On parabolic inequalities for generators of diffusions with jumps

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Abstract We prove absolute continuity of space-time probabilities satisfying certain parabolic inequalities for generators of diffusions with jumps. As an application, we prove absolute continuity of transition probabilities of singular diffusions with jumps under minimal conditions that ensure absolute continuity of the corresponding diffusions without jumps.

Keywords diffusion with jumps, transition probability, absolute continuity

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Let us a consider a diffusion process ξ_t in $\mathbb{R}^d$ with jumps (see, e.g., [3], [15], [16], and [26]) having time-dependent generator

$$Lu(x,t) = L_0 u(x,t) + \int_{\mathbb{R}^d\backslash\{0\}} [u(x+y,t) - u(x,t) - \langle y, \nabla_x u(x,t)\rangle]\, \nu_{x,t}(dy),$$

where

$$L_0 u(x,t) = a^{ij}(x,t)\partial_{x_i}\partial_{x_j} u(x,t) + b^i(x,t)\partial_{x_i} u(x,t)$$

with summation over repeated indices, $\langle\,\cdot\,,\,\cdot\,\rangle$ is the inner product in $\mathbb{R}^d$, $\nabla_x u(x,t) = (\partial_{x_1} u(x,t),\ldots,\partial_{x_d} u(x,t))$. The result of this work gives a simple sufficient condition for the absolute continuity of the transition probabilities $P(t,dy)$ of this diffusion (distributions of ξ_t) with respect to Lebesgue measure for almost all t. It says that, under minimal natural assumptions about the diffusion matrix $A=(a^{ij})$, the drift $b=(b^i)$, and the intensity measure ν_t, the space-time measure $(\det A)^{1/(d+1)}P(t,dy)\,dt$ is absolutely continuous on $\mathbb{R}^d\times(0,T)$. Hence $P(t,dy)$ is absolutely continuous for almost every t if $\det A>0$. Moreover, our result is concerned with a parabolic inequality related to the Fokker–Planck–Kolmogorov equation

$$\partial_t P = L^* P$$

satisfied by the transition probabilities. In particular, it even does not assume existence of a diffusion. The case without jumps ($\nu = 0$) was considered in [7]. The main feature of our hypotheses (as in [7]) is that they allow rather singular coefficients, no local boundedness or Lebesgue integrability is assumed. Informally the conclusion is that the condition known to be sufficient for absolute continuity without jumps remains sufficient in the presence of jumps. Certainly, this conclusion does not come as a surprise and is quite expected for the experts, but the main point is to justify this under minimal possible assumptions, which is not straightforward, as we shall see. Under considerably less general assumptions (in particular, including existence of certain stochastic integrals) a result of this type for the space-time measure can be derived from [1], [2], and [21] (although formally is not stated there), where Krylov-type estimates of stochastic integrals were derived; another difference of a more principal character between our framework and that of [1], [2], and [21] is the fact that we deal with an inequality (in place of equations) defined in the weak sense and involving only the second order part of the operator. It

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should be noted that there are other types of sufficient conditions that are expressed either entirely in terms of the jump part and apply without A or involve the Malliavin calculus (which requires smoothness of the coefficients) to give absolute continuity in the case of degenerate A (see [4], [6], [9], [10], [12], [13], [19], [20], [22], [23], [24], and [27]). Existence of densities in the case of absolutely continuous initial distributions is considered in [28]; in our setting no initial distribution is involved since the parabolic equation (or inequality) is considered on $\mathbb{R}^d \times (0,T)$.

We consider a more general situation. Why it covers the previous case will be explained in Example 2. Let $T > 0$ be fixed. Suppose that for every $(x,t) \in \mathbb{R}^d \times (0,T)$ we are given a nonnegative symmetric matrix $A(x,t) = (a^{ij}(x,t))_{1\le i,j\le d}$ and a locally bounded nonnegative Borel measure $K(x,t,dy)$ on $\mathbb{R}^d \setminus \{x\}$.

We assume that for every $(x,t) \in \mathbb{R}^d \times (0,T)$ and every $\psi \in C_0^\infty(\mathbb{R}^d)$ the mapping

$$(x,t) \mapsto \int_{\mathbb{R}^d\setminus\{x\}} (|x-y| \wedge 1)^2\, \psi(y)\, K(x,t,dy)$$

is Borel measurable, where $\alpha \wedge \beta = \min(\alpha,\beta)$ and $|\cdot|$ is the norm in $\mathbb{R}^d$.

A function $\varphi \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ with $0 \le \varphi \le 1$ is called a local unit if

(i) $\varphi(x,y) = 1$ for every (x,y) in a neighborhood of the diagonal $x = y$,

(ii) for every ball $U \subset \mathbb{R}^d$ there exists a ball $U' \subset \mathbb{R}^d$ such that the mappings $y \mapsto \varphi(x,y)$, where $x \in U$, have support in U'.

Let us fix such a function φ. Note that φ need not be symmetric.

Let $U(x,R)$ denote the closed ball of radius R centered at x in $\mathbb{R}^d$.

For every $u \in C_0^\infty(\mathbb{R}^d \times (0,T))$ let

$$\mathcal{A}u(x,t) = \sum_{i,j=1}^d a^{ij}(x,t)\partial_{x_i}\partial_{x_j}u(x,t),$$

$$\mathcal{K}u(x,t) = \int_{\mathbb{R}^d\setminus\{x\}} [u(y,t) - u(x,t)\varphi(x,y) - \langle y-x, \nabla_x u(x,t)\rangle\varphi(x,y)]\, K(x,t,dy).$$

Our main result is the following theorem. Let $C_0^\infty(\mathbb{R}^d \times (0,T))$ denote the class of all infinitely differentiable functions on $\mathbb{R}^d \times (0,T)$ with compact support.

Theorem 1. *Assume that a nonnegative locally finite Borel measure μ on $\mathbb{R}^d \times (0,T)$ is such that $a^{ij} \in L^1_{loc}(\mu)$, for every $\psi \in C_0^\infty(\mathbb{R}^d)$ the mapping*

$$(x,t) \mapsto \int_{\mathbb{R}^d\setminus\{x\}} (|x-y| \wedge 1)^2\psi(y)\, K(x,t,dy)$$

is integrable with respect to μ on $\mathbb{R}^d \times (0,T)$, and for every compact set $Q \subset \mathbb{R}^d \times (0,T)$ there exists a constant $C > 0$ such that

$$\int_{\mathbb{R}^d\times(0,T)} [\partial_t u + \mathcal{A}u + \mathcal{K}u]\, d\mu \le C\Big(\max_{x,t} |\nabla_x u(x,t)| + \max_{x,t} |u(x,t)|\Big) \qquad (1)$$

for all nonnegative functions $u \in C_0^\infty(\mathbb{R}^d \times (0,T))$ with $\operatorname{supp} u \subset Q$. Then we have $(\det A)^{1/(d+1)} \cdot \mu = \varrho\, dx\, dt$, where $\varrho \in L^{(d+1)'}_{loc}(\mathbb{R}^d \times (0,T))$, $(d+1)' := (d+1)/d = 1 + 1/d$.

Proof. The method of our proof is similar to [7, Theorem 3.1], but some additional technicalities arise. Let us fix a nonnegative function $\zeta \in C_0^\infty(\mathbb{R}^d \times (0,T))$ of the form $\zeta(x,t) = \eta(x)\theta(t)$, where $\eta \in C_0^\infty(\mathbb{R}^d)$, $\theta \in C_0^\infty((0,T))$, $\operatorname{supp}\eta \subset U(x_0,R')$, $x_0 \in \mathbb{R}^d$, $\operatorname{supp}\theta \subset J$, J is a closed interval in $(0,T)$, positive numbers $R > R'$ are such that for every x in the ball $U(x_0,R')$ the support of $y \mapsto \varphi(x,y)$ belongs to $U(x_0,R)$.

Set $u(x,t) = h(x,t)\zeta(x,t)$, where h is a nonnegative function in $C^\infty(U(x_0,R)\times J)$, i.e., h is C^∞ in a neighborhood of the compact set $W := U(x_0,R)\times J$. Note that

$$\begin{aligned}
u(y,t) - u(x,t)\varphi(x,y) - \langle y-x, \nabla_x u(x,t)\rangle\varphi(x,y) = \\
= [h(y,t) - h(x,t) - \langle y-x, \nabla_x h(x,t)\rangle]\varphi(x,y)\zeta(x,t) + \\
+ h(y,t)\zeta(y,t)(1-\varphi(x,y)) + \\
+ h(y,t)\varphi(x,y)[\zeta(y,t) - \zeta(x,t) - \langle y-x, \nabla_x\zeta(x,t)\rangle] + \\
+ (h(y,t) - h(x,t))\varphi(x,y)\langle y-x, \nabla_x\zeta(x,t)\rangle.
\end{aligned}$$

Let us now assume that h is convex with respect to x in $U(x_0,R)$. This means that

$$h(y,t) - h(x,t) - \langle y-x, \nabla_x h(x,t)\rangle \geq 0.$$

Therefore,

$$\begin{aligned}
u(y,t) - u(x,t)\varphi(x,y) - \langle y-x, \nabla_x u(x,t)\rangle\varphi(x,y) \geq \\
\geq -\zeta(y,t)(1-\varphi(x,y)) \max_{(x,t)\in W}|h(x,t)| - \\
- \varphi(x,y)|\zeta(y,t) - \zeta(x,t) - \langle y-x, \nabla_x\zeta(x,t)\rangle| \max_{(x,t)\in W}|h(x,t)| - \\
- |x-y|^2\varphi(x,y)|\nabla_x\zeta(x,t)| \max_{(x,t)\in W}|\nabla_x h(x,t)|.
\end{aligned}$$

The next step is to show that there exists a constant $C_1 > 0$ such that

$$\int_{\mathbb{R}^d\times(0,T)} \mathcal{K}u(x,t)\,d\mu \geq -C_1\Big(\max_{(x,t)\in W}|\nabla_x h(x,t)| + \max_{(x,t)\in W}|h(x,t)|\Big). \tag{2}$$

To this end we observe that there is a nonnegative function $\psi \in C_0^\infty(\mathbb{R}^d)$ such that

$$\begin{aligned}
\zeta(y,t)(1-\varphi(x,y)) + \varphi(x,y)|\zeta(y,t) - \zeta(x,t) - \langle y-x, \nabla_x\zeta(x,t)\rangle| + \\
+ |x-y|^2\varphi(x,y)|\nabla_x\zeta(x,t)| \leq (|x-y|\wedge 1)^2\psi(y).
\end{aligned}$$

This can be verified separately for each term. Since by our choice $\zeta(y,t) = \eta(y)\theta(t)$, we can deal with $\eta(y)$ in place of $\zeta(y,t)$. The first term is estimated by using Taylor's expansion and the fact that $\varphi - 1$ vanishes at the diagonal with all derivatives. Taylor's expansion also helps to handle the second term, but here one has to consider x belonging to a ball U containing the neighborhood of radius 1 around $\operatorname{supp}\eta$, and in this case condition (ii) on φ applies; for other x and y with $|y-x| \leq 1$ the second term vanishes and if $|y-x| > 1$, then this term is estimated by $\eta(y)\max_t\theta(t)$. The last term is estimated even simpler. Hence we obtain (2). It is clear that

$$\partial_t u + \mathcal{A}u = [\partial_t h + \mathcal{A}h]\zeta + [\partial_t\zeta + \mathcal{A}\zeta]h + 2\langle A\nabla_x h, \nabla_x\zeta\rangle.$$

Hence there exists a constant $C_2 > 0$ (depending on ζ but not on h) such that

$$\int_{\mathbb{R}^d\times(0,T)} [\partial_t u + \mathcal{A}u]\,d\mu \geq \int_{\mathbb{R}^d\times(0,T)} [\partial_t h + \mathcal{A}h]\zeta\,d\mu - C_2\Big(\max_{(x,t)\in W}|\nabla_x h(x,t)| + \max_{(x,t)\in W}|h(x,t)|\Big).$$

Summing the obtained estimates and using condition (1) we obtain

$$\int_{\mathbb{R}^d\times(0,T)} [\partial_t h + \mathcal{A}h]\zeta\,d\mu \leq C_3\Big(\max_{(x,t)\in W}|\nabla_x h(x,t)| + \max_{(x,t)\in W}|h(x,t)|\Big). \tag{3}$$

We shall now construct a suitable function h. According to Krylov's result (see [17, Theorem 2] or [18, Chapter III, § 2]), there exist constants $C(R,d) > 0$ and $\kappa(R,d) > 0$ such that for every nonnegative function $f \in C_0^\infty(U(x_0,R)\times J)$ there exists a bounded function $z \le 0$ on $U(x_0, 4R) \times \mathbb{R}^1$ such that $x \mapsto z(x,t)$ is convex on $U(x_0, 4R)$ for each fixed $t \in J$ and has the following two properties: for any smooth probability density g with support in the centered unit ball of $\mathbb{R}^{d+1}$, letting $g_\varepsilon(y,t) = \varepsilon^{-d-1} g(y/\varepsilon, t/\varepsilon)$, $f_\varepsilon = f * g_\varepsilon$, and $z_\varepsilon = z * g_\varepsilon$, $\varepsilon > 0$, one has

$$C(R,d)\big(\det(\alpha^{ij})\big)^{1/(d+1)} f_\varepsilon(x,t) \le \partial_t z_\varepsilon(x,t) + \alpha^{ij}\partial_{x_i}\partial_{x_j} z_\varepsilon(x,t),$$
$$|z(x,t)| \le \kappa(R,d)\|f\|_{L^{d+1}(U(x_0,R)\times J)} \quad \forall\,(x,t) \in U(x_0,2R) \times J$$

for every $\varepsilon \in (0,R)$ and each nonnegative symmetric matrix (α^{ij}). It should be noted that in [17] there is minus in front of $\partial_t z_\varepsilon(x,t)$, but this makes no difference since we can reverse the variable t. For all sufficiently small ε we have

$$|z_\varepsilon(x,t)| \le 2\kappa(R,d)\|f\|_{L^{d+1}(U(x_0,R)\times J)} \quad \forall\,(x,t) \in U(x_0,2R) \times J.$$

We observe that z_ε is nonpositive and convex with respect to x on $U(x_0,3R) \times J$ for any $\varepsilon < R$. Moreover,

$$\max_{x\in U(x_0,R),t\in J} |\nabla_x z_\varepsilon(x,t)| \le 2R^{-1} \max_{x\in U(x_0,2R),t\in J} |z_\varepsilon(x,t)| \le 2R^{-1}\kappa(R,d)\|f\|_{L^{d+1}(U(x_0,R)\times J)}.$$

Substituting $h(x,t) = z_\varepsilon(x,t) + \max_{(y,t)\in W} |z_\varepsilon(y,t)|$ in (3), which is possible since $h \ge 0$ on W and $x \mapsto h(x,t)$ is convex on $U(x_0,R)$, we obtain the estimate

$$\int_{\mathbb{R}^d\times(0,T)} |\det A|^{1/(d+1)} f_\varepsilon \zeta\, d\mu \le C_4 \|f\|_{L^{d+1}}.$$

Letting $\varepsilon \to 0$, we obtain the same estimate for f, which yields that

$$|\det A|^{1/(d+1)}\zeta\, d\mu = \varrho_\zeta\, dx\, dt, \quad \varrho_\zeta \in L^{d+1}(U(x_0,R)\times J).$$

Since $\zeta(x,t) = \eta(x)\theta(t)$, where η and θ were arbitrary smooth functions with compact support, we arrive at the desired conclusion for the measure $|\det A|^{1/(d+1)} \cdot \mu$. $\square$

Remark 1. We observe that the exponent $(d+1)' = (d+1)/d$ cannot be replaced by a larger one. Indeed, it is well-known that the class $W^{1,1}_{loc}(\mathbb{R}^d \times (0,1))$ is embedded into $L^{(d+1)'}_{loc}(\mathbb{R}^d \times (0,1))$, but not into $L^p_{loc}(\mathbb{R}^d \times (0,T))$ with $p > (d+1)'$. So let $\mu = \varrho\, dx\, dt$, where $\varrho \in W^{1,1}_{loc}(\mathbb{R}^d \times (0,1))$ and $\varrho \ge 0$, but ϱ does not belong to any $L^p_{loc}(\mathbb{R}^d \times (0,1))$ with $p > (d+1)'$. Then, for every compact set $Q \subset \mathbb{R}^d \times (0,1)$, there exists a constant $C > 0$ such that

$$\int_{\mathbb{R}^d\times(0,1)} \big[\partial_t u + \Delta u\big]\, d\mu \le C\Big(\max_{x,t} |\nabla_x u(x,t)| + \max_{x,t} |u(x,t)|\Big)$$

for every nonnegative function $u \in C_0^\infty(\mathbb{R}^d \times (0,1))$ with $\operatorname{supp} u \subset Q$, since

$$\int_{\mathbb{R}^d\times(0,1)} \big[\partial_t u + \Delta u\big]\, d\mu \le \int_Q [|u||\partial_t\varrho| + |\nabla_x u||\nabla_x \varrho|]\, dx\, dt \le$$
$$\le \|\varrho\|_{W^{1,1}(Q)}\Big(\max_{x,t} |\nabla_x u(x,t)| + \max_{x,t} |u(x,t)|\Big).$$

It is even possible to construct a counter-example where the exact equation is fulfilled: the measure $\mu = e^{-|x|^2}\, dx\, dt$ trivially satisfies the equation $\partial\mu/\partial t = L^*\mu$ with $\mathcal{K} = 0$ and $A = \alpha(t)e^{|x|^2} I$, where $\alpha \in L^1[0,1]$ is such that its restrictions to all closed intervals $J \subset (0,1)$ belong to no $L^p(J)$ with $p > 1$; then $(\det A)^{1/(d+1)}\mu = \alpha(t)^{d/(d+1)} e^{-|x|^2/(d+1)}\, dx\, dt$, where $\alpha(t)^{d/(d+1)} e^{-|x|^2/(d+1)}$ is not in $L^p_{loc}(\mathbb{R}^d \times (0,1))$ whenever $p > (d+1)/d$. In this example the density itself is nice, so the negative effect is due to the factor $(\det A)^{1/(d+1)}$, but it is easy to modify this construction in such a way that also the density of μ will not be locally integrable to powers larger that 1: to this end, we take a positive function $\beta \in L^1(\mathbb{R}^d)$ belonging to no $L^p_{loc}(\mathbb{R}^d)$ with $p > 1$ and pass to the measure $\beta \cdot \mu$ and the diffusion matrix A/β. For $d = 1$ another simple example is this: $\mu = \varrho(x)\,dx\,dt$, where ϱ is a probability density such that $1/\varrho \in L^1_{loc}(\mathbb{R})$, but $1/\varrho \notin L^p_{loc}(\mathbb{R})$ if $p > 1$; then μ satisfies the equation with $A = \varrho^{-1}$ and $A^{1/2}\mu$ has density $\varrho^{-1/2}$ not belonging to $L^p_{loc}(\mathbb{R})$ if $p > 2$.

Let us consider some examples.

Example 1. Let

$$Lu = a^{ij}\partial_{x_i}\partial_{x_j}u + b^i\partial_{x_i}u + cu$$

and let a nonnegative locally bounded Borel measure μ on $\mathbb{R}^d \times (0,T)$ satisfy the equation

$$\partial_t\mu = L^*\mu$$

in the sense of the identity

$$\int_{\mathbb{R}^d\times(0,T)} [\partial_t u + Lu]\,d\mu = 0 \quad \forall u \in C_0^\infty(\mathbb{R}^d \times (0,T)),$$

where we assume that $a^{ij}, b^i, c \in L^1_{loc}(\mu)$. Then the measure $|\det A|^{1/(d+1)} \cdot \mu$ has a density $\varrho \in L^{(d+1)'}_{loc}(\mathbb{R}^d \times (0,T))$ with respect to Lebesgue measure on $\mathbb{R}^d \times (0,T)$. This result was obtained in [7, Theorem 3.1]. For the proof it suffices to note that

$$\int_{\mathbb{R}^d\times(0,T)} [\partial_t u + \mathcal{A}u]\,d\mu = -\int_{\mathbb{R}^d\times(0,T)} [\langle b, \nabla_x u\rangle + cu]\,d\mu,$$

and for every compact $Q \subset \mathbb{R}^d \times (0,T)$ there exists a constant $C > 0$ such that

$$\left|\int_{\mathbb{R}^d\times(0,T)} [\langle b, \nabla_x u\rangle + cu]\,d\mu\right| \le C\Big(\max_{x,t} |\nabla_x u(x,t)| + \max_{x,t} |u(x,t)|\Big).$$

for every $u \in C_0^\infty(\mathbb{R}^d \times (0,T))$ with support in Q.

Example 2. Let μ be a finite nonnegative measure on $\mathbb{R}^d \times (0,T)$ satisfying the equation

$$\partial_t\mu = L^*\mu$$

in the sense of the integral identity

$$\int_{\mathbb{R}^d\times(0,T)} [\partial_t u + Lu]\,d\mu = 0 \quad \forall u \in C_0^\infty(\mathbb{R}^d \times (0,T)),$$

where

$$Lu(x,t) = a^{ij}(x,t)\partial_{x_i}\partial_{x_j}u(x,t) + b^i(x,t)\partial_{x_i}u(x,t) + c(x,t)u(x,t) + \\ + \int_{\mathbb{R}^d\setminus\{0\}} [u(x+y,t) - u(x,t) - \langle y, \nabla_x u(x,t)\rangle]\,\nu_{x,t}(dy),$$

$a^{ij}, b^i, c \in L^1_{loc}(\mu)$, and, for each $t \in (0,T)$ and $x \in \mathbb{R}^d$, $\nu_{x,t}$ is a nonnegative locally bounded Borel measure on $\mathbb{R}^d \setminus \{0\}$ such that the function

$$M(x,t) := \int_{\mathbb{R}^d\setminus\{0\}} |y|^2 \wedge |y|\,\nu_{x,t}(dy)$$

is μ-integrable (under these hypotheses Lu is μ-integrable for every function u with compact support in $\mathbb{R}^d \times (0,T)$). Then the locally finite measure $|\det A|^{1/(d+1)} \cdot \mu$ has a density $\varrho \in L^{(d+1)'}_{loc}(\mathbb{R}^d \times (0,T))$ with respect to Lebesgue measure on $\mathbb{R}^d \times (0,T)$. Hence μ itself is absolutely continuous on the set $\{\det A \neq 0\}$.

If the measure μ is given by a family of probability measures μ_t on $\mathbb{R}^d$, i.e., $\mu = \mu_t(dx)\,dt$, then a sufficient condition for the μ-integrability of M is the integrability of $\sup_x M(x,t)$ with respect to Lebesgue measure on $(0,T)$.

Proof. We take a function $\varphi_0 \in C_0^\infty(\mathbb{R}^d)$ such that $0 \le \varphi_0 \le 1$ and $\varphi_0(x) = 1$ if $|x| \le 1$. It is clear that the function $\varphi(x,y) := \varphi_0(y-x)$ is a local unit function. Set

$$\widehat{b^i}(x,t) = b^i(x,t) - \int_{\mathbb{R}^d\setminus\{0\}} y_i(1-\varphi_0(y))\,\nu_{x,t}(dy),$$

$$\widehat{c}(x,t) = c(x,t) - \int_{\mathbb{R}^d\setminus\{0\}} (1-\varphi_0(y))\,\nu_{x,t}(dy).$$

Note that $\widehat{b}, \widehat{c} \in L^1_{loc}(\mu)$. Set $K(x,t,B) := \nu_{x,t}(B-x)$ and

$$\mathcal{K}u(x,t) = \int_{\mathbb{R}^d\setminus\{x\}} [u(y,t) - \varphi(x,y)u(x,t) - \varphi(x,y)\langle y-x, \nabla_x u(x,t)\rangle]\,K(x,t,dy).$$

Then the function

$$\int_{\mathbb{R}^d\setminus\{x\}} (|x-y|\wedge 1)^2\psi(y)\,K(x,t,dy) = \int_{\mathbb{R}^d\setminus\{0\}} (|y|\wedge 1)^2\psi(y+x)\,\nu_{x,t}(dy)$$

is μ-integrable if ψ is bounded. In addition,

$$\int_{\mathbb{R}^d\times(0,T)} [\partial_t u + \mathcal{A}u + \mathcal{K}u]\,d\mu = -\int_{\mathbb{R}^d\times(0,T)} [\langle \widehat{b}, \nabla_x u\rangle + \widehat{c}u]\,d\mu.$$

In order to apply Theorem 1 we should only note that for every compact $Q \subset \mathbb{R}^d \times (0,T)$ there exists a constant $C > 0$ such that

$$\left|\int_{\mathbb{R}^d\times(0,T)} [\langle \widehat{b}, \nabla_x u\rangle + \widehat{c}u]\,d\mu\right| \le C\Big(\max_{x,t}|\nabla_x u(x,t)| + \max_{x,t}|u(x,t)|\Big).$$

for every $u \in C_0^\infty(\mathbb{R}^d \times (0,T))$ with support in Q. $\square$

Let us observe that the same proof applies to some other frequently used operators (see [5] and [26]).

Remark 2. (i) Let us consider the operator

$$Lu(x,t) = a^{ij}(x,t)\partial_{x_i}\partial_{x_j}u(x,t) + b^i(x,t)\partial_{x_i}(x,t) + c(x,t)u(x,t) + \\ + \int_{\mathbb{R}^d\setminus\{0\}} \left[u(x+y,t) - u(x,t) - \frac{\langle y, \nabla_x u(x,t)\rangle}{1+|y|^2}\right]\nu_{x,t}(dy).$$

If a finite nonnegative measure μ on $\mathbb{R}^d \times (0,T)$ satisfies the equation $\partial_t\mu = L^*\mu$ and

$$\int_{\mathbb{R}^d\times(0,T)}\int_{\mathbb{R}^d\setminus\{0\}} \frac{|y|^2}{1+|y|^2}\,\nu_{x,t}(dy)\,\mu(dx\,dt) < \infty,$$

then the locally finite measure $|\det A|^{1/(d+1)}\cdot\mu$ has a density $\varrho \in L^{(d+1)'}_{loc}(\mathbb{R}^d\times(0,T))$ with respect to Lebesgue measure on $\mathbb{R}^d\times(0,T)$. Hence μ itself is absolutely continuous on the set $\{\det A \neq 0\}$. This follows by the same reasoning as in Example 2 letting

$$\widehat{b^i}(x,t) = b^i(x,t) - \int_{\mathbb{R}^d\setminus\{0\}} y_i((1+|y|^2)^{-1} - \varphi_0(y))\,\nu_{x,t}(dy),$$

$$\widehat{c}(x,t) = c(x,t) - \int_{\mathbb{R}^d\setminus\{0\}} (1-\varphi_0(y))\,\nu_{x,t}(dy).$$

(ii) Let us consider the operator

$$Lu(x,t) = a^{ij}(x,t)\partial_{x_i}\partial_{x_j}u(x,t) + b^i(x,t)\partial_{x_i}(x,t) + c(x,t)u(x,t) + \\ + \int_{\mathbb{R}^d\setminus\{0\}} \Big[u(x+y,t) - u(x,t) - \langle y, \nabla_x u(x,t)\rangle I_{\{|y|<1\}}(y)\Big]\, \nu_{x,t}(dy),$$

where $I_{\{|y|<1\}}$ is the indicator of the open unit ball. Let μ be a finite nonnegative measure on $\mathbb{R}^d \times (0,T)$ satisfying the equation $\partial_t\mu = L^*\mu$. Then the same conclusion holds if

$$\int_{\mathbb{R}^d\times(0,T)} \int_{\mathbb{R}^d\setminus\{0\}} |y|^2 \wedge 1\, \nu_{x,t}(dy)\, \mu(dx\, dt) < \infty.$$

Indeed, letting

$$\widehat{b^i}(x,t) = b^i(x,t) - \int_{\mathbb{R}^d\setminus\{0\}} y_i(I_{\{|y|<1\}}(y) - \varphi_0(y))\, \nu_{x,t}(dy),$$

$$\widehat{c}(x,t) = c(x,t) - \int_{\mathbb{R}^d\setminus\{0\}} (1-\varphi_0(y))\, \nu_{x,t}(dy),$$

we can also apply the same reasoning as in Example 2. If μ is given by a family of probability measures $\mu_t(dx)$, then the above condition holds for $\nu_{x,t}(dy) = q(x,t,y)|y|^{-d-\alpha(x)}\, dy$, where $0 < \alpha(x) < 2$ and

$$\int_0^T \sup_x \bigg(\int_{\mathbb{R}^d\setminus\{0\}} \frac{(|y|^2\wedge 1)q(x,t,y)}{|y|^{d+\alpha(x)}}\, dy\bigg)\, dt < \infty.$$

In particular, it is fulfilled if $q(x,t,y)$ is bounded and $0 < \alpha_1 \le \alpha(x) \le \alpha_2 < 2$.

In the case where the coefficients of L do not depend on t one can impose somewhat weaker assumptions.

Example 3. We recall that according to a remarkable theorem of Courrège (see [11]) a linear operator L from $C_0^\infty(\mathbb{R}^d)$ to the space of locally bounded Borel functions satisfies the positive maximum principle (i.e., $Lf(x) \le 0$ once $f \ge 0$ and x is a point of maximum of f) if and only if it has the form

$$Lu(x) = a^{ij}(x)\partial_{x_i}\partial_{x_j}u(x) + b^i(x)\partial_{x_i}u(x) + c(x)u(x) + \\ + \int_{\mathbb{R}^d\setminus\{x\}} [u(y) - \varphi(x,y)u(x) - \varphi(x,y)\langle y-x, \nabla_x u(x)\rangle]\, \nu(x,dy),$$

where b^i and c are locally bounded Borel functions, $A(x) = (a^{ij}(x))$ is a positive symmetric matrix, φ is a local unit function, $\nu(x,dy)$ is a Lévy kernel, i.e., $\nu(x,dy)$ is a Borel nonnegative measure on $\mathbb{R}^d \setminus \{x\}$ for each x and the mapping

$$x \mapsto \int_{\mathbb{R}^d\setminus\{x\}} |x-y|^2\psi(y)\, \nu(x,dy)$$

is Borel measurable and locally bounded for every $\psi \in C_0^\infty(\mathbb{R}^d)$, $\nu(x\colon |x| > \delta) < \infty$ for every $\delta > 0$, and

$$c(x) + \int_{\mathbb{R}^d\setminus\{x\}} (1-\varphi(x,y))\, \nu(x,dy) \le 0.$$

It is worth noting that generators of Feller processes satisfy the positive maximum principle and have the above form.

Let μ be a finite nonnegative Borel measure on $\mathbb{R}^d \times (0,T)$. Assume that $a^{ij} \in L^1_{loc}(\mu)$ and that the mapping

$$x \mapsto \int_{\mathbb{R}^d\backslash\{x\}} |x-y|^2 \psi(y)\,\nu(x,dy)$$

is integrable with respect to μ on $\mathbb{R}^d \times (0,T)$ for each function $\psi \in C_0^\infty(\mathbb{R}^d)$. If μ satisfies the parabolic equation $\partial_t\mu = L^*\mu$, then the measure $|\det A|^{1/(d+1)} \cdot \mu$ has a density $\varrho \in L^{(d+1)'}_{loc}(\mathbb{R}^d \times (0,T))$ with respect to Lebesgue measure on $\mathbb{R}^d \times (0,T)$.

These examples show that if there is a diffusion with jumps having generator L, then its space-time law is absolutely continuous on the set $\{\det A \neq 0\}$. In the case of a homogeneous diffusion (with time independent coefficients) and $\det A > 0$ we obtain the absolute continuity of the transition probability $P(t,dy)$ for almost every $t > 0$ (independently of the initial distribution). However, even for diffusions without jumps and $\det A > 0$ examples are known (see [14], [25]) where $P(t,dy)$ may be singular for some fixed t.

As already noted above, the papers [1], [2], and [21] contain Krylov-type estimates of certain stochastic integrals involving diffusion and jump parts; from these estimates one can derive absolute continuity on the set $\{\det A > 0\}$ of the space-time measure of the process satisfying the corresponding stochastic equation. However, the existence of such stochastic integrals and processes requires much stronger global assumptions about all coefficients a^{ij}, b^i, c, and $\nu_{x,t}$. More specifically, in [21] the coefficients a^{ij} and b^i are uniformly bounded, $c = 0$, and there are some uniform bounds for ν_x; in [1] and [2] existence of certain stochastic integrals is assumed along with some additional global estimates, in particular, for bounded A the drift b must be also bounded. The proofs in the papers cited also use Krylov's powerful results (more precisely, in [21] and [1] in place of [17] some earlier results of Krylov are employed), but in a different way.

Clearly, our paper focuses on the extreme case where jumps a posteriori have no influence; it would be interesting to find other combinations of contributions of the continuous and jump parts leading to absolute continuity in the case of nonsmooth coefficients.

It would be also interesting to study signed solutions of the Fokker–Planck–Kolmogorov equation with L as in [7] (see also [8]).

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References

[1] Anulova, S.V.: On processes with Lévy generating operator in a half-space. Izv. Akad. Nauk SSSR, Ser. Mat. **42**, 708–750 (1978) (in Russian); English transl.: Math. USSR Izvestiya **13**(1), 9–51 (1979)

[2] Anulova, S., Pragarauskas, G.: Weak Markov solutions of stochastic equations. Litov. Mat. Sb./Lietuv. Mat. Rink. **17**(2), 5–26 (1977) (in Russian); English transl.: Lithuanian Math. J. **17**(2), 141–155 (1977)

[3] Applebaum, D.: Lévy processes and stochastic calculus. Cambridge Univ. Press, Cambridge (2004)

[4] Bally, V., Clement, E.: Integration by parts formula and applications to equations with jumps. Probab. Theory Related Fields **151**(3-4), 613–657 (2011)

[5] Bass, R.F.: Stochastic differential equations with jumps. Probab. Surveys **1**, 1–19, (2004)

[6] Bichteler, K., Gravereaux, J.-B., Jacod, J.: Malliavin calculus for processes with jumps. Gordon and Breach, New York (1987)

[7] Bogachev, V.I., Krylov, N.V., Röckner, M.: On regularity of transition probabilities and invariant measures of singular diffusions under minimal conditions. Comm. Partial Diff. Eq. **26**(11-12), 2037–2080 (2001)

[8] Bogachev, V.I., Krylov, N.V., Röckner, M.: Elliptic and parabolic equations for measures. Uspehi Mat. Nauk **64**(6), 5–116 (2009) (in Russian); English transl.: Russian Math. Surveys **64**(6), 973–1078 (2009)

[9] Bouleau, N., Denis, L.: Application of the lent particle method to Poisson-driven SDEs. Probab. Theory Related Fields **151**(3-4), 403–433 (2011)

[10] Cass, T.: Smooth densities for solutions to stochastic differential equations with jumps. Stochastic Process. Appl. **119**(5), 1416–1435 (2009)

[11] Courrège, P.: Sur la forme intégro-différentielle des opérateurs de C_k^∞ dans C satisfaisant au principe du maximum. Séminaire Brelot–Choquet–Deny. Théorie du Potentiel, t. 10, n 1, exposé 2, 1–38 (1965/1966)

[12] Denis, L.: A criterion of density for solutions of Poisson-driven SDEs. Probab. Theory Related Fields **118**(3), 406–426 (2000)

[13] Di Nunno, G., Øksendal, B., Proske, F.: Malliavin calculus for Lévy processes with applications to finance. Springer-Verlag, Berlin (2009)

[14] Fabes, E.B., Kenig, C.E.: Examples of singular parabolic measures and singular transition probability densities. Duke Math. J. **48**, 845–856 (1981)

[15] Ikeda, N., Watanabe, S.: Stochastic differential equations and diffusion processes. 2nd ed., North-Holland/Kodansha, Amsterdam – Tokyo (1989)

[16] Jacob, N.: Pseudo-differential operators and Markov processes. Akademic Verlag, Berlin (1996)

[17] Krylov, N.V.: Sequences of convex functions and estimates of the maximum of the solution of a parabolic equation. Sibirskii Mat. Zhurn. **17**(2), 290–303 (1976) (in Russian); English transl.: Siberian J. Math. **17**(2), 226–236 (1976)

[18] Krylov, N.V.: Nonlinear elliptic and parabolic equations of second order. Nauka, Moscow (1985) (in Russian); English transl.: Reidel, Dordrecht (1987)

[19] Kulik, A.M.: On a regularity of distribution for solution of SDE of a jump type with arbitrary Lévy measure of the noise. Ukrain. Mat. Zh. **57**(9), 1261–1283 (2005); English transl.: Ukrainian Math. J. **57**(9), 1477–1501 (2005)

[20] Kunita, H.: Smooth density of canonical stochastic differential equation with jumps. Asterisque No. 327, 69–91 (2009)

[21] Lepeltier, J.-P., Marchal, B.: Problème des martingales et équations différentielles stochastiques associées à un opérateur intégro-différentiel. Ann. Inst. H. Poincaré, Nouv. Sér., Sect. B **12**, 43–103 (1976)

[22] Nourdin, I., Simon, T.: On the absolute continuity of Lévy processes with drift. Ann. Probab. **34**(3), 1035–1051 (2006)

[23] Osswald, H.: Malliavin calculus for Lévy processes and infinite-dimensional Brownian motion. An introduction. Cambridge University Press, Cambridge (2012)

[24] Picard, J.: On the existence of smooth densities for jump processes. Probab. Theory Related Fields **105**(4), 481–511 (1996)

[25] Safonov, M.V.: An example of a diffusion process with the singular distribution at a fixed moment. In: Abstracts of Communications of the Third International Vilnius Conference on Probability Theory and Mathematical Statistics, Vol. II, p. 133–134. Vilnius (1981)

[26] Stroock, D.: Diffusion processes associated with Lévy generators. Wahr. theor. verw. Geb. **32**, 209–244 (1975)

[27] Takeuchi, A.: Absolute continuity for solutions to stochastic functional differential equations with jumps. Stoch. Dyn. **7**(2), 153–185 (2007)

[28] Zhang, X.: Degenerate irregular SDEs with jumps and application to integro-differential equations of Fokker–Planck type. Arxiv Math. 1008.1884v2 (2011)