A note on stochastic semilinear equations and their associated Fokker-Planck equations *

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Abstract

The main purpose of this paper is to prove existence and uniqueness of (probabilistically weak and strong) solutions to stochastic differential equations (SDE) on Hilbert spaces under a new approximation condition on the drift, recently proposed in [BDR10] to solve Fokker-Planck equations (FPE), extended in this paper to a considerably larger class of drifts. As a consequence we prove existence of martingale solutions to the SDE (whose time marginals then solve the corresponding FPE). Applications include stochastic semilinear partial differential equations with white noise and a non-linear drift part which is the sum of a Burgers-type part and a reaction diffusion part. The main novelty is that the latter is no longer assumed to be of at most linear, but of at most polynomial growth. This case so far had not been covered by the existing literature. We also give a direct and more analytic proof for existence of solutions to the corresponding FPE, extending the technique from [BDR10] to our more general framework, which in turn requires to work on a suitable Gelfand triple rather than just the Hilbert state space.

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1 Introduction

Let H be a separable real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and corresponding norm $|\cdot|$. $L(H)$ denotes the set of all bounded linear operators on H, $\mathcal{B}(H)$ its Borel σ-algebra.

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Consider the following type of non-autonomous stochastic differential equations on H and time interval $[0,T]$:

$$\begin{cases} dX(t) = (AX(t) + F(t, X(t)))dt + \sqrt{G}dW(t), \\ X(s) = x \in H, t \geq s. \end{cases} \tag{1.1}$$

Here $W(t), t \geq 0$, is a cylindrical Wiener process on H defined on a stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$, G is a linear symmetric positive definite operator in H, $D(F) \in \mathcal{B}([0,T] \times H)$, $F : D(F) \subset [0,T] \times H \to H$ is a Borel measurable map, and $A : D(A) \subset H \to H$ is the infinitesimal generator of a C_0-semigroup $e^{tA}, t \geq 0$, on H.

Even in this case, where the noise is additive, it is a fundamental question in the theory of stochastic differential equations (SDE) in infinite dimensional state spaces with numerous applications to concrete (non-linear) stochastic partial differential equations (SPDE), whether there exists a (unique) weak or strong (in the probabilistic sense) solution to SDE (1.1). In [BDR10], for a large class of semigroup generators A and in the fully elliptic case, i.e., where G has an inverse $G^{-1} \in L(H)$ (in particular including the case of space-time white noise), a quite general approximation condition on F was identified, which implies that (at least) the corresponding Fokker-Planck equation (FPE) has a solution, which is also unique under some L^2-conditions on F (see [BDR11] and the recent paper [BDRS13]). The purpose of this paper is to generalize this result under the same approximation condition on F (see Hypothesis 2.2 (i), (ii)) in two ways:

(a) We prove that (1.1) has indeed a weak (=martingale) solution (in the sense of Stroock-Varadhan). In particular, its time marginals solve the corresponding FPE.

(b) We prove **(a)** in a more general framework, namely allowing (as is usual in the variational approach to SDE on Hilbert spaces, see e.g. [G98], [GR00] and also [PR07]) that F takes values in a larger space, more precisely in $D((-A)^{1/2})^*$, assuming (as in [BDR10]) that A is negative definite and self-adjoint. In short: we shall work in a Gelfand triple.

This is done in Section 2 of this paper and the corresponding main result is summarized in Theorem 2.3 there. In order to include degenerate cases, where e.g. $\mathrm{TrG} < \infty$, we assume, instead of the requirement $G^{-1} \in L(H)$ from [BDR10], that the approximating equations have martingale solutions (see Hypothesis 2.2 (iii) below), which can be easily checked in many applications.

We would like to stress, however, that, though **(b)** above may hint in this direction, our result is not at all covered by the variational approach to SDE on Hilbert spaces (see e.g. [PR07]), since, first, there is no monotonicity condition on F and, second, the noise coefficient operator G is not assumed to have finite trace.

We would also like to stress that in our main application (see Section 4 below) by a standard result from the seminal paper [MR99] we can also prove uniqueness of the martingale solutions. This, however, by principle cannot generally imply uniqueness of solutions to the corresponding FPE, because the latter might have solutions which are not the time marginals of a martingale solution. However, it is well-known that uniqueness for FPE implies uniqueness of martingale problems (see [SV79]). Therefore, as FPE are concerned our uniqueness results in this paper are much weaker and, in fact, far from those in [BDR11] and [BDRS13] for FPE.

Our more general framework, indicated under **(b)** above, considerably widens the range of applications in comparison with those in [BDR10].

Let us briefly describe a class of examples, which we present in detail in Section 4 of this paper and which have been studied intensively in the literature, however, under more stringent assumptions (on the function f in (1.2) below).

Consider the stochastic semilinear partial differential equation (SPDE)

$$dX(t) = (\frac{\partial^2}{\partial \xi^2}X(t) + f(t, X(t)) + \frac{\partial}{\partial \xi}g(t, X(t)))dt + \sqrt{G}dW(t), \tag{1.2}$$

on $H := L^2(0,1)$ with Dirichlet boundary condition

$$X(t,0) = X(t,1) = 0, t \in [0,T],$$

and initial condition

$$X(0) = x \in H,$$

where $f(\xi,t,z), g(\xi,t,z)$ are Borel measurable functions of $(\xi,t,z) \in [0,1] \times \mathbb{R}_+ \times \mathbb{R}$, W is a cylindrical Wiener process on H and G is a linear symmetric positive definite operator in H.

This kind of stochastic partial differential equations has been studied intensively. If $f = 0$ and $g = \frac{1}{2}r^2$, the above equation is just the stochastic Burgers equation and has been investigated in many papers (see e.g. [DDT94], [DZ96] and the references therein). When $g = 0$ then the above equation is a stochastic reaction-diffusion equation which has also attracted a lot of attention (see e.g. [DZ92], [D04], [BDR10] and the references therein). In the general case, this kind of equations has been studied e.g. in [G98], [GR00], where, however, f was assumed to be of at most linear growth. We stress that the linear growth of f cannot be dropped in [G98], [GR00], since the approximation technique used there requires this assumption.

As an application of our main result (Theorem 2.3 below) we obtain that (1.2) has a martingale solution which under a natural integrability condition is unique (see Theorem 4.2), where we assume the usual conditions on the "Burgers-part" g of the drift, but in contrast to [G98], [G00] we can allow f to be of polynomial growth. We, however, pay a price for considering such more general f, because we do not recover all results from [G98], [G00] where e.g. (1.2) with multiplicative noise is included under certain assumptions, g is allowed to be of polynomial growth in [G00] and under local Lipschitz assumptions on f (and g) also existence and uniqueness of strong solutions is shown. If, however, we assume one sided local Lipschitz assumption on f (see (4.14) below), we also get existence and uniqueness of strong solutions under only polynomial growth conditions on f (see Theorem 4.7 below) by proving pathwise uniqueness and applying the Yamada-Watanable Theorem. We also stress that our condition for f is more general than the one imposed in the corresponding applications in [BDR10] (see condition (f2) in Section 4), which allows us to take more general f (see Example 4.0).

At least if $\mathrm{TrG} < \infty$, we can also apply our framework to a lot of other stochastic semilinear equations, as e.g. the stochastic 2D Navier-Stokes equation (see Remark 4.9). Since in this case there are many known existence results (cf. [GRZ09] and the references therein) based on Itô's formula and the Burkholder-Davis-Gundy inequality to obtain the estimates required for tightness of the distributions of the approximations, we do not give details here, but concentrate on (1.2) in our applications.

Though, as mentioned above, our Theorem 2.3 implies the existence of solutions to the FPE associated to (1.1), we nevertheless also give an alternative direct proof for the latter which is more analytic in nature and a generalization of the corresponding one in [BDR10]. We think that this proof is of sufficient independent interest. Therefore, we include it here, stressing the modifications needed in our (in comparison with that in [BDR10]) more general framework.

We mention here that recently, there has been quite an interest in Fokker-Planck equations with irregular coefficients in finite dimensions (see e.g. [A04], [DPL89], [F08], [BDR08a] and the references therein). In [BDR08b], [BDR09] and [BDR10], Bogachev, Da Prato and the authors have started the study of Fokker-Planck equations also in infinite dimensions, more precisely, on Hilbert spaces. Let us briefly present the formulation of the FPE corresponding to (1.1) in our framework.

The Kolmogorov operator L_0 corresponding to (1.1) reads as follows:

$$L_0u(t,x) := D_tu(t,x) + \frac{1}{2}\mathrm{Tr}[\mathrm{GD^2u(t,x)}] + \langle \mathrm{x, A^*Du(t,x)} \rangle + {}_{\mathrm{V^*}}\langle \mathrm{F(t,x), Du(t,x)} \rangle_{\mathrm{V}},\ \ \mathrm{(t,x) \in D(F)},$$

where D_t denotes the derivative in time and D, D^2 denote the first- and second-order Frechet derivatives in space, i.e., in $x \in H$, respectively. Furthermore, $V := D((-A)^{1/2})$, V^* is its dual and ${}_{V^*}\langle \cdot, \cdot \rangle_V$ denotes their dualization, assuming again that A is negative definite and self-adjoint. The operator L_0 is defined on the space $D(L_0) := \mathcal{E}_A([0,T] \times H)$, defined to be the linear span of all real and imaginary parts of all functions $u_{\phi,h}$ of the form

$$u_{\phi,h}(t,x) = \phi(t)e^{i\langle x, h(t)\rangle}, t \in [0,T], x \in H,$$

where $\phi \in C^1([0,T])$, $\phi(T) = 0$, $h \in C^1([0,T]; D(A^*))$ and A^* denotes the adjoint of A.

For a fixed initial time $s \in [0,T]$ the Fokker-Planck equation is an equation for measures $\mu(dt,dx)$ on $[s,T] \times H$ of the type

$$\mu(dt,dx) = \mu_t(dx)dt,$$

with $\mu_t \in \mathcal{P}(H)$ for all $t \in [s,T]$, and $t \mapsto \mu_t(A)$ measurable on $[s,T]$ for all $A \in \mathcal{B}(H)$, i.e. $\mu_t(dx), t \in [s,T]$, is a probability kernel from $([s,T], \mathcal{B}([s,T]))$ to $(H, \mathcal{B}(H))$. Then the FPE corresponding to (1.1) for an initial condition $\zeta \in \mathcal{P}(H)$ reads as follows: $\forall u \in D(L_0)$

$$\int_H u(t,y)\mu_t(dy) = \int_H u(s,y)\zeta(dy) + \int_s^t ds' \int_H L_0u(s',y)\mu_{s'}(dy), \text{ for } dt - a.e. t \in [s,T], \quad (1.3)$$

where the dt-zero set may depend on u.

In Section 3 of this paper, we prove directly the existence of solutions to FPE (1.3) within the same framework as in Section 2, which generalizes the results in [BDR10] . In Section 4 as an application we prove the existence of solutions for the FPE associated with concrete SPDE of type (1.2), i.e. allowing polynomially growing nonlinearities for the reaction-diffusion part f and Burgers type nonlinearities g at the same time (see Theorem 4.3 below), which can not be handled within the framework of [BDR10].

Finally, we recall that our work covers the case $G^{-1} \in L(H)$, i.e. the case of full (including white) noise. If $\mathrm{TrG} < \infty$, there are many other known existence results for FPE (cf. [BDR08b, BDR09]), based on the method of constructing Lyapunov functions with weakly compact level sets for the Kolmogorov operator L_0. These techniques so far could, however, not be used when $\mathrm{TrG} = \infty$.

2 Existence of martingale solutions

Let us start with formulating our assumptions on the coefficients of SDE (1.1).

Hypothesis 2.1 (i) A is self-adjoint such that there exists $\omega \in (-\infty, 0)$ such that $\langle Ax, x\rangle \leq \omega|x|^2, x \in D(A)$, and A^{-1} is compact on H.

(ii) $G \in L(H)$ is symmetric, nonnegative.

(iii) There exists $\delta, \delta_1 > 0$ such that

$$\int_0^T \mathrm{Tr}[(-A)^\delta e^{rA} G(-A)^\delta e^{rA}]dr < \infty, \quad \int_0^1 r^{-2\delta_1}\mathrm{Tr}[e^{rA}Ge^{rA}]dr < \infty.$$

Under Hypothesis 2.1, there exists an orthonormal basis $\{e_k\}_{k\geq 0}$ for H consisting of eigenfunctions of $-A$ such that the associated sequence of eigenvalues $\{\lambda_k\}$ form an increasing unbounded sequence. It is well known (see [D04, Theorem 2.9]) that under Hypothesis 2.1 (iii) the stochastic convolution

$$W_A(t) = \int_0^t e^{(t-r)A}\sqrt{G}dW(r), \quad t \geq 0,$$

is a well-defined continuous process in H with values in $D((-A)^\delta)$ and

$$\sup_{t\in[0,T]} E|(-A)^\delta W_A(t)|^2 \leq \int_0^T \mathrm{Tr}[(-A)^\delta e^{rA} G(-A)^\delta e^{rA}]dr < \infty. \tag{2.1}$$

Remark If $(-A)^{2\delta-1}$ is of trace-class for some $\delta > 0$ and $G \in L(H)$, Hypothesis 2.1 (iii) is obviously satisfied. We would like to point out here that there is a misprint in Hypothesis 2.1 (iii) in [BDR10], where $(-A)^{-2\delta}$ should be replaced by $(-A)^{2\delta-1}$. Likewise in the right hand side of inequality (2.1) in [BDR10].

We weaken resp. modify Hypotheses 2.2, 2.3 in [BDR10] as follows: let $V := D((-A)^{1/2})$ and consider the following Gelfand triple:

$$D(A) \subset V \subset H \cong H^* \subset V^* \subset D(A)^*,$$

where V^* and $D(A)^*$ are the dual of $V, D(A)$ respectively and ${}_{D(A)^*}\langle\cdot,\cdot\rangle_{D(A)} = {}_{V^*}\langle\cdot,\cdot\rangle_V = \langle\cdot,\cdot\rangle$ if restricted to $H \times D(A)$. We have the following formulas for the norm in V, V^*,

$$|\cdot|_V^2 = \sum_k \lambda_k|\langle\cdot, e_k\rangle|^2, \quad |\cdot|_{V^*}^2 = \sum_k \lambda_k^{-1}|\langle\cdot, e_k\rangle|^2.$$

Furthermore, we relax the assumptions on F in (1.1) to be just V^*-valued. More precisely, let $F : D(F) \subset [0,T] \times H \to V^*$ be Borel measurable. Then the Kolmogorov operator is given as follows

$$L_0 u(t,x) := D_t u(t,x) + \frac{1}{2}\mathrm{Tr}[GD^2u(t,x)] + \langle x, ADu(t,x)\rangle + {}_{V^*}\langle F(t,x), Du(t,x)\rangle_V,$$

for $u \in D(L_0)$. Below we fix $s \in [0,T]$ as starting time.

Hypothesis 2.2 There exist measurable maps $F_\alpha : [s,T] \times H \to D(A)^*, \alpha \in (0,1]$, $K > 0$ and a lower semicontinuous function $J : [s,t] \times H \to [1,\infty]$, such that the following four conditions are satisfied for all $\alpha \in (0,1]$:

(i) for all $(t,x) \in D(F)$ and all $h \in D(A)$

$$F_\alpha(t,x) \in V^*, \quad |F_\alpha(t,x)|_{V^*} \le J(t,x) < \infty,$$

$$|{}_{V^*}\langle F(t,x) - F_\alpha(t,x), h\rangle_V| \le \alpha c(h) J^2(t,x),$$

for some constant $c(h) > 0$.

(ii) $(t,x) \mapsto {}_{D(A)^*}\langle F_\alpha(t,x), h\rangle_{D(A)}$ is continuous on $[s,T] \times H, \forall h \in D(A), \alpha \in (0,1]$.

(iii) The following approximating stochastic equations for $\alpha \in (0,1]$

$$dX_\alpha(t) = [AX_\alpha(t) + F_\alpha(t, X_\alpha(t))]dt + \sqrt{G}dW(t), X_\alpha(s) = x \in H, \tag{2.2}$$

have a martingale solution which we denote by $X_\alpha(\cdot, s, x)$, i.e. there exists a stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[s,T]}, P)$, a cylindrical Wiener process W on H and a progressively measurable process $X_\alpha : [s,T] \times \Omega \to H$, such that for P-a.e. $\omega \in \Omega$ and $\phi \in D(A)$,

$$X_\alpha(\cdot,\omega) \in L^2([s,T]; H) \cap C([s,T]; D(A)^*),$$

$$\langle X_\alpha(t) - x, \phi\rangle = \int_s^t (\langle X_\alpha(\tau), A\phi\rangle + {}_{D(A)^*}\langle F_\alpha(\tau, X_\alpha(\tau)), \phi\rangle_{D(A)})d\tau + \int_s^t \langle \phi, \sqrt{G}dW(\tau)\rangle, \quad \forall t \in [s,T].$$

(iv)$|F|_{V^*} \le J$ on $[s,T] \times H$, where we set $|F|_{V^*} := +\infty$ on $[s,T] \times H \backslash D(F)$, and setting

$$P^\alpha_{s,t}\varphi(x) := E[\varphi(X_\alpha(t,s,x))], \quad 0 \le s < t \le T, \varphi \in \mathcal{B}_b(H),$$

we have

$$\int_s^t P^\alpha_{s,s'} J^2(s',\cdot)(x) ds' \le K \int_s^t J^2(s',x)ds', \forall x \in H, t \in [s,T], \alpha \in (0,1].$$

Remark (i) Since $J \equiv \infty$ on $[s,T] \times H \setminus D(F)$, the latter inequality obviously holds if it holds on $D(F)$. Therefore, if we can find a function which is a Lyapunov function for $P^\alpha_{s,t}$ uniformly in α i.e.

$$P^\alpha_{s,t} J^2(t,\cdot)(x) \le K J^2(t,x), \qquad \forall (t,x) \in D(F), t \in [s,T], \alpha \in (0,1],$$

Hypothesis 2.2 (iv) is satisfied.

(ii) If G has a bounded inverse and if the approximation in Hypothesis 2.2 can be chosen such that F_α are bounded measurable maps, then we can use Girsanov's theorem to obtain the existence of a martingale solution. For the case that TrG $< \infty$, we could choose $F_\alpha = P_{[\frac{1}{\alpha}]+1} F$, where P_n is the orthogonal projection onto the linear space spanned by the first n eigenvectors e_k. Then we can apply the results in [PR07, Chapter 4] to the equation

$$dX_\alpha(t) = [AX_\alpha(t) + F_\alpha(t, X_\alpha(t))]dt + GdW(t),$$

provided F_α satisfies the monotonicity assumptions specified there, and obtain the existence of a martingale solution required in Hypothesis 2.2 (iii).

Theorem 2.3 Assume Hypotheses 2.1, 2.2. Then for every $x \in B := \{x \in H : \int_s^T J^2(t,x)dt < \infty\}$, there exists a martingale solution to (1.1), i.e. there exists a stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[s,T]}, P)$, a cylindrical Wiener process W on H and a progressively measurable process $X : [s,T]\times\Omega \to H$, such that for P-a.e. $\omega \in \Omega$ and $\phi \in D(A)$,

$$X(\cdot,\omega) \in L^2([s,T];H) \cap C([s,T];D(A)^*),$$

and

$$\langle X(t) - x, \phi\rangle = \int_s^t (\langle X(\tau), A\phi\rangle + {}_{D(A)^*}\langle F(\tau, X(\tau)), \phi\rangle_{D(A)})d\tau + \int_s^t \langle \phi, \sqrt{G}dW(\tau)\rangle \quad \forall t \in [0,T].$$

Moreover, for $\delta_2 := \delta \wedge \frac{1}{2}$ with δ as in Hypothesis 2.1

$$E\int_s^T (J^2(s', X(s')) + |(-A)^{\delta_2}X(s')|^2 + |X(s')|^2)ds' \le C\int_s^T (J^2(s',x) + |x|^2)ds'.$$

Proof For simplicity we assume $s = 0$. For $\alpha \in (0,1]$, set $X_\alpha(t) := X_\alpha(t,0,x), x \in B$, and

$$Y_\alpha(t) := X_\alpha(t) - W_A(t), \quad t \ge 0.$$

Then for $\phi \in D(A)$,

$$\langle Y_\alpha(t) - x, \phi\rangle = \int_0^t (\langle Y_\alpha(s'), A\phi\rangle + {}_{D(A)^*}\langle F_\alpha(s', X_\alpha(s')), \phi\rangle_{D(A)})ds' \quad \forall t \in [0,T].$$

Choosing $\phi = e_k$ in the above equation, we obtain

$$\langle Y_\alpha(t), e_k\rangle^2 = \langle x, e_k\rangle^2 + 2\int_0^t \langle Y_\alpha(s'), e_k\rangle(\langle Y_\alpha(s'), Ae_k\rangle + {}_{D(A)^*}\langle F_\alpha(s', X_\alpha(s')), e_k\rangle_{D(A)})ds', \quad \forall t \in [0,T].$$

Then by the Cauchy-Schwarz inequality and, since $Ae_k = -\lambda_k e_k$, we have

$$\langle Y_\alpha(t), e_k\rangle^2 + \int_0^t \lambda_k \langle Y_\alpha(s'), e_k\rangle^2 ds' \le \langle x, e_k\rangle^2 + \int_0^t \lambda_k^{-1} |{}_{D(A)^*}\langle F_\alpha(s', X_\alpha(s')), e_k\rangle_{D(A)}|^2 ds'.$$

Summing over k we get

$$|Y_\alpha(t)|^2 + \int_0^t |(-A)^{1/2}Y_\alpha(s')|^2 ds' \le |x|^2 + \int_0^t |F_\alpha(s', X_\alpha(s'))|_{V^*}^2 ds',$$

where we set $|F_\alpha|_{V^*} := +\infty$ on $[0,T] \times H\backslash D(F)$. Taking expectation and applying Hypothesis 2.2 yield

$$E|Y_\alpha(t)|^2 + \int_0^t E|(-A)^{1/2}Y_\alpha(s')|^2 ds' \le |x|^2 + K\int_0^t J^2(s',x)ds', \quad t \ge 0. \tag{2.3}$$

Then we deduce that for any $\varepsilon > 0$ there exists $R_1 > 0$ such that

$$P(\int_0^T |(-A)^{1/2}Y_\alpha(s')|^2 ds' > R_1) < \varepsilon, \quad \forall \alpha \in (0,1].$$

Since by Hypothesis 2.2 we have

$$E\int_0^T |F_\alpha(s',X_\alpha(s'))|_{V^*}^2 ds' \le E\int_0^T J^2(s',X_\alpha(s'))ds' \le \int_0^T J^2(s',x)ds', \tag{2.4}$$

and

$$E\int_0^T |(-A)Y_\alpha(s')|_{V^*}^2 ds' = E\int_0^T |(-A)^{1/2}Y_\alpha(s')|^2 ds',$$

we deduce that for any $\varepsilon > 0$ there exists $R_2 > 0$ such that

$$P(\int_0^T \left|\frac{dY_\alpha}{dt}\right|_{V^*}^2 ds' > R_2) < \varepsilon \quad \forall \alpha \in (0,1].$$

Then by the compactness Theorems 2.1 and 2.2 in [FG95], the laws of $X_\alpha = Y_\alpha + W_A$ are tight in $L^2([0,T];H)\cap C([0,T];D(A)^*)$. Thus, by Skorokhod's representation theorem there exists a subsequences n_k and a sequence of random elements $\hat{X}_k, k=1,2,3,...$ in $L^2([0,T];H)\cap C([0,T];D(A)^*)$, on some probability space $(\hat{\Omega},\hat{\mathcal{F}},\hat{P})$, such that $\hat{X}_k$ converges almost surely in $L^2([0,T];H)\cap C([0,T];D(A)^*)$ to a random element $\hat{X}$ for $k\to\infty$ and the distributions of $\hat{X}_k$ and $X_{\frac{1}{n_k}}$ coincide. Then the second inequality in (2.4) holds for $\hat{X}_k$ and $\hat{X}$ by the lower semicontinuity of J. Define for $\phi \in D(A)$,

$$\hat{M}_k(\phi)(t) := \langle \hat{X}_k(t)-x,\phi\rangle - \int_0^t \langle \hat{X}_k(s'),A\phi\rangle ds' - \int_0^t {}_{D(A)^*}\langle F_{1/n_k}(s',\hat{X}_k(s')),\phi\rangle_{D(A)} ds'.$$

$\hat{M}_k(\phi)$ is a family of martingales with respect to the filtration

$$\mathcal{G}_t^k = \sigma(\hat{X}_k(r), r\le t).$$

For all $r \le t \in [0,T]$ and all bounded continuous functions φ on $L^2([0,r];H)\cap C([0,r];D(A)^*)$ we have

$$\hat{E}((\hat{M}_k(\phi)(t)-\hat{M}_k(\phi)(r))\varphi(\hat{X}_k|_{[0,r]}))=0,$$

and

$$\hat{E}[(\hat{M}_k(\phi)(t)^2-\hat{M}_k(\phi)(r)^2-\int_r^t |\sqrt{G}\phi|_H^2 ds')\varphi(\hat{X}_k|_{[0,r]})]=0.$$

By the Burkholder-Davis-Gundy inequality we have that for $1<p<\infty$ there exists $C_p\in(0,\infty)$ such that

$$\sup_k \hat{E}|\hat{M}_k(\phi)(t)|^{2p} \le C_p\hat{E}(\int_0^t |\sqrt{G}\phi|_H^2 dr)^p < \infty. \tag{2.5}$$

Now we prove the following estimate: for fixed $\eta > 0$

$$\hat{E}\int_0^t |{}_{D(A)^*}\langle F_\eta(s',\hat{X}_k(s'))-F_\eta(s',\hat{X}(s')),\phi\rangle_{D(A)}|ds' \to 0, \quad k\to\infty. \tag{2.6}$$

Indeed, we set $G_R(t,x) := {}_{D(A)^*}\langle F_\eta(t,x),\phi\rangle_{D(A)}\chi_R({}_{D(A)^*}\langle F_\eta(t,x),\phi\rangle_{D(A)})$, where $\chi_R \in C_0^\infty : \mathbb{R}\to[0,1]$ is a cutoff function with $\chi_R(r)=1$ when $|r|\le R$ and $\chi_R(r)=0$ when $|r|>2R$. Then by the dominated convergence theorem we obtain

$$\lim_{k\to\infty}\hat{E}\int_0^t |G_R(s',\hat{X}_k(s'))-G_R(s',\hat{X}(s'))|ds' = 0.$$

Then we have

$$
\begin{aligned}
&\lim_{R\to\infty}\sup_k \hat{E}\int_0^t |_{D(A)^*}\langle F_\eta(s',\hat{X}_k(s')),\phi\rangle_{D(A)} - G_R(s',\hat{X}_k(s'))|ds' \\
&\leq 2\lim_{R\to\infty}\sup_k \hat{E}\int_0^t |_{D(A)^*}\langle F_\eta(s',\hat{X}_k(s')),\phi\rangle_{D(A)}|1_{\{|_{D(A)^*}\langle F_\eta(s',\hat{X}_k(s')),\phi\rangle_{D(A)}|>R\}}ds' \\
&\leq C\lim_{R\to\infty}\sup_k \hat{E}\int_0^t J^2(s',\hat{X}_k(s'))ds'/R = 0,
\end{aligned}
$$

where we used Hypothesis 2.2 in the second inequality and (2.4) to deduce the last convergence. The above convergence also holds for $\hat{X}$. Combining the above estimates (2.6) follows.

By Hypothesis 2.2 we have

$$
\begin{aligned}
&\hat{E}\int_0^t |_{D(A)^*}\langle F_{1/n_k}(s',\hat{X}_k(s')) - F(s',\hat{X}(s')),\phi\rangle_{D(A)}|ds' \\
\leq&\hat{E}\int_0^t |_{D(A)^*}\langle F_{1/n_k}(s',\hat{X}_k(s')) - F(s',\hat{X}_k(s')),\phi\rangle_{D(A)}|ds' \\
&+\hat{E}\int_0^t |_{D(A)^*}\langle F(s',\hat{X}_k(s')) - F_\eta(s',\hat{X}_k(s')),\phi\rangle_{D(A)}|ds' \\
&+\hat{E}\int_0^t |_{D(A)^*}\langle F(s',\hat{X}(s')) - F_\eta(s',\hat{X}(s')),\phi\rangle_{D(A)}|ds' \\
&+\hat{E}\int_0^t |_{D(A)^*}\langle F_\eta(s',\hat{X}_k(s')) - F_\eta(s',\hat{X}(s')),\phi\rangle_{D(A)}|ds' \\
\leq& C\hat{E}\int_0^t \frac{1}{n_k}J^2(s',\hat{X}_k(s'))ds' + C\hat{E}\int_0^t \eta J^2(s',\hat{X}_k(s'))ds' + C\hat{E}\int_0^t (\eta J^2(s',\hat{X}(s'))ds' \\
&+\hat{E}\int_0^t |_{D(A)^*}\langle F_\eta(s',\hat{X}_k(s')) - F_\eta(s',\hat{X}(s')),\phi\rangle_{D(A)}|ds' \\
\to& 0, k\to\infty,
\end{aligned}
\tag{2.7}
$$

where in the second inequality we use Hypothesis 2.2 and the last convergence follows by (2.4) for $\hat{X}_k$ and $\hat{X}$ and (2.6). In fact, we could choose η_0 small enough such that the second term and the third term in the right hand side of last inequality converge to 0. Then for such η_0 we could find k large enough such that the first term and the last term converge to 0. Then by (2.5) and (2.7) we obtain

$$\lim_{k\to\infty}\hat{E}|\hat{M}_k(\phi)(t) - M(\phi)(t)| = 0$$

and

$$\lim_{k\to\infty}\hat{E}|\hat{M}_k(\phi)(t) - M(\phi)(t)|^2 = 0,$$

where

$$M(\phi)(t) := \langle\hat{X}(t) - x,\phi\rangle - \int_0^t\langle\hat{X}(s'),A\phi\rangle ds' - \int_0^t\langle F(s,\hat{X}(s')),\phi\rangle ds'.$$

Taking the limit we obtain that for all $r\leq t\in[0,T]$ and all bounded continuous functions φ on $L^2([0,r];H)\cap C([0,r];D(A)^*)$

$$\hat{E}((M(\phi)(t) - M(\phi)(r))\varphi(\hat{X}\restriction_{[0,r]})) = 0.$$

and

$$\hat{E}((M(\phi)(t)^2 - M(\phi)(r)^2 - \int_r^t |\sqrt{G}v|_H^2 ds)\varphi(\hat{X} \restriction_{[0,r]})) = 0.$$

Thus the existence of a martingale solution for (1.1) follows by a martingale representation theorem (cf. [DZ92, Theorem 8.2],[O05, Theorem 2]). The last inequality follows by (2.1), (2.3), (2.4) and the lower semicontinuity of $J^2 + |(-A)^{\delta_2} \cdot|^2 + |\cdot|^2$. □

Set

$$P_{s,t}\varphi(x) := E[\varphi(X(t,s,x))], \quad 0 \le s < t \le T, \varphi \in \mathcal{B}_b(H),$$

and

$$\mu_t(dx) := (P_{s,t})^* \zeta(dx),$$

where $\zeta \in \mathcal{P}(H)$ such that

$$\int_s^T \int_H (J^2(s', x) + |x|^2)\zeta(dx)ds' < \infty.$$

Now Itô's formula implies that this is a solution to the corresponding Fokker-Planck equation, i.e. $\forall u \in D(L_0)$

$$\int_H u(t,y)\mu_t(dy) = \int_H u(s,y)\zeta(dy) + \int_s^t ds' \int_H L_0 u(s', y)\mu_{s'}(dy), \text{ for all } t \in [s,T].$$

3 Existence of solutions to the Fokker-Planck equation

In this section we prove directly the existence of solutions for the Fokker-Planck equation (1.3) under the same conditions as in Section 2.

Set

$$W_A(t,s) = \int_s^t e^{(t-s')A}\sqrt{G}dW(s'), \quad t \ge s.$$

The Kolmogorov operator L_α corresponding to (2.2) is given by

$$\begin{aligned} L_\alpha u(t,x) :=& D_t u(t,x) + \frac{1}{2}\mathrm{Tr}[\mathrm{GD^2u(t,x)}] + \langle \mathrm{x, ADu(t,x)} \rangle \\ &+ {}_{D(A)^*}\langle F_\alpha(t,x), Du(t,x)\rangle_{D(A)}, \ (t,x) \in [0,T] \times H, \quad u \in D(L_0). \end{aligned}$$

Fix $s \in [0,T)$ and set

$$\mu_t^\alpha(dx) := (P_{s,t}^\alpha)^* \zeta(dx),$$

where $\zeta \in \mathcal{P}(H)$ is the initial condition, at $t = s$.

Now Itô's formula implies that this is a solution to the corresponding Fokker-Planck equation, i.e. $\forall u \in D(L_0)$

$$\int_H u(t,y)\mu_t^\alpha(dy) = \int_H u(s,y)\zeta(dy) + \int_s^t ds' \int_H L_\alpha u(s', y)\mu_{s'}^\alpha(dy), \text{ for all } t \in [s,T], \tag{3.1}$$

Theorem 3.1 Assume Hypotheses 2.1, 2.2 and let $\zeta \in \mathcal{P}(H)$ be such that

$$\int_s^T \int_H (J^2(s', x) + |x|^2)\zeta(dx)ds' < \infty.$$

Then there exists a solution $\mu_t(dx)dt$ to the Fokker-Planck equation (1.3) such that

$$\sup_{t\in[s,T]}\int_H |x|^2\mu_t(dx)<\infty,$$

and

$$t\mapsto \int_H u(t,x)\mu_t(dx)$$

is continuous on $[s,T]$ for all $u\in D(L_0)$. Finally, for some $C>0$ and for $\delta_2:=\delta\wedge\frac{1}{2}$ with δ as in Hypothesis 2.1 one has

$$\int_s^T\int_H (J^2(s',x)+|(-A)^{\delta_2}x|^2+|x|^2)\mu_{s'}(dx)ds'\le C\int_s^T\int_H (J^2(s',x)+|x|^2)\zeta(dx)ds'. \tag{3.2}$$

Proof For $\alpha\in(0,1]$, set $X_\alpha(t):=X_\alpha(s,t,x), x\in H$, and

$$Y_\alpha(t):=X_\alpha(t)-W_A(t,s),\quad t\ge s.$$

By the same arguments to obtain (2.3) we also have here that

$$E|Y_\alpha(t)|^2+\int_s^t E|(-A)^{1/2}Y_\alpha(s')|^2ds'\le |x|^2+K\int_s^t J^2(s',x)ds',\quad t\ge s. \tag{3.3}$$

Then for $s\le t\le T$ we obtain

$$E|X_\alpha(t)|^2\le 2|x|^2+2K\int_s^T J^2(s',x)ds'+2\kappa,$$

where $\kappa:=\sup_{t\in[s,T]}E|W_A(t)|^2<\infty$. Now we integrate with respect to ζ over $x\in H$ and obtain for all $s\le t\le T$ and some $C\in(0,\infty)$ that

$$\int_H |x|^2\mu_t^\alpha(dx)\le C[1+\int_s^T\int_H (J^2(s',x)+|x|^2)\zeta(dx)ds']. \tag{3.4}$$

Hence we can use Prohorov' theorem (see [B07, Theorem 8.6.7]) to obtain that for each $t\in[s,T]$, there exists a sub-sequence $\{\alpha_n\}$ (possibly depending on t) such that the measures $\mu_t^{\alpha_n}$ converge τ_w-weakly to a measure $\tilde{\mu}_t\in\mathcal{P}(H)$ as $n\to\infty$, where τ_w denotes the weak topology on H.

Now we have that for $\varphi\in\mathcal{E}_A(H)$, defined to be the set of all linear combinations of all real parts of functions of the form $x\mapsto e^{i\langle x,h\rangle}, h\in D(A)$,

$$t\mapsto \mu_t^\alpha(\varphi):=\int_H\varphi(x)\mu_t^\alpha(dx), \alpha\in(0,1] \text{ are equicontinuous on } [s,T]. \tag{3.5}$$

In fact, for $s\le t_1\le t_2\le T$

$$\begin{aligned}|\mu_{t_2}^\alpha(\varphi)-\mu_{t_1}^\alpha(\varphi)|\le&\frac{1}{2}\|\mathrm{Tr}[\mathrm{G}D^2\varphi]\|_\infty|\mathrm{t}_2-\mathrm{t}_1|\\&+|t_2-t_1|^{1/2}\|AD\varphi\|_\infty(\int_{t_1}^{t_2}\int_H|x|^2\mu_{s'}^\alpha(dx)ds')^{1/2}\\&+|t_2-t_1|^{1/2}\|(-A)^{1/2}D\varphi\|_\infty(\int_{t_1}^{t_2}\int_H J^2(s',x)\zeta(dx)ds')^{1/2},\end{aligned}$$

where $\|\cdot\|_\infty$ denotes the sup-norm on H. By (3.4) and Hypothesis 2.2, (3.5) follows.

Then by the same arguments as in the proof of [BDR10, Theorem 2.6], we can construct a measure μ_t and a subsequence $\{\alpha_n\}$ such that $\mu_t^{\alpha_n}$ converge τ_w-weakly to μ_t for all $t \in [0,T]$. Indeed, by a diagonal argument we can choose $\{\alpha_n\}$ such that $\mu_t^{\alpha_n} \to \tilde{\mu}_t$ τ_ω-weakly as $n \to \infty$ for every rational $t \in [s,T]$. Moreover (3.4) holds for $\tilde{\mu}_t$ in place of μ_t^α for $t \in [s,T] \cap \mathbb{Q}$. Hence by [B07, Theorem 8.6.7], for each $t \in [s,T] \setminus \mathbb{Q}$ there exists $r_n(t) \in [s,T] \cap \mathbb{Q}, n \in \mathbb{N}$ converging to t and $\mu_t \in \mathcal{P}(H)$ such that $\tilde{\mu}_{r_n(t)} \to \mu_t$ τ_w-weakly as $n \to \infty$. Now for fix $t \in [s,T] \backslash \mathbb{Q}$ suppose $\{\mu_t^{\alpha_n}\}$ does not weakly converge to μ_t. Then by (3.4) and [B07, Theorem 8.6.7] there exists a subsequence $\{\alpha_{n_k}\}$, $\varphi \in \mathcal{E}_A(H)$ and $\nu \in \mathcal{P}(H)$ such that $\mu_t^{\alpha_{n_k}} \to \nu$ τ_w-weakly as $k \to \infty$ and $\mu_t(\varphi) \neq \nu(\varphi)$. On the other hand, for all $n, k \in \mathbb{N}$

$$\begin{aligned}|\nu(\varphi) - \mu_t(\varphi)| \leq & |\nu(\varphi) - \mu_t^{\alpha_{n_k}}(\varphi)| + \sup_{l \in \mathbb{N}} |\mu_t^{\alpha_{n_l}}(\varphi) - \mu_{r_n(t)}^{\alpha_{n_l}}(\varphi)| \\ & + |\mu_{r_n(t)}^{\alpha_{n_k}}(\varphi) - \tilde{\mu}_{r_n(t)}(\varphi)| + |\tilde{\mu}_{r_n(t)}(\varphi) - \mu_t(\varphi)|.\end{aligned}$$

Letting $k \to \infty$ and then $n \to \infty$ it follows from (3.5) that $\mu_t(\varphi) = \nu(\varphi)$. Letting $\mu_t := \tilde{\mu}_t$ for $t \in [s,T] \cap \mathbb{Q}$, we have that $\mu_t^{\alpha_n}$ converge τ_w-weakly to μ_t for all $t \in [0,T]$.

(3.4) and Lebesgue's dominated convergence theorem imply that $t \mapsto \int_H u(t,x)\mu_t(dx)$ is continuous on $[s,T]$ for all $u \in D(L_0)$.

Now for $\delta_2 := \delta \wedge \frac{1}{2}$ with δ as in Hypothesis 2.1 (iii) by (3.3) and (2.1) we obtain

$$\int_s^T \int_H |(-A)^{\delta_2} x|^2 \mu_t^\alpha(dx)dt \leq C[1 + \int_s^T \int_H (J(s',x)^2 + |x|^2)\zeta(dx)ds'], \tag{3.6}$$

which implies that $\mu_t^{\alpha_n}(dx)dt$ converge weakly to $\mu_t(dx)dt$ on $[s,T] \times H$ by the compactness of $(-A)^{-\delta_2}$. Now (3.2) follows from (3.4), (3.6) and the lower semicontinuity of $J^2 + |(-A)^{\delta_2} \cdot|^2 + |\cdot|^2$.

It remains to prove that $\mu_t(dx)dt$ solves the Fokker-Planck equation (1.3). Since every $h \in C^1([0,T]; D(A))$ can be written as a uniform limit of piecewise affine $h_n \in C([0,T]; D(A)), n \in \mathbb{N}$, it follows by (3.2) and linearity that $\mu_t(dx)dt$ satisfies the Fokker-Planck equation (1.3) if and only if it does so for all $u \in D(L_0)$ such that $u(t,x) = \phi(t)e^{i\langle h(t),x\rangle}, x \in H, t \in [0,T]$, with $\phi \in C^1([0,T]), \phi(T) = 0$ and piecewise affine $h \in C([0,T]; D(A))$. Fix such a function $u \in D(L_0)$, by (3.1) we have

$$\int_s^T \int_H L_{\alpha_n} u(t,x)\mu_t^{\alpha_n}(dx)dt = -\int_s^T u(s,x)\zeta(dx),$$

with α_n as above.

Since we already know that $\mu_t^{\alpha_n}(dx)dt \to \mu_t(dx)dt$ weakly and since the coefficient of the second order part of L_{α_n} is just G (hence independent of n), it now suffices to prove that for all $g \in C_b([s,T] \times H)$ and all piecewise affine $h \in C([0,T]; D(A))$,

$$\lim_{n\to\infty} \int_s^T \int_H F_{\alpha_n}^h(t,x)g(t,x)\mu_t^{\alpha_n}(dx)dt = \int_s^T \int_H F^h(t,x)g(t,x)\mu_t(dx)dt, \tag{3.7}$$

where

$$F_\alpha^h(t,x) := {}_{D(A)^*}\langle F_\alpha(t,x), h(t)\rangle_{D(A)} + \frac{\langle Ah(t),x\rangle}{1+\alpha|\langle Ah(t),x\rangle|},$$

$$F^h(t,x) := {}_{D(A)^*}\langle F(t,x), h(t)\rangle_{D(A)} + \langle Ah(t), x\rangle.$$

For $\eta \in (0,1]$ we have

$$\begin{aligned}
&|\int_s^T \int_H F^h_{\alpha_n}(t,x)g(t,x)\mu_t^{\alpha_n}(dx)dt - \int_s^T \int_H F^h(t,x)g(t,x)\mu_t(dx)dt| \\
\leq &\|g\|_\infty \int_s^T \int_H |F^h_{\alpha_n}(t,x) - F^h(t,x)|\mu_t^{\alpha_n}(dx)dt \\
&+ \|g\|_\infty \int_s^T \int_H |F^h(t,x) - F^h_\eta(t,x)|\mu_t^{\alpha_n}(dx)dt \\
&+ \|g\|_\infty \int_s^T \int_H |F^h(t,x) - F^h_\eta(t,x)|\mu_t(dx)dt \\
&+ |\int_s^T \int_H F^h_\eta(t,x)g(t,x)\mu_t^{\alpha_n}(dx)dt - \int_s^T \int_H F^h_\eta(t,x)g(t,x)\mu_t(dx)dt|.
\end{aligned} \tag{3.8}$$

By Hypothesis 2.2 we have for all $\alpha, \beta \in (0,1]$

$$\begin{aligned}
\int_s^T \int_H |F^h_\beta(t,x) - F^h(t,x)|\mu_t^\alpha(dx)dt \leq &\beta C(h) \int_s^T \int_H (J^2(t,x) + |x|^2)\mu_t^\alpha(dx)dt \\
\leq &\beta C(h)C(1 + \int_s^T \int_H (J^2(t,x) + |x|^2)\zeta(dx)dt),
\end{aligned} \tag{3.9}$$

where C is a constant independent of α, β and we used Hypothesis 2.2 and (3.4) in the last step. This implies that if $n \to \infty$ and $\eta \to 0$ the first two terms in (3.8) converge to zero. Since (3.9) holds for μ_t in place of μ_t^α, we deduce that the third term converges to zero if $\eta \to 0$. Now we consider the last summand. Since F^h_η is continuous on $[s,T] \times H$ by our assumption, there exists a continuous function $\tilde{G}_R$ on $[s,T] \times H$ satisfying $\|\tilde{G}_R\|_\infty \leq R$, and $\tilde{G}_R(t,x) = F^h_\eta(t,x)$ on B_R, for $B_R := \{|F^h_\eta| \leq R\}$. By the weak convergence we obtain

$$\lim_{n\to\infty} \int_s^T \int_H \tilde{G}_R(t,x)g(t,x)\mu_t^{\alpha_n}(dx)dt = \int_s^T \int_H \tilde{G}_R(t,x)g(t,x)\mu_t(dx)dt.$$

By the above estimate we get

$$\begin{aligned}
&\int_s^T \int_H |\tilde{G}_R(t,x) - F^h_\eta(t,x)|\mu_t^{\alpha_n}(dx)dt \\
\leq &CR \int_{B_R^c} \mu_t^{\alpha_n}(dx)dt + CC(h) \int_{B_R^c} (|F_\eta(t,x)|_{V^*} + |x|)\mu_t^{\alpha_n}(dx)dt \\
\leq &CR^{-1} \int_s^T \int_H (J^2(t,x) + |x|^2)\mu_t^{\alpha_n}(dx)dt + C\gamma(h) \int_{B_R^c} (J(t,x) + |x|)\mu_t^{\alpha_n}(dx)dt,
\end{aligned}$$

where in the last inequality we used Hypothesis 2.2. Then the last summand converges to zero if $R \to \infty$ and $n \to \infty$. Hence (3.7) is verified and the assertion follows. □

4 Application

Consider the stochastic semilinear partial differential equation

$$dX(t) = (\frac{\partial^2}{\partial \xi^2} X(t) + f(t, X(t)) + \frac{\partial}{\partial \xi} g(t, X(t)))dt + \sqrt{G} dW(t), \tag{4.1}$$

with Dirichlet boundary condition

$$X(t, 0) = X(t, 1) = 0, t \in [0, T], \tag{4.2}$$

and the initial condition

$$X(0) = x, \tag{4.3}$$

on $H = L^2(0, 1) := L^2((0, 1), d\xi)$, with $d\xi =$ Lebesgue measure. Here $f, g : (0, 1) \times [0, T] \times \mathbb{R} \to \mathbb{R}$ are functions such that for every $\xi \in (0, 1)$ the maps $f(\xi, \cdot, \cdot), g(\xi, \cdot, \cdot)$ are continuous on $[0, T] \times \mathbb{R}$ and satisfy the following conditions:
(f1) There exist $m \in \mathbb{N}$ (without loss of generality $m \geq 2$) and a nonnegative function $c_1 \in L^2(0, T)$ such that for all $t \in [0, T], z \in \mathbb{R}, \xi \in (0, 1)$

$$|f(\xi, t, z)| \leq c_1(t)(1 + |z|^m).$$

(f2) There exists a nonnegative function $c_2 \in L^1(0, T)$ and $m_1 \in (0, \infty)$ such that for all $t \in [0, T], z_1, z_2 \in \mathbb{R}, \xi \in (0, 1)$

$$(f(\xi, t, z_1 + z_2) - f(\xi, t, z_1))z_2 \leq c_2(t)(|z_2|^2 + |z_1|^{m_1} + 1).$$

(g1) The function g is of the form $g(\xi, t, z) = g_1(\xi, t, z) + g_2(t, z)$, where g_1 and g_2 are Borel functions of $(\xi, t, z) \in (0, 1) \times [0, T] \times \mathbb{R}$ and of $(t, z) \in [0, T] \times \mathbb{R}$, respectively. The function g_1 satisfies a linear growth and the function g_2 a quadratic growth condition, i.e. there is a constant K such that

$$|g_1(\xi, t, z)| \leq K(1 + |z|), \quad |g_2(t, z)| \leq K(1 + |z|^2),$$

for all $t \in [0, T], \xi \in (0, 1), z \in \mathbb{R}$.
(g2) g is a locally Lipschitz function with linearly growing Lipschitz constant, i.e. there exists a constant L such that

$$|g(\xi, t, z_1) - g(\xi, t, z_2)| \leq L(1 + |z_1| + |z_2|)|z_1 - z_2|,$$

for all $t \in [0, T], \xi \in (0, 1), z_1, z_2 \in \mathbb{R}$.

Example 4.0 Now we give examples for f satisfying (f1) (f2). Let $f : (0, 1) \times [0, T] \times \mathbb{R} \to \mathbb{R}$ be a function such that for every $\xi \in (0, 1)$ the maps $f(\xi, \cdot, \cdot)$ are continuous on $[0, T] \times \mathbb{R}$. Moreover $f = f_1 + f_2$ satisfies the polynomial growth condition (f1) for some $m \geq 2$ and there exists a constant C such that

$$f_1(\xi, t, \cdot) \in C^1(\mathbb{R}), \quad \partial_z f_1(\xi, t, z) \leq C, \quad (\xi, t, z) \in (0, 1) \times [0, T] \times \mathbb{R},$$

$$f_2(\xi, t, z)z \leq C[1 + |z|^2], \quad |f_2(\xi, t, z)| \leq C(1 + |z|^{2 - \frac{1}{m}}) \quad (\xi, t, z) \in (0, 1) \times [0, T] \times \mathbb{R}.$$

It immediately follows from the mean value theorem that f_1 satisfies (f2). Now we check (f2) for f_2: for $t \in [0,T], z_1, z_2 \in \mathbb{R}, \xi \in (0,1)$

$$\begin{aligned}(f_2(\xi,t,z_1+z_2)-f_2(\xi,t,z_1))z_2 \leq& f_2(\xi,t,z_1+z_2)(z_1+z_2)-f_2(\xi,t,z_1+z_2)z_1+(1+|z_1|^2)|z_2| \\ \leq& C+C(z_1+z_2)^2+C(1+|z_1+z_2|^{2-\frac{1}{m}})|z_1|+(1+|z_1|^m)|z_2| \\ \leq& |z_2|^2+C(1+|z_1|^{2m}).\end{aligned}$$

Let $A : D(A) \subset H \to H$ be defined by

$$Ax(\xi) = \frac{\partial^2}{\partial \xi^2} x(\xi), \xi \in (0,1), \quad D(A) = H^2(0,1) \cap H_0^1(0,1).$$

Then $V = H_0^1(0,1)$. Let $D(F) := [0,T] \times L^{2m}(0,1)$ and for $(t,x) \in D(F)$

$$F := F_1 + F_2, \quad F_1(t,x)(\xi) := f(\xi,t,x(\xi)), \quad F_2(t,x)(\xi) := \partial_\xi g(\xi,t,x(\xi)), \xi \in (0,1),$$

where F_2 takes values in V^*.

Finally, let $G \in L(H)$ be symmetric, nonnegative and such that $G^{-1} \in L(H)$ and there exist $\theta, q \geq 0$ with $\frac{1}{2q} + 2\theta < 1$ such that

$$\|(\sum_k (A^{-\theta}\sqrt{G}(e_k))^2)^{1/2}\|_{L^q} < \infty, \tag{G.1}$$

where $\{e_k\}$ is an orthonormal basis of H.

If $G = Id$, (G.1) is obviously satisfied. By (G.1), [B97, Corollary 3.5] and [D04, Exercise 2.16] we know that W_A is a Gaussian random variable in $C([0,T] \times [0,1])$.

It is easily checked that A, G satisfy Hypothesis 2.1 with $\delta, \delta_1 \in (0, \frac{1}{4})$.

For $\alpha \in (0,1]$ and $(t,x) \in [0,T] \times H$ we define $F_\alpha : [0,T] \times H \to D(A)^*$,

$$F_\alpha := F_1^\alpha + F_2, \quad F_1^\alpha(t,x)(\xi) := \frac{F_1(t,x)(\xi)}{1+\alpha|F_1(t,x)(\xi)|}, \xi \in [0,1].$$

If $F_1 \equiv 0$, there exists a unique (probabilistically) strong solution to (4.1) by [G98, Theorem 2.1]. Since F_1^α is bounded, by Girsanov's Theorem (cf. [MR99, Theorem 3.1], [DFPR12, Theorem 13]) , we obtain that there exists a stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, P)$, a cylindrical Wiener process W on H and a progressively measurable process $X_\alpha : [s,T] \times \Omega \to H$ as in Hypothesis 2.2 (iii) satisfying the following stochastic differential equation

$$dX_\alpha(t) = [AX_\alpha(t) + F_\alpha(t, X_\alpha(t))]dt + \sqrt{G}dW(t), X_\alpha(s) = x, s \leq t, \tag{4.4}$$

for all $x \in H$.

Define for $m \geq 2$ as in (f1)

$$J(t,x) := \begin{cases} 2(c_1(t)+K)(1+|x|^m_{L^{2m}(0,1)}), & \text{if } (t,x) \in D(F) \\ +\infty, & \text{otherwise.} \end{cases}$$

By (g1) we have

$$|F_2(t,x)|_{V^*} \leq 2K(1+|x|^2_{L^4}) \leq J(t,x) < \infty \quad \forall (t,x) \in D(F) = [0,T] \times L^{2m}(0,1).$$

By (f1) we obtain

$$|F(t,x)|_{V^*} \le J(t,x) < \infty \quad \forall (t,x) \in D(F) = [0,T] \times L^{2m}(0,1).$$

One also easily checks that F_α satisfies Hypothesis 2.2 (i)-(iii). It remains to check the last part of Hypothesis 2.2 (iv), which, however, immediately follows from the following proposition.

Proposition 4.1 For any $s \in [0,T)$, there exists $C \in (0,\infty)$, such that for $\alpha \in (0,1], x \in L^{2m}(0,1)$

$$E(|X_\alpha(t,s,x)|^{2m}_{L^{2m}(0,1)}) \le C(1+|x|^{2m}_{L^{2m}(0,1)}), \quad \forall t \in [s,T].$$

Proof Set $Y_\alpha(t) := X_\alpha(t,s,x) - W_A(s,t), t \in [s,T]$. Then we obtain for $\phi \in D(A), t \in [s,T]$

$$\langle Y_\alpha(t) - x, \phi\rangle = \int_s^t (\langle Y_\alpha(s'), A\phi\rangle + {}_{D(A)^*}\langle F_\alpha(s', X_\alpha(s')), \phi\rangle_{D(A)})ds'. \tag{4.5}$$

Since the trajectories of W_A can be approximated by functions $W_A^n := (1-\frac{1}{n}A)^{-1}W_A$ from $C([s,T],H^2)$ in $L^2([s,T],H) \cap C([s,T],H^{-2})$, we can replace W_A by smooth functions W_A^n. Moreover, we can approximate g by smooth functions $g_n := \varphi_n * \chi_n(g)$ for smooth functions φ_n on $[0,1]\times\mathbb{R}$ with supp$\varphi_n \subset [-\frac{1}{n},\frac{1}{n}]^2$ and $\chi_n : \mathbb{R} \to [0,n]$ is a smooth function on $\mathbb{R}$ satisfying $\chi_n(r) = r$ if $|r| \le n$, $\chi_n(r) = 0$ if $|r| > 2n$ and $|\chi_n'| \le C$ for a constant C independent of n. We also approximate x by smooth functions x_n such that $|x_n|_{L^{2m}} \le |x|_{L^{2m}}$. Then each g_n has bounded derivative with respect to ξ and z and satisfies (g1), (g2) with K, L replaced by $2K, 3CL$ respectively. By a standard method (see e.g. [GRZ09, Theorem 4.6]) we obtain that there exists a stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ and a pair process $(Y_\alpha^n, \bar{W}_A^n)$ such that

$$Y_\alpha^n \in L^\infty([s,T],H) \cap L^2([s,T],H_0^1) \cap C([s,T],H^{-2}) \quad P-a.s.$$

and $\bar{W}_A^n$ has the same distribution as W_A^n and for $\phi \in H_0^1, t \in [s,T]$

$$\langle Y_\alpha^n(t) - x_n, \phi\rangle = \int_s^t ({}_{V^*}\langle A\phi, Y_\alpha^n(s')\rangle_V + {}_{V^*}\langle F_1^\alpha(s', Y_\alpha^n(s') + \bar{W}_A^n(s')) + F_2^n(s', Y_\alpha^n(s') + \bar{W}_A^n(s')), \phi\rangle_V)ds', \tag{4.6}$$

where $F_2^n(t,x)(\xi) := \partial_\xi g_n(\xi, t, x(\xi))$. Below we denote W_A^n as $\bar{W}_A^n$ if there's no confusion. Now taking $\phi = \lambda_k e_k$ and e_k as in (4.6) and by the product rule for $\lambda_k\langle Y_\alpha^n(t), e_k\rangle$ and $\langle Y_\alpha^n(t), e_k\rangle$ we have

$$\begin{aligned}
&\lambda_k\langle Y_\alpha^n(t), e_k\rangle^2 + \int_s^t \lambda_k^2 \langle Y_\alpha^n(s'), e_k\rangle^2 ds' \\
\le &\lambda_k\langle x_n, e_k\rangle^2 + \int_s^t |{}_{V^*}\langle F_1^\alpha(s', Y_\alpha^n(s') + W_A^n(s')) + F_2^n(s', Y_\alpha^n(s') + W_A^n(s')), e_k\rangle_V|^2 ds'.
\end{aligned}$$

Then taking sum we have the following estimate since g_n has bounded derivative,

$$\begin{aligned}
&|Y_\alpha^n(t)|_V^2 + \int_s^t |AY_\alpha^n(s')|^2 ds' \\
\le &|x_n|_V^2 + \int_s^t C_\alpha + C_n(1 + |Y_\alpha^n(s')|_V^2 + |W_A^n(s')|_V^2) ds',
\end{aligned}$$

which combining with Gronwall's lemma implies that $Y_\alpha^n \in L^\infty([s,T],H_0^1)\cap L^2([s,T],H^2)$.

Moreover, (4.6) can be easily extended to $\phi \in \{u \in L^2([0,T],H_0^1) : \frac{du}{dt} \in L^2([0,T],V^*)\}$:

$$\begin{aligned}\langle Y_\alpha^n(t),\phi(t)\rangle - \langle x_n,\phi(s)\rangle = &\int_s^t ({}_{V^*}\langle \frac{d\phi}{dt}(s'),Y_\alpha^n(s')\rangle_V + {}_{V^*}\langle A\phi(s'),Y_\alpha^n(s')\rangle_V \\ &+ {}_{V^*}\langle F_1^\alpha(s',Y_\alpha^n(s')+W_A^n(s'))+F_2^n(s',Y_\alpha^n(s')+W_A^n(s')),\phi(s')\rangle_V)ds'\end{aligned}$$

Since $Y_\alpha^n \in L^\infty([s,T],H_0^1)\cap L^2([s,T],H^2)$, we can choose $\phi = (Y_\alpha^n(t))^{2m-1}$ and obtain for $t \in [s,T]$

$$\begin{aligned}&\frac{1}{2m}\frac{d}{dt}\int |Y_\alpha^n(t)|^{2m}d\xi + (2m-1)\int |Y_\alpha^n(t)|^{2m-2}|\partial_\xi Y_\alpha^n(t)|^2 d\xi \\ =&\int F_1^\alpha(t,Y_\alpha^n(t)+W_A^n(s,t))Y_\alpha^n(t)^{2m-1}d\xi + {}_{V^*}\langle F_2^n(t,Y_\alpha^n(t)+W_A^n(s,t)),Y_\alpha^n(t)^{2m-1}\rangle_V \\ :=&I_1+I_2.\end{aligned}$$

Let us estimate I_2. We have

$$\begin{aligned}&{}_{V^*}\langle F_2^n(t,Y_\alpha^n(t)+W_A^n(s,t)),Y_\alpha^n(t)^{2m-1}\rangle_V \\ =&{}_{V^*}\langle [F_2^n(t,Y_\alpha^n(t)+W_A^n(s,t))-F_2^n(t,Y_\alpha^n(t))],Y_\alpha^n(t)^{2m-1}\rangle_V + {}_{V^*}\langle F_2^n(t,Y_\alpha^n(t)),Y_\alpha^n(t)^{2m-1}\rangle_V.\end{aligned} \tag{4.7}$$

For the first term on the right hand side of (4.7), we have by (g2), and Young's inequality

$$\begin{aligned}&{}_{V^*}\langle [F_2^n(t,Y_\alpha^n(t)+W_A^n(s,t))-F_2^n(t,Y_\alpha^n(t))],Y_\alpha^n(t)^{2m-1}\rangle_V \\ \leq& C\int (1+|Y_\alpha^n(t)|+|W_A^n(s,t)|)|W_A^n(s,t)||Y_\alpha^n(t)|^{2m-2}|\partial_\xi Y_\alpha^n(t)|d\xi \\ \leq&\frac{1}{2}\int |Y_\alpha^n(t)|^{2m-2}|\partial_\xi Y_\alpha^n(t)|^2 d\xi + C\int |W_A^n(s,t)|^2|Y_\alpha^n(t)|^{2m}d\xi \\ &+C\int (1+|W_A^n(s,t)|)|W_A^n(s,t)||Y_\alpha^n(t)|^{2m-2}|\partial_\xi Y_\alpha^n(t)|d\xi \\ \leq&\int |Y_\alpha^n(t)|^{2m-2}|\partial_\xi Y_\alpha^n(t)|^2 d\xi + C|W_A^n(s,t)|_{L^{4m}}^{4m} + C|W_A^n(s,t)|_{L^{2m}}^{2m} + (C|W_A^n(s,t)|_{L^\infty}^2+C)|Y_\alpha^n(t)|_{L^{2m}}^{2m}.\end{aligned}$$

For the second term on the right hand side of (4.7), we have

$$\int_0^1 g_2^n(t,Y_\alpha^n)Y_\alpha^n(t)^{2m-2}\partial_\xi Y_\alpha^n(t)d\xi = \int_0^1 \partial_\xi g_3(t,Y_\alpha^n)d\xi = 0,$$

where $g_3(t,r) = \int_0^r g_2^n(t,z)z^{2m-2}dz$. Then we obtain by (g1)

$$\begin{aligned}{}_{V^*}\langle F_2^n(t,Y_\alpha^n(t)),Y_\alpha^n(t)^{2m-1}\rangle_V =& -(2m-1)\int g_1^n(\xi,t,Y_\alpha^n)Y_\alpha^n(t)^{2m-2}\partial_\xi Y_\alpha^n(t)d\xi \\ \leq& C\int (1+|Y_\alpha^n(t)|)|Y_\alpha^n(t)|^{2m-2}|\partial_\xi Y_\alpha^n(t)|d\xi \\ \leq&\int (|Y_\alpha^n(t)|^{2m-2}|\partial_\xi Y_\alpha^n(t)|^2 + C|Y_\alpha^n(t)|^{2m}+C)d\xi.\end{aligned}$$

Now we consider I_1. We note that by (f1), (f2), for all $y, z \in \mathbb{R}, t \in [0,T], \xi \in (0,1)$,

$$\begin{aligned}
&|f(\xi,t,y+z)y| \\
=&|(f(\xi,t,y+z) - f(\xi,t,z))y + f(\xi,t,z)y| \\
\leq& c_2(t)(|y|^2 + |z|^{m_1} + 1) + c_1(t)(1+|z|^m)|y| \\
\leq& c(t)(1 + |y|^2 + |z|^{m_1} + |z|^m|y|),
\end{aligned}$$

where $c(t) = c_1(t) + c_2(t)$. Then

$$I_1 \leq c(t) \int [1 + |Y_\alpha^n(t)|^2 + |W_A^n(s,t)|^m |Y_\alpha^n(t)| + |W_A^n(s,t)|^{m_1}] |Y_\alpha^n(t)|^{2m-2} d\xi.$$

Now we obtain

$$\begin{aligned}
&\frac{1}{2m}\frac{d}{dt}\int |Y_\alpha^n(t)|^{2m} d\xi + \int |Y_\alpha^n(t)|^{2m-2} |\partial_\xi Y_\alpha^n(t)|^2 d\xi \\
\leq& c(t) \int [1 + (2 + \frac{2m-1}{2m})|Y_\alpha^n(t)|^{2m} + \frac{1}{2m}|W_A^n(s,t)|^{2m^2} + \frac{1}{m}|W_A^n(s,t)|^{mm_1}] d\xi \\
&+ C|W_A^n(s,t)|_{L^{4m}}^{4m} + C|W_A^n(s,t)|_{L^{2m}}^{2m} + C + (C|W_A^n(t)|_{L^\infty}^2 + c)|Y_\alpha^n(t)|_{L^{2m}}^{2m}.
\end{aligned} \tag{4.8}$$

Then $c(\cdot) \in L^1([0,T])$ by (f1), (f2) and Gronwall's lemma yields that

$$\begin{aligned}
|Y_\alpha^n(t)|_{L^{2m}}^{2m} \leq& e^{\int_s^t C|c(t')| + C(|W_A^n(s,t')|_{L^\infty}^2 + 1)dt'} (|x_n|_{L^{2m}}^{2m} + C\int_s^t (|c(t')|(|W_A^n(s,t')|_{L^{2m^2}}^{2m^2} \\
&+ |W_A^n(s,t')|_{L^{mm_1}}^{mm_1}) + |W_A^n(s,t')|_{L^{4m}}^{4m} + |W_A^n(s,t')|_{L^{2m}}^{2m} + 1)dt') \\
\leq& e^{\int_s^t C|c(t')| + C(|W_A(s,t')|_{L^\infty}^2 + 1)dt'} (|x|_{L^{2m}}^{2m} + C\int_s^t (|c(t')|(|W_A(s,t')|_{L^{2m^2}}^{2m^2} \\
&+ |W_A(s,t')|_{L^{mm_1}}^{mm_1}) + |W_A(s,t')|_{L^{4m}}^{4m} + |W_A(s,t')|_{L^{2m}}^{2m} + 1)dt').
\end{aligned} \tag{4.9}$$

By (G.1), [B97, Corollary 3.5] and [D04, Exercise 2.16] we know that W_A is a Gaussian random variable in $C([s,T] \times [0,1])$. Then we have for any $p > 1$

$$E \sup_{(t,\xi)\in[s,T]\times(0,1)} |W_A(s,t)(\xi)|^p < \infty,$$

and by Fernique's Theorem (cf. [DZ92, Theorem 2.6]) there exists a constant $\varepsilon > 0$ independent of s such that

$$E e^{\varepsilon \sup_{(t,\xi)\in[s,T]\times(0,1)} |W_A(s,t)(\xi)|^2} < \infty, \tag{4.10}$$

where $\varepsilon > 0$ can be chosen independent of s. Indeed, since by the Markov property of W_A we have for any $r > 0$ that

$$P(\sup_{t\in[s,T]} |W_A(s,t)|_{L^\infty} \leq r) = P(\sup_{t\in[s,T]} |W_A(t-s)|_{L^\infty} \leq r) \geq P(\sup_{t\in[0,T]} |W_A(t)|_{L^\infty} \leq r),$$

we can choose common ε and r such that

$$\log(\frac{1 - P(\sup_{t\in[0,T]} |W_A(t)|_{L^\infty} \leq r)}{P(\sup_{t\in[0,T]} |W_A(t)|_{L^\infty} \leq r)}) + 32\varepsilon r^2 \leq -1.$$

Then (4.10) follows from Fernique's Theorem.

Taking expectation in (4.9) we obtain for $s \leq t \leq t_0$ such that $t_0 - s$ is small enough,

$$E|Y_\alpha^n(t)|_{L^{2m}}^{2m} \leq C|x|_{L^{2m}}^{2m} + C, \tag{4.11}$$

where C is a constant independent of α, n. By (4.8) and (4.9) we have

$$E|Y_\alpha^n(t)|_{L^2([s,t_0],H^1)}^2 \leq C|x|_{L^{2m}}^{2m} + C.$$

Moreover, since by (f1) (g1) we have

$$\begin{aligned}
&\int_s^{t_0} (|AY_\alpha^n(s')|_{H^{-1}}^2 + |F_1^\alpha(s', Y_\alpha^n(s') + W_A^n(s'))|_{H^{-1}}^2 + |F_2^n(s', Y_\alpha^n(s') + W_A^n(s'))|_{H^{-1}}^2) ds' \\
\leq &C \int_s^{t_0} |Y_\alpha^n(s')|_{H^1}^2 ds' + C \int_s^{t_0} (c_1(s')^2 + K^2)(1 + |Y_\alpha^n(s') + W_A^n(s')|_{L^{2m}}^{2m}) ds',
\end{aligned}$$

we obtain

$$E|Y_\alpha^n(t)|_{W^{1,2}([s,t_0],H^{-1})}^2 \leq C|x|_{L^{2m}}^{2m} + C.$$

Thus by [FG95, Theorems 2.1, 2.2] we get Y_α^n in $L^2([s,t_0],H) \cap C([s,t_0],H^{-2})$ are tight. Also W_A^n in $L^2([s,t_0],H) \cap C([s,t_0],H^{-2})$ are tight. Therefore, we have (Y_α^n, W_A^n) are tight in $(L^2([s,t_0],H) \cap C([s,t_0],H^{-2})) \times (L^2([s,t_0],H) \cap C([s,t_0],H^{-2}))$. Hence there exists a subsequence (still denoted by (Y_α^n, W_A^n)) converging in distribution. By Skorohod's embedding theorem, there exist a stochastic basis $(\tilde{\Omega}, \tilde{\mathcal{F}}, \{\tilde{\mathcal{F}}_t\}_{t\in[s,t_0]}, \tilde{P})$ and, on this basis, $L^2([s,t_0];H) \cap C([s,t_0],H^{-2})$-valued random variables $\tilde{Y}_\alpha^n, \tilde{Y}_\alpha, \tilde{W}_A^n, \tilde{W}_A, n \geq 1$, such that for every $n \in \mathbb{N}$, $(\tilde{Y}_\alpha^n, \tilde{W}_A^n)$ has the same law as (Y_α^n, W_A^n) on $(L^2([s,t_0];H) \cap C([s,t_0],H^{-2})) \times (L^2([s,t_0];H) \cap C([s,t_0],H^{-2}))$, and $\tilde{Y}_\alpha^n \to \tilde{Y}_\alpha, \tilde{W}_A^n \to \tilde{W}_A$ in $L^2([s,t_0];H) \cap C([s,t_0],H^{-2})$, $\tilde{P}$ -a.s.. Then (4.11) holds for $\tilde{Y}_\alpha^n, \tilde{Y}_\alpha$. For each $n \geq 1$, define the process

$$\begin{aligned}
\tilde{M}_n(t) :=& \tilde{Y}_\alpha^n(t) + \tilde{W}_A^n(t) - x_n - \int_s^t A\tilde{Y}_\alpha^n(s')ds' - \int_s^t A\tilde{W}_A^n(s')ds' - \int_s^t F_1^\alpha(s', \tilde{Y}_\alpha^n(s') + \tilde{W}_A^n(s'))ds' \\
&- \int_s^t F_2^n(s', \tilde{Y}_\alpha^n(s') + \tilde{W}_A^n(s'))ds'.
\end{aligned}$$

In fact $\tilde{M}_n$ is a square integrable martingale with respect to the filtration

$$\{\mathcal{G}_n\}_t = \sigma\{\tilde{Y}_\alpha^n(r), \tilde{W}_A^n(r), r \leq t\}.$$

For all $r \leq t \in [s,t_0]$, all bounded continuous functions ϕ on $(C([s,r];H^{-2}) \cap L^2([s,r];H)) \times (C([s,r];H^{-2}) \cap L^2([s,r];H))$ and all $v \in C^\infty([0,1])$, we have

$$\tilde{E}(\langle \tilde{M}_n(t) - \tilde{M}_n(r), v\rangle \phi(\tilde{Y}_\alpha^n \restriction_{[s,r]}, \tilde{W}_A^n \restriction_{[s,r]})) = 0$$

and

$$\tilde{E}((\langle \tilde{M}_n(t), v\rangle^2 - \langle \tilde{M}_n(r), v\rangle^2 - \int_r^t |(1 - \frac{1}{n}A)^{-1}\sqrt{G}v|_H^2 ds)\phi(\tilde{Y}_\alpha^n \restriction_{[s,r]}, \tilde{W}_A^n \restriction_{[s,r]})) = 0.$$

By the Burkholder-Davis-Gundy inequality we have for $1 < p < \infty$

$$\sup_n \tilde{E}|\langle \tilde{M}_n(t), v\rangle|^{2p} \leq C \sup_n E(\int_0^t |(1-\frac{1}{n}A)^{-1}\sqrt{G}v|_H^2 dr)^p < \infty.$$

Since $\tilde{Y}_\alpha^n \to \tilde{Y}_\alpha, \tilde{W}_A^n \to \tilde{W}_A$ in $L^2([s,t_0];H) \cap C([s,t_0],H^{-\beta})$, we have

$$\begin{aligned}
&\tilde{E}\int_s^t |\langle F_2^n(s',\tilde{Y}_\alpha^n(s')+\tilde{W}_A^n(s'))-F_2(s',\tilde{Y}_\alpha(s')+\tilde{W}_A(s')),v\rangle|ds' \\
\leq&\tilde{E}\int_s^t |\langle F_2^n(s',\tilde{Y}_\alpha^n(s')+\tilde{W}_A^n(s'))-F_2^n(s',\tilde{Y}_\alpha(s')+\tilde{W}_A(s')),v\rangle|ds' \\
&+\tilde{E}\int_s^t |\langle F_2^n(s',\tilde{Y}_\alpha(s')+\tilde{W}_A(s'))-F_2(s',\tilde{Y}_\alpha(s')+\tilde{W}_A(s')),v\rangle|ds' \\
\leq&C\tilde{E}\int_s^t (|\tilde{Y}_\alpha^n(s')|+|\tilde{W}_A^n(s')|+|\tilde{W}_A(s')|+1+|\tilde{Y}_\alpha(s')|)[|\tilde{Y}_\alpha^n(s')-\tilde{Y}_\alpha(s')|+|\tilde{W}_A^n(s')-\tilde{W}_A(s')|]ds' \\
&+\tilde{E}\int_s^t |\langle F_2^n(s',\tilde{Y}_\alpha(s')+\tilde{W}_A(s'))-F_2(s',\tilde{Y}_\alpha(s')+\tilde{W}_A(s')),v\rangle|ds' \\
\to&0, \text{ as } n\to\infty,
\end{aligned}$$

where in the second inequality we used (g2) and we used (4.11) to obtain the convergence. The other terms can be estimated similarly, which altogether implies that

$$\lim_{n\to\infty} \tilde{E}|\langle \tilde{M}_n(t)-M(t),v\rangle| = 0$$

and

$$\lim_{n\to\infty} \tilde{E}|\langle \tilde{M}_n(t)-M(t),v\rangle|^2 = 0,$$

where

$$M(t) := \tilde{Y}_\alpha(t)+\tilde{W}_A(t)-x-\int_s^t A\tilde{Y}_\alpha(s')ds'-\int_s^t A\tilde{W}_A(s')ds'-\int_s^t F_\alpha(s',\tilde{Y}_\alpha(s')+\tilde{W}_A(s'))ds.$$

Taking the limit we obtain that for all $r \leq t \in [s,t_0]$, all bounded continuous functions on $(C([s,r];H^{-\beta}) \cap L^2([s,r];H)) \times (C([s,r];H^{-\beta}) \cap L^2([s,r];H))$ and $v \in C^\infty([0,1])$,

$$\tilde{E}(\langle M(t)-M(r),v\rangle \phi(\tilde{Y}_\alpha \restriction_{[s,r]}, \tilde{W}_A \restriction_{[s,r]})) = 0.$$

and

$$\tilde{E}((\langle M(t),v\rangle^2-\langle M(r),v\rangle^2-\int_r^t |\sqrt{G}v|_H^2 ds')\phi(\tilde{Y}_\alpha \restriction_{[s,r]}, \tilde{W}_A \restriction_{[s,r]})) = 0.$$

Thus, the existence of a martingale solution for (4.4) follows by a martingale representation theorem (cf. [DZ92, Theorem 8.2],[O05, Theorem 2]). Now we obtain $\tilde{X}_\alpha = \tilde{Y}_\alpha + \tilde{W}_A$ is a martingale solution of (4.4) in $[s,t_0]$. Thus, by Girsanov's Theorem and the pathwise uniqueness of the solution to (4.4) when $f \equiv 0$, we obtain the uniqueness of (the distributions for) the martingale solution of (4.4), which implies that $\tilde{X}_\alpha$ has the same distribution as X_α. By this and (4.11) we have for $s \leq t \leq t_0$,

$$E|Y_\alpha(t)|_{L^{2m}(0,1)}^{2m} \leq C|x|_{L^{2m}(0,1)}^{2m} + C.$$

Moreover,

$$E|X_\alpha(t,s,x)|^{2m}_{L^{2m}(0,1)} \le C|x|^{2m}_{L^{2m}(0,1)} + C, \tag{4.12}$$

where C is a constant independent of α, s. Furthermore, by [EK86, Theorem 4.2] and the uniqueness of the distributions for the martingale solution of (4.4) we obtain that the laws of the martingale solutions $X_\alpha(t,s,x)$ of (4.4) form a Markov process. We use $\mu^\alpha_{s,t}(x,dy)$ to denote the distribution of $X_\alpha(t,s,x)$, $x \in H$. Then by the Markov property we have for $0 \le s \le t_1 \le t_2 \le T, x \in H$

$$\mu^\alpha_{s,t_2}(x,dz) = \int_H \mu^\alpha_{s,t_1}(x,dy)\mu^\alpha_{t_1,t_2}(y,dz).$$

By this and (4.12) we obtain by iteration that for any $t \in [s,T]$

$$\int |z|^{2m}_{L^{2m}}\mu^\alpha_{s,t}(x,dz) = \int\int |z|^{2m}_{L^{2m}}\mu^\alpha_{t_1,t}(y,dz)\mu^\alpha_{s,t_1}(x,dy) \le C|x|^{2m}_{L^{2m}(0,1)} + C,$$

which is exactly our assertion. □

Since $c_1 \in L^2([0,T])$ by (f1), the set B in Theorem 2.3 is $L^{2m}(0,1)$. By Theorem 2.3 we now obtain the following:

Theorem 4.2 Suppose that (f1), (f2), (g1), (g2), (G.1) hold. For each initial value $x \in L^{2m}(0,1)$ there exists a martingale solution to problem (4.1)-(4.3), i.e. there exists a stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, P)$, a cylindrical Wiener process W on H and a progressively measurable process $X : [0,T] \times \Omega \to H$, such that for P-a.e. $\omega \in \Omega$,

$$X(\cdot,\omega) \in L^\infty([0,T];L^2(0,1)) \cap C([0,T];H^{-2})$$

and for all $\phi \in C^2([0,1])$

$$\begin{aligned}\langle X(t),\phi\rangle =&\langle x,\phi\rangle + \int_0^t \langle X(r), \partial^2_\xi\phi\rangle dr + \int_0^t \langle f(r,X(r)),\phi\rangle dr\\ &- \int_0^t \langle g(r,X(r)),\partial_\xi\phi\rangle dr + \int_0^t \langle \phi, \sqrt{G}dW(r)\rangle \quad \forall t \in [0,T] \quad P-a.s..\end{aligned}$$

We also have

$$X - W_A \in L^2([0,T],H^1) \quad P-a.s., \quad E\int_0^T |X(t)|^{2m}_{L^{2m}}dt < \infty. \tag{4.13}$$

Moreover, if P, P' are the laws of two martingale solutions on $C([0,T];H^{-2})$ to problem (4.1)-(4.3) with the same initial value $x \in L^{2m}$ and

$$\int_0^T |\omega(t)|^{2m}_{L^{2m}}dt < \infty \quad P + P' - a.s.,$$

then $P = P'$, where $\omega(\cdot)$ is the canonical process on $C([0,T];H^{-2})$.

Proof (4.13) follows from (4.8)-(4.11). The weak uniqueness follows by (f1), [MR99, Theorem 3.3] and the pathwise uniqueness of the solution of (4.4) when $f \equiv 0$. Here we can extend a solution to $C([0,\infty),H^{-2})$ by taking $X(t) = X(T), t \ge T$ and apply the results in [MR99]. □

Likewise, Theorem 3.1 applies to all $\zeta \in \mathcal{P}(H)$ such that

$$\int_H |x|^{2m}_{L^{2m}(0,1)} \zeta(dx) < \infty.$$

More precisely, we have:

Theorem 4.3 Let $\zeta \in \mathcal{P}(H)$ be such that

$$\int_H |x|^{2m}_{L^{2m}} \zeta(dx) < \infty.$$

Then there exists a solution $\mu_t(dx)dt$ to the Fokker-Planck equation (1.3) such that

$$\sup_{t\in[s,T]} \int_H |x|^2 \mu_t(dx) < \infty$$

and

$$t \mapsto \int_H u(t,x)\mu_t(dx)$$

is continuous on $[s,T]$ for all $u \in D(L_0)$. Finally, for some $C > 0$ and $\delta \in (0, \frac{1}{4})$ as in Hypothesis 2.1 (iii)

$$\int_s^T \int_H (|x|^{2m}_{L^{2m}} + |(-A)^\delta x|^2 + |x|^2)\mu_r(dx)dr \le C \int_H |x|^{2m}_{L^{2m}} \zeta(dx).$$

Remark 4.4 (i) Here we choose the L^{2m}-norm as a Lyapunov function J in Hypothesis 2.2. In [RS06], the first named author of this paper and Sobol studied the above semilinear stochastic partial differential equations with time independent coefficients. They also choose the L^{2m}-norm as a Lyapunov function with weakly compact level sets for the Kolmogorov operator L_0 and by analyzing the resolvent of the operator L they constructed a unique martingale solution to this problem if the noise is trace-class. In this paper, we concentrate on space-time white noise for which the method of constructing Lyapunov functions with weakly compact level sets for the Kolmogorov operator L_0 is more delicate than in the case, where $\mathrm{TrG} < \infty$.

(ii) If $g \equiv 0$, we can obtain the uniqueness of the solution to the Fokker-Planck equation by [BDR11, Theorem 4.1].

To obtain pathwise uniqueness, we additionally assume that f satisfies the following inequality: for $t \in [0,T], \xi \in [0,1], z_1, z_2 \in \mathbb{R}$,

$$\langle f(\xi,t,z_1) - f(\xi,t,z_2), z_1 - z_2 \rangle \le L(1 + |z_1|^{m-1} + |z_2|^{m-1})|z_1 - z_2|^2. \tag{4.14}$$

Now we give the definition of a (probabilistically) strong solution to (4.1)-(4.3).

Definition 4.5 We say that there exists a (probabilistically) strong solution to (4.1)-(4.3) over the time interval $[0,T]$ if for every probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, P)$ with an $\mathcal{F}_t$-Wiener process W, there exists an $\mathcal{F}_t$-adapted process $X : [0,T] \times \Omega \to H$ such that for $P-a.s.\ \omega \in \Omega$

$$X(\cdot,\omega) \in L^\infty([0,T]; L^2(0,1)) \cap C([0,T]; H^{-2}),$$

and for all $\phi \in C^2([0,1])$ we have P-a.s.

$$\begin{aligned}\langle X(t),\phi\rangle =&\langle X_0,\phi\rangle + \int_0^t \langle X(r),\partial_\xi^2\phi\rangle dr + \int_0^t \langle f(r,X(r)),\phi\rangle dr \\ &- \int_0^t \langle g(r,X(r)),\partial_\xi\phi\rangle dr + \int_0^t \langle \phi,\sqrt{G}dW(r)\rangle \quad \forall t\in[0,T].\end{aligned}$$

Theorem 4.6 Suppose that f satisfies (4.14). Then there exists at most one probabilistically strong solution to (4.1)-(4.3) such that

$$\int_0^T |X(t)|_{L^{2m}}^{2m} dt < \infty, \quad P-a.s.,$$

and

$$X - W_A \in L^2([0,T],H_0^1) \quad P-a.s..$$

Proof Consider two solutions X_1, X_2 of (4.1)-(4.3) in the interval $[0,T]$. Since $X - W_A \in L^2([0,T],H_0^1)$ P-a.s. and $X_1 - X_2 \in L^2([0,T],H_0^1)$ P-a.s., we have

$$\begin{aligned}\langle X_1(t)-X_2(t),\phi\rangle =& \int_0^t \langle X_1(r)-X_2(r),\partial_\xi^2\phi\rangle dr + \int_0^t \langle f(r,X_1(r))-f(r,X_2(r)),\phi\rangle dr \\ &- \int_0^t \langle g(r,X_1(r))-g(r,X_2(r)),\partial_\xi\phi\rangle dr, \quad \forall t\in[0,T], \quad P-a.s..\end{aligned}$$

Taking $\phi = e_k$ we obtain

$$\begin{aligned}\langle X_1(t)-X_2(t),e_k\rangle^2 =&2\int_0^t \langle X_1(r)-X_2(r),e_k\rangle[\langle X_1(r)-X_2(r),\partial_\xi^2 e_k\rangle \\ &+ \langle f(r,X_1(r))-f(r,X_2(r)),e_k\rangle \\ &- \langle g(r,X_1(r))-g(s,X_2(r)),\partial_\xi e_k\rangle]dr, \quad \forall t\in[0,T], \quad P-a.s..\end{aligned}$$

Summing over k, we obtain

$$\begin{aligned}&|X_1(t)-X_2(t)|^2 + 2\int_0^t |\nabla(X_1(r)-X_2(r))|^2 dr \\ \leq& 2\int_0^t \langle f(r,X_1(r))-f(r,X_2(r)),X_1(r)-X_2(r)\rangle dr \\ &- 2\int_0^t \langle g(r,X_1(r))-g(s,X_2(r)),\partial_\xi(X_1(r)-X_2(r))\rangle dr.\end{aligned}$$

For the first term on the right hand side by (4.14) we have

$$\begin{aligned}
&\int_0^t \langle f(r,X_1(r))-f(r,X_2(r)),X_1(r)-X_2(r)\rangle dr\\
\leq &C\int_0^t |X_1(r)-X_2(r)|_{L^4}^2(1+|X_1(r)|_{L^{2m}}^{m-1}+|X_2(r)|_{L^{2m}}^{m-1})dr\\
\leq &C\int_0^t |X_1(r)-X_2(r)|_{H^{1/4}}^2(1+|X_1(r)|_{L^{2m}}^{m-1}+|X_2(r)|_{L^{2m}}^{m-1})dr\\
\leq &C\int_0^t |X_1(r)-X_2(r)|_{H^1}^{1/2}|X_1(r)-X_2(r)|^{3/2}(1+|X_1(r)|_{L^{2m}}^{m-1}+|X_2(r)|_{L^{2m}}^{m-1})dr\\
\leq &\int_0^t \varepsilon|X_1(r)-X_2(r)|_{H^1}^2+C|X_1(r)-X_2(r)|^2(1+|X_1(r)|_{L^{2m}}^{2m}+|X_2(r)|_{L^{2m}}^{2m})dr,
\end{aligned}$$

where we used Holder's inequality in the first inequality, $H^{1/4}\subset L^4$ in the second inequality, the interpolation inequality in the third inequality and Young's inequality in the last inequality. For the second term on the right hand side we have

$$\begin{aligned}
&\int_0^t \langle g(r,X_1(r))-g(r,X_2(r)),\partial_\xi(X_1(r)-X_2(r))\rangle dr\\
\leq &C\int_0^t |\partial_\xi(X_1(r)-X_2(r))||X_1(r)-X_2(r)|_{L^4}(1+|X_1(r)|_{L^4}+|X_2(r)|_{L^4})dr\\
\leq &C\int_0^t |\partial_\xi(X_1(r)-X_2(r))|^{5/4}|X_1(r)-X_2(r)|^{3/4}(1+|X_1(r)|_{L^4}+|X_2(r)|_{L^4})dr\\
\leq &\int_0^t \varepsilon|X_1(r)-X_2(r)|_{H^1}^2+C|X_1(r)-X_2(r)|^2(1+|X_1(r)|_{L^{2m}}^{2m}+|X_2(r)|_{L^{2m}}^{2m})dr.
\end{aligned}$$

where we used $H^{1/4}\subset L^4$ and the interpolation inequality in the second inequality and Young's inequality in the last inequality. Combining the above three inequalities and using Gronwall-Bellman's inequality, $X_1=X_2$ follows. □

Combining Theorems 4.3 and 4.6 we obtain the following existence and uniqueness result by using the Yamada-Watanabe Theorem (cf. [Ku07, Theorem 3.14]).

Theorem 4.7 Suppose that (f1), (f2), (4.14), (g1), (g2), (G.1) hold. Then for each initial condition $X_0\in L^{2m}(0,1)$, there exists a pathwise unique probabilistically strong solution X of equation (4.1) over $[0,T]$ with initial condition $X(0)=X_0$ such that

$$\int_0^T |X(t)|_{L^{2m}}^{2m}dt<\infty \quad P-a.s.$$

and

$$X-W_A\in L^2([0,T],H^1) \quad P-a.s.. \tag{4.15}$$

Remark 4.8 If c_1 in (f1) is bounded and (4.14) is modified to the following stronger local Lipschitz condition

$$|f(\xi,t,z_1)-f(\xi,t,z_2)|\leq L(1+|z_1|^{m-1}+|z_2|^{m-1})|z_1-z_2|,$$

condition (4.15) can be dropped. Then we can also prove that there exists a unique probabilistically strong solution $X \in C([0,T], L^{2m})$ by considering mild solutions and using similar arguments as in [G98].

Remark 4.9 If TrG $< \infty$, we can apply Theorems 2.3 and 3.1 to other stochastic semilinear equations and to higher dimension. For example, we can consider the 2D stochastic Navier-Stokes equation. Let O be a bounded domain in $\mathbb{R}^2$ with smooth boundary. Define

$$V := \{v \in H_0^1(O; \mathbb{R}^2), \mathrm{div} v = 0 \text{ a.e. in } O\},$$

and H to be the closure of V with respect to L^2-norm. The linear operator P_H (Helmhotz-Hodge projection) and A (Stokes operator with viscosity ν) are defined by

$$P_H : L^2(O, \mathbb{R}^2) \to H \text{ orthogonal projection ;}$$

$$A : H^2(O, \mathbb{R}^2) \cap V \to H : Ax = \nu P_H \Delta x.$$

The nonlinear operator $F : V \to V^*$ is defined by $F(x) := -P_H[x \cdot \nabla x]$. Then if G is a trace-class symmetric non-negative operator, Hypothesis 2.1 is satisfied. For Hypothesis 2.2 we choose $F_\alpha = P_{\frac{1}{[\alpha]+1}} F$ as in Remark (ii) before Theorem 2.3 and $J(t,x) := |x||x|_V + 1$. Then by Itô's formula we know that Hypothesis 2.2 (iv) is satisfied. Consequently, we obtain the existence of a martingale solution for the stochastic 2D Navier-Stokes equation. Of course, as said in the introduction, this result is well-known and not the best possible for the 2D Navier-Stokes equation. Therefore, we omit the details here.

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