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ON UNIQUENESS PROBLEMS RELATED TO THE FOKKER–PLANCK–KOLMOGOROV EQUATION FOR MEASURES

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Dedicated to Professor N. V. Krylov on occasion of his 70th birthday

We survey recent results related to uniqueness problems for parabolic equations for measures. We consider equations of the form $\partial_t\mu = L^*\mu$ for bounded Borel measures on $\mathbb{R}^d\times[0,T)$, where L is a second order elliptic operator, for example, $Lu = \Delta_x u+(b, \nabla_x u)$, and the equation is understood as the identity

$$\int (\partial_t u + Lu)\, d\mu = 0$$

for all smooth functions u with compact support in $\mathbb{R}^d \times (0,T)$. Our study are motivated by equations of such a type, namely, the Fokker–Planck–Kolmogorov equations for transition probabilities of diffusion processes. Solutions are considered in the class of probability measures and in the class of signed measures with integrable densities. We present some recent positive results, give counterexamples, and formulate open problems. Bibliography: 34 *titles.*

1 Introduction

This paper is a survey of recent results on the uniqueness of probability and integrable solutions to the Cauchy problem for the Fokker–Planck–Kolmogorov equation. We give sufficient uniqueness conditions and construct examples of nonuniqueness.

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Let $T > 0$, and let

$$Lu(x,t) = \sum_{i,j=1}^{d} a^{ij}(x,t)\partial_{x_i}\partial_{x_j}u(x,t) + \sum_{i=1}^{d} b^i(x,t)\partial_{x_i}u(x,t),$$

where a^{ij} andb^i are Borel functions on $\mathbb{R}^d \times (0,T)$ such that $A = (a^{ij})_{1\leqslant i,j\leqslant d}$ is a nonnegative symmetric matrix.

We say that a Borel locally finite measure μ on $\mathbb{R}^d \times (0,T)$ (possibly signed) is defined by a family of Borel locally finite measures $(\mu_t)_{0<t<T}$ on $\mathbb{R}^d$ if for every bounded Borel set B the mapping $t \mapsto \mu_t(B)$ is measurable and $\mu(dxdt) = \mu_t(dx)\,dt$. We will deal with measures of bounded variation. The variation of μ is denoted by $|\mu|$.

A Borel locally finite measure μ on $\mathbb{R}^d \times (0,T)$ defined by a family of measures $(\mu_t)_{0<t<T}$ on $\mathbb{R}^d$ satisfies the *Fokker–Planck–Kolmogorov equation*

$$\partial_t\mu = L^*\mu \tag{1.1}$$

if $a^{ij}, b^i \in L^1_{\text{loc}}(|\mu|, \mathbb{R}^d \times (0,T))$ and for every function $u \in C_0^\infty(\mathbb{R}^d \times (0,T))$

$$\int_0^T \int_{\mathbb{R}^d} \big[\partial_t u(x,t) + Lu(x,t)\big]\, d\mu_t\, dt = 0.$$

If a solution μ is given by a density ϱ with respect to the Lebesgue measure on $\mathbb{R}^d \times (0,T)$, then Equation (1.1) can be written as an equation for the density:

$$\partial_t\varrho = \partial_{x_i}\partial_{x_j}(a^{ij}\varrho) - \partial_{x_i}(b^i\varrho).$$

Throughout the paper, we assume that A satisfies the following condition:

(H1) for every ball $U \subset \mathbb{R}^d$ there exist numbers $\gamma = \gamma(U) > 0$ and $M = M(U) > 0$ such that

$$(A(x,t)y,y) \geqslant \gamma|y|^2, \quad \|A(x,t)\| \leqslant M$$

for all $(x,t) \in U \times [0,T]$ and $y \in \mathbb{R}^d$.

In the case of nonnegative measures, condition (H1) ensures the existence of densities (cf. [1]).

Let ν be a locally finite Borel measure on $\mathbb{R}^d$. We say that a Borel locally finite measure μ defined by a family of Borel locally finite measures $(\mu_t)_{0<t<T}$ satisfies the *initial condition* $\mu\big|_{t=0} = \nu$ if for every $\zeta \in C_0^\infty(\mathbb{R}^d)$

$$\lim_{t\to 0} \int_{\mathbb{R}^d} \zeta(x)\, d\mu_t = \int_{\mathbb{R}^d} \zeta(x)\, d\nu.$$

Thus, we study the uniqueness problem for solutions to the Cauchy problem

$$\partial_t\mu = L^*\mu, \quad \mu\big|_{t=0} = \nu, \tag{1.2}$$

and we are interested in the two classes of solutions: probability and integrable.

A *probability solution* is a solution μ defined by a family of probability measures $(\mu_t)_{0<t<T}$, i.e., $\mu_t \geqslant 0$ and $\mu_t(\mathbb{R}^d) = 1$. The set of all probability solutions $\mu = \mu_t(dx)\,dt$ such that $|b| \in L^2(\mu, U \times [0,T])$ for every ball $U \subset \mathbb{R}^d$ is denoted by $\mathscr{P}_\nu$.

An *integrable solution* is a solution μ defined by a family of finite measures $(\mu_t)_{0<t<T}$ such that $\sup_t \|\mu_t\| < \infty$. If a measure μ is given by a density ϱ with respect to the Lebesgue measure on $\mathbb{R}^d \times (0,T)$, then the latter condition can be written as

$$\sup_{t\in(0,T)} \int_{\mathbb{R}^d} |\varrho(x,t)|\,dx < \infty.$$

The set of all integrable solutions $\mu = \mu_t(dx)\,dt$ such that for every ball $U \subset \mathbb{R}^d$ one has $|b| \in L^p(|\mu|, U \times [0,T])$ for some $p > d+2$ is denoted by $\mathscr{I}_\nu$.

We emphasize that, in the definitions of the classes $\mathscr{P}_\nu$ and $\mathscr{I}_\nu$, we assume that

$$|b| \in L^2(\mu, U \times [0,T])$$

in the case of a probability solution and

$$|b| \in L^p(|\mu|, U \times [0,T]) \quad \text{with} \quad p > d+2$$

in the case of an integrable solution. So, if the drift b is bounded on $U \times [0,T]$, these conditions are automatically fulfilled. Moreover, in place of these conditions, one can assume that the solution μ is given by a density ϱ and $\varrho \in L^r(U \times [0,T])$, $b \in L^s(U \times [0,T])$, where $2/r + 1/s = 1$ in the case of the class $\mathscr{P}_\nu$ and $p?r + 1/s = 1$ in the case of the class $\mathscr{I}_\nu$.

We present several methods of proving the fact that the set $\mathscr{P}_\nu$ consists of at most one element. In addition, we construct an example of an operator L with unit matrix A and an infinitely differentiable vector field b such that the Cauchy problem (1.2) has an infinite-dimensional simplex of probability solutions. We also find sufficient conditions for the uniqueness of integrable solutions. In particular, we show that the uniqueness conditions for the class I_ν differ essentially from those for the class $\mathscr{P}_\nu$. For example, let $A = I$ (the unit matrix), and let b be a locally bounded vector field. Then for the uniqueness of a probability solution to the Cauchy problem it suffices to have a function $V \in C^2(\mathbb{R}^d)$ with $\lim\limits_{|x|\to\infty} V(x) = +\infty$ and $|\nabla V(x)| \leqslant C_1$ such that $LV(x,t) \leqslant C_2$, while for the uniqueness of an integrable solution the inequality $LV(x,t) \geqslant -C_2$ is sufficient. In the case of a radial function V, such conditions actually mean that for the uniqueness of a probability solution the quantity $(b(x,t),x)$ should not tend too quickly to $+\infty$ and for the uniqueness of an integrable solution $(b(x,t),x)$ should not tend too quickly to $-\infty$. Such a function V is called a *Lyapunov function*.

Conditions involving Lyapunov functions are well known in probability theory, for example, the Hasminskii condition [2] for the existence of global solutions to stochastic equations with unbounded coefficients. Sufficient conditions for the existence of a probability solution to the Cauchy problem (1.2) in terms of Lyapunov functions were obtained in [3, 4] and sufficient conditions for uniqueness were found in [5, 6]. In [7, 8], sufficient conditions for the uniqueness in the class $C([0,T], L^1(\mathbb{R}^d))$ were expressed in terms of the behavior of the function $(b(x),x)$ as $|x| \to \infty$. In [9], sufficient conditions for the uniqueness of integrable solutions were obtained in terms of Lyapunov functions.

The uniqueness problem in various classes for solutions to parabolic equations, in particular, for the Fokker–Planck–Kolmogorov equation was actively studied for several decades. In the classical works of Tychonoff [10] and Widder [11], the uniqueness of solutions to the Cauchy problem for the heat equation was established in the class of functions growing not faster than $e^{C|x|^2}$ and in the class of nonnegative functions. In addition, some examples of nonuniqueness were constructed there. Sufficient conditions for uniqueness in the classes of integrable solutions and nonnegative solutions and the Tychonoff class for general second order parabolic equations in divergence form were obtained by Aronson and Besala [12, 13] and, in nondivergence form, by Friedman (cf. for example, [14]). The existence and uniqueness of integrable solutions to the Cauchy problem for the Fokker–Planck–Kolmogorov equation was studied in [7, 8, 15, 16]. Surveys of recent results on the uniqueness of nonnegative solutions to the Cauchy problem for parabolic equations can be found in [17, 18]. The papers [15, 19, 20] are devoted to the Fokker–Planck–Kolmogorov equation with degenerate matrices A.

In this paper, survey the recent results obtained in [5, 6, 9] and give new sufficient conditions for the uniqueness of probability solutions in the case where the functions a^{ij} belong to the class VMO_x.

In some papers (cf., for example, [16]), the definition of a solution $\mu = \mu_t(dx)\,dt$ (probability or integrable) to the Cauchy problem somewhat differs from the above definition. Namely, it is assumed that for every function $\zeta \in C_0^\infty(\mathbb{R}^d)$ the mapping

$$t \mapsto \int\limits_{\mathbb{R}^d} \zeta(x)\, d\mu_t$$

is continuous on $[0, T)$ and for every $t \in [0, T)$

$$\int\limits_{\mathbb{R}^d} \zeta(x)\, d\mu_t = \int\limits_{\mathbb{R}^d} \zeta(x)\, d\nu + \int\limits_0^t \int\limits_{\mathbb{R}^d} L\zeta(x, s)\, d\mu_s\, ds.$$

Certainly, in this case, one has to assume that $a^{ij}, b^i \in L^1(|\mu|, U \times [0, T])$ for every ball U. The following lemma clarifies the relation between these two definitions of a solution.

Lemma 1.1. *Let $\mu = \mu_t(dx)\,dt$ be a solution to the Cauchy problem* (1.2) *such that*

$$\sup_{t\in(0,1)} \|\mu_t\| < \infty.$$

Suppose that $a^{ij}, b^i \in L^1(|\mu|, U \times [0, T])$ for every ball $U \subset \mathbb{R}^d$. Then for every function $\varphi \in C_b(\mathbb{R}^d \times [0, T]) \bigcap C_b^{1,2}(\mathbb{R}^d \times (0, T))$

$$\int\limits_{\mathbb{R}^d} \varphi(x, t)\, d\mu_t = \int\limits_{\mathbb{R}^d} \varphi(x, 0)\, d\nu + \int\limits_0^t \int\limits_{\mathbb{R}^d} \partial_t\varphi + L\varphi\, d\mu_s\, ds \quad \textit{for a.a. } t \in [0, T]. \tag{1.3}$$

Proof. It suffices to prove this equality in the case $\varphi(x, t) = 0$ for $|x| > R$ for some numbers $R > 0$ and all $t \in [0, T]$. Let $\eta \in C_0^\infty((0, T))$. By definition,

$$\int\limits_0^1 \int\limits_{\mathbb{R}^d} \partial_t(\varphi\eta) + L(\varphi\eta)\, d\mu_t\, dt = 0.$$

Thus,

$$-\int_0^1 \eta'(t) \int_{\mathbb{R}^d} \varphi(x,t)\, d\mu_t\, dt = \int_0^1 \eta(t) \int_{\mathbb{R}^d} \partial_t\varphi + L\varphi\, d\mu_t\, dt.$$

Therefore, the function

$$t \to \int_{\mathbb{R}^d} \varphi(x,t)\, d\mu_t$$

has an absolutely continuous version and

$$\frac{d}{dt}\int_{\mathbb{R}^d} \varphi(x,t)\, d\mu_t = \int_{\mathbb{R}^d} \partial_t\varphi + L\varphi\, d\mu_t.$$

Then for some number $C \in \mathbb{R}$

$$\int_{\mathbb{R}^d} \varphi(x,t)\, d\mu_t = C + \int_0^t \int_{\mathbb{R}^d} \partial_s\varphi + L\varphi\, \mu_s\, ds \quad \text{for a.a. } t \in [0,T].$$

We observe that $\varphi(x,t)$ converges uniformly to $\varphi(x,0)$ as $t \to 0$. In addition, we have

$$\lim_{t\to 0}\int_{\mathbb{R}^d} \varphi(x,0)\, \mu_t(\,dx) = \int_{\mathbb{R}^d} \varphi(x,0)\, \nu(dx).$$

Therefore,

$$C = \int_{\mathbb{R}^d} \varphi(x,0)\, d\nu.$$

□

Remark 1.2. Let I be the set of all points t for which the equality (1.3) holds. Then for all $\tau, t \in I$, $\tau < t$,

$$\int_{\mathbb{R}^d} \varphi(x,t)\, d\mu_t = \int_{\mathbb{R}^d} \varphi(x,\tau)\, d\mu_\tau + \int_\tau^t \int_{\mathbb{R}^d} \partial_t\varphi + L\varphi\, d\mu_s\, ds.$$

Indeed, it suffices to subtract the equality (1.3) for τ from an analogous equality for t.

Remark 1.3. If $\varphi(\,\cdot\,,t) = \psi \in C_b^2(\mathbb{R}^d)$ for all $t \in [0,T]$, then we can write (1.3) as follows:

$$\int_{\mathbb{R}^d} \psi(x)\, d\mu_t = \int_{\mathbb{R}^d} \psi(x)\, d\nu + \int_0^t \int_{\mathbb{R}^d} L\psi(x,s)\, d\mu_s\, ds \quad \text{for a.a. } t \in [0,T]. \tag{1.4}$$

Moreover, if J_ψ^μ is the set of all points $t \in [0,T]$ where the equality (1.4) holds, then J_ψ^μ is a full measure set in $[0,T]$ and the following mapping is continuous on J_ψ^μ:

$$t \mapsto \int_{\mathbb{R}^d} \psi(x)\, \mu_t(dx).$$

Remark 1.4. Let $T \in J^{\mu}_{\varphi(\,\cdot\,,T)}$. Then the equality (1.3) is fulfilled with $t = T$. Indeed, $\varphi(x,t)$ converges uniformly to $\varphi(x,T)$ as $t \to T$. Let I be the set of all points $t \in [0,T]$ where the equality (1.3) holds. Let $t_n \in J^{\mu}_{\varphi(\,\cdot\,,T)} \bigcap I$, and let $\lim\limits_{n\to\infty} t_n = T$. Then

$$\lim_{n\to\infty} \int\limits_{\mathbb{R}^d} \varphi(x,t_n)\,d\mu_{t_n} = \int\limits_{\mathbb{R}^d} \varphi(x,T)\,d\mu_T$$

and the equality (1.3) is fulfilled for every t_n. Letting $n \to \infty$, we obtain (1.3) with $t = T$.

The Fokker–Planck–Kolmogorov equation arises naturally in the study of diffusion processes. We consider several examples.

Example 1.5 (the classical definition of a diffusion)**.** The analytic theory of diffusion processes goes back to the celebrated work of Kolmogorov [21], where differential equations for transition densities were investigated. Let $U(x,\varepsilon) = \{y\colon\ |x-y| < \varepsilon\}$, and let $V(x,\varepsilon) = \{y\colon\ |x-y| > \varepsilon\}$. We recall that a Markov process in $\mathbb{R}^d$ with transition probability $P(s,x,t,B)$ is called a *diffusion* if the following conditions are fulfilled (cf., for example, [22]):

(i) for all $\varepsilon > 0$, $t \geqslant 0$ and $x \in \mathbb{R}^d$

$$\lim_{h\to 0} h^{-1} P(t,x,t+h,V(x,\varepsilon)) = 0,$$

(ii) for some $\varepsilon > 0$ and all $t \geqslant 0$, $x \in \mathbb{R}^d$

$$\lim_{h\to 0} h^{-1} \int\limits_{U(x,\varepsilon)} (y-x) P(t,x,t+h,dy) = b(x,t),$$

(iii) for some $\varepsilon > 0$ and all $t \geqslant 0$, $x, z \in \mathbb{R}^d$

$$\lim_{h\to 0} h^{-1} \int\limits_{U(x,\varepsilon)} (y-x,z)^2 P(t,x,t+h,dy) = 2(A(x,t)z,z).$$

Suppose that all the listed limit relationships are fulfilled locally uniformly in x and the functions a^{ij}, b^i are locally bounded. Then it is known (cf., for example, [22, Chapter 1, §1, Theorem 7]) that the transition probabilities satisfy the Fokker–Planck–Kolmogorov equation (1.1) in the above sense. Let us recall the proof. Let $f \in C_0^\infty(\mathbb{R}^d)$. Then

$$\frac{d}{dt} \int\limits_{\mathbb{R}^d} f(y) P(s,x,t,\,dy) = \lim_{h\to 0} h^{-1} \Biggl(\int\limits_{\mathbb{R}^d} f(y) P(s,x,t+h,\,dy) - \int\limits_{\mathbb{R}^d} f(z) P(s,x,t,\,dz) \Biggr).$$

Applying the Kolmogorov–Chapman equation, we get

$$\frac{d}{dt} \int\limits_{\mathbb{R}^d} f(y) P(s,x,t,\,dy) = \lim_{h\to 0} \int\limits_{\mathbb{R}^d} P(s,x,t,dz) h^{-1} \int\limits_{\mathbb{R}^d} (f(y)-f(z)) P(t,z,t+h,dy).$$

Using conditions (i)–(iii) and the Taylor expansion for f, we obtain the equality

$$\lim_{h\to 0} h^{-1} \int\limits_{\mathbb{R}^d} (f(y)-f(z)) P(t,z,t+h,dy) = a^{ij}(z,t)\partial_{z_i}\partial_{z_j} f(z) + b^i(z,t)\partial_{z_i} f(z).$$

Since the convergence as $h \to 0$ is uniform in z, we have

$$\frac{d}{dt}\int_{\mathbb{R}^d} f(y)P(s,x,t,\,dy) = \lim_{h\to 0}\int_{\mathbb{R}^d} \bigl(a^{ij}(z,t)\partial_{z_i}\partial_{z_j}f(z) + b^i(z,t)\partial_{z_i}f(z)\bigr)P(s,x,t,dz).$$

So, we have proved that the transition probabilities satisfy Equation (1.1). In addition, condition (i) yields that for every function $\zeta \in C_0^\infty(\mathbb{R}^d)$

$$\lim_{h\to 0}\int_{\mathbb{R}^d} \zeta(y)P(s,x,s+h,dy) = \zeta(x),$$

i.e., $P(s,x,t,dy)$ satisfies the condition $P\big|_{s=t} = \delta_x$, where δ_x is the Dirac measure at $x \in \mathbb{R}^d$.

Let ν be a finite Borel measure on $\mathbb{R}^d$, and let

$$\mu_t(dx) = \int_{\mathbb{R}^d} P(s,y,t,dx)\nu(dy).$$

It is readily verified that the measure $\mu = \mu_t(dx)\,dt$ satisfies the Cauchy problem for Equation (1.1) with the initial condition $\mu\big|_{t=s} = \nu$.

One can consider diffusion processes in a broader sense, for instance, almost surely continuous Markov processes in $\mathbb{R}^d$ such that their transition probabilities $P(s,x,t,dy)$ satisfy Equation (1.1) with the initial condition $P\big|_{t=s} = \delta_x$. Such processes are called *quasidiffusions*.

Let us note that since the distribution of a Markov process is completely determined by its initial distribution and transition probabilities, the uniqueness of a probability solution to the Cauchy problem for the Fokker–Planck–Kolmogorov equation implies the weak uniqueness of a diffusion process whose transition probabilities solve the Fokker–Planck equation.

Example 1.6 (martingale problems)**.** Assume that the coefficients a^{ij} and b^i are locally bounded. Let $C([0,+\infty))$ be the space of continuous functions on $[0,+\infty)$. Let $\mathscr{F}_t^s$ denote the minimal σ-algebra containing all sets of the form $\{x \in C([0,+\infty))\colon\ x(\tau) \in B\}$, where $\tau \in [s,t]$ and B is a Borel set in $\mathbb{R}^d$. Let $\mathscr{F}_\infty^s$ denote the minimal σ-algebra containing all classes $\mathscr{F}_t^s$ with $t \geqslant s$. Following [23], we say that a probability measure $P_{s,z}$ on the space $C([0,+\infty))$, where $(z,s) \in \mathbb{R}^d \times [0,T)$, is a *solution to the martingale problem for the operator* L if

(i) $P_{s,z}(x(s) = z) = 1$,

(ii), for every function $f \in C_0^\infty(\mathbb{R}^d)$ the expression

$$f(x(t)) - f(x(s)) - \int_s^t Lf(x(\tau))\,d\tau$$

is a martingale with respect to $(P_{s,z}, \mathscr{F}_t^s)$.

Let $\mathbb{E}_{s,z}$ denote the expectation with respect to the measure $P_{s,z}$. By the definition of $P_{s,z}$ and the properties of martingales, for every $t \in [s,T]$ we have

$$\mathbb{E}_{s,z}f(x(t)) = \mathbb{E}_{s,z}f(x(s)) + \int_s^t \mathbb{E}_{s,z}Lf(x(\tau))\,d\tau.$$

We set $\mu_t(B) = P_{s,z}(x\colon\ x(t) \in B)$. Then the change of variables formula yields

$$\int\limits_{\mathbb{R}^d} f(x)\, d\mu_t = \int\limits_{\mathbb{R}^d} f(x)\, d\mu_s + \int\limits_s^t \int\limits_{\mathbb{R}^d} Lf(x,\tau)\, d\mu_\tau\, d\tau.$$

Note also that $\mu_s = \delta_z$. Thus, the measure $\mu = \mu_t(dx)\, dt$ satisfies the Cauchy problem for Equation (1.1) with the initial condition $\mu\big|_{t=s} = \delta_z$.

The proof of the uniqueness of a solution to the martingale problem consists usually of two steps. At the first step, for a sufficiently large class of functions f, for example, $f \in C_0^\infty(\mathbb{R}^d)$, one proves the equality

$$\mathbb{E}^1_{s,z} f(x(t)) = \mathbb{E}^2_{s,z} f(x(t)) \tag{1.5}$$

for every $t \in [s,T]$, where $\mathbb{E}^1_{s,z}$ and $\mathbb{E}^1_{s,z}$ are the expectations with respect to two solutions to the martingale problem $P^1_{s,z}$ and $P^2_{s,z}$. The equality (1.5) is equivalent to the coincidence of $P^1_{s,z}$ and $P^2_{s,z}$ on all sets of the form $\{x\colon\ x(t) \in B\}$, where B is an arbitrary Borel set and $t \in [s,T]$. The second step is concerned with deriving the equality $P^1_{s,z} = P^2_{s,z}$ from (1.5).

We observe that the equality (1.5) is equivalent to the equality of the solutions μ^1 and μ^2 to the Cauchy problem for Equation (1.1) with the initial condition $\mu\big|_{t=s} = \delta_z$ corresponding to two solutions to the martingale problem $P^1_{s,z}$ and $P^2_{s,z}$. So, the proof of (1.5) often reduces to that of the uniqueness of a probability solution to the Cauchy problem for the Fokker–Planck–Kolmogorov equation. Moreover, as shown in [16] in the case of bounded coefficients, any probability solution to Equation (1.5) is given by some solution to the martingale problem.

If the coefficients are unbounded, the situation becomes more complicated. If one derives the uniqueness of a probability solution to the Fokker–Planck–Kolmogorov equation from the uniqueness of a solution to the martingale problem, then it is necessary to prove, at least under the same assumptions providing the uniqueness for the martingale problem, that every solution to the Cauchy problem is generated by a solution to the martingale problem. It is also clear that if we have managed to prove the uniqueness for the Cauchy problem under the same assumptions under which it holds for the martingale problem, then a unique probability solution will be automatically generated by a solution to the martingale problem.

Finally, we note that the proof of the well-posedness of the martingale problem with unbounded coefficients often employs the following observation (cf. [23, Theorem 10.1.1]), which, in a sense, localizes the uniqueness problem.

Suppose that Ω is a domain in $\mathbb{R}^d \times [0,+\infty)$, $(s,z) \in \Omega$, and $\tau = \inf\{t \geqslant s\colon (t, x(t)) \notin \Omega\}$. Let the martingale problem with bounded coefficients a^{ij} and b^i be well posed, and let $P_{s,z}$ be its solution. Let functions $\overline{a}^{ij}$ and $\overline{b}^i$ be locally bounded. If $\overline{a}^{ij} = a^{ij}$ and $\overline{b}^i = b^i$ on Ω and the measure $\overline{P}_{s,z}$ is some solution to the martingale problem with coefficients $\overline{a}^{ij}$ and $\overline{b}^i$, then $P_{s,z} = \overline{P}_{s,z}$ on $\mathscr{F}^s_\tau$, where $\mathscr{F}^s_\tau$ is the σ-algebra of events $A \in \mathscr{F}^s_\infty$ such that $A \bigcap \{\tau \leqslant t\} \in \mathscr{F}^s_t$. There is no analog of this property for probability solutions to the Fokker–Planck–Kolmogorov equation, which complicates finding sufficient conditions for the uniqueness of solutions to the Cauchy problem.

Example 1.7 (kernels of semigroups)**.** Suppose that a^{ij} and b^i depend only on x and belong to the class $C^\infty(\mathbb{R}^d)$. Suppose also that the matrix $A = (a^{ij})$ satisfies condition (H1). Following [24] (cf. also [25]), we construct a semigroup on $C_b(\mathbb{R}^d)$ whose generator extends L. We set $U_R = U(0,R)$. It is well known that for every function $f \in C_b(\mathbb{R}^d)$ there exists a unique function u_R such that

$$\begin{aligned}
&\partial_t u_R = L u_R, \quad x \in U_R,\ t > 0,\\
&u_R(x,t) = 0, \quad x \in \partial U_R,\ t > 0,\\
&u_R(x,0) = f(x), \quad x \in \overline{U_R}.
\end{aligned}$$

Moreover, the function u_R is given by a semigroup $\{T_t^R\}_{t\geqslant 0}$ on $C_b(U_R)$, i.e., $u_R = T_t^R f$. According to the maximum principle, $\max_x |u_R(x,t)| \leqslant \max_x |f(x)|$ and, if $f \geqslant 0$, then $u_{R_1}(x,t) \leqslant u_{R_2}(x,t)$ whenever $x \in U_R$ and $R < R_1 < R_2$. We set

$$T_t f(x) = \lim_{R\to\infty} u_R(x,t).$$

As is shown in [24, 25], the constructed semigroup possesses the following properties. Let $f \in C_b(\mathbb{R}^d)$ and $u(x,t) = T_t f(x)$. Then

(i) $u \in C^\infty(\mathbb{R}^d \times (0,+\infty))$ and $\partial_t u = Lu$,

(ii) $T_t f(x) \to f(x)$ as $t \to 0$ uniformly on compacts sets in $\mathbb{R}^d$,

(iii) if $f \in C_0^\infty(\mathbb{R}^d)$, then $\partial_t T_t f = T_t L f$,

(iv) there exists a positive function $p \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d \times (0,+\infty))$ such that

$$T_t f(x) = \int\limits_{\mathbb{R}^d} p(x,y,t) f(y)\, dy$$

By these properties, the function $(y,t) \to p(x,y,t)$ is a subprobability kernel and satisfies the equation $\partial_t p = L^* p$. Let ν be a finite Borel measure on $\mathbb{R}^d$. Then the measure $\mu = \mu_t(dx)\,dt$, where

$$\mu_t(B) = \int\limits_{\mathbb{R}^d} \nu(dx) \int\limits_B p(x,y,t)\, dy,$$

is a solution to the Cauchy problem for Equation (1.1) with the initial condition $\mu\big|_{t=0} = \nu$. Indeed, let $\varphi \in C_0^\infty(\mathbb{R}^d \times (0,+\infty))$. Then

$$\int\limits_0^{+\infty} \int\limits_{\mathbb{R}^d} \big[\partial_t\varphi + L\varphi\big]\, d\mu = \int\limits_{\mathbb{R}^d} \nu(dx) \int\limits_0^{+\infty} \int\limits_{\mathbb{R}^d} \big[\partial_t\varphi + L\varphi\big] p(x,y,t)\, dy = 0.$$

Let $\zeta \in C_0^\infty(\mathbb{R}^d)$. Taking into account property (ii), we obtain

$$\lim_{t\to 0} \int\limits_{\mathbb{R}^d} \zeta(y)\mu_t(dy) = \lim_{t\to 0} \int\limits_{\mathbb{R}^d} T_t\zeta(x)\nu(dx) = \int\limits_{\mathbb{R}^d} \zeta(x)\nu(dx).$$

We use the semigroup considered in this example for constructing examples of nonuniqueness.

2 Examples of Nonuniqueness

We show that the set of probability solutions may consist of several elements. It is well known that if $A \equiv 0$ and the vector field b is just continuous, then the Cauchy problem may have more than one solution. Indeed, let $b(x)=x^{2/3}$. Then the Cauchy problem for the ordinary equation $\dot{x}=b(x)$, $x(0)=0$ has different solutions $x_1(t)=t^3/3$ and $x_2(t)=0$. The measures $\delta_{x_1(t)}$ and $\delta_{x_2(t)}$ are different solutions to the corresponding Cauchy problem (1.2).

The question arises about nonuniqueness in the case $A=I$ and $b\in C^\infty(\mathbb{R}^d\times[0,T])$. It turns out that, in this case, there is also a nonuniqueness example a construction of which is based not on the local nonregularity of the drift b, but on the rapid growth of $|b(x,t)|$ at infinity.

To construct such an example , we need several auxiliary results on the stationary Fokker–Planck–Kolmogorov equation, which we present in the following three propositions.

Proposition 2.1. *Let $d\geqslant 2$. There exists a vector field $B\in C^\infty(\mathbb{R}^d)$ such that the stationary Fokker–Planck–Kolmogorov equation $L^*\mu=0$ with the operator $L=\Delta+(B(x),\nabla)$ has an infinite-dimensional simplex of probability solutions. Any solution is given by a positive density of the class $C^\infty(\mathbb{R}^d)$ with respect to the Lebesgue measure.*

Such examples are constructed in [26]–[28]. For a survey of recent results related to the uniqueness for the stationary Fokker–Planck–Kolmogorov equation we refer to [29].

Proposition 2.2. *Assume that $d\geqslant 2$ and $L=\Delta+(B(x),\nabla)$, where B is an infinitely differentiable vector field on $\mathbb{R}^d$. Let $\{T_t\}_{t\geqslant 0}$ be the semigroup corresponding to the operator L in Example* 1.7*, and let μ be an arbitrary probability solution to the equation $L^*\mu=0$. Then for every nonnegative function $\varphi\in C_0^\infty(\mathbb{R}^d)$*

$$\int\limits_{\mathbb{R}^d} T_t\varphi\,d\mu\leqslant\int\limits_{\mathbb{R}^d}\varphi\,d\mu.$$

Proof. Let $\varphi\in C_0^\infty(\mathbb{R}^d)$, $\varphi\geqslant 0$, and let a number $R>0$ be such that $\operatorname{supp}\varphi\subset U_R$, where U_R is the ball $U(0,R)$. As in Example 1.7, let the function u_R be the solution to the mixed problem

$$\begin{aligned}&\partial_t u_R=Lu_R, \quad x\in U_R,\ t>0,\\ &u_R(x,t)=0, \quad x\in\partial U_R,\ t>0,\\ &u_R(x,0)=\varphi(x), \quad x\in\overline{U_R}.\end{aligned}$$

By [14, Chapter 3, § 5, Theorem 12], $u_R\in C^\infty(\overline{U_R}\times[0,+\infty))$. Moreover, by the maximum principle, $u_R\geqslant 0$ and $u_R(x,t)>0$ whenever $(x,t)\in U_R\times(0,+\infty)$. We set

$$v_{k,\varepsilon}(\,\cdot\,)=(u_R(\,\cdot\,,t)+k)^{1+\varepsilon}, \quad k>0, \quad \varepsilon>0.$$

We note that the measure μ is given by a smooth density ϱ and the equation $L^*\mu=0$ can be written as an equation for the density ϱ as follows:

$$\operatorname{div}\bigl(\nabla\varrho-b\varrho\bigr)=0.$$

Multiplying the last equality by the function $v_{k,\varepsilon}-k^{1+\varepsilon}$ and integrating by parts, we obtain

$$0=\int\limits_{U_R}\bigl[(b,\nabla v_{k,\varepsilon})+\Delta v_{k,\varepsilon}\bigr]\varrho\,dx+\int\limits_{\partial U_R}\varrho(\nabla v_{k,\varepsilon},n)dS,$$

where n is the outward normal vector to ∂U_R. Note that

$$\nabla v_{k,\varepsilon} = (1+\varepsilon)(u_R+k)^{\varepsilon}\nabla u_R,$$
$$\Delta v_{k,\varepsilon} = \varepsilon(1+\varepsilon)(u_R+k)^{\varepsilon-1}|\nabla u_R|^2 + (1+\varepsilon)(u_R+k)^{\varepsilon}\Delta u_R.$$

Dropping the positive term $\varepsilon(1+\varepsilon)(u_R+k)^{\varepsilon-1}|\nabla u_R|^2$, taking into account that $u_R(x,t)=0$ if $x \in \partial U_R$ and $u_R(x,t) > 0$ at all inner points $x \in U_R$, and first letting $k \to 0$ and then letting $\varepsilon \to 0$, we arrive at the inequality

$$\int\limits_{U_R} Lu_R \varrho\, dx \leqslant 0.$$

Since $\partial_t u_R = Lu_R$, we have

$$\frac{d}{dt}\int\limits_{U_R} u_R(x,t)\varrho(x)\, dx \leqslant 0.$$

Let us extend u_R by zero outside U_R. Then

$$\int\limits_{\mathbb{R}^d} u_R(x,t)\, d\mu \leqslant \int\limits_{\mathbb{R}^d} \varphi\, d\mu$$

and it remains to observe that $T_t\varphi(x) = \lim\limits_{R\to\infty} u_R(x,t)$. □

Proposition 2.3. *Let $\tau > 0$. Let the assumptions of PRoposition 2.2 be satisfied. Assume that the measure μ is invariant under the family of operators $\{T_t\}_{0\leqslant t<\tau}$, i.e.,*

$$\int\limits_{\mathbb{R}^d} T_t\varphi\, d\mu = \int\limits_{\mathbb{R}^d} \varphi\, d\mu \tag{2.1}$$

for every $t \in [0,\tau)$ and $\varphi \in C_0^\infty(\mathbb{R}^d)$. Then μ is a unique probability solution to the equation $L^*\mu = 0$.

Proof. We first show that μ is indeed a solution to our equation. By property (iii) in Example 1.7, we have $\partial_t T_t\varphi(x) = T_t L\varphi(x)$. Therefore,

$$\max_{x,t} t^{-1}|T_t\varphi(x) - \varphi(x)| \leqslant \max_x |L\varphi(x)|.$$

By the dominated convergence theorem, we have

$$0 = \lim_{t\to 0}\int\limits_{\mathbb{R}^d} t^{-1}(T_t\varphi(x) - \varphi(x))\, d\mu = \int\limits_{\mathbb{R}^d} L\varphi(x)\, d\mu,$$

which is equivalent to the equality $L^*\mu = 0$. Since $B \in C^\infty(\mathbb{R}^d)$, the solution μ is given by a strictly positive infinitely differentiable density ϱ with respect to the Lebesgue measure.

Let ν be another probability solution to the equation $L^*\mu = 0$. According to Proposition 2.2, for every nonnegative function $\varphi \in C_0^\infty(\mathbb{R}^d)$

$$\int\limits_{\mathbb{R}^d} T_t\varphi\, d\nu \leqslant \int\limits_{\mathbb{R}^d} \varphi\, d\nu,$$

which enables us to extend the operator T_t to a continuous linear operator on $L^1(\nu)$ such that the inequality holds for all nonnegative functions in $L^1(\nu)$. In particular, $T_t 1 \leqslant 1$ ν-a.e. We note that $\mu = v\nu$, where the positive function v is the ratio of densities of μ and ν. The equality (2.1) yields

$$\int_{\mathbb{R}^d} (T_t 1 - 1) v \, d\nu = 0,$$

which along with the inequality $T_t 1 \leqslant 1$ implies the equality $T_t 1 = 1$. On the space $L^\infty(\mu)$, we have the adjoint operator T_t^* for which the equality $T_t^* 1 = 1$ holds as well. Thus,

$$\int_{\mathbb{R}^d} T_t \varphi \, d\nu = \int_{\mathbb{R}^d} \varphi T_t^* 1 \, d\nu = \int_{\mathbb{R}^d} \varphi \, d\nu,$$

i.e., (2.1) is fulfilled for ν. Hence the measures μ and ν are invariant with respect to the semigroup $\{T_t\}_{t \geqslant 0}$. Therefore, the measure $\mu - \nu$ is also invariant, which yields the invariance of the measure $|\mu - \nu|$ (cf., for example, [30]). As shown above, the measure $|\mu - \nu|$ satisfies the equation $L^*\mu = 0$ and either vanishes identically or possesses a strictly positive density, which is impossible since both μ and ν are probability measures. Hence $\mu = \nu$. □

Corollary 2.4. *Let B be a vector field from Proposition* 2.1*, and let ν be a probability solution to the equation $L^*\mu = 0$, where $L = \Delta + (B(x), \nabla)$. Let $\{T_t\}_{t \geqslant 0}$ be the semigroup specified in Example* 1.7 *corresponding to the operator L. Let*

$$\sigma_t(B) = \int_{\mathbb{R}^d} \nu(dx) \int_B p(x, y, t) \, dy,$$

where p is the kernel of the semigroup $\{T_t\}_{t \geqslant 0}$. Then the measure $\sigma = \sigma_t(dx)\, dt$ solves the Cauchy problem (1.2) *with the initial condition $\sigma\big|_{t=0} = \nu$. Moreover, $\sigma_t \leqslant \nu$ if $t > 0$ and for every $T > 0$ one has $\sigma_t \not\equiv \nu$ if $t \in (0, T)$.*

Proof. Only the last two assertions require a proof. The inequality $\sigma_t \leqslant \nu$ follows from Proposition 2.2. Indeed, for every nonnegative function $\varphi \in C_0^\infty(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d} \varphi(x) \, d\sigma_t = \int_{\mathbb{R}^d} T_t \varphi(x) \, d\nu \leqslant \int_{\mathbb{R}^d} \varphi(x) \, d\nu.$$

If $\sigma_t = \nu$ for all $t \in (0, T)$, then ν is an invariant measure for the semigroup $\{T_t\}_{t \geqslant 0}$ and, by Proposition 2.3, it is a unique probability solution to the equation $L^*\mu = 0$, which contradicts our assumptions. □

The existence of a solution $\sigma = \sigma_t(dx)\, dt$ to the Cauchy problem with the initial condition $\sigma\big|_{t=0} = \nu$ such that ν is a nonunique solution to the equation $L^*\nu = 0$, $\sigma_t \leqslant \nu$, and $\sigma_t \not\equiv \nu$ can be also deduced from the results of [26] (cf. [9]).

We now proceed by constructing an example of the Cauchy problem with several probability solutions.

Example 2.5. We split our construction in several steps.

Step I. Let $b(x,y)=(B(x),C(y))$, where $x,y\in\mathbb{R}^2$ and $B=(b^1,b^2)$, $C=(c^1,c^2)$ are infinitely differentiable vector fields on $\mathbb{R}^2$ such that the elliptic equations

$$L_1^*\nu=0,\quad L_2^*\sigma=0$$

with respect to measures on $\mathbb{R}^2$, where

$$L_1u=\Delta_x u+(B,\nabla_x u),\quad L_2u=\Delta_y u+(C,\nabla_y u),$$

have at least two linearly independent probability solutions. Such vector fields B and C exist according to Proposition 2.1. We set

$$Lu=L_1u+L_2u=\Delta_x u+\Delta_y u+(B,\nabla_x u)+(C,\nabla_y u)=\Delta u+(b,\nabla u).$$

Step II. Let ν be a probability solution to the first equation $L_1^*\nu=0$. According to Corollary 2.4 the measure $\mu^\nu(\,dx\,dt)=\nu_t(\,dx)\,dt$, where

$$\nu_t(B)=\int\limits_{\mathbb{R}^d}\nu(dx)\int\limits_B p(x,y,t)\,dy,$$

is a solution to the Cauchy problem with operator L_1 and initial condition ν. Moreover, $\nu_t\leqslant\nu$ and $\nu_t\not\equiv\nu$. We set $\widehat{\nu}_t=\nu-\nu_t$ and observe that the measure $\widehat{\nu}(\,dx\,dt)=\widehat{\nu}_t(\,dx)\,dt$ is nonnegative and is a nonzero solution to the Cauchy problem $\partial_t\widehat{\nu}=L_1^*\widehat{\nu}$, $\widehat{\nu}\big|_{t=0}=0$.

Step III. Let σ_1 and σ_2 be two linearly independent probability solutions to the equation $L_2^*\sigma=0$. We set $\mu_t=\nu\cdot\sigma_1+\widehat{\nu}_t\cdot(\sigma_2-\sigma_1)$. We show that the measures $\nu\cdot\sigma_1(\,dx)\,dt$ and $\mu(\,dx\,dt)=\mu_t(\,dx)\,dt$ are probability solutions to the Cauchy problem $\partial_t\mu=L^*\mu$, $\mu\big|_{t=0}=\nu\cdot\sigma_1$. Indeed, we have the equality $L^*(\nu\cdot\sigma_1)=\sigma_1L_1^*\nu+\nu L_2^*\sigma_1=0$ and $\nu\cdot\sigma_1$ does not depend on t. Similarly,

$$L^*(\widehat{\nu}_t\cdot(\sigma_2-\sigma_1))=(\sigma_2-\sigma_1)L_1^*\widehat{\nu}_t=(\sigma_2-\sigma_1)\partial_t\widehat{\nu}_t=\partial_t(\widehat{\nu}_t\cdot(\sigma_2-\sigma_1)).$$

Since $\nu-\widehat{\nu}_t=\nu_t\geqslant 0$, we have

$$\mu_t=\nu\cdot\sigma_1+\widehat{\nu}_t\cdot(\sigma_2-\sigma_1)=(\nu-\widehat{\nu}_t)\sigma_1+\widehat{\nu}_t\sigma_2\geqslant 0.$$

Taking into account that $\widehat{\nu}|_{t=0}=0$, we obtain $\mu\big|_{t=0}=\nu\cdot\sigma_1$. It remains to observe that μ_t is a probability measure for every t since $(\sigma_2-\sigma_1)(\mathbb{R}^2)=0$.

Note that, by Proposition 2.1, the vector fields B and C can be chosen in such a way that the corresponding stationary Kolmogorov equations $L_1^*\nu=0$ and $L_2^*\sigma=0$ will have infinite-dimensional simplices of probability solutions. Therefore, changing the measure σ_2 in this example, we obtain an infinite-dimensional simplex of probability solutions to the Cauchy problem.

Remark 2.6. If, in place of the strip $\mathbb{R}^d\times(0,T)$, we consider $\mathbb{R}^d\times\mathbb{R}$ and do not impose any conditions as $t\to-\infty$, then there exist simple examples of the equation $\partial_t\mu=L^*\mu$ with several probability solutions (cf., for example, [5]). Assume that $d=1$, $A(x)=1$, and $b(x,t)=-x$. Then for any number α the measure $\mu^\alpha=\mu_t^\alpha(dx)\,dt$, where

$$\mu_t^\alpha(dx)=(2\pi)^{-1/2}e^{-\frac{(x+e^{-t}\alpha)^2}{2}}\,dx,$$

is a probability solution to our equation.

Concluding this section, we give an example of the Cauchy problem which has a unique probability solution that is not a unique integrable or nonnegative solution.

Example 2.7. Assume that $d = 1$ and $A(x) = 1$. The Cauchy problem for the measure $\mu = v(x,t)\,dx\,dt$ with the initial condition $\nu = u(x)\,dx$ can be written as the Cauchy problem for the density v as follows:

$$v_t = (v_x - bv)_x, \quad v(0,x) = u(x).$$

We seek a nonnegative solution v in the form $e^t\Phi'(x)$. Substituting into the equation we obtain

$$\Phi' = (\Phi'' - b\Phi')'.$$

Integrating this equality, we arrive at the relationship

$$\Phi = \Phi'' - b\Phi' + \mathrm{const}\,.$$

We set $\Phi(x) = \operatorname{arc\,tg} x$ and

$$b(x) = \frac{\Phi''(x) - \Phi(x)}{\Phi'(x)}.$$

Then $b(x) = -2x(1+x^2)^{-1} - (1+x^2)\operatorname{arc\,tg} x$. Let $u(x) = (\pi(1+x^2))^{-1}$. Since $b(x)x \leqslant 0$, from Theorem 3.4 it follows that there exists a unique probability solution to the Cauchy problem. However, there is yet another integrable and nonnegative solution; namely, $e^t u(x)\,dx\,dt$.

3 The Uniqueness of Probability Solutions if the Diffusion Matrix Coefficients Belong to VMO_x

We begin our study of the uniqueness of probability solutions with the classical Holmgren principle, the main idea of which can be illustrated by the following trivial example. Let A be a bounded linear operator on a Hilbert space H. Then $\ker A = (\mathrm{Im}\, A^*)^\perp$. Therefore, to prove the uniqueness of a solution to the equation $Ax = f$, one has to verify that the range of A^* is dense in H. In the general case, such an approach requires high smoothness of the coefficients of the differential operator L. For instance, if a^{ij} and b^i are continuous in (x,t) and have continuous bounded derivatives in x up to the second order, then, as shown in [23] and [14] (cf. also [19]), for every $t > 0$ and every function $\psi \in C_0^\infty(\mathbb{R}^d)$ there exists a function $f \in C_b([0,t]\times\mathbb{R}^d)\bigcap C_b^{1,2}((0,t)\times\mathbb{R}^d)$ such that

$$\begin{aligned}&\partial_t f(x,t) + Lf(x,t) = 0,\\ &f(t,x) = \psi(x),\end{aligned}$$

Using the fact that μ_t is a probability measure and applying Lemma 1.1, we obtain the equality

$$\int\limits_{\mathbb{R}^d} \psi(x)\,d\mu_t = \int\limits_{\mathbb{R}^d} f(x,t)\,d\nu$$

which immediately yields the uniqueness of a solution to the Cauchy problem. Indeed, for any two solutions μ^1 and μ^2 we have

$$\int\limits_{\mathbb{R}^d} \psi(x)\,d\mu_t^1 = \int\limits_{\mathbb{R}^d} \psi(x)\,d\mu_t^2$$

for every function $\psi \in C_0^\infty(\mathbb{R}^d)$, which implies that $\mu^1 = \mu^2$.

If the diffusion matrix is nonsingular, then the assumptions on coefficients can be considerably weakened. It is well known (cf., for example, [23]) that, in the nondegenerate case, it suffices to have the Hölder continuity of coefficients. In this section, we show that, combining the results of [1] and [31], one can obtain a considerably stronger result.

The following assertion is proved in [1].

Proposition 3.1. *Let $Q = (q^{ij})$ be a mapping from $\mathbb{R}^d \times (0,T)$ to the set of symmetric nonnegative matrices. Let a locally finite nonnegative Borel measure μ on $\mathbb{R}^d \times (0,T)$ be such that $q^{ij} \in L^1_{\mathrm{loc}}(\mu, \mathbb{R}^d \times [0,T])$, and let for every nonnegative function $\varphi \in C_0^\infty(\mathbb{R}^d \times (0,T))$*

$$\int\limits_{\mathbb{R}^d\times(0,T)} \bigl[\partial_t\varphi + q^{ij}\partial_{x_i}\partial_{x_j}\varphi\bigr]\,d\mu \leqslant C\bigl(\sup_{\mathbb{R}^d\times(0,T)}|\varphi| + \sup_{\mathbb{R}^d\times(0,T)}|\nabla_x\varphi|\bigr).$$

Then the measure $(\det Q)^{1/(d+1)}\mu$ has a density of the class $L^{(d+1)'}_{\mathrm{loc}}(\mathbb{R}^d \times (0,T))$ with respect to the Lebesgue measure on $\mathbb{R}^d \times (0,T)$.

Corollary 3.2. *If a locally finite nonnegative Borel measure μ on $\mathbb{R}^d \times (0,T)$ satisfies the equation $\partial_t\mu = L^*\mu$ and condition (H1) is fulfilled, then $\mu = \varrho\,dx\,dt$ and $\varrho \in L^{(d+1)'}_{\mathrm{loc}}(\mathbb{R}^d \times (0,T))$.*

Let g be a bounded function on $\mathbb{R}^{d+1}$. We set

$$O(g,R) = \sup_{(x,t)\in\mathbb{R}^{d+1}}\sup_{r\leqslant R} r^{-2}|U(x,r)|^{-2}\int\limits_t^{t+r^2}\int\int\limits_{y,z\in U(x,r)} |g(y,s)-g(z,s)|\,dy\,dz\,ds.$$

If $\lim\limits_{R\to 0} O(g,R) = 0$, then the function g is said to belong to the class $VMO_x(\mathbb{R}^{d+1})$.

If $g \in VMO_x(\mathbb{R}^{d+1})$, then one can always assume that $O(g,R) \leqslant w(R)$ for all $R > 0$, where w is a continuous function on $[0,+\infty)$ and $w(0) = 0$.

Suppose that a function g is defined on $\mathbb{R}^d \times [0,T]$ and is bounded on $U \times [0,T]$ for every ball U. Let us extend g by zero to the entire space $\mathbb{R}^{d+1}$. If for every function $\zeta \in C_0^\infty(\mathbb{R}^d)$ the function $g\zeta$ belongs to the class $VMO_x(\mathbb{R}^{d+1})$, then we say that g belongs to the class $VMO_{x,\mathrm{loc}}(\mathbb{R}^d \times [0,T])$.

Remark 3.3. Assume that $\omega \in C_0^\infty(\mathbb{R}^1)$, $\omega \geqslant 0$, $\|\omega\|_{L^1} = 1$, and

$$\omega_\varepsilon(x,t) = \varepsilon^{-d-1}\omega(|x|/\varepsilon)\omega(|t|/\varepsilon).$$

If $g \in VMO_x(\mathbb{R}^{d+1})$, then $g_\varepsilon = g * \omega_\varepsilon \in VMO_x(\mathbb{R}^{d+1})$ and $O(g_\varepsilon, R) \leqslant O(g,R)$. It suffices to observe that

$$r^{-2}|U(x,r)|^{-2}\int\limits_t^{t+r^2}\int\int\limits_{y,z\in U(x,r)} |g_\varepsilon(y,s)-g_\varepsilon(z,s)|\,dy\,dz\,ds$$

$$\leqslant \int\limits_{\mathbb{R}^{d+1}} \omega(|\xi|)\omega(|\tau|)\,d\xi\,d\tau\, r^{-2}|U(x+\varepsilon\xi,r)|^{-2}\int\limits_{t+\varepsilon\tau}^{t+\varepsilon\tau+r^2}\int\int\limits_{y,z\in U(x+\varepsilon\xi,r)} |g(y,s)-g(z,s)|\,dy\,dz\,ds,$$

where the right-hand side is estimated by the quantity $O(g,R)$. Note also that $g_\varepsilon \to g$ in $L^p_{\mathrm{loc}}(\mathbb{R}^{d+1})$ for every $p \geqslant 1$. Therefore, for every function g of class $VMO_x(\mathbb{R}^{d+1})$ one can find a sequence of infinitely differentiable functions g_k converging to g in $L^p_{\mathrm{loc}}(\mathbb{R}^{d+1})$ for every $p \geqslant 1$ such that $O(g_k,R) \leqslant O(g,R)$.

Remark 3.4. Let functions a^{ij} be defined on $\mathbb{R}^{d+1}$, and let a symmetric matrix $A = (a^{ij})$ be such that for some numbers $\gamma > 0$ and $M > 0$

$$(A(x,t)y,y) \geqslant \gamma|y|^2, \quad |A(x,t)| \leqslant M$$

for all $(x,t) \in \mathbb{R}^{d+1}$ and $y \in \mathbb{R}^d$. Then these inequalities with the same numbers γ and M are fulfilled for the matrix $A_\varepsilon = (a^{ij}_\varepsilon)$, where $a^{ij}_\varepsilon = a^{ij} * \omega_\varepsilon$ and the function ω_ε is defined as in Remark 3.3. It suffices to observe that $\|\omega\|_{L^1(\mathbb{R})} = 1$ and

$$(A_\varepsilon(x,t)y,y) = \int\limits_{\mathbb{R}^{d+1}} (A(x+\varepsilon\xi, t+\varepsilon\tau)y,y)\omega(|\xi|)\omega(|\tau|)\,d\xi\,d\tau \quad \forall\, y.$$

Let $W^{1,2}_p(\mathbb{R}^d \times (-1,T))$ be the space of functions $u \in L^p(\mathbb{R}^d \times (-1,T))$ having Sobolev derivatives $\partial_t u$, $\partial_{x_i} u$ and $\partial_{x_i}\partial_{x_j} u$ in $L^p(\mathbb{R}^d \times (-1,T))$ and finite norm

$$\begin{aligned}\|u\|_{W^{1,2}_p(\mathbb{R}^d\times(-1,T))} &= \|u\|_{L^p(\mathbb{R}^d\times(-1,T))} + \|\partial_{x_i} u\|_{L^p(\mathbb{R}^d\times(-1,T))} \\ &\quad + \|\partial_{x_i}\partial_{x_j} u\|_{L^p(\mathbb{R}^d\times(-1,T))} + \|\partial_t u\|_{L^p(\mathbb{R}^d\times(-1,T))}.\end{aligned}$$

The next result follows from [31].

Proposition 3.5. *Let $q^{ij}, h^i \in C^\infty(\mathbb{R}^{d+1})$ and*

$$\sup_{x,t} |q^{ij}(x,t)| + |h^i(x,t)| \leqslant M,$$

where the matrix $Q = (q^{ij})$ is symmetric, and let for some number $\kappa > 0$

$$(Q(x,t)y,y) \geqslant \kappa|y|^2$$

for all $(x,t) \in \mathbb{R}^{d+1}$ and $y \in \mathbb{R}^d$. Suppose that there exists a continuous function w on $[0,+\infty)$ such that $w(0) = 0$ and $O(q^{ij},R) \leqslant w(R)$ for all i, j. Assume that $T > 0$ and $p > d+1$. Then the Cauchy problem

$$\begin{aligned} \partial_t f + q^{ij}\partial_{x_i}\partial_{x_j} f + h^i \partial_{x_i} f = 0, \\ f\big|_{t=T} = \psi, \end{aligned} \tag{3.1}$$

where $\psi \in C^\infty_0(\mathbb{R}^d)$, has a unique solution $f \in W^{1,2}_p(\mathbb{R}^d \times (-1,T))$ and there is a number $C > 0$, depending only on the numbers d, p, M, κ, T, and the functions ψ, w, such that

$$\|f\|_{W^{1,2}_p(\mathbb{R}^d\times(-1,T))} \leqslant C.$$

Moreover, $f \in C^\infty(\mathbb{R}^d \times (-1,T)) \bigcap C_b(\mathbb{R}^d \times [-1,T])$.

Proof. According to [31, Theorem 2.1], there exists a unique solution to the Cauchy problem

$$\partial_t g + q^{ij}\partial_{x_i}\partial_{x_j} g + h^i\partial_{x_i} g = -q^{ij}\partial_{x_i}\partial_{x_j}\psi - h^i\partial_{x_i}\psi,$$
$$g\big|_{t=T} = 0$$

in the class $W_p^{1,2}(\mathbb{R}^d \times (-1,T))$. Moreover,

$$\|g\|_{W^{1,2}(\mathbb{R}^d\times(-1,T))} \leqslant C_1\|q^{ij}\partial_{x_i}\partial_{x_j}\psi + h^i\partial_{x_i}\psi\|_{L^p(\mathbb{R}^d\times(-1,T))},$$

where C_1 depends only on d, p, M, κ, T, and w. The function $f = g + \psi$ is a unique solution to the Cauchy problem (3.1) in the Sobolev class $W_p^{1,2}$. Since the matrix Q is nonsingular and $q^{ij}, h^i \in C^\infty(\mathbb{R}^{d+1})$, we have $g \in C^\infty(\mathbb{R}^d \times (-1,T))$ (cf., for example, [32, Theorem 12.2]). Consequently, $f \in C^\infty(\mathbb{R}^d \times (-1,T))$. Let U be the unit ball in $\mathbb{R}^d$. Since $p > d+1$, the Sobolev embedding theorem yields $f \in C^\alpha(U \times [-1,T])$ and

$$\|f\|_{C^\alpha(U\times[-1,T])} \leqslant C_2\|f\|_{W_p^{1,2}(\mathbb{R}^d\times[-1,T])},$$

where $\alpha = 1-(d+1)/p$ and C_2 depends only on d and p. In particular, we have the inclusion $f \in C_b(\mathbb{R}^d \times [-1,T])$. □

We present the main result of this section.

Theorem 3.6. *Suppose that $a^{ij} \in VMO_{x,\mathrm{loc}}(\mathbb{R}^d \times [0,T])$ and the matrix $A = (a^{ij})$ satisfies condition* (H1). *Then the set*

$$\mathscr{M}_\nu = \{\mu \in \mathscr{P}_\nu\colon\ a^{ij}, b^i \in L^1(\mu, \mathbb{R}^d \times [0,T])\}$$

consists of at most one element.

Proof. Let $\varphi_N(x) = \eta(x/N)$, where $\eta \in C_0^\infty(\mathbb{R}^d)$ is a nonnegative function such that $\eta(x) = 1$ if $|x| \leqslant 1$ and $\eta(x) = 0$ if $|x| > 2$, $0 \leqslant \eta \leqslant 1$ and there exists a number $K > 0$ such that for all x the inequality $|\nabla\eta(x)|^2\eta^{-1}(x) \leqslant K$ is fulfilled.

We fix $N \in \mathbb{N}$. Let U be an open ball in $\mathbb{R}^d$ containing the support of φ_N, and let $\zeta \in C_0^\infty(\mathbb{R}^d)$ be a function such that $0 \leqslant \zeta \leqslant 1$ and $\zeta(x) = 1$ if $x \in U$. We set

$$\overline{A}(x,t) = \zeta(x)A(x,t) + (1-\zeta(x))I,$$

where I is the unit matrix. Let $\overline{A}(x,t) = I$ if $t < 0$ or $t > T$. Then $\overline{a}^{ij} \in VMO_x(\mathbb{R}^{d+1})$ and for some numbers $\kappa > 0$ and $M_1 > 0$

$$(\overline{A}(x,t)y, y) \geqslant \kappa|y|^2, \quad |\overline{a}^{ij}(x,t)| \leqslant M_1$$

for all $(x,t) \in \mathbb{R}^{d+1}$ and $y \in \mathbb{R}^d$. Moreover, $\overline{A}(x,t) = A(x,t)$ for all $(x,t) \in U \times [0,T]$.

Using Remarks 3.3 and 3.4, we find a sequence of matrices $Q_n = (q_n^{ij})$ such that $q_n^{ij} \in C^\infty(\mathbb{R}^{d+1})$ and for all $(x,t) \in \mathbb{R}^{d+1}$ and $y \in \mathbb{R}^d$ the following inequality holds:

$$(Q_n(x,t)y, y) \geqslant \kappa|y|^2, \quad |q_n^{ij}(x,t)| \leqslant M_1,$$

one has $O(q_n^{ij}, R) \leqslant Q(\overline{a^{ij}}, R)$ and $\lim\limits_{n\to\infty}\|\overline{A} - Q_n\|_{L^r(U\times[0,T])} = 0$, where $r = 2(d+1)$.

Let $\sigma^1 = \sigma^1_t(dx)\,dt$ and $\sigma^2 = \sigma^2_t(dx)\,dt$ belong to the class $\mathscr{M}_\nu$. Then $\sigma = (\sigma^1+\sigma^2)/2$ also belongs to $\mathscr{M}_\nu$. By assumption, $|b| \in L^2(\mu, U\times[0,T])$. Let us find a sequence of vector fields $(h^i_m)_{1\leqslant i\leqslant d}$ on $\mathbb{R}^{d+1}$ such that $h^i_m \in C^\infty(\mathbb{R}^{d+1})$, $|h_m(x,t)| \leqslant M_2(m)$ for all $(x,t)\in\mathbb{R}^{d+1}$, and

$$\lim_{m\to\infty}\int_0^T\int_U |b(x,t)-h_m(x,t)|^2\,d\sigma_t\,dt = 0.$$

We set

$$L_{n,m} = q^{ij}_n\partial_{x_i}\partial_{x_j} + h^i_m\partial_{x_i}.$$

Let $\psi\in C_0^\infty(\mathbb{R}^d)$ with $\max_x|\psi(x)|\leqslant 1$. Let J denote the set of all points $t\in[0,T]$ for which equality (1.4) is fulfilled for all functions $\psi\varphi_k$ and $\psi^2\varphi_k$, $k\in\mathbb{N}$, and both measures σ^1 and σ^2. It is clear that J is a full measure set in $[0,T]$ and Remarks 1.3 and 1.4 are applicable to it. Let $t\in J$.

According to Proposition 3.5, there exists a solution $f_{n,m}\in C^\infty(\mathbb{R}^d\times(-1,t))\bigcap C_b(\mathbb{R}^d\times[-1,t])$ to the Cauchy problem

$$\partial_s f_{n,m} + L_{n,m}f_{n,m} = 0,$$
$$f_{n,m}\big|_{s=t} = \psi.$$

Let I be the set of all points $s\in[0,t]$ where the equality (1.3) of Lemma 1.1 holds for all functions $f_{n,m}\varphi_N$ and $f^2_{n,m}\varphi_N$, where $k\in\mathbb{N}$, and both measures σ^1 and σ^2. We observe that $t\in I$ and I is a full measure set in $[0,t]$. The measure $\mu=\sigma^1-\sigma^2$ satisfies the Cauchy problem (1.2) with zero initial condition. Assume that $\tau\in I$ and $\tau<t$. Applying Lemma 1.1 to the function $f_{n,m}\varphi_N$, we obtain

$$\int_{\mathbb{R}^d}\psi(x)\varphi_N(x)\,d\mu_t - \int_{\mathbb{R}^d} f_{n,m}(x,\tau)\varphi_N(x)\,d\mu_\tau$$
$$= \int_\tau^t\int_{\mathbb{R}^d}\bigl[\varphi_N(L-L_{n,m})f_{n,m} + (A\nabla f_{n,m},\nabla\varphi_N) + f_{n,m}L\varphi_N\bigr]\,d\mu_s\,ds. \tag{3.2}$$

Let us estimate the quantity

$$\int_\tau^t\int_{\mathbb{R}^d}\varphi_N|\sqrt{A}\nabla f_{n,m}|^2\,d\sigma_s\,ds.$$

For this purpose, we apply Lemma 1.1 to the function $f^2_{n,m}\varphi_N$ and the measure σ. We obtain

$$\int_{\mathbb{R}^d}\psi^2(x)\varphi_N(x)\,d\sigma_t - \int_{\mathbb{R}^d} f^2_{n,m}(x,\tau)\varphi_N(x)\,d\sigma_\tau$$
$$= 2\int_\tau^t\int_{\mathbb{R}^d}|\sqrt{A}\nabla f_{n,m}|^2\varphi_N\,d\sigma_s\,ds$$
$$+ \int_\tau^t\int_{\mathbb{R}^d}\bigl[2f_{n,m}\varphi_N(L-L_{n,m})f_{n,m} + 2f_{n,m}(A\nabla f_{n,m},\nabla\varphi_N) + f^2_{n,m}L\varphi_N\bigr]\,d\sigma_s\,ds.$$

We observe that $0 \leqslant \varphi_N(x) \leqslant 1$, $|\psi(x)| \leqslant 1$ and, by the maximum principle, $|f_{n,m}(x,s)| \leqslant 1$. Applying the inequality $2\alpha\beta \leqslant \theta\alpha^2 + \theta^{-1}\beta^2$ with $\theta > 0$, we find that

$$
\begin{aligned}
&2\int_\tau^t \int_{\mathbb{R}^d} |(b-h_m, \nabla f_{n,m})||f_{n,m}|\varphi_N \, d\sigma_s \, ds \\
&\leqslant \theta^{-1} \int_\tau^t \int_{\mathbb{R}^d} |A^{-1/2}(b-h_m)|^2 \varphi_N \, d\sigma_s \, ds + \theta \int_\tau^t \int_{\mathbb{R}^d} |\sqrt{A}\nabla f_{n,m}|^2 \varphi_N \, d\sigma_s \, ds.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&2\int_\tau^t \int_{\mathbb{R}^d} |f_{n,m}||(A\nabla f_{n,m}, \nabla\varphi_N)| \, d\sigma_s \, ds \\
&\leqslant \theta^{-1} \int_\tau^t \int_{\mathbb{R}^d} \varphi_N^{-1}|\sqrt{A}\nabla\varphi_N|^2 \, d\sigma_s \, ds + \theta \int_\tau^t \int_{\mathbb{R}^d} |\sqrt{A}\nabla f_{n,m}|^2 \varphi_N \, d\sigma_s \, ds.
\end{aligned}
$$

Let $\theta = 1/2$. Using the obtained inequalities and taking into account condition (H1), we obtain the estimate

$$
\int_\tau^t \int_{\mathbb{R}^d} \varphi_N |\sqrt{A}\nabla f_{n,m}|^2 \, d\sigma_s \, ds \leqslant 1 + I + R,
$$

where

$$
\begin{aligned}
I &= 2\int_\tau^t \int_{\mathbb{R}^d} |a^{ij} - q^{ij}||\partial_{x_i}\partial_{x_j} f_{n,m}|\varphi_N \, d\sigma_s \, ds + 2\kappa \int_\tau^t \int_{\mathbb{R}^d} |b^i - h^i_m|^2 \varphi_N \, d\sigma_s \, ds, \\
R &= \int_0^T \int_{\mathbb{R}^d} |L\varphi_N| + 2\varphi_N^{-1}|\sqrt{A}\nabla\varphi_N|^2 \, d\sigma_s \, ds.
\end{aligned}
$$

Coming back to the equality (3.2), applying the last estimate, and taking into account that $|\mu| \leqslant 2\sigma$, we get

$$
\begin{aligned}
&\int_{\mathbb{R}^d} \psi(x)\varphi_N(x) \, d\mu_t - \int_{\mathbb{R}^d} f_{n,m}(x,\tau)\varphi_N(x) \, d\mu_\tau \\
&\leqslant 2\int_\tau^t \int_{\mathbb{R}^d} |a^{ij} - q^{ij}||\partial_{x_i}\partial_{x_j} f_{n,m}|\varphi_N \, d\sigma_s \, ds \\
&+ 2\kappa^{1/2}(1+I+R)^{1/2}\biggl(\int_0^T \int_{\mathbb{R}^d} |b^i - h^i_m|^2 \varphi_N \, d\sigma_s \, ds\biggr)^{1/2} \\
&+ 2(1+I+R)^{1/2}\biggl(\int_0^T \int_{\mathbb{R}^d} \varphi^{-1}|\sqrt{A}\nabla\varphi_N| \, d\sigma_s \, ds\biggr)^{1/2} + 2\int_0^T \int_{\mathbb{R}^d} |L\varphi_N| \, d\sigma_s \, ds. \qquad (3.3)
\end{aligned}
$$

By Corollary 3.2, the measure σ has density $\varrho \in L^{(d+1)'}_{\rm loc}(\mathbb{R}^d\times(0,T))$. Let $p=2(d+1)$. According to Proposition 3.5, there is a number C, depending only on p, d, κ, M_1, T, U, w, ψ, h_m and independent of n, such that

$$\max_{i,j}\|\partial_{x_i}\partial_{x_j}f_{n,m}\|_{L^p(U\times[0,T])}\leqslant C.$$

We recall that $A=\overline{A}$ on U. Applying the Hölder inequality, we obtain

$$\int_\tau^t\int_{\mathbb{R}^d}|a^{ij}-q^{ij}||\partial_{x_i}\partial_{x_j}f_{n,m}|\varphi_N\,d\sigma_s\,ds$$
$$\leqslant\|\overline{A}-Q_n\|_{L^r(U\times(0,T))}\|\partial_{x_i}\partial_{x_j}f_{n,m}\|_{L^p(U\times(0,T)}\|\varrho\|_{L^{(d+1)'}(U\times[\tau,t])}.$$

Therefore, the expression

$$\int_\tau^t\int_{\mathbb{R}^d}|a^{ij}-q^{ij}||\partial_{x_i}\partial_{x_j}f_{n,m}|\varphi_N\,d\sigma_s\,ds$$

tends to zero as $n\to\infty$ provided that τ and m are fixed.

According to Proposition 3.5 and the Sobolev embedding theorem, there exists a number C_1 independent of n such that

$$\|f_{n,m}\|_{C^\alpha(\overline{U}\times[0,t])}\leqslant C_1,$$

where $\overline{U}$ is the closure of the ball U. Therefore, on the compact set $\overline{U}\times[0,t]$, the family of functions $(f_{n,m})_n$ is uniformly bounded and equicontinuous, which enables us to select a uniformly convergent subsequence. Keeping the same indices n, we assume that $f_{n,m}\rightrightarrows f_m$ as $n\to\infty$ uniformly on $\overline{U}\times[0,t]$. It is clear that the function f_m is continuous on $\overline{U}\times[0,t]$. Keeping τ and m fixed, we let $n\to+\infty$ in the inequality (3.3). We obtain

$$\int_{\mathbb{R}^d}\psi(x)\varphi_N(x)\,d\mu_t-\int_{\mathbb{R}^d}f_m(x,\tau)\varphi_N(x)\,d\mu_\tau$$
$$\leqslant 2(1+I_m+R)^{1/2}I_m^{1/2}+2(1+I_m+R)^{1/2}R^{1/2}+2R \qquad (3.4)$$

where

$$I_m=2\kappa\int_0^T\int_{\mathbb{R}^d}|b^i-h^i_m|^2\varphi_N\,d\sigma_s\,ds.$$

Let us pick a sequence $\tau_k\in I\bigcap(0,t)$ such that $\lim\limits_{k\to\infty}\tau_k=0$. Since f_m is continuous on $\overline{U}\times[0,t]$, the functions $f_m(x,\tau_k)$ converge to $f_m(x,0)$ uniformly in x as $k\to\infty$. In addition, since the initial condition is zero, one has

$$\lim_{k\to0}\int_{\mathbb{R}^d}f_m(x,0)\varphi_N(x)\,d\mu_{\tau_k}=0.$$

Therefore,

$$\lim_{k\to0}\int_{\mathbb{R}^d}f_m(x,\tau_k)\varphi_N(x)\,d\mu_{\tau_k}=0.$$

Replacing τ in (3.4) by τ_k and letting first $k \to \infty$ and then $m \to \infty$, we arrive at the equality

$$\int\limits_{\mathbb{R}^d} \psi(x)\varphi_N(x)\, d\mu_t \leqslant 2(1+R)^{1/2}R^{1/2} + 2R.$$

Finally, letting $N \to \infty$ and using the fact that $a^{ij}, b^i \in L^1(\mu, \mathbb{R}^d \times [0,T])$, we obtain

$$\int\limits_{\mathbb{R}^d} \psi(x)\, d\mu_t \leqslant 0.$$

Replacing ψ by $-\psi$ we obtain the opposite inequality. Hence for almost all $t \in [0,T]$

$$\int\limits_{\mathbb{R}^d} \psi(x)\, d\mu_t = 0.$$

Since ψ is an arbitrary function in $C_0^\infty(\mathbb{R}^d)$ satisfying $|\psi| \leqslant 1$, we have $\mu_t = 0$. Therefore, $\sigma^1 = \sigma^2$. $\square$

Remark 3.7. (i) It is easily seen from the proof of the theorem that one can weaken the condition $a^{ij}, b^i \in L^1(\mu, \mathbb{R}^d \times [0,T])$ restricting the class of probability solutions in which uniqueness is proved. Let $V \in C^2(\mathbb{R}^d)$ be a positive function such that $\lim\limits_{|x|\to\infty} V(x) = +\infty$. Suppose, as above, that $a^{ij} \in VMO_{x,\mathrm{loc}}(\mathbb{R}^d \times [0,T])$ and the matrix $A = (a^{ij})$ satisfies condition (H1). Then the set $\mathscr{M}_\nu$ consisting of measures $\mu \in \mathscr{P}_\nu$ such that

$$\lim_{N\to\infty} \int\limits_0^T \int\limits_{N\leqslant V\leqslant N+1} \frac{|LV|}{V} + \frac{|\sqrt{A}\nabla V|^2}{V^2}\, d\mu = 0$$

contains at most one element.

(ii) It can be also seen from the proof of the theorem that the conditions on the matrix A can be weakened as follows. The theorem remains valid if we assume that the symmetric matrix $A = (a^{ij})$, where the functions a^{ij} are Borel measurable on $\mathbb{R}^d \times [0,T]$, satisfies condition (H1) and for each ball $U \subset \mathbb{R}^d$ there is a sequence of symmetric positive definite matrices $A_k = (a_k^{ij})_{1\leqslant i,j\leqslant d}$ and numbers N, p, q with $p^{-1} + q^{-1} = (d+1)^{-1}$ such that

(i) $a_k^{ij} \in C^\infty(U \times [0,T])$, $\sup\limits_k \sup\limits_{U\times[0,T]} |A_k(x,t)| < \infty$, and $\lim\limits_{k\to\infty} \|A - A_k\|_{L^q(U\times(0,T))} = 0$,

(ii) the following estimate holds:

$$\|\varphi\|_{W_p^{1,2}(U\times(0,T))} \leqslant N\|\partial_t\varphi + a_k^{ij}\partial_{x_i}\partial_{x_j}\varphi\|_{L^p(U\times(0,T))}$$

for every $k \in \mathbb{N}$ and $\varphi \in C^\infty(U \times [0,T])$ such that $\varphi(\,\cdot\,,T) = 0$ and $\operatorname{supp} \varphi(\,\cdot\,,t) \subset U$ for all $t \in [0,T]$.

Example 3.8. Using estimates with Lyapunov functions (cf., for example, [33]), one can obtain the following sufficient uniqueness condition. Let the matrix A satisfy condition (H1), and let $a^{ij} \in VMO_{x,\mathrm{loc}}(\mathbb{R}^d \times [0,T])$. Assume that $\alpha > 0$ and $r > 2$. Suppose that for all $(x,t) \in \mathbb{R}^d \times [0,T]$ and some positive numbers c_1, c_2, c_3 and $c_4 < \alpha$, $c_5 < \alpha$

(i) $|x|^{r-2}$ trace $A(x,t) + (r-2)|x|^{r-4}(A(x,t)x,x) + \alpha r|x|^{2r-4}(A(x,t)x,x)$
$\quad + |x|^{r-2}(b(x,t),x) \leqslant c_1$,

(ii) $|a^{ij}(x,t)| \leqslant c_2 \exp\bigl(c_4|x|^r\bigr)$ and $|b^i(x,t)| \leqslant c_3 \exp\bigl(c_5|x|^r\bigr)$.

Suppose also that ν is a probability measure on $\mathbb{R}^d$ such that $\exp\bigl(\alpha|x|^r\bigr) \in L^1(\nu)$. Then the set $\mathscr{P}_\nu$ consists of at most one element. Indeed, inequality (i) and the condition on the initial distribution ν enable us to conclude that $\exp\bigl(\alpha|x|^r\bigr) \in L^1(\mu, \mathbb{R}^d \times [0,T])$ for every probability solution μ to the Cauchy problem (1.2). Condition (ii) ensures that $a^{ij}, b^i \in L^1(\mu, \mathbb{R}^d \times [0,T])$. Therefore, Theorem 3.6 applies.

It should be noted that uniqueness is established under very broad assumptions on the diffusion matrix and drift. However, our global conditions impose restrictions on the whole class of probability solution for which we prove uniqueness. In addition, the verification of the assumptions of the theorem by means of Lyapunov functions involves restrictions on the growth of $|b(x,t)|$.

4 The Uniqueness of Probability Solutions if the Diffusion Matrix is Lipschitzian in x

In this section, we discuss a method of proving the uniqueness of probability solutions that requires stronger local regularity of the coefficients of the differential operator L, but enables one to weaken considerably the global assumptions on the coefficients and solutions. The principal results of this section were obtained in [6].

We assume that along with condition (H1) the following condition is fulfilled:

(H2) for every ball $U \subset \mathbb{R}^d$ there exists $\Lambda = \Lambda(U) > 0$ such that for all $x, y \in U$ and $t \in [0,T]$

$$|a^{ij}(x,t) - a^{ij}(y,t)| \leqslant \Lambda|x-y|;$$

As already noted in the introduction, condition (H1) implies the existence of a density ϱ for any probability solution μ with respect to the Lebesgue measure. Moreover, if along with (H1) and (H2) we have $b \in L^p_{\mathrm{loc}}(\mathbb{R}^d \times (0,T))$ for some $p > d+2$, then one can choose a continuous version of ϱ on $\mathbb{R}^d \times (0,T)$ such that for almost every $t \in (0,T)$ the function $\varrho(\,\cdot\,,t)$ belongs to $W^{1,p}(U)$ for every ball $U \subset \mathbb{R}^d$. Since for almost all $t \in (0,T)$ the measure $\mu_t(dx) = \varrho(x,t)\,dx$ is a probability measure on $\mathbb{R}^d$, the Harnack inequality yields that for every ball U in $\mathbb{R}^d$ and any interval $J \subset (0,T)$ there exists $C > 0$ such that $\varrho(x,t) \geqslant C$ for all $(x,t) \in U \times J$.

If we choose a continuous version of the density ϱ, then $\varrho(\,\cdot\,,t)$ may be a probability density not for every $t \in (0,T)$, but just for almost every t with respect to the Lebesgue measure on $[0,T]$. By the Fatou theorem, for every t

$$\int\limits_{\mathbb{R}^d} \varrho(x,t)\,dx \leqslant 1.$$

Note also that for every $\zeta \in C_0^\infty(\mathbb{R}^d)$ the function

$$t \mapsto \int\limits_{\mathbb{R}^d} \zeta(x)\varrho(x,t)\,dx$$

is continuous on $(0,T)$ and for almost all t it coincides with the function

$$t \mapsto \int\limits_{\mathbb{R}^d} \zeta(x)\, d\mu_t.$$

Therefore,

$$\lim_{t\to 0} \int\limits_{\mathbb{R}^d} \zeta(x)\varrho(x,t)\, dx = \int\limits_{\mathbb{R}^d} \zeta(x)\, d\nu.$$

Below, we will deal with the continuous version of the density ϱ.

For every measure μ given by a Sobolev class density ϱ with respect to the Lebesgue measure, its logarithmic gradient β_μ with respect to the metric generated by the matrix A is defined by the following formula:

$$\beta_\mu^i = \sum_{j=1}^{d} \bigl(\partial_{x_j} a^{ij} + a^{ij}\varrho^{-1}\partial_{x_j}\varrho\bigr).$$

Throughout this section, we assume that b is locally integrable to power $p > d+2$ with respect to the Lebesgue measure on $\mathbb{R}^d \times (0,T)$ and conditions (H1) and (H2) are fulfilled.

Suppose that there are two solutions to the Cauchy problem (1.2) in the class $\mathscr{P}_\nu$ given by densities σ and ϱ with respect to the Lebesgue measure. Then these densities are continuous on $\mathbb{R}^d \times (0,T)$. In addition, the functions σ and ϱ are strictly positive. Let $v(x,t) = \sigma(x,t)/\varrho(x,t)$. The function v is continuous and positive on $\mathbb{R}^d \times (0,T)$.

Lemma 4.1. *Suppose that for every $\lambda > 0$*

$$\int\limits_{\mathbb{R}^d} e^{\lambda(1-v(x,t))}\varrho(x,t)\, dx \leqslant 1 \tag{4.1}$$

for almost all $t \in (0,T)$. Then $v \equiv 1$, i.e., $\sigma = \varrho$.

Proof. Let t be such that $\varrho(\,\cdot\,,t)$ and $\sigma(\,\cdot\,,t)$ are probability densities and the inequality (4.1) holds for all natural numbers λ. We observe that the set of points t where this is not true is a set of zero Lebesgue measure. If there is a ball $U \subset \mathbb{R}^d$ such that $v(x,t) \leqslant 1-\delta$ for each $x \in U$ and some $\delta > 0$, then

$$e^{\lambda\delta} \int\limits_U \varrho\, dx \leqslant \int\limits_U e^{\lambda(1-v(x,t))}\varrho(x,t)\, dx \leqslant 1.$$

Letting $\lambda \to \infty$, we obtain a contradiction. Therefore, $v(x,t) \geqslant 1$ for all $x \in \mathbb{R}^d$. Assume now that there is a ball $V \subset \mathbb{R}^d$ such that for each $x \in V$ and some $\gamma > 0$ we have $v(x,t) \geqslant 1+\gamma$. Then

$$\begin{aligned} 1 = \int\limits_{\mathbb{R}^d} v(x,t)\varrho(x,t)\, dx &= \int\limits_V v(x,t)\varrho(x,t)\, dx + \int\limits_{\mathbb{R}^d\setminus V} v(x,t)\varrho(x,t)\, dx \\ &\geqslant (1+\gamma)\int\limits_V \varrho(x,t)\, dx + \int\limits_{\mathbb{R}^d\setminus V} \varrho(x,t)\, dx = 1 + \gamma \int\limits_V \varrho(x,t)\, dx. \end{aligned}$$

We again obtain a contradiction. $\square$

The following lemma is the main step in our approach.

Lemma 4.2. *Assume that $\psi \in C_0^\infty(\mathbb{R}^d)$, $\psi \geqslant 0$, and $0 < t < T$. Let f be one of the functions $e^{\lambda(1-z)}$ and $e^{\lambda(1-z)} - e^\lambda$. Then*

$$\int\limits_{\mathbb{R}^d} f(v(x,t))\varrho(x,t)\psi(x)\,dx \leqslant f(1)\int\limits_{\mathbb{R}^d}\psi(x)\,d\nu + \int\limits_0^t\int\limits_{\mathbb{R}^d}\varrho f(v)L\psi\,dx\,ds. \tag{4.2}$$

If, in addition, $(b-\beta_\mu)\varrho \in L^1(U\times(0,1))$ for every ball $U \subset \mathbb{R}^d$, then

$$\int\limits_{\mathbb{R}^d} f(v(x,t))\varrho(x,t)\psi(x)\,dx \leqslant f(1)\int\limits_{\mathbb{R}^d}\psi(x)\,d\nu + \frac{1}{2}\int\limits_0^t\int\limits_{\mathbb{R}^d}\varrho(A\nabla\psi,\nabla\psi)\psi^{-1}|f'(v)|^2 f''(v)^{-1}\,dx\,ds$$
$$+\int\limits_0^t\int\limits_{\mathbb{R}^d} f(v)(b-\beta_\mu,\nabla\psi)\varrho\,dx\,ds. \tag{4.3}$$

A complete proof of the lemma can be found in [6], so we only give an informal reasoning, which, however, becomes rigorous if we deal with the class of smooth functions on $\mathbb{R}^d\times[0,T]$ and assume that the initial condition is given by a strictly positive density.

Proof of Lemma 4.2. We set

$$h^i = b^i - \sum_{j=1}^d \partial_{x_j} a^{ij}.$$

We recall certain relations following from the Leibniz formula and the chain rule. For every ξ,η in $C^\infty(\mathbb{R}^d\times(0,T))$ and $\Phi\in C^\infty(\mathbb{R})$ we have the following equalities:

$$L^*\Phi(\xi) = \Phi'(\xi)L^*\xi + \Phi''(\xi)(A\nabla\xi,\nabla\xi) + (\xi\Phi'(\xi)-\Phi(\xi))\operatorname{div} h,$$
$$L^*(\xi\cdot\eta) = \eta L^*\xi + \xi L^*\eta + 2(A\nabla\xi,\nabla\eta) + \xi\eta\operatorname{div} h.$$

Thus, $\sigma = v\varrho$ and

$$\partial_t\sigma = L^*\sigma \quad\text{and}\quad \partial_t\varrho = L^*\varrho.$$

Multiplying the equation $\partial_t\varrho = L^*\varrho$ by v and subtracting the obtained equality from the equation $\partial_t\sigma = L^*\sigma$, we arrive at the following equation for the function v:

$$\varrho\partial_t v = \varrho L^* v + 2(A\nabla\varrho,\nabla v) + \varrho v\operatorname{div} h.$$

Multiplying the latter relationship by the function $f'(v)$ and taking into account the equalities $\partial_t f(v) = f'(v)\partial_t v$ and $\nabla f(v) = f'(v)\nabla v$, we obtain

$$\varrho\partial_t(f(v)) = \varrho f'(v)L^*v + 2(A\nabla\varrho,\nabla f(v)) + \varrho v f'(v)\operatorname{div} h.$$

Since

$$f'(v)L^*v = L^*f(v) - f''(v)(A\nabla v,\nabla v) - (vf'(v)-f(v))\operatorname{div} h,$$

we have

$$\varrho\partial_t(f(v)) = \varrho L^*f(v) + 2(A\nabla\varrho,\nabla f(v)) + \varrho f(v)\operatorname{div} h - \varrho f''(v)(A\nabla v,\nabla v).$$

Adding the last equality to the equality $f(v)\partial_t\varrho = f(v)L^*\varrho$, we find that

$$\partial_t(\varrho f(v)) = L^*(\varrho f(v)) - \varrho f''(v)(A\nabla v, \nabla v).$$

Multiplying this equation by the function ψ and integrating, we obtain

$$\int_0^t\int_{\mathbb{R}^d} \partial_t(f(v)\varrho)\psi\,dx\,ds + \int_0^t\int_{\mathbb{R}^d} \varrho(A\nabla v, \nabla v)f''(v)\psi\,dx\,ds = \int_0^t\int_{\mathbb{R}^d} \psi L^*(\varrho f(v))\,dx\,ds.$$

Applying the Newton–Leibniz formula and taking into account the trivial equality $v(x,0)\equiv 1$, we FIND

$$\int_0^t\int_{\mathbb{R}^d} \partial_t(f(v)\varrho)\psi\,dx\,ds = \int_{\mathbb{R}^d} f(v(x,t))\varrho(x,t)\psi(x)\,dx - f(1)\int_{\mathbb{R}^d}\psi(x)\,d\nu.$$

Since L^* is adjoint to L, one has

$$\int_0^t\int_{\mathbb{R}^d} \psi L^*(\varrho f(v))\,dx\,ds = \int_0^t\int_{\mathbb{R}^d} \varrho f(v)L\psi\,dx\,ds.$$

Thus, we have the following equality:

$$\int_{\mathbb{R}^d} f(v(x,t))\varrho(x,t)\psi(x)\,dx + \int_0^t\int_{\mathbb{R}^d} \varrho(A\nabla v, \nabla v)f''(v)\psi\,dx\,ds$$
$$= f(1)\int_{\mathbb{R}^d}\psi(x)\,d\nu + \int_0^t\int_{\mathbb{R}^d}\varrho f(v)L\psi\,dx\,ds. \tag{4.4}$$

Taking into account condition (H1) and the inequalities $f''\geqslant 0$ and $\psi\geqslant 0$, we obtain the estimate (4.2). To derive the estimate (4.3), we need some additional transformations in (4.4). Integrating by parts, we obtain

$$\sum_{i,j=1}^d\int_0^t\int_{\mathbb{R}^d}\varrho f(v)a^{ij}\partial_{x_i}\partial_{x_j}\psi\,dx\,ds = -\int_0^t\int_{\mathbb{R}^d}\varrho f'(v)(A\nabla v,\nabla\psi)\,dx\,ds - \int_0^t\int_{\mathbb{R}^d}\varrho f(v)(\beta_\mu,\nabla\psi)\,dx\,ds.$$

The Cauchy inequality yields

$$|(A\nabla v,\nabla\psi)| \leqslant \sqrt{(A\nabla v,\nabla v)}\cdot\sqrt{(A\nabla\psi,\nabla\psi)}.$$

Applying the inequality $qr\leqslant 2^{-1}(q^2+r^2)$ with

$$q = |f''(v)|^{1/2}|\psi|^{1/2}(A\nabla v,\nabla v)^{1/2},$$
$$r = |f''(v)|^{-1/2}|\psi|^{-1/2}(A\nabla\psi,\nabla\psi)^{1/2}|f'(v)|,$$

we obtain

$$\int_0^t \int_{\mathbb{R}^d} f'(v)(A\nabla v, \nabla\psi)\varrho\, dx\, ds \leqslant \frac{1}{2}\int_0^t \int_{\mathbb{R}^d} \varrho(A\nabla v, \nabla v) f''(v)\psi\, dx\, ds$$
$$+ \frac{1}{2}\int_0^t \int_{\mathbb{R}^d} \varrho(A\nabla\psi, \nabla\psi)\psi^{-1}|f'(v)|^2 f''(v)^{-1}\, dx\, ds.$$

Thus,

$$\int_{\mathbb{R}^d} f(v(x,t))\varrho(x,t)\psi(x)\, dx + \frac{1}{2}\int_0^t \int_{\mathbb{R}^d} \varrho(A\nabla v, \nabla v) f''(v)\psi\, dx\, ds$$
$$\leqslant f(1)\int_{\mathbb{R}^d} \psi(x)\, d\nu + \frac{1}{2}\int_0^t \int_{\mathbb{R}^d} \varrho(A\nabla\psi, \nabla\psi)\psi^{-1}|f'(v)|^2 f''(v)^{-1}\, dx\, ds + \int_0^t \int_{\mathbb{R}^d} f(v)(b-\beta_\mu, \nabla\psi)\, dx\, ds.$$

Since $f'' \geqslant 0$ and condition (H1) holds, the last inequality implies the estimate (4.3). $\square$

The following result shows that, in the case of sufficiently regular coefficients a^{ij} and b^i in Theorem 3.6, one can considerably weaken the restrictions on the class of probability solutions in which uniqueness is established.

Theorem 4.3. *Suppose that conditions* (H1), (H2) *hold and* $b \in L^p_{\mathrm{loc}}(\mathbb{R}^d \times (0,T))$ *for some* $p > d+2$. *Assume also that for some measure* μ *in the class* $\mathscr{P}_\nu$

$$a^{ij}, b^i \in L^1(\mu, \mathbb{R}^d \times (0,T)) \quad \forall\ 1 \leqslant i,\ j \leqslant d.$$

Then the set $\mathscr{P}_\nu$ *consists of exactly one element* μ.

Proof. Let the measure μ be given by a density ϱ with respect to the Lebesgue measure. Suppose that there is yet another measure in $\mathscr{P}_\nu$ given by a density σ. As above, put $v = \sigma/\varrho$. Let $\psi(x) = \zeta(x/N)$, where $\zeta \in C_0^\infty(\mathbb{R}^d)$ is a nonnegative function such that $\zeta(x) = 1$ if $|x| \leqslant 1$ and $\zeta(x) = 0$ if $|x| > 2$, and there exists a number $K > 0$ such that for all x

$$|\zeta(x)| \leqslant K, \quad |\nabla\zeta(x)| \leqslant K, \quad \sum_{i,j=1}^d |\partial_{x_i}\partial_{x_j}\zeta(x)| \leqslant K.$$

Let $f(z) = e^{\lambda(1-z)}$. It is clear that $|f(z)| \leqslant e^\lambda$ for $z \geqslant 0$. By the inequality (4.2) of Lemma 4.2,

$$\int_{\mathbb{R}^d} e^{\lambda(1-v(x,t))}\varrho(x,t)\zeta(x/N)\, dx$$
$$\leqslant \int_{\mathbb{R}^d} \zeta(x/N)\, d\nu + e^\lambda K N^{-2} \sum_{i,j=1}^d \int_0^t \int_{N<|x|<2N} |a^{ij}|\varrho\, dx\, ds + e^\lambda K N^{-1} \sum_{i=1}^d \int_0^t \int_{N<|x|<2N} |b^i|\varrho\, dx\, ds.$$

By assumption, $|a^{ij}|\varrho, |b^i|\varrho \in L^1(\mathbb{R}^d \times (0,T))$. Letting $N \to \infty$, we obtain (4.1). Therefore, Lemma 4.1 yields the required assertion. $\square$

Remark 4.4. The conditions on the coefficients in this theorem can be weakened as follows. Let $V \in C^2(\mathbb{R}^d)$ be a positive function such that $\lim\limits_{|x|\to\infty} V(x) = +\infty$. If, in addition to conditions (H1) and (H2) and the inclusion $b \in L^p_{\mathrm{loc}}(\mathbb{R}^d \times (0,T)$ with some $p > d+2$, there is a measure $\mu \in \mathscr{P}_\nu$ satisfying the condition

$$\lim_{N\to\infty} \int\limits_0^T \int\limits_{N\leqslant V\leqslant N+1} \frac{|LV|}{V} + \frac{|\sqrt{A}\nabla V|^2}{V^2}\, d\mu = 0,$$

then μ is the only element in $\mathscr{P}_\nu$.

Theorem 4.5. *Suppose that conditions* (H1) *and* (H2) *hold and* $b \in L^p_{\mathrm{loc}}(\mathbb{R}^d \times (0,T))$ *for some* $p > d+2$. *Assume also that for some measure* μ *in the class* $\mathscr{P}_\nu$

$$a^{ij}, b^i - \beta^i_\mu \in L^1(\mu, \mathbb{R}^d \times (0,T)) \quad \forall\ 1 \leqslant i,\ j \leqslant d.$$

Then the set $\mathscr{P}_\nu = 1$ *consists of this element* μ.

Proof. Let the measure μ be given by a density ϱ with respect to the Lebesgue measure. Suppose that there is yet another measure in $\mathscr{P}_\nu$ given by a density σ. As above, we set $v = \sigma/\varrho$. Let $\psi(x) = \zeta(x/N)$, where $\zeta \in C_0^\infty(\mathbb{R}^d)$ is a nonnegative function such that $\zeta(x) = 1$ if $|x| \leqslant 1$ and $\zeta(x) = 0$ if $|x| > 2$, and there exists a number $K > 0$ such that for all x

$$|\zeta(x)| \leqslant K, \quad |\nabla\zeta(x)| \leqslant K, \quad |\nabla\zeta(x)|^2\zeta^{-1}(x) \leqslant K.$$

Let $f(z) = e^{\lambda(1-z)}$. Then $|f(z)| \leqslant e^\lambda$ for $z \geqslant 0$. Applying the inequality (4.3) of Lemma 4.2, we obtain

$$\int\limits_{\mathbb{R}^d} e^{\lambda(1-v(x,t))}\varrho(x,t)\zeta(x/N)\,dx$$
$$\leqslant \int\limits_{\mathbb{R}^d} \zeta(x/N)\,d\nu + e^\lambda K N^{-2} \sum_{i,j=1}^d \int\limits_0^t \int\limits_{\mathbb{R}^d} |a^{ij}|\varrho\,dx\,ds + e^\lambda K N^{-1} \sum_{i=1}^d \int\limits_0^t \int\limits_{\mathbb{R}^d} |b^i - \beta^i_\mu|\varrho\,dx\,ds.$$

By assumption, $|a^{ij}|\varrho, |b^i - \beta^i_\mu|\varrho \in L^1(\mathbb{R}^d \times (0,T))$. Letting $N \to \infty$ and applying Lemma 4.1, we obtain the required assertion. $\square$

Theorem 4.6. *Let conditions* (H1) *and* (H2) *hold, and let* $b \in L^p_{\mathrm{loc}}(\mathbb{R}^d \times (0,T))$ *for some* $p > d+2$. *Suppose that there exists a positive function* $V \in C^2(\mathbb{R}^d)$ *such that* $V(x) \to +\infty$ *as* $|x| \to +\infty$ *and for some number* $C > 0$ *and all* $(x,t) \in \mathbb{R}^d \times [0,T]$

$$LV(x,t) \leqslant C, \quad |\sqrt{A(x,t)}\nabla V(x)| \leqslant C.$$

Then $\mathscr{P}_\nu \leqslant 1$ *consists of at most one element.*

Proof. Suppose that the class $\mathscr{P}_\nu$ contains two measures given by densities σ and ϱ with respect to the Lebesgue measure. We set $v = \sigma/\varrho$. Let $\psi(x) = \zeta(N^{-1}V(x))$, where $\zeta \in C_0^\infty(\mathbb{R})$ is a nonnegative function such that $\zeta(z) = 1$ if $|z| < 1$ and $\zeta(z) = 0$ if $|z| > 2$, and, in addition, $\zeta'(z) \leqslant 0$ if $z > 0$, and there exists a number $K > 0$ such that for all z

$$|\zeta(z)| \leqslant K, \quad |\zeta'(z)| \leqslant K, \quad |\zeta''(z)| \leqslant K.$$

Let $f(z) = e^{\lambda(1-z)} - e^{\lambda}$. Then $f(z) \leqslant 0$ and $|f(z)| \leqslant 2e^{\lambda}$ for $z \geqslant 0$. We observe that

$$f(v)\zeta' LV \leqslant Cf(v)\zeta'$$

since $f(v)\zeta' \geqslant 0$. Using the inequality (4.2) of Lemma 4.2, we obtain

$$\int\limits_{\mathbb{R}^d} (e^{\lambda(1-v(x,t))} - e^{\lambda})\varrho(x,t)\zeta(N^{-1}V(x))\,dx$$

$$\leqslant (1-e^{\lambda}) \int\limits_{\mathbb{R}^d} \zeta(N^{-1}V(x))\,d\nu + 2e^{\lambda}CK(N^{-1}+CN^{-2}) \int\limits_0^t \int\limits_{\mathbb{R}^d} \varrho\,dx\,ds.$$

Letting $N \to \infty$, we find

$$\int\limits_{\mathbb{R}^d} (e^{\lambda(1-v(x,t))} - e^{\lambda})\varrho(x,t)\,dx \leqslant (1-e^{\lambda}) \int\limits_{\mathbb{R}^d} d\nu.$$

Since for almost all $t \in (0,T)$ the function $\varrho(\,\cdot\,,t)$ is a probability density and ν is a probability measure, for almost all t we have

$$\int\limits_{\mathbb{R}^d} e^{\lambda(1-v(x,t))}\varrho(x,t)\,dx \leqslant 1.$$

Applying Lemma 4.1 we complete the proof. □

Remark 4.7. In the assumptions of the theorem, the inequalities $LV \leqslant C$ and $|\sqrt{A}\nabla V| \leqslant C$ can be replaced by $LV \leqslant CV$ and $|\sqrt{A}\nabla V| \leqslant CV$ respectively. Indeed, if $LV \leqslant CV$ and $|\sqrt{A}\nabla V| \leqslant CV$, then, replacing V with $\ln V$, we obtain the inequalities

$$L(\ln V) = V^{-1}LV - V^{-2}|\sqrt{A}\nabla V|^2 \leqslant C,$$
$$|\sqrt{A}\nabla(\ln V)| = V^{-1}|\sqrt{A}\nabla V| \leqslant C.$$

Let us consider an application of the last theorem.

Example 4.8. Let $V(x) = \ln(\ln(1+|x|))$ if $|x| > 1$. Then, whenever $|x| > 1$, we have

$$|\sqrt{A(x,t)}\nabla V(x)|^2 = (A(x,t)\nabla V(x), \nabla V(x)) = \frac{(A(x,t)x,x)}{|x|^2(|x|+1)^2\ln^2(|x|+1)}.$$

Let us calculate $LV(x,t)$ for $|x| > 1$:

$$LV(x,t) = -\frac{(A(x,t)x,x)}{|x|^2(1+|x|)^2\ln(1+|x|)}\Big(1 + \frac{1}{\ln(1+|x|)} + \frac{1+|x|}{|x|}\Big)$$
$$+ \frac{\operatorname{trace} A(x,t)}{|x|(1+|x|)\ln(1+|x|)} + \frac{(b(x,t),x)}{|x|(|x|+1)\ln(|x|+1)}.$$

To ensure the assumptions of the theorem, it suffices to have the estimates

$$\operatorname{trace} A(x,t) \leqslant C + C|x|^2\ln(1+|x|),$$
$$(A(x,t)x,x) \leqslant C + C|x|^4\ln^2(1+|x|),$$
$$(b(x,t),x) \leqslant C + C|x|^2\ln(1+|x|)$$

for all $(x,t) \in \mathbb{R}^d \times (0,T)$.

Combining Theorem 4.6, the last example and Theorem 2.6.1 on the existence of a probability solution to the Cauchy problem (1.2) in [33], we obtain the following existence and uniqueness theorem.

Theorem 4.9. *Suppose that conditions* (H1) *and* (H2) *hold and for every ball* $U \subset \mathbb{R}^d$

$$\sup_{(x,t)\in U\times[0,1]} |b(x,t)| < \infty.$$

Suppose also that there exists a number $C > 0$ *such that*

$$\begin{aligned} \operatorname{trace} A(x,t) &\leqslant C + C|x|^2 \ln(1+|x|), \\ (A(x,t)x,x) &\leqslant C + C|x|^4 \ln^2(1+|x|), \\ (b(x,t),x) &\leqslant C + C|x|^2 \ln(1+|x|) \end{aligned}$$

for all $(x,t) \in \mathbb{R}^d \times (0,T)$. *Then for every Borel probability measure* ν *on* $\mathbb{R}^d$ *there exists a unique measure* μ *defined by a family of probability measures* $(\mu_t)_{0<t<1}$ *on* $\mathbb{R}^d$ *satisfying the Cauchy problem* (1.2).

Apart from the considered methods of proving the uniqueness of a probability solution to the Cauchy problem (1.2), one should note yet another one suggested in [5]. The main idea is as follows. Let $P = \partial_t + L$. Suppose that $\mathscr{K}_\nu$ is a convex subset of the set of probability solutions to the Cauchy problem with initial condition ν such that for every measure $\mu \in \mathscr{K}_\nu$ we have

$$\overline{P(C_0^\infty([0,T)\times\mathbb{R}^d))} = L^1(\mu,(0,T)\times\mathbb{R}^d),$$

where $\overline{P(\,\cdot\,)}$ denotes L^1-closure. Then the set $\mathscr{K}_\nu$ consists of at most one element. Indeed, if $\mu^1, \mu^2 \in \mathscr{K}_\nu$, then $\mu = (\mu^1+\mu^2)/2 \in \mathscr{K}_\nu$ and $\mu^1 = \varrho^1\mu$, $\mu^2 = \varrho^2\mu$, and the functions ϱ^1 and ϱ^2 are bounded. In addition, one has the equality

$$\int_0^T \int_{\mathbb{R}^d} Pu(x,t)(\varrho^1(x,t) - \varrho^2(x,t))\,d\mu = 0 \quad \forall u \in C_0^\infty([0,T)\times\mathbb{R}^d),$$

which implies that $\varrho^1(x,t) - \varrho^2(x,t) = 0$ for almost all (x,t) with respect to the measure μ. In [5], there are examples of such sets $\mathscr{K}_\nu$. In particular, if the diffusion matrix is the unit matrix and $b \in L^p(U\times(0,T))$ for every ball U and some $p > d+2$, then for $\mathscr{K}_\nu$ one can take the set of all probability solutions μ to the Cauchy problem such that $b \in L^1(\mu, U\times(0,T))$ for every ball $U \in \mathbb{R}^d$ and

$$b - \beta_\mu \in L^1(\mu, \mathbb{R}^d \times (0,T)), \tag{4.5}$$

where β_μ is the logarithmic derivative of the measure μ. It is readily seen that Theorem 4.5 considerably reinforces the last assertion. It turns out that if the inclusion (4.5) is fulfilled for at least one probability solution, then there are no other probability solutions independently of whether they a priori satisfy this inclusion or not.

5 Uniqueness of Integrable Solutions

In the previous sections, we discussed the uniqueness of probability solutions. However, many papers (cf., for example, [7, 8, 15, 16]) are concerned with the uniqueness of integrable solutions and the uniqueness of a probability solution is obtained as a consequence. As we see from Example 2.7, this approach does not always lead to precise results: the uniqueness of a probability solution can take place also in the case where there are several linearly independent integrable solutions. Moreover, the conditions for uniqueness in the class $\mathscr{I}_\nu$ differ principally from those in the class $\mathscr{P}_\nu$.

In this section, we obtain sufficient conditions for the uniqueness of integrable solutions to the Cauchy problem (1.2) in terms of Lyapunov functions, which enables us to deal with operators L with rapidly growing coefficients. The principal results presented below were obtained in [9] in the case of a unit diffusion matrix. However, the case of a nonconstant diffusion matrix under conditions (H1) and (H2) can be treated in a similar way.

We recall that, according to [1], every locally finite measure μ satisfying the equation $\partial_t\mu = L^*\mu$ with $b \in L^p_{\mathrm{loc}}(|\mu|, \mathbb{R}^d \times (0,T))$, where $p > d+2$, possesses a continuous density ϱ. As in the case of a probability solution, we need several auxiliary lemmas.

Lemma 5.1. *Suppose that conditions* (H1) *and* (H2) *hold. Let a measure $\mu = \varrho\,dx\,dt$ belong to the class $\mathscr{I}_\nu$. Then for every nonnegative function $\psi \in C_0^\infty(\mathbb{R}^d)$ and any numbers s and t in $(0,T)$ with $s < t$*

$$\int\limits_{\mathbb{R}^d} \psi(x)|\varrho(x,t)|\,dx \leqslant \int\limits_{\mathbb{R}^d} \psi(x)|\varrho(x,s)|\,dx + \int\limits_s^t\int\limits_{\mathbb{R}^d} L\psi(x,\tau)|\varrho(x,\tau)|\,dx\,d\tau.$$

We begin with an informal reasoning, clarifying this assertion and becoming rigorous in the case of smooth coefficients. Then a complete proof will be given.

Assume that $f \in C^2(\mathbb{R})$ and $f'' \geqslant 0$. Let $\psi \in C_0^\infty(\mathbb{R}^d)$, and let $\psi \geqslant 0$. We set

$$h^i = b^i - \sum_{j=1}^d \partial_{x_j} a^{ij}.$$

We have the equality

$$L^* f(\varrho) = f'(\varrho)L^*\varrho + f''(\varrho)(A\nabla\varrho, \nabla\varrho) + (\varrho f'(\varrho) - f(\varrho))\,\mathrm{div}\,h.$$

Since $f'' \geqslant 0$ and $\partial_t\varrho = L^*\varrho$,

$$L^* f(\varrho) \geqslant \partial_t(f(\varrho)) + (\varrho f'(\varrho) - f(\varrho))\,\mathrm{div}\,h.$$

Multiplying the last inequality by ψ and integrating, we find

$$\begin{aligned}\int\limits_{\mathbb{R}^d} \psi(x) f(\varrho(x,t))\,dx \leqslant \int\limits_{\mathbb{R}^d} \psi(x) f(\varrho(x,s))\,dx + \int\limits_s^t\int\limits_{\mathbb{R}^d} L\psi(x,\tau) f(\varrho(x,\tau))\,dx\,d\tau \\ - \int\limits_s^t\int\limits_{\mathbb{R}^d} (\varrho(x,\tau) f'(\varrho(x,\tau)) - f(\varrho(x,\tau)))\,\mathrm{div}\,h\,dx\,d\tau.\end{aligned}$$

This inequality does not involve the second order derivative of the function f. Therefore, it remains valid for functions that are convex downwards and Lipschitz. Let us take $f(z) = |z|$. Then $zf'(z) - f(z) = 0$, and we obtain the desired inequality

$$\int\limits_{\mathbb{R}^d} \psi(x)|\varrho(x,t)|\,dx \leqslant \int\limits_{\mathbb{R}^d} \psi(x)|\varrho(x,s)|\,dx + \int\limits_s^t \int\limits_{\mathbb{R}^d} L\psi(x,\tau)|\varrho(x,\tau)|\,dx\,d\tau.$$

Now, we give the rigorous proof.

Proof of Lemma 5.1. Let $\omega \in C_0^\infty(\mathbb{R}^d)$, $\omega \geqslant 0$, $\|\omega\|_{L^1} = 1$, $\omega_\varepsilon(x) = \varepsilon^{-d}\omega(x/\varepsilon)$. Let g_ε denote the convolution $g * \omega_\varepsilon$ for any locally integrable function g. The function ϱ_ε is smooth with respect to x on every compact set in $\mathbb{R}^d \times (0,T)$. We observe that for all $x \in \mathbb{R}^d$ the function $t \mapsto \varrho_\varepsilon(x,t)$ is absolutely continuous on every closed interval $J \subset (0,T)$. Indeed, for all $x \in \mathbb{R}^d$ we have the equality

$$\partial_t \varrho_\varepsilon = \partial_{x_i}\partial_{x_j}(a^{ij}\varrho)_\varepsilon - \partial_{x_i}(b^i\varrho)_\varepsilon. \tag{5.1}$$

in the sense of the theory of distributions on $t \in (0,T)$. Moreover, the right-hand side of this equality is integrable on J because $b\varrho$ and ϱ are integrable on $U \times J$ for every ball U. Let

$$\mathscr{A}\varphi = \sum_{i,j=1}^d a^{ij}\partial_{x_i}\partial_{x_j}\varphi, \quad \mathscr{A}^*\varphi = \sum_{i,j=1}^d \partial_{x_i}\partial_{x_j}(a^{ij}\varphi).$$

Let also

$$R^i_{\varrho,\varepsilon} = \sum_{j=1}^d (\partial_{x_j}(a^{ij}\varrho)_\varepsilon - \partial_{x_j}(a^{ij}\varrho_\varepsilon))$$

Then we can write (5.1) as follows:

$$\partial_t \varrho_\varepsilon = \mathscr{A}^*\varrho_\varepsilon - \operatorname{div}\big((b\varrho)_\varepsilon - R_{\varrho,\varepsilon}\big).$$

Let $\delta > 0$. We set $g_\delta \in C^1(\mathbb{R})$, $g_\delta(-r) = -g_\delta(r)$, $g_\delta(r) = \operatorname{sgn} r$ for $|r| > \delta$, $|g_\delta| \leqslant 1$ and $|g'_\delta| \leqslant C\delta^{-1}$ for some number $C > 0$. Let

$$f_\delta(r) = \int\limits_0^r g_\delta(y)\,dy, \quad r \in \mathbb{R}.$$

Note that $f_\delta \in C^2(\mathbb{R})$ and $\lim\limits_{\delta\to 0} f_\delta(r) = |r|$, $\lim\limits_{\delta\to 0} f'_\delta(r) = \operatorname{sgn} r$. Let $\psi \in C_0^\infty(\mathbb{R}^d)$. Integrating by parts, we obtain the equality

$$\int\limits_s^t \int\limits_{\mathbb{R}^d} f_\delta(\varrho_\varepsilon)\mathscr{A}\psi\,dx\,d\tau = \int\limits_s^t \int\limits_{\mathbb{R}^d} \mathscr{A}\big(\psi f'_\delta(\varrho_\varepsilon)\big)\varrho_\varepsilon + f''_\delta(\varrho_\varepsilon)(A_d\nabla\varrho_\varepsilon, \nabla\varrho_\varepsilon)\psi\,dx\,d\tau - W_\varepsilon - V_\varepsilon,$$

where

$$W_\varepsilon = \int\limits_s^t \int\limits_{\mathbb{R}^d} (\varrho_\varepsilon f'_\delta(\varrho_\varepsilon) - f_\delta(\varrho_\varepsilon))\partial_{x_i}a^{ij}\partial_{x_j}\psi\,dx\,dt,$$

$$V_\varepsilon = \int\limits_s^t \int\limits_{\mathbb{R}^d} \psi f''_\delta(\varrho_\varepsilon)\varrho_\varepsilon\partial_{x_i}\varrho_\varepsilon\partial_{x_i}a^{ij}\,dx\,dt.$$

Note that

$$\int_s^t \int_{\mathbb{R}^d} \mathscr{A}\big(\psi f_\delta'(\varrho_\varepsilon)\big)\varrho_\varepsilon \, dx\, d\tau = \int_s^t \int_{\mathbb{R}^d} \partial_t (f_\delta(\varrho_\varepsilon))\psi \, dx\, d\tau$$

$$- \int_s^t \int_{\mathbb{R}^d} \big[((b\varrho)_\varepsilon - R_{\varrho,\varepsilon}, \nabla\psi) f_\delta'(\varrho_\varepsilon) + ((b\varrho)_\varepsilon - R_{\varrho,\varepsilon}, \nabla\varrho_\varepsilon) f_\delta''(\varrho_\varepsilon)$$

Thus, we have the equality

$$\int_{\mathbb{R}^d} \psi(x) f_\delta(\varrho_\varepsilon(x,t))\, dx + \int_s^t \int_{\mathbb{R}^d} f_\delta''(\varrho_\varepsilon)(A_d\nabla\varrho_\varepsilon, \nabla\varrho_\varepsilon)\psi \, dx\, d\tau$$

$$= \int_{\mathbb{R}^d} \psi(x) f_\delta(\varrho_\varepsilon(x,s))\, dx + \int_s^t \int_{\mathbb{R}^d} f_\delta(\varrho_\varepsilon)\mathscr{A}\psi + f_\delta'(\varrho_\varepsilon)((b\varrho)_\varepsilon, \nabla\psi)\, dx\, d\tau + W_\varepsilon + V_\varepsilon + Z_\varepsilon,$$

where

$$Z_\varepsilon = -\int_s^t \int_{\mathbb{R}^d} \big[(R_{\varrho,\varepsilon}, \nabla\psi) f_\delta'(\varrho_\varepsilon) + ((b\varrho)_\varepsilon - R_{\varrho,\varepsilon}, \nabla\varrho_\varepsilon) f_\delta''(\varrho_\varepsilon)\big]\psi \, dx\, d\tau.$$

Let a ball U contain the support of the function ψ. Let $\gamma = \gamma(U) > 0$, and let $\Lambda = \Lambda(U) > 0$ be numbers from conditions (H1) and (H2). Recall that $f_\delta'' \geqslant 0$. Then

$$\int_s^t \int_{\mathbb{R}^d} f_\delta''(\varrho_\varepsilon)(A\nabla\varrho_\varepsilon, \nabla\varrho_\varepsilon)\psi \, dx\, d\tau \geqslant \gamma \int_s^t \int_{\mathbb{R}^d} f_\delta''(\varrho_\varepsilon)|\nabla\varrho_\varepsilon|^2\psi \, dx\, d\tau.$$

Let us estimate W_ε. Note that $|rf_\delta'(r) - f_\delta(r)| \leqslant C\delta$. We obtain the inequality

$$|W_\varepsilon| \leqslant d\Lambda\delta T|U| \max_x |\nabla\psi(x)|.$$

Let us estimate V_ε. Since $f_\delta''(r) = 0$ for $|r| > \delta$ and $|f_\delta''(r)| \leqslant C\delta^{-1}$, we have

$$|V_\varepsilon| \leqslant \frac{\gamma}{3} \int_s^t \int_{\mathbb{R}^d} f_\delta''(\varrho_\varepsilon)|\nabla\varrho_\varepsilon|^2\psi \, dx\, d\tau + 3\gamma^{-1} d\Lambda^2\delta T|U| \max_x |\psi|.$$

Finally, let us estimate Z_ε. We have

$$|Z_\varepsilon| \leqslant \int_s^t \int_{\mathbb{R}^d} |R_{\varrho,\varepsilon}||\nabla\psi| + 3\gamma^{-1}\delta^{-1}\psi|R_{\varrho,\varepsilon}|^2 \, dx\, d\tau$$

$$+ 3\gamma^{-1} \int_s^t \int_{\mathbb{R}^d} f_\delta''(\varrho_\varepsilon)|(b\varrho)_\varepsilon|^2\psi \, dx\, d\tau + \frac{\gamma}{3} \int_s^t \int_{\mathbb{R}^d} f_\delta''(\varrho_\varepsilon)|\nabla\varrho_\varepsilon|^2\psi \, dx\, d\tau.$$

Combining the obtained estimates, we arrive at the inequality

$$\int\limits_{\mathbb{R}^d} \psi(x) f_\delta(\varrho_\varepsilon(x,t))\,dx = \int\limits_{\mathbb{R}^d} \psi(x) f_\delta(\varrho_\varepsilon(x,s))\,dx + \int\limits_s^t \int\limits_{\mathbb{R}^d} f_\delta(\varrho_\varepsilon)\mathscr{A}\psi + f'(\varrho_\varepsilon)((b\varrho)_\varepsilon, \nabla\psi)\,dx\,d\tau$$
$$+ \delta \cdot C(d,\Lambda,\gamma,\psi) + \int\limits_s^t \int\limits_{\mathbb{R}^d} |R_{\varrho,\varepsilon}||\nabla\psi| + 3\gamma^{-1}\delta^{-1}\psi|R_{\varrho,\varepsilon}|^2\,dx\,d\tau$$
$$+ 3\gamma^{-1} \int\limits_s^t \int\limits_{\mathbb{R}^d} f''_\delta(\varrho_\varepsilon)|(b\varrho)_\varepsilon|^2\psi\,dx\,d\tau.$$

Letting $\varepsilon \to 0$ and taking into account that

$$\lim_{\varepsilon\to 0} \int\limits_s^t \int\limits_U |R_{\varrho,\varepsilon}|^2\,dx\,d\tau = 0,$$

we obtain the inequality

$$\int\limits_{\mathbb{R}^d} \psi(x) f_\delta(\varrho(x,t))\,dx = \int\limits_{\mathbb{R}^d} \psi(x) f_\delta(\varrho(x,s))\,dx + \int\limits_s^t \int\limits_{\mathbb{R}^d} f_\delta(\varrho)\mathscr{A}\psi + f'_\delta(\varrho)(b\varrho, \nabla\psi)\,dx\,d\tau$$
$$+ \delta \cdot C(d,\Lambda,\gamma,\psi) + 3\gamma^{-1} \int\limits_s^t \int\limits_{\mathbb{R}^d} f''_\delta(\varrho)|b\varrho|^2\psi\,dx\,d\tau.$$

Recall that $b \in L^p(|\mu|, U\times[0,T])$ and

$$\left|\int\limits_s^t \int\limits_{\mathbb{R}^d} f''_\delta(\varrho)|b\varrho|^2\psi\,dx\,d\tau\right| \leqslant C\delta^{(p-2)/p} \int\limits_s^t \int\limits_{\mathbb{R}^d} |b|^2|\varrho|^{2/p}\psi\,dx\,d\tau$$
$$\leqslant C\delta^{(p-2)/p} \biggl(\int\limits_s^t \int\limits_{\mathbb{R}^d} |b|^p|\varrho|\psi\,dx\,d\tau\biggr)^{2/p} \biggl(\int\limits_s^t \int\limits_{\mathbb{R}^d} \psi\,dx\,d\tau\biggr)^{2/(p-2)}.$$

Hence

$$\lim_{\delta\to 0} \int\limits_s^t \int\limits_{\mathbb{R}^d} f''_\delta(\varrho)|b\varrho|^2\psi\,dx\,d\tau = 0.$$

Letting $\delta \to 0$, we obtain the inequality

$$\int\limits_{\mathbb{R}^d} \psi(x)|\varrho(x,t)|\,dx \leqslant \int\limits_{\mathbb{R}^d} \psi(x)|\varrho(x,s)|\,dx + \int\limits_s^t \int\limits_{\mathbb{R}^d} \bigl[\mathscr{A}\psi + (b,\nabla\psi)\bigr]|\varrho(x,\tau)|\,dx\,d\tau. \qquad \square$$

The following lemma gives some information about the behavior of the solution as $t \to 0$.

Lemma 5.2. *Suppose that conditions* (H1) *and* (H2) *hold and the functions* a^{ij} *are continuous on* $\mathbb{R}^d \times [0,T]$. *Let the measure* $\mu = \varrho(x,t)\,dx\,dt$ *belong to* $\mathscr{I}_\nu$. *If* $\nu = 0$, *then for every ball* $U \subset \mathbb{R}^d$

$$\lim_{t\to 0} \int\limits_U |\varrho(x,t)|\,dx = 0.$$

Proof. The case $A = I$ is considered in [9]. The reasoning is similar in the present more general situation. Let us set $A(x,t) = A(x,-t)$ for $t \in [-T,0]$. Then the functions a^{ij} are continuous on $\mathbb{R}^d \times [-T,T]$ and satisfy conditions (H1) and (H2). Assume also that $\psi \in C_0^\infty(\mathbb{R}^d)$, $\psi \geqslant 0$, and U is a ball containing the support of the function ψ. We can redefine the functions a^{ij} outside U in such a way that conditions (H1) and (H2) will hold on $\mathbb{R}^d \times [0,T]$ with the same numbers $\gamma = \gamma(U)$, $M = M(U)$, and $\Lambda = \Lambda(U)$. We set $g_\delta(r) = \delta^{-1} r$ if $|r| < \delta$ and $g_\delta(r) = \operatorname{sgn} r$ if $|r| \geqslant \delta$.

Let $0 < t < T$. By [34, Theorem 1.3], there exists a solution Q to the Cauchy problem

$$\begin{aligned} \partial_s Q + A_0 Q = 0, \\ Q\big|_{s=t} = g_\delta(\varrho(\,\cdot\,,t)), \end{aligned} \tag{5.2}$$

on $\mathbb{R}^d \times (-1,t)$, where

$$A_0 Q = \sum_{i,j=1}^d a^{ij} \partial_{x_i} \partial_{x_j} Q.$$

The function $g_\delta(\varrho(\,\cdot\,,t))$ is continuous and bounded. Moreover, $|g_\delta(\varrho(x,t))| \leqslant 1$. According to [34, Theorems 1.1 and 1.3] (cf. also [14]), we have the inclusion

$$Q \in C_b((-T,t] \times \mathbb{R}^d) \bigcap C^{1,2}((-T,t) \times \mathbb{R}^d)$$

and the inequalities

$$|Q(x,u)| \leqslant C, \quad |\nabla Q(x,u)| \leqslant C(t-u)^{-1/2}$$

with some number C depending on γ, Λ, and d. Let $s < t$ be fixed. Applying Lemma 1.1 with $\varphi(x,u) = Q(x,u)\psi(x)$ on $[0,s] \times \mathbb{R}^d$, which is possible since φ belongs to the respective class, after integration by parts (which is again possible due to the stated properties of Q and the compactness of the support of ψ) we obtain

$$\begin{aligned} \int\limits_{\mathbb{R}^d} Q(x,s)\psi(x)\varrho(x,s)\,dx = \int\limits_0^s \int\limits_{\mathbb{R}^d} \Big[&(A(x,\tau)\nabla Q(x,\tau), \nabla\psi(x)) + Q(x,\tau) A_0 \varphi(x) \\ &+ (b(x,\tau), \nabla Q(x,\tau))\psi(x) + (b(x,\tau), \nabla\psi(x)) Q(x,\tau) \Big] \varrho(x,\tau)\,dx\,d\tau. \end{aligned} \tag{5.3}$$

We estimate the right=hand side of the equality by the expression

$$C \sup_{x\in\mathbb{R}^d} \big(|\varphi(x)| + |A\nabla\varphi(x)| + |A_0\varphi(x)|\big) \times \int\limits_0^s \int\limits_{\mathbb{R}^d} \big(1 + (t-\tau)^{-1/2}\big)\big(1 + |b(y,\tau)|\big) |\varrho(x,\tau)|\,dx\,d\tau.$$

Recall that $\varrho \in L^\infty((0,1), L^1(\mathbb{R}^d))$ and $b \in L^p(|\mu|, \mathbb{R}^d \times [0,1])$. Applying the Hölder inequality to the functions $|b|$ and $(t-\tau)^{-1/2}$, we obtain

$$\int\limits_{\mathbb{R}^d} Q(x,s)\varphi(x)\varrho(x,s)\,dx \leqslant C_1 t^{(2-p')/2p'},$$

where C_1 does not depend on s and t. Since ϱ and Q are continuous functions on $\mathbb{R}^d \times (0,t]$, letting $s \to t$, we obtain

$$\int\limits_{\mathbb{R}^d} \varphi(x) g_\delta(\varrho(x,t))\varrho(x,t)\,dx \leqslant C_1 t^{(2-p')/2p'}.$$

Letting $\delta \to 0$, we arrive at the inequality

$$\int\limits_{\mathbb{R}^d} \varphi(x)|\varrho(x,t)|\,dx \leqslant C_1 t^{(2-p')/2p'}.$$

Since $p > 2$, we have $p' < 2$ and $t^{(2-p')/2p'} \to 0$ as $t \to 0$. $\square$

Finally, we prove our main result on the uniqueness of integrable solutions.

Theorem 5.3. *Let conditions* (H1) *and* (H2) *hold , and let the functions a^{ij} be continuous on $\mathbb{R}^d \times [0,T]$. Let a measure $\mu = \varrho(x,t)\,dx\,dt$ in $\mathscr{I}_\nu$ be such that*

$$\frac{a^{ij}}{1+|x|^2}, \frac{b^i}{1+|x|} \in L^1(|\mu|, \mathbb{R}^d \times [0,T]).$$

Then the function

$$t \mapsto \int\limits_{\mathbb{R}^d} |\varrho(x,t)|\,dx$$

is decreasing on $(0,T)$. Moreover, the set

$$\mathscr{L}_\nu = \left\{\mu \in \mathscr{I}_\nu \colon \frac{|a^{ij}|}{1+|x|^2}, \frac{|b^i|}{1+|x|} \in L^1(|\mu|, \mathbb{R}^d \times [0,T])\right\}$$

consists of at most one element.

Proof. We set $\psi(x) = \zeta(x/N)$, where the function $\zeta \in C_0^\infty(\mathbb{R}^d)$ is such that $\zeta \geqslant 0$, $\zeta(x) = 1$ if $|x| \leqslant 1$ and $\zeta(x) = 0$ if $|x| > 2$, and for some number $K > 0$ and all x

$$|\zeta(x)| \leqslant K, \quad |\partial_{x_i}\zeta(x)| \leqslant K, \quad |\partial_{x_i}\partial_{x_j}\zeta(x)| \leqslant K.$$

Applying Lemma (5.1), we obtain the inequality

$$\int\limits_{\mathbb{R}^d} \psi_N(x)|\varrho(x,t)|\,dx \leqslant \int\limits_{\mathbb{R}^d} \psi_N(x)|\varrho(x,s)|\,dx + KN^{-2}(1+N^2)\sum_{i,j=1}^d \int\limits_0^t \int\limits_{N<|x|<2N} \frac{|a^{ij}|}{1+|x|^2}|\varrho|\,dx\,ds$$
$$+ KN^{-1}(1+N^{-1})\sum_{i=1}^d \int\limits_0^t \int\limits_{N<|x|<2N} \frac{|b^i|}{1+|x|}|\varrho|\,dx\,ds.\,dx\,d\tau.$$

Letting $N \to \infty$, we obtain

$$\int\limits_{\mathbb{R}^d} |\varrho(x,t)|\,dx \leqslant \int\limits_{\mathbb{R}^d} |\varrho(x,s)|\,dx.$$

Assume that there are two measures $\mu_1, \mu_2 \in \mathscr{L}_\nu$. Then the measure $\mu_1 - \mu_2$ satisfies the Cauchy problem with zero initial condition and

$$\frac{a^{ij}}{1+|x|^2}, \frac{b^i}{1+|x|} \in L^1(|\mu_1 - \mu_2|, \mathbb{R}^d \times [0,T]).$$

Consequently, whenever $t > s$ we have

$$\int\limits_{\mathbb{R}^d} |\varrho_1(x,t) - \varrho_2(x,t)|\,dx \leqslant \int\limits_{\mathbb{R}^d} |\varrho_1(x,s) - \varrho_2(x,s)|\,dx.$$

Applying Lemma 5.2, we obtain

$$\int\limits_{\mathbb{R}^d} |\varrho_1(x,t) - \varrho_2(x,t)|\,dx \leqslant 0$$

It follows that $\mu_1 = \mu_2$. □

Remark 5.4. The conditions on the coefficients in this theorem can be weakened as follows: the inclusions

$$\frac{a^{ij}}{1+|x|^2}, \quad \frac{b^i}{1+|x|} \in L^1(|\mu|, \mathbb{R}^d \times [0,T])$$

can be replaced with the equality

$$\lim_{N\to\infty} \int\limits_0^T \int\limits_{N\leqslant V\leqslant N+1} \frac{|LV|}{V} + \frac{|\sqrt{A}\nabla V|^2}{V^2}\,d|\mu| = 0,$$

where $V \in C^2(\mathbb{R}^d)$ is a positive function such that $\lim\limits_{|x|\to\infty} V(x) = +\infty$.

Theorem 5.5. *Let conditions* (H1) *and* (H2) *hold, and let the functions* a^{ij} *be continuous on* $\mathbb{R}^d \times [0,T]$. *Suppose that there exists a positive function* $V \in C^2(\mathbb{R}^d)$ *such that* $V(x) \to +\infty$ *as* $|x| \to +\infty$ *and for some number* $C > 0$ *and all* $(x,t) \in \mathbb{R}^d \times (0,1)$

$$LV(x,t) \geqslant -C, \quad |\sqrt{A}\nabla V(x)| \leqslant C.$$

Then for every measure $\mu = \varrho(x,t)\,dx\,dt$ *in* $\mathscr{I}_\nu$ *the function*

$$t \mapsto \int\limits_{\mathbb{R}^d} |\varrho(x,t)|\,dx$$

is decreasing on $(0,T)$. *Moreover, if the initial distribution* ν *vanishes, then* $\mu = 0$. *Therefore, the set* I_ν *consists of at most one element.*

Proof. By Lemma 5.1, for every nonnegative function $\psi \in C_0^\infty(\mathbb{R}^d)$ and $0 < s < t < T$

$$\int\limits_{\mathbb{R}^d} \psi(x)|\varrho(x,t)|\,dx \leqslant \int\limits_{\mathbb{R}^d} \psi(x)|\varrho(x,s)|\,dx + \int\limits_s^t \int\limits_{\mathbb{R}^d} L\psi(x,\tau)|\varrho(x,\tau)|\,dx\,d\tau. \tag{5.4}$$

We set $\psi(x) = \zeta(N^{-1}V(x))$, where ζ is a function such that $\zeta \geqslant 0$, $\zeta(z) = 1$ if $|z| \leqslant 1$ and $\zeta(z) = 0$ if $|z| > 2$, $\zeta'(z) \leqslant 0$ if $z > 0$, and for some number $M > 0$ and all x

$$|\zeta(x)| \leqslant M, \quad |\zeta'(x)| \leqslant M, \quad |\zeta''(x)| \leqslant M.$$

Applying (5.4), we arrive at the inequality

$$\int\limits_{\mathbb{R}^d} \psi_N(x)|\varrho(x,t)|\,dx \leqslant \int\limits_{\mathbb{R}^d} \psi(x)|\varrho(x,s)|\,dx$$
$$+\int\limits_0^t\int\limits_{\mathbb{R}^d} \big[N^{-1}\zeta'(N^{-1}V(x))LV(x,\tau) + N^{-2}\zeta''(N^{-1}V(x))|\nabla V(x)|^2|\varrho(x,\tau)|\big]\,dx\,d\tau.$$

We observe that $\zeta' LV \leqslant MC$. Therefore,

$$\int\limits_{\mathbb{R}^d} \psi_N(x)|\varrho(x,t)|\,dx \leqslant \int\limits_{\mathbb{R}^d} \psi(x)|\varrho(x,s)|\,dx + MC(N^{-1} + CN^{-2})\int\limits_s^t\int\limits_{\mathbb{R}^d} |\varrho(x,\tau)|\,dx\,d\tau.$$

Since $\lim\limits_{N\to\infty} \psi_N(x) = 1$, letting $N \to \infty$, we obtain

$$\int\limits_{\mathbb{R}^d} |\varrho(x,t)|\,dx \leqslant \int\limits_{\mathbb{R}^d} |\varrho(x,s)|\,dx \tag{5.5}$$

and the assertion about monotonicity is proved. If $\nu = 0$, then Lemma 5.2 yields the equality

$$\lim_{s\to 0}\int\limits_U |\varrho(x,s)|\,dx = 0.$$

Using this inequality along with (5.5), we obtain

$$\int\limits_{\mathbb{R}^d} |\varrho(x,t)|\,dx = 0 \quad \forall t \in (0,T). \qquad \square$$

Let us consider several examples.

Example 5.6. Let $V(x) = \ln(\ln(1+|x|))$ if $|x| > 1$. Then, whenever $|x| > 1$, we have

$$|\sqrt{A(x,t)}\nabla V(x)|^2 = (A(x,t)\nabla V(x), \nabla V(x)) = \frac{(A(x,t)x,x)}{|x|^2(|x|+1)^2\ln^2(|x|+1)}.$$

Let us calculate $LV(x,t)$ for $|x| > 1$:

$$LV(x,t) = -\frac{(A(x,t)x,x)}{|x|^2(1+|x|)^2\ln(1+|x|)}\Big(1 + \frac{1}{\ln(1+|x|)} + \frac{1+|x|}{|x|}\Big)$$
$$+\frac{\operatorname{trace} A(x,t)}{|x|(1+|x|)\ln(1+|x|)} + \frac{(b(x,t),x)}{|x|(|x|+1)\ln(|x|+1)}.$$

To ensure the assumptions of the theorem, it suffices to have the estimates

$$(A(x,t)x,x) \leqslant C + C|x|^4\ln(1+|x|),$$
$$(b(x,t),x) \geqslant -C|x|^2\ln(1+|x|) - C$$

for all $(x,t) \in \mathbb{R}^d \times (0,1)$.

Let us give an example with a Lyapunov function that is not radial.

Example 5.7. Let A be the unit matrix, and let

$$V(x)=\sum_{i=1}^{d}\ln(\ln(2+x_i^2)).$$

To ensure the condition $LV(x,t)\geqslant -C$, it suffices to have the estimate

$$\sum_{i=1}^{d}\frac{2x_ib^i(x,t)}{(2+x_i^2)\ln(2+x_i^2)}\geqslant -C_1$$

for some numbers $C_1>0$ and all $(x,t)\in\mathbb{R}^d\times(0,1)$.

Let $d=2$. Then

$$V(x,y)=\ln\ln(2+x^2)+\ln\ln(2+y^2).$$

We set

$$b^1(x,y)=y(2+x^2)\ln(2+x^2),$$
$$b^2(x,y)=-x(2+y^2)\ln(2+y^2).$$

Then

$$\frac{2xb^1(x,y)}{(2+x^2)\ln(2+x^2)}+\frac{2yb^2(x,y)}{(2+y^2)\ln(2+y^2)}=0.$$

Therefore, there exists at most one integrable solution to the corresponding Cauchy problem. Since $|LV|\leqslant C$ in this case, for any probability initial distribution there exists a unique probability solution to the Cauchy problem, which is also a unique integrable solution.

We observe that

$$b^1(x,y)x+b^2(x,y)y=xy[(2+x^2)\ln(2+x^2)-(2+y^2)\ln(2+y^2)].$$

Hence it is clear that $b^1(x,y)x+b^2(x,y)y$ cannot be estimated from below by an expression of the form $-C(\sqrt{x^2+y^2})^3-C$. Therefore, sufficient conditions of the form

$$\frac{(b^1(x,y)x+b^2(x,y)y)}{\sqrt{x^2+y^2}}\geqslant\beta(\sqrt{x^2+y^2}),$$

expressed in terms of the function β, do not work in this example.

Finally, let us mention several open problems related to the above discussion.

- Is a probability solution to the Cauchy problem on $\mathbb{R}^1\times[0,T]$ with the unit diffusion coefficient and an infinitely differentiable drift unique? Recall that we have constructed a counterexample in dimension $d=4$.
- Is it true that under merely locally assumptions about the coefficients (for example, locally bounded) every solution to the Fokker–Planck–Kolmogorov equation is generated by a solution to the martingale problem or it may happen that the Cauchy problem has a probability solution whereas the martingale problem is not solvable? It would be interesting to investigate the connections between the two problems under local assumptions.

- Is it possible to obtain the results of Sections 4 and 5 under weaker assumptions about the diffusion matrix, for instance, assuming that it is Hölder continuous in x?
- Our theorems involving Lyapunov functions impose the additional restriction on the Lyapunov function V of the form $|\sqrt{A}\nabla V(x)| \leqslant C$. Is it really needed? We observe that analogous results for martingale problems do not use this restriction.
- It would be interesting to find conditions ensuring that the Cauchy problem has no probability or integrable solutions.

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