The short-wave model for the Camassa–Holm equation: a Riemann–Hilbert approach

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Abstract. We present the inverse scattering transform approach to the Cauchy problem on the line for the short-wave model for the Camassa–Holm equation

$$u_{txx} - 2u_x + 2u_x u_{xx} + u u_{xxx} = 0$$

in the form of an associated Riemann–Hilbert problem. This approach allows us to give a representation of the classical (smooth) solutions, describe their asymptotics as $t \to \infty$, and describe cuspons — non-smooth soliton solutions with a cusp.

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1. Introduction

We consider the partial differential equation

$$u_{txx} - 2\omega u_x + 2u_x u_{xx} + u u_{xxx} = 0, \tag{1.1}$$

where $\omega > 0$ is a parameter and $u \equiv u(x,t)$ is real-valued. This equation stems from the short-wave limit of the Camassa–Holm (CH) equation

$$u_t - u_{txx} + 2\omega u_x + 3u u_x = 2u_x u_{xx} + u u_{xxx}. \tag{1.2}$$

Indeed, introducing new space and time variables (x', t') and a scaling of u by

$$x' = \frac{x}{\varepsilon}, \qquad t' = \varepsilon t, \qquad u' = \frac{u}{\varepsilon^2}$$

where ε is a small positive parameter, then (1.1) is the leading term of (1.2) as $\varepsilon \to 0$.

Equation (1.1) is a model for short capillary waves propagating under the action of gravity [15, 5]. On the other hand, it is a member of the Dym hierarchy

[3, 1, 17]. For $\omega = 0$, (1.1) reduces to the Hunter–Saxton equation [16], so (1.1) is sometimes called the *modified Hunter–Saxton* (mHS) equation [18].

Equation (1.1) is (at least, formally) integrable: it possesses a Lax pair representation

$$\psi_{xx} = \lambda(m+\omega)\psi, \tag{1.3a}$$

$$\psi_t = \Big(\frac{1}{2\lambda} - u\Big)\psi_x + \frac{1}{2}u_x\psi, \tag{1.3b}$$

where $\psi \equiv \psi(x,t,\lambda)$ and

$$m := -u_{xx}. \tag{1.4}$$

A correspondence between the hierarchies of equation (1.1) and of the Korteweg–de Vries (KdV) equation is presented in [18] with the help of the Liouville transformation. In [2], weak piecewise solutions were studied using algebraic-geometric methods. In [19], the multi-cusp solutions of (1.1) — non-smooth soliton solutions with a cusp — were obtained taking a scaling limit in the Hirota-type formulas for the smooth multi-soliton solution of the Camassa–Holm equation.

In this paper we present a Riemann–Hilbert (RH) approach to equation (1.1). In Section 2 we study the initial value problem on the line $x \in (-\infty, \infty)$ assuming that the initial data $u(x,0) = u_0(x)$ are smooth and decay sufficiently fast as $|x| \to \infty$ and satisfy $-u_{0xx} + \omega > 0$ for all x. The solution of the Cauchy problem is presented in terms of the solution of an associated Riemann–Hilbert problem. The obtained representation then allows us, in Section 3, to apply the nonlinear steepest descent method for oscillatory Riemann–Hilbert problems and to obtain a detailed description for the leading term of the asymptotics of the Cauchy problem. In Section 4 we show how the cusp solitons, which were presented in [19] using a direct method, can be retrieved in the framework of our Riemann–Hilbert approach.

2. Riemann–Hilbert formalism

Without loss of generality, in what follows we assume that $\omega = 1$. We consider the Cauchy problem

$$u_{txx} - 2u_x + 2u_x u_{xx} + u u_{xxx} = 0, \qquad x \in (-\infty,\infty),\ t > 0, \tag{2.1a}$$

$$u(x,0) = u_0(x), \qquad x \in (-\infty,\infty), \tag{2.1b}$$

where

- $u_0(x)$ is smooth,
- $u_0(x)$ decays sufficiently fast as $|x| \to \infty$,
- $-u_{0xx}(x) + 1 > 0$ for all x.

Equation (2.1a) resembles the Camassa–Holm equation (1.2) in many aspects. Particularly, in analogy with the CH equation, one can show that $-u_{xx}(x,t) + 1 > 0$ for all (x,t).

For solving the Cauchy problem, we propose an inverse scattering formalism, where the solution is represented in terms of the solution of an associated Riemann–Hilbert problem in the complex plane of the spectral parameter. In the case of the CH equation, a similar approach was developed in [9]. One of the main advantages of such representation is the possibility to use it efficiently in studying the long-time behavior of the solution using the nonlinear steepest descent method [7, 8].

Similarly to the case of the Camassa–Holm equation [9], it is convenient to introduce the inverse scattering formalism for the Lax pair in the form of a system of first order matrix-valued linear equations, which allows controlling well the behavior of the dedicated (Jost) solutions of this system as functions of the spectral parameter.

2.1. Lax pairs

The coefficients of the original Lax pair (1.3) have singularities at $k=\infty$ and also at $k=0$. In order to have a good control on the behavior of eigenfunctions at $k=\infty$ and at $k=0$ we introduce new forms of (1.3), the first one appropriate at $k=\infty$, the second one at $k=0$.

Proposition 2.1. *The mHS equation* (2.1a) *is the compatibility condition of the system of* 2×2 *linear equations:*

$$\tilde\Phi_x + \mathrm{i}k\sqrt{m+1}\sigma_3\tilde\Phi = U\tilde\Phi, \tag{2.2a}$$

$$\tilde\Phi_t - \mathrm{i}k\left\{\frac{1}{2k^2} + u\sqrt{m+1}\right\}\sigma_3\tilde\Phi = V\tilde\Phi, \tag{2.2b}$$

where $\tilde\Phi \equiv \tilde\Phi(x,t,k)$,

$$m = -u_{xx}, \tag{2.3a}$$

$$U = \frac{m_x}{4(m+1)}\sigma_1, \tag{2.3b}$$

$$V = -\frac{u m_x}{4(m+1)}\sigma_1 - \frac{1}{4\mathrm{i}k}\left[\sqrt{m+1} + \frac{1}{\sqrt{m+1}} - 2\right]\sigma_3, \tag{2.3c}$$

and σ_1, σ_2, σ_3 *are the Pauli matrices*

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Since $U \equiv U(x,t,k)$ and $V \equiv V(x,t,k)$ are bounded at $k=\infty$, this Lax pair (2.2)-(2.3) is appropriate for controlling the behavior of its solutions for large k.

Proof. Let k be the spectral parameter defined by $\lambda = -k^2$, and let $\tilde\Phi \equiv \tilde\Phi(x,t,k)$ be the vector-valued function defined by

$$\tilde\Phi = G\begin{pmatrix} \psi \\ \psi_x \end{pmatrix}, \tag{2.4a}$$

where

$$G(x,t,k) = G_0(k)\begin{pmatrix}(m(x,t)+1)^{\frac{1}{4}} & 0\\ 0 & (m(x,t)+1)^{-\frac{1}{4}}\end{pmatrix} \tag{2.4b}$$

$$G_0(k) = \frac{1}{2}\begin{pmatrix}1 & -\frac{1}{\mathrm{i}k}\\ 1 & \frac{1}{\mathrm{i}k}\end{pmatrix}. \tag{2.4c}$$

Then the system (2.2)-(2.3) for $\tilde\Phi$ follows from the Lax pair (1.3) for ψ. □

Proposition 2.2. *The mHS equation* (2.1a) *is the compatibility condition of the system of* 2×2 *linear equations:*

$$\tilde\Phi_{0x} + \mathrm{i}k\sigma_3\tilde\Phi_0 = U_0\tilde\Phi_0, \tag{2.5a}$$

$$\tilde\Phi_{0t} + \frac{1}{2\mathrm{i}k}\sigma_3\tilde\Phi_0 = V_0\tilde\Phi_0, \tag{2.5b}$$

where $\tilde\Phi_0 \equiv \tilde\Phi_0(x,t,k)$,

$$m = -u_{xx}, \tag{2.6a}$$

$$U_0 = -\frac{\mathrm{i}k}{2}m(\mathrm{i}\sigma_2+\sigma_3), \tag{2.6b}$$

$$V_0 = \frac{u_x}{2}\sigma_1 + \mathrm{i}ku\left[\sigma_3 + \frac{m}{2}(\mathrm{i}\sigma_2+\sigma_3)\right]. \tag{2.6c}$$

Since $U_0 \equiv U_0(x,t,k)$ and $V_0 \equiv V_0(x,t,k)$ are bounded at $k=0$, this Lax pair (2.5)-(2.6) is appropriate for controlling the behavior of its solutions as $k\to 0$.

Moreover, $U_0(x,t,0) = 0$ for all (x,t). This property will be used in Section 2.4, in establishing the relationship between the solution of the associated Riemann–Hilbert problem and the solution of equation (2.1a).

Proof. Let k be the spectral parameter defined by $\lambda = -k^2$, and let $\tilde\Phi_0 \equiv \tilde\Phi_0(x,t,k)$ be the vector-valued function defined by

$$\tilde\Phi_0 = G_0\begin{pmatrix}\psi\\ \psi_x\end{pmatrix},$$

where G_0 is as in (2.4c). Then (2.5)-(2.6) for $\tilde\Phi_0$ follows from (1.3) for ψ. □

2.2. Jost solutions

Now assume that $u(x,t)$ is a solution of the Cauchy problem (2.1) and define the Jost solutions of the systems (2.2) and (2.5), taking into account that U, V, U_0, and V_0 all decay to 0 as $|x|\to\infty$.

2.2.1. The l.h.s. of (2.2) suggest introducing the function

$$p(x,t,k) = x - \int_x^\infty \left(\sqrt{m(\xi,t)+1} - 1\right)\mathrm{d}\xi - \frac{t}{2k^2}, \tag{2.7}$$

so that

$$p_x = \sqrt{m+1},$$
$$p_t = -\frac{1}{2k^2} - u\sqrt{m+1},$$

the last equality being a consequence of (2.1a). Then the 2×2 matrix-valued Jost solutions $\tilde{\Phi}_\pm$ are defined as

$$\tilde{\Phi}_\pm(x,t,k) := \Phi_\pm(x,t,k)\mathrm{e}^{-\mathrm{i}kp(x,t,k)\sigma_3},$$

where $\Phi_\pm$ satisfy the integral equations (equivalent to (2.2) with boundary conditions $\Phi_\pm \to I$ as $x \to \pm\infty$, I being the 2×2 identity matrix)

$$\Phi_\pm(x,t,k) = I + \int_{\pm\infty}^{x} \mathrm{e}^{-\mathrm{i}k\int_y^x \sqrt{m(\xi,t)+1}\mathrm{d}\xi\hat{\sigma}_3}(U\Phi_\pm)(y,t,\xi)\mathrm{d}y, \tag{2.8}$$

where $\mathrm{e}^{\hat{\sigma}_3}A := \mathrm{e}^{\sigma_3}A\mathrm{e}^{-\sigma_3}$ for a 2×2 matrix A.

It follows from (2.8) that the columns $\Phi_\pm^{(1)}$, $\Phi_+^{(2)}$ of those 2×2 matrix-valued functions $\Phi_\pm$ are analytic and bounded in the corresponding half-planes:

- $\Phi_-^{(1)}$ and $\Phi_+^{(2)}$ are analytic and bounded for $\operatorname{Im} k > 0$;
- $\Phi_+^{(1)}$ and $\Phi_-^{(2)}$ are analytic and bounded for $\operatorname{Im} k < 0$;
- they are continuous up to the real axis.

2.2.2. In order to control the behavior of Jost solutions as $k \to 0$, it is convenient to use the Lax pair (2.5). Let us define the Jost solutions $\tilde{\Phi}_{0\pm}$ of (2.5) by

$$\tilde{\Phi}_{0\pm}(x,t,k) := \Phi_{0\pm}(x,t,k)\mathrm{e}^{-\mathrm{i}kx-\frac{t}{2\mathrm{i}k}\sigma_3},$$

where $\Phi_{0\pm}$ satisfy the integral equations (equivalent to (2.5) with the boundary conditions $\Phi_{0\pm} \to I$ as $x \to \pm\infty$)

$$\Phi_{0\pm}(x,t,k) = I - \frac{\mathrm{i}k}{2}\int_{\pm\infty}^{x} \mathrm{e}^{-\mathrm{i}k(x-y)\hat{\sigma}_3}\left[m(y,t)(\mathrm{i}\sigma_2+\sigma_3)\Phi_{0\pm}(y,t,\xi)\right]\mathrm{d}y. \tag{2.9}$$

An important consequence of (2.9) is that the expansions in powers of k of the columns of $\Phi_{0\pm}$ have the form

$$\Phi_0(x,t,k) = I + \frac{\mathrm{i}k}{2}u_x(x,t)(\mathrm{i}\sigma_2+\sigma_3) - (\mathrm{i}k)^2u(x,t)\sigma_1 + \mathrm{O}(k^3), \quad k\to 0, \tag{2.10}$$

which is to be understood columnwise, $\Phi_-^{(1)}$, $\Phi_+^{(2)}$ in the upper half-plane $\operatorname{Im} k > 0$, $\Phi_+^{(1)}$, $\Phi_-^{(2)}$ in the lower half-plane $\operatorname{Im} k < 0$, with limiting paths transverse to the real axis.

2.2.3. Now we notice that the eigenfunctions $\tilde{\Phi}$ and $\tilde{\Phi}_0$, being related to the same Lax pair (1.3), must be related to each other as

$$\Phi_\pm(x,t,k) = G(x,t,k)G_0^{-1}(k)\Phi_{0\pm}\mathrm{e}^{-(\mathrm{i}kx+\frac{t}{2\mathrm{i}k}\sigma_3}C_\pm(k)\mathrm{e}^{\mathrm{i}kp(x,t,k)\sigma_3} \tag{2.11}$$

with $C_\pm(k)$ independent of x and t. Evaluating (2.11) as $x\to\pm\infty$ gives $C_+(k)\equiv I$ and $C_-(k)=\mathrm{e}^{\mathrm{i}kc\sigma_3}$, where

$$c = \int_{-\infty}^{\infty}\left(\sqrt{m(\xi,t)+1}-1\right)\mathrm{d}\xi$$

is conserved under the dynamics governed by (2.1a). Taking into account the definition of p and the equality $Q=GG_0^{-1}$, we arrive at

Proposition 2.3. *The functions $\Phi_\pm$ and $\Phi_{0\pm}$ are related as follows:*

$$\Phi_+(x,t,k) = Q(x,t)\Phi_{0+}(x,t,k)\mathrm{e}^{-\mathrm{i}k\int_x^\infty\left(\sqrt{m(\xi,t)}-1\right)\mathrm{d}\xi\sigma_3}, \tag{2.12a}$$

$$\Phi_-(x,t,k) = Q(x,t)\Phi_{0-}(x,t,k)\mathrm{e}^{\mathrm{i}k\int_{-\infty}^x\left(\sqrt{m(\xi,t)}-1\right)\mathrm{d}\xi\sigma_3}, \tag{2.12b}$$

where

$$Q(x,t) = \frac{1}{2}\begin{pmatrix} q+\frac{1}{q} & q-\frac{1}{q} \\ q-\frac{1}{q} & q+\frac{1}{q}\end{pmatrix} \tag{2.13a}$$

$$q(x,t) = (m(x,t)+1)^{\frac{1}{4}}. \tag{2.13b}$$

This proposition 2.3 together with (2.10) allows expressing the coefficients of the expansions in powers of k of $\Phi_\pm$ in terms of u.

2.3. Scattering matrix

Now define the scattering matrix $S(k)$ as a matrix relating the Jost solutions:

$$\Phi_+(x,t,k) = \Phi_-(x,t,k)\mathrm{e}^{-\mathrm{i}kp\sigma_3}S(k)\mathrm{e}^{\mathrm{i}kp\sigma_3}, \quad k\in\mathbb{R}. \tag{2.14}$$

Due to the symmetries of the underlying system (2.2) it has the form

$$S(k) = \begin{pmatrix} \bar{a}(k) & b(k) \\ \bar{b}(k) & a(k)\end{pmatrix}.$$

The scattering coefficient $a(k)$ is analytic in the half-plane $\operatorname{Im} k>0$: it can be expressed as the determinant of a matrix whose entries are analytic in $\operatorname{Im} k>0$:

$$a(k) = \begin{vmatrix}\Phi_-^{(1)} & \Phi_+^{(2)}\end{vmatrix}.$$

Combining (2.12)-(2.13) with (2.10), we obtain the expansion of $a(k)$ at $k=0$:

$$a(k) = 1+\mathrm{i}kc+(\mathrm{i}k)^2\frac{c^2}{2}+\mathrm{O}(k^3). \tag{2.15}$$

Proposition 2.4 (scattering coefficient). $a(k)\neq 0$ *for all k such that* $\operatorname{Im} k\geq 0$.

Proof. First, notice that since $\det S(k) \equiv 1$, $|a(k)|^2 = 1 + |b(k)|^2 > 0$ for all $k \in \mathbb{R}$. Second, a zero k_0 of $a(k)$ with $\operatorname{Im} k_0 > 0$ would correspond to an eigenvalue of the Dirac equation

$$\hat{\Phi}_y + \mathrm{i}k\sigma_3\hat{\Phi} = \frac{\widehat{m}_y}{4(\widehat{m}+1)}\sigma_1\hat{\Phi}. \tag{2.16}$$

This follows from the scattering relation (2.14) and the differential equation (2.2a) under the change of variables

$$\begin{aligned} &y = x - \int_x^\infty \left(\sqrt{m(\xi)+1} - 1\right) \mathrm{d}\xi, \\ &\hat{\Phi}(y,t,k) = \tilde{\Phi}(x,t,k), \\ &\widehat{m}(y,t) = m(x,t). \end{aligned}$$

But for real-valued m, (2.16) is self-adjoint and thus has no nonreal eigenvalues. □

2.4. Inverse scattering problem

Now return to the scattering relation (2.14) and consider it as an inverse scattering problem:

Problem (inverse scattering). *Given the scattering matrix $S(k)$, find 2×2 matrix-valued functions $\Phi_\pm(x,t,k)$ satusfying the scattering relation* (2.14).

This inverse problem is directly related to the Cauchy problem (2.1). Indeed, setting in (2.8) and (2.14) $t = 0$ shows that given $u(x,0)$, $S(k)$ can be calculated via the solutions of (2.8) with $t = 0$, where $m(x,0)$ is to be replaced by $m_0(x) = -u_{0xx}(x)$. On the other hand, as follows from Proposition 2.3 and (2.10), $\Phi_\pm(x,t,k)$ evaluated at $k = 0$ contain all information needed for obtaining $u(x,t)$ for all t.

It is convenient to reformulate the inverse scattering problem as a Riemann–Hilbert problem, where the jump contour is $\mathbb{R}$ and the jump matrix is expressed in terms of $S(k)$.

2.5. Riemann–Hilbert formulation

Still assuming u to be a solution of equation (2.1a), define a piecewise (respective to $\mathbb{R}$) analytic, 2×2 matrix-valued function M by

$$M(x,t,k) = \begin{cases} \begin{pmatrix} \frac{\Phi_-^{(1)}(x,t,k)}{a(k)} & \Phi_+^{(2)}(x,t,k) \end{pmatrix}, & \operatorname{Im} k > 0, \\ \begin{pmatrix} \Phi_+^{(1)}(x,t,k) & \frac{\Phi_-^{(2)}(x,t,k)}{\overline{a(\bar{k})}} \end{pmatrix}, & \operatorname{Im} k < 0. \end{cases} \tag{2.17}$$

Then for each $x \in \mathbb{R}$ and $t \geq 0$, $M(x,t,k)$ can be characterized by the following properties:

(i) *Jump relation across* $\mathbb{R}$. The boundary values $M_\pm$ of M as k approaches $\mathbb{R}$ from the side $\pm \operatorname{Im} k > 0$ are connected by the jump matrix J:

$$M_-(x,t,k) = M_+(x,t,k)J(x,t,k), \quad k \in \mathbb{R}, \tag{2.18a}$$

with

$$J(x,t,k) = \mathrm{e}^{-\mathrm{i}kp(x,t,k)\sigma_3} J_0(k)\mathrm{e}^{\mathrm{i}kp(x,t,k)\sigma_3} \tag{2.18b}$$

$$J_0(k) = \begin{pmatrix} 1 & \bar{r}(k) \\ -r(k) & 1-|r(k)|^2 \end{pmatrix}, \tag{2.18c}$$

$$r(k) = -\frac{\bar{b}(k)}{a(k)}. \tag{2.18d}$$

(ii) *Behavior at* $k=\infty$.

$$M(x,t,k) = I + \mathrm{O}(1/k) \text{ as } k \to \infty. \tag{2.19}$$

(iii) *Symmetry.*

$$\overline{M(x,t,\bar{k})} = M(x,t,-k) = \sigma_1 M(x,t,k)\sigma_1. \tag{2.20}$$

An important characteristic property of M is its behavior as $k \to 0$. From (2.10), (2.12a), and (2.15) it follows that

$$\begin{aligned} M(x,t,k) = Q(x,t)\Bigg[I + \mathrm{i}k\left\{\frac{u_x(x,t)}{2}(\mathrm{i}\sigma_2+\sigma_3) - N(x,t)\sigma_3\right\} \\ + (\mathrm{i}k)^2\left\{-u(x,t)\sigma_1 - \frac{u_x(x,t)}{2}N(x,t)(I-\sigma_1) + \frac{1}{2}N^2(x,t)I\right\} \\ + \mathrm{O}(k^3)\Bigg], \qquad k \to 0, \end{aligned} \tag{2.21}$$

where

$$N(x,t) = \int_x^\infty \left(\sqrt{m(\xi,t)+1} - 1\right) \mathrm{d}\xi.$$

The above characterization of M suggests the following interpretation.

- Properties (2.18)-(2.20) suggest introducing a family of Riemann–Hilbert problems of piecewise analytic factorization of a matrix given for $k \in \mathbb{R}$ in terms of the reflection coefficient $r(k)$.

In order to have explicit dependence of these Riemann–Hilbert problems on parameters, we introduce a new scale (y,t) by

$$y = y(x,t) = x - \int_x^\infty \left(\sqrt{m(\xi,t)+1} - 1\right) \mathrm{d}\xi \equiv x - N(x,t). \tag{2.22}$$

This makes the jump matrix J explicitly dependent on the parameters (y,t):

$$J(x,t,k) = \hat{J}(y(x,t),t,k), \tag{2.23a}$$

$$\hat{J}(y,t,k) = \mathrm{e}^{-\mathrm{i}k\left(y-\frac{t}{2k^2}\right)\sigma_3} J_0(k)\mathrm{e}^{\mathrm{i}k\left(y-\frac{t}{2k^2}\right)\sigma_3}. \tag{2.23b}$$

- Property (2.21) is used in order to obtain u — in the scale (y,t) — as well as the relationship between the original variables (x,t) and the new variables in terms of the solution of the RH problem above evaluated at $k=0$.

Concerning the last property, let us introduce the row vector-valued function

$$\begin{pmatrix}\mu_1 & \mu_2\end{pmatrix} = \begin{pmatrix}1 & 1\end{pmatrix} M.$$

Then from (2.21) we have as $k \to 0$:

$$\mu_1(x,t,k) = q(x,t)\Big(1 - \mathrm{i}kN(x,t) + (\mathrm{i}k)^2\Big\{-u(x,t) + \frac{1}{2}N^2(x,t)\Big\} + \mathrm{O}(k^3)\Big),$$

$$\mu_2(x,t,k) = q(x,t)\Big(1 + \mathrm{i}kN(x,t) + (\mathrm{i}k)^2\Big\{-u(x,t) + \frac{1}{2}N^2(x,t)\Big\} + \mathrm{O}(k^3)\Big),$$

which yields

$$\frac{\mu_1}{\mu_2}(x,t,k) = 1 - 2\mathrm{i}kN(x,t) + \mathrm{O}(k^2), \tag{2.24a}$$

$$\mu_1(x,t,k)\mu_2(x,t,k) = \sqrt{m(x,t)+1}\left(1 + 2k^2u(x,t) + \mathrm{O}(k^3)\right). \tag{2.24b}$$

Equations (2.24) show that the vector-valued function μ contains all necessary information for representing (parametrically) the solution of the Cauchy problem (2.1) in terms of the solution of a vector-valued RH problem:

Theorem 2.5 (Riemann–Hilbert formulation). *Let $r(k)$ be the reflection coefficient associated with the initial data $u_0(x)$ via the scattering relation* (2.14) *connecting the solutions of the integral equations* (2.8) *taken at $t = 0$, with $m(x,0)$ replaced by $m_0(x) = -u_{0xx}(x)$.*

Let the following family of (2×1) Riemann–Hilbert problems $\mathrm{RH}_{y,t}$, indexed by $\{(y,t) \mid y \in \mathbb{R},\ t > 0\}$: Given $r(k)$ for $k \in \mathbb{R}$, find a vector-valued function

$$\hat\mu(y,t;k) = \left(\hat\mu_1(y,t;k) \quad \hat\mu_2(y,t;k)\right), \quad k \in \mathbb{C}$$

satisfying the following four conditions:

(a) Analyticity*: $\hat\mu(\,\cdot\,,\cdot\,;k)$ is analytic in the two open half-planes $\operatorname{Im} k > 0$ and $\operatorname{Im} k < 0$, and continuous up to the boundary $\operatorname{Im} k = 0$.*

(b) Jump relation*: The two limiting values*

$$\hat\mu_\pm(\,\cdot\,,\cdot\,;k) = \lim_{\varepsilon\to+0}\hat\mu_\pm(\,\cdot\,,\cdot\,;k\pm\varepsilon), \quad k \in \mathbb{R}$$

are related by:

$$\hat\mu_+(y,t;k) = \hat\mu_-(y,t;k)\hat J(y,t;k), \quad k \in \mathbb{R}, \tag{2.25a}$$

where the jump matrix is

$$\hat J(y,t;k) = \mathrm{e}^{(-\mathrm{i}ky - \frac{t}{2\mathrm{i}k}\sigma_3}J_0(k)\mathrm{e}^{(\mathrm{i}ky + \frac{t}{2\mathrm{i}k}\sigma_3} \tag{2.25b}$$

with

$$J_0(k) = \begin{pmatrix}1-|r(k)|^2 & -\overline{r(k)}\\ r(k) & 1\end{pmatrix}. \tag{2.25c}$$

(c) Normalization*:*

$$\hat\mu(y,t;k) \to (1 \quad 1) \quad \textit{as } k \to \infty. \tag{2.25d}$$

(d) Symmetry*:*

$$\hat\mu_1(\,\cdot\,,\cdot\,;-k) = \hat\mu_2(\,\cdot\,,\cdot\,;k). \tag{2.25e}$$

Then

(i) *each Riemann–Hilbert problem* $\mathrm{RH}_{y,t}$ *has a unique solution,*

(ii) *the solution* $u(x,t)$ *of the initial value problem* (2.1a) *can be expressed, in parametric form, in terms of the solutions of these Riemann–Hilbert problems:*

$$u(x,t) = \hat{u}(y(x,t),t), \tag{2.26}$$

where

$$x(y,t) = y - \lim_{k\to 0}\left(\frac{\hat{\mu}_1(y,t;k)}{\hat{\mu}_2(y,t;k)} - 1\right)\frac{1}{2\mathrm{i}k}, \tag{2.27a}$$

$$\hat{u}(y,t) = \lim_{k\to 0}\left(\frac{\hat{\mu}_1(y,t;k)\hat{\mu}_2(y,t;k)}{\hat{\mu}_1(y,t;0)\hat{\mu}_2(y,t;0)} - 1\right)\frac{1}{2k^2}. \tag{2.27b}$$

Remark 2.6. Alternatively, $\hat{u}(y,t)$ can be expressed as

$$\hat{u}(y,t) = \frac{\partial x}{\partial t}(y,t). \tag{2.28}$$

Moreover, from (2.24b) it follows that $m \equiv -u_{xx}$ can be obtained (in the variables (y,t)) as

$$\widehat{m}(y,t) = \hat{\mu}_1^2(y,t,0)\hat{\mu}_2^2(y,t,0) - 1. \tag{2.29}$$

Proof. (i) Since the analytic structure of the RH problem (2.25) exactly follows that in the case of the KdV or CH equation [4, 7] — only the dependence on the parameters (y,t) is different — its unique solvability is a consequence of the same "vanishing lemma" as in the case of the KdV equation.

(ii) Formulas (2.27), as well as (2.28) and (2.29), follow directly from (2.22) and (2.24), taking into account the uniqueness of solution of the RH problem (2.25). □

3. Long-time asymptotics

The existence of the representation of the solution u to the Cauchy problem (2.1) in terms of a solution of the associated Riemann–Hilbert problem (2.25) makes it possible to study the long-time behavior of the former problem via the long-time analysis of the latter, applying the nonliner steepest descent method introduced by Deift and Zhou [12] (the application of this method to the Camassa–Holm equation can be found in [7, 8, 10]. A key feature of this method is the deformation of the original RH problem according to the "signature table" for the phase function θ in the jump matrix $\hat{J}$ written in the form

$$\hat{J}(y,t;k) = \mathrm{e}^{-\mathrm{i}t\theta(\hat{\zeta},k)\sigma_3}J_0(k)\mathrm{e}^{\mathrm{i}t\theta(\hat{\zeta},k)\sigma_3}, \tag{3.1}$$

where

$$\theta(\hat{\zeta},k) = \hat{\zeta}k - \frac{1}{2k}, \tag{3.2}$$

$$\hat{\zeta} := \frac{y}{t}. \tag{3.3}$$

The signature table is the distribution of signs of $\operatorname{Im}\theta(\hat{\zeta},k)$ in the k-plane, depending on the values of $\hat{\zeta}$. For equation (2.1a),

$$\operatorname{Im}\theta(\hat{\zeta},k) = \operatorname{Im} k\Big(\hat{\zeta} + \frac{1}{2|k|^2}\Big),$$

and thus two cases are to be distinguished:

1. *Case* $\hat{\zeta} \geq 0$. In this case the set $\{k \mid \operatorname{Im}\theta(\hat{\zeta},k) = 0\}$ concides with the real axis $\operatorname{Im} k = 0$ and $\pm\operatorname{Im}\theta > 0$ for $\pm\operatorname{Im} k > 0$.
2. *Case* $\hat{\zeta} < 0$. In this case

$$\{k \mid \operatorname{Im}\theta(\hat{\zeta},k) = 0\} = \{k \mid \operatorname{Im} k = 0\} \cup \{k \mid |k| = (2\hat{\zeta})^{-1/2}\}.$$

The distribution of signs of $\operatorname{Im}\theta$ is shown in Figure 1.

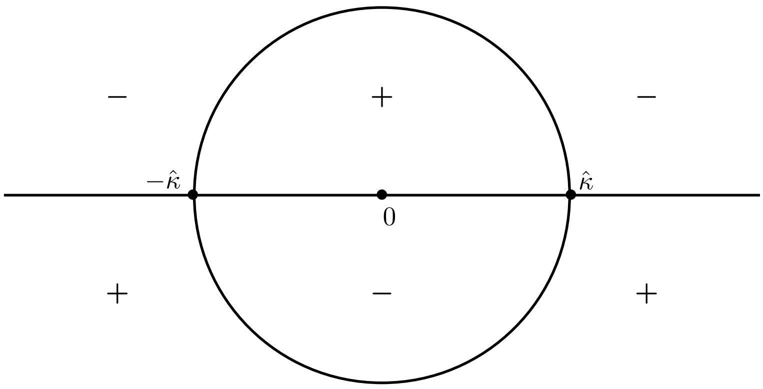

FIGURE 1. Sign distribution of $\operatorname{Im}\theta$ in the k-plane in the case $\hat{\zeta} \geq 0$

Accordingly, the long-time behavior of u turns out to be qualitatively different in two ranges of values of

$$\zeta \equiv \frac{x}{t}\,. \tag{3.4}$$

3.1. Range $\zeta > \varepsilon$

In a domain of the form $\zeta > \varepsilon$ for any $\varepsilon > 0$, the signature table suggests the use of the following factorization of the jump matrix for all $k \in \mathbb{R}$:

$$\hat{J} = \begin{pmatrix} 1 & -\bar{r}(\bar{k})\mathrm{e}^{-2\mathrm{i}t\theta} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ r(k)\mathrm{e}^{2\mathrm{i}t\theta} & 1 \end{pmatrix}. \tag{3.5}$$

Indeed, the triangular factors in (3.5) can be absorbed into the new RH problem for $\mu^{(1)}(y,t;k)$ in exactly the same way as in the case, for example, of the CH equation [7, 8]:

$$\mu_1 = \begin{cases} \hat\mu \begin{pmatrix} 1 & 0 \\ -r(k)\mathrm{e}^{2\mathrm{i}t\theta} & 1 \end{pmatrix}, & 0 < \operatorname{Im} k < \varepsilon, \\ \hat\mu \begin{pmatrix} 1 & -\bar r(\bar k)\mathrm{e}^{-2\mathrm{i}t\theta} \\ 0 & 1 \end{pmatrix}, & -\varepsilon < \operatorname{Im} k < 0, \\ \hat\mu, & \text{otherwise.} \end{cases}$$

This reduces the RH problem to that with exponentially decaying (in t) to the identity matrix jump matrix. Since this RH problem is holomorphic (there is no residue conditions), its solution decays fast to I and consequently $\hat u(y,t)$ decays fast to 0 while the y approaches fast x.

3.2. Range $\zeta < -\varepsilon$

In a domain of the form $\zeta < -\varepsilon$ for any $\varepsilon > 0$, the signature table dictates the use of two factorizations. Let $\pm\hat\kappa$ be the points where the distribution of signs is changing:

$$\hat\kappa = \frac{1}{\sqrt{2|\hat\zeta|}}\,. \tag{3.6}$$

(i) For $k \in (-\hat\kappa, \hat\kappa)$ we consider again the factorization (3.5)

$$\hat J = \begin{pmatrix} 1 & -\bar r(\bar k)\mathrm{e}^{-2\mathrm{i}t\theta} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ r(k)\mathrm{e}^{2\mathrm{i}t\theta} & 1 \end{pmatrix}.$$

(ii) For $k \in (-\infty, -\hat\kappa) \cup (\hat\kappa, \infty)$ we consider a factorization with triangular factors in reverse order:

$$\hat J = \begin{pmatrix} 1 & 0 \\ \frac{r(k)}{1-|r(k)|^2}\mathrm{e}^{2\mathrm{i}t\theta} & 1 \end{pmatrix} \begin{pmatrix} 1-|r(k)|^2 & 0 \\ 0 & \frac{1}{1-|r(k)|^2} \end{pmatrix} \begin{pmatrix} 1 & -\frac{\bar r(\bar k)}{1-|r(k)|^2}\mathrm{e}^{-2\mathrm{i}t\theta} \\ 0 & 1 \end{pmatrix} \tag{3.7}$$

Similarly to the previous case, the appropriate sequence of deformations of the RH problem is the same as in the case of the CH equation, so we will follow it giving details mainly for items specific to the considered equation.

The deformations involve the removal of the diagonal factor in (3.7) and the consequent absorption of the triagonal factors, leading, after an appropriate rescaling, to a model RH problem on the contour consisting of two crosses centered at $k = \pm\hat\kappa$, see [7, 8], which finally leads to the asymptotics in the form of modulated decaying (of the order $\mathrm{O}(t^{-1/2})$) oscillations. The diagonal term is removed introducing $\mu^{(1)} = \hat\mu\delta^{\sigma_3}$, where

$$\delta(k;\hat\zeta) = \exp\left\{\frac{1}{2\pi\mathrm{i}}\left(\int_{-\infty}^{-\hat\kappa} + \int_{\hat\kappa}^{\infty}\right)\log(1-|r(s)|^2)\frac{\mathrm{d}s}{s-k}\right\}. \tag{3.8}$$

solves a scalar RH problem whose jump condition is

$$\delta_+ = \delta_-(1-|r(k)|^2)$$

across the contour $(-\infty, -\hat\kappa) \cup (\hat\kappa, \infty)$.

The triangular factors are absorbed into the RH problem for $\mu^{(2)}$:

$$\mu^{(2)} = \begin{cases} \mu^{(1)} \begin{pmatrix} 1 & 0 \\ -r\delta^{-2}\mathrm{e}^{2\mathrm{i}t\theta} & 1 \end{pmatrix}, & \operatorname{Im} k > 0,\ k \text{ near } (-\hat\kappa, \hat\kappa), \\ \mu^{(1)} \begin{pmatrix} 1 & \frac{\bar r}{1-|r|^2}\delta_+^2\mathrm{e}^{-2\mathrm{i}t\theta} \\ 0 & 1 \end{pmatrix}, & \operatorname{Im} k > 0,\ k \text{ near } \mathbb{R}\setminus[-\hat\kappa, \hat\kappa], \\ \mu^{(1)} \begin{pmatrix} 1 & -\bar r\delta^2\mathrm{e}^{-2\mathrm{i}t\theta} \\ 0 & 1 \end{pmatrix}, & \operatorname{Im} k < 0,\ k \text{ near } (-\hat\kappa, \hat\kappa), \\ \mu^{(1)} \begin{pmatrix} 1 & 0 \\ \frac{r}{1-|r|^2}\delta_-^{-2}\mathrm{e}^{2\mathrm{i}t\theta} & 1 \end{pmatrix}, & \operatorname{Im} k < 0,\ k \text{ near } \mathbb{R}\setminus[-\hat\kappa, \hat\kappa]. \end{cases} \tag{3.9}$$

Now we notice that since the solution $\hat u$ is related to the solution of the RH problem evaluated at $k=0$, it is affected by the above two transformations above. However, the latter one turns out not to affect the terms in the expansion of the solution of the RH problem at $k=0$ up at least to the terms of order k^2 and thus it does not really affect $\hat u$. This is due to the following fact.

Proposition 3.1 (reflection coefficient). $r(k) = \mathrm{O}(k^3)$ *as* $k \to 0$.

Proof. Follows from (2.15) and from the identity $|r(k)|^2 = 1 - |a(k)|^{-2}$. □

Now, in order to reduce the RH problem for $\mu^{(2)}$, as $t\to\infty$, to a model problem whose solution can be given explicitly in terms of parabolic cylinder functions, see [12, 8], the leading term of the factor $\delta(k)\mathrm{e}^{-\mathrm{i}t\theta(k)}$ as $k\to\pm\hat\kappa$ is to be evaluated. One has

$$\delta(k) = \left(\frac{\hat\kappa - k}{\hat\kappa + k}\right)^{-\mathrm{i}h} \mathrm{e}^{\chi(k)} \tag{3.10}$$

with

$$h \equiv h(\hat\kappa) = -\frac{1}{2\pi}\log\bigl(1 - |r(\hat\kappa)|^2\bigr), \tag{3.11}$$

$$\chi(k) = -\frac{1}{2\pi\mathrm{i}}\int_{\mathbb{R}\setminus[-\hat\kappa,\hat\kappa]} \log|k-s|\,\mathrm{d}\log(1-|r(s)|^2), \tag{3.12}$$

and

$$\theta(k) = -\frac{1}{\hat\kappa} + \frac{1}{\hat\kappa^3}(k-\hat\kappa)^2(1+\mathrm{O}(k-\hat\kappa)). \tag{3.13}$$

Therefore, introducing the scaled spectral variable $\hat k$ by

$$k - \hat\kappa = \frac{\hat k}{\sqrt{2\hat\kappa^{-3}t}},$$

the factor $\delta(k)\mathrm{e}^{-\mathrm{i}t\theta(k)}$ can be approximated as

$$\delta(k)\mathrm{e}^{-\mathrm{i}t\theta(k)} \approx \tilde\delta\hat k^{\mathrm{i}h}\mathrm{e}^{-\mathrm{i}\hat k^2/4}, \tag{3.14}$$

where

$$\tilde\delta = \left(\frac{8t}{\hat\kappa}\right)^{\frac{\mathrm{i}h}{2}} \mathrm{e}^{\frac{\mathrm{i}t}{\hat\kappa}}\mathrm{e}^{\chi(\hat\kappa)}. \tag{3.15}$$

Similarly for k near $-\hat{\kappa}$.

The model RH problem for $\mu^{(2)}$, which is formulated in the $\hat{k}$-plane on the cross centered at $\hat{k}=0$ and has a constant (in $\hat{k}$) jump matrix, can be solved solved in terms of parabolic cylinder functions, see, e.g., [12, 8]. It gives rise to a quantity

$$\beta = r(\hat{\kappa})\frac{\Gamma(-\mathrm{i}h)h}{\sqrt{2\pi}\,\mathrm{e}^{\mathrm{i}\pi/4}\,\mathrm{e}^{-\pi h/2}} \tag{3.16}$$

(where Γ is the Euler Gamma function), in terms of which the main term, as $t\to\infty$, of the solution $\hat{\mu}$ of the original RH problem can be expressed: tracing back the deformations of the RH problem gives, similarly to [8],

$$\begin{aligned}\hat{\mu}_1(k)\delta^{-1}(k) &= \left(1+\frac{1}{\sqrt{2\hat{\kappa}^{-3}t}}\left[\frac{\mathrm{i}\beta\tilde{\delta}^2}{k+\hat{\kappa}}+\frac{-\mathrm{i}\bar{\beta}\tilde{\delta}^{-2}}{-k+\hat{\kappa}}\right]\right)\left(1+\mathrm{O}(t^{-\varepsilon})\right),\\ \hat{\mu}_2(k)\delta(k) &= \left(1+\frac{1}{\sqrt{2\hat{\kappa}^{-3}t}}\left[\frac{-\mathrm{i}\bar{\beta}\tilde{\delta}^{-2}}{k+\hat{\kappa}}+\frac{\mathrm{i}\beta\tilde{\delta}^2}{-k+\hat{\kappa}}\right]\right)\left(1+\mathrm{O}(t^{-\varepsilon})\right).\end{aligned} \tag{3.17}$$

In view of (2.27), in order to obtain u from the solution of the RH problem, we need expansions, as $k\to 0$, of $\delta(k)$ (up to the order $\mathrm{O}(k)$) as well as of the r.h.s. of (3.17) (up to the order $\mathrm{O}(k^2)$). From (3.8) it follows that

$$\delta(k) = 1-\frac{\mathrm{i}k}{\pi}\int_{\hat{\kappa}}^{\infty}\frac{\log(1-|r(s)|^2)}{s^2}\mathrm{d}s+\mathrm{O}(k^2), \tag{3.18}$$

while the formulas (3.17) reduce to

$$\begin{aligned}\hat{\mu}_1(k)\delta^{-1}(k) &= 1+\sqrt{\frac{2\hat{\kappa}}{t}}A_R-\mathrm{i}\sqrt{\frac{2}{\hat{\kappa}t}}A_I k+\sqrt{\frac{2}{\hat{\kappa}^3 t}}A_R k^2+\mathrm{O}(k^3t^{-1/2-\varepsilon}),\\ \hat{\mu}_2(k)\delta(k) &= 1+\sqrt{\frac{2\hat{\kappa}}{t}}A_R+\mathrm{i}\sqrt{\frac{2}{\hat{\kappa}t}}A_I k+\sqrt{\frac{2}{\hat{\kappa}^3 t}}A_R k^2+\mathrm{O}(k^3t^{-1/2-\varepsilon}),\end{aligned} \tag{3.19}$$

where

$$\begin{aligned}A_R &= \mathrm{Re}\big(\mathrm{i}\beta\tilde{\delta}^2\big),\\ A_I &= \mathrm{Im}\big(\mathrm{i}\beta\tilde{\delta}^2\big).\end{aligned} \tag{3.20}$$

Consequently we have

$$\frac{\hat{\mu}_1(k)}{\hat{\mu}_2(k)} = 1-2\mathrm{i}k\left(\sqrt{\frac{2}{\hat{\kappa}t}}A_I+\int_{\hat{\kappa}}^{\infty}\frac{\log(1-|r(s)|^2)}{s^2}\,\mathrm{d}s\right)+\mathrm{O}(k^2t^{-1/2-\varepsilon}).$$

Thus, in view of (2.27a), y and x are asymptotically related for large t by

$$\begin{aligned}y &= x+\Delta(\hat{\kappa})+\mathrm{o}(1),\\ \Delta(\hat{\kappa}) &= \int_{\hat{\kappa}}^{\infty}\frac{\log(1-|r(s)|^2)}{s^2}\mathrm{d}s.\end{aligned} \tag{3.21}$$

Hence, the domains $\hat{\zeta}<-\varepsilon$ and $\zeta<-\varepsilon$ coincide asymptotically.

The asymptotics for $\hat{u}$ comes from the expansion

$$\hat{\mu}_1(k)\hat{\mu}_2(k) = 1 + \sqrt{\frac{8\hat{\kappa}}{t}}A_R + \sqrt{\frac{8}{\hat{\kappa}^3 t}}A_R k^2 + \mathrm{O}(k^3 t^{-1/2-\varepsilon}),$$

which, in view of (2.27b), gives

$$\hat{u}(y,t) = \sqrt{\frac{2}{\hat{\kappa}^3 t}}A_R(1+\mathrm{o}(1)) \tag{3.22}$$

as well as (see (2.29))

$$\widehat{m}(y,t) = \sqrt{\frac{32\hat{\kappa}}{t}}A_R(1+\mathrm{o}(1)). \tag{3.23}$$

Finally, substituting (3.11), (3.12), (3.15), (3.16) and (3.20) into (3.22), introducing

$$\kappa = \frac{1}{\sqrt{2|\zeta|}} = \left(2\,\frac{|x|}{t}\right)^{-1/2},$$

and taking into account the shift Δ, see (3.21), when passing from y to x, one obtains

$$u(x,t) = \sqrt{\frac{2h(\kappa)}{\kappa^3 t}}\cos\left\{\frac{2}{\kappa}t + h(\kappa)\log\left(\frac{8t}{\kappa}\right) + \phi_0(\kappa)\right\}(1+\mathrm{o}(1)), \tag{3.24}$$

where

$$\begin{aligned}\phi_0(\kappa) = {} & \frac{\pi}{4} + \arg(r(\kappa)) + \arg\Gamma(\mathrm{i}h) \\ & + \frac{1}{\pi}\int_{\mathbb{R}\setminus[-\kappa,\kappa]} \log|k-s|\,\mathrm{d}\log(1-|r(s)|^2) \\ & + 2\kappa\int_{\kappa}^{\infty}\frac{\log(1-|r(s)|^2)}{s^2}\,\mathrm{d}s. \end{aligned} \tag{3.25}$$

Theorem 3.2 (asymptotics). *Let $u(x,t)$ be the solution of the Cauchy problem* (2.1). *Then the behavior of u as $t\to\infty$ is described as follows. Let ε be any small positive number.*

(i) *In the domain $\zeta \equiv x/t > \varepsilon$, $u(x,t)$ tends to* 0 *with fast decay.*

(ii) *In the domain $\zeta \equiv x/t < -\varepsilon$, $u(x,t)$ exhibits decaying (of the order* $\mathrm{O}(t^{-1/2})$*) modulated oscillations given by* (3.24)*, where the coefficients $h(\kappa)$ and $\phi_0(\kappa)$ are functions of $\kappa = 1/\sqrt{2\zeta}$ given in terms of the associated reflection coefficient $r(k)$ by* (3.11) *and* (3.25)*.*

Remark 3.3. The sectors of different asymptotic behavior match, as $\varepsilon\to 0$, through the fast decay. Indeed, as $x/t \to 0-$, $\kappa\to\infty$ and $h(\kappa)\to 0$ and thus the amplitude in (3.24) decays faster. This is in contrast with the case of the Camassa–Holm equation, where the matching between the soliton (fast decay) sector and the oscillatory sector is expressed in terms of the Painlevé II transcendents (and possibly modulated elliptic waves), see [6, 10].

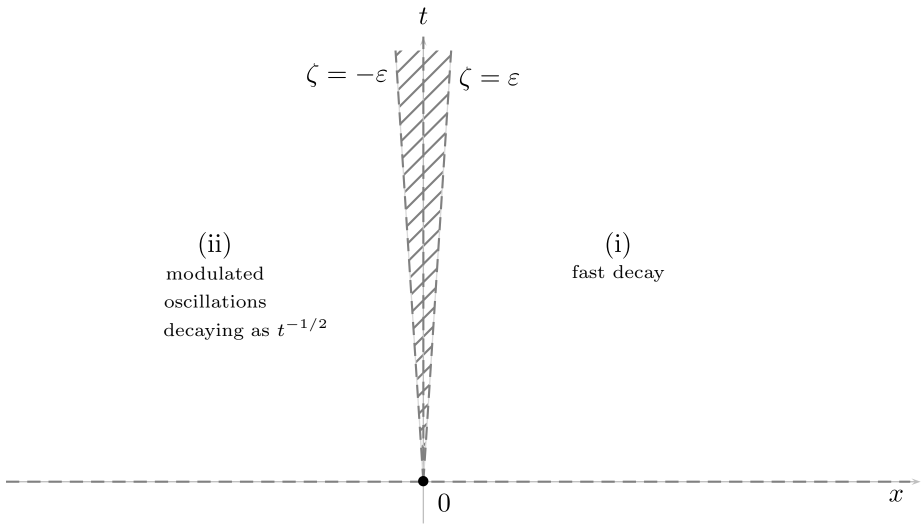

FIGURE 2. The different regions of the (x,t)-half-plane, $\zeta = \frac{x}{t}$

4. Cuspons

4.1. Introduction

In the context of the inverse scattering transform method, nonreal zeros of the scattering coefficient $a(k)$ correspond to eigenvalues of the "x-equation" of the Lax pair and consequently to multi-soliton solutions of the underlying nonlinear equation (see, e.g., [14, 9]. These solutions dominate in the long-time asymptotics of solutions of initial boundary value problems with general initial conditions [13, 7]. Moreover, its construction, in the framework of the RH method, corresponds to solving a meromorphic RH problem, with trivial jump conditions ($r(k) \equiv 0$ and thus $J \equiv I$) and with nontrivial residue conditions at zeros of $a(k)$. Since in this case solving the RH problem reduces to solving a system of linear algebraic equations, this construction leads to explicit formulas for multi-solitons.

In the case of the mHS equation (2.1a), we see that $a(k)$ has no zeros, which indicates that this equation has no classical (smooth) soliton solutions. This is in contrast with the Camassa–Holm equation (1.2), where a finite number of zeros may exist, all lying on an interval of the imaginary axis. These zeros correspond to smooth solitons moving to the right with velocities greater than a critical one depending on ω [11, 9]: for $\omega = 1$, the interval is $(0, \frac{\mathrm{i}}{2})$, and the soliton velocities are greater than 2.

On the other hand, the mHS equation (2.1a) has non-classical (non-smooth) soliton-like travelling wave solutions. In [19], they were constructed taking an appropriate scaling limit in formulas for smooth, multi-soliton solutions of the Camassa–Holm equation. These solutions are called "cuspons" because they have a cusp singularity.

4.2. Cuspons via Riemann–Hilbert approach

Let us show that these cuspon solutions can be obtained in the framework of our Riemann–Hilbert approach. To do this we set $r(k) \equiv 0$ and "force" $a(k)$ to have zeros in the upper half-plane. Formally this means that we consider a meromorphic Riemann–Hilbert problem, with no jumps along contours but with residue conditions at these zeros. This does not contradict the properties of "true" $a(k)$ presented in Section 2 because, as we will see, the associated solutions of (2.1a) are not classical ones.

To fix ideas, we assume that there is only one such zero; the generalization to the case of multiple zeros is straightforward, assuming that all zeros are purely imaginary and that the corresponding residue constants have an appropriate sign: for $\nu_j > 0$, $\gamma_j < 0$,

$$\operatorname{Res}_{k=\mathrm{i}\nu_j} \hat{\mu}_1(y,t;k) = \mathrm{i}\gamma_j \mathrm{e}^{-2\nu_j y - \frac{t}{\nu_j}} \hat{\mu}_2(y,t;\mathrm{i}\nu_j). \tag{4.1}$$

• In the case of a single residue condition,

$$\operatorname{Res}_{k=\mathrm{i}\nu} \hat{\mu}_1(y,t;k) = \mathrm{i}\gamma \mathrm{e}^{-2\nu y - \frac{t}{\nu}} \hat{\mu}_2(y,t;\mathrm{i}\nu), \qquad \nu > 0, \tag{4.2}$$

the solution of the associated RH problem obviously has the form

$$\begin{pmatrix} \hat{\mu}_1 & \hat{\mu}_2 \end{pmatrix} = \begin{pmatrix} \frac{k-B}{k-\mathrm{i}\nu} & \frac{k+B}{k+\mathrm{i}\nu} \end{pmatrix}. \tag{4.3}$$

Here $B = B(y,t)$ is to be calculated using the residue condition (4.2), which gives for B the equation

$$\mathrm{i}\nu - B = g(\mathrm{i}\nu + B), \tag{4.4a}$$

$$g(y,t) = \frac{\gamma}{2\nu} \mathrm{e}^{-2\nu\left(y + \frac{t}{2\nu^2}\right)}, \tag{4.4b}$$

and thus

$$B(y,t) = \mathrm{i}\nu \frac{1 - g(y,t)}{1 + g(y,t)}. \tag{4.5}$$

• In case of multiple residue conditions, instead of (4.4a), we would have a system of linear equations.

Now, in accordance with the procedure presented in Theorem 2.5, we evaluate (4.3) as $k \to 0$ using (4.4b) and (4.5); this gives

$$\begin{aligned} \hat{\mu}_1(y,t,k) &= \frac{B}{\mathrm{i}\nu}\Big(1 + k\Big\{\frac{1}{\mathrm{i}\nu} - \frac{1}{B}\Big\} + k^2\Big\{-\frac{1}{\nu^2} + \frac{\mathrm{i}}{\nu B}\Big\} + \mathrm{O}(k^3)\Big), \\ \hat{\mu}_2(y,t,k) &= \frac{B}{\mathrm{i}\nu}\Big(1 - k\Big\{\frac{1}{\mathrm{i}\nu} - \frac{1}{B}\Big\} + k^2\Big\{-\frac{1}{\nu^2} + \frac{\mathrm{i}}{\nu B}\Big\} + \mathrm{O}(k^3)\Big). \end{aligned} \tag{4.6}$$

Then (2.27) and (2.29) give, repectively

$$x(y,t) = y - \frac{2}{\nu}\frac{1+g(y,t)}{1-g(y,t)}\,, \tag{4.7a}$$

$$\hat{u}(y,t) = \frac{2}{\nu^2}\frac{g(y,t)}{(1-g(y,t))^2}\,, \tag{4.7b}$$

$$\widehat{m}(y,t) = \left(\frac{1-g(y,t)}{1+g(y,t)}\right)^4 - 1. \tag{4.7c}$$

4.3. Comments

Now we make some observations.

4.3.1. Formulas (4.4b) and (4.7) indicate that in order to have real-valued $u(x,t)$, the parameters ν and γ must be real.

4.3.2. Let $\nu > 0$. If $\gamma > 0$, then $g > 0$ and, moreover, $g = 1$ for certain values of y and t, which implies that the associated solution $u(x,t)$ is growing to infinity at certain finite x and t.

4.3.3. If $\gamma < 0$ and $\nu > 0$, then $g < 0$ and thus $u(x,t)$ is bounded for all x and t; moreover, it decays exponentially as $x \to \pm\infty$ for every fixed t. On the other hand, its derivative at its minimum (when g approaches -1) is unbounded (and so is for the second derivative, as indicated by (4.7c)). Introducing

$$\phi(y,t) = -\nu\Big(y + \frac{t}{2\nu^2} - y_0\Big),$$

$$y_0 = \frac{1}{2\nu}\log\left(-\frac{\gamma}{2\nu}\right),$$

so that $g = -\mathrm{e}^{2\phi}$, we can rewrite (4.7) in the form

$$x(y,t) = y + \frac{1}{\nu}\tanh\phi(y,t) + \frac{1}{\nu}, \tag{4.8a}$$

$$\hat{u}(y,t) = -\frac{1}{2\nu^2}\frac{1}{\cosh^2\phi(y,t)}, \tag{4.8b}$$

$$\widehat{m}(y,t) = \coth^4\phi(y,t) - 1. \tag{4.8c}$$

Thus the cuspon solution is negative and is moving to the left with velocity $v = -\frac{1}{2\nu^2}$, the cusp corresponding to $\phi = 0$. The three formulas (4.8) are consistent with those obtained in [19], except that the sign of the cuspon is different, due to the fact that in [19] the mHS depends on a parameter κ which is related to ω by $\omega = -\kappa^2$. Indeed, formulas (17a)–(17c) in [19] for the 1-cusp soliton solution correspond to (4.8) with $\kappa = 1$, if one sets $\nu = -\frac{k_1}{2}$ and makes the change of variables $(u,t) \mapsto (-u,-t)$.

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