Approximation of eigenvalues for unbounded Jacobi matrices using finite submatrices

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Abstract. We consider an infinite Jacobi matrix with off-diagonal entries dominated by the diagonal entries going to infinity. The corresponding self-adjoint operator J has discrete spectrum and our purpose is to present results on the approximation of eigenvalues of J by eigenvalues of its finite submatrices.

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1. General presentation

1.1. Introduction

This work is motivated by recent advances in the asymptotic analysis of large eigenvalues of infinite Jacobi matrices with discrete spectrum. We are interested in possible generalisations of results on the approximation of eigenvalues of an infinite Jacobi matrix by eigenvalues of its finite submatrices obtained in the paper of V. Volkmer [10]. The purpose is to investigate more general Jacobi matrices and to control the approximation of the n-th eigenvalue $\lambda_n(J)$ when $n \to \infty$. This purpose has been achieved in the paper of M. Malejki [9] developing the Rayleigh–Ritz approach from [10].

In this paper we present an alternative approach based on exploiting decay properties of resolvent kernels to control functions of J expressed by means of the Helffer–Sjöstrand formula. As indicated in Section 2.2 our method allows us to recover the results from [9] and to obtain results of similar type for larger classes of Jacobi matrices. On the other hand stronger hypotheses on the entries of J allow us to use submatrices of smaller size while the approximation of the spectrum is limited to suitable intervals as described in Theorem 2.3. This type of results is of great importance in a paper in preparation [4].

It should be noted that we consider a very large class of Jacobi matrices. In particular we do not require that the sequence of diagonal entries is increasing. But, if the sequence of diagonal entries $(d_k)_{k=1}^{\infty}$ is increasing and the off-diagonal entries form a sequence small with respect to $(d_{k+1} - d_k)_{k=1}^{\infty}$, then more precise estimates of large eigenvalues are proved in [3]. We also cite the papers [1, 2, 8, 6, 7] which contain results of similar type and more references.

1.2. A class of infinite Jacobi matrices

In this paper we consider infinite tridiagonal matrices

$$\begin{pmatrix} d_1 & \bar{b}_1 & 0 & 0 & 0 & \dots \\ b_1 & d_2 & \bar{b}_2 & 0 & 0 & \dots \\ 0 & b_2 & d_3 & \bar{b}_3 & 0 & \dots \\ 0 & 0 & b_3 & d_4 & \bar{b}_4 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{1.1}$$

satisfying the following two conditions

- $(d_k)_{k=1}^{\infty}$ is a sequence of real numbers satisfying

$$d_k \xrightarrow[k\to+\infty]{} +\infty, \tag{1.2a}$$

- $(b_k)_{k=1}^{\infty}$ is a sequence of complex numbers satisfying

$$\frac{|b_{k-1}| + |b_k|}{d_k} \xrightarrow[k\to+\infty]{} 0. \tag{1.2b}$$

It is well known ([5]) that a matrix (1.1) satisfying conditions (1.2) defines a self-adjoint operator J in the Hilbert space l^2 of complex valued sequences $x = (x_k)_{k=1}^{\infty}$ such that $\sum_{k=1}^{\infty} |x_k|^2 < \infty$. More precisely, the domain of J is

$$\mathcal{D} := \big\{(x_k)_{k=1}^{\infty} \in l^2 \mid (d_k x_k)_{k=1}^{\infty} \in l^2\big\}, \tag{1.3}$$

and $J\colon \mathcal{D} \to l^2$ is given by

$$Jx = (d_k x_k + \bar{b}_k x_{k+1} + b_{k-1} x_{k-1})_{k=1}^{\infty}, \tag{1.4}$$

where, by convention $x_0 = 0$ and $b_0 = 0$.

Moreover, the spectrum $\sigma(J)$ is discrete and bounded from below. Hence, there is an orthonormal basis $(v_n)_{n=1}^{\infty}$ satisfying $Jv_n = \lambda_n(J) v_n$, where $(\lambda_n(J))_{n=1}^{\infty}$ is the sequence of eigenvalues of J arranged in increasing order and repeated according to multiplicity:

$$\lambda_1(J) \le \dots \le \lambda_n(J) \le \lambda_{n+1}(J) \le \dots$$

1.3. Approximation by finite submatrices

From now on, J is as in Section 1.2. For two integers $1 \le j \le k$ we denote by $J_{[j,k]}$ the square matrix of size $k-j+1$ given by

$$J_{[j,k]} = \begin{pmatrix} d_j & \bar{b}_j & 0 & \dots & 0 & 0 & 0 \\ b_j & d_{j+1} & \bar{b}_{j+1} & \dots & 0 & 0 & 0 \\ 0 & b_{j+1} & d_{j+2} & \dots & 0 & 0 & 0 \\ \vdots & \vdots & & \ddots & & \vdots & \vdots \\ 0 & 0 & 0 & & d_{k-2} & \bar{b}_{k-2} & 0 \\ 0 & 0 & 0 & \dots & b_{k-2} & d_{k-1} & \bar{b}_{k-1} \\ 0 & 0 & 0 & \dots & 0 & b_{k-1} & d_k \end{pmatrix} \tag{1.5}$$

Our purpose is to show that the spectrum of J in the interval $[\lambda',\, \lambda]$ is well approximated by the spectrum of the finite submatrix

$$J_{\lambda',\lambda} := J_{[m(\lambda'),\, k(\lambda)]} \tag{1.6}$$

provided the two integers $1 \le m(\lambda') \le k(\lambda)$ are well chosen. More precisely, for any $\nu > 0$ the estimate

$$\operatorname{dist}\bigl(\sigma(J) \cap [\lambda',\, \lambda],\, \sigma(J_{\lambda',\lambda}) \cap [\lambda',\, \lambda]\bigr) \le c_\nu \lambda^{-\nu} \tag{1.7}$$

with a constant c_ν independent of λ', λ can be obtained by comparing the counting functions

$$\mathcal{N}(\lambda, J) := \operatorname{card}\{n \in \mathbb{N}^* \mid \lambda_n(J) \le \lambda\} \tag{1.8a}$$

$$\mathcal{N}(\lambda', \lambda, J) := \operatorname{card}\{n \in \mathbb{N}^* \mid \lambda' < \lambda_n(J) \le \lambda\} \tag{1.8b}$$

with analog quantities for $J_{\lambda',\lambda}$. From the point of view of applications we distinguish two cases, according to the choice of λ'.

Case 1. We consider λ' close to λ. Then we prove that for every $\nu > 0$ one can find $\lambda(\nu)$ such that for $\lambda \ge \lambda(\nu)$, $\lambda' < \lambda$ one has the estimates (Theorem 2.3)

$$\mathcal{N}(\lambda' + \lambda^{-\nu}, \lambda - \lambda^{-\nu}, J_{\lambda',\lambda}) \le \mathcal{N}(\lambda', \lambda, J) \le \mathcal{N}(\lambda' - \lambda^{-\nu}, \lambda + \lambda^{-\nu}, J_{\lambda',\lambda}). \tag{1.9}$$

Case 2. We fix $\lambda' < \inf \sigma(J)$, hence $\mathcal{N}(\lambda', \lambda, J) = \mathcal{N}(\lambda, J)$. This is convenient to estimate all eigenvalues lower than λ. Let $\nu > 0$ be fixed. If

$$J_\lambda := J_{[1,k(\lambda)]} \tag{1.10}$$

then for $\lambda \ge \lambda(\nu)$ we obtain the estimates (Theorem 2.2)

$$\mathcal{N}(\lambda - \lambda^{-\nu}, J_\lambda) \le \mathcal{N}(\lambda, J) \le \mathcal{N}(\lambda + \lambda^{-\nu}, J_\lambda). \tag{1.11}$$

These estimates allow us to find (Theorem 2.1) a sequence $(k_N)_{N=1}^\infty$ such that the first N eigenvalues of J are close to the corresponding eigenvalues of a square submatrix of size k_N:

$$\sup_{1 \le n \le N} |\lambda_n(J) - \lambda_n(J_{[1,\, k_N]})| = \mathrm{O}(N^{-\nu}), \quad N \to \infty, \tag{1.12}$$

where $(\lambda_n(J_{[1,\,k_N]}))_{n=1}^{k_N}$ are the eigenvalues of $J_{[1,\,k_N]}$ arranged in non-decreasing order and repeated according to multiplicity:

$$\lambda_1(J_{[1,\,k_N]}) \le \cdots \le \lambda_n(J_{[1,\,k_N]}) \le \lambda_{n+1}(J_{[1,\,k_N]}) \le \cdots \le \lambda_{k_N}(J_{[1,\,k_N]}).$$

1.4. Contents

All main results, stated in Section 2, derived from Theorem 5.3 as follows:

$$\text{Theorem 5.3} \begin{array}{l} \Longrightarrow \text{Theorem 2.2} \Longrightarrow \text{Theorem 2.1} \\ \Longrightarrow \text{Theorem 2.3} \end{array}$$

Theorem 5.3, stated in Section 5, follows from a series of lemmas:

$$\begin{array}{l} \begin{array}{l}\text{Lemma 3.1}\\ \text{Lemma 3.2}\end{array} \Longrightarrow \begin{array}{l}\text{Lemma 3.3} \Longrightarrow \text{Lemma 3.4}\\ \text{Lemma 4.1} \Longrightarrow \text{Lemma 4.2}\end{array} \Longrightarrow \text{Lemma 4.3} \Longrightarrow \text{Lemma 4.4} \\ \qquad\qquad\qquad \text{Lemma 4.2} \Longrightarrow \begin{array}{l}\text{Lemma 5.1}\\ \text{Lemma 4.4}\end{array} \Longrightarrow \text{Lemma 5.2} \Longrightarrow \text{Theorem 5.3} \end{array}$$

The proofs of Theorems 2.1, 2.2, 2.3, and of Lemma 4.1 are given in Section 6. All preliminary results are stated and proved in Sections 3–5.

2. Main results

2.1. Estimates of eigenvalues lower than λ

We consider the Jacobi operator $J\colon \mathcal{D} \to l^2$ defined in Section 1. We assume $(d_k)_{k=1}^{\infty}$ and $(b_k)_{k=1}^{\infty}$ behave asymptotically as follows, when $k \to \infty$:

$$d_k = ck^\alpha + \mathrm{O}(k^\beta), \tag{2.1a}$$

$$|b_k| = \mathrm{O}(k^\beta), \tag{2.1b}$$

where $c > 0$, α, β are fixed real numbers such that

$$0 \le \beta < \alpha < 1 + \beta. \tag{2.1c}$$

We will prove

Theorem 2.1. *Let J be the Jacobi operator defined by* (1.1) *with assumptions* (2.1). *Let $C_0 > 0$ be a large enough constant, $\lambda \ge 1$ and*

$$\kappa(\lambda) := c^{-1/\alpha}(\lambda + C_0\lambda^{\beta/\alpha})^{1/\alpha}. \tag{2.2}$$

If for every $\lambda \ge 1$, $k(\lambda) \ge 1$ is an integer $\ge \kappa(\lambda)$, then

$$\sup_{n \le \mathcal{N}(\lambda, J)} |\lambda_n(J) - \lambda_n(J_{[1,\,k(\lambda)]})| = \mathrm{O}(\lambda^{-\infty}), \quad \lambda \to \infty. \tag{2.3}$$

Moreover, for any given $\nu > 0$, (1.12) *holds with $k_N = N + \hat{C}N^{1+\beta-\alpha}$ provided the constant $\hat{C}$ is large enough:*

$$\sup_{1 \le n \le N} |\lambda_n(J) - \lambda_n(J_{[1,\,k_N]})| = \mathrm{O}(N^{-\nu}), \quad N \to \infty,$$

Proof. See Section 6.2. The proof is based on a comparison of two infinite matrices and on estimates described in Theorem 2.2 below. □

Notations. We denote by D, $B\colon \mathcal{D} \to l^2$ the diagonal and off-diagonal parts of J:

$$Dx = (d_k x_k)_{k=1}^{\infty}, \tag{2.4}$$

$$Bx = (\bar{b}_k x_{k+1} + b_{k-1} x_{k-1})_{k=1}^{\infty}. \tag{2.5}$$

For $\lambda \geq 1$ we also denote by $B_\lambda\colon \mathcal{D} \to l^2$ the operator defined by

$$B_\lambda x = (\bar{b}_{k,\lambda} x_{k+1} + b_{k-1,\lambda} x_{k-1})_{k=1}^{\infty}, \tag{2.6}$$

where $(b_{k,\lambda})_{k=1}^{\infty}$ is a complex valued sequence such that

$$|b_{k,\lambda}| \leq |b_k|. \tag{2.7}$$

Theorem 2.2. *Let D, B, B_λ be as above, satisfying* (2.1) *and* (2.7)*, and consider*

$$J = D + B, \tag{2.8}$$

$$J_\lambda = D + B_\lambda. \tag{2.9}$$

Let $\kappa(\lambda)$ be as in Theorem 2.1*. Then, under the additional assumption*

$$b_k = b_{k,\lambda} \quad \textit{when} \quad k \leq \kappa(\lambda) \tag{2.10}$$

and for any $\nu > 0$, one can find $\lambda(\nu) \geq 1$ such that (1.11) *holds for $\lambda \geq \lambda(\nu)$,* i.e.,

$$\mathcal{N}(\lambda - \lambda^{-\nu}, J_\lambda) \leq \mathcal{N}(\lambda, J) \leq \mathcal{N}(\lambda + \lambda^{-\nu}, J_\lambda).$$

Proof. See Section 6.1. □

2.2. Comments

(i) Conditions (2.1a) and (2.1b) with $c > 0$ and $0 \leq \beta < \alpha$ ensure (1.2).

(ii) We observe that $d_k \sim ck^\alpha \sim \lambda \iff k \sim \kappa(\lambda)$. More precisely,

$$\kappa(\lambda) = c^{-1/\alpha} \lambda^{1/\alpha} (1 + C_0 \lambda^{(\beta-\alpha)/\alpha})^{1/\alpha}$$

has the following expansion

$$\kappa(\lambda) = c^{-1/\alpha} \lambda^{1/\alpha} + c^{-1/\alpha} (C_0/\alpha) \lambda^{(1+\beta-\alpha)/\alpha} + \mathrm{o}(\lambda^{(1+\beta-\alpha)/\alpha}) \tag{2.11}$$

when $\lambda \to \infty$.

(iii) The condition (2.1c) ensures $\alpha < \beta + 1$, hence (2.11) implies

$$\kappa(\lambda) - c^{-1/\alpha} \lambda^{1/\alpha} \to \infty \text{ when } \lambda \to \infty.$$

(iv) The condition (2.1a) does not imply the monotonicity of $(d_k)_{k=k_0}^{\infty}$ unless $\alpha > \beta + 1$. However, using (2.11) with $\alpha > \beta + 1$ one obtains $\kappa(\lambda) - c^{-1/\alpha} \lambda^{1/\alpha} \to 0$ when $\lambda \to \infty$ and the statement of Theorem 2.1 is in general false in the case $\alpha > \beta + 1$. The case $\alpha > \beta + 1$ is investigated in [3] where we show that (1.11) holds for a given $\nu > 0$ if one uses $\kappa(\lambda) = c^{-1/\alpha} \lambda^{1/\alpha} + C_{\alpha,\beta} \nu$.

(v) The structure of $\kappa(\lambda)$ is in general optimal modulo the choice of the constant

C_0. In our proof we do not look at the best value of C_0, giving just a sufficient condition on C_0 below. If C_1, $C_2 > 0$ are such that for $k \geq k_0$ one has

$$ck^\alpha - C_1 k^\beta \leq d_k \leq ck^\alpha + C_1 k^\beta, \tag{2.12}$$

$$|b_k| \leq C_2 k^\beta, \tag{2.13}$$

then the assertion of Theorem 2.1 is ensured by taking $C_0 > c^{-\beta/\alpha}(C_1 + 12C_2)$.

(vi) If in Theorem 2.2 we choose

$$b_{k,\lambda} = \begin{cases} b_k & \text{if } k \leq k(\lambda), \\ 0 & \text{if } k > k(\lambda), \end{cases} \tag{2.14}$$

where $k(\lambda) \in \mathbb{N}$ satisfies $k(\lambda) \geq \kappa(\lambda)$, then

$$J_\lambda = \begin{pmatrix} J_{[1,k(\lambda)]} & 0 & 0 & 0 & \cdots \\ 0 & d_{1+k(\lambda)} & 0 & 0 & \cdots \\ 0 & 0 & d_{2+k(\lambda)} & 0 & \cdots \\ 0 & 0 & 0 & d_{3+k(\lambda)} & \\ \vdots & \vdots & \vdots & & \ddots \end{pmatrix} \tag{2.15}$$

and $\sigma(J_\lambda) = \sigma(J_{[1,k(\lambda)]}) \cup \{d_k\}_{k>k(\lambda)}$. This observation is the key point to deduce Theorem 2.1 from Theorem 2.2.

(vii) In Sections 3–5 we present the proof in a slightly more general framework — the condition (2.1b) is replaced by (3.1) and we use $\kappa(\lambda) = \kappa_0(\lambda) + 2\lambda^\varepsilon$ where $\kappa_0(\lambda)$ satisfies (4.5) and $\varepsilon > 0$ is an arbitrary fixed number. Then the assertion of Theorem 2.1 follows when we take $\delta = \beta/\alpha$ and $\varepsilon < 1 + \beta - \alpha$.

(viii) The proof described in Sections 3–5 can be adapted to cover more general behaviour of the diagonal entries. In particular we can consider the case when (2.1b) holds and there exist $\alpha > 0$, $0 < c < c'$ and $k_0 \in \mathbb{N}$ such that one has

$$ck^\alpha \leq d_k \leq c'k^\alpha \text{ if } k > k_0. \tag{2.16}$$

We can use $\delta = 1$ in the proof and obtain similar assertions if

$$\kappa(\lambda) = ((1 + C_0)\lambda/c)^{1/\alpha}, k_N = (1 + \hat{C})N \tag{2.17}$$

with suitable values of $C_0 > 0$ and $\hat{C} > 0$. In the case

$$d_k = ck^\alpha + \mathrm{o}(k^\alpha) \text{ when } k \to \infty \tag{2.18}$$

we can take arbitrary constants $C_0 > 0$ and $\hat{C} > 0$, covering the result of M. Malejki [9]. Finally we can replace (2.16) by

$$ck^\alpha \leq d_k \leq c'k^{\alpha'} \text{ if } k > k_0 \tag{2.19}$$

and obtain similar assertions if $k_N = \hat{C}N^{\alpha'/\alpha}$.

2.3. Estimates of eigenvalues in the interval $[\lambda', \lambda]$

For $1 \leq \lambda' \leq \lambda$ we consider a complex valued sequence $(b_{k,\lambda',\lambda})_{k=1}^{\infty}$ satisfying

$$|b_{k,\lambda',\lambda}| \leq |b_k| \tag{2.20}$$

and define $B_{\lambda',\lambda} \colon \mathcal{D} \to l^2$ by

$$B_{\lambda',\lambda} x = (\bar{b}_{k,\lambda',\lambda} x_{k+1} + b_{k-1,\lambda',\lambda} x_{k-1})_{k=1}^{\infty}. \tag{2.21}$$

Theorem 2.3. *Let* $(d_k)_{k=1}^{\infty}$, $(b_k)_{k=1}^{\infty}$, *and* $(b_{k,\lambda',\lambda})_{k=1}^{\infty}$, $1 \leq \lambda' \leq \lambda$ *be complex-valued sequences satisfying* (2.1) *and* (2.20). *We denote*

$$J_{\lambda',\lambda} = D + B_{\lambda',\lambda} \tag{2.22}$$

with $D = \operatorname{diag}(d_k)$ *and* $B_{\lambda',\lambda}$ *given by* (2.21). *Assume moreover*

$$b_k = b_{k,\lambda',\lambda} \quad \textit{when} \quad \kappa(\lambda', \lambda) \leq k \leq \kappa(\lambda), \tag{2.23}$$

where $\kappa(\lambda)$ *is given by* (2.2) *and*

$$\kappa(\lambda', \lambda) := c^{-1/\alpha}(\lambda' - C_0 \lambda^{\beta/\alpha})^{1/\alpha}. \tag{2.24}$$

Then for any $\nu > 0$ *there is a constant* $c_\nu > 0$ *such that* (1.9) *holds:*

$$\mathcal{N}(\lambda' + \lambda^{-\nu}, \lambda - \lambda^{-\nu}, J_{\lambda',\lambda}) \leq \mathcal{N}(\lambda', \lambda, J) \leq \mathcal{N}(\lambda' - \lambda^{-\nu}, \lambda + \lambda^{-\nu}, J_{\lambda',\lambda}).$$

Proof. See Section 6.4. □

3. Preliminary results

We will deduce Theorem 2.1 from an analysis of a larger class of matrices.

Assumption. In Sections 3–5 we assume that (1.2a), (2.7) hold and there exist some constants

$$0 < \delta < 1 \quad \text{and} \quad C,\, C' > 0$$

such that the following estimate holds for all $k \in \mathbb{N}^*$:

$$|b_k| + |b_{k-1}| \leq C(|d_k|^{\delta} + C'). \tag{3.1}$$

Notations. To begin we introduce an auxiliary family of operators

$$J_\lambda^+ := D_\lambda^+ + B_\lambda \tag{3.2}$$

where B_λ is given by (2.6) and $D_\lambda^+ \colon \mathcal{D} \to l^2$ is the self-adjoint operator given by

$$D_\lambda^+ := \max\{|D|,\, \lambda + 6C\lambda^{\delta}\} \tag{3.3}$$

i.e. $D_\lambda^+ = \operatorname{diag}(d_{k,\lambda}^+)_{k \in \mathbb{N}^*}$ with

$$d_{k,\lambda}^+ = \max\{|d_k|,\, \lambda + 6C\lambda^{\delta}\}. \tag{3.4}$$

Lemma 3.1. *Let* C, δ *be as in* (3.1). *If* $\lambda \geq (6C)^{1/(1-\delta)}$ *then for every* $k \in \mathbb{N}^*$

$$d_{k,\lambda}^+ \geq \lambda + 3C(d_{k,\lambda}^+)^{\delta}. \tag{3.5}$$

Proof. By (3.4) we have $\lambda(1+6C\lambda^{\delta-1}) \le d^+_{k,\lambda}$, hence

$$\lambda \le d^+_{k,\lambda}(1+6C\lambda^{\delta-1})^{-1} \le d^+_{k,\lambda}(1-3C\lambda^{\delta-1}), \tag{3.6}$$

where the last inequality follows from the assuption $6C\lambda^{\delta-1} \le 1$. To complete the proof of (3.5) it suffices to write $-\lambda^{\delta-1} \le -(d^+_{k,\lambda})^{\delta-1}$ in the right hand side of (3.6). □

Notations. The scalar product in l^2 is given by $\langle x, y\rangle = \sum_{k=1}^{\infty} \bar{x}_k y_k$. The canonical basis $(\mathrm{e}_n)_{n=1}^{\infty}$ is defined by $\mathrm{e}_n = (\delta_{k,n})_{k=1}^{\infty}$ where $\delta_{n,n} = 1$ and $\delta_{k,n} = 0$ for $k \ne n$. If $P_1, P_2 \colon \mathcal{D} \to l^2$ are two operators such that $P_2 - P_1$ is self-adjoint then we write

$$P_1 \le P_2 \iff \langle (P_2 - P_1)x,\, x\rangle \ge 0 \text{ for all } x \in \mathcal{D}\,.$$

If $(v_k)_{k=1}^{\infty}$ is a complex-valued sequence, then $V = \mathrm{diag}(v_k)_{k\in\mathbb{N}^*}$ denotes the closed linear operator satisfying $V\,\mathrm{e}_k = v_k\,\mathrm{e}_k$ for every integer $k \ge 1$.

Lemma 3.2. *If B is given by* (2.5) *with $b_0 = 0$, then*

$$\pm B \le \mathrm{diag}(|b_k| + |b_{k-1}|)_{k\in\mathbb{N}^*}. \tag{3.7}$$

Proof. For $x \in \mathcal{D}$ we can rewrite the expression

$$\langle x, Bx\rangle = \sum_j b_j x_{j+1}\bar{x}_j + \sum_k \bar{b}_{k-1} x_{k-1}\bar{x}_k \tag{3.8}$$

with $k = j+1$ and estimate $|\langle x, Bx\rangle|$ by

$$\sum_j 2|b_j|\,|x_{j+1}x_j| \le \sum_j |b_j|(|x_{j+1}|^2 + |x_j|^2). \tag{3.9}$$

Therefore the right hand side of (3.9) can be written in the form

$$\sum_k |b_k||x_{k+1}|^2 + \sum_j |b_j||x_j|^2 = \sum_j (|b_{j-1}| + |b_j|)|x_j|^2 \tag{3.10}$$

and the proof of (3.7) is complete. □

We apply Lemma 3.1 and Lemma 3.2 in the following

Lemma 3.3. *Let C, C', δ be as in* (3.1). *If $\lambda \ge \max\{(6C)^{1/(1-\delta)},\, C'^{1/\delta}\}$, then*

$$J^+_\lambda \ge \lambda + C(D^+_\lambda)^\delta \ge \lambda + C\lambda^\delta. \tag{3.11}$$

Proof. Lemma 3.2 applied to $B_\lambda = J^+_\lambda - D^+_\lambda$ gives the estimate

$$J^+_\lambda - \lambda \ge \mathrm{diag}(d^+_{k,\lambda} - \lambda - |b_{k,\lambda}| - |b_{k-1,\lambda}|)_{k\in\mathbb{N}^*}. \tag{3.12}$$

The assumption on λ and Lemma 3.1 imply the estimates

$$C' \le \lambda^\delta \le (d^+_{k,\lambda})^\delta.$$

Using these estimates, (3.1) and (3.4) we obtain

$$|b_{k,\lambda}| + |b_{k-1,\lambda}| \le C(|d_k|^\delta + C') \le 2C(d^+_{k,\lambda})^\delta, \tag{3.13}$$

hence, by (3.13) and (3.5),

$$d_{k,\lambda}^{+} - \lambda - |b_{k,\lambda}| - |b_{k-1,\lambda}| \geq C(d_{k,\lambda}^{+})^{\delta},$$

which completes the proof due to (3.12). $\square$

Notations. We fix

$$\hat{\lambda} := \max\{(6C)^{1/(1-\delta)},\, C'^{1/\delta},\, (6/C)^{1/\delta}\} \tag{3.14}$$

and introduce

$$\hat{D} := D_{\hat{\lambda}}^{+} = \max\{D,\, \hat{\lambda} + 6C\hat{\lambda}^{\delta}\}. \tag{3.15}$$

Then we denote

$$R_{\lambda}^{+} := (J_{\lambda}^{+} - z)^{-1} \tag{3.16}$$

for $z \in \mathbb{C} \setminus \sigma(J_{\lambda}^{+})$, and define

$$\Gamma(\lambda) := \{z \in \mathbb{C} \mid \operatorname{Re} z < \lambda + 2,\ 0 < |\operatorname{Im} z| < 2\}. \tag{3.17}$$

Lemma 3.4. *There exists a constant $c_0 > 0$ such that for any $z \in \Gamma(\lambda)$*

$$\|\hat{D}^{\delta/2} R_{\lambda}^{+}(z) \hat{D}^{\delta/2}\| \leq c_0. \tag{3.18}$$

Proof. We assume $\lambda \geq \hat{\lambda}$. By definition of $\hat{\lambda}$ we can apply Lemma 3.3 and write

$$J_{\lambda}^{+} \geq \lambda + \frac{1}{2}C\lambda^{\delta} + \frac{1}{2}C(D_{\lambda}^{+})^{\delta}.$$

The definition of $\hat{\lambda}$ ensures moreover $\frac{1}{2}C\hat{\lambda}^{\delta} \geq 3$ and allows us to write

$$J_{\lambda}^{+} - \operatorname{Re} z \geq \lambda - \operatorname{Re} z + 3 + \frac{1}{2}C\hat{D}^{\delta}.$$

Further on we assume $\operatorname{Re} z \leq \lambda + 2$ and observe that

$$J_{\lambda}^{+} - \operatorname{Re} z \geq \frac{1}{2}C\hat{D}^{\delta} \implies R_{\lambda}^{+}(\operatorname{Re} z) \leq 2C^{-1}\hat{D}^{-\delta}.$$

Thus we have proved

$$\operatorname{Re} z \leq \lambda + 2 \implies \|\hat{D}^{\delta/2} R_{\lambda}^{+}(\operatorname{Re} z) \hat{D}^{\delta/2}\| \leq 2C^{-1}.$$

To complete the proof it suffices to observe that the resolvent series

$$\sum_{n=0}^{\infty} (\mathrm{i}\operatorname{Im} z)^{n} (R_{\lambda}^{+}(\operatorname{Re} z))^{n+1} = R_{\lambda}^{+}(z)$$

converges uniformly in $\Gamma(\lambda)$ due to $R_{\lambda}^{+}(\operatorname{Re} z) \leq 2C^{-1}\hat{D}^{-\delta} \leq 2C^{-1}\hat{\lambda}^{-\delta} \leq \frac{1}{3}$. $\square$

4. Decay of resolvent kernels

For $r \in \mathbb{R}$ we define the orthogonal projections

$$P_r^+ = \sum_{\substack{k \in \mathbb{N}^* \\ k > r}} \langle x, \mathrm{e}_k \rangle \, \mathrm{e}_k, \tag{4.1}$$

$$P_r^- = I - P_r^+ \tag{4.2}$$

and consider the following version of the standard decay property of the kernel $\langle \mathrm{e}_j, R_\lambda^+(z) \, \mathrm{e}_k \rangle$ when $|k - j| \to \infty$:

Lemma 4.1. *For $0 < r < s$, $\lambda \geq 1$ and $z \in \mathbb{C} \setminus \mathbb{R}$ let*

$$Q_{\delta,\lambda}^{r,s}(z) := \hat{D}^{\delta/2} P_s^+ \, R_\lambda^+(z) P_r^- \, \hat{D}^{\delta/2}. \tag{4.3}$$

Then for any $\nu \geq 0$ one can find $c_\nu > 0$ such that one has

$$||Q_{\delta,\lambda}^{r,s}(z)|| \leq c_\nu (s - r)^{-\nu} \text{ if } z \in \Gamma(\lambda) \tag{4.4}$$

Proof. The proof of Lemma 4.1 is given in Section 6.3. □

The remaining part of this section is devoted to consequences of Lemma 4.1.

Assumption. From now on we assume that $\kappa_0 \colon \mathbb{R} \to \mathbb{R}$ satisfies the condition

$$j \geq \kappa_0(\lambda) \implies d_j \geq \lambda + 6C\lambda^\delta. \tag{4.5}$$

Notations. We also fix $\varepsilon > 0$ and for $\rho \geq 0$ we denote

$$\kappa_\rho(\lambda) = \kappa_0(\lambda) + \rho\lambda^\varepsilon. \tag{4.6}$$

By (3.4), $d_j \geq \lambda + 6C\lambda^\delta \implies d_{j,\lambda}^+ = d_j$. Thus, condition (4.5) ensures

$$P_{\kappa_0(\lambda)}^+ (D - D_\lambda^+) = 0, \tag{4.7}$$

which allows us to compare $R_\lambda^+(z)$ with

$$R_\lambda(z) = (D + B_\lambda - z)^{-1} \tag{4.8}$$

in the following

Lemma 4.2. *Denote*

$$Q_{\delta,\lambda}(z) := \hat{D}^{\delta/2} P_{\kappa_1(\lambda)}^+ \big(R_\lambda^+(z) - R_\lambda(z) \big). \tag{4.9}$$

Then for any $\nu \geq 0$ one can find a constant c_ν such that one has

$$||Q_{\delta,\lambda}(z)|| \leq c_\nu \lambda^{-\nu} |\mathrm{Im}\, z|^{-1} \text{ if } z \in \Gamma(\lambda). \tag{4.10}$$

Proof. Using $D - D_\lambda^+ = P_{\kappa_0(\lambda)}^- (D - D_\lambda^+)$ in the equality

$$R_\lambda(z) - R_\lambda^+(z) = R_\lambda^+(z)(D - D_\lambda^+) R_\lambda(z)$$

we find

$$Q_{\delta,\lambda}(z) = Q_{\delta,\lambda}^{\kappa_0(\lambda),\kappa_1(\lambda)}(z) \hat{D}^{-\delta/2} (D - D_\lambda^+) R_\lambda(z).$$

Since $0 \leq D_\lambda^+ - D \leq \lambda + 6C\lambda^\delta$ and $||R_\lambda(z)|| \leq |\mathrm{Im}\, z|^{-1}$, we complete the proof using Lemma 4.1 with $r = \kappa_0(\lambda)$, $s = \kappa_1(\lambda)$. □

Lemma 4.3. *Denote*

$$\hat{Q}_{\delta,\lambda}(z) := \hat{D}^{\delta/2} P^{+}_{\kappa_2(\lambda)} R_\lambda(z) P^{-}_{\kappa_1(\lambda)}. \tag{4.11}$$

Then for any $\nu \geq 0$ one can find a constant c_ν such that, for any $z \in \Gamma(\lambda)$,

$$||\hat{D}^{\delta/2} P^{+}_{\kappa_1(\lambda)} R_\lambda(z)|| \leq c_0 |\operatorname{Im} z|^{-1}, \tag{4.12}$$

$$||\hat{Q}_{\delta,\lambda}(z)|| \leq c_\nu \lambda^{-\nu} |\operatorname{Im} z|^{-1}. \tag{4.13}$$

Proof. Combine Lemma 4.2 with Lemma 3.4 and Lemma 4.1. □

Assumption. We consider the assumption

$$B_\lambda - B = (B_\lambda - B) P^{+}_{\kappa_2(\lambda)} = P^{+}_{\kappa_2(\lambda)} (B_\lambda - B) \tag{4.14}$$

in order to compare $R_\lambda(z)$ with

$$R(z) = (D + B - z)^{-1}. \tag{4.15}$$

Lemma 4.4. *Assume that* (4.14) *holds. Then for any $\nu > 0$ one can find a constant c_ν such that, for any $z \in \Gamma(\lambda)$,*

$$\|(R_\lambda(z) - R(z)) P^{-}_{\kappa_1(\lambda)}\| \leq c_\nu \lambda^{-\nu} |\operatorname{Im} z|^{-2}. \tag{4.16}$$

Proof. Using $B - B_\lambda = P^{+}_{\kappa_2(\lambda)}(B - B_\lambda)$ we can express

$$(R_\lambda(z) - R(z)) P^{-}_{\kappa_1(\lambda)} = R(z)(B - B_\lambda) R_\lambda(z) P^{-}_{\kappa_1(\lambda)}$$

in the form

$$\tilde{Q}_{\delta,\lambda}(\bar{z})^* \hat{D}^{-\delta/2} (B - B_\lambda) \hat{D}^{-\delta/2} \hat{Q}_{\delta,\lambda}(z)$$

where $\hat{Q}_{\delta,\lambda}(z)$ is given by (4.11) and

$$\tilde{Q}_{\delta,\lambda}(\bar{z}) = \hat{D}^{\delta/2} P^{+}_{\kappa_2(\lambda)} R(\bar{z}).$$

Using (4.12) in the special case $b_{k,\lambda} = b_k$ for all $k \in \mathbb{N}^*$ we obtain, for $z \in \Gamma(\lambda)$,

$$\|\tilde{Q}_{\delta,\lambda}(\bar{z})\| \leq c_0 |\operatorname{Im} z|^{-1}.$$

We complete the proof using $\|\hat{D}^{-\delta/2}(B - B_\lambda)\hat{D}^{-\delta/2}\| \leq c_0$ and (4.13). □

5. Application of Helffer–Sjöstrand formula

Consider $g \in C_0^\infty(\mathbb{R})$ and a self-adjoint operator A. Then we express

$$g(A) = \frac{\mathrm{i}}{2\pi} \int_{\operatorname{supp} \tilde{g}} \bar{\partial} \tilde{g}(w) (A - w)^{-1} \mathrm{d}w \wedge \mathrm{d}\bar{w}, \tag{5.1}$$

where $\tilde{g} \in C_0^\infty(\mathbb{C})$ is an "almost analytic" extension of g, i.e.

(i) $\tilde{g}(t) = g(t)$ for $t \in \mathbb{R}$,

(ii) for any $\nu \in \mathbb{N}$ there is a constant c_ν such that one has the estimate

$$|\bar{\partial} \tilde{g}(w)| \leq c_\nu |\operatorname{Im} w|^\nu \text{ for } w \in \mathbb{C}, \tag{5.2}$$

where $\bar{\partial} \tilde{g}(t + \mathrm{i}s) = \frac{1}{2}(\partial_t + \mathrm{i}\partial_s) \tilde{g}(t + \mathrm{i}s)$ for $s, t \in \mathbb{R}$,

(iii) $\operatorname{supp}\tilde{g} \subset \{w \in \mathbb{C} \mid \operatorname{Re} w \in \operatorname{supp} g,\ |\operatorname{Im} w| \le 1\}$.

Lemma 5.1. *Let $g \in C_0^\infty(\mathbb{R})$ and denote*

$$G_{\nu,\lambda}(\tau) = P^+_{\kappa_1(\lambda)}\big(g(\lambda^\nu(J^+_\lambda - \tau)) - g(\lambda^\nu(J_\lambda - \tau))\big) \tag{5.3}$$

where $\lambda \ge 1$, $\tau \in \mathbb{R}$ and $\nu \ge 0$. Then

$$\sup_{\tau \le \lambda} \|G_{\nu,\lambda}(\tau)\| = \mathrm{O}(\lambda^{-\infty}). \tag{5.4}$$

Proof. Let λ_0 be a large enough constant. We assume $\lambda \ge \lambda_0$ and $\tau \le \lambda$. Then

$$z := \tau + \lambda^{-\nu} w \in \Gamma(\lambda) \tag{5.5}$$

holds for $w \in \operatorname{supp}\tilde{g}$. Writing (5.1) with $A = \lambda^\nu(J_\lambda - \tau)$ we can use

$$(A - w)^{-1} = \lambda^{-\nu}(J_\lambda - z)^{-1}$$

with z given by (5.5). Similarly, writing (5.1) with $A = \lambda^\nu(J^+_\lambda - \tau)$ and z as before, we find the expression

$$G_{\nu,\lambda}(\tau) = \frac{\mathrm{i}}{2\pi}\int_{\Gamma(\lambda)} \overline{\partial}\tilde{g}(\lambda^\nu(z-\tau))Q_{0,\lambda}(z)\lambda^\nu \mathrm{d}z \wedge \mathrm{d}\bar{z} \tag{5.6}$$

where $Q_{0,\lambda}(z)$ is given by (4.9) with $\delta = 0$. To complete the proof it remains to use (4.10) with 2ν instead of ν. ☐

Notations. From now on, $g \in C_0^\infty((-1,1))$. For $1 \le \lambda' < \lambda$ and $\nu \ge 0$ we introduce

$$f_{\nu,\lambda',\lambda}(t) = \lambda^\nu \int_{\lambda'}^{\lambda} g(\lambda^\nu(t-\tau))\mathrm{d}\tau. \tag{5.7}$$

Lemma 5.2. *If J, J_λ and $f_{\nu,\lambda',\lambda}$ are as in* (2.8), (2.9) *and* (5.7), *respectively, then*

$$||f_{\nu,\lambda',\lambda}(J_\lambda) - f_{\nu,\lambda',\lambda}(J)|| = \mathrm{O}(\lambda^{-\infty}). \tag{5.8}$$

Proof. Due to Lemma 5.1 we find that the norm of

$$P^+_{\kappa_1(\lambda)}\big(f_{\nu,\lambda',\lambda}(J_\lambda)) - f_{\nu,\lambda',\lambda}(J^+_\lambda))\big) = \lambda^\nu \int_{\lambda'}^{\lambda} G_{\nu,\lambda}(\tau)\mathrm{d}\tau$$

is $\mathrm{O}(\lambda^{-\infty})$ and since for $\lambda \ge \lambda_0$ one has

$$\operatorname{supp} f_{\nu,\lambda',\lambda} \cap \sigma(J^+_\lambda) \subset [\lambda' - \lambda^{-\nu},\ \lambda + \lambda^{-\nu}] \cap [\lambda + C\lambda^\delta, \infty) = \varnothing,$$

we conclude that $f_{\nu,\lambda',\lambda}(J^+_\lambda) = 0$, hence

$$||P^+_{\kappa_1(\lambda)} f_{\nu,\lambda',\lambda}(J_\lambda)|| = \mathrm{O}(\lambda^{-\infty}). \tag{5.9}$$

In the case $b_{\lambda,k} = b_k$ for all $k \in \mathbb{N}^*$ one has $B = B_\lambda$, hence

$$||P^+_{\kappa_1(\lambda)} f_{\nu,\lambda',\lambda}(J)|| = \mathrm{O}(\lambda^{-\infty}). \tag{5.10}$$

Due to (5.9) and (5.10) it suffices to show that the norm of

$$F_{\nu,\lambda',\lambda} = \big(f_{\nu,\lambda',\lambda}(J_\lambda)) - f_{\nu,\lambda',\lambda}(J)\big)P^-_{\kappa_1(\lambda)} \tag{5.11}$$

is $\mathrm{O}(\lambda^{-\infty})$. However using (4.16) with 3ν instead of ν we find that the norm of

$$F_{\nu,\lambda',\lambda} = \frac{\mathrm{i}\lambda^{2\nu}}{2\pi}\int_{\lambda'}^{\lambda}\mathrm{d}\tau\int_{\Gamma(\lambda)}\overline{\partial}\tilde{g}(\lambda^{\nu}(z-\tau))(R_{\lambda}(z)-R(z))P^{-}_{\kappa_1(\lambda)}\mathrm{d}z\wedge\mathrm{d}\bar{z}$$

is $\mathrm{O}(\lambda^{-\nu})$. □

Theorem 5.3. *Let J, J_λ be as before and $\nu \geq 0$. If $\lambda(\nu)$ is large enough, then*

$$\mathcal{N}(\lambda'+\lambda^{-\nu},\lambda-\lambda^{-\nu},J_\lambda) \leq \mathcal{N}(\lambda',\lambda,J) \leq \mathcal{N}(\lambda'-\lambda^{-\nu},\lambda+\lambda^{-\nu},J_\lambda) \tag{5.12}$$

holds for any $\lambda \geq \lambda(\nu)$ and $\lambda' \leq \lambda$.

Proof. Let $\mathbf{1}_Z : \mathbb{R} \to \{0,\,1\}$ denote the characteristic function of $Z \subset \mathbb{R}$. We denote by $\mathbf{1}_{[c,\,c']}(J)$ the spectral projector of J with respect to the interval $[c,\,c']$. Assume $\operatorname{supp} g \subset [0,1]$, $g \geq 0$ and $\int_{\mathbb{R}} g = 1$. Then

$$\mathbf{1}_{[\lambda'+\lambda^{-\nu},\lambda]}(t) \leq f_{\nu,\lambda',\lambda}(t) \leq \mathbf{1}_{[\lambda',\lambda+\lambda^{-\nu}]}(t) \tag{5.13}$$

and due to Lemma 5.2 one can find $\lambda(\nu)$ such that

$$\lambda \geq \lambda(\nu) \implies ||f_{\nu,\lambda',\lambda}(J) - f_{\nu,\lambda',\lambda}(J_\lambda)|| < 1. \tag{5.14}$$

Assume that the second inequality (5.12) is false. Then one has

$$\dim\operatorname{Ran}\mathbf{1}_{[\lambda'+\lambda^{-\nu},\lambda]}(J) > \dim\operatorname{Ran}\mathbf{1}_{[\lambda',\lambda+\lambda^{-\nu}]}(J_\lambda) \tag{5.15}$$

for a certain $\lambda \geq \lambda(\nu)$ and one can find $x \in l^2 \setminus \{0\}$ such that

$$x \in \operatorname{Ran}\mathbf{1}_{[\lambda'+\lambda^{-\nu},\lambda]}(J) \cap \operatorname{Ran}\mathbf{1}_{[\lambda',\lambda+\lambda^{-\nu}]}(J_\lambda)^{\perp}. \tag{5.16}$$

However (5.16) ensures $x = \mathbf{1}_{(\lambda'+\lambda^{-\nu},\lambda]}(J)x$ and $f_{\nu,\lambda',\lambda}(J)x = x$ follows from (5.13). Then $x \in \operatorname{Ran}\mathbf{1}_{[\lambda',\lambda+\lambda^{-\nu}]}(J_\lambda)^{\perp} = \ker\mathbf{1}_{[\lambda',\lambda+\lambda^{-\nu}]}(J_\lambda) \implies f_{\nu,\lambda',\lambda}(J_\lambda)x = 0$ and

$$\big(f_{\nu,\lambda',\lambda}(J) - f_{\nu,\lambda',\lambda}(J_\lambda)\big)x = x$$

gives a contradiction with (5.14). By exchange of J and J_λ in the above reasoning we obtain the first inequality (5.12). □

6. Proofs of main results

6.1. Proof of Theorem 2.2

In order to deduce Theorem 2.2 from Theorem 5.3 we assume that (2.12) and (2.13) hold. Then

$$k > k_0 \implies |b_{k-1}| + |b_k| \leq 2C_2 k^{\beta} \sim 2C_2 c^{-\beta/\alpha} d_k^{\beta/\alpha}$$

and it is clear that for any $C > 2C_2 c^{-\beta/\alpha}$ we can find C' such that (3.1) holds with $\delta = \beta/\alpha$. We introduce

$$\kappa_0(\lambda) := c^{-1/\alpha}(\lambda + C'_0\lambda^{\delta})^{1/\alpha},$$
$$\kappa_2(\lambda) := \kappa_0(\lambda) + 2\lambda^{\varepsilon} \text{ with } \varepsilon < 1+\beta-\alpha.$$

Then $\kappa_2(\lambda) \le \kappa(\lambda)$ holds if $C_0' < C_0$, $\kappa(\lambda)$ is given by (2.2) and $\lambda \ge \lambda_0$. Since

$$j \ge \kappa_0(\lambda) \implies d_k \ge c\kappa_0(\lambda)^\alpha - C_1\kappa_0(\lambda)^\beta \ge \lambda + (C_0' - C_1 c^{-\delta})\lambda^\delta + \mathrm{o}(\lambda^\delta)$$

it is clear that $j \ge \kappa_0(\lambda) \implies d_j \ge \lambda + 6C\lambda^\delta$ holds if $C_0' > C_1 c^{-\delta} + 6C$. Thus the assumption $C_0 > c^{-\delta}(C_1 + 12C_2)$ allows us to choose C_0' and C such that (4.14) holds and Theorem 5.3 can be applied.

6.2. Proof of Theorem 2.1

Let J_λ be as in Remark (vi) in Section 2.2. We claim that

$$n \le \mathcal{N}(\lambda, J_\lambda) \implies \lambda_n(J_\lambda) = \lambda_n(J_{[1,k(\lambda)]}). \tag{6.1}$$

Indeed, $\sigma(J_\lambda) \setminus \sigma(J_{[1,k(\lambda)]}) = \{d_j\}_{j>k(\lambda)}$ and (6.1) follows by combining

$$n \le \mathcal{N}(\lambda, J_\lambda) \implies \lambda_n(J_\lambda) \le \lambda \tag{6.2}$$

with the property $j > k(\lambda) \implies d_j \ge \lambda + 6C\lambda^\delta > \lambda$. Thus we obtain (2.3) if we check that for any $\nu > 0$ one has

$$n \le \mathcal{N}(\lambda, J_\lambda) \implies \lambda_n(J_\lambda) - \lambda^{-\nu} \le \lambda_n(J) \le \lambda_n(J_\lambda) + \lambda^{-\nu} \tag{6.3}$$

if $\lambda \ge \lambda(\nu)$. However using (5.12) with λ' small enough and (6.2) with J instead of J_λ we find

$$n \le \mathcal{N}(\lambda_n(J_\lambda), J_\lambda) \le \mathcal{N}(\lambda_n(J_\lambda) + \lambda^{-\nu}, J) \implies \lambda_n(J) \le \lambda_n(J_\lambda) + \lambda^{-\nu}.$$

The remaining inequality of (6.3) follows by similar arguments if J and J_λ are exchanged. To prove the last assertion we observe that $N = \mathcal{N}(\lambda_N(J), J)$ it suffices to show that

$$k_N = N(1 + \hat{C}N^{\beta-\alpha}) > \kappa(\lambda_N(J)) \tag{6.4}$$

holds if $\hat{C}$ is fixed large enough. Let us assume that (2.12) and (3.1) hold. Then introducing $\Lambda := \mathrm{diag}(k)_{k\in\mathbb{N}^*}$ we can write the inequalities

$$c\Lambda^\alpha - (C_1 + 2C_2)\Lambda^\beta - \tilde{C} \le J \le c\Lambda^\alpha + (C_1 + 2C_2)\Lambda^\beta + \tilde{C}.$$

However $\lambda_n(c\Lambda^\alpha \pm (C_1+2C_2)\Lambda^\beta) = cn^\alpha \pm (C_1+2C_2)n^\beta$ and the min-max principle implies

$$cn^\alpha - (C_1 + 2C_2)n^\beta - \tilde{C} \le \lambda_n(J) \le cn^\alpha + (C_1 + 2C_2)n^\beta + \tilde{C}.$$

Therefore

$$\begin{aligned}\kappa(\lambda_N(J)) &= c^{-1/\alpha}\lambda_N(J)^{1/\alpha}(1 + \mathrm{O}(\lambda_N(J))^{(\beta-\alpha)/\alpha}))\\ &= c^{-1/\alpha}(cN^\alpha + \mathrm{O}(N^\beta))^{1/\alpha}(1 + \mathrm{O}(N^{\beta-\alpha}))\end{aligned}$$

ensures $\kappa(\lambda_N(J)) = N(1 + \mathrm{O}(N^{\beta-\alpha}))$, completing the proof of (6.4).

6.3. Proof of Lemma 4.1

Let $f \in C^\infty(\mathbb{R})$ be such that for every $t \in \mathbb{R}$ one has

$$\mathbf{1}_{(-\infty,\,0]}(t) \le f(t) \le \mathbf{1}_{(-\infty,\,1]}(t) \tag{6.5}$$

and define $V_{r,s} = \mathrm{diag}(v_{k,r,s})_{k=1}^\infty$ with $v_{k,r,s} := f((k-r)/(s-r))$ for $k \in \mathbb{N}^*$. Then $f(t) = 1$ for $t \le 0$ ensures $P_r^- = V_{r,s} P_r^-$ and $f(t) = 0$ for $t \ge 1$ ensures $P_s^+ V_{r,s} = 0$, hence

$$P_s^+ R_\lambda^+(z) P_r^- = P_s^+ [R_\lambda^+(z), V_{r,s}] P_r^- = P_s^+ R_\lambda^+(z) [V_{r,s}, B_\lambda] R_\lambda^+(z) P_r^- . \tag{6.6}$$

Thus denoting $B'_{r,s,\lambda} := [\mathrm{i}V_{r,s}, B_\lambda]$ we can estimate

$$||Q_{\delta,\lambda}^{r,s}(z)|| \le C_0 ||\hat{D}^{-\delta/2} B'_{r,s,\lambda} \hat{D}^{-\delta/2}|| \tag{6.7}$$

and for $x \in \mathcal{D}$ we compute

$$\mathrm{i}[V_{r,s}, B_\lambda] x = (\bar{b}'_{k,r,s,\lambda} x_{k+1} + b'_{k-1,r,s,\lambda} x_{k-1})_{k=1}^\infty \tag{6.8}$$

with $b'_{k,r,s,\lambda} = (v_{k+1,r,s} - v_{k,r,s}) b_{k,\lambda}$. Since there is a constant $C_1 > 0$ such that

$$|v_{k+1,r,s} - v_{k,r,s}| \le C_1 |s-r|^{-1},$$

Lemma 3.1 ensures $\pm B'_{r,s,\lambda} \le C'_1 \hat{D}^\delta$ for a certain constant $C'_1 > 0$ and it is clear that the right hand side of (6.7) can be estimated by $c_1|s-r|^{-1}$. Next we assume that (4.4) holds for a given $\nu \ge 1$ and show that it still holds with 2ν instead of ν. For this purpose we observe that

$$B_\lambda P_t^+ - P_t^+ B_\lambda = (B_\lambda P_t^+ - P_t^+ B_\lambda) P_{t-1}^+ \tag{6.9}$$

ensures $-[B_\lambda,\, P_t^-] = [B_\lambda,\, P_t^+] = [B_\lambda,\, P_t^+] P_{t-1}^+$ and

$$[B_\lambda,\, P_t^-] = B_\lambda P_t^- - P_t^- B_\lambda = P_{t+1}^- (B_\lambda P_t^- - P_t^- B_\lambda) \tag{6.10}$$

Assuming $s - t \ge 4$ and taking $t = (s-r)/2$ we can express $-P_s^+ R_\lambda^+(z) P_r^-$ as

$$P_s^+ [P_t^-,\, R_\lambda^+(z)] P_r^- = P_s^+ R_\lambda^+(z) P_{t+1}^- [B_\lambda,\, P_t^-] P_{t-1}^+ R_\lambda^+(z) P_r^- \tag{6.11}$$

and we complete the proof estimating

$$||Q_{\delta,\lambda}^{r,t}(z)|| \le c'_\nu (s-t-1)^{-\nu} (t-1-r)^{-\nu} \le c''_\nu (s-r)^{-2\nu}.$$

6.4. Proof of Theorem 2.3

Let $J = D + B_\lambda$, $J_{\lambda',\lambda} = D + B_{\lambda',\lambda}$ and assume

$$b_{k,\lambda} = b_{k,\lambda',\lambda} = 0 \text{ if } d_k \ge \lambda + 6C\lambda^\delta. \tag{6.12}$$

We introduce

$$d^-_{k,\lambda',\lambda} = \min\{d_k,\, \lambda' - 6C\lambda^\delta\}$$

and denote

$$D^-_{\lambda',\lambda} := \min\{D,\, \lambda' - 6C\lambda^\delta\} = \mathrm{diag}(d^-_{k,\lambda',\lambda})_{k\in\mathbb{N}^*}, \tag{6.13}$$

$$J^-_{\lambda',\lambda} := D^-_{\lambda',\lambda} + B_{\lambda',\lambda} \tag{6.14}$$

Then $J^-_{\lambda',\lambda} - \lambda' \le -C\lambda^\delta$ and we introduce $\kappa\colon \mathbb{R}^2 \to \mathbb{R}$ satisfying

$$j \le \kappa(\lambda',\lambda) \implies d_j \le \lambda' - 6C\lambda^\delta, \tag{6.15}$$

hence $P^-_{\kappa(\lambda',\lambda)}(D - D^-_{\lambda',\lambda}) = 0$. Then we fix $\varepsilon > 0$ and consider

$$\kappa_\rho(\lambda',\lambda) = \kappa(\lambda',\lambda) - \rho\lambda^\varepsilon \tag{6.16}$$

with $\rho \ge 0$. In order to compare

$$R_{\lambda',\lambda}(z) := (J_{\lambda',\lambda} - z)^{-1}, \tag{6.17}$$

$$R^-_{\lambda',\lambda}(z) := (J^-_{\lambda',\lambda} - z)^{-1}, \tag{6.18}$$

we observe that reasoning similarly as in Section 6.3 we obtain

$$\sup_{z\in\Gamma'(\lambda')} \|P^+_s R^-_{\lambda',\lambda}(z) P^-_r\| \le c_\nu (s-r)^{-\nu} \text{ if } s > r \tag{6.19}$$

with $\Gamma'(\lambda') := \{z \in \mathbb{C} \mid \operatorname{Re} z > \lambda' - 2,\ 0 < |\operatorname{Im} z| < 2\}$ and we deduce

$$\|P^-_{\kappa_1(\lambda',\lambda)}(R^-_{\lambda',\lambda}(z) - R_{\lambda',\lambda}(z))\| \le c_\nu \lambda^{-\nu} |\operatorname{Im} z|^{-1} \text{ if } z \in \Gamma'(\lambda'), \tag{6.20}$$

$$\|P^-_{\kappa_2(\lambda',\lambda)} R_{\lambda',\lambda}(z) P^+_{\kappa_1(\lambda',\lambda)}\| \le c_\nu \lambda^{-\nu} |\operatorname{Im} z|^{-1} \text{ if } z \in \Gamma'(\lambda'), \tag{6.21}$$

similarly as in Section 4. Next we assume

$$B_\lambda - B_{\lambda',\lambda} = (B_\lambda - B_{\lambda',\lambda}) P^-_{\kappa_2(\lambda',\lambda)} = P^-_{\kappa_2(\lambda',\lambda)} (B_\lambda - B_{\lambda',\lambda}). \tag{6.22}$$

and reasoning as in Section 4 we obtain

$$||(R_\lambda(z) - R_{\lambda',\lambda}(z)) P^+_{\kappa_1(\lambda',\lambda)}|| \le c_\nu \lambda^{-\nu} |\operatorname{Im} z|^{-2} \text{ if } z \in \Gamma'(\lambda'). \tag{6.23}$$

Then taking g and $f_{\nu,\lambda',\lambda}$ as in Section 5 we obtain

$$\|P^-_{\kappa_1(\lambda',\lambda)}(g(\lambda^\nu(J^-_{\lambda',\lambda} - \tau)) - g(\lambda^\nu(J_{\lambda',\lambda} - \tau)))\| = \mathrm{O}(\lambda^{-\infty}) \tag{6.24}$$

and using

$$\operatorname{supp} f_{\nu,\lambda',\lambda} \cap \sigma(J^-_{\lambda',\lambda}) \subset [\lambda' - C\lambda^{-\nu},\ \lambda + C\lambda^{-\nu}] \cap (-\infty, \lambda' - C\lambda^\delta] = \varnothing$$

we deduce $f_{\nu,\lambda',\lambda}(J^-_{\lambda',\lambda}) = 0$ and

$$\|P^-_{\kappa_1(\lambda',\lambda)} f_{\nu,\lambda',\lambda}(J_{\lambda',\lambda})\| + \|P^-_{\kappa_1(\lambda',\lambda)} f_{\nu,\lambda',\lambda}(J_\lambda)\| = \mathrm{O}(\lambda^{-\infty}). \tag{6.25}$$

Finally, reasoning as in the proof of Lemma 5.2 we obtain

$$\|f_{\nu,\lambda',\lambda}(J_{\lambda',\lambda}) - f_{\nu,\lambda',\lambda}(J_\lambda)\| = \mathrm{O}(\lambda^{-\infty}) \tag{6.26}$$

and similarly as in the proof of Theorem 5.3 we obtain

$$\mathcal{N}(\lambda' + \lambda^{-\nu}, \lambda - \lambda^{-\nu}, J_{\lambda,\lambda'}) \le \mathcal{N}(\lambda', \lambda, J_\lambda) \le \mathcal{N}(\lambda' - \lambda^{-\nu}, \lambda + \lambda^{-\nu}, J_{\lambda,\lambda'}). \tag{6.27}$$

To complete the proof we observe that in (6.27) we can replace J_λ by J due to Theorem 5.3.

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