroup

$$P_t^\beta f := e^{-\beta t} \sum_{k=0}^{\infty} \frac{(\beta t)^k}{k!} (\beta R_\beta)^k f \quad \forall \ t \geq 0, \ f \in \mathcal{B}_b(Y_\Delta), \tag{8}$$

i.e. we have for all $t, s \geq 0, f \in \mathcal{B}_b(Y_\Delta)$

$$E[f(X_{t+s}^\beta) \mid \sigma(X_u^\beta, u \leq t)] = (P_s^\beta f)(X_t^\beta). \tag{9}$$

Here (9) easily follows from [1, Chapter 4 (2.14)]. Furthermore from the formula (8) one can see that $(P_t^\beta)_{t\geq 0}$ is a strongly continuous contraction semigroup on the Banach space $(\mathcal{B}_b(Y_\Delta), \|\cdot\|_\infty)$. The corresponding generator is

$$L^\beta f(x) = \frac{d}{dt} P_t^\beta f(x)\big|_{t=0} = \beta(\beta R_\beta f(x) - f(x)), \quad f \in \mathcal{B}_b(Y_\Delta). \tag{10}$$

Define the forms $\mathcal{E}^{(\beta)}$, $\beta > 0$, by

$$\mathcal{E}^{(\beta)}(u, v) := \beta(u - \beta G_\beta u, v)_{\mathcal{H}}, \quad u, v \in \mathcal{H},$$

where we recall that $(G_\beta)_{\beta>0}$ is the L^2-resolvent of $\mathcal{E}$. It is known (see e.g. [3, Chapter I]), that the C_0-semigroup of submarkovian contractions on $L^2(E; m)$ that is associated to $\mathcal{E}^{(\beta)}$ is given by

$$T_t^\beta f = e^{-\beta t} \sum_{j=0}^{\infty} \frac{(\beta t)^j}{j!} (\beta G_\beta)^j f, \quad f \in \mathcal{H}. \tag{11}$$

From (8), (9), and (11) it follows that (X_t^β) is associated with $\mathcal{E}^{(\beta)}$. Since $R_\beta f$ is an m-version of $G_\beta f$ for any measurable $f \in \mathcal{H}$, by Example 2.1 we see that $\mathcal{E}^{(\beta)}$ is a generalized Dirichlet form and

$$\mathcal{E}^{(\beta)}(u, v) = (-L^\beta u, v)_{\mathcal{H}}, \quad u, v \in \mathcal{H}.$$

For an arbitrary subset $M \subset E_\Delta$ let

$$\Omega_M := D_M[0, \infty)$$

be the space of all càdlàg functions from $[0, \infty)$ to M. Let $(X_t)_{t\geq 0}$ be the coordinate process on Ω_{E_Δ}, i.e. $X_t(\omega) = \omega(t)$ for $\omega \in \Omega_{E_\Delta}$. Ω_{E_Δ} is equipped with the Skorokhod topology (see [1, Chapter 3]). Let P_x^β be the law of X^β on Ω_{E_Δ} with initial distribution δ_x if $x \in Y_\Delta$, and if $x \in E_\Delta \setminus Y_\Delta$ let P_x^β be the Dirac measure on Ω_{E_Δ} such that $P_x^\beta[X_t = x \text{ for all } t \geq 0] = 1$. Finally, let $(\mathcal{F}_t^\beta)_{t\geq 0}$ be the completion w.r.t. $(P_x^\beta)_{x\in E_\Delta}$ of the natural filtration of $(X_t)_{t\geq 0}$.

Proposition 3.1 $\mathbb{M}^\beta := (\Omega_{E_\Delta},\ (X_t)_{t\geq 0},\ (\mathcal{F}_t^\beta)_{t\geq 0},\ (P_x^\beta)_{x\in E_\Delta})$ *is a Hunt process associated with* $\mathcal{E}^{(\beta)}$*, i.e. for all* $t \geq 0$ *and any* m*-version of* $u \in L^2(E; m)$*,* $x \mapsto \int u(X_t)\, dP_x^\beta$ *is an* m*-version of* $T_t^\beta u$*.*

Proof By construction it is clear that $\mathbb{M}^\beta$ is a right process that has left limits in E_Δ. So we only have to prove the quasi-left continuity up to ∞. This can be shown as in [3, IV.3.21], see also [6, Section 4].

□

The next aim is to prove the relative compactness of the family $\{P_x^\beta \mid \beta \in \mathbb{Q}_+^*\}$. We make use of the same compactification method as in [6]. Let $J = \{u_n \mid n \in \mathbb{N}\}$ and

$$g_n := R_1 u_n, \; n \in \mathbb{N}.$$

Define for all $x, y \in Y_\Delta$

$$\rho(x,y) = \sum_{n=1}^{\infty} \frac{1}{2^n} |g_n(x) - g_n(y)| \wedge 1.$$

By Lemma 2.11(ii) and Lemma 2.12 $\{g_n \mid n \in \mathbb{N}\}$ separates the points of Y_Δ and hence ρ defines a metric on Y_Δ. Since Y_Δ is a Lusin topological space, it follows by [9, Lemma 18, p.108] that $\mathcal{B}(Y_\Delta) = \sigma(g_n \mid n \in \mathbb{N}) = (\rho\text{–})\mathcal{B}(Y_\Delta)$. Now define

$$\overline{E} := \overline{Y}_\Delta^\rho.$$

$(\overline{E}, \rho)$ is a compact metric space by Tychonoff's theorem.

We extend the kernel $(R_\alpha)_{\alpha \in \mathbb{Q}_+^*}$ to the space $\overline{E}$ by setting for $\alpha \in \mathbb{Q}_+^*, \; A \in \mathcal{B}(\overline{E})$,

$$R_\alpha(x, A) := \begin{cases} R_\alpha(x, A \cap Y_\Delta), & x \in Y_\Delta \\ \frac{1}{\alpha} 1_A(x), & x \in \overline{E} \setminus Y_\Delta. \end{cases} \tag{12}$$

We may regard $(X_t^\beta)_{t \geq 0}$ as a càdlàg process with state space $\overline{E}$ and use the same notation as before: P_x^β denotes hence the law of $(X_t^\beta)_{t \geq 0}$ in $\Omega_{\overline{E}}$ with initial distribution δ_x. Each g_n is ρ-uniformly continuous and extends therefore uniquely to a continuous function on $\overline{E}$ which we denote again by g_n.

Theorem 3.2 $\{P_x^\beta \mid \beta \in \mathbb{Q}_+^*\}$ *is relatively compact for any* $x \in \overline{E}$.

Proof (cf. [6, Theorem 3.2]) We first show that assumptions of [1, Chapter 4, 9.4 Theorem] are fulfilled with $C_a = C(\overline{E})$ (where C_a is as in [1, Chapter 4, 9.4 Theorem]). Since $g_n \in D(L^\beta)$ it follows that

$$\left(g_n(X_t^\beta) - \int_0^t L^\beta g_n(X_s^\beta)\, ds \right)_{t \geq 0}$$

is an $(P_x^\beta, (\mathcal{F}_t^\beta)_{t \geq 0})$-martingale for any $x \in \overline{E}$. Since

$$L^\beta g_n = 1_{Y_\Delta} \beta R_\beta (g_n - u_n).$$

we have for all $n \in \mathbb{N}$

$$\sup_{\beta \in \mathbb{Q}_+^*} \|L^\beta g_n\|_\infty = \sup_{\beta \in \mathbb{Q}_+^*} \|1_{Y_\Delta} \beta R_\beta (g_n - u_n)\|_\infty \leq \|1_{Y_\Delta} (g_n - u_n)\|_\infty < +\infty.$$

So, we proved that $R_1 J := \{g_n \mid n \in \mathbb{N}\} \subset D$ where $D \subset C(\overline{E})$ is the linear space from [1, Chapter 4, 9.4 Theorem].
Since for any $u \in J$

$$R_1 J \ni R_1\left(\alpha(u - \alpha R_{\alpha+1} u)\right)(x) = \alpha R_{\alpha+1} u(x) \nearrow u(x), \quad \mathbb{Q}_+^* \ni \alpha \to \infty, \forall\, x \in Y_\Delta,$$

we see by Dini's theorem that every $u \in J$ has a unique (ρ-uniformly) continuous extension to $\overline{E}$ that is again denoted by u. Thus we may and do consider J as a subset of $C(\overline{E})$. In particular, if $\overline{A}^{\|\cdot\|_\infty}$ denotes the uniform closure of $A \subset C(\overline{E})$, we have

$$\overline{J-J}^{\|\cdot\|_\infty} \subset \overline{R_1(J-J)}^{\|\cdot\|_\infty} \subset \overline{D}^{\|\cdot\|_\infty} \subset C(\overline{E}).$$

Since $J-J$ contains the constant functions, is inf-stable and separates the points of $\overline{E}$ we obtain that $J-J$ is dense in $C(\overline{E})$ by the Stone-Weierstraß theorem. Hence $\overline{D}^{\|\cdot\|_\infty} = C(\overline{E})$ and so by [1, Chapter 4, 9.4 Theorem] $\{f \circ X^\beta \mid \beta \in \mathbb{Q}_+^*\}$ is relatively compact for all $f \in C(\overline{E})$. Since $\overline{E}$ is compact, the compact containment condition trivially holds and so by [1, Chapter 4, 9.1 Theorem] $\{X^\beta \mid \beta \in \mathbb{Q}_+^*\}$ is relatively compact as desired.

□

4 Limiting process associated with the strictly quasi-regular generalized Dirichlet form

For a Borel subset $S \subset Y$, we write $S_\Delta := S \cup \{\Delta\}$. The topology on S_Δ is, except otherwise stated, the one induced by the metric ρ. In particular the ρ-topology and the original one generate the same Borel σ-algebra on S_Δ.
The proof of Theorem 4.5 below relies on Lemma 4.4 below and on Lemma 2.15 above. Lemma 4.4 tells us that any s.$\mathcal{E}$-nest is a pointwise strict $\mathcal{E}^\beta$-nest.
Let $\mathbb{M}^\beta := (\Omega_{E_\Delta},\ (X_t)_{t\geq 0},\ (\mathcal{F}_t^\beta)_{t\geq 0},\ (P_x^\beta)_{x\in E_\Delta})$ be the Hunt process from Proposition 3.1, R_α and $\overline{E}$ be as in Section 3, and S and $\widehat{e}_n$ be as in Lemma 2.15.

Lemma 4.1 *Let $S \in \mathcal{B}(E)$ be as in Lemma 2.15. Then*

$$P_x^\beta[X_t \in S_\Delta,\ X_{t-} \in S_\Delta\ \forall\ t \geq 0] = 1\ \forall\ x \in S_\Delta.$$

Proof In view of Lemma 2.15(i) the proof is the same as in [6, Lemma 3.7].

□

Lemma 4.2 *Let $S \in \mathcal{B}(E)$ be as in Lemma 2.15. Let $\beta \in \mathbb{Q}_+^*,\ \beta \geq 2,\ n \geq 1$. Then $\widehat{e}_n$ is a (P_t^β)-2-excessive function on S_Δ, i.e.*

$$e^{-2t}P_t^\beta \widehat{e}_n(x) \leq \widehat{e}_n(x) \quad \textit{and}$$

$$\lim_{t\to 0} e^{-2t}P_t^\beta \widehat{e}_n(x) = \widehat{e}_n(x)\ \forall\ x \in S_\Delta.$$

Proof In view of Lemma 2.15(i) and (iii) the proof is the same as in [6, Lemma 3.8].

□

Define for $n \in \mathbb{N}$ the stopping time

$$\tau_n := \inf\{t \geq 0 \mid X_t \in U_n\},$$

called the first entry time of (X_t) in U_n.

Remark 4.3 *The proof of [6, Lemma 3.7] (resp. [5, Lemma 3.2]) contains an inaccuracy, namely it is not true that $X_{\tau_n} \in U_n$ P_x^β-a.s. on $\{\tau_n < \infty\}$ (by right-continuity X_{τ_n} will be in general in the closure $\overline{U}_n$ of U_n). This leads to a wrong argument so that the proof of [6, Lemma 3.7] (resp. [5, Lemma 3.2]) cannot be maintained. Following the proof of Lemma 4.4 below one can easily see how this inaccuracy can be corrected. Therefore the statement of [6, Lemma 3.7] (resp. [5, Lemma 3.2]) remains true and no results of [6] (resp. [5]) are affected.*

Lemma 4.4 *Let $S \in \mathcal{B}(E)$ and $\widehat{e}_n$ be as in Lemma 2.15. Let $\beta \in \mathbb{Q}_+^*$, $\beta \geq 2$ and $\mathbb{M}^\beta := (\Omega_{\overline{E}}, (X_t)_{t\geq 0}, (P_x^\beta)_{x\in\overline{E}})$ be the canonical realization of the Markov process (X_t^β). Then*

$$E_x^\beta[e^{-2\tau_n}] \leq \widehat{e}_n(x), \ \forall \ x \in S_\Delta.$$

Proof Since by Lemma 4.1 S_Δ is invariant set of $\mathbb{M}^\beta$, the restriction $\mathbb{M}^\beta_{S_\Delta}$ of $\mathbb{M}^\beta$ to S_Δ is still a Hunt process. Since $\widehat{e}_n$ is a (P_t^β)-2-excessive function on S_Δ we have that $(e^{-2t}\widehat{e}_n(X_t))_{t\geq 0}$ is a positive right-continuous $(P_x^\beta, (\mathcal{F}_t^\beta)_{t\geq 0})$-supermartingale for all $x \in S_\Delta$. By the optional sampling theorem and normality we have

$$E_x^\beta[e^{-2\tau_n}\widehat{e}_n(X_{\tau_n})] \leq \widehat{e}_n(x), \ x \in S_\Delta.$$

By Lemma 2.15(ii) we have that $\widehat{e}_n(x) \geq 1$ for all $x \in U_n$. Hence, by right-continuity for all $x \in S_\Delta$ we have $\widehat{e}_n(X_{\tau_n}) = \lim_{\mathbb{Q}_+^* \ni t_n \downarrow \tau_n} \widehat{e}_n(X_{t_n}) \geq 1$ P_x^β-a.s. on $\{\tau_n < \infty\}$. (As usual we let $X_\infty := \Delta$, and $f(\Delta) := 0$ for any function f). It follows that for all $x \in S_\Delta$

$$E_x^\beta[e^{-2\tau_n}] \leq E_x^\beta[e^{-2\tau_n}\widehat{e}_n(X_{\tau_n})] \leq \widehat{e}_n(x).$$

□

The following theorem provides two Borel sets Z and Ω with the property that all paths from Ω take their values and left-limits in Z_Δ. Z and Ω are big enough, in the sense that $E \setminus Z$ is strictly $\mathcal{E}$-exceptional and $P_x[\Omega] = 1$. Restricting our process to Ω in Theorem 4.9 we obtain a Hunt process.

Theorem 4.5 *There exists a Borel subset $Z \subset Y$ and a Borel subset $\Omega \subset \Omega_{\overline{E}}$ with the following properties:*

(i) *$E \setminus Z$ is strictly $\mathcal{E}$-exceptional.*

(ii) *$R_\alpha(x, \overline{E} \setminus Z_\Delta) = 0, \ \forall \ x \in Z_\Delta, \ \alpha \in \mathbb{Q}_+^*$.*

(ii) *If $\omega \in \Omega$, then $\omega_t, \ \omega_{t-} \in Z_\Delta$ for all $t \geq 0$. Moreover, each $\omega \in \Omega$ is cadlag in the original topology of Y_Δ and $\omega_{t-}^0 = \omega_{t-}$ for all $t > 0$, where ω_{t-}^0 denotes the left limit in the original topology.*

(iv) *If $x \in Z_\Delta$ and P_x is a weak limit of some sequence $(P_x^{\beta_j})_{j\in\mathbb{N}}$ with $\beta_j \in \mathbb{Q}_+^*, \ \beta_j \uparrow \infty$, then $P_x[\Omega] = 1$.*

Proof In view of Lemma 4.4, Lemma 2.15, the proof is the same as in [6, Theorem 3.3].

□

Since the identities (9) and (10) carry over to $B_b(\overline{E})$ the proof of the following Lemma is the same as in [6, Lemma 4.1].

Lemma 4.6 *Define for* $\alpha, \beta \in \mathbb{Q}_+^*$

$$R_\alpha^\beta f(x) := E_x^\beta \left[\int_0^\infty e^{-\alpha t} f(X_t)\, dt \right], \quad f \in \mathcal{B}_b(\overline{E}),\ x \in \overline{E}.$$

Then

$$R_\alpha^\beta f = \left(\frac{\beta}{\alpha + \beta} \right)^2 R_{\frac{\alpha\beta}{\alpha+\beta}} f + \frac{1}{\alpha + \beta} f. \tag{13}$$

Lemma 4.7 *Let* $x \in \overline{E}$ *and let* P_x *be a weak limit of a subsequence* $(P_x^{\beta_j})_{j \geq 1}$ *with* $\beta_j \uparrow \infty,\ \beta_j \in \mathbb{Q}_+^*$*. Define the kernel*

$$P_t f(x) := E_x[f(X_t)] \quad \forall\ f \in \mathcal{B}_b(\overline{E}).$$

Then

$$\int_0^\infty e^{-\alpha t} P_t f(x)\, dt = R_\alpha f(x), \quad \forall\ f \in \mathcal{B}_b(\overline{E}),\ \alpha \in \mathbb{Q}_+^*. \tag{14}$$

In particular, the kernels $P_t,\ t \geq 0$*, are independent of the subsequence* $(P_x^{\beta_j})_{j \geq 1}$.

Proof The proof is the same as in [6, Lemma 4.2].

□

Theorem 4.8 *Let* Z *be as specified in Theorem 4.5. For every* $x \in Z_\Delta$ *the relatively compact set* $\{P_x^\beta \mid \beta \in \mathbb{Q}_+^*\}$ *has a unique limit* P_x *for* $\beta \uparrow \infty$*. The process* $(\Omega_{\overline{E}}, (X_t)_{t \geq 0}, (P_x)_{x \in Z_\Delta})$ *is a Markov process with the transition semigroup* $(P_t)_{t \geq 0}$ *determined by (14). Moreover,*

$$P_x[X_t \in Z_\Delta, X_{t-} \in Z_\Delta \ \textit{for all}\ t \geq 0] = 1$$

for all $x \in Z_\Delta$.

Proof The proof is the same as in [6, Theorem.3].

□

In what follows let $(P_x)_{x \in Z_\Delta}$ be as in Theorem 4.8. Let Ω and Z_Δ be specified by Theorem 4.5. Since $P_x[\Omega] = 1$ for all $x \in Z_\Delta$, we may restrict P_x and the coordinate process $(X_t)_{t \geq 0}$ to Ω. Let $(\mathcal{F}_t)_{t \geq 0}$ be the natural filtration of $(X_t)_{t \geq 0}$.

Theorem 4.9 $\mathbb{M}^Z := (\Omega, (X_t)_{t \geq 0}, (\mathcal{F}_t)_{t \geq 0}, (P_x)_{x \in Z_\Delta})$ *is a Hunt process with respect to both the* ρ*-topology and the original topology.*

Proof The proof that $\mathbb{M}^Z$ is a Hunt process is the same as in [6, Theorem 4.4.]

□

Remark 4.10 *Let* $\mathbb{M}$ *be the trivial extension of* $\mathbb{M}^Z$ *to* E_Δ *(see [3, IV. (3.48)], or [10, IV. (2.18)]). Then* $\mathbb{M}$ *is again a Hunt process and strictly properly associated in the resolvent sense with* $(\mathcal{E}, \mathcal{F})$ *by (R2) and (14). The Hunt process* $\mathbb{M}$ *is unique up to the equivalence described in [3, IV. 6.3]. I this sense* $\mathbb{M}$ *is the same process as the one constructed in [12, Theorem 3] under the condition* **SD3**.

Theorem 4.11 *Let* $\mathcal{E}$ *be a strictly quasi-regular generalized Dirichlet form satisfying* **D3**. *Then there exists a strictly* m*-tight Hunt process which is strictly properly associated in the resolvent sense with* $\mathcal{E}$.

Proof For the existence of the Hunt process which is strictly properly associated in the resolvent sense with $\mathcal{E}$ see Remark 4.10. The m-tightness is a direct consequence of the existence of a strict $\mathcal{E}$-nest like in Definition 2.7(i) and the representation of the capacity from [12, Lemma 1(i)].

□

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