Markov Jump Dynamics with Additive Intensities in Continuum: State Evolution and Vlasov Scaling

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Abstract

We investigate a stochastic (conservative) non-equilibrium jump dynamics of interacting particles in continuum. The corresponding evolution of correlation functions as well as a mesoscopic scaling (Vlasov scaling) is studied. We derive a kinetic equation for the particle density which is a Vlasov-type equation for the considered model.

Keywords Interacting particle system, Jump dynamics, Non-equilibrium evolution, Vlasov scaling, Kinetic equation

1 Introduction

It is useful to describe the position of particles in continuum by locally finite subsets of $\mathbb{R}^d$. In that case, the configurations space of a continuous interacting particle system is given by

$$\Gamma \equiv \Gamma(\mathbb{R}^d) := \{\gamma \subset \mathbb{R}^d \big| \ |\gamma \cap K| < \infty \text{ for any compact } K \subset \mathbb{R}^d \ \}. \tag{1}$$

The elements of $\gamma = \{x_1, x_2, \dots\} \in \Gamma$ describe the location of the particles.

In this paper, we present a particular stochastic (conservative) jump dynamics in continuum. Generally speaking, the time evolution of a large class of jump dynamics in continuum is given by the following mechanism: if we fix a time point $t > 0$ and assume that the system is at this time point in the configuration $\gamma \in \Gamma$, every particle $x \in \gamma$ jumps during the infinitesimal time interval $[t, t + dt]$ to a point $y \in \mathbb{R}^d$ according to a (probability) rate $c(x, y, \gamma)$ depending on $x \in \gamma$, $y \in \mathbb{R}^d$ and the actual state $\gamma \in \Gamma$. This leads to a change of the state γ to $\gamma \backslash x \cup y$. The transition rates $c(x, y, \gamma)$ contains all information about the evolution of the dynamics. The concrete form of these rates appears from modeling a concrete system. As motivated above, the infinitesimal generator should have on observables (i.e. functions F on Γ) the following form:

$$(LF)(\gamma) = \sum_{x\in\gamma} \int_{\mathbb{R}^d} dy \ c(x, y, \gamma)\big(F(\gamma \backslash x \cup y) - F(\gamma)\big). \tag{2}$$

Since this expression involves infinite sums, the generator (2) has at this place only a heuristic signification. The task is now to give the above generator a rigorous meaning and to construct a semigroup $(e^{tL})_{t>0}$ which has L

as infinitesimal generator. By means of this semigroup it is possible to solve the Kolmogorov equation, which describes the time evolution of observables according to the dynamics. The corresponding dual dynamics describes the time evolution of states (i.e. probability measures on Γ). If this semigroup is a Markov semigroup, we get even a probabilistic object, namely a stochastic process $(X_t)_{t>0}$ associated to that semigroup. In the equilibrium case, the above described program can be realized for a quite general class of transition rates (via the Dirichlet form approach, see [25]). Here, equilibrium case means, that is only possible to start the dynamics (almost) in some equilibrium, which means, that the evolution of observables is (almost) time independent. More interesting (from the point of view of applications) is the construction of a non-equilibrium dynamics since within the non-equilibrium framework it is possible to start the dynamics in a state (far away from some equilibrium) and investigate the time evolution of the dynamics (maybe it approaches some equilibrium after long time). It turns out that in the non-equilibrium framework, it is demanding to show that a generator of type (2) induces a semigroup. Indeed this could only be shown for a system of particles without interactions (see [27]). Nevertheless it is sometimes still possible to construct the time evolution of correlation functions. This is the aim of the first part of this work for a particular form of transition rates. These transition rates have the following form:

$$c(x, y, \gamma) := a(x - y)\big(1 + \langle \gamma, c_{x,y}\rangle\big), \quad x, y \in \mathbb{R}^d, \gamma \in \Gamma, \tag{3}$$

where a is a non-negative function in $L^1(dx)$ with $\|a\|_1 = 1$ and $\langle \gamma, c_{x,y}\rangle := \sum_{\tilde{x}\in\gamma} c_{x,y}(\tilde{x})$. Here, $c_{x,y}$ is a non-negative function in $L^1(dx)$ which depends on x and y. We will assume that $c_{x,y}(\tilde{x})$ is given by on of the following three expressions:

$$c_{x,y}(\tilde{x}) = \kappa(x - \tilde{x}), \tag{4}$$

$$c_{x,y}(\tilde{x}) = \kappa(y - \tilde{x}) \tag{5}$$

or

$$c_{x,y}(\tilde{x}) = \frac{1}{2}\big(\kappa(x - \tilde{x}) + \kappa(y - \tilde{x})\big), \tag{6}$$

where κ is a bounded and non-negative function in $L^1(dx)$. The rates (3) are linear in γ and for $\kappa \equiv 0$ we obtain the free dynamics, i.e. a dynamics of non-interacting particles. This dynamics is fully studied, see [20]. In the case (6), the rate are symmetric in x and y. We refer to this case as the symmetric case. The free dynamics has a Poisson measure as equilibrium state. The symmetric case has this property as well. In this sense, this model describes particles with a minimal interaction.

The second part of this paper discusses a mesoscopic scaling of the considered model, namely Vlasov scaling. The limit of the scaled dynamics leads to a kinetic equation for the microscopic model. It is worth noting that the Vlasov scaling coincides with the well-known Lebowitz-Penrose scaling (see e.g. [36]). This fact as well as the definitions of the corresponding scalings will be discussed in Section 8. The complex evolution of a many-body system is often approximately described by kinetic equations, see e.g. [43, 44]. Besides the Boltzmann equation (which describes the evolution of the particle density of a dilute gas, see e.g. [30]), the Vlasov equation plays an important role in physics. This equation is a good approximation in situations where long range forces (i.e.

forces caused by the collective effects of a large number of particles over relatively long distance) are present and short range forces (i.e. forces caused by collisions) are neglectable. Such circumstances are (approximately) valid in a plasma (due to long range Coulomb forces), see e.g. [17]. One can derive the Vlasov equation from the BBGKY-hierarchy by assuming that propagation of chaos holds, see e.g. [16]. In this situation the equation for the particle density is a closed equation since the second correlation function factorizes and this yields the Vlasov equation. In [6] the authors have shown that in the mean field scaling limit for Hamiltonian dynamics the empirical distribution of the particles has at every time $t > 0$ a Lebesgue density (if so, for $t = 0$) and this density satisfies a Vlasov-type equation. More general deterministic dynamical systems where considered in [7]. Note that the resulting Vlasov-type equations for particle densities are considered in the class of finite measures (in the weak form) or integrable functions (in the strong form). The latter implies, in fact, that we are restricted to the case of finite-volume systems or systems with zero mean density in an infinite volume. A detailed analysis of Vlasov-type equations for integrable functions is presented in the recent work [29].

For the model considered in this work, the approaches mentioned above are not applicable since a description in terms of proper stochastic evolution equations for particle motion is, generally speaking, absent. For that reason we have to follow in this work a general approach, proposed in [9], to study the Vlasov-type scaling for some classes of stochastic evolutions in continuum. The first step is to derive hierarchical equations for the evolution of correlation functions which generalizes the BBGKY-hierarchy from Hamiltonian to the dynamics considered here [10]. Then, we perform the scaling, which, roughly speaking, assures that on the one hand, the interaction gets weaker and on the other hand, the correlations between particles gets stronger. The limiting hierarchy posses a chaos preservation property. Namely, if we start with an initial correlation function which corresponds to a (non-homogeneous) Poisson state of the system, then this property will be preserved during the time evolution. This special property of the virtual Vlasov system allows us to derive a non-linear evolutionary equation for the evolving Poisson state which is the macroscopic Vlasov-type equation derived from the microscopic infinite-particle system. We remark, that we are working in an infinite volume with non-zero averaged density. The zero density case corresponds to a different physical situation of the underlying microscopic model, see e.g. [4].

2 General Facts and Notations

Let $\mathcal{B}(\mathbb{R}^d)$ be the family of all Borel sets in the d-dimensional Euclidean space $\mathbb{R}^d$ and $\mathcal{B}_b(\mathbb{R}^d)$ the subfamily of all bounded Borel sets. The n-particle space is defined by

$$\Gamma_0^{(n)} := \{\eta \subset \mathbb{R}^d \,|\, |\eta| = n\}, \quad n \in \mathbb{N} = \{0, 1, 2, \dots\},$$

where $|\cdot|$ means the cardinality of a finite set. For $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$ one defines the set $\Gamma_{0,\Lambda}^{(n)} \equiv \Gamma_\Lambda^{(n)}$ analogue. Further, one introduces for every $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$ a map $N_\Lambda : \Gamma_\Lambda^{(n)} \to \mathbb{N}$, $N_\Lambda(\eta) := |\eta \cap \Lambda|$. For short we write $\eta_\Lambda := \eta \cap \Lambda$. We can identify the set $\Gamma_0^{(n)}$ with the symmetrization of

$$\widetilde{\left(\mathbb{R}^d\right)^n} := \{(x_1, \ldots, x_n) \in \left(\mathbb{R}^d\right)^n \big| x_i \neq x_j \text{ if } i \neq j\},$$

i.e. $\Gamma_0^{(n)} \cong \widetilde{\left(\mathbb{R}^d\right)^n}/S_n$, where S_n denotes the permutation group over $\{1, \ldots, n\}$. Due to this identification we can introduce a topology $\mathcal{T}(\Gamma_0^{(n)})$ on $\Gamma_0^{(n)}$. The corresponding Borel σ-algebra $\mathcal{B}(\Gamma_0^{(n)})$ coincides with $\sigma(N_\Lambda | \Lambda \in \mathcal{B}_b(\mathbb{R}^d))$. The space of finite particle configurations is defined by

$$\Gamma_0 := \bigsqcup_{n \in \mathbb{N}} \Gamma_0^{(n)}.$$

This set is equipped with the topology $\mathcal{T}(\Gamma_0)$ of disjoint unions. The space $\Gamma_{0,\Lambda} = \Gamma_\Lambda$, $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$ is defined analogue.

The configuration space (space of infinite particle configurations) is defined by

$$\Gamma = \{\gamma \subset \mathbb{R}^d \big| |\gamma \cap \Lambda| < \infty, \text{ for all } \Lambda \in \mathcal{B}_b(\mathbb{R}^d)\}.$$

The space Γ is equipped with the vague topology, i.e. the smallest topology for which all mappings

$$\Gamma \ni \gamma \mapsto \langle \gamma, f \rangle := \sum_{x \in \gamma} f(x) \in \mathbb{R}$$

are continuous for any function f on $\mathbb{R}^d$ with compact support; note that the summation in $\sum_{x \in \eta} f(x)$ is taken over finitely many points of γ which belongs to the support of f. In [21], it was shown that Γ with the vague topology may be metrizable and becomes a Polish space (i.e. a complete separable metric space). Corresponding to this topology, the Borel σ-algebra $\mathcal{B}(\Gamma)$ is the smallest σ-algebra for which all mappings $N_\Lambda : \Gamma \to \mathbb{N}$, $N_\Lambda(\gamma) = |\gamma \cap \Lambda|$ are measurable, i.e.

$$\mathcal{B}(\Gamma) = \sigma(N_\Lambda | \Lambda \in \mathcal{B}(\mathbb{R}^d).$$

For every $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$ one can define a projection

$$p_\Lambda : \Gamma \to \Gamma_\Lambda, \; p_\Lambda(\gamma) := \gamma \cap \Lambda$$

and with respect to this projection, Γ is the projective limit of the spaces $\{\Gamma_\Lambda\}_{\Lambda \in \mathcal{B}_b(\mathbb{R}^d)}$, see [1] and the references therein.

On $\Gamma_0^{(n)}$ we introduce a measure $\lambda^{(n)}$ by

$$\lambda^{(n)} := \frac{1}{n!} \sigma^{(n)},$$

where $\sigma^{(n)}$ is the restriction of the Lebesgue product measure $(dx)^n$ to $(\Gamma_0^{(n)}, \mathcal{B}(\Gamma_0^{(n)}))$. The combinatorial $\frac{1}{n!}$ factor takes into account the indistinguishability of the n particles. We extend the measures $\lambda^{(n)}$ to a measure λ on Γ_0 by setting

$$\lambda|_{\Gamma_0^{(n)}} = \lambda^{(n)},$$

i.e. $\lambda = \sum_{n \in \mathbb{N}} \frac{1}{n!} \sigma^{(n)}$. The measure λ is called the Lebesgue-Poisson measure. For any $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$ the restriction of λ to Γ_Λ will be denoted by λ_Λ. It holds

$\lambda_\Lambda(\Gamma_\Lambda) = e^{m(\Lambda)}$, where $m(\Lambda)$ denotes the Lebesgue measure of $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$. We define a probability measure π_Λ on Γ_Λ by $\pi_\Lambda := e^{-m(\Lambda)}\lambda_\Lambda$. The Fourier-transformation $\widehat{\pi_\Lambda}(f)$, f a smooth function with compact support on $\mathbb{R}$ is equals to

$$\begin{aligned}\widehat{\pi_\Lambda}(f) &= \int_\Gamma e^{i\langle\gamma,f\rangle}\pi_\Lambda(d\gamma)\\ &= \exp\left(\int_\Lambda e^{i(f(x)-1)}dx\right).\end{aligned}$$

Since in the above expression there is no need for a restriction to a space of finite volume Λ, we can extend the measures π_Λ to a probability measure π on Γ, see [1] for details and references therein.

By $L^0_{\mathrm{ls}}(\Gamma_0)$ we denote the set of all measurable functions on Γ_0 with bounded support, i.e. $G \in L^0_{\mathrm{ls}}(\Gamma_0)$ iff $G|_{\Gamma_0\setminus\Gamma_\Lambda} \equiv 0$ for some $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$. A set $M \in \mathcal{B}(\Gamma_0)$ is called bounded if it exists a $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$ and $N \in \mathbb{N}$ such that $M \subset \bigsqcup_{n=0}^N \Gamma_\Lambda^{(n)}$. We denote the set of all bounded and measurable functions with bounded support by $B_{\mathrm{bs}}(\Gamma_0)$, i.e. $G \in B_{\mathrm{bs}}(\Gamma_0)$ if $G|_{\Gamma_0\setminus M} \equiv 0$ for some bounded $M \in \mathcal{B}_b(\Gamma_0)$. We also consider the set $\mathcal{F}_{\mathrm{cyl}}(\Gamma)$ of all cylinder functions on Γ. Each $F \in \mathcal{F}_{\mathrm{cyl}}(\Gamma)$ is characterized by the following property: $F(\gamma) = F|_{\Gamma_\Lambda}(\gamma_\Lambda)$ for some $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$. Further, by $\mathcal{F}_{\mathrm{cyl}}\mathcal{P}(\Gamma)$ we denote the subspace of all cylinder functions which are polynomially bounded, i.e. $F \in \mathcal{F}_{\mathrm{cyl}}\mathcal{P}(\Gamma)$, iff $F \in \mathcal{F}_{\mathrm{cyl}}(\Gamma)$ (i.e. $F(\gamma) = F|_{\Gamma_\Lambda}(\gamma_\Lambda)$ for some $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$) and there exists a polynomial P on $\mathbb{R}$ such that $|F(\gamma_\Lambda)| \le P(|\gamma_\Lambda|)$.

For any measurable function $f : \mathbb{R}^d \to \mathbb{R}$ we define a Lebesgue-Poisson coherent state corresponding to the one particle function f by

$$e_\lambda(f,\eta) := \prod_{x\in\eta} f(x), \quad \eta \in \Gamma_0.$$

Using the Fock space isomorphism between $L^2(\Gamma_0, d\lambda)$, see e.g. [14], [18] one sees, that, $e_\lambda(f,\cdot)$ corresponds indeed to the usual Fock coherent state

$$e(f) := \left(\frac{f^{\otimes n}}{n!}\right)_{n\in\mathbb{N}}.$$

There is the following mapping from $L^0_{\mathrm{ls}}(\Gamma_0)$ to $\mathcal{F}_{\mathrm{cyl}}(\Gamma)$ which plays a key role in our further considerations:

$$KG(\eta) := \sum_{\eta\Subset\gamma} G(\eta), \quad \gamma \in \Gamma, \tag{7}$$

where $G \in B_{\mathrm{bs}}(\Gamma_0)$. This mapping can be interpreted as a combinatorial version of the Fourier transform and is called K-transform, see [19], [31], [32] for details. The summation in (7) is taken over all finite subconfigurations $\eta \in \Gamma_0$ of the (infinite) configuration $\gamma \in \Gamma$; we denote this by the symbol $\eta \Subset \gamma$. The K-transform is linear, positivity preserving and invertible, with

$$K^{-1}F(\eta) := \sum_{\xi\subset\eta} (-1)^{|\eta\setminus\xi|} F(\xi), \quad \eta \in \Gamma_0. \tag{8}$$

Here and in the sequel inclusions like $\xi \subset \eta$ holds for $\xi = \emptyset$ as well as for $\xi = \eta$. The expression (8) for the inverse K-transform is obtained by an application of

Möbius inversion formula, see e.g. [45]. Further, the K-transform maps $B_{\mathrm{bs}}(\Gamma_0)$ into $\mathcal{F}_{\mathrm{cyl}}\mathcal{P}(\Gamma)$.

For two measurable functions G_1, G_2 on Γ_0 we define a convolution by

$$\begin{aligned}(G_1 * G_2)(\eta) &:= \sum_{(\eta_1,\eta_2,\eta_3)\in\mathcal{P}_3(\eta)} G_1(\eta_1\cup\eta_2)G_2(\eta_2\cup\eta_3) \\ &= \sum_{\xi\subset\eta} G_1(\xi)\sum_{\zeta\subset\xi} G_2((\eta\backslash\xi)\cup\eta), \quad \eta\in\Gamma_0,\end{aligned} \tag{9}$$

where $\mathcal{P}_3(\eta)$ denotes the family of all partitions of η in three parts which may be empty, see e.g. [19]. It is easy ti verify that the space of all measurable functions on Γ_0 endowed with this product hast the structure of a commutative algebra with unit element $e_\lambda(0,\cdot)$. Furthermore, for $G_1, G_2 \in L^0_{\mathrm{ls}}(\Gamma_0)$ we have $G_1 * G_2 \in L^0_{\mathrm{ls}}(\Gamma_0)$ and

$$K(G_1 * G_2) = (KG_1)\cdot(KG_2), \tag{10}$$

cf. [19].

Let $\mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ be the set of all probability measures μ which have finite local moments of all order, i.e. $\int_\Gamma |\gamma_\Lambda|^n\mu(d\gamma) < \infty$ for all $\Lambda\in\mathcal{B}(\mathbb{R}^d)$ and $n\in\mathbb{N}$. A measure ρ on Γ_0 is called locally finite iff $\rho(M)<\infty$ for all bounded sets $M\in\mathcal{B}(\Gamma_0)$. The set of such measures is denoted by $\mathcal{M}_{\mathrm{lf}}(\Gamma_0)$. One can define a transform $K^* : \mathcal{M}^1_{\mathrm{fm}}(\Gamma)\to\mathcal{M}_{\mathrm{lf}}(\Gamma_0)$ which is dual to the K-transform, i.e. for every $\mu\in\mathcal{M}^1_{\mathrm{fm}}(\Gamma)$, $G\in\mathcal{B}_{\mathrm{bs}}(\Gamma_0)$ holds

$$\int_\Gamma KG(\gamma)\mu(d\gamma) = \int_{\Gamma_0} G(\eta)(K^*\mu)(d\eta).$$

The measure $\rho_\mu := K^*\mu$ is called correlation measure of μ. If ρ_μ has a density with respect to (w.r.t. for short) the Lebesgue-Poisson measure λ i.e. $d\rho_\mu = k_\mu d\lambda$, the functions

$$k_\mu^{(n)} : \left(\mathbb{R}^d\right)^n \to \mathbb{R}_+, \quad n\in\mathbb{N},$$

$$k_\mu^{(n)}(x_1,\dots,x_n) := \begin{cases} k_\mu(\{x_1,\dots,x_n\}) & \text{if } (x_1,\dots,x_n)\in\widetilde{\left(\mathbb{R}^d\right)^n} \\ 0 & \text{otherwise.}\end{cases}$$

are the well-known correlation functions of statistical physics, see e.g. [39], [40].

As shown in [19], for $\mu\in\mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ and $G\in L^1(\Gamma_0,\rho_\mu)$, the series

$$KG(\eta) := \sum_{\eta\Subset\gamma} G(\eta)$$

is μ-a.s. absolutely convergent. Furthermore, $KG\in L^1(\Gamma,\mu)$ and

$$\int_\Gamma KG(\gamma)\mu(d\gamma) = \int_{\Gamma_0} G(\eta)(K^*\mu)(d\eta).$$

Thus, we can extend the K-transform to a mapping

$$K_\mu : L^1(\Gamma_0, d\rho_\mu)\to L^1(\Gamma, d\mu). \tag{11}$$

Finally, we recall the so-called Milnos Lemma which plays a very important role in our calculations (cf. [28])

Lemma 1 *Let $n \in \mathbb{N}$, $n \geq 2$. Then*

$$\int_{\Gamma_0} \dots \int_{\Gamma_0} G(\eta_1 \cup \dots \cup \eta_n) H(\eta_1, \dots, \eta_n) \lambda(d\eta_1) \dots \lambda(d\eta_n)$$
$$= \int_{\Gamma_0} G(\eta) \sum_{(\eta_1,\dots,\eta_n) \in \mathcal{P}_n(\eta)} H(\eta_1, \dots, \eta_n) \lambda(d\eta)$$

for all measurable functions $G : \Gamma \to \mathbb{R}$ and $H : \Gamma_0 \times \dots \times \Gamma_0 \to \mathbb{R}$ with respect to which both sides of the equality make sense. Here $\mathcal{P}_n(\eta)$ denotes the set of all ordered partitions of η in n parts, which may be empty.

3 Hierarchical Equations

In this section we derive the hierarchical equations for the considered jump dynamics which are the analogue of the BBGKY-hierarchy for Hamiltonian dynamics. These equations describe the time evolution of correlation functions. We consider the generator L of the jump dynamics, acting on observables F, which is heuristically given by

$$(LF)(\gamma) = \sum_{x \in \gamma} \int_{\mathbb{R}^d} dy \; a(x-y)(1 + \langle \gamma, c_{x,y} \rangle)\big(F(\gamma \backslash x \cup y) - F(\gamma)\big). \tag{12}$$

In order to give the above generator a rigorous meaning, we proceed in the following way : As already mentioned, the K-transform can be regarded as a combinatorial Fourier transform. It is well known that a differential operator on $\mathbb{R}^d$ is in Fourier representation simply given by multiplication with a polynomial. More general, a pseudo-differential operator is in Fourier representation given by multiplication with a symbol, see e.g. [41], [13]. Within our framework, we can proceed in an analog way, since we have an analogue to the Fourier transform, namely the K-transform. In the sequel, we define an operator $\widehat{L} := K^{-1}LK$, which we will also call the symbol corresponding to L. The advantage will be, that the symbol acts on quasi-observables, i.e. on functions depending only on finitely many coordinates. The symbol will be a well defined object. The following informal consideration links the symbol with the infinitesimal generator for correlation functions:

The evolution of the initial state $\mu_0 \in \mathcal{M}^1(\Gamma)$ of the system is of primary interest. It is informally given by the solution to the following Cauchy problem

$$\frac{d\mu_t}{dt} = L^* \mu_t, \qquad \mu_t|_{t=0} = \mu_0,$$

where L^* is an heuristic adjoint to L with respect to the duality

$$< F, \mu > := \int_\Gamma F d\mu.$$

As it was shown in [10], the corresponding evolution of the correlation functions is given by

$$\frac{dk_t}{dt} = L^{\triangle} k_t, \qquad k_t|_{t=0} = k_0, \tag{13}$$

where $L^{\triangle}$ is the dual operator to $\widehat{L} := K^{-1}LK$ with respect to the duality

$$\int_{\Gamma_0} \lambda(d\eta)\ \big(\widehat{L}G\big)(\eta)k(\eta) = \int_{\Gamma_0} \lambda(d\eta)\ G(\eta)\big(L^{\triangle}k\big)(\eta).$$

The hierarchical structure of (13) is described by the countable infinite system of equations

$$\frac{\partial}{\partial t}k_t^{(n)} = \big(L^{\triangle}k_t\big)^{(n)}, \quad k_t^{(n)} := k_t|_{\Gamma_0^{(n)}},\ \big(L^{\triangle}k_t\big)^{(n)} := \big(L^{\triangle}k_t\big)|_{\Gamma_0^{(n)}},\ n \in \mathbb{N}.$$

It was also shown in [10] that the operator $\widehat{L}$ is given by the following formula

$$\begin{aligned}
(\widehat{L}G)(\eta) = &\sum_{x\in\eta}\int_{\mathbb{R}^d} dya(x-y)(G(\eta\backslash x\cup y) - G(\eta)) \\
&+\sum_{x\in\eta}\sum_{\tilde{x}\in\eta\backslash x}\int_{\mathbb{R}^d} dya(x-y)c_{x,y}(\tilde{x})(G(\eta\backslash\{x,\tilde{x}\}\cup y) - G(\eta\backslash\tilde{x})) \\
&+\sum_{x\in\eta}\sum_{\tilde{x}\in\eta\backslash x}\int_{\mathbb{R}^d} dya(x-y)c_{x,y}(\tilde{x})(G(\eta\backslash x\cup y) - G(\eta)),\ G\in B_{\mathrm{bs}}(\Gamma_0).
\end{aligned} \tag{14}$$

Moreover,

$$\begin{aligned}
(L^{\triangle}k)(\eta) = &\sum_{y\in\eta}\int_{\mathbb{R}^d} d\tilde{x}\int_{\mathbb{R}^d} dx\ k(\tilde{x}\cup(\eta\backslash y)\cup x)a(x-y)c_{x,y}(\tilde{x}) \\
&- \int_{\mathbb{R}^d} d\tilde{x}k(\eta\cup\tilde{x})\sum_{x\in\eta}\int_{\mathbb{R}^d} dy\ a(x-y)c_{x,y}(\tilde{x}) \\
&+\sum_{y\in\gamma}\int_{\mathbb{R}^d} dxk((\eta\backslash y)\cup x)(a(x-y) + \sum_{\tilde{x}\in\eta\backslash y} a(x-y)c_{x,y}(\tilde{x})) \\
&- k(\eta)\sum_{x\in\eta}\int_{\mathbb{R}^d} dy(a(x-y) + a(x-y)\sum_{\tilde{x}\in\eta\backslash x} c_{x,y}(\tilde{x})), \quad \eta\in\Gamma_0.
\end{aligned} \tag{15}$$

In the sequel, we want to construct the time evolution of correlation functions for the considered model, i.e. we have to solve the Cauchy problem (13).

4 Evolution Equation for Quasi-Observables

The evolutionary equation (13) is analogue to the BBGKY-hierarchy for Hamiltonian dynamics, see e.g. [3], [5]. As in the case for (infinite) Hamiltonian dynamics, the computation of the n-th correlation function requires the knowledge of the $(n+1)$-th correlation function. But the dual evolution equation (which is given by the symbol $\widehat{L}$), which describes the evolution of quasi-observables, has the feature that the computation of the n-th component of a quasi-observable requires the knowledge of the components of order less than n. This makes a recursive computation of the evolution of the components of quasi-observables possible. The duality between quasi-observables and correlation functions allows us to transfer this evolution to correlation functions.

We consider the following evolution equation (Kolmogorov equation for quasi-observables)

$$\begin{cases} \frac{d}{dt} G_t = \widehat{L} G_t \\ G_t|_{t=0} = G_0, \end{cases} \tag{16}$$

where $\widehat{L}$ is given by (14). In the sequel, we seek a solution of the evolution equation for quasi-observables with G_0 given in some proper space. We recall, that we denote by λ the Lebesgue-Poisson measure on Γ_0. We define for $n \in \mathbb{N}$ and a function G on Γ_0 a symmetric function $G^{(n)}$ on $\mathbb{R}^{nd}$ by

$$G^{(n)} := G|_{\Gamma_0^{(n)}}.$$

We refer to the sequence $(G^{(n)})_{n\in\mathbb{N}}$ as components of the function G and we also remark, that the space $L^1(\Gamma_0, d\lambda)$ has the following Fock-type structure:

$$\begin{aligned} L^1(\Gamma_0, d\lambda) &\cong \bigoplus_n L^1_{sym}(\mathbb{R}^{nd}, \frac{1}{n!} d^n x) \\ G &\mapsto (G^{(n)})_{n\in\mathbb{N}}, \end{aligned}$$

since for $G \in L^1(\Gamma_0, d\lambda)$ holds

$$\|G\|_{L^1(\Gamma_0,\lambda)} = \sum_{n=0}^{\infty} \frac{1}{n!} \int \dots \int |G^{(n)}(x_1, \dots, x_n)| dx_1 \dots dx_n.$$

We observe the following: $(\widehat{L}G)^{(n)}$ is given by

$$(\widehat{L}G)^{(n)} = D^{(n)} G^{(n)} + R^{(n-1)} G^{(n-1)} \tag{17}$$

with

$$\begin{aligned} &(D^{(n)} G^{(n)})(\eta) \\ &= \sum_{x\in\eta} \int_{\mathbb{R}^d} dy\ a(x-y)(G^{(n)}(\eta\backslash x \cup y) - G^{(n)}(\eta)) \\ &\quad + \sum_{x\in\eta} \sum_{\tilde{x}\in\eta\backslash x} \int_{\mathbb{R}^d} dy\ a(x-y) c_{x,y}(\tilde{x})(G^{(n)}(\eta\backslash x \cup y) - G^{(n)}(\eta)) \end{aligned} \tag{18}$$

and

$$\begin{aligned} &(R^{(n-1)} G^{(n-1)})(\eta) \\ &= \sum_{x\in\eta} \sum_{\tilde{x}\in\eta\backslash x} \int_{\mathbb{R}^d} dy\ a(x-y) c_{x,y}(\tilde{x}) \\ &\quad \times (G^{(n-1)}(\eta\backslash\{x,\tilde{x}\} \cup y) - G^{(n-1)}(\eta\backslash\tilde{x})), \quad \eta \in \Gamma_0^{(n)}. \end{aligned}$$

Now equation (16) reads in components as follows:

$$\frac{\partial}{\partial t} G_t^{(n)} = D^{(n)} G_t^{(n)} + R^{(n-1)} G_t^{(n-1)}. \tag{19}$$

Due to this observation the following strategy for the construction of a solution of (16) should be reasonable:

Fix $n \in \mathbb{N}$ and assume that $D^{(n)}$ generates a semigroup (in some proper Banach space). If $G_t^{(n-1)}$ is already known, the solution of the system (19) is given by

$$G_t^{(n)} = e^{tD^{(n)}} G_0^{(n)} + \int_0^t e^{(t-s)D^{(n)}} R^{(n-1)} G_s^{(n-1)} ds, \quad t > 0, \tag{20}$$

where the above integral is to interpret in Bochners sense. Hence, given G_0, we can compute the components of the solution G_t of (16) successively. In the following we try to realize this approach.

Before we solve equation (16) we have to analyze the operators $D^{(n)}$ and $R^{(n-1)}$. The solution of (16) should be in a proper L^1-space, because in this case, the corresponding dual evolution will be in a L^∞-space (we want to work in a non-zero density framework). Especially, we will need that the operators $D^{(n)}$ induce contraction semigroups in proper L^1-spaces.

We recall, that we denote by $\sigma^{(n)}$ the projection of the $n \cdot d$-dimensional Lebesgue-measure to $\Gamma_0^{(n)}$.

Lemma 2 *Let $c_{x,y}(\tilde{x}) = \kappa(x - \tilde{x})$ Under the condition*

$$\int_{\mathbb{R}^d} a(x-y)\kappa(x-\tilde{x})dx \le \kappa(y - \tilde{x}), \tag{21}$$

the operator $D^{(n)}$ generates a contraction semigroup in $L^1(\Gamma_0^{(n)}, d\sigma^{(n)})$.

Proof: It is easily seen (by means of the Minlos Lemma), that the operator $D^{(n)}$ is bounded in $L^1(\Gamma_0^{(n)}, d\sigma^{(n)})$. Thus, $D^{(n)}$ induces a semigroup in $L^1(\Gamma_0^{(n)}, d\sigma^{(n)})$. In order to show that this semigroup is contractive, we use the Lumer-Phillips Theorem, see [35]. Thus, we have to show

- it exists some $\lambda > 0$ such that the range of $D^{(n)} - \lambda \mathbf{1}$ is the whole space $L^1(D^{(n)}, d\sigma^{(n)})$
- for all $\lambda > 0$ holds $\|(D^{(n)} - \lambda \mathbf{1})G^{(n)}\| \ge \lambda \|G^{(n)}\|$.

To the first point: this is obvious since $D^{(n)}$ is bounded an therefore $\sigma(D^{(n)}) \subset B_{D^{(n)}}(0)$. Now to the second point: let $\lambda > 0$. We write $D^{(n)} = D_1^{(n)} + D_2^{(n)}$ with

$$(D_1^{(n)} G^{(n)})(\eta) := -\Big(|\eta| + E^\kappa(\eta)\Big) G^{(n)}(\eta),$$

where

$$E^\kappa(\eta) := \sum_{x \in \eta} \sum_{\tilde{x} \in \eta \setminus x} \kappa(x - \tilde{x}),$$

and

$$D_2^{(n)} := D^{(n)} - D_1^{(n)}.$$

It holds

$$
\begin{aligned}
\|(D^{(n)} - \lambda\mathbf{1})G^{(n)}\| &= \|(D_1^{(n)} + D_2^{(n)} - \lambda\mathbf{1})G^{(n)}\| \\
&\geq \|(D_1^{(n)} - \lambda\mathbf{1})G^{(n)}\| - \|D_2^{(n)}G^{(n)}\| \\
&= \int_{\Gamma_0^{(n)}} \Big(\lambda + |\eta| + E^{\kappa}(\eta)\Big)|G^{(n)}(\eta)|\sigma^{(n)}(d\eta) \\
&\quad - \|D_2^{(n)}G^{(n)}\|
\end{aligned}
\tag{22}
$$

Now we write $D_2^{(n)}$ as $D_2^{(n)} := D_{2,1}^{(n)} + D_{2,2}^{(n)}$ with

$$
(D_{2,1}^{(n)}G^{(n)})(\eta) := \sum_{x\in\eta}\int_{\mathbb{R}^d} dy\ a(x-y)G^{(n)}(\eta\backslash x \cup y)
$$

and

$$
(D_{2,2}^{(n)}G^{(n)})(\eta) = \sum_{x\in\eta}\sum_{\tilde{x}\in\eta\backslash x}\kappa(x-\tilde{x})\int_{\mathbb{R}^d} dy\ a(x-y)G^{(n)}(\eta\backslash x\cup y).
$$

Obviously,

$$
\|D_{2,1}^{(n)}\| \leq \int_{\Gamma_0^{(n)}} \sigma^{(n)}(d\eta)|\eta||G^{(n)}(\eta)|. \tag{23}
$$

For the norm of $D_{2,2}^{(n)}$ holds (we use Minlos Lemma and (21)):

$$
\begin{aligned}
\|D_{2,2}^{(n)}\| &\leq \int_{\Gamma_0^{(n)}} \sigma^{(n)}(d\eta)\sum_{x\in\eta}\sum_{\tilde{x}\in\eta\backslash x}\kappa(x-\tilde{x})\int_{\mathbb{R}^d} dy\ a(x-y)|G^{(n)}(\eta\backslash x\cup y)| \\
&= \int_{\Gamma_0^{(n)}} \sigma^{(n)}(d\eta)\sum_{y\in\eta}\sum_{\tilde{x}\in\eta\backslash y}\int_{\mathbb{R}^d} dx\ a(x-y)\kappa(x-\tilde{x})|G^{(n)}(\eta)| \\
&\leq \int_{\Gamma_0^{(n)}} \sigma^{(n)}(d\eta)E^{\kappa}(\eta)|G^{(n)}(\eta)|.
\end{aligned}
\tag{24}
$$

Combining (22), (23) and (24) yields

$$
\|(D^{(n)} - \lambda\mathbf{1})G^{(n)}\| \geq \lambda\|G^{(n)}\|. \quad \square
$$

Lemma 3 *Let $c_{x,y}(\tilde{x}) = \kappa(y-\tilde{x})$. Under the condition*

$$
\int_{\mathbb{R}^d} a(x-y)\kappa(y-\tilde{x}) \geq \kappa(x-\tilde{x}), \tag{25}
$$

the operator $D^{(n)}$ generates a contraction semigroup in $L^1(\Gamma_0^{(n)}, d\sigma^{(n)})$.

Proof: The proof is analogous to Lemma 2. $\square$

Lemma 4 *Let $c_{x,y}(\tilde{x}) = \frac{1}{2}\big(\kappa(x-\tilde{x}) + \kappa(y-\tilde{x})\big)$. Then $D^{(n)}$ generates a contraction semigroup in $L^1(\Gamma_0^{(n)}, d\sigma^{(n)})$.*

Proof: The symmetry of $c_{x,y}(\tilde{x})$ in x and y and Minlos Lemma yields

$$\begin{aligned}
&\|(D^{(n)} - \lambda\mathbf{1})G^{(n)}\| \\
&\geq \left| \int_{\Gamma_0^{(n)}} \sigma^{(n)}(d\eta) \sum_{x\in\eta} \int_{\mathbb{R}^d} dy\ a(x-y)G^{(n)}(\eta\backslash x \cup y) - G^{(n)}(\eta) - \lambda G^{(n)}(\eta) \right| \\
&\geq \lambda \|G^{(n)}\|. \quad \square
\end{aligned}$$

Now we investigate the operators $R^{(n-1)}$ more detailed. To this end we introduce the notation $X_n := L^1(\Gamma_0^{(n)}, d\sigma^{(n)})$ and we define for a symmetric function $G^{(n)}$ on $\mathbb{R}^{nd}$ a function $\widetilde{G^{(n)}}$ on Γ_0 by

$$\widetilde{G^{(n)}}(\eta) = \begin{cases} G^{(n)}(x_1, \ldots, x_n) \text{ if } \eta = \{x_1, \ldots, x_n\} \\ 0 \text{ else.} \end{cases}$$

Then, we can write the L^1-norm of X_n as

$$\|G^{(n)}\|_{X_n} = n! \int_{\Gamma_0} \lambda(d\eta) |\widetilde{G^{(n)}}(\eta)|$$

The next Lemma holds for all the three cases which we want to consider for $c_{x,y}(\tilde{x})$. We present the proof only for the case $c_{x,y}(\tilde{x}) = \frac{1}{2}\big(\kappa(x-\tilde{x}) + \kappa(y-\tilde{x})\big)$. The other cases are analogue.

Lemma 5 *We can regard $R^{(n-1)}$ as an operator*

$$R^{(n-1)} : X_{n-1} \to X_n.$$

This operator is continuous, moreover

$$\|R^{(n-1)}G^{(n-1)}\|_{X_n} \leq Bn(n-1)\|G^{(n-1)}\|_{X_{n-1}} \tag{26}$$

for all $G^{(n-1)} \in X_{n-1}$ where $B > 0$ is given by

$$B = 2\langle\kappa\rangle\langle a\rangle. \tag{27}$$

Proof: We write $R^{(n-1)} = R_1^{(n-1)} + R_2^{(n-1)}$ with

$$\begin{aligned}
(R_1^{(n-1)}G^{(n-1)})(\eta) := &\sum_{x\in\eta} \sum_{\tilde{x}\in\eta\backslash x} \int_{\mathbb{R}^d} dy a(x-y) c_{x,y}(\tilde{x}) \\
&\times (G^{(n-1)}(\eta\backslash\{x,\tilde{x}\} \cup y)
\end{aligned}$$

and

$$\begin{aligned}
(R_2^{(n-1)}G^{(n-1)})(\eta) := &\sum_{x\in\eta} \sum_{\tilde{x}\in\eta\backslash x} \int_{\mathbb{R}^d} dy a(x-y) c_{x,y}(\tilde{x}) \\
&\times G^{(n-1)}(\eta\backslash\tilde{x}).
\end{aligned}$$

For a bounded function $G^{(n-1)} \in X_{n-1}$ holds (we use Milnos Lemma):

$$
\begin{aligned}
&\|R_1^{(n-1)}G^{(n-1)}\|_{X_n} \\
&\le n! \int_{\Gamma_0} \lambda(d\eta) \mathbf{1}_{\Gamma_0^{(n)}}(\eta) \sum_{x\in\eta} \sum_{\tilde{x}\in\eta\setminus x} \int_{\mathbb{R}^d} dy\ a(x-y) \\
&\quad \times c_{x,y}(\tilde{x}) |\widetilde{G^{(n-1)}}(\eta\setminus\{x,\tilde{x}\}\cup y)| \\
&= n! \int_{\Gamma_0} \lambda(d\eta) \int_{\mathbb{R}^d} dx\ \mathbf{1}_{\Gamma_0^{(n)}}(\eta\cup x) \sum_{\tilde{x}\in\eta} \int_{\mathbb{R}^d} dy\ a(x-y) \\
&\quad \times c_{x,y}(\tilde{x}) |\widetilde{G^{(n-1)}}(\eta\setminus\tilde{x}\cup y)| \\
&= n! \int_{\Gamma_0} \lambda(d\eta) \int_{\mathbb{R}^d} d\tilde{x} \int_{\mathbb{R}^d} dx\ \mathbf{1}_{\Gamma_0^{(n-1)}}(\eta\cup\tilde{x}) \int_{\mathbb{R}^d} dy a(x-y) \\
&\quad \times c_{x,y}(\tilde{x}) |\widetilde{G^{(n-1)}}(\eta\cup y)| \\
&= n! \int_{\Gamma_0} \lambda(d\eta) \sum_{y\in\eta} \mathbf{1}_{\Gamma_0^{(n-2)}}(\eta\setminus y) \int_{\mathbb{R}^d} dx\ a(x-y) \\
&\quad \times \int_{\mathbb{R}^d} d\tilde{x}\ \ \frac{1}{2}(\kappa(x-\tilde{x})+\kappa(y-\tilde{x})) |\widetilde{G^{(n-1)}}(\eta)| \\
&= \langle\kappa\rangle n! \int_{\Gamma_0} \lambda(d\eta) \sum_{y\in\eta} \mathbf{1}_{\Gamma_0^{(n-1)}}(\eta) \int_{\mathbb{R}^d} dx a(x-y) |\widetilde{G^{(n-1)}}(\eta)| \\
&= \langle\kappa\rangle\langle a\rangle n! \int_{\Gamma_0} \lambda(d\eta) \sum_{y\in\eta} \mathbf{1}_{\Gamma_0^{(n-1)}}(\eta) |\widetilde{G^{(n-1)}}(\eta)| \\
&= \langle\kappa\rangle\langle a\rangle n! \int_{\Gamma_0} \lambda(d\eta) \mathbf{1}_{\Gamma_0^{(n-1)}}(\eta) |\eta| |\widetilde{G^{(n-1)}}(\eta)| \\
&= \langle\kappa\rangle\langle a\rangle (n-1) n (n-1)! \int_{\Gamma_0} \lambda(d\eta) |\widetilde{G^{(n-1)}}(\eta)| \\
&= \langle\kappa\rangle\langle a\rangle n(n-1) \|G^{(n-1)}\|_{X_{n-1}}.
\end{aligned}
\tag{28}
$$

Hence, we obtain

$$
\|R_1^{(n-1)}G^{(n-1)}\|_{X_n} \le \langle\kappa\rangle\langle a\rangle n(n-1)\|G^{(n-1)}\|_{X_{n-1}} \tag{29}
$$

For $\|R_2^{(n-1)} G^{(n-1)}\|_{X_n}$ we get:

$$\begin{aligned}
&\|R_2^{(n-1)} G^{(n-1)}\|_{X_n} \\
&\le n! \int_{\Gamma_0} \lambda(d\eta) \mathbf{1}_{\Gamma_0^{(n)}}(\eta) \sum_{x\in\eta} \sum_{\tilde{x}\in\eta\backslash x} \int_{\mathbb{R}^d} dy a(x-y) c_{x,y}(\tilde{x}) |\widetilde{G^{(n-1)}}(\eta\backslash\tilde{x})| \\
&= n! \int_{\Gamma_0} \lambda(d\eta) \int_{\mathbb{R}^d} dx\ \mathbf{1}_{\Gamma_0^{(n)}}(\eta\cup x) \sum_{\tilde{x}\in\eta} \int_{\mathbb{R}^d} dy\ a(x-y) \\
&\quad \times c_{x,y}(\tilde{x}) |\widetilde{G^{(n-1)}}(\eta\backslash\tilde{x}\cup x)| \\
&= n! \int_{\Gamma_0} \lambda(d\eta) \int_{\mathbb{R}^d} dx\ \mathbf{1}_{\Gamma_0^{(n-1)}}(\eta) \sum_{\tilde{x}\in\eta} \int_{\mathbb{R}^d} dy\ a(x-y) \\
&\quad \times c_{x,y}(\tilde{x}) |\widetilde{G^{(n-1)}}(\eta\backslash\tilde{x}\cup x)| \\
&= n! \int_{\Gamma_0} \lambda(d\eta) \mathbf{1}_{\Gamma_0^{(n-2)}}(\eta) \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy\ a(x-y) |\widetilde{G^{(n-1)}}(\eta\cup x)| \\
&\quad \times \int_{\mathbb{R}^d} d\tilde{x}\ \frac{1}{2}(\kappa(x-\tilde{x}) + \kappa(y-\tilde{x})) \\
&= n! \langle a\rangle\langle\kappa\rangle \int_{\Gamma_0} \lambda(d\eta) \mathbf{1}_{\Gamma_0^{(n-2)}}(\eta) \int_{\mathbb{R}^d} dx\ |\widetilde{G^{(n-1)}}(\eta\cup x)| \\
&= n! \langle a\rangle\langle\kappa\rangle \int_{\Gamma_0} \lambda(d\eta) \mathbf{1}_{\Gamma_0^{(n-1)}}(\eta) \sum_{x\in\eta} |\widetilde{G^{(n-1)}}(\eta)| \\
&= n(n-1)\ \langle a\rangle\langle\kappa\rangle \|G^{(n-1)}\|_{X_{n-1}}
\end{aligned} \tag{30}$$

i.e.

$$\|R_2^{(n-1)} G^{(n-1)}\|_{X_n} \le \langle a\rangle\langle\kappa\rangle n(n-1) \|G^{(n-1)}\|_{X_{n-1}} \tag{31}$$

Altogether, we derive by means of (29) and (31):

$$\|R^{(n-1)} G^{(n-1)}\|_{X_n} \le Bn(n-1) \|G^{(n-1)}\|_{X_{n-1}}. \tag{32}$$

with $B = 2\langle a\rangle\langle\kappa\rangle$. This shows the assertions of Lemma 5. $\Box$

Now we are able to construct the evolution of quasi-observables. First of all, we have to introduce the spaces in which the evolution will live.

To this end, let α and $C > 0$. By $\mathcal{I}_{\alpha,C}$ we denote the functional Banach space, consisting of all Fock-type vectors $G = (G^{(n)})_{n\in\mathbb{N}} \in \bigoplus_{n\in\mathbb{N}} X_n$ for which the norm

$$\|G\|_{\mathcal{I}_{\alpha,C}} := \sup_{n\in\mathbb{N}} \frac{\|G^{(n)}\|_{X_n} C^n}{\alpha^n n!} \tag{33}$$

is less than infinite.

The next theorem concerns the evolution of quasi-observables. Since we are primary interested in the evolution of correlation function, we formulate the theorem below in a way suitable to transport the dynamics of quasi-observables to correlation functions. The statement of the next Theorem holds in all the tree case which we want to consider. In the non symmetric case we have to assume (21) resp. (25).

Theorem 1 *Consider the evolution problem*

$$\begin{cases} \frac{d}{dt} G_t = \widehat{L} G_t \\ G_t|_{t=0} = G_0. \end{cases} \tag{34}$$

where $\widehat{L}$ *is given by* (14) *and the component-wise solution*

$$\begin{aligned} G_t^{(n)} &= e^{tD^{(n)}} G_0^{(n)} + \int_0^t e^{(t-s)D^{(n)}} R^{(n-1)} G_s^{(n-1)} ds, \quad t > 0 \\ G_t^{(0)} &= G_0^{(0)}. \end{aligned} \tag{35}$$

Let α, $C > 0$ *and define* C_t, $t \in \mathbb{R}$ *by*

$$C_t := \frac{C}{1 + \frac{tBC}{\alpha}}, \tag{36}$$

where B *is given by* (27)*. Then the following holds: If* G_0 *is an element of* $\mathcal{I}_{\alpha,C}$ *then,* G_t *is an element of* $\mathcal{I}_{\alpha,C_t}$*,* $t > 0$ *and the bound*

$$\|G_t\|_{\mathcal{I}_{\alpha,C_t}} \leq \|G_0\|_{\mathcal{I}_{\alpha,C}} \tag{37}$$

holds for all $t > 0$*.*

Proof: We proof by induction that for all $n \in \mathbb{N}$ holds:

$$\|G_t^{(n)}\|_{X_n} \leq \|G_0\|_{\mathcal{I}_{\alpha,C}} n! \frac{\alpha^n}{C^n} \left(1 + \frac{tBC}{\alpha}\right)^n, \quad t > 0. \tag{38}$$

Clearly, from (38) follows (37). Since $G_0 \in \mathcal{I}_{\alpha,C}$, it follows

$$\|G_0^{(n)}\|_{X_n} \leq \|G_0\|_{\mathcal{I}_{\alpha,C}} \left(\frac{\alpha}{C}\right)^n n!. \tag{39}$$

We assume that (38) holds for $k < n$. By iterating the formula

$$G_t^{(n)} = e^{tD^{(n)}} G_0^{(n)} + \int_0^t e^{(t-s)D^{(n)}} R^{(n-1)} G_s^{(n-1)} ds,$$

we obtain

$$G_t^{(n)} = \sum_{k=0}^{n} A_{k,n}(t) G_0^{(n-k)}$$

with

$$\begin{aligned} A_{k,n}(t) &:= \int_0^t \int_0^{s_1} \ldots \int_0^{s_{k-1}} e^{(t-s_1)D^{(n)}} R^{(n-1)} e^{(s_1-s_2)D^{(n-1)}} R^{(n-2)} \ldots \\ &\quad \times R^{(n-k)} e^{e^{s_k} D^{(n-k)}} ds_k \ldots ds_1 \\ A_{0,n}(t) &: = e^{tD^{(n)}}. \end{aligned}$$

By means of the contraction property of $D^{(n)}$ (Lemma 2-Lemma 4) and Lemma 5 we derive:

$$\begin{aligned} \|A_{k,n}(t) G_0^{(n-k)}\|_{X_n} &\leq \frac{t^k}{k!} B^k n(n-1)(n-1)(n-2)\ldots \\ &\quad \times (n-k+1)(n-k) \|G_0^{(n-k)}\|_{X_{n-k}} \\ &= \frac{t^k}{k!} B^k \frac{n!}{(n-k)!} \frac{(n-1)!}{(n-k-1)!} \|G_0^{(n-k)}\|_{X_{n-k}}. \end{aligned}$$

Thus, we obtain

$$\begin{aligned}\|G_t^{(n)}\|_{X_n} &\leq \sum_{k=0}^{n} \frac{t^k B^k}{k!} \frac{n!}{(n-k)!} \frac{(n-1)!}{(n-k-1)!} \|G_0^{(n-k)}\|_{X_{n-k}} \\ &= \sum_{k=0}^{n} (tB)^{n-k} \frac{n!(n-1)!}{(n-k)!k!(k-1)!} \|G_0^{(k)}\|_{X_k}.\end{aligned}$$

Now we use (39) and obtain

$$\begin{aligned}\|G_t^{(n)}\|_{X_n} &\leq \|G_0\|_{\mathcal{I}_{\alpha,C}} n! \sum_{k=0}^{n} (tB)^{n-k} \binom{n}{k} \frac{k}{n} \frac{\alpha^k}{C^k} \\ &\leq \|G_0\|_{\mathcal{I}_{\alpha,C}} n! \frac{\alpha^n}{C^n} \sum_{k=0}^{n} \left(\frac{tBC}{\alpha} \right)^{n-k} \binom{n}{k} \\ &= \|G_0\|_{\mathcal{I}_{\alpha,C}} n! \frac{\alpha^n}{C^n} \left(1 + \frac{tBC}{\alpha} \right)^n.\end{aligned}$$

This shows (38). $\square$

Remark 1 *We can define a propagator* $\widehat{P}_t$ *by*

$$\widehat{P}_t : \mathcal{I}_{\alpha,C} \to \mathcal{I}_{\alpha,C_t}, \quad \widehat{P}_t G := G_t$$

where G_t *is the solution of* (34) *with initial data* G*. This propagator describes the time evolution of quasi-observables.*

5 The Evolution of Correlation Functions

In this section we construct the evolution for correlation functions. Since we are working in a non-zero density framework, the natural spaces in which this evolution takes place, are of the type

$$\mathcal{K}_C := \{k : \Gamma_0 \to \mathbb{R} | k \cdot C^{-|\cdot|} \in L^\infty(\Gamma_0, d\lambda)\}, \quad C > 0,$$

cf. [22]. The space $\mathcal{K}_C$ is the dual space of

$$\mathcal{L}_C := L^1(\Gamma_0, C^{|\cdot|} d\lambda),$$

where the duality is given by the following expression:

$$\langle\langle k, G \rangle\rangle := \int_{\Gamma_0} k \cdot G \, d\lambda, \quad G \in \mathcal{L}_C.$$

It is clear that $\mathcal{K}_C$ is a Banach space with the norm

$$\|k\|_C := \|k C^{-|\cdot|}\|_{L^\infty(\Gamma_0, d\lambda)}.$$

Note also, that $k \cdot C^{-|\cdot|} \in L^\infty(\Gamma_0, d\lambda)$ means that the function k satisfies the bound

$$|k(\eta)| \leq const\ C^{|\eta|} \quad \lambda - \text{a.e.}$$

We remind, that the space $\mathcal{I}_{\alpha,C}$ consists of all Fock-type vectors $G = (G^{(n)})_{n\in\mathbb{N}} \in \bigoplus_{n\in\mathbb{N}} X_n$, s.t. $\|G\|_{\mathcal{I}_{\alpha,C}} < \infty$ holds (see. (33)). For $\alpha \in (0,1)$, we obtain the inclusion

$$\mathcal{L}_{\frac{C}{\alpha}} \subset \mathcal{I}_{\alpha,C} \subset \mathcal{L}_C \tag{40}$$

since it holds firstly:

$$\frac{C^n}{n!}\|G^{(n)}\|_{X_n} \leq \alpha^n \|G\|_{\mathcal{I}_{\alpha,C}}, \quad n\in\mathbb{N}, G\in\mathcal{I}_{\alpha,C},$$

which implies

$$\|G\|_{\mathcal{L}_C} \leq \frac{1}{1-\alpha}\|G\|_{\mathcal{I}_{\alpha,C}} < \infty.$$

Because of that, we obtain

$$\mathcal{I}_{\alpha,C} \subset \mathcal{L}_C. \tag{41}$$

Secondly, it holds for $G \in \mathcal{L}_{\frac{C}{\alpha}}$

$$\|G\|_{\mathcal{I}_{\alpha,C}} \leq \|G\|_{\mathcal{L}_{\frac{C}{\alpha}}},$$

hence

$$\mathcal{L}_{\frac{C}{\alpha}} \subset \mathcal{I}_{\alpha,C}.$$

Altogether, we obtain (40). We consider also a functional space $\mathcal{J}_{\alpha,C}$, which consists of all Fock-type vectors $k = (k^{(n)})_{n\in\mathbb{N}} \in \bigoplus_{n\in\mathbb{N}} X_n^*$, $X_n^* := L^\infty(\Gamma_0^{(n)}, d\sigma^{(n)})$ for which

$$\|k\|_{\mathcal{J}_{\alpha,C}} := \sum_{n=0}^{\infty} \frac{\alpha^n}{C^n}\|k^{(n)}\|_{X_n^*} < \infty$$

holds. Let $G \in \mathcal{I}_{\alpha,C}$ and $k \in \mathcal{J}_{\alpha,C}$. It follows

$$\begin{aligned}
|\langle\langle k, G\rangle\rangle| &\leq \sum_{n=0}^{\infty} \frac{1}{n!}\int_{\Gamma_0^{(n)}} |k^{(n)}||G^{(n)}| d\sigma^{(n)} \\
&\leq \sum_{n=0}^{\infty} \frac{1}{n!}\|G^{(n)}\|_{X_n}\|k^{(n)}\|_{X_n^*} \\
&= \sum_{n=0}^{\infty} \frac{C^n}{\alpha^n n!}\|G^{(n)}\|_{X_n}\frac{\alpha^n}{C^n}\|k^{(n)}\|_{X_n^*} \\
&\leq \left(\sup_{n\in\mathbb{N}} \|G^{(n)}\|_{X_n}\frac{C^n}{\alpha^n n!}\right)\left(\sum_{n=0}^{\infty}\frac{\alpha^n}{C^n}\|k^{(n)}\|_{X_n^*}\right) \\
&= \|G\|_{\mathcal{I}_{\alpha,C}}\|k\|_{\mathcal{J}_{\alpha,C}}.
\end{aligned} \tag{42}$$

Hence, $G \mapsto \langle\langle k, G\rangle\rangle$ is a bounded linear functional on $\mathcal{I}_{\alpha,C}$ and therefore

$$\mathcal{J}_{\alpha,C} \subset (\mathcal{I}_{\alpha,C})^*.$$

Next, we observe that for $k \in \mathcal{J}_{\alpha,C}$ holds:

$$\frac{\alpha^n}{C^n}\|k^{(n)}\|_{X_n^*} \leq \|k\|_{\mathcal{J}_{\alpha,C}}, \quad n \in \mathbb{N}.$$

It follows

$$\|k\|_{\mathcal{K}_{\frac{C}{\alpha}}} \leq \|k\|_{\mathcal{J}_{\alpha,C}}$$

which implies

$$\mathcal{J}_{\alpha,C} \subset \mathcal{K}_{\frac{C}{\alpha}}$$

Otherwise, for $k \in \mathcal{K}_C$ holds

$$\|k^{(n)}\|_{X_n^*} \leq C^n \|k\|_{\mathcal{K}_C}, \quad n \in \mathbb{N}.$$

Using this, we conclude

$$\|k\|_{\mathcal{J}_{\alpha,C}} = \sum_{n=0}^{\infty} \frac{\alpha^n}{C^n} \|k^{(n)}\|_{X_n^*} \leq \sum_{n=0}^{\infty} \frac{\alpha^n}{C^n} C^n \|k\|_{\mathcal{K}_C} = \frac{1}{1-\alpha} \|k\|_{\mathcal{K}_C},$$

therefore,

$$\mathcal{K}_C \subset \mathcal{J}_{\alpha,C} \subset \mathcal{K}_{\frac{C}{\alpha}}. \tag{43}$$

Further, by means of (40), we get

$$\mathcal{K}_C \subset (\mathcal{I}_{\alpha,C})^* \subset \mathcal{K}_{\frac{C}{\alpha}}. \tag{44}$$

Altogether,

$$\mathcal{K}_C \subset \mathcal{J}_{\alpha,C} \subset (\mathcal{I}_{\alpha,C})^* \subset \mathcal{K}_{\frac{C}{\alpha}}.$$

Now to the construction of the evolution of correlation functions:

Theorem 2 *Let $C_0 > 0$, $\alpha \in (0,1)$ and define a time horizon $T > 0$ by*

$$T := \frac{\alpha}{C_0 B},$$

where B is given by (27). Further, we define C_t^* by

$$C_t^* := \frac{C_0}{1 - \frac{C_0 B t}{\alpha}}. \tag{45}$$

It holds: for all $t < T$ exists a mapping

$$\langle\langle P_t^{\triangle} \cdot | : \mathcal{K}_{C_0} \to \mathcal{I}^*_{\alpha,C_t^*} \subset \mathcal{K}_{\frac{C_t^*}{\alpha}}$$

with the following properties: for all $G \in \mathcal{I}_{\alpha,C_t^*}$ holds

$$\langle\langle P_t^{\triangle} k_0 | G \rangle\rangle = \langle\langle k_0, G_t \rangle\rangle, \tag{46}$$

where G_t is the solution $(G_s)_{s\geq 0}$ of (34) with initial data $G_0 = G$ evaluated at $s = t$.

Proof: Let $k_0 \in \mathcal{K}_{C_0} \subset \mathcal{I}^*_{\alpha,C_0}$ and $t < T$. We define the mapping $\langle\langle P_t^{\triangle} \cdot | : \mathcal{K}_{C_0} \to \mathcal{I}^*_{\alpha,C_t^*}$ by

$$\langle\langle P_t^{\triangle} k_0 | G_0 \rangle\rangle := \langle\langle k_0, G_t \rangle\rangle, \tag{47}$$

where G_t is the solution $(G_s)_{s\geq 0}$ of (34) with initial data $G_0 = G$ evaluated at $s = t$. Since $G \in \mathcal{I}_{\alpha,C_t^*}$ it follows by Theorem 1 that G_t is an element of $\mathcal{I}_{\alpha,C_t}$ where

$$C_t = \frac{C_t^*}{1 + \frac{tBC_t^*}{\alpha}}.$$

By plug in the definition (45) of C_t^* together with a simple calculation shows that $C_t = C_0$. Thus, $\langle\langle P_t^\triangle \cdot |$ is well defined. Moreover, since $k_0 \in \mathcal{K}_{C_0} \subset \mathcal{J}_{\alpha,C_0}$ and and $G_t \in \mathcal{I}_{\alpha,C_0}$, we obtain by means of (42) and (37):

$$\begin{aligned}
|\langle\langle P_t^\triangle k_0 | G_0\rangle\rangle| &= |\langle\langle k_0, G_t\rangle\rangle| \\
&\leq \|k_0\|_{\mathcal{J}_{\alpha,C_0}} \|G_t\|_{\mathcal{I}_{\alpha,C_0}} \\
&\leq \|k_0\|_{\mathcal{J}_{\alpha,C_0}} \|G_0\|_{\mathcal{I}_{\alpha,C_t^*}}.
\end{aligned}$$

That is why $\langle\langle P_t^\triangle k_0 | \in \mathcal{I}^*_{\alpha,C_t^*}$. But since $\mathcal{I}^*_{\alpha,C_t^*} \subset \mathcal{K}_{\frac{C_t^*}{\alpha}}$ (cf. (44)), we can regard $\langle\langle P_t^\triangle \cdot |$ as a mapping

$$\langle\langle P_t^\triangle \cdot | : \mathcal{K}_{C_0} \to \mathcal{K}_{\frac{C_t^*}{\alpha}}.$$

By definition, this mapping has the property

$$\langle\langle P_t^\triangle k_0 | G\rangle\rangle = \langle\langle k_0, G_t\rangle\rangle. \quad \square$$

Remark 2 *The evolution* $k_t := P_t^\triangle k_0$, $t \in [0,T)$ *describes the time evolution of the initial correlation function* k_0. *We can regard the mapping* $P_t^\triangle$ *as the dual propagator to* $\widehat{P}_t$ *(cf. Remark 1), because it holds*

$$\langle\langle P_t^\triangle k_0, G\rangle\rangle = \langle\langle k_0, \widehat{P}_t G\rangle\rangle.$$

6 Weak Solution

Now, let us consider our model of jumping particles and an initial correlation function $k_0 \in \mathcal{K}_{C_0}$. We fix a time point $t < T = \frac{\alpha}{C_0 B}$ and consider the above constructed evolution k_t. Let $G \in \mathcal{I}_{\alpha,C_t^*}$ and $(G_s)_{s\geq 0}$ be the solution of (34) with initial data $G_0 = G$. Using the contractivity of the diagonal part (Lemma 2-Lemma 4) of $\widehat{L}$ and Lemma 5 we get:

$$\begin{aligned}
&\int_0^t \sum_{n=0}^\infty \frac{1}{n!} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} k_0^{(n)}(x_1,\dots,x_n) \big| \big(\widehat{L}G_s\big)^{(n)}(x_1,\dots,x_n)\big| dx_1 \dots dx_n ds \\
&\leq \|k_0\|_{C_0} \int_0^t \sum_{n=0}^\infty \frac{C_0^n}{n!} \|\big(\widehat{L}G_s\big)^{(n)}\|_{X_n} ds \\
&\leq \|k_0\|_{C_0} \int_0^t \sum_{n=0}^\infty \frac{C_0^n}{n!} \big(\|G_s^{(n)}\|_{X_n} + Bn(n-1)\|G_s^{(n-1)}\|_{X_{n-1}}\big) ds.
\end{aligned}$$

Due to Theorem 1 it holds $G_s \in \mathcal{I}_{\alpha,C_s}$ with

$$C_s = \frac{C_t^*}{1 + \frac{sBC_t^*}{\alpha}}.$$

Therefore

$$\|G_s^{(n)}\|_{X_n} \leq \|G\|_{\mathcal{I}_{\alpha,C_s}} n! \frac{\alpha^n}{C_s^n} \leq \|G\|_{\mathcal{I}_{\alpha,C_t^*}} n! \frac{\alpha^n}{C_s^n}.$$

Using this, we conclude:

$$\begin{aligned}
&\int_0^t \sum_{n=0}^{\infty} \frac{C_0^n}{n!} \big(\|G_s^{(n)}\|_{X_n} + Bn(n-1)\|G_s^{(n-1)}\|_{X_{n-1}}\big) ds \\
&\leq \|G\|_{\mathcal{I}_{\alpha,C_t^*}} \int_0^t \left(\sum_{n=0}^{\infty} \left(\frac{\alpha C_0}{C_s}\right)^n + B \sum_{n=0}^{\infty} (n-1) \left(\frac{\alpha C_0}{C_s}\right)^{n-1} \right) ds \\
&< \infty.
\end{aligned}$$

In the last step we have used $\frac{C_0}{C_s} < 1$. This is true since C_s is decreasing and $C_t = C_0$ (cf. the Proof of Theorem 2).

Therefore, we obtain by means of Fubini's theorem:

$$\begin{aligned}
\langle\langle k_t, G\rangle\rangle &= \langle\langle k_0, G_t\rangle\rangle \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\mathbb{R}^d} \dots \int_{\mathbb{R}^d} k_0^{(n)}(x_1, \dots, x_n) \Big(G_0^{(n)}(x_1, \dots, x_n) \\
&\quad + \int_0^t \big(\widehat{L} G_s\big)^{(n)}(x_1, \dots, x_n) \Big) dx_1 \dots dx_n ds \\
&= \langle\langle k_0, G\rangle\rangle + \int_0^t \langle\langle k_0, \widehat{L} G_s\rangle\rangle ds.
\end{aligned} \tag{48}$$

We remark that the function G depends implicitly on t because G belongs to the space $\mathcal{I}_{\alpha,C_t^*}$. But for a function

$$G \in \bigcap_{0<t<T-\epsilon} \mathcal{I}_{\alpha,C_t^*} = \mathcal{I}_{\alpha,C_{T-\epsilon}^*}, \quad \epsilon > 0,$$

the relation (48) holds uniformly in t and we can differentiate:

$$\frac{d}{dt} \langle\langle k_t, G\rangle\rangle = \langle\langle k_0, \widehat{L} G_t\rangle\rangle.$$

The latter fact means that k_t is a weak solution of

$$\begin{cases} \frac{d}{dt} k_t = L^{\triangle} k_t \\ k_t|_{t=0} = k_0. \end{cases} \tag{49}$$

7 Invariant Distribution

The considerations in this Section are only valid for the symmetric case, cf. (6). Let us start the dynamics in a Poisson state $\mu_0(d\gamma) = \pi_z(d\gamma)$ with activity $z > 0$. The initial correlation function $k_0 = k_{\pi_z}$ of π_z is given by

$$k_0(\eta) = z^{|\eta|}, \quad \eta \in \Gamma_0.$$

Let us fix $t \in [0, T)$ and $G \in \mathcal{I}_{\alpha,C_t^*}$. As in (48), we get:

$$\langle\langle k_t, G\rangle\rangle = \langle\langle k_0, G\rangle\rangle + \int_0^t \int_{\Gamma_0} \lambda(d\eta) z^{|\eta|} \big(\widehat{L} G_s\big)(\eta). \tag{50}$$

Using the symmetry of $c_{x,y}(\tilde{x})$ in x and y and Milnos Lemma, we conclude

$$\int_{\Gamma_0} \lambda(d\eta) z^{|\eta|} \big(\widehat{L} G_s\big)(\eta) = 0. \tag{51}$$

Combining (50) and (51) yields:

$$\langle\langle k_t, G\rangle\rangle = \langle\langle k_0, G\rangle\rangle, \quad G \in \mathcal{I}_{\alpha, C_t^*}.$$

Thus, $k_t = k_0$, $t \in [0, T]$. This shows that, the time evolution of the jump dynamics leaves the Poisson measure π_z invariant.

Commonly, the measure π_z refers to the free case, i.e. a dynamic without interaction. In a physical context, the measure π_z describes an ideal gas in equilibrium with activity $z > 0$ which depends on the chemical potential. Since the jump dynamics still has the state π_z as invariant measure, we have an interacting-particles system with a minimal interaction.

8 Vlasov Scaling

This section is devoted to the study of the Vlasov scaling limit of the dynamics. We consider the model of jumping particles with rates (3), but we replace the function κ by $\epsilon\kappa$. Here, $\epsilon > 0$ is a scaling parameter which describes the strength of the interaction. We are interested in the behavior of the system when ϵ tends to 0. For small ϵ, the interaction becomes weaker. To compensate this, we perform an additional scaling, which assures that the correlations between the particles become stronger.

A detailed description of the Vlasov scaling for stochastic dynamics of continuous system is given in [9]. Below some informal considerations to motivate the scaling: assume that we are interested in the time evolution of an initial Poison state π_ϱ w.r.t. a (inhomogeneous) density ϱ, i.e. we have to study the evolution of the corresponding correlation function $k_{\pi_\varrho}(\eta) = e_\lambda(\varrho, \eta)$. Now, let $L_\epsilon^\triangle$ be the operator given by (15) with κ replaced by $\epsilon\kappa$. Set $P_{t,\epsilon}^\triangle$ to be the corresponding evolution operator of correlation functions defined by (47). In order to make the correlations stronger, we replace the density ϱ by $\epsilon^{-1}\varrho$ (i.e. the system becomes more dense). It holds $e_\lambda(\epsilon^{-1}\varrho, \eta) = \big(R_\epsilon e_\lambda(\varrho, \cdot)\big)(\eta)$ where $\big(R_\epsilon k\big)(\eta) := \epsilon^{-|\eta|} k(\eta)$. Now, we let this dense system with weak interaction evolve, i.e. we consider $\big(e^{tL_\epsilon^\triangle} R_\epsilon\big) e_\lambda(\varrho, \cdot)$, where $e^{tL_\epsilon^\triangle}$ is an heuristic notation for $P_{t,\epsilon}^\triangle$. Afterwards we reverse the effect of increasing the density, i.e. we consider

$$\big(R_\epsilon^{-1} e^{tL_\epsilon^\triangle} R_\epsilon\big) e_\lambda(\varrho, \cdot) = \big(e^{tR_\epsilon^{-1} L_\epsilon^\triangle R_\epsilon}\big) e_\lambda(\varrho, \cdot).$$

Motivated by these heuristic calculations, we introduce an operator

$$L_{\epsilon,\mathrm{ren}}^\triangle := R_\epsilon^{-1} L_\epsilon^\triangle R_\epsilon. \tag{52}$$

It describes (for small ϵ) a dense and weakly interacting system. Clearly, on quasi-observables, we have to consider the operator

$$\widehat{L}_{\epsilon,\mathrm{ren}} := R_\epsilon \widehat{L} R_\epsilon^{-1}.$$

Remark 3 *As mentioned in the introduction, the Vlasov scaling limit coincides with the Lebowitz-Penrose scaling limit. Let $\tilde{L}_\epsilon$ be the generator $L^\triangle$ with a and κ replaced by $\epsilon^d a(\epsilon\cdot)$ and $\epsilon^d\kappa(\epsilon\cdot)$, respectively. For a function k, we define an operator $S_\epsilon k$ by*

$$(S_\epsilon k)^{(n)}(x_1,\ldots,x_n) := k^{(n)}(\epsilon x_1,\ldots,\epsilon x_n).$$

The rescaled dynamics according to the Lebowitz-Penrose scaling (cf. [36]) is described by the generator $L_{\epsilon,LP}$, which is defined by

$$L_{\epsilon,LP} = S_\epsilon^{-1}\tilde{L}_\epsilon S_\epsilon.$$

An easy computation shows that $L_{\epsilon,LP} = L^\triangle_{\epsilon^d,ren}$.

In the next section we analyze the Vlasov scaling limit on quasi-observables, later we transport this scaling to correlation functions.

9 Scaling on Quasi-Observables

We observe that the the components of the operator $\widehat{L}_{\epsilon,\mathrm{ren}}$ are given by

$$(L_{\epsilon,\mathrm{ren}}G)^{(n)} = D_\epsilon^{(n)}G^{(n)} + R^{(n-1)}G^{(n-1)} \tag{53}$$

where $D_\epsilon^{(n)}$ is given by

$$\begin{aligned}
&(D^{(n)}G^{(n)})(\eta)\\
&= \sum_{x\in\eta}\int_{\mathbb{R}^d} dy\ a(x-y)(G^{(n)}(\eta\backslash x\cup y) - G^{(n)}(\eta))\\
&\quad + \epsilon\sum_{x\in\eta}\sum_{\tilde{x}\in\eta\backslash x}\int_{\mathbb{R}^d} dy\ a(x-y)c_{x,y}(\tilde{x})(G^{(n)}(\eta\backslash x\cup y) - G^{(n)}(\eta)),
\end{aligned}$$

i.e. $D_\epsilon^{(n)}$ is given by (18) but κ replaced by $\epsilon\kappa$. We also introduce an operator $\widehat{L}_V$ by

$$\begin{aligned}
(\widehat{L}_V G)(\eta) = &\sum_{x\in\eta}\int_{\mathbb{R}^d} dya(x-y)(G(\eta\backslash x\cup y) - G(\eta))\\
&+ \sum_{x\in\eta}\sum_{\tilde{x}\in\eta\backslash x}\int_{\mathbb{R}^d} dya(x-y)c_{x,y}(\tilde{x})(G(\eta\backslash\{x,\tilde{x}\}\cup y) - G(\eta\backslash\tilde{x})).
\end{aligned}$$

Clearly, the components of L_V are given by

$$(L_V G)^{(n)} = L_0^{(n)}G^{(n)} + R^{(n-1)}G^{(n-1)}$$

where L_0 is the generator of a free jump dynamics w.r.t. the kernel a (cf. [20]). The operator $\widehat{L}_V$ is the componentwise limit of $\widehat{L}_{\epsilon,\mathrm{ren}}$ for $\epsilon\to 0$, i.e. the following holds:

$$\left(\widehat{L}_{\epsilon,\mathrm{ren}}G\right)^{(n)} \to \left(\widehat{L}_V G\right)^{(n)} \quad \text{in } X_n \text{ for } \epsilon\to 0,\ G\in X_n. \tag{54}$$

Since both $L_0^{(n)}$ and $D_\epsilon^{(n)}$ induce contraction semigroups in X_n, we can solve the equations

$$\begin{cases} \frac{d}{dt} G_{t,\epsilon} = \widehat{L}_{\epsilon,\mathrm{ren}} G_{t,\epsilon} \\ G_{t,\epsilon}|_{t=0} = G_0. \end{cases} \tag{55}$$

respectively

$$\begin{cases} \frac{d}{dt} G_{t,V} = L_V G_{t,V} \\ G_{t,V}|_{t=0} = G_0. \end{cases} \tag{56}$$

The statements of Theorem 1 also holds for the above two evolutionary problems. We remark that for $0 < \epsilon < 1$ and $G_0 \in \mathcal{I}_{\alpha,C}$ holds $G_{t,\epsilon} \in \mathcal{I}_{\alpha,C_{t,\epsilon}}$ with

$$C_{t,\epsilon} = \frac{C}{1 + \frac{\epsilon BCt}{\alpha}}$$

and $G_{t,V} \in \mathcal{I}_{\alpha,C_t}$ with

$$C_t = \frac{C}{1 + \frac{BCt}{\alpha}}.$$

Since $\mathcal{I}_{\alpha,C_{t,\epsilon}} \subset \mathcal{I}_{\alpha,C_t}$, $G_{t,\epsilon}$ and $G_{t,V}$ belong to the same space.

Now we can state the following

Theorem 3 *Let $G_0 \in \mathcal{I}_{\alpha,C}$ and consider the solutions $(G_{t,\epsilon})_{t>0}$, $(G_{t,V})_{t>0}$ of* (55) *resp.* (56) *with initial condition G_0. Then, it holds for all $n \in \mathbb{N}, t > 0$:*

$$\lim_{\epsilon\to 0} G_{t,\epsilon}^{(n)} = G_{t,V}^{(n)} \quad \textit{in} \quad X_n. \tag{57}$$

Proof: Using the representation of $G_{t,\epsilon}^{(n)}, G_{t,V}^{(n)}$ by the recurrent relation, we conclude

$$\begin{aligned}
&\|G_{t,\epsilon}^{(n)} - G_{t,V}^{(n)}\|_{X_n} \\
&\le \|e^{tD_\epsilon^{(n)}} G_0^{(n)} - e^{tL_0^{(n)}} G_0^{(n)}\|_{X_n} \\
&\quad + \|\int_0^t e^{(t-s)D_\epsilon^{(n)}} R^{(n-1)} G_{s,\epsilon}^{(n-1)} - e^{(t-s)L_0^{(n)}} R^{(n-1)} G_{s,V}^{(n-1)}\, ds\|_{X_n} \\
&\le \|e^{tD_\epsilon^{(n)}} G_0^{(n)} - e^{tL_0^{(n)}} G_0^{(n)}\|_{X_n} \\
&\quad + \|\int_0^t e^{(t-s)D_\epsilon^{(n)}} R^{(n-1)} (G_{s,\epsilon}^{(n-1)} - G_{s,V}^{(n-1)})\, ds\|_{X_n} \\
&\quad + \|\int_0^t e^{(t-s)D_\epsilon^{(n)}} R^{(n-1)} G_{s,V}^{(n-1)} - e^{(t-s)L_0^{(n)}} R^{(n-1)} G_{s,V}^{(n-1)}\, ds\|_{X_n}.
\end{aligned} \tag{58}$$

Since $\lim_{\epsilon\to 0} D_\epsilon^{(n)} G_0^{(n)} = L_0^{(n)} G_0^{(n)}$ in X_n and since $L_0^{(n)}$ is the generator of a contraction semigroup in X_n, it follows (see [35])

$$\lim_{\epsilon\to 0} e^{tD_\epsilon^{(n)}} G^{(n)} = e^{tL_0^{(n)}} G^{(n)} \quad \text{in } X_n.$$

By the same arguments we get

$$\lim_{\epsilon\to 0} e^{(t-s)D_\epsilon^{(n)}} R^{(n-1)} G_{s,V}^{(n-1)} = e^{(t-s)L_0^{(n)}} R^{(n-1)} G_{s,V}^{(n-1)} \quad \text{in } X_n. \tag{59}$$

Further, using the bound

$$
\begin{aligned}
&\|e^{(t-s)D_\epsilon^{(n)}} R^{(n-1)} G_{s,V}^{(n-1)} - e^{(t-s)L_0^{(n)}} R^{(n-1)} G_{s,V}^{(n-1)} \|_{X_n} \\
&\leq 2\|R^{(n-1)} G_{s,V}^{(n-1)}\|_{X_n} \\
&\leq 2Bn(n-1)\|G_{s,V}^{(n-1)}\|_{X_{n-1}} \\
&\leq 2Bn(n-1)\left(\frac{\alpha}{C_s}\right)^{n-1} \|G_0\|_{\mathcal{I}_{\alpha,C}} \in L^1([0,t],\, ds)
\end{aligned}
\tag{60}
$$

and dominated convergence, we obtain

$$
\lim_{\epsilon\to 0} \| \int_0^t e^{(t-s)D_\epsilon^{(n)}} R^{(n-1)} G_{s,V}^{(n-1)} - e^{(t-s)L_0^{(n)}} R^{(n-1)} G_{s,V}^{(n-1)}\, ds\|_{X_n} = 0. \tag{61}
$$

Similar reasoning yields

$$
\lim_{\epsilon\to 0} \| \int_0^t e^{(t-s)D_\epsilon^{(n)}} W^{(n-1)} (G_{s,\epsilon}^{(n-1)} - G_{s,V}^{(n-1)})\, ds\|_{X_n} = 0, \tag{62}
$$

if we suppose that,

$$
\lim_{\epsilon\to 0} \|G_{t,\epsilon}^{(n-1)} - G_{t,V}^{(n-1)}\|_{X_{n-1}} = 0
$$

holds. Thus, the statement follows by induction. $\square$

10 Scaling Limit for Correlation Functions

Now we can investigate the Vlasov scaling limit for correlation functions. We consider the operator $L^{\triangle}_{\epsilon,\mathrm{ren}}$ (see (52)). The limiting operator $L^{\triangle}_V := \lim_{\epsilon\to 0} L^{\triangle}_{\epsilon,\mathrm{ren}}$ is given by

$$
\begin{aligned}
\big(L_V k\big)(\eta) = &\sum_{x\in\eta} \int_{\mathbb{R}^d} d\tilde{x} \int_{\mathbb{R}^d} dy\; k(\eta\backslash x \cup \{y,\tilde{x}\}) a(x-y) c_{x,y}(\tilde{x}) \\
&- \sum_{x\in\eta} \int_{\mathbb{R}^d} d\tilde{x}\; k(\eta\cup\tilde{x}) \int_{\mathbb{R}^d} dy\; a(x-y) c_{x,y}(\tilde{x}) \\
&+ \sum_{x\in\eta} \int_{\mathbb{R}^d} dy\; a(x-y)\big(k(\eta\backslash x\cup y) - k(\eta)\big).
\end{aligned}
$$

Now, let $k_0 \in \mathcal{K}_{C_0}$. Proceeding analog as in the previous sections (Theorem 2), we obtain evolutions $k_{t,\epsilon}$ resp. $r_t \in \mathcal{I}^*_{\alpha,C_t^*}$ of k_0 which are dual to the evolutions $G_{t,\epsilon}$ resp. $G_{t,V}$ for $t \in [0,T)$. We stress that the time interval is independent of ϵ.

Theorem 4 *Let $k_0 \in \mathcal{K}_{C_0}$ for some $C_0 > 0$.Then, for fixed $t \in [0,T)$, $k_{t,\epsilon}$ converges weakly to r_t for $\epsilon \to 0$ i.e.*

$$
\lim_{\epsilon\to 0} \langle\langle k_{t,\epsilon}, G\rangle\rangle = \langle\langle r_t, G\rangle\rangle \tag{63}
$$

for all $G_0 \in \mathcal{I}_{\alpha,C_t^*}$

Proof: Let $G \in \mathcal{I}_{\alpha,C_t^*}$. Then, according to Theorem 1, $G_{t,\epsilon}, G_{t,V} \in \mathcal{I}_{\alpha,C_0}$ (cf. also the proof of Theorem 2). It holds:

$$\begin{aligned}
&|\langle\langle k_{t,\epsilon}, G\rangle\rangle - \langle\langle r_t, G\rangle\rangle| \\
&= |\langle\langle k_0, G_{t,\epsilon}\rangle\rangle - \langle\langle k_0, G_{t,V}\rangle\rangle \\
&\leq \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\mathbb{R}^{nd}} |k_0^{(n)}||G_{t,\epsilon}^{(n)} - G_{t,V}^{(n)}| d^n x \\
&\leq \sum_{n=0}^{\infty} \frac{C_0^n}{n!} \int_{\mathbb{R}^{nd}} |G_{t,\epsilon}^{(n)} - G_{t,V}^{(n)}| d^n x \\
&= \sum_{n=0}^{\infty} \frac{C_0^n}{n!} \|G_{t,\epsilon}^{(n)} - G_{t,V}^{(n)}\|_{X_n}.
\end{aligned} \tag{64}$$

Using that $G_{t,\epsilon}, G_{t,V} \in \mathcal{I}_{\alpha,C_0}$ we obtain

$$\frac{C_0^n}{n!} \|G_{t,\epsilon}^{(n)} - G_{t,V}^{(n)}\|_{X_n} \leq 2\|G\|_{\mathcal{I}_{\alpha,C_t^*}}, \left(\alpha \frac{C_0}{C_0}\right)^n. \tag{65}$$

Since $\alpha < 1$, the right side of the above estimate is summable. This completes the proof. $\square$

11 The Vlasov Equation

Now we consider a coherent state $k_0 = e_\lambda(\varrho, \cdot)$ as initial correlation function. Here, ϱ is a (bounded) one particle density. We choose $C_0 > 0$, such that, $\varrho \leq C_0$ holds. According to Theorem 2, we obtain evolutions $k_{t,\epsilon}$ and r_t of $k_0 = e_\lambda(\varrho, \cdot)$ under the dynamics described by $\widehat{L}_{\epsilon,\mathrm{ren}}$ resp. L_V and $r_t = \lim_{\epsilon\to 0} k_{t,\epsilon}$ is a weak solution of the equation

$$\begin{cases} \frac{d}{dt} r_t = L_V^{\triangle} r_t \\ r_t|_{t=0} = e_\lambda(\varrho, \cdot). \end{cases} \tag{66}$$

This Cauchy-problem describes the time evolution of a virtual interacting particle system and has the following chaos preservation property: if ϱ_t is a solution of the non-local equation

$$\begin{cases} \frac{d}{dt} \varrho_t = v(\varrho_t) \\ \varrho_t|_{t=0} = \varrho, \end{cases}$$

where $v(\varrho_t)$ is in the case (4) equals to

$$v(\varrho_t) = \big(\kappa * \varrho_t\big)(x)\big(a * \varrho_t\big)(x) - \varrho_t(x)\big(a * \kappa * \varrho_t\big)(x) + (\varrho_t * a)(x) - \langle a\rangle \varrho_t(x)$$
$$v(\varrho_t) = \Big(\big(\kappa * \varrho_t)\varrho_t\big) * a\Big)(x) - \langle a\rangle \varrho_t(x)\big(\kappa * \varrho_t\big)(x) + (\varrho_t * a)(x) - \langle a\rangle \varrho_t(x),$$

while in the case (5), it is equals to

$$v(\varrho_t) = \Big(\big(\kappa * \varrho_t)\varrho_t\big) * a\Big)(x) - \langle a\rangle \varrho_t(x)\big(\kappa * \varrho_t\big)(x) + (\varrho_t * a)(x) - \langle a\rangle \varrho_t(x),$$

and in the symmetric case (cf. (6)), $v(\varrho_t)$ is given by

$$\begin{aligned} v(\varrho_t) = \frac{1}{2}\Big(\big((\kappa * \varrho_t)\varrho_t \big) * a \Big)(x) + \frac{1}{2}(\kappa * \varrho_t)(x)\big(a * \varrho_t\big)(x) + (\varrho_t * a)(x) \\ - \frac{1}{2}\varrho_t(x)\big(a * \kappa * \varrho_t\big)(x) - \frac{1}{2}\langle a\rangle \varrho_t(x)\big(\kappa * \varrho_t\big)(x) - \langle a\rangle \varrho_t(x), \end{aligned}$$

then $r_t = e_\lambda(\varrho_t, \cdot)$ is a solution of (66). This can be seen by using the formula

$$\frac{\partial}{\partial t} e_\lambda(\varrho_t, \eta) = \sum_{x\in\eta} e_\lambda(\varrho_t, \eta\backslash x) \frac{\partial}{\partial t}\varrho_t(x),$$

see also [9]. Thus, we have derived a mesoscopic (deterministic) kinetic equation from the microscopic (stochastic) particle evolution. The infinite linear chain of equations (66) reduces to one non-linear equation for ϱ_t.

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