Strong solutions for SPDE with locally monotone coefficients driven by Lévy noise

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Abstract

Motivated by applications to a manifold of semilinear and quasilinear stochastic partial differential equations (SPDEs) we establish the existence and uniqueness of strong solutions to coercive and locally monotone SPDEs driven by Lévy processes. We illustrate the main result of our paper by showing how it can be applied to the following SPDEs: stochastic reaction-diffusion equations, Burgers type equations, 2D Navier-Stokes equations, p-Laplace equations and porous media equations with locally monotone perturbations.

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1 Introduction and Main Results

In recent years, Stochastic Partial Differential Equations (SPDEs) driven by jump type noise such as Lévy-type or Poisson-type perturbations become extremely popular for modeling financial, physical and biological phenomena. In some circumstances, purely Brownian motion perturbation has many imperfections while capturing some large moves and unpredictable events. Therefore, Lévy-type perturbations come to the stage to reproduce the performance of those natural phenomena in some real world models. The existence and uniqueness of solutions for SPDEs driven by jump type noise has already been intensively investigated by many authors, see e.g. Kallianpur and Xiong [24], Albeverio et al [2], Mueller et al [40, 41], Applebaum and Wu [3], Mytnik [42], Truman and Wu [53], Knoche [25], Hausenblas [22, 23],

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Mandrekar and Rüdiger [36], Röckner and Zhang [48], Dong and Xu [14], Marinelli and Röckner [37], Bo et al [4], Dong et al [16], Brzeźniak and Hausenblas [7], Brzeźniak and Zhu [10], Brzeźniak et al [8], and the recent monograph by Peszat and Zabczyk [44]. The last reference can also be used for more detailed expositions and references.

In this paper, we aim to establish a framework in which one can treat a large number of SPDEs driven by Lévy type noise including stochastic reaction-diffusion equations, Burgers type equations, 2D Navier-Stokes equations, p-Laplace equation and porous media equation. In the case of Wiener noise the second named author and Röckner in [35] investigated such problems as special cases of SPDEs with locally monotone coefficients. They showed that the method first used by Menaldi and Sritharan [38] (and later used by Sritharan and Sundar [51], Chueshov and Millet [11] for various stochastic equations of hydrodynamics) can be generalized to such an extent that it covers all the above listed problems. On the other hand, there are no many papers studying non-Lipschitz SPDEs driven by Lévy type noises with small jumps. The first and third named author proved in [9] the existence and uniqueness of solutions to stochastic nonlinear beam equation driven by Lévy type noise. They together with Hausenblas extended in [8] (see also [15]) the work of Menaldi and Sritharan by showing that their method yields the existence and uniqueness of solutions to stochastic 2D Navier Stokes equations driven by a Lévy type noise. There is also the work of the first named author and Hausenblas [7] in which by means of generalized compactness method the existence of solutions to stochastic reaction diffusion equations driven by a Lévy type noise was investigated. This paper is a generalization of both papers [35] and [8].

The line of investigation proposed in this paper began with the celebrated works by Pardoux [43] and Krylov and Rozovskii [27], and later it was further developed by many authors, see Gyöngy and Krylov [19], Gyöngy [21]. Ren et al [46], Röckner and Wang [47] and Zhang [54]. Roughly speaking, for stochastic equations in finite dimensional spaces, the existence and uniqueness result was obtained under the local monotonicity assumption for the coefficients, see [27] for SDEs driven by Brownian motion and [19] for SDEs driven by (possibly discontinuous) locally square integrable martingale. However, concerning the existence and uniqueness of strong solutions to SPDEs in infinite dimensional spaces driven by Wiener process or local martingale, all results were established for the **globally** monotone coefficients SPDE (cf. [27, 21, 46, 54]). It was the breakthrough paper by Menaldi and Sritharan [38] which used certain **local** monotonicity of the 2D stochastic Navier-Stokes equations to show the existence of strong solutions to such equations (see also [51, 11]).

Recently, this variational framework has been extended by the second named author and Röckner in [35] for SPDE driven by Wiener process in Hilbert space with locally monotone coefficients. They showed that many fundamental examples of SPDEs can be included into their framework, for instance the stochastic Burgers type equations and the stochastic 2D Navier-Stokes equations. Moreover, the examples investigated in [11, 12, 34], such as magneto-hydrodynamic equations, the Boussinesq model for the Bénard convection, 2D magnetic Bénard problem and stochastic 3D Leray-α model, can be also included into that framework. What we do in the present paper is to confirm the natural conjecture that the framework in [35] works not only for locally monotone SPDEs driven by multiplicative Gaussian noise but also by multiplicative Lévy type noise. However, we should point out that our results are not applicable to evolution equations with general space time white noise, see for instance [9, 7] and the references therein. The reason is that the solutions of SPDEs with general space time white noise are not regular enough to fit in the variational framework.

The main contribution of this work is that we establish a unified framework for a large class of semilinear and quasilinear SPDE driven by general Lévy noises, which generalizes many previous works [43, 27, 21, 35]. The main result is applied to various types of concrete examples (see Section 4 for details). It also recovers and improves many known results in the literature, see for instance [14, 38, 48, 15, 11, 8]. In a recent work [10] by the first and third named author, a type of stochastic nonlinear beam equations with Poisson-type noises was studied and the existence and uniqueness of solutions was established by following a natural route of constructing a local mild solutions and proving, with the help of the Khasminski test, that this solution is a global one. In contrast to [10], the approach used in this paper is different. We will follow the lines in [8, 35] and the technique involves the use of the Galerkin approximation, local monotonicity arguments but not, as opposed to [7], compactness argument. We shall use the result from [19, 1] for finite dimensional case to construct a sequence of solutions of approximated equations and obtain a prior estimate for those approximated solutions. Then we show the limit of those approximated solutions solves the original equation by using the local monotonicity arguments.

Now let us describe the framework in more detail. Let

$$V \subset H \equiv H^* \subset V^*$$

be a Gelfand triple, *i.e.* $(H, \langle\cdot,\cdot\rangle_H)$ is a separable Hilbert space which is identified with its dual space by the Riesz Lemma, V is a reflexive Banach space that is continuously and densely embedded into H. If ${}_{V^*}\langle\cdot,\cdot\rangle_V$ denotes the duality between V and its dual space V^*, then we have

$${}_{V^*}\langle u, v\rangle_V = \langle u, v\rangle_H,\ \ u \in H, v \in V.$$

Let $(\Omega, \mathbb{P}, \mathbb{F}, \mathcal{F})$, where $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$, be a filtered probability space, $(Z, \mathcal{Z})$ be a measurable space, and ν be a σ-finite measure on it. Let

$$\tilde{N}((0,t] \times B) = N((0,t] \times B) - t\nu(B),\ \ t \geq 0,\ \ B \in \mathcal{Z}$$

be a compensated Poisson random measure on $[0,T] \times \Omega \times Z$ associated with a stationary Poisson point process p (see Section 2 for more details). A typical example of N is a Poisson random measure associated with a Lévy process taking values in a separable Banach space. Let U be a separable Hilbert space and let us denote by $(\mathcal{T}_2(U;H), \|\cdot\|_2)$ the Hilbert space of all Hilbert-Schmidt operators from U to H. Let us assume that $\{W_t\}_{t\geq 0}$ be a U-valued cylindrical Wiener process on the probability space $(\Omega, \mathbb{P}, \mathbb{F}, \mathcal{F})$. Let $\mathcal{P}$ be a predictable σ-field, i.e. the σ-field generated by all left continuous and $\mathbb{F}$-adapted real-valued processes on $[0,T] \times \Omega$. We shall denote by $\mathcal{BF}$ the σ-field of the progressively measurable sets on $[0,T] \times \Omega$, i.e.

$$\mathcal{BF} = \{A \subset [0,T] \times \Omega : \forall\ t \in [0,T], A \cap ([0,t] \times \Omega) \in \mathcal{B}([0,t]) \otimes \mathcal{F}_t\}.$$

Now we consider a type of SPDEs driven by Lévy processes of the following form:

$$
\begin{aligned}
\mathrm{d}X_t &= A(t, X_t)\mathrm{d}t + B(t, X_t)\mathrm{d}W_t \\
&\quad + \int_{D^c} f(t, X_{t-}, z)\tilde{N}(\mathrm{d}t, \mathrm{d}z) + \int_D g(t, X_{t-}, z)N(\mathrm{d}t, \mathrm{d}z), \qquad (1.1)\\
X_0 &= x,
\end{aligned}
$$

where x is an $\mathcal{F}_0$-measurable random variable, $A : [0, T]\times\Omega\times V \to V^*$ and $B : [0, T]\times\Omega\times V \to \mathcal{T}_2(U; H)$ are both $\mathcal{BF} \otimes \mathcal{B}(V)$-measurable functions, $D \in \mathcal{Z}$ with $\mathbb{E}N((0, t] \times D) < \infty$ for every $0 < t \leq T$, and $f, g : [0, T] \times \Omega \times V \times Z \to H$ are $\mathcal{P} \otimes \mathcal{B}(V) \otimes \mathcal{Z}$-measurable functions.

The main aim of this work is to establish the existence and uniqueness of strong solutions to (1.1) under the coercivity and local monotonicity conditions. For this purpose, let us first formulate the main assumptions on the coefficients.

Suppose there exist constants $\alpha > 1$, $\beta \geq 0$, $\theta > 0$, $K > 0$, a positive $\mathbb{F}$-adapted process F and a measurable, bounded on balls function $\rho : V \to [0, +\infty)$ such that the following conditions hold for all $v, v_1, v_2 \in V$ and $(t, \omega) \in [0, T] \times \Omega$:

$(H1)$ (Hemicontinuity) The map $s \mapsto {}_{V^*}\langle A(t, v_1 + sv_2), v\rangle_V$ is continuous on $\mathbb{R}$.

$(H2)$ (Local monotonicity)

$$
\begin{aligned}
&2_{V^*}\langle A(t, v_1) - A(t, v_2), v_1 - v_2\rangle_V + \|B(t, v_1) - B(t, v_2)\|_2^2 \\
&+ \int_{D^c} \|f(t, v_1, z) - f(t, v_2, z)\|_H^2 \nu(\mathrm{d}z) \leq (K + \rho(v_2)) \|v_1 - v_2\|_H^2,
\end{aligned}
$$

$(H3)$ (Coercivity)

$$
2_{V^*}\langle A(t, v), v\rangle_V + \|B(t, v)\|_2^2 + \theta\|v\|_V^\alpha \leq F_t + K\|v\|_H^2.
$$

$(H4)$ (Growth)

$$
\|A(t, v)\|_{V^*}^{\frac{\alpha}{\alpha-1}} \leq (F_t + K\|v\|_V^\alpha) \left(1 + \|v\|_H^\beta\right).
$$

Definition 1.1. (Solution of SEE) An H-valued càdlàg $\mathbb{F}$-adapted process $\{X_t\}_{t\in[0,T]}$ is called a solution of (1.1), if for its $\mathrm{d}t \times \mathbb{P}$-equivalent class $\bar{X}$ we have

(1) $\bar{X} \in L^\alpha([0, T]; V) \cap L^2([0, T]; H)$, $\mathbb{P}$-a.s.;

(2) the following equality holds $\mathbb{P}$-a.s.:

$$
\begin{aligned}
X_t = x &+ \int_0^t A(s, \bar{X}_s)\mathrm{d}s + \int_0^t B(s, \bar{X}_s)\mathrm{d}W_s \\
&+ \int_0^t \int_{D^c} f(s, \bar{X}_{s-}, z)\tilde{N}(\mathrm{d}s, \mathrm{d}z) + \int_0^t \int_D g(s, \bar{X}_{s-}, z)N(\mathrm{d}s, \mathrm{d}z),\ t \in [0, T].
\end{aligned}
$$

Remark 1.1. The integrability of all terms in the above equality are implicitly required in the definition and it will be all justified in the proof of existence of solutions. Note that $A(s, \bar{X}_s)$ is a V^*-valued process according to the definition, however, the integral with respect to $\mathrm{d}s$ in the above equality is initially a V^*-valued Bochner integral which turns out to be in fact H-valued.

Now we can present the main result of this paper.

Theorem 1.2. *Suppose that conditions $(H1)-(H4)$ hold for $F \in L^{\frac{\beta+2}{2}}([0,T]\times\Omega; \mathrm{d}t\times\mathbb{P})$, and there exists constants $\gamma < \frac{\theta}{2\beta}$ and $C > 0$ such that for all $t\in[0,T], \omega\in\Omega$ and $v \in V$ we have*

$$\|B(t,v)\|_2^2 + \int_{D^c} \|f(t,v,z)\|_H^2 \nu(\mathrm{d}z) \le F_t + C\|v\|_H^2 + \gamma\|v\|_V^\alpha; \tag{1.2}$$

$$\int_{D^c} \|f(t,v,z)\|_H^{\beta+2} \nu(\mathrm{d}z) \le F_t^{\frac{\beta+2}{2}} + C\|v\|_H^{\beta+2}; \tag{1.3}$$

$$\rho(v) \le C(1+\|v\|_V^\alpha)(1+\|v\|_H^\beta). \tag{1.4}$$

(i) Then for any $x \in L^{\beta+2}(\Omega, \mathcal{F}_0, \mathbb{P}; H)$, equation (1.1) has a unique solution $\{X_t\}_{t\in[0,T]}$.
(ii) If $g \equiv 0$, then there exists a constant C such that

$$\sup_{t\in[0,T]} \mathbb{E}\|X_t\|_H^{\beta+2} + \mathbb{E}\int_0^T \|X_t\|_H^\beta \|X_t\|_V^\alpha \mathrm{d}t \le C\left(\mathbb{E}\|x\|_H^{\beta+2} + \mathbb{E}\int_0^T F_t^{(\beta+2)/2}\mathrm{d}t\right). \tag{1.5}$$

(iii) If $g \equiv 0$ and γ is small enough, then we have

$$\mathbb{E}\left(\sup_{t\in[0,T]} \|X_t\|_H^{\beta+2}\right) + \mathbb{E}\int_0^T \|X_t\|_H^\beta \|X_t\|_V^\alpha \mathrm{d}t \le C\left(\mathbb{E}\|x\|_H^{\beta+2} + \mathbb{E}\int_0^T F_t^{(\beta+2)/2}\mathrm{d}t\right). \tag{1.6}$$

Remark 1.3. (1) If $f = g \equiv 0$ in (1.1) (i.e. Wiener noise case), then Theorem 1.2 recovers the main result in [35]. Moreover, we improve [35, Theorem 1.1] for allowing a positive constant γ in (1.2), which means that the diffusion coefficient B can also depend on some gradient term of solution in applications. We also want to emphasize that $(H2)$ is essentially weaker than the classical monotonicity condition used extensively in the literature (i.e. $\rho \equiv 0$, see e.g. [43, 27, 45, 48, 46, 18]). The typical examples are the stochastic Burgers equations and 2D Navier-Stokes equation on a bounded or unbounded domain, which satisfies $(H2)$ but does not satisfy the standard monotonicity condition (see Section 3 or [35, 34] for more examples).

(2) If $\rho \equiv 0$ in $(H2)$ and $\beta = 0$ in $(H4)$, then the existence and uniqueness of strong solutions to (1.1) follows from the general result of Gyöngy [21].

(3) If the noise is zero (or additive type) in (1.1), then the existence and uniqueness of solutions is established in [34] by replacing $(H2)$ with the following more general local monotonicity condition:

$${}_{V^*}\langle A(t,v_1) - A(t,v_2), v_1 - v_2\rangle_V \le (K + \eta(v_1) + \rho(v_2))\, \|v_1 - v_2\|_H^2,$$

where $\eta, \rho : V \to [0, +\infty)$ are measurable functions and locally bounded in V.

(4) In general, the estimates (1.5) and (1.6) might not hold anymore if we have large jumps term in the equation. However, if we assume the Lévy measure has finite moment of certain order, then it is still possible to obtain some similar estimates and this will be investigated in a separated work.

Remark 1.4. (1) Note that if $\beta = 0$ in $(H4)$, then one can just take any $\gamma < \infty$ in (1.2). In this case, the assumption on B in (1.2) can be removed since it follows directly from $(H3)$ and $(H4)$ (cf. [45, Remark 4.1.1]).

(2) If f satisfies the following growth condition for some fixed $p \geq \beta + 2$:

$$\|f(t,v,z)\|_H^p \leq h(z)^p (F_t^{\frac{p}{2}} + C\|v\|_H^p), \quad (t,v,z) \in [0,T] \times V \times D^c,$$

where $\int_{D^c} \left[h(z)^{\beta+2} + h(z)^2\right] \nu(dz) < \infty$, then it is easy to show that conditions (1.3) and (1.2) hold.

In particular, if f satisfying the following conditions:

$$\begin{aligned} &\|f(t,x,z) - f(t,y,z)\|_H \leq C\|x-y\|_H \|z\|, \ t \in [0,T], \ x,y \in V, \ z \in D^c; \\ &\|f(t,x,z)\|_H \leq C(1 + \|x\|_H)\|z\|, \ t \in [0,T], \ x,y \in V, \ z \in D^c, \end{aligned}$$

where $\int_{D^c} \|z\|^2 \nu(\mathrm{d}z) < \infty$, then $(H2)$, (1.2) and (1.3) are all satisfied.

The rest of the paper is organized as follows: in the next section we will recall some preliminaries on the Poisson random measure and its corresponding stochastic integral. The proof of the main result will be given in Section 3 and some concrete examples of SPDE will be studied in Section 4 as applications.

2 Some Preliminaries on Poisson Random Measure

As a preparation, we begin with a brief review of some terminology and results of Poisson random measures. Let $(S, \mathcal{S})$ be a measurable space, $\mathbb{N} = \{0, 1, 2, \cdots\}$ and $\bar{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$. Let $\mathbb{M}_{\bar{\mathbb{N}}}(S)$ denote the space of all $\bar{\mathbb{N}}$-valued measures on $(S, \mathcal{S})$. We use the symbol $\mathcal{B}(\mathbb{M}_{\bar{\mathbb{N}}}(S))$ to denote the smallest σ-field on $\mathbb{M}_{\bar{\mathbb{N}}}(S)$ such that all mappings $i_B : \mathbb{M}_{\bar{\mathbb{N}}}(S) \ni \mu \mapsto \mu(B) \in \bar{\mathbb{N}}$, $B \in \mathcal{S}$ are measurable.

Definition 2.1. A map $N : \Omega \times \mathcal{S} \to \bar{\mathbb{N}}$ is called an $\bar{\mathbb{N}}$-valued **random measure** if for each $\omega \in \Omega$, $N(\omega, \cdot) \in \mathbb{M}_{\bar{\mathbb{N}}}(S)$ and for each $A \in \mathcal{S}$, $N(\cdot, A)$ is an $\bar{\mathbb{N}}$-valued random variable on the probability space $(\Omega, \mathbb{P}, \mathcal{F})$. We will often write $N(A)$ instead of $N(\cdot, A)$ for simplicity of notation.

Definition 2.2. An $\bar{\mathbb{N}}$-valued random measure N is called a **Poisson random measure** if

(1) for any $B \in \mathcal{S}$ provided $\mathbb{E}[N(B)] < \infty$, $N(B)$ is a random variable of Poisson distribution with the parameter $\eta(B) = \mathbb{E}[N(B)]$;

(2) for any pairwise disjoint sets $B_1, \cdots, B_n \in \mathcal{S}$, the random variables

$$N(B_1),\ \cdots,\ N(B_n)$$

are independent.

Let $(Z, \mathcal{Z})$ be a measurable space. A **point function** α on $(Z, \mathcal{Z})$ is a mapping $\alpha : \mathcal{D}(\alpha) \to Z$, where the domain $\mathcal{D}(\alpha) \subset (0, \infty)$ of α is a countable subset. Let Π_Z be the set of all point functions on Z. For each point function, we define a counting measure N_α by

$$N_\alpha(U) := \sharp\{s \in \mathcal{D}(\alpha) :\ (s, \alpha(s)) \in U\},\quad U \in \mathcal{B}((0,\infty)) \otimes \mathcal{Z}.$$

Denote by $\mathcal{Q}$ the σ-field on Π_Z generated by all the subsets $\{\alpha \in \Pi_Z :\ N_\alpha(U) = k\}$, $U \in \mathcal{Z}$, $k = 0, 1, 2, \cdots$. A function $p : \Omega \to \Pi_Z$ is called a **point process** on Z if it is $\mathcal{F}/\mathcal{Q}$-measurable. Let p be a point process in $(\Pi_Z, \mathcal{Q})$. Analogously, we may define for every $\omega \in \Omega$, the counting measure N_p associated with p by

$$N_p(U, \omega) := \sharp\{s \in \mathcal{D}(p(\omega)) : (s, p(s, \omega)) \in U\},\quad U \in \mathcal{B}((0,\infty)) \otimes \mathcal{Z}. \tag{2.1}$$

In particular, we have

$$N_p((0,t] \times A, \omega) = \sharp\{s \in (0,t] \cap \mathcal{D}(p(\omega)) : p(s, \omega) \in A\},\quad A \in \mathcal{Z},\quad 0 < t \leq T. \tag{2.2}$$

It is also useful to introduce the shifted point process $\theta_t p$ defined by

$$\begin{aligned}(\theta_t p)(s) &= p(s+t),\ s > 0;\\ \mathcal{D}(\theta_t p) &= \{s \in (0,\infty) : s + t \in \mathcal{D}(p)\}.\end{aligned}$$

and the stopped point process $\alpha_t p$ defined by

$$\begin{aligned}(\alpha_t p)(s) &= p(s),\ \text{for } s \in \mathcal{D}(\alpha_t p);\\ \mathcal{D}(\alpha_t p) &= (0,t] \cap \mathcal{D}(p).\end{aligned}$$

Definition 2.3. A point process p is said to be **finite** if $\mathbb{E}N_p((0,t] \times Z) < \infty$ for every $0 < t \leq T$.

A point process p is said to be **σ-finite** if there exists an increasing sequence $\{D_n\}_{n\in\mathbb{N}} \subset \mathcal{Z}$ such that $\cup_n D_n = Z$ and $\mathbb{E}N_p((0,t] \times D_n) < \infty$ for all $0 < t \leq T$ and $n \in \mathbb{N}$.

A point process p is said to be **stationary** if for every $t > 0$, p and $\theta_t p$ have the same probability laws.

A point process p is said to be **renewal** if it is stationary and for every $0 < t < \infty$, the point processes $\alpha_t p$ and $\theta_t p$ are independent.

A point process p is said to be **adapted** to the filtration $\mathbb{F}$ if for every $t > 0$ and $A \in \mathcal{Z}$, its counting measure $N_p((0,t] \times A)$ is $\mathcal{F}_t$-measurable.

A point process p is called a **Poisson point process** if $N_p(\cdot)$ defined by (2.1) is a Poisson random measure on $((0,\infty) \times Z, \mathcal{B}((0,\infty)) \otimes \mathcal{Z})$.

Remark 2.1. It can be shown that if a point process p is σ-finite and renewal, then N_p defined by (2.1) is a Poisson random measure (cf. [31, Theorem 3.1],). It is easy to verify that a Poisson point process is stationary if and only if there exists a nonnegative measure ν on $(Z, \mathcal{Z})$ such that

$$\mathbb{E}N_p((0,t] \times A) = t\nu(A), \quad t \geq 0, \quad A \in \mathcal{Z}. \tag{2.3}$$

In such a case, we say that the Poisson random measure N_p is time homogenous. At this point, it should be mentioned that, in the literature, some authors may use the above property (2.3) as an alternative definition of stationary property of a Poisson point process. In fact, this is consistent with our definition of a stationary point process here.

Let $\mathcal{M}^q_T(\mathcal{P} \otimes \mathcal{Z}, \mathrm{d}t \times \mathbb{P} \times \nu; H)$, $q \in [1, \infty)$, be the space of all (equivalence classes of) $\mathcal{P} \otimes \mathcal{Z}$-measurable functions $f : [0, T] \times \Omega \times Z \to H$ such that

$$\mathbb{E} \int_0^T \int_Z \|f(s, \cdot, z)\|^q_H \nu(\mathrm{d}z)\mathrm{d}s < \infty. \tag{2.4}$$

Let $\mathcal{M}_T(\mathcal{P} \otimes \mathcal{Z}, N; H)$ be the space of all $\mathcal{P} \otimes \mathcal{Z}$-measurable functions $f : [0, T] \times \Omega \times Z \to H$ such that

$$\mathbb{E} \int_0^T \int_Z \|f(s, \cdot, z)\|_H N(\mathrm{d}s, \mathrm{d}z) < \infty. \tag{2.5}$$

Here $\int_0^T \int_Z \|f(s, \cdot, z)\|_H N(\mathrm{d}s, \mathrm{d}z)(\omega)$ is understood to be the Lebesgue integral *w.r.t.* the measure $N(\cdot, \cdot)(\omega)$ for every $\omega \in \Omega$ and is equal to the convergent sum (cf. [29]),

$$\int_0^T \int_Z \|f(s, \omega, z)\|_H N(\mathrm{d}s, \mathrm{d}z) = \sum_{s \in (0,T] \cap \mathcal{D}(p(\omega))} \|f(s, \omega, p(s, \omega))\|_H.$$

It should come as no surprise that if $f : [0, T] \times \Omega \times Z \to H$ is a $\mathcal{B}([0, T]) \otimes \mathcal{F}_T \otimes \mathcal{Z}$-measurable function and $\mathbb{E} \int_0^T \int_Z \|f(s, \cdot, z)\|_H N(ds, dz) < \infty$, then for every $\omega \in \Omega$, $f(\cdot, \omega, \cdot)$ is $\mathcal{B}([0, T]) \otimes \mathcal{Z}$-measurable and $\int_0^T \int_Z \|f(s, \omega, z)\|_H N(\mathrm{d}s, \mathrm{d}z)(\omega) < \infty$, $\mathbb{P}$-a.s., hence for almost all $\omega \in \Omega$, $f(\cdot, \omega, \cdot)$ is Bochner integrable with respect to $N(\mathrm{d}s, \mathrm{d}z)(\omega)$ and we have for every $t \leq T$

$$\int_0^t \int_Z f(s, \omega, z) N(\mathrm{d}s, \mathrm{d}z)(\omega) = \sum_{s \in (0,t] \cap \mathcal{D}(p(\omega))} f(s, \omega, p(s, \omega)), \ \mathbb{P}\text{-a.s.} \tag{2.6}$$

Now we state some important properties of the stochastic integrals w.r.t. the compensated Poisson random measures for further reference. Proofs of these properties and detailed discussions can be found in [29] (see also [6, 49, 56]).

Proposition 2.2. *Let $f \in \mathcal{M}^2_T(\mathcal{P} \otimes \mathcal{Z}, \mathrm{d}t \times \mathbb{P} \times \nu; H)$.*

(i) *The stochastic integral process* $\int_0^t \int_Z f(s,\cdot,z)\tilde{N}(\mathrm{d}s,\mathrm{d}z)$, $t \in [0,T]$ *is a càdlàg 2-integrable martingale. More precisely, it has a modification which has càdlàg trajectories;*

(ii) *The following isometry property holds:*

$$\mathbb{E}\Big\| \int_0^t \int_Z f(s,\cdot,z)\tilde{N}(\mathrm{d}s,\mathrm{d}z)\Big\|_H^2 = \mathbb{E}\int_0^t \int_Z \|f(s,\cdot,z)\|_H^2 \nu(\mathrm{d}z)\mathrm{d}s, \quad t \in (0,T]; \tag{2.7}$$

(iii) *If* $D \in \mathcal{Z}$ *with* $\mathbb{E}(N((0,t]\times D)) < \infty$, *then for every* $t \in [0,T]$, $\mathbb{P}$*-a.s.,*

$$\int_0^t \int_D f(s,\cdot,z)\tilde{N}(\mathrm{d}s,\mathrm{d}z) = \sum_{s\in(0,t]\cap\mathcal{D}(p)} f(s,\cdot,p(s))1_D(p(s)) - \int_0^t \int_D f(s,\cdot,z)\nu(\mathrm{d}z)\mathrm{d}s; \tag{2.8}$$

(iv) *If* $f \in \mathcal{M}_T^2(\mathcal{P}\otimes\mathcal{Z}, \mathrm{d}t\times\mathbb{P}\times\nu; H) \cap \mathcal{M}_T^1(\mathcal{P}\otimes\mathcal{Z}, \mathrm{d}t\times\mathbb{P}\times\nu; H)$, *then we have for each* $t \in [0,T]$, $\mathbb{P}$*-a.s.*

$$\int_0^t \int_Z f(s,\cdot,z)\tilde{N}(\mathrm{d}s,\mathrm{d}z) = \sum_{s\in(0,t]\cap\mathcal{D}(p)} f(s,\cdot,p(s)) - \int_0^t \int_Z f(s,\cdot,z)\nu(\mathrm{d}z)\mathrm{d}s. \tag{2.9}$$

Remark 2.3. (1) We may extend the stochastic integral to $\mathcal{P}\otimes\mathcal{Z}$-measurable functions f satisfying

$$\int_0^T \int_Z \|f(s,\cdot,z)\|_H^2 \nu(\mathrm{d}z)\mathrm{d}s < \infty, \ \mathbb{P}\text{-a.s.}.$$

In this case, the stochastic integral process $\int_0^t \int_Z f(s,\cdot,z)\tilde{N}(\mathrm{d}s,\mathrm{d}z)$, $t \in [0,T]$ is a càdlàg 2-integrable local martingale and for every stopping time $\tau \le T$, we have

$$\int_0^{t\wedge\tau} \int_Z f(s,\cdot,z)\tilde{N}(\mathrm{d}s,\mathrm{d}z) = \int_0^t \int_Z 1_{[0,\tau]} f(s,\cdot,z)\tilde{N}(\mathrm{d}s,\mathrm{d}z).$$

(2) From now on, when we talk about the stochastic integral process $\int_0^t \int_Z f(s,\cdot,z)\tilde{N}(\mathrm{d}s,\mathrm{d}z), t \in [0,T]$, we implicitly assume that the stochastic integral process has càdlàg trajectories, unless stated otherwise, in which case, the stochastically equivalence coincides with the $\mathbb{P}$-equivalence.

(3) For Banach spaces martingale type p ($1 < p \le 2$), one has, instead of the Itô isometry property (2.7), the following continuity property (C_p is some constant):

$$\mathbb{E}\Big\| \int_0^T \int_Z f(s,\cdot,z)\tilde{N}(\mathrm{d}s,\mathrm{d}z)\Big\|_H^p \le C_p\mathbb{E}\int_0^T \int_Z \|f(s,\cdot,z)\|_H^p \nu(\mathrm{d}z)\mathrm{d}s.$$

(4) Even though there are close connections between predictable processes and progressively measurable processes, the predictability requirement of the function f in Proposition 2.2 (*iii*) and (*iv*) is necessary. In fact, one can find a progressively measurable but not predictable function such that identities (2.8) and (2.9) no longer hold (cf. [56]).

One should note that another important and widely used class of Poisson random measures are the one associated to a Lévy process, which is actually a special type of Poisson random measures associated to a Poisson point process as we discussed before. More precisely, let $L := (L_t)_{t\geq 0}$ be an Z-valued Lévy process (in this case Z need to be a separable Banach space). Without loss of generality we may always assume that the Lévy process L is càdlàg, even if we don't impose the càdlàg property in the definition of a Lévy process, see e.g. [13, Theorem 16.1]. Hence, for every $\omega \in \Omega$, $L_\cdot(\omega)$ has at most countable number of jumps on $[0,t]$. So it is easy to see that for every $\omega \in \Omega$,

$$\triangle L_\cdot(\omega) : [0,\infty) \to Z;\ \triangle L_s(\omega) := L_s(\omega) - L_{s-}(\omega)$$

is a point function in $(Z \setminus \{0\}, \mathcal{B}(Z \setminus \{0\}))$. Let us define

$$N(A,\omega) = \sharp\{s \in (0,\infty) : (s, \triangle L_s(\omega)) \in A\}, \quad A \in \mathcal{B}((0,\infty)) \otimes \mathcal{B}(Z \setminus \{0\}),\ \omega \in \Omega. \tag{2.10}$$

By the definition of a point process, it is relatively straightforward to show that $\triangle L : \Omega \to \Pi_Z$ is $\mathcal{F}/\mathcal{Q}$-measurable. This means that $\triangle L$ is a point process. Since the Lévy process L has independent and stationary increments, one can show that the point process $\triangle L$ is stationary and renewal. Obviously, by taking $D_n = \{x \in Z : \|x\| > \frac{1}{n}\}$, we find that the point process $\triangle L$ is σ-finite. On the basis of Remark 2.1, we know that N defined by (2.10) is a stationary Poisson random measure with a nonnegative measure $\nu(\cdot)$ such that

$$\mathbb{E}N((0,t] \times A) = t\nu(A),\ t > 0,\ A \in \mathcal{B}(Z \setminus \{0\}).$$

In such a case, N is called the Poisson random measure associated to the Lévy process L.

3 Proof of The Main Theorem

3.1 Without large jumps

First of all we note that since $\nu(D) < \infty$, for almost all $\omega \in \Omega$, the set

$$\{s \in (0,T] \cap \mathcal{D}(p) : p(s,\omega) \in D\}$$

contains only finitely many points. Hence we may denote these points according to their magnitude by

$$0 < \tau_1(\omega) < \tau_2(\omega) < \cdots < \tau_m(\omega) < \cdots$$

In other words, we put

$$\begin{aligned}\tau_1 &= \inf\{s \in (0,\infty) \cap \mathcal{D}(p) : p(s) \in D\} \wedge T;\\ \tau_m &= \inf\{s \in (0,\infty) \cap \mathcal{D}(p) : p(s) \in D; s > \tau_{m-1}\} \wedge T,\ m \geq 2.\end{aligned}$$

The random times $\tau_1, \tau_2, \cdots$ form a random configuration of points in $(0,T]$ with $p(\tau_i) \in D$ and it is a sequence of jump times of the Poisson process $N(t,D)$, $t \in [0,T]$. So, it is easy to see that $\tau_m \uparrow T$ as $m \to \infty$ $\mathbb{P}$-a.s. and for each m, the random time τ_m is a stopping time. Indeed, for every $u > 0$, we have by definition

$$\{\tau_m \leq u\} = \{N(u, D) \geq m\} \in \mathcal{F}_u.$$

Note that since $\int_0^t g(s, X_{s-}, z) N(\mathrm{d}s, \mathrm{d}z) = 0$ for $t \in [0, \tau_1)$, the equation (1.1) can be rewritten into the following type of equation on the interval $[0, \tau_1)$:

$$\begin{aligned} \mathrm{d}X_t &= A(t, X_t)\mathrm{d}t + B(t, X_t)\mathrm{d}W_t + \int_{D^c} f(t, X_{t-}, z)\tilde{N}(\mathrm{d}t, \mathrm{d}z), \\ X_0 &= x. \end{aligned} \tag{3.1}$$

Actually, by means of interlacing procedure (which will be introduced in Section 3.2), for the proof of Theorem 1.2 it is sufficient to show the existence and uniqueness of solutions to (3.1).

Theorem 3.1. *Under the same assumptions as in Theorem 1.2, for every* $x \in L^{\beta+2}(\Omega, \mathcal{F}_0, \mathbb{P}; H)$, *there exists a unique càdlàg* H*-valued* $\mathbb{F}$*-adapted process* (X_t) *such that*

(i) *the following equality holds* $\mathbb{P}$*-a.s.:*

(3.2)
$$X_t = x + \int_0^t A(s, \bar{X}_s)\mathrm{d}s + \int_0^t B(s, \bar{X}_s)\mathrm{d}W_s + \int_0^t \int_{D^c} f(s, \bar{X}_{s-}, z)\tilde{N}(\mathrm{d}s, \mathrm{d}z),\ t \in [0, T],$$

where $\bar{X} \in L^\alpha([0,T] \times \Omega, \mathrm{d}t \times \mathbb{P}; V) \cap L^2([0,T] \times \Omega, \mathrm{d}t \times \mathbb{P}; H)$ *and it is* $\mathrm{d}t \times \mathbb{P}$*-equivalent to* X*;*

(ii) *we have*

$$\sup_{t \in [0,T]} \mathbb{E}\|X_t\|_H^{\beta+2} + \mathbb{E}\int_0^T \|X_t\|_H^\beta \|X_t\|_V^\alpha \mathrm{d}t \leq C\left(\mathbb{E}\|x\|_H^{\beta+2} + \mathbb{E}\int_0^T F_t^{(\beta+2)/2}\mathrm{d}t\right).$$

The proof of Theorem 3.1 is divided into three steps. Assume that $\{e_1, e_2, \cdots\} \subset V$ is an orthonormal basis of H such that $span\{e_1, e_2, \cdots\}$ is dense in V. Denote $H_n := span\{e_1, \cdots, e_n\}$. Let $P_n : V^* \to H_n$ be defined by

$$P_n y := \sum_{i=1}^n {}_{V^*}\langle y, e_i\rangle_V e_i,\ y \in V^*.$$

It is easy to see that $P_n|_H$ is just the orthogonal projection onto H_n in H and we have

$${}_{V^*}\langle P_n A(t,u), v\rangle_V = \langle P_n A(t,u), v\rangle_H = {}_{V^*}\langle A(t,u), v\rangle_V,\ u \in V, v \in H_n.$$

Let $\{g_1, g_2, \cdots\}$ be an orthonormal basis of U and

$$W_t^{(n)} := \sum_{i=1}^n \langle W_t, g_i\rangle_U g_i = \tilde{P}_n W_t,$$

where $\tilde{P}_n$ is the orthogonal projection onto $span\{g_1, \cdots, g_n\}$ in U.

For each finite $n \in \mathbb{N}$, we consider the following stochastic equation on H_n:

$$
\begin{aligned}
\mathrm{d}X_t^{(n)} &= P_n A(t, X_t^{(n)})\mathrm{d}t + P_n B(t, X_t^{(n)})\mathrm{d}W_t^{(n)} + \int_{D^c} P_n f(t, X_{t-}^{(n)}, z)\tilde{N}(\mathrm{d}t, \mathrm{d}z), \\
X_0^{(n)} &= P_n x.
\end{aligned} \tag{3.3}
$$

According to [19, Theorem 1] (cf. also [1, Theorem 3.1]), (3.3) has a unique strong solution of the form

$$
\begin{aligned}
X_t^{(n)} =& P_n x + \int_0^t P_n A(s, X_s^{(n)})\mathrm{d}t + \int_0^t P_n B(s, X_s^{(n)})\mathrm{d}W_s^{(n)} \\
&+ \int_0^t \int_{D^c} P_n f(s, X_{s-}^{(n)}, z)\tilde{N}(\mathrm{d}s, \mathrm{d}z), \quad t \in [0, T].
\end{aligned} \tag{3.4}
$$

In order to construct the solution of (3.1), we need a priori estimate for $X^{(n)}$.

Lemma 3.2. *Under the same assumptions as in Theorem 1.2, there exists $C > 0$ such that*

$$
\begin{aligned}
&\sup_{n\in\mathbb{N}} \Big(\sup_{t\in[0,T]} \mathbb{E}\|X_t^{(n)}\|_H^{\beta+2} + \mathbb{E}\int_0^T \|X_t^{(n)}\|_H^\beta \|X_t^{(n)}\|_V^\alpha \mathrm{d}t \Big) \\
\leq& C\left(\mathbb{E}\|x\|_H^{\beta+2} + \mathbb{E}\int_0^T F_t^{(\beta+2)/2}\mathrm{d}t \right).
\end{aligned} \tag{3.5}
$$

Proof. For any given $n \in \mathbb{N}$, we define

$$\tau_R^{(n)} := \inf\{t \geq 0 : \|X_t^{(n)}\|_H > R\} \wedge T.$$

Since the solution $(X_t^{(n)})_{0\leq t\leq T}$ is right continuous and $\mathbb{F}$-adapted, $\tau_R^{(n)}$ is a stopping time for every $R \in \mathbb{N}$. Moreover, since the trajectories $t \mapsto X_t^{(n)}(\omega)$ are right continuous with left limits $\mathbb{P}$-a.s., the process $X^{(n)}$ is bounded on every compact intervals, hence we see that $\tau_R^{(n)} \uparrow T$, $\mathbb{P}$-a.s. and $\mathbb{P}\{\tau_R^{(n)} < T\} = 0$ as $R \to \infty$.

For the simplicity of notations we take $p = \beta + 2$. By applying the Itô formula (cf. [39]) to the function $\|\cdot\|_H^p$ and the process $X_t^{(n)}$ we have

$$
\begin{aligned}
\|X_t^{(n)}\|_H^p =& \|X_0^{(n)}\|_H^p + p(p-2)\int_0^t \|X_{s-}^{(n)}\|_H^{p-4} \|(P_n B(s, X_s^{(n)})\tilde{P}_n)^* X_{s-}^{(n)}\|_H^2 \mathrm{d}s \\
&+ \frac{p}{2}\int_0^t \|X_{s-}^{(n)}\|_H^{p-2} \left(2{}_{V^*}\langle A(s, X_s^{(n)}), X_{s-}^{(n)}\rangle_V + \|P_n B(s, X_s^{(n)})\tilde{P}_n\|_2^2 \right) \mathrm{d}s \\
&+ \int_0^t p\|X_{s-}^{(n)}\|_H^{p-2} \langle X_{s-}^{(n)}, P_n B(s, X_s^{(n)})\mathrm{d}W_s^{(n)}\rangle_H \\
&+ \int_0^t \int_{D^c} p\|X_{s-}^{(n)}\|_H^{p-2} \langle X_{s-}^{(n)}, P_n f(s, X_{s-}^{(n)}, z)\rangle_H \tilde{N}(\mathrm{d}s, \mathrm{d}z) \\
&+ \int_0^t \int_{D^c} \Big[\|X_{s-}^{(n)} + P_n f(s, X_{s-}^{(n)}, z)\|_H^p - \|X_{s-}^{(n)}\|_H^p \\
&\quad - p\|X_{s-}^{(n)}\|_H^{p-2} \langle X_{s-}^{(n)}, P_n f(s, X_{s-}^{(n)}, z)\rangle_H \Big] N(\mathrm{d}s, \mathrm{d}z), \quad t \in [0, T].
\end{aligned} \tag{3.6}
$$

Then conditions $(H3)$ and (1.2) give

$$\begin{aligned}
&\|X_t^{(n)}\|_H^p + \frac{p\theta}{2}\int_0^t \|X_s^{(n)}\|_H^{p-2}\|X_s^{(n)}\|_V^\alpha \mathrm{d}s \\
\le&\|x\|_H^p + p(p-2)\int_0^t \left(C\|X_s^{(n)}\|_H^p + F_s\cdot\|X_s^{(n)}\|_H^{p-2} + \gamma\|X_s^{(n)}\|_H^{p-2}\|X_s^{(n)}\|_V^\alpha\right)\mathrm{d}s \\
&+\frac{p}{2}\int_0^t \left(K\|X_s^{(n)}\|_H^p + F_s\cdot\|X_s^{(n)}\|_H^{p-2}\right)\mathrm{d}s + Y(t) + Z(t) + I(t),
\end{aligned}$$

where,

$$\begin{aligned}
Y(t) &= \int_0^t p\|X_{s-}^{(n)}\|_H^{p-2}\langle X_{s-}^{(n)}, P_nB(s,X_s^{(n)})\mathrm{d}W_s^{(n)}\rangle_H; \\
Z(t) &= \int_0^t\int_{D^c} p\|X_{s-}^{(n)}\|_H^{p-2}\langle X_{s-}^{(n)}, P_nf(s,X_{s-}^{(n)},z)\rangle_H\tilde{N}(\mathrm{d}s,\mathrm{d}z); \\
I(t) &= \int_0^t\int_{D^c}\Big|\|X_{s-}^{(n)} + P_nf(s,X_{s-}^{(n)},z)\|_H^p - \|X_{s-}^{(n)}\|_H^p \\
&\quad - p\|X_{s-}^{(n)}\|_H^{p-2}\langle X_{s-}^{(n)}, P_nf(s,X_{s-}^{(n)},z)\rangle_H\Big|N(\mathrm{d}s,\mathrm{d}z).
\end{aligned}$$

Note that $\|X_t^{(n)}\|_H \le R$, for $t < \tau_R^{(n)}$. Since $X_t^{(n)}$ takes values in the finite dimensional space H_n, there exists a constant C such that

$$\|X_t^{(n)}\|_V \le CR, \quad t < \tau_R^{(n)}.$$

Hence by (1.2) we have

$$\mathbb{E}\int_0^{t\wedge\tau_R^{(n)}} \|X_{s-}^{(n)}\|_H^{2(p-1)}\|B(s,X_s^{(n)})\|_2^2\ \mathrm{d}s < \infty$$

$$\mathbb{E}\int_0^{t\wedge\tau_R^{(n)}}\int_{D^c}\|X_{s-}^{(n)}\|_H^{2(p-1)}\|f(s,X_{s-}^{(n)},z)\|_H^2\ \nu(\mathrm{d}z)\mathrm{d}s < \infty.$$

Therefore, the processes $Y_{t\wedge\tau_R^{(n)}}$ and $Z_{t\wedge\tau_R^{(n)}}$ are martingales. Denote, for notational simplicity, the stopped process $X_{t\wedge\tau_R^{(n)}}^{(n)}$ etc again by $X_t^{(n)}$ etc. Then by Young's inequality and martingale property we have

$$\begin{aligned}
&\mathbb{E}\|X_t^{(n)}\|_H^p + \left(\frac{p\theta}{2} - \gamma p(p-2)\right)\mathbb{E}\int_0^t \|X_s^{(n)}\|_H^{p-2}\|X_s^{(n)}\|_V^\alpha \mathrm{d}s \\
\le&\mathbb{E}\|x\|_H^p + C\mathbb{E}\int_0^t\left(\|X_s^{(n)}\|_H^p + F_s^{p/2}\right)\mathrm{d}s + \mathbb{E}I(t),
\end{aligned}$$

where C is some constant.

The Taylor formula implies that for some constant C_p ($p \geq 2$) we have

$$\big| \|x+h\|_H^p - \|x\|_H^p - p\|x\|_H^{p-2}\langle x, h\rangle_H \big| \leq C_p(\|x\|_H^{p-2}\|h\|_H^2 + \|h\|_H^p), \ x, h \in H_n. \tag{3.7}$$

In particular, if $p = 2$, the above inequality can be replaced by the equality with $C_p = 1$, i.e.

$$\big| \|x+h\|_H^2 - \|x\|_H^2 - 2\langle x, h\rangle_H \big| = \|h\|_H^2, \quad \text{for all } x, h \in H_n.$$

Then it follows from (1.3) and (3.7) that

$$\begin{aligned}
\mathbb{E}I(t) \leq & \ \mathbb{E}\int_0^t \int_{D^c} \Big| \|X_{s-}^{(n)} + P_n f(s, X_{s-}^{(n)}, z)\|_H^p - \|X_{s-}^{(n)}\|_H^p \\
& \quad - p\|X_{s-}^{(n)}\|_H^{p-2}\langle X_{s-}^{(n)}, P_n f(s, X_{s-}^{(n)}, z)\rangle_H \Big| N(\mathrm{d}s, \mathrm{d}z) \\
= & \ \mathbb{E}\int_0^t \int_{D^c} \Big| \|X_s^{(n)} + P_n f(s, X_s^{(n)}, z)\|_H^p - \|X_s^{(n)}\|_H^p \\
& \quad - p\|X_s^{(n)}\|^{p-2}\langle X_s^{(n)}, P_n f(s, X_s^{(n)}, z)\rangle_H \Big| \nu(\mathrm{d}z)\mathrm{d}s \\
\leq & \ C\mathbb{E}\int_0^t \|X_s^{(n)}\|_H^{p-2}\|f(s, X_s^{(n)}, z)\|_H^2 \mathrm{d}s + C\mathbb{E}\int_0^t \|f(s, X_s^{(n)}, z)\|_H^p \mathrm{d}s \\
\leq & \ C\mathbb{E}\int_0^t \left(F_t^{p/2} + \|X_s^{(n)}\|_H^p \right) \mathrm{d}s,
\end{aligned} \tag{3.8}$$

where C is some constant may change from line to line.

Combining the above estimates we get

$$\begin{aligned}
& \mathbb{E}\|X_t^{(n)}\|_H^p + \left(\frac{p\theta}{2} - \gamma p(p-2)\right)\mathbb{E}\int_0^t \|X_s^{(n)}\|_H^{p-2}\|X_s^{(n)}\|_V^\alpha \mathrm{d}s \\
\leq & \mathbb{E}\|x\|_H^p + C\mathbb{E}\int_0^t \left(\|X_s^{(n)}\|_H^p + F_s^{p/2}\right)\mathrm{d}s,
\end{aligned}$$

where C is some constant.

By Gronwall's lemma we have

$$\mathbb{E}\|X_{t\wedge\tau_R^{(n)}}^{(n)}\|_H^p + \mathbb{E}\int_0^{T\wedge\tau_R^{(n)}} \|X_s^{(n)}\|_H^{p-2}\|X_s^{(n)}\|_V^\alpha \mathrm{d}s \leq C\left(\mathbb{E}\|x\|_H^p + \mathbb{E}\int_0^T F_s^{p/2}\mathrm{d}s\right), \ n \geq 1.$$

Here the constant C is independent of n and the stopping times $\tau_R^{(n)}$. Therefore, applying Fatou's lemma yields the desired inequality (3.5). □

If we assume the same assumptions as for Theorem 1.2, but with the condition (1.3) replaced by a weaker assumption

$$\int_{D^c} \|f(t, v, z)\|_H^{\beta+2}\nu(\mathrm{d}z) \leq F_t^{(\beta+2)/2} + C\|v\|_H^{\beta+2} + \gamma\|v\|_H^\beta\|v\|_V^\alpha, \tag{3.9}$$

we arrive at the following Lemma.

Lemma 3.3. *There exists a constant γ_0 such that if* (3.9) *is satisfied with $\gamma < \gamma_0$, then we have*

$$\begin{aligned}&\sup_{n\in\mathbb{N}}\left(\mathbb{E}\sup_{t\in[0,T]}\|X_t^{(n)}\|_H^{\beta+2}+\mathbb{E}\int_0^T\|X_t^{(n)}\|_H^{\beta}\|X_t^{(n)}\|_V^{\alpha}\mathrm{d}t\right)\\ \leq&C\left(\mathbb{E}\|x\|_H^{\beta+2}+\mathbb{E}\int_0^T F_t^{(\beta+2)/2}\mathrm{d}t\right).\end{aligned}\tag{3.10}$$

Proof. Let $p=\beta+2$ as before. By (3.6), $(H3)$ and (1.2), we find

$$\begin{aligned}&\sup_{s\in[0,t\wedge\tau_R^{(n)}]}\|X_s^{(n)}\|_H^p+\frac{p\theta}{2}\int_0^{t\wedge\tau_R^{(n)}}\|X_s^{(n)}\|_H^{p-2}\|X_s^{(n)}\|_V^{\alpha}\mathrm{d}s\\ \leq&\|x\|_H^p+p(p-2)\int_0^{t\wedge\tau_R^{(n)}}\left(C\|X_s^{(n)}\|_H^p+F_s\cdot\|X_s^{(n)}\|_H^{p-2}+\gamma\|X_s^{(n)}\|_H^{p-2}\|X_s^{(n)}\|_V^{\alpha}\right)\mathrm{d}s\\ &+\frac{p}{2}\int_0^{t\wedge\tau_R^{(n)}}\left(K\|X_s^{(n)}\|_H^p+F_s\cdot\|X_s^{(n)}\|_H^{p-2}\right)\mathrm{d}s+I_1(t)+I_2(t)+I_3(t)\\ \leq&\|x\|_H^p+\gamma p(p-2)\int_0^{t\wedge\tau_R^{(n)}}\|X_s^{(n)}\|_H^{p-2}\|X_s^{(n)}\|_V^{\alpha}\mathrm{d}s\\ &+C\int_0^{t\wedge\tau_R^{(n)}}\left(\|X_s^{(n)}\|_H^p+F_s^{p/2}\right)\mathrm{d}s+I_1(t)+I_2(t)+I_3(t),\end{aligned}$$

where C is some constant which may change from line to line, $\tau_R^{(n)}$ are the stopping times defined in the proof of the proceeding Lemma and

$$\begin{aligned}I_1(t)&:=p\sup_{r\in[0,t\wedge\tau_R^{(n)}]}\left|\int_0^r\|X_s^{(n)}\|_H^{p-2}\langle X_s^{(n)},P_nB(s,X_s^{(n)})\mathrm{d}W_s^{(n)}\rangle_H\right|;\\ I_2(t)&:=p\sup_{r\in[0,t\wedge\tau_R^{(n)}]}\left|\int_0^r\int_{D^c}\|X_s^{(n)}\|_H^{p-2}\langle X_{s-}^{(n)},P_nf(s,X_{s-}^{(n)},z)\rangle_H\tilde{N}(\mathrm{d}s,\mathrm{d}z)\right|;\\ I_3(t)&:=\sup_{r\in[0,t\wedge\tau_R^{(n)}]}\Big|\int_0^r\int_{D^c}\Big[\|X_{s-}^{(n)}+P_nf(s,X_{s-}^{(n)},z)\|_H^p-\|X_{s-}^{(n)}\|_H^p\\ &\qquad-p\|X_{s-}^{(n)}\|_H^{p-2}\langle X_{s-}^{(n)},P_nf(s,X_{s-}^{(n)},z)\rangle\Big]N(\mathrm{d}z,\mathrm{d}s)\Big|.\end{aligned}$$

On the basis of the Burkholder-Davis-Gundy inequality, (1.2), Cauchy-Schwartz inequality and Young's inequality, we have for any $\varepsilon > 0$,

(3.11)

$$\begin{aligned}
&\mathbb{E}I_1(t)\\
=&p\mathbb{E}\sup_{r\in[0,t\wedge\tau_R^{(n)}]}\left|\int_0^r \|X_s^{(n)}\|_H^{p-2}\langle X_s^{(n)}, P_nB(s,X_s^{(n)})\mathrm{d}W_s^{(n)}\rangle_H\right|\\
\leq&3p\mathbb{E}\left[\int_0^{t\wedge\tau_R^{(n)}} \|X_s^{(n)}\|_H^{2p-2}\|B(s,X_s^{(n)})\|_2^2\mathrm{d}s\right]^{1/2}\\
\leq&3p\mathbb{E}\left[\sup_{s\in[0,t\wedge\tau_R^{(n)}]}\|X_s^{(n)}\|_H^p\cdot\Big(\int_0^{t\wedge\tau_R^{(n)}}\|X_s^{(n)}\|_H^{p-2}\left(F_s+C\|X_s^{(n)}\|_H^2+\gamma\|X_s^{(n)}\|_V^\alpha\right)\mathrm{d}s\Big)\right]^{1/2}\\
\leq&3p\left[\varepsilon\mathbb{E}\sup_{s\in[0,t]}\|X_s^{(n)}\|_H^p\right]^{1/2}\left[\frac{1}{\varepsilon}\mathbb{E}\Big(\int_0^{t\wedge\tau_R^{(n)}}\|X_s^{(n)}\|_H^{p-2}\left(F_s+C\|X_s^{(n)}\|_H^2+\gamma\|X_s^{(n)}\|_V^\alpha\right)\mathrm{d}s\Big)\right]^{1/2}\\
\leq&\varepsilon\mathbb{E}\sup_{s\in[0,t\wedge\tau_R^{(n)}]}\|X_s^{(n)}\|_H^p+C_{\varepsilon,p}\mathbb{E}\Big(\int_0^{t\wedge\wedge\tau_R^{(n)}}\|X_s^{(n)}\|_H^{p-2}\left(F_s+C\|X_s^{(n)}\|_H^2+\gamma\|X_s^{(n)}\|_V^\alpha\right)\mathrm{d}s\Big)\\
\leq&\varepsilon\mathbb{E}\sup_{s\in[0,t\wedge\tau_R^{(n)}]}\|X_s^{(n)}\|_H^p+\gamma C_{\varepsilon,p}\mathbb{E}\int_0^{t\wedge\tau_R^{(n)}}\|X_s^{(n)}\|_H^{p-2}\|X_s^{(n)}\|_V^\alpha ds\\
&+C_{\varepsilon,p}\mathbb{E}\int_0^{t\wedge\tau_R^{(n)}}\left(\|X_s^{(n)}\|_H^p+F_s^{p/2}\right)\mathrm{d}s.
\end{aligned}$$

Similarly, using the Burkholder-Davis inequality (cf. [28]), (1.2) and Young's inequality we have

(3.12)

$$\begin{aligned}
&\mathbb{E}I_2(t)\\
&=p\,\mathbb{E}\sup_{r\in[0,t\wedge\tau_R^{(n)}]}\left|\int_0^r\int_{D^c}\|X_s^{(n)}\|_H^{p-2}\langle X_{s-}^{(n)}, P_nf(s,X_{s-}^{(n)},z)\rangle_H\tilde{N}(\mathrm{d}s,\mathrm{d}z)\right|\\
&\leq C\mathbb{E}\left[\int_0^{t\wedge\tau_R^{(n)}}\int_{D^c}\|X_s^{(n)}\|_H^{2p-2}\|P_nf(s,X_s^{(n)},z)\|_H^2\nu(\mathrm{d}z)\mathrm{d}s\right]^{\frac{1}{2}}\\
&\leq C\mathbb{E}\left[\sup_{s\in[0,t\wedge\tau_R^{(n)}]}\|X_s^{(n)}\|_H^p\left(\int_0^{t\wedge\tau_R^{(n)}}\|X_s^{(n)}\|_H^{p-2}(F_s+C\|X_s^{(n)}\|_H^2+\gamma\|X_s^{(n)}\|_V^\alpha)\mathrm{d}s\right)\right]^{\frac{1}{2}}\\
&\leq\varepsilon\mathbb{E}\sup_{s\in[0,t\wedge\tau_R^{(n)}]}\|X_s^{(n)}\|_H^p+C_{\varepsilon,p}\mathbb{E}\Big(\int_0^{t\wedge\tau_R^{(n)}}\|X_s^{(n)}\|^{p-2}\left(F_s+\|X_s^{(n))}\|_H^2+\gamma\|X_s^{(n))}\|_V^\alpha\right)\mathrm{d}s\Big)
\end{aligned}$$

$$\le \varepsilon\mathbb{E}\sup_{s\in[0,t\wedge\tau_R^{(n)}]}\|X_s^{(n)}\|_H^p+\gamma C_{\varepsilon,p}\mathbb{E}\int_0^{t\wedge\tau_R^{(n)}}\|X_s^{(n)}\|_H^{p-2}\|X_s^{(n)}\|_V^\alpha ds$$

$$+C_{\varepsilon,p}\mathbb{E}\int_0^{t\wedge\tau_R^{(n)}}\left(\|X_s^{(n)}\|_H^p+F_s^{p/2}\right)\mathrm{d}s,$$

where $C_{\varepsilon,p}$ is not necessarily the same number from line to line.

For the term $I_3(t)$, by (3.9), (1.2) and (3.7), we have

$$\begin{aligned}
\mathbb{E}I_3(t)\le{}&\mathbb{E}\int_0^{t\wedge\tau_R^{(n)}}\int_{D^c}\Big|\|X_{s-}^{(n)}+P_nf(s,X_{s-}^{(n)},z)\|_H^p-\|X_{s-}^{(n)}\|_H^p\\
&-p\|X_{s-}^{(n)}\|_H^{p-2}\langle X_{s-}^{(n)},P_nf(s,X_{s-}^{(n)},z)\rangle_H\Big|N(\mathrm{d}s,\mathrm{d}z)\\
={}&\mathbb{E}\int_0^{t\wedge\tau_R^{(n)}}\int_{D^c}\Big|\|X_s^{(n)}+P_nf(s,X_s^{(n)},z)\|_H^p-\|X_s^{(n)}\|_H^p\\
&-p\|X_s^{(n)}\|^{p-2}\langle X_s^{(n)},P_nf(s,X_s^{(n)},z)\rangle_H\Big|\nu(\mathrm{d}z)\mathrm{d}s\\
\le{}&C_p\mathbb{E}\int_0^{t\wedge\tau_R^{(n)}}\int_{D^c}\Big(\|X_s^{(n)}\|_H^{p-2}\|f(s,X_s^{(n)},z)\|_H^2+\|f(s,X_s^{(n)},z)\|_H^p\Big)\nu(\mathrm{d}z)\mathrm{d}s\\
\le{}&\gamma C_p\mathbb{E}\int_0^{t\wedge\tau_R^{(n)}}\|X_s^{(n)}\|_H^{p-2}\|X_s^{(n)}\|_V^\alpha\mathrm{d}s+C_p\mathbb{E}\int_0^{t\wedge\tau_R^{(n)}}(F_t^{p/2}+\|X_s^{(n)}\|_H^p)\mathrm{d}s.
\end{aligned}\tag{3.13}$$

Combining the estimates (3.11)-(3.13) we get

$$\begin{aligned}
&\mathbb{E}(I_1(t)+I_2(t)+I_3(t))\\
\le{}&2\varepsilon\mathbb{E}\sup_{s\in[0,t\wedge\tau_R^{(n)}]}\|X_s^{(n)}\|_H^p+\gamma C_{\varepsilon,p}\mathbb{E}\int_0^{t\wedge\tau_R^{(n)}}\|X_s^{(n)}\|_H^{p-2}\|X_s^{(n)}\|_V^\alpha ds\\
&+C_{\varepsilon,p}\mathbb{E}\int_0^{t\wedge\tau_R^{(n)}}\|X_s^{(n)}\|_H^p\mathrm{d}s+C_{\varepsilon,p}\mathbb{E}\int_0^T F_s^{p/2}\mathrm{d}s.
\end{aligned}$$

Let $\varepsilon=\frac{1}{3}$, then we have

$$\begin{aligned}
&\frac{1}{3}\mathbb{E}\sup_{s\in[0,t\wedge\tau_R^{(n)}]}\|X_s^{(n)}\|_H^p+\left(\frac{p\theta}{2}-3\gamma C_0\right)\mathbb{E}\int_0^{t\wedge\tau_R^{(n)}}\|X_s^{(n)}\|_H^{p-2}\|X_s^{(n)}\|_V^\alpha\mathrm{d}s\\
\le{}&\mathbb{E}\|x\|_H^p+C_0\mathbb{E}\int_0^{t\wedge\tau_R^{(n)}}\|X_s^{(n)}\|_H^p\mathrm{d}s+C_0\mathbb{E}\int_0^T F_s^{p/2}\mathrm{d}s,
\end{aligned}$$

where C_0 is some constant.

Observe that $\|X_s^{(n)}\|_H\le R$, for $s<\tau_R^{(n)}$. Then we see that the right-hand side of the above inequality is finite. Therefore, if γ is small enough (e.g. $\gamma<\gamma_0:=\frac{p\theta}{6C_0}$), we may apply

Gronwall's lemma to get

$$\mathbb{E}\sup_{t\in[0,T\wedge\tau_R^{(n)}]}\|X_t^{(n)}\|_H^p+\mathbb{E}\int_0^{T\wedge\tau_R^{(n)}}\|X_s^{(n)}\|_H^{p-2}\|X_s^{(n)}\|_V^\alpha\mathrm{d}s\le C\left(\mathbb{E}\|x\|_H^p+\mathbb{E}\int_0^T F_s^{p/2}\mathrm{d}s\right),\ n\ge 1,$$

where C is a constant independent of n. Recall that $\tau_R^{(n)}\uparrow T$, $\mathbb{P}$-a.s. and $\mathbb{P}\{\tau_R^{(n)}<T\}=0$ as $R\to\infty$. It then follows from Fatou's lemma that

$$\begin{aligned}
&\mathbb{E}\sup_{t\in[0,T]}\|X_t^{(n)}\|_H^p+\mathbb{E}\int_0^T\|X_s^{(n)}\|_H^{p-2}\|X_s^{(n)}\|_V^\alpha\mathrm{d}s\\
&\le\liminf_{R\to\infty}\left(\mathbb{E}\sup_{t\in[0,T\wedge\tau_R^{(n)}]}\|X_t^{(n)}\|_H^p+\mathbb{E}\int_0^{T\wedge\tau_R^{(n)}}\|X_s^{(n)}\|_H^{p-2}\|X_s^{(n)}\|_V^\alpha\mathrm{d}s\right)\\
&\le C\left(\mathbb{E}\|X_0\|_H^p+\mathbb{E}\int_0^T F_s^{p/2}\mathrm{d}s\right),\ \text{for all } n\ge 1.
\end{aligned}$$

This completes the proof. □

For the simplicity of notations, we introduce the following spaces:

$$\begin{aligned}
&K=L^\alpha([0,T]\times\Omega,\mathrm{d}t\times\mathbb{P};V);\\
&K^*=L^{\frac{\alpha}{\alpha-1}}([0,T]\times\Omega,\mathrm{d}t\times\mathbb{P};V^*);\\
&J=L^2([0,T]\times\Omega,\mathrm{d}t\times\mathbb{P};\mathcal{T}_2(U;H));\\
&\mathcal{M}=\mathcal{M}_T^2(\mathcal{P}\otimes\mathcal{Z},\mathrm{d}t\times\mathbb{P}\times\nu;H).
\end{aligned}$$

Lemma 3.4. *Under the same assumptions as in Theorem 1.2, there exists a subsequence (n_k) and an element $\bar{X}\in K\cap L^\infty([0,T];L^p(\Omega;H))$ such that*

(i) $X^{(n_k)}\to\bar{X}$ weakly in K and weakly star in $L^\infty([0,T];L^p(\Omega;H))$;

(ii) $Y^{(n_k)}:=P_{n_k}A(\cdot,X^{(n_k)})\to Y$ weakly in K^*;

(iii) $Z^{(n_k)}:=P_{n_k}B(\cdot,X^{(n_k)})\to Z$ weakly in J and

$$\int_0^\cdot P_{n_k}B(s,X_s^{(n_k)})\mathrm{d}W_s^{(n_k)}\to\int_0^\cdot Z_s\mathrm{d}W_s$$

weakly in $L^\infty([0,T],\mathrm{d}t;L^2(\Omega,\mathbb{P};H))$;

(iv) $F^{(n_k)}:=P_{n_k}f(\cdot,X^{(n_k)},\cdot)1_{D^c}\to F1_{D^c}$ weakly in $\mathcal{M}$.

Proof. Applying Lemma 3.2 with $p=2$(i.e. $\beta=0$) we have

$$\sup_n\mathbb{E}\int_0^T\|X_t^{(n)}\|_V^\alpha dt<\infty. \tag{3.14}$$

Since the space K is reflexive, we can find a weakly convergent subsequence $\{X^{(n_k)}\}$ and $\bar{X}\in K$ such that $X^{(n_k)}$ converges to $\bar{X}$ weakly in K.

Similarly, since $L^\infty([0,T];L^p(\Omega;H)) = (L^1([0,T];L^{\frac{p}{p-1}}(\Omega;H)))^*$, by the Banach-Alaoglu theorem, (3.5) allows us to get another weakly star convergent subsequence (for simplicity we still denote it by the same notation $\{X^{(n_k)}\}$) and $\bar{X} \in K \cap L^p(\Omega; L^\infty([0,T];H))$ such that assertion (i) holds. Meanwhile, by $(H4)$ and (3.5) we have

$$
\begin{aligned}
&\sup_n \mathbb{E}\int_0^T \|A(t,X_t^{(n)})\|_{V^*}^{\frac{\alpha}{\alpha-1}}\mathrm{d}t \\
\le&\sup_n \mathbb{E}\int_0^T (F_t + C\|X_t^{(n)}\|_V^\alpha)(1+\|X_t^{(n)}\|_H^\beta)\mathrm{d}t \\
\le&C\sup_n \mathbb{E}\int_0^T \left(F_t + \|X_t^{(n)}\|_V^\alpha + F_t^{\frac{\beta+2}{2}} + \|X_t^{(n)}\|_H^{\beta+2} + \|X_t^{(n)}\|_H^\beta\|X_t^{(n)}\|_V^\alpha\right)\mathrm{d}t \\
<&\infty.
\end{aligned}
$$

Therefore, claim (ii) also holds.

Also, note that by (1.2)

$$
\begin{aligned}
&\sup_n \mathbb{E}\int_0^T \|P_nB(t,X_t^{(n)})\|_2^2\mathrm{d}t \\
\le&\sup_n \mathbb{E}\int_0^T \left(F_t + C\|X_t^{(n)}\|_H^2 + \gamma\|X_t^{(n)}\|_V^\alpha\right)\mathrm{d}t < \infty.
\end{aligned}
$$

Hence by taking a subsequence we have that $P_{n_k}B(t,X_t^{(n_k)})$ converges to Z weakly in J.

Recall that $\tilde{P}_n$ is the orthogonal projection onto $span\{g_1,\cdots,g_n\}$ in U, one may assume $P_{n_k}B(t,X_t^{(n_k)})\tilde{P}_n$ also converges to Z weakly in J without loss of generality. Since

$$
\int_0^\cdot P_nB(s,X_t^{(n_k)})\mathrm{d}W_s^{n_k} = \int_0^\cdot P_{n_k}B(s,X_s^{(n_k)})\tilde{P}_{n_k}\mathrm{d}W_s
$$

and weakly convergence is preserved under the linear continuous mapping

$$
I : \phi \in J \mapsto I(\phi) := \int \phi \ \mathrm{d}W \in L^2([0,T]\times\Omega;H),
$$

we know that

$$
\int_0^\cdot P_nB(s,X_s^{(n_k)})\tilde{P}_n\mathrm{d}W_s
$$

converges weakly to $\int_0^\cdot Z_s\mathrm{d}W_s$, i.e. (iii) holds.

Similarly, by (1.2) we have

$$
\begin{aligned}
&\sup_n \mathbb{E}\int_0^T\int_{D^c} \|P_nf(s,X_{s-}^{(n)},z)\|_H^2\nu(\mathrm{d}z)\mathrm{d}s \\
\le&\sup_n \int_0^T \left(F_t + C\|X_s^{(n)}\|_H^2 + \gamma\|X_s^{(n)}\|_V^\alpha\right)\mathrm{d}s \\
<&\infty,
\end{aligned}
$$

which yields claim (iv). □

Proof of Theorem 3.1. Existence of solutions: Now we define a V^*-valued process X by

$$X_t := X_0 + \int_0^t Y_s \mathrm{d}s + \int_0^t Z_s \mathrm{d}W_s + \int_0^t \int_{D^c} F(s,z)\tilde{N}(\mathrm{d}s,\mathrm{d}z),\ t \in [0,T]. \tag{3.15}$$

By Lemma 3.4, it is easy to see that X is a V^*-valued modification of the V-valued process $\bar{X}$, i.e. $X = \bar{X}$ $\mathrm{d}t \times \mathbb{P}$-a.e. in V. Moreover, we have

$$\sup_{t\in[0,T]} \mathbb{E}\|X_t\|_H^p + \mathbb{E}\int_0^T \|X_t\|_V^\alpha \mathrm{d}t < \infty.$$

By [20], we know that X is an H-valued càdlàg $\mathbb{F}$-adapted process satisfying

$$\begin{aligned}\|X_t\|_H^2 =& \|X_0\|_H^2 + \int_0^t \Big(2{}_{V^*}\langle Y_s, \bar{X}_s\rangle_V + \|Z_s\|_2^2\Big)\mathrm{d}s + 2\int_0^t \langle \bar{X}_s, Z_s \mathrm{d}W_s\rangle_H \\ &+ 2\int_0^t\int_{D^c} \langle \bar{X}_s, F(s,z)\rangle_H \tilde{N}(\mathrm{d}s,\mathrm{d}z) + \int_0^t\int_{D^c} \|F(s,z)\|_H^2 N(\mathrm{d}s,\mathrm{d}z).\end{aligned} \tag{3.16}$$

Therefore, it remains to verify that

$$\begin{aligned} A(\cdot,\bar{X}) = Y,\ B(\cdot,\bar{X}) = Z, \quad \mathrm{d}t\times\mathbb{P} - a.e.; \\ \text{and } f(s,\bar{X}_{s-},z) = F(s,z), \quad \mathrm{d}t\times\mathbb{P}\times\nu - a.e.. \end{aligned}$$

Define

$$\mathcal{N} = \Big\{\phi : \phi \text{ is a } V\text{-valued } \mathbb{F}\text{-adapted process such that } \mathbb{E}\int_0^T \rho(\phi_s) ds < \infty\Big\}.$$

For $\phi \in K \cap \mathcal{N} \cap L^\infty([0,T]; L^p(\Omega;H))$, by applying the Itô formula to the process $X^{(n_k)}$, see [50, proof of Theorem 4.1] we have

$$\begin{aligned} & e^{-\int_0^t (K+\rho(\phi_s))\mathrm{d}s}\|X_t^{(n_k)}\|_H^2 \\ =& \|X_0^{(n_k)}\|_H^2 + \int_0^t e^{-\int_0^s (K+\rho(\phi_r))\mathrm{d}r}\Big(2{}_{V^*}\langle A(s,X_s^{(n_k)}), X_{s-}^{(n_k)}\rangle_V \\ &+ \|P_{n_k}B(s,X_s^{(n_k)})\tilde{P}_{n_k}\|_2^2 - (K+\rho(\phi_s))\|X_s^{(n_k)}\|_H^2\Big)\mathrm{d}s\Big] \\ &+ 2\int_0^t e^{-\int_0^s (K+\rho(\phi_r))\mathrm{d}r}\langle X_{s-}^{(n_k)}, P_{n_k}B(s,X_s^{(n_k)})\mathrm{d}W_s^{n_k}\rangle_H \\ &+ 2\int_0^t\int_{D^c} e^{-\int_0^s (K+\rho(\phi_r))\mathrm{d}r}\langle X_{s-}^{(n_k)}, P_{n_k} f(s,X_{s-}^{(n_k)},z)\rangle_H \tilde{N}(\mathrm{d}s,\mathrm{d}z) \\ &+ \int_0^t\int_{D^c} e^{-\int_0^s (K+\rho(\phi_r))\mathrm{d}r}\|P_{n_k} f(s,X_{s-}^{(n_k)},z)\|_H^2 N(\mathrm{d}s,\mathrm{d}z). \end{aligned}$$

Then by taking the expectation to both sides and $(H2)$ we get

$$
\begin{aligned}
&\mathbb{E}\left(e^{-\int_0^t (K+\rho(\phi_s))\mathrm{d}s}\|X_t^{(n_k)}\|_H^2\right)-\mathbb{E}\left(\|X_0^{(n_k)}\|_H^2\right)\\
=&\mathbb{E}\Big[\int_0^t e^{-\int_0^s (K+\rho(\phi_r))\mathrm{d}r}\Big(2_{V^*}\langle A(s,X_s^{(n_k)}),X_{s-}^{(n_k)}\rangle_V\\
&+\|P_{n_k}B(s,X_s^{(n_k)})\tilde{P}_{n_k}\|_2^2-(K+\rho(\phi_s))\|X_s^{(n_k)}\|_H^2\Big)\mathrm{d}s\Big]\\
&+\mathbb{E}\Big[\int_0^t\int_{D^c} e^{-\int_0^s (K+\rho(\phi_r))\mathrm{d}r}\|P_{n_k}f(s,X_{s-}^{(n_k)},z)\|_H^2\nu(\mathrm{d}z)\mathrm{d}s\Big]\\
\leq&\mathbb{E}\Big[\int_0^t e^{-\int_0^s (K+\rho(\phi_r))\mathrm{d}r}\Big(2_{V^*}\langle A(s,X_s^{(n_k)})-A(s,\phi_s),X_s^{(n_k)}-\phi_s\rangle_V\\
&+\|B(s,X_s^{(n_k)})-B(s,\phi_s)\|_2^2-(K+\rho(\phi_s))\|X_s^{(n_k)}-\phi_s\|_H^2\\
&+\int_{D^c}\|f(s,X_s^{(n_k)},z)-f(s,\phi_s,z)\|_H^2\nu(\mathrm{d}z)\Big)\mathrm{d}s\Big]\\
&+\mathbb{E}\Big[\int_0^t e^{-\int_0^s (K+\rho(\phi_r))\mathrm{d}r}\Big(2_{V^*}\langle A(s,X_s^{(n_k)})-A(s,\phi_s),\phi_s\rangle_V+2_{V^*}\langle A(s,\phi_s),X_s^{(n_k)}\rangle_V\\
&-\|B(s,\phi_s)\|_2^2+2\langle B(s,X_s^{(n_k)}),B(s,\phi_s)\rangle_{\mathcal{T}_2(U,H)}-2(K+\rho(\phi_s))\langle X_s^{(n_k)},\phi_s\rangle_H\\
&+(K+\rho(\phi_s))\|\phi_s\|_H^2+\int_{D^c}\left(2\langle f(s,X_s^{(n_k)},z),f(s,\phi_s,z)\rangle_H-\|f(s,\phi_s,z)\|_H^2\right)\nu(\mathrm{d}z)\Big)\mathrm{d}s\Big]\\
\leq&\mathbb{E}\Big[\int_0^t e^{-\int_0^s (K+\rho(\phi_r))\mathrm{d}r}\Big(2_{V^*}\langle A(s,X_s^{(n_k)})-A(s,\phi_s),\phi_s\rangle_V+2_{V^*}\langle A(s,\phi_s),X_s^{(n_k)}\rangle_V\\
&-\|B(s,\phi_s)\|_2^2+2\langle B(s,X_s^{(n_k)}),B(s,\phi_s)\rangle_{\mathcal{T}_2(U,H)}-2(K+\rho(\phi_s))\langle X_s^{(n_k)},\phi_s\rangle_H\\
&+(K+\rho(\phi_s))\|\phi_s\|_H^2+\int_{D^c}\left(2\langle f(s,X_s^{(n_k)},z),f(s,\phi_s,z)\rangle_H-\|f(s,\phi_s,z)\|_H^2\right)\nu(\mathrm{d}z)\Big)\mathrm{d}s\Big].
\end{aligned}
$$

Hence for any nonnegative function $\psi\in L^\infty([0,T];\mathrm{d}t)$ we have

$$
\begin{aligned}
&\mathbb{E}\left[\int_0^T\psi_t\left(e^{-\int_0^t (K+\rho(\phi_s))\mathrm{d}s}\|X_t\|_H^2-\|X_0\|_H^2\right)\mathrm{d}t\right]\\
&\leq\liminf_{k\to\infty}\mathbb{E}\left[\int_0^T\psi_t\left(e^{-\int_0^t (K+\rho(\phi_s))\mathrm{d}s}\|X_t^{(n_k)}\|_H^2-\|X_0^{(n_k)}\|_H^2\right)\mathrm{d}t\right]\\
&\leq\liminf_{k\to\infty}\mathbb{E}\Big[\int_0^T\psi_t\Big(\int_0^t e^{-\int_0^s (K+\rho(\phi_r))\mathrm{d}r}\Big(2_{V^*}\langle A(s,X_s^{(n_k)})-A(s,\phi_s),\phi_s\rangle_V\\
&\quad+2_{V^*}\langle A(s,\phi_s),X_s^{(n_k)}\rangle_V-\|B(s,\phi_s)\|_2^2+2\langle B(s,X_s^{(n_k)}),B(s,\phi_s)\rangle_{\mathcal{T}_2(U,H)}\\
&\quad-2(K+\rho(\phi_s))\langle X_s^{(n_k)},\phi_s\rangle_H+(K+\rho(\phi_s))\|\phi_s\|_H^2\\
&\quad+\int_{D^c}\left(2\langle f(s,X_s^{(n_k)},z),f(s,\phi_s,z)\rangle_H-\|f(s,\phi_s,z)\|_H^2\right)\nu(\mathrm{d}z)\Big)\mathrm{d}s\Big)\mathrm{d}t\Big]
\end{aligned}
\tag{3.17}
$$

$$
\begin{aligned}
=&\mathbb{E}\Big[\int_0^T \psi_t\Big(\int_0^t e^{-\int_0^s (K+\rho(\phi_r))\mathrm{d}r}\Big(2_{V^*}\langle Y_s - A(s,\phi_s),\phi_s\rangle_V + 2_{V^*}\langle A(s,\phi_s),\bar{X}_s\rangle_V \\
&- \|B(s,\phi_s)\|_2^2 + 2\langle Z_s, B(s,\phi_s)\rangle_{\mathcal{T}_2(U,H)} - 2(K+\rho(\phi_s))\langle \bar{X}_s,\phi_s\rangle_H \\
&+ (K+\rho(\phi_s))\|\phi_s\|_H^2 + \int_{D^c}\big(2\langle F(s,z), f(s,\phi_s,z)\rangle_H - \|f(s,\phi_s,z)\|_H^2\big)\,\nu(\mathrm{d}z)\Big)\mathrm{d}s\Big)\mathrm{d}t\Big].
\end{aligned}
$$

On the other hand, by (3.16) we have for $\phi \in K \cap \mathcal{M} \cap L^\infty([0,T];L^p(\Omega;H))$,

$$
\begin{aligned}
&\mathbb{E}\left(e^{-\int_0^t (K+\rho(\phi_s))\mathrm{d}s}\|X_t\|_H^2\right) - \mathbb{E}\left(\|X_0\|_H^2\right) \\
=&\mathbb{E}\Big[\int_0^t e^{-\int_0^s (K+\rho(\phi_r))\mathrm{d}r}\Big(2_{V^*}\langle Y_s,\bar{X}_s\rangle_V + \|Z_s\|_2^2 \\
&- (K+\rho(\phi_s))\|X_s\|_H^2 + \int_{D^c}\|F(s,z)\|_H^2\nu(\mathrm{d}z)\Big)\mathrm{d}s\Big].
\end{aligned} \tag{3.18}
$$

Combining (3.18) with (3.17) we have

$$
\begin{aligned}
&\mathbb{E}\Big[\int_0^T \psi_t\Big(\int_0^t e^{-\int_0^s (K+\rho(\phi_r))\mathrm{d}r}\Big(2_{V^*}\langle Y_s - A(s,\phi_s),\bar{X}_s-\phi_s\rangle_V - (K+\rho(\phi_s))\|\bar{X}_s-\phi_s\|_H^2 \\
&+ \|B(s,\phi_s)-Z_s\|_2^2 + \int_{D^c}\|f(s,\phi_s,z)-F(s,z)\|_H^2\nu(\mathrm{d}z)\Big)\mathrm{d}s\Big)\mathrm{d}t\Big] \le 0.
\end{aligned} \tag{3.19}
$$

Therefore, if we take $\phi = \bar{X}$ in (3.19), we obtain that $Z = B(\cdot,\bar{X})$ in J, $F(\cdot,\cdot) = f(\cdot,\bar{X}_\cdot,\cdot)$ in $\mathcal{M}$.

Note that (3.19) also implies that

$$
\begin{aligned}
&\mathbb{E}\Big[\int_0^T \psi_t\Big(\int_0^t e^{-\int_0^s (K+\rho(\phi_r))\mathrm{d}r}\Big(2_{V^*}\langle Y_s - A(s,\phi_s),\bar{X}_s-\phi_s\rangle_V \\
&\qquad - (K+\rho(\phi_s))\|\bar{X}_s-\phi_s\|_H^2\Big)\mathrm{d}s\Big)\mathrm{d}t\Big] \le 0.
\end{aligned} \tag{3.20}
$$

If we take $\phi = \bar{X} - \varepsilon\tilde{\phi}v$ in (3.20) for $\tilde{\phi} \in L^\infty([0,T]\times\Omega;\mathrm{d}t\times\mathbb{P};\mathbb{R})$ and $v \in V$, divide both sides by ε and let $\varepsilon \to 0$, then we have

$$
\mathbb{E}\Big[\int_0^T \psi_t\Big(\int_0^t e^{-\int_0^s (K+\rho(\bar{X}_r))\mathrm{d}r}\Big(2\tilde{\phi}_{s\,V^*}\langle Y_s - A(s,\bar{X}_s), v\rangle_V\Big)\mathrm{d}s\Big)\mathrm{d}t\Big] \le 0.
$$

Then the claim $Y = A(\cdot,\bar{X})$ follows immediately.

Therefore, the process $X = \{X_t\}_{t\ge 0}$ is a solution to (3.1). Furthermore, the estimates (1.5) and (1.6) follows directly from Lemma 3.2, 3.3 and 3.4.

Uniqueness of solutions: We finally proceed with showing the uniqueness of solutions to (3.1).

Suppose X_t, Y_t are the solutions of (3.1) with initial conditions X_0, Y_0 respectively, i.e.

(3.21)

$$X_t = X_0 + \int_0^t A(s, X_s)\mathrm{d}s + \int_0^t B(s, X_s)\mathrm{d}W_s + \int_0^t \int_{D^c} f(s, X_{s-}, z)\tilde{N}(\mathrm{d}s, \mathrm{d}z),\ t \in [0, T];$$
$$Y_t = Y_0 + \int_0^t A(s, Y_s)\mathrm{d}s + \int_0^t B(s, Y_s)\mathrm{d}W_s + \int_0^t \int_{D^c} f(s, Y_{s-}, z)\tilde{N}(\mathrm{d}s, \mathrm{d}z),\ t \in [0, T].$$

We define the following stopping times:

$$\sigma_N := \inf\{t \in [0, T] : \|X_t\|_H \geq N\} \wedge \inf\{t \in [0, T] : \|Y_t\|_H \geq N\} \wedge T.$$

Applying again the Schmalfuss [50] trick, by means of the Itô formula (3.16) we have

$$\begin{aligned}
& e^{-\int_0^{t\wedge\sigma_N}(K+\rho(Y_s))\mathrm{d}s}\|X_{t\wedge\sigma_N} - Y_{t\wedge\sigma_N}\|_H^2 - \|X_0 - Y_0\|_H^2 \\
=& \int_0^{t\wedge\sigma_N} e^{-\int_0^s(K+\rho(Y_r))\mathrm{d}r}\Big(2{}_{V^*}\langle A(s, X_s) - A(s, Y_s), X_s - Y_s\rangle_V \\
& + \|B(s, X_s) - B(s, Y_s)\|_2^2 - (K + \rho(Y_s))\|X_s - Y_s\|_H^2\Big)\mathrm{d}s \\
& + 2\int_0^{t\wedge\sigma_N} e^{-\int_0^s(K+\rho(Y_r))\mathrm{d}r}\langle X_s - Y_s, B(s, X_s)\mathrm{d}W_s - B(s, Y_s)\mathrm{d}W_s\rangle_H \\
& + 2\int_0^{t\wedge\sigma_N}\int_{D^c} e^{-\int_0^s(K+\rho(Y_r))\mathrm{d}r}\langle X_s - Y_s, f(s, X_{s-}, z) - f(s, Y_{s-}, z)\rangle_H\tilde{N}(\mathrm{d}s, \mathrm{d}z) \\
& + \int_0^{t\wedge\sigma_N}\int_{D^c} e^{-\int_0^s(K+\rho(Y_r))\mathrm{d}r}\|f(s, X_{s-}, z) - f(s, Y_{s-}, z)\|_H^2 N(\mathrm{d}s, \mathrm{d}z).
\end{aligned}$$

It then follows from $(H2)$ that

$$\begin{aligned}
& \mathbb{E}\left[e^{-\int_0^{t\wedge\sigma_N}(K+\rho(Y_s))\mathrm{d}s}\|X_t - Y_t\|_H^2\right] - \mathbb{E}\|X_0 - Y_0\|_H^2 \\
=& \mathbb{E}\Big[\int_0^{t\wedge\sigma_N} e^{-\int_0^s(K+\rho(Y_r))\mathrm{d}r}\Big(2{}_{V^*}\langle A(s, X_s) - A(s, Y_s), X_s - Y_s\rangle_V \\
& + \|B(s, X_s) - B(s, Y_s)\|_2^2 - (K + \rho(Y_s))\|X_s - Y_s\|_H^2 \\
& + \int_{D^c}\|f(s, X_{s-}, z) - f(s, Y_{s-}, z)\|_H^2\nu(\mathrm{d}z)\Big)\mathrm{d}s\Big] \\
\leq& 0.
\end{aligned}$$

Hence if $X_0 = Y_0$ $\mathbb{P}$-a.s., then

$$\mathbb{E}\left[e^{-\int_0^{t\wedge\sigma_N}(K+\rho(Y_s))\mathrm{d}s}\|X_t - Y_t\|_H^2\right] = 0,\ t \in [0, T].$$

Note that by (1.4) and (1.5) (see Lemma 3.2) we have

$$\int_0^T (K + \rho(Y_s))\mathrm{d}s < \infty,\ \mathbb{P}\text{-a.s.}.$$

Therefore, by letting $N \to \infty$ (hence $\sigma_N \uparrow T$) we have that $X_t = Y_t$, $\mathbb{P}$-a.s., $t \in [0,T]$. Then the pathwise uniqueness follows from the path càdlàg property of X, Y in H.

This completes the proof of Theorem 3.1. $\square$

3.2 With large jumps

Let τ be a stopping time such that $\tau < \infty$ a.s.. We define

$$\begin{aligned} W^\tau(t) &= W(t+\tau) - W(\tau); \\ p^\tau(t) &= p(t+\tau), t \in \mathcal{D}(p^\tau), \end{aligned} \tag{3.22}$$

where $\mathcal{D}(p^\tau) = \{t \in (0,\infty) : t + \tau \in \mathcal{D}(p)\}$. Let $\mathcal{F}_t^\tau = \mathcal{F}_{t+\tau}$, $t \in [0,T]$. The following result is a direct extension from Theorem II6.4 and II6.5 in [29].

Proposition 3.5. *The process W^τ defined by (3.22) is a cylindrical $\mathcal{F}_t^\tau$-Wiener process and p^τ is a stationary $\mathcal{F}_t^\tau$-Poisson point process with the intensity measure ν.*

Clearly W^τ is independent of $\mathcal{F}_\tau$ and W^τ, p^τ enjoy the same properties as W, p.

Corollary 3.6. *Let τ be a stopping time on $[0,T]$ and X_τ be an $\mathcal{F}_\tau$-measurable random variable. Under the same assumptions as in Theorem 1.2, there exists a unique càdlàg H-valued $\mathbb{F}$-adapted process (X_t) and a process $\bar{X} \in L^\alpha([\tau,T];V) \cap L^2([\tau,T];H)$, $\mathbb{P}$-a.s. which is $\mathrm{d}t \times \mathbb{P}$-equivalent to X such that the equality holds $\mathbb{P}$-a.s.:*

(3.23)
$$X_t = X_\tau + \int_\tau^t A(s,\bar{X}_s)\mathrm{d}s + \int_\tau^t B(s,\bar{X}_s)\mathrm{d}W_s + \int_\tau^t \int_{D^c} f(s,\bar{X}_{s-},z)\tilde{N}(\mathrm{d}s,\mathrm{d}z),\ t \in [\tau,T].$$

Moreover, if $X_\tau \in L^{\beta+2}(\Omega,\mathcal{F}_\tau,\mathbb{P};H)$, then we have

$$\bar{X} \in L^\alpha([\tau,T]\times\Omega,\mathrm{d}t\times\mathbb{P};V) \cap L^{\beta+2}([\tau,T]\times\Omega,\mathrm{d}t\times\mathbb{P};H).$$

Proof. We first assume $X_\tau = h \in H$, then it is obvious that $X_\tau \in L^{\beta+2}(\Omega,\mathcal{F}_\tau,\mathbb{P};H)$.

Let N^τ be the compensated Poisson random measure associated to the Poisson point process p^τ. As an immediate consequence of Theorem 3.1, there exists a unique $(\mathcal{F}_t^\tau)$-adapted H-valued càdlàg process $X^{\tau,h}$ such that

$$\begin{aligned} X_t^{\tau,h} =& h + \int_0^t A(s+\tau,\bar{X}_s^{\tau,h})ds + \int_0^t B(s+\tau,\bar{X}_s^{\tau,h})\mathrm{d}W_s^\tau \\ &+ \int_0^t \int_{D^c} f(s+\tau,\bar{X}_{s-}^{\tau,h},z)\tilde{N}^\tau(\mathrm{d}s,\mathrm{d}z),\ t \in [0,T], \end{aligned}$$

where as before $\bar{X}^{\tau,h}$ is the $\mathrm{d}t\times\mathbb{P}$-equivalent class of $X^{\tau,h}$. Indeed, this follows along the same lines of the proof of Theorem 3.1 in such a way that all computations involving the expectations are replaced by conditional expectations with respect to $\mathcal{F}_\tau$.

Since for any $h \in H$, the solution $X_t^{\tau,h}$ is a measurable function of h, by replacing h with the $\mathcal{F}_\tau$-measurable random variable X_τ, where X_τ, W^τ and p^τ are mutually independent, we obtain an unique solution X^τ satisfying

$$\begin{aligned} X_t^\tau =& X_\tau + \int_0^t A(s+\tau, \bar{X}_s^\tau) ds + \int_0^t B(s+\tau, \bar{X}_s^\tau) \mathrm{d}W_s^\tau \\ &+ \int_0^t \int_{D^c} f(s+\tau, \bar{X}_{s-}^\tau, z) \tilde{N}^\tau(\mathrm{d}s, \mathrm{d}z),\ t \in [0,T]. \end{aligned}$$

Set $X_t := X_{t-\tau}^\tau$, then it is straightforward to see that X is the unique solution to equation (3.23) with initial condition X_τ. □

For convenience, we use $X'_{\tau,t}(\xi)$, $t \in [0,T]$ to denote the solution to equation (3.23) on $[0,T]$ with initial condition ξ at time τ and $X_{0,t}(x)$, $t \in [0,T]$ to denote the solution to equation (1.1) on $[0,T]$ with initial condition x at time 0.

Theorem 3.1 tells us that equation (3.1) with initial condition x at time 0 has a unique H-valued càdlàg solution $X' := (X'_{0,t}(x))_{t\in[0,T]}$ on $[0,T]$, that is

$$\begin{aligned} X'_{0,t}(x) =& x + \int_0^t A(s, \bar{X}'_{0,s}(x)) \mathrm{d}s + \int_0^t B(s, \bar{X}'_{0,s}(x)) \mathrm{d}W_s \\ &+ \int_0^t \int_{D^c} f(s, \bar{X}'_{0,s-}(x), z) \tilde{N}(\mathrm{d}s, \mathrm{d}z),\ t \in [0,T]. \end{aligned}$$

Here $\bar{X}'_{0,\cdot}(x) \in L^\alpha([0,T]\times\Omega, \mathrm{d}t\times\mathbb{P}; V) \cap L^2([0,T]\times\Omega, \mathrm{d}t\times\mathbb{P}; H)$ and it is $\mathrm{d}t\times\mathbb{P}$-equivalent to $X'_{0,\cdot}(x)$. Recall that $\{\tau_n\}$ are the arrival times for the jumps of the Poisson process $N(t,D)$, $t \in [0,T]$. Now we may construct a solution to (1.1) on $[0,\tau_1]$ as follows:

$$X_{0,t}(x) = \begin{cases} X'_{0,t}(x), & \text{for } 0 \le t < \tau_1; \\ X'_{0,\tau_1-}(x) + g(\tau_1, \bar{X}'_{0,\tau_1-}(x), p(\tau_1)), & \text{for } t = \tau_1. \end{cases}$$

We note that since the process $X'_{0,t}(x), t \in [0,T]$ has no jumps occurring at time τ_1, we infer $X_{0,\tau_1-}(x) = X'_{0,\tau_1-}(x) = X'_{0,\tau_1}(x)$. Set $\bar{X}_{0,t}(x) = \bar{X}'_{0,t}(x)$ on $[0,\tau_1]$. Then it is easy to see that $\bar{X}_{0,t}(x)$ is $\mathrm{d}t \times \mathbb{P}$-equivalent to $X_{0,t}(x)$ on $[0,\tau_1]$. Hence we have

$$\begin{aligned} X_{0,\tau_1}(x) &= X'_{0,\tau_1-}(x) + g(\tau_1, \bar{X}'_{0,\tau_1-}(x), p(\tau_1)) \\ &= x + \int_0^{\tau_1} A(s, \bar{X}'_{0,s}(x)) \mathrm{d}s + \int_0^{\tau_1} B(s, \bar{X}'_{0,s}(x)) \mathrm{d}W_s \\ &\quad + \int_0^{\tau_1} \int_{D^c} f(s, \bar{X}'_{0,s-}(x), z) \tilde{N}(\mathrm{d}s, \mathrm{d}z) + g(\tau_1, \bar{X}'_{0,\tau_1-}(x), p(\tau_1)) \\ &= x + \int_0^{\tau_1} A(s, \bar{X}_{0,s}(x)) \mathrm{d}s + \int_0^{\tau_1} B(s, \bar{X}_{0,s}(x)) \mathrm{d}W_s \\ &\quad + \int_0^{\tau_1} \int_{D^c} f(s, \bar{X}_{0,s-}(x), z) \tilde{N}(\mathrm{d}s, \mathrm{d}z) + g(\tau_1, \bar{X}_{0,\tau_1-}(x), p(\tau_1)). \end{aligned}$$

Also, since τ_1 is the time at which the first jump of the process $N(t, D)$, $t \in [0, T]$ happened, we infer

$$\int_0^t \int_D g(s, \bar{X}'_{0,s-}(x), z) N(\mathrm{d}s, \mathrm{d}z) = \begin{cases} 0, & t \in [0, \tau_1), \\ g(\tau_1, \bar{X}'_{0,\tau_1-}(x), p(\tau_1)), & t \in [\tau_1, \tau_2). \end{cases}$$

It follows that for $t \in [0, \tau_1]$ we have

$$\begin{aligned} X_{0,t}(x) =& x + \int_0^t A(s, \bar{X}_{0,s}(x))\mathrm{d}s + \int_0^t B(s, \bar{X}_{0,s}(x))\mathrm{d}W_s \\ &+ \int_0^t \int_{D^c} f(s, \bar{X}_{0,s-}(x), z)\tilde{N}(\mathrm{d}s, \mathrm{d}z) + \int_0^t \int_D g(s, \bar{X}_{0,s-}(x), z) N(\mathrm{d}s, \mathrm{d}z), \end{aligned}$$

which shows that the process $X_{0,t}(x)$ is an H-valued solution to the equation (1.1) on $[0, \tau_1]$.

Since the valued of $g(\cdot, X, \cdot)$ at time τ_1 depends only on the valued of X_{τ_1-} strictly prior to the time τ_1, the uniqueness of the solution $X'_{0,t}(x)$ on $[0, \tau_1)$ implies the uniqueness of the solution $X_{0,t}(x)$ on $[0, \tau_1]$.

Next, let $X'_{\tau_1,t}(X_{0,\tau_1}(x))$ be the unique solution to the equation (3.1) with initial condition $X_{0,\tau_1}(x)$ at time τ_1, then there exists a $\mathrm{d}t \times \mathbb{P}$-equivalent class $\bar{X}'_{\tau_1,t}(X_{0,\tau_1}(x))$, $t \in [\tau_1, T]$ satisfying

$$\begin{aligned} X'_{\tau_1,t}(X_{0,\tau_1}(x)) = X_{0,\tau_1}(x) &+ \int_{\tau_1}^t A(s, \bar{X}'_{\tau_1,s}(X_{0,\tau_1}(x)))\mathrm{d}s + \int_{\tau_1}^t B(s, \bar{X}'_{\tau_1,s}(X_{0,\tau_1}(x)))\mathrm{d}W_s \\ &+ \int_{\tau_1}^t \int_{D^c} f(s, \bar{X}'_{\tau_1,s-}(X_{0,\tau_1}(x)), z)\tilde{N}(\mathrm{d}s, \mathrm{d}z), \quad t \in [\tau_1, T]. \end{aligned}$$

We define

$$X_{0,t}(x) = \begin{cases} X_{0,t}(x), & \text{for } 0 \le t \le \tau_1; \\ X'_{\tau_1,t}(X_{0,\tau_1}(x)), & \text{for } \tau_1 < t < \tau_2; \\ X'_{\tau_1,\tau_2-}(X_{0,\tau_1}(x)) + g(\tau_2, \bar{X}'_{\tau_1,\tau_2-}(X_{0,\tau_1}(x)), p(\tau_2)), & \text{for } t = \tau_2, \end{cases}$$

and

$$\bar{X}_{0,t}(x) = \begin{cases} \bar{X}_{0,t}(x), & \text{for } t \in [0, \tau_1]; \\ \bar{X}'_{\tau_1,t}(X_{0,\tau_1}(x)), & \text{for } t \in [\tau_1, \tau_2]. \end{cases}$$

It is easy to see that $\bar{X}_{0,s}(x) = X_{0,s}(x), \mathrm{d}t \times \mathbb{P}$ on $[0, \tau_2]$. Then we have for $t \in (\tau_1, \tau_2)$,

$$\begin{aligned} &X_{0,t}(x) = X'_{\tau_1,t}(X_{0,\tau_1}(x)) \\ =& x + \int_0^{\tau_1} A(s, \bar{X}_{0,s}(x))\mathrm{d}s + \int_0^{\tau_1} B(s, \bar{X}_{0,s}(x))\mathrm{d}W_s \\ &+ \int_0^{\tau_1} \int_{D^c} f(s, \bar{X}_{0,s-}(x), z)\tilde{N}(\mathrm{d}s, \mathrm{d}z) + g(\tau_1, \bar{X}_{0,\tau_1-}(x), p(\tau_1)) \\ &+ \int_{\tau_1}^t A(s, \bar{X}'_{\tau_1,s}(X_{0,\tau_1}(x)))\mathrm{d}s + \int_{\tau_1}^t B(s, \bar{X}'_{\tau_1,s}(X_{0,\tau_1}(x)))\mathrm{d}W_s \end{aligned}$$

$$
\begin{aligned}
&+\int_{\tau_1}^t\int_{D^c} f(s,\bar{X}'_{\tau_1,s-}(X_{0,\tau_1}(x)),z)\tilde{N}(\mathrm{d}s,\mathrm{d}z)\\
=&x+\int_0^t A(s,\bar{X}_{0,s}(x))\mathrm{d}s+\int_0^t B(s,\bar{X}'_{0,s}(x))\mathrm{d}W_s\\
&+\int_0^t\int_{D^c} f(s,\bar{X}_{0,s-}(x),z)\tilde{N}(\mathrm{d}s,\mathrm{d}z)+\int_0^t\int_D g(s,\bar{X}_{0,s-}(x),z)N(\mathrm{d}s,\mathrm{d}z).
\end{aligned}
$$

As we known that $X'_{\tau_1,\tau_2-}(X_{0,\tau_1}(x)) = X'_{\tau_1,\tau_2}(X_{0,\tau_1}(x))$, a similar argument as above gives

$$
\begin{aligned}
X_{0,\tau_2}(x) =&x+\int_0^{\tau_2} A(s,\bar{X}_{0,s}(x))\mathrm{d}s+\int_0^{\tau_2} B(s,\bar{X}_{0,s}(x))\mathrm{d}W_s\\
&+\int_0^{\tau_2}\int_{D^c} f(s,\bar{X}_{0,s-}(x),z)\tilde{N}(\mathrm{d}s,\mathrm{d}z)+\int_0^{\tau_2}\int_D g(s,\bar{X}_{0,s-}(x),z)N(\mathrm{d}s,\mathrm{d}z).
\end{aligned}
$$

In particular,

$$
\begin{aligned}
\int_0^{\tau_2}\int_D g(s,\bar{X}_{0,s-}(x),z)N(\mathrm{d}s,\mathrm{d}z) &= g(\tau_1,\bar{X}_{0,\tau_1-},p(\tau_1))+g(\tau_2,\bar{X}_{0,\tau_2-},p(\tau_2))\\
&= g(\tau_1,\bar{X}'_{0,\tau_1-}(x),p(\tau_1))+g(\tau_2,\bar{X}'_{\tau_1,\tau_2-}(X_{0,\tau_1}(x)),p(\tau_2)).
\end{aligned}
$$

Therefore, $X_{0,t}(x)$ is a solution of (1.1) on $[0,\tau_2]$ and the uniqueness of the solution on $[0,\tau_2]$ follows from the uniqueness of the solutions $X'_{0,t}(x)$ and $X'_{\tau_1,t}(X_{0,\tau_1})(x)$.

By using this type of interlacing structure, one can construct a unique solution recursively to the equation (1.1) in the time interval $[0,\tau_n]$ for every $n\in\mathbb{N}$.

Now the proof is complete. □

4 Application and Examples

Theorem 1.2 gives a unified framework for a very large class of SPDE driven by general Lévy noise, which generalizes both the classical result in [27, 45] and the recent result in [35, 11]. Within this framework, the issue of existence and uniqueness of solutions to a large class of stochastic evolution equations with monotone coefficients (cf. [45, 27] for the stochastic porous medium equation and stochastic p-Laplace equation) and with locally monotone coefficients (cf. [35] for stochastic generalized Burgers equations and stochastic 2D Navier-Stokes equations) driven by more general Lévy processes instead of Wiener processes can be treated.

For the simplicity of notation we use D_i to denote the spatial derivative $\frac{\partial}{\partial x_i}$, and $\Lambda\subseteq\mathbb{R}^d$ is an open bounded domain with smooth boundary. For the standard Sobolev space $W_0^{1,p}(\Lambda)$ ($p\geq 2$) we always use the following (equivalent) Sobolev norm:

$$
\|u\|_{1,p} := \left(\int_\Lambda |\nabla u(x)|^p dx\right)^{1/p}.
$$

For $d = 2$, we recall the following well-known estimate on $\mathbb{R}^2$ (cf. [52]):

$$\|u\|_{L^4}^4 \le C\|u\|_{L^2}^2 \|\nabla u\|_{L^2}^2, \ u \in W_0^{1,2}(\Lambda). \tag{4.1}$$

W also recall the following estimate on $\mathbb{R}^3$:

$$\|u\|_{L^4}^4 \le C\|u\|_{L^2} \|\nabla u\|_{L^2}^3, \ u \in W_0^{1,2}(\Lambda), \tag{4.2}$$

We first recall the following lemma in [35], which is used to verify the local monotonicity condition $(H2)$ in examples.

Lemma 4.1. *Consider the Gelfand triple*

$$V := W_0^{1,2}(\Lambda) \subseteq H := L^2(\Lambda) \subseteq W^{-1,2}(\Lambda)$$

and the operator

$$A(u) = \Delta u + \sum_{i=1}^{d} f_i(u) D_i u,$$

where $f_i : \mathbb{R} \to \mathbb{R}$ ($i = 1, \cdots, d$) are bounded Lipschitz functions.

(1) *If $d < 3$, then there exists a constant $K > 0$ such that*

$$2_{V^*}\langle A(u) - A(v), u - v\rangle_V \le -\|u - v\|_V^2 + \left(K + K\|v\|_V^2\right) \|u - v\|_H^2, \ u, v \in V.$$

(2) *If $d = 3$, then there exists a constant $K > 0$ such that*

$$2_{V^*}\langle A(u) - A(v), u - v\rangle_V \le -\|u - v\|_V^2 + \left(K + K\|v\|_V^4\right) \|u - v\|_H^2, \ u, v \in V.$$

(3) *If f_i are independent of u for $i = 1, \cdots, d$, i.e.*

$$A(u) = \Delta u + \sum_{i=1}^{d} f_i \cdot D_i u,$$

then for any $d \ge 1$ we have

$$2_{V^*}\langle A(u) - A(v), u - v\rangle_V \le -\|u - v\|_V^2 + K\|u - v\|_H^2, \ u, v \in V.$$

Remark 4.2. (1) Note that the coercivity condition $(H3)$ directly follows from the local monotonicity $(H2)$ above by taking $v = 0$.

(2) One should note that the boundedness assumption of f_i might be removed in some cases. For example, if

$$d = 1, \Lambda = [0, 1], A(u) = \Delta u + \frac{\partial F(u)}{\partial x},$$

where F satisfies $F(x) - F(y) = c(x - y)^2 + f(y)(x - y)$ for some constant c and function f with at most linear growth.

Then by Hölder's inequality, (4.1) and Young's inequality we have the following local monotonicity:

$$
\begin{aligned}
&{}_{V^*}\langle A(u)-A(v),u-v\rangle_V\\
&=-\|u-v\|_V^2+\int_\Lambda\left(\frac{\partial F(u)}{\partial x}-\frac{\partial F(v)}{\partial x}\right)(u-v)\,dx\\
&=-\|u-v\|_V^2-\int_\Lambda\left(F(u)-F(v)\right)\frac{\partial}{\partial x}(u-v)\,dx\\
&=-\|u-v\|_V^2-\frac{c}{3}\int_\Lambda\frac{\partial}{\partial x}(u-v)^3\,dx+\int_\Lambda f(v)(u-v)\frac{\partial}{\partial x}(u-v)\,dx \qquad (4.3)\\
&=-\|u-v\|_V^2+\int_\Lambda f(v)(u-v)\frac{\partial}{\partial x}(u-v)\,dx\\
&\le-\|u-v\|_V^2+\|f(v)\|_{L^4}\|u-v\|_{L^4}\|u-v\|_V\\
&\le-\|u-v\|_V^2+Cn\|f(v)\|_{L^4}\|u-v\|_H^{1/2}\|u-v\|_V^{3/2}\\
&\le-\frac{3}{4}\|u-v\|_V^2+(K+K\|v\|_{L^4}^4)\|u-v\|_H^2,\ u,v\in V,
\end{aligned}
$$

where K is a generic constant that may change from line to line.

One simple example is $F(x)=c_1x^2+c_2x$, in this case $f(x)(=f_1(x))=2c_1x+c_2$ is not bounded but the local monotonicity still holds. Hence the result also covers the case of Burgers type equations (see also [35]).

For all examples below in this section, we will only state the result on the existence and uniqueness of solutions. But we should remark that one can also obtain those regularity estimates (1.5) and (1.6) by Theorem 1.2 if we don't have the large jumps term in our equations (i.e. $g=0$).

4.1 Semilinear type SPDEs

Example 4.3. *(Stochastic Burgers type equations) Let Λ be an open bounded domain in $\mathbb{R}^d$ with smooth boundary. We consider the following semilinear stochastic equation*

$$
\begin{aligned}
\mathrm{d}X_t=&\left(\Delta X_t+\sum_{i=1}^d f_i(X_t)D_iX_t+f_0(X_t)\right)\mathrm{d}t+B(X_t)\mathrm{d}W_t\\
&+\int_{D^c}f(X_{t-},z)\tilde{N}(\mathrm{d}t,\mathrm{d}z)+\int_D g(X_{t-},z)N(\mathrm{d}t,\mathrm{d}z); \qquad (4.4)\\
X_0=&x.
\end{aligned}
$$

Suppose the coefficients satisfy the following conditions:
(i) f_i are bounded Lipschitz functions on $\mathbb{R}$ for $i=1,\cdots,d$;
(ii) f_0 is a continuous function on $\mathbb{R}$ such that

$$
\begin{aligned}
&|f_0(x)|\le C(|x|^r+1),\ x\in\mathbb{R};\\
&(f_0(x)-f_0(y))(x-y)\le C(1+|y|^s)(x-y)^2,\ x,y\in\mathbb{R}.
\end{aligned} \qquad (4.5)
$$

where C, r, s are some positive constants;

(iii) the function $B : W_0^{1,2}(\Lambda) \to \mathcal{T}_2(U; L^2(\Lambda))$ satisfies the following condition:

$$\|B(v_1) - B(v_2)\|_2^2 \le C \int_\Lambda |v_1 - v_2|^2 \mathrm{d}x, \;\; v_1, v_2 \in W_0^{1,2}(\Lambda).$$

(iv) $f, g : \mathbb{R} \times Z \to \mathbb{R}$ such that for all $v, v_1, v_2 \in W_0^{1,2}(\Lambda)$,

$$\begin{aligned}
&\int_{D^c} \int_\Lambda |f(v_1, z) - f(v_2, z)|^2 \mathrm{d}x \nu(\mathrm{d}z) \le C \int_\Lambda |v_1 - v_2|^2 \mathrm{d}x; \\
&\int_{D^c} \int_\Lambda |f(v, z)|^2 \mathrm{d}x \nu(\mathrm{d}z) \le C(1 + \int_\Lambda |v|^2 \mathrm{d}x); \\
&\int_{D^c} \left(\int_\Lambda |f(v, z)|^2 \mathrm{d}x \right)^3 \nu(\mathrm{d}z) \le C \left(1 + \left(\int_\Lambda |v|^2 \mathrm{d}x \right)^3 \right).
\end{aligned} \tag{4.6}$$

Then we have the following result:

(1) If $d = 1, r = 3, s = 2$, then for any $x \in L^6(\Omega, \mathcal{F}_0, \mathbb{P}; H)$, (4.4) has a unique solution $\{X_t\}_{t \in [0,T]}$.

(2) If $d = 2, r = \frac{7}{3}, s = 2$, then for any $x \in L^6(\Omega, \mathcal{F}_0, \mathbb{P}; H)$, (4.4) has a unique solution $\{X_t\}_{t \in [0,T]}$.

(3) If $d = 3, r = \frac{7}{3}, s = \frac{4}{3}$ and $f_i, i = 1, \cdots, d$ are bounded measurable functions which are independent of X_t, then for any $x \in L^6(\Omega, \mathcal{F}_0, \mathbb{P}; H)$, (4.4) has a unique solution $\{X_t\}_{t \in [0,T]}$.

Proof. We consider the following Gelfand triple

$$V := W_0^{1,2}(\Lambda) \subseteq H := L^2(\Lambda) \subseteq (W_0^{1,2}(\Lambda))^*$$

and define the operator

$$A(u) = \Delta u + \sum_{i=1}^d f_i(u) D_i u + f_0(u), \;\; u \in V.$$

One can show that A, B satisfies $(H1) - (H4)$ with $\alpha = 2, \beta = 4$ (see [35, Example 3.2]).

Moreover, it is easy to show that f also satisfies the required conditions (i.e. $(H2)$, (1.2) and (1.3)) by (4.6).

Then all assertions follow from Theorem 1.2. □

Remark 4.4. (1) As mentioned in Remark 4.2, if $d = 1$, one may take $f_1(x) = x$ such that Theorem 1.2 can be applied to classical stochastic Burgers equation (i.e. (4.4) with $f_0 \equiv 0$). Therefore, the above example improves the main result in [14] (Theorem 2.2) in the sense that we allow the coefficient B in front of Wiener noise to be non-additive type. Another improvement is that we also allow a polynomial perturbation term f_0 in the drift of (4.4). For example, one can take $f_0(x) = -x^3 + c_1 x^2 + c_2 x (c_1, c_2 \in \mathbb{R})$ and show that (4.5) holds. Hence (4.4) also covers some stochastic reaction-diffusion type equations driven by certain type of a Lévy noise (cf. [7]).

(2) If $Z = \mathbb{R}^d$, $D^c = \{z \in \mathbb{R}^d : |z| \leq 1\}$ and ν is a Lévy measure on $\mathbb{R}^d$, then one simple sufficient condition for f satisfying (4.6) is to assume

$$
\begin{aligned}
&|f(x,z) - f(y,z)| \leq C|x-y||z|,\ x,y \in \mathbb{R},\ z \in D^c;\\
&|f(x,z)| \leq C(1+|x|)|z|,\ x,y \in \mathbb{R},\ z \in D^c.
\end{aligned}
$$

(3) One should note that in the example 4.3, B is assumed to be Lipschitz from $W_0^{1,2}(\Lambda)$ (w.r.t. $\|\cdot\|_H$) to $\mathcal{T}_2(U; L^2(\Lambda))$ only for simplicity. Actually, the Lipschitz condition on B can even be weakened to the requirement

$$
\|B(v_1) - B(v_2)\|_2^2 \leq \|v_1 - v_2\|_V^2 + \left(K + K\|v_2\|_V^2\right)\|v_1 - v_2\|_H^2.
$$

4.2 Quasi-linear type SPDEs

Besides from the example of semilinear SPDE above, we can also apply the main result to the following quasi-linear SPDE on $\mathbb{R}^d$ ($d \geq 3$) driven by Lévy noise.

Example 4.5. *(Stochastic p-Laplace equations) We consider the following equation on $\mathbb{R}^d$ for $p > 2$*

$$
\begin{aligned}
\mathrm{d}X_t =& \left(\sum_{i=1}^d D_i\left(|D_i X_t|^{p-2} D_i X_t\right) + f_0(X_t)\right)\mathrm{d}t + B(X_t)\mathrm{d}W_t\\
&+ \int_{D^c} f(X_{t-}, z)\tilde{N}(\mathrm{d}t, \mathrm{d}z) + \int_D g(X_{t-}, z) N(\mathrm{d}t, \mathrm{d}z);\\
X_0 =& x.
\end{aligned} \tag{4.7}
$$

Suppose the following conditions hold:

(i) f_0 is a continuous function on $\mathbb{R}$ such that

$$
\begin{aligned}
f_0(x)x &\leq C(|x|^{\frac{p}{2}+1} + 1),\ x \in \mathbb{R};\\
|f_0(x)| &\leq C(|x|^r + 1),\ x \in \mathbb{R};\\
(f_0(x) - f_0(y))(x-y) &\leq C(1+|y|^t)|x-y|^s,\ x,y \in \mathbb{R},
\end{aligned} \tag{4.8}
$$

where $C > 0$ and $r, s, t \geq 1$ are some constants.

(ii) $B : W_0^{1,p}(\Lambda) \to \mathcal{T}_2(U; L^2(\Lambda))$ satisfying the following condition:

$$
\|B(v_1) - B(v_2)\|_2^2 \leq C \int_\Lambda |v_1 - v_2|^2 \mathrm{d}x,\ v_1, v_2 \in W_0^{1,p}(\Lambda).
$$

(iv) $f, g : \mathbb{R} \times Z \to \mathbb{R}$ such that for all $v, v_1, v_2 \in W_0^{1,p}(\Lambda)$,

$$
\begin{aligned}
&\int_{D^c}\int_\Lambda |f(v_1,z) - f(v_2,z)|^2 \mathrm{d}x\nu(\mathrm{d}z) \leq C\int_\Lambda |v_1 - v_2|^2 \mathrm{d}x;\\
&\int_{D^c}\int_\Lambda |f(v,z)|^2 \mathrm{d}x\nu(\mathrm{d}z) \leq C(1 + \int_\Lambda |v|^2 \mathrm{d}x);\\
&\int_{D^c}\left(\int_\Lambda |f(v,z)|^2 \mathrm{d}x\right)^3 \nu(\mathrm{d}z) \leq C\left(1 + \left(\int_\Lambda |v|^2 \mathrm{d}x\right)^3\right).
\end{aligned} \tag{4.9}
$$

Then we have

(1) if $d < p$, $s = 2$, $r = p + 1$ and $t \le p$, then for any $x \in L^6(\Omega, \mathcal{F}_0, \mathbb{P}; H)$, (4.7) has a unique solution.

(2) if $d > p$, $2 < s < p$, $r = \frac{2p}{d} + p - 1$ and $t \le \min\left\{\frac{p^2(s-2)}{(d-p)(p-2)}, \frac{p(p-s)}{p-2}\right\}$, for any $x \in L^6(\Omega, \mathcal{F}_0, \mathbb{P}; H)$ (4.7) has a unique solution.

Proof. (1) Define the following Gelfand triple

$$V := W_0^{1,p}(\Lambda) \subseteq H := L^2(\Lambda) \subseteq W^{-1,q}(\Lambda),$$

then it is well known that $\sum_{i=1}^d D_i\left(|D_i u|^{p-2} D_i u\right)$ satisfy $(H1)$-$(H4)$ with $\alpha = p$ (cf. [32, 33]). In particular, there exists a constant $\delta > 0$ such that

$$\sum_{i=1}^d {}_{V^*}\langle D_i\left(|D_i u|^{p-2} D_i u\right) - D_i\left(|D_i v|^{p-2} D_i v\right), u - v\rangle_V \le -\delta \|u-v\|_V^p, \ u, v \in W_0^{1,p}(\Lambda). \tag{4.10}$$

Recall that for $d < p$ we have the following Sobolev embedding

$$W_0^{1,p}(\Lambda) \subseteq L^\infty(\Lambda).$$

Hence by (4.8) we have

$$\begin{aligned} {}_{V^*}\langle f_0(u) - f_0(v), u - v\rangle_V &\le C \int_\Lambda \left(1 + |v|^t\right)|u - v|^2 dx \\ &\le C\left(1 + \|v\|_{L^\infty}^t\right)\|u - v\|_{L^2}^2 \\ &\le C\left(1 + \|v\|_V^t\right)\|u - v\|_H^2, \ u, v \in V, \end{aligned} \tag{4.11}$$

where C is a constant may change from line to line.

Hence $(H2)$ holds with $\rho(v) = C\|v\|_V^t$.

Note that from (4.8) we have

$$\begin{aligned} {}_{V^*}\langle f_0(u), u\rangle_V &\le C \int_\Lambda (1 + |u|^{\frac{p}{2}+1}) dx \\ &\le C\left(1 + \|u\|_{L^\infty}^{p/2}\|u\|_H\right) \\ &\le \frac{\delta}{2}\|u\|_V^p + C\left(1 + \|u\|_H^2\right), \ u \in V. \end{aligned} \tag{4.12}$$

Therefore, (4.12) together with (4.10) verify $(H3)$ with $\alpha = p$.

$(H4)$ with $\beta = 4$ (in fact one may take $\beta = \frac{2p}{p-1} < 4$) follows from the following estimate:

$$\|f_0(u)\|_{V^*} \le C\left(1 + \|u\|_{L^{p+1}}^{p+1}\right) \le C\left(1 + \|u\|_{L^\infty}^{p-1}\|u\|_H^2\right) \le C\left(1 + \|u\|_V^{p-1}\|u\|_H^2\right), \ u \in V.$$

Then combining with (4.9) we know that the assertions follow from Theorem 1.2.

(2) Note that for $d > p$ we have the following Sobolev embedding

$$W_0^{1,p}(\Lambda) \subseteq L^{p_0}(\Lambda),\ p_0 = \frac{dp}{d-p}.$$

Let $t_0 = \frac{p(s-2)}{s(p-2)} \in (0,1)$ and $p_1 \in (2, p_0)$ such that

$$\frac{1}{p_1} = \frac{1-t_0}{2} + \frac{t_0}{p_0}.$$

Then we have the following interpolation inequality:

$$\|u\|_{L^{p_1}} \le \|u\|_{L^2}^{1-t_0} \|u\|_{L^{p_0}}^{t_0},\ u \in W_0^{1,p}(\Lambda).$$

Since $2 < s < p$, it is easy to show that $s < p_1$.

Let $p_2 = \frac{p_1}{p_1 - s}$, then by (4.8) we have

$$\begin{aligned} {}_{V^*}\langle f_0(u) - f_0(v), u - v\rangle_V &\le C \int_\Lambda \left(1 + |v|^t\right) |u-v|^s dx \\ &\le C \left(1 + \|v\|_{L^{tp_2}}^t\right) \|u-v\|_{L^{p_1}}^s \\ &\le C \left(1 + \|v\|_{L^{tp_2}}^t\right) \|u-v\|_{L^2}^{s(1-t_0)} \|u-v\|_{L^{p_0}}^{st_0} \\ &\le \varepsilon \|u-v\|_{L^{p_0}}^p + C_\varepsilon \left(1 + \|v\|_{L^{tp_2}}^{tb}\right) \|u-v\|_{L^2}^2, \end{aligned} \tag{4.13}$$

where $\varepsilon, C_\varepsilon$ are some constants and the last step follows from the following Young inequality

$$xy \le \varepsilon x^a + C_\varepsilon y^b,\ x, y \in \mathbb{R},\ a = \frac{p-2}{s-2},\ b = \frac{p-2}{p-s}.$$

With some calculations, one have

$$\frac{s}{p_1} = \frac{p-s}{p-2} + \frac{p(s-2)}{p_0(p-2)},\ p_2 = \frac{p_0(p-2)}{(p_0-p)(s-2)}.$$

Hence if $t \le \frac{(p_0-p)(s-2)}{p-2}$, then

$$\|u\|_{L^{tp_2}} \le C\|u\|_{L^{p_0}} \le C\|u\|_V,\ v \in V.$$

Therefore, $(H2)$ follows from (4.10) and (4.13).

$(H3)$ can be verified for $\alpha = p$ in a similar manner.

For $r = \frac{2p}{d} + p - 1$, by the interpolation inequality we have

$$\|f_0(u)\|_{V^*} \le C\left(1 + \|u\|_{L^{rp_0'}}^r\right) \le C\left(1 + \|u\|_{p_0}^{p-1} \|u\|_H^\theta\right),\ u \in V,$$

where

$$\frac{1}{p_0} + \frac{1}{p_0'} = 1,\quad \theta = \frac{2p}{d}.$$

Therefore, $(H4)$ also holds with $\beta = 4$.

Then all assertions follow from Theorem 1.2. $\square$

Remark 4.6. One further generalization is to replace $\sum_{i=1}^{d} D_i\left(|D_i u|^{p-2} D_i u\right)$ by more general quasi-linear differential operator

$$\sum_{|\alpha|\le m} (-1)^{|\alpha|} D_\alpha A_\alpha(x, Du(x,t); t),$$

where $Du = (D_\beta u)_{|\beta|\le m}$. Under certain assumptions (cf. [55, Proposition 30.10]) this operator also satisfies the monotonicity and coercivity conditions. Then by a similar argument, according to Theorem 1.2, we can obtain the existence and uniqueness of solutions to this type of quasi-linear SPDE driven by Lévy noise.

4.3 Stochastic hydrodynamical systems

The next example is the stochastic 2D Navier-Stokes equation driven by Lévy noise (cf. [5, 17, 38, 35] for Wiener noise case). The classical Navier-Stokes equation is a very important model in fluid mechanics to describe the time evolution of incompressible fluids, it can be formulated as follows (2D case):

$$\begin{aligned} &\partial_t u(t) = \nu \Delta u(t) - (u(t)\cdot\nabla)\, u(t) - \nabla p(t) + f(t), \\ &\nabla\cdot u(t) = 0, \end{aligned}$$

where $u(t,x) = (u^1(t,x), u^2(t,x))$ represents the velocity field, ν is the viscosity constant, $p(t,x)$ denotes the pressure and f is an external force field acting on the fluid.

Let Λ be a bounded domain in $\mathbb{R}^2$ with smooth boundary. Define

$$V = \left\{ v \in W_0^{1,2}(\Lambda, \mathbb{R}^2) : \nabla\cdot v = 0 \ a.e. \text{ in } \Lambda \right\},\ \|v\|_V := \left(\int_\Lambda |\nabla v|^2 dx \right)^{1/2},$$

and H is the closure of V in the following norm

$$\|v\|_H := \left(\int_\Lambda |v|^2 dx \right)^{1/2}.$$

The linear operator P_H (the Helmholtz-Leray projection) and A (Stokes operator with viscosity constant ν) are defined by

$$P_H : L^2(\Lambda, \mathbb{R}^2) \to H \quad \text{orthogonal projection;}$$

$$A : W^{2,2}(\Lambda, \mathbb{R}^2) \cap V \to H,\ Au = \nu P_H \Delta u.$$

It is well known that the Navier-Stokes equation can be reformulated as follows:

$$u' = Au + F(u) + f_0,\ u(0) = u_0 \in H, \tag{4.14}$$

where $f_0 \in L^2(0,T;V^*)$ denotes some external force and

$$F : \mathcal{D}_F \subset H \times V \to H,\ F(u,v) = -P_H\left[(u\cdot\nabla)\, v\right], F(u) = F(u,u).$$

It is standard that in the framework of the Gelfand triple

$$V \subseteq H \equiv H^* \subseteq V^*,$$

one can show that the following mappings

$$A : V \to V^*, \ F : V \times V \to V^*$$

are well defined. In particular, we have

$${}_{V^*}\langle F(u,v), w\rangle_V = -{}_{V^*}\langle F(u,w), v\rangle_V, \ {}_{V^*}\langle F(u,v), v\rangle_V = 0, \ u, v, w \in V.$$

Now we consider the stochastic 2D Navier-Stokes equation driven by Lévy noise:

$$\begin{aligned} \mathrm{d}X_t =& (AX_t + F(X_t) + f_0(t))\,\mathrm{d}t + B(X_t)\mathrm{d}W_t \\ &+ \int_{D^c} f(X_{t-}, z)\tilde{N}(\mathrm{d}t, \mathrm{d}z) + \int_D g(X_{t-}, z)N(\mathrm{d}t, \mathrm{d}z); \\ X_0 =& x. \end{aligned} \tag{4.15}$$

Example 4.7. *(Stochastic 2D Navier-Stokes equation) Suppose that $B : V \to \mathcal{T}_2(U; H)$ and $f, g : \mathbb{R} \times Z \to \mathbb{R}$ satisfy the following conditions:*

$$\begin{aligned} &\|B(v_1) - B(v_2)\|_2^2 + \int_{D^c} \|f(v_1, z) - f(v_2, z)\|_H^2 \nu(\mathrm{d}z) \le C\|v_1 - v_2\|_H^2; \\ &\int_{D^c} \|f(v, z)\|_H^2 \nu(\mathrm{d}z) \le C(1 + \|v\|_H^2); \\ &\int_{D^c} \|f(v, z)\|_H^4 \nu(\mathrm{d}z) \le C(1 + \|v\|_H^4), \end{aligned} \tag{4.16}$$

where C is some constant.

Then for any $x \in L^4(\Omega, \mathcal{F}_0, \mathbb{P}; H)$, (4.15) has a unique solution $\{X_t\}_{t\in[0,T]}$.

Proof. The hemicontinuity $(H1)$ is obvious since F is a bilinear map.

Note that ${}_{V^*}\langle F(v), v\rangle_V = 0$, it is also easy to show $(H3)$ with $\alpha = 2$:

$$\begin{aligned} {}_{V^*}\langle Av + F(v) + F_t, v\rangle_V &\le -\nu\|v\|_V^2 + \|F_t\|_{V^*}\|v\|_V \le -\frac{\nu}{2}\|v\|_V^2 + C\|F_t\|_{V^*}^2, \ v \in V, \\ \|B(v)\|_2^2 &\le 2K\|v\|_H^2 + 2\|B(0)\|_2^2, \ v \in V. \end{aligned}$$

Recall the following estimates (cf. e.g.[38, Lemmas 2.1, 2.2])

$$\begin{aligned} &|{}_{V^*}\langle F(w), v\rangle_V| \le 2\|w\|_{L^4(\Lambda;\mathbb{R}^2)}\|v\|_V; \\ &|{}_{V^*}\langle F(w), v\rangle_V| \le 2\|w\|_V^{3/2}\|w\|_H^{1/2}\|v\|_{L^4(\Lambda;\mathbb{R}^2)}, v, w \in V. \end{aligned} \tag{4.17}$$

Then we have

$$\begin{aligned} {}_{V^*}\langle F(u) - F(v), u - v\rangle_V &= -{}_{V^*}\langle F(u, u - v), v\rangle_V + {}_{V^*}\langle F(v, u - v), v\rangle_V \\ &= -{}_{V^*}\langle F(u - v), v\rangle_V \\ &\le 2\|u - v\|_V^{3/2}\|u - v\|_H^{1/2}\|v\|_{L^4(\Lambda;\mathbb{R}^2)} \\ &\le \frac{\nu}{2}\|u - v\|_V^2 + \frac{32}{\nu^3}\|v\|_{L^4(\Lambda;\mathbb{R}^2)}^4\|u - v\|_H^2, \ u, v \in V. \end{aligned} \tag{4.18}$$

Hence we have the local monotonicity:

$$ {}_{V^*}\langle Au + F(u) - Av - F(v), u - v\rangle_V \le -\frac{\nu}{2}\|u-v\|_V^2 + \frac{32}{\nu^3}\|v\|_{L^4(\Lambda;\mathbb{R}^2)}^4\|u-v\|_H^2. $$

Combining with (4.16) we know that $(H2)$ holds with $\rho(v) = C\|v\|_{L^4(\Lambda;\mathbb{R}^2)}^4$.

(4.17) and (4.1) imply that $(H4)$ holds with $\beta = 2$.

Then it is easy to see that the existence and uniqueness of solutions to (4.15) follows from Theorem 1.2. □

Remark 4.8. As we mentioned in the introduction, besides the stochastic 2D Navier-Stokes equation, many other hydrodynamical systems also satisfy the local monotonicity and coercivity conditions that we assumed. For example, in a recent work of Chueshov and Millet [11], they have studied the well-posedness and large deviation principle for an abstract stochastic semilinear equation (driven by Wiener noise) which covers a wide class of fluid dynamical models. In fact, the Condition (C1) and (C2) in [11] implies that the assumptions in Theorem 1.2 hold. More precisely, (2.2) in [11] implies the coercivity $(H3)$ holds, and the local monotonicity $(H2)$ follows from (2.4) (or (2.8)) in [11]. Other assumptions in Theorem 1.2 can be also verified easily.

Therefore, Theorem 1.2 can be applied to show the well-posedness of all hydrodynamical models in [11] driven by general Lévy noise, e.g. stochastic magneto-hydrodynamic equations, stochastic Boussinesq model for the Bénard convection, stochastic 2D magnetic Bénard problem and stochastic 3D Leray-α model driven by Lévy noise.

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