THE FINITE SPEED OF PROPAGATION FOR SOLUTIONS TO NONLINEAR STOCHASTIC WAVE EQUATIONS DRIVEN BY MULTIPLICATIVE NOISE

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Abstract

We prove that the solutions to the stochastic wave equation in $\mathcal{O} \subset \mathbb{R}^d$, $d\dot{X}_t - \Delta X_t dt + g(X_t)dt = \sigma(X_t)dW_t$, for $d = 1, 2, 3$, where g is a C^1-function with polynomial growth less than 3 and σ is Lipschitz with $\sigma(0) = 0$, propagate with finite speed. This result resembles the classical finite speed of propagation result for the solution to the Klein-Gordon equation and extends to equations with dissipative damping. A similar result follows for the equation with additive noise of the form $F(t)dW_t$ where $F(t) = F(t, \xi)$ has compact support (in ξ) for each $t > 0$.

Keywords: wave equation, stochastic equation, Wiener process, Sobolev spaces.

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1 Introduction

It is well known that relativistic equations, such as nonlinear wave and Klein-Gordon equations, have the finite speed of propagation property and this is a distinctive feature of hyperbolic equations. We shall prove here that this property remains valid for nonlinear wave equations driven by multiplicative Wiener noise.

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More precisely, we here consider the stochastic equation

$$
\begin{aligned}
& d\dot{X}(t) - \Delta X(t)dt + g(X(t))dt = \sigma(X(t))dW(t) \\
& \qquad\qquad\qquad\qquad\qquad\qquad \text{in } (0,\infty)\times\mathcal{O}, \\
& X(0,\xi) = x(\xi),\ \xi\in\mathcal{O},\quad \dot{X}(0,\xi) = y(\xi),\ \xi\in\mathcal{O}, \\
& X(t) = 0 \text{ on } (0,\infty)\times\partial\mathcal{O},
\end{aligned} \tag{1.1}
$$

where $\mathcal{O}$ is a bounded and open domain in $\mathbb{R}^d$, $d \geq 1$, $\sigma : \mathbb{R} \to \mathbb{R}$ is Lipschitz, $\sigma(0) = 0$ and $W(t)$ is a spatial Gaussian noise, that is, white in time. More precisely,

$$
W(t,\xi) = \sum_{j=1}^{\infty} \mu_j e_j(\xi)\beta_j(t), \quad t \geq 0,\ \xi \in \mathcal{O}. \tag{1.2}
$$

Here $\{\beta_j\}_{j=1}^{\infty}$ are mutually independent Brownian motions on a probability space $\{\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}\}$, $\{\mathcal{F}\}_t$ is the natural filtration induced by $\{\beta_j(t)\}_j$ and $\{e_j\}$ is an orthonormal system in the space $L^2(\mathcal{O})$. For simplicity, we take $\{e_j\}$ the orthonormal system of eigenfunctions of the Laplace operator Δ with Dirichlet boundary conditions, that is,

$$
-\Delta e_j = \lambda_j e_j \ \text{ in } \ \mathcal{O}; \quad e_j = 0 \ \text{ on } \ \partial\mathcal{O}. \tag{1.3}
$$

The function $g : \mathbb{R} \to \mathbb{R}$ is assumed to satisfy

(H1) $g \in C^1(\mathbb{R})$, *g is increasing and* $g(0) = 0$, $\forall r \in \mathbb{R}$ *and*

$$
|g'(r)| \leq L(1 + |r|^p), \quad \forall r \in \mathbb{R}, \tag{1.4}
$$

where $0 \leq p \leq 2$ *if* $d = 3$ *and* $p \geq 0$ *if* $d = 1, 2$.

In the special case of the normalized Klein-Gordon equation from quantum field theory, i.e. $g(r) = \alpha|r|^p r$, assumption (H1) is satisfied (cf. [15]). As regards the Wiener process (1.2), we assume that

$$
\sum_{j=1}^{\infty} \mu_j^2 |e_j|_\infty^2 < \infty \tag{1.5}
$$

and in $3-D$, by virtue of (1.3), this holds if

$$
\sum_{j=1}^{\infty} \mu_j^2 \lambda_j^2 < \infty. \tag{1.5'}
$$

The existence theory for nonlinear stochastic wave equations was treated e.g. in [3], [4], [6], [7], [9], [10], [11], [12], [14]. The existence of an invariant measure and ergodic properties of the transition semigroups corresponding to stochastic wave equations with dissipative damping and additive noise were studied in [1], [2], [3], [5]. Here we shall prove that, for equation (1.1) with probability one, the speed of propagation is with velocity less or equal to 1. This localization result is new in the stochastic case we consider here. The standard strategy to prove this property for the deterministic Klein-Gordon equation is based on the Paley-Wiener theorem combined with fix point arguments (cf. [15]), but this does not seem applicable here, and so we shall use a different approach inspired by Tartar's energy method (see [16]).

2 Preliminaries

Everywhere in the following, $\mathcal{O}$ is a bounded and open subset of $\mathbb{R}^d$ with boundary $\partial\mathcal{O}$. Denote by $W^{1,p}(\mathcal{O})$, $1 \le p \le \infty$, $H^k(\mathcal{O})$, $k = 1, 2$, and $H_0^1(\mathcal{O})$ the standard Sobolev spaces on $\mathcal{O}$. The norm in the space $L^p(\mathcal{O})$ of Lebesgue p-integrable functions on $\mathcal{O}$ is denoted by $|\cdot|_p$, the scalar product by $\langle\cdot,\cdot\rangle_2$ and $|\cdot|$ is the Euclidean norm in $\mathbb{R}^d$.

Given a Hilbert space with the norm $|\cdot|_{\mathcal{H}}$, denote by $L^q([0,T]; L^p(\Omega;\mathcal{H}))$, $p, q \in [1, +\infty]$, the space of all q-integrable functions $u : [0,T] \to L^p(\Omega;\mathcal{H})$. $C([0,T]; L^p(\Omega;\mathcal{H}))$ denotes the corresponding space of maps which are p-mean continuous, with norm

$$\|u\|_{C([0,T];L^p(\Omega;\mathcal{H}))} = \sup\{\mathbb{E}|u(t)|^p_{\mathcal{H}};\ t \in [0,T]\}^{1/p}.$$

We set $V = H_0^1(\mathcal{O})$ with the norm $\|u\|_V^2 = \int_{\mathcal{O}} |\nabla u|^2 d\xi$ and scalar product $\langle u, v\rangle_V = \int_{\mathcal{O}} \nabla u \cdot \nabla v \, d\xi$. Denote by H the space $L^2(\mathcal{O})$. We write problem(1.1) as an infinite dimensional system setting $Y(t) = \dot{X}(t)$,

$$\begin{cases} dX(t) = Y(t)dt, \\ dY(t) = -(AX(t) + g(X(t))dt + \sigma(X(t))dW(t), \\ X(0) = x, \quad Y(0) = y, \end{cases} \tag{2.1}$$

where A is the realization of the Laplace operator with Dirichlet boundary conditions in $L^2(\mathcal{O})$, that is, $Ax = -\Delta x$, $\forall x \in D(A) := H^2(\mathcal{O}) \cap H_0^1(\mathcal{O})$.

We consider the product Hilbert space $\mathcal{H} = V \times H$ with scalar product

$$\langle (u,v), (\widetilde{u}, \widetilde{v})\rangle_{\mathcal{H}} = \langle u, \widetilde{u}\rangle_V + \langle v, \widetilde{v}\rangle_2 .$$

We can rewrite problem (2.2) in the space $V \times H$ as

$$
\begin{cases} dZ = \mathcal{A}Z\,dt - \mathcal{F}(Z)dt + \mathcal{B}(Z)dW(t), \\ Z(0) = \begin{pmatrix} x \\ y \end{pmatrix}, \end{cases} \tag{2.2}
$$

where

$$
Z(t) = \begin{pmatrix} X(t) \\ Y(t) \end{pmatrix}, \qquad \mathcal{A} = \begin{pmatrix} 0 & 1 \\ -A & 0 \end{pmatrix},
$$

$$
\mathcal{F}(Z) = \begin{pmatrix} 0 \\ g(X) \end{pmatrix}, \quad \mathcal{B}(Z)W = \begin{pmatrix} 0 \\ \sigma(X)W \end{pmatrix}.
$$

It is well known (see, e.g., [13], po. 220) that $\mathcal{A}$ is the infinitesimal generator of a strongly continuous group in $V \times H$ given by (the wave kernel)

$$
e^{t\mathcal{A}} = \begin{pmatrix} \cos(A^{\frac{1}{2}}t) & A^{-\frac{1}{2}}\sin(A^{\frac{1}{2}}t) \\ -A^{\frac{1}{2}}\sin(A^{\frac{1}{2}}t) & \cos(A^{\frac{1}{2}}t) \end{pmatrix}.
$$

Definition 2.1 An $\{\mathcal{F}_t\}$-adapted stochastic process $Z(t) = \binom{X(t)}{Y(t)}$ with $\mathbb{P}$-a.s.-continuous sample paths in $V \times H$ is called a *mild solution* to system (2.1) if, for any $T > 0$, we have

(i) $X \in C([0,T]; L^2(\Omega; V))$, $Y \in C([0,T]; L^2(\Omega; H))$,

(ii) $j(X) \in L^\infty(0,T; L^1(\Omega \times \mathcal{O}))$,

(iii) for all $t \geq 0$, $\mathbb{P}$-a.s.

$$
\begin{cases} X(t) = x + \int_0^t Y(s)ds, \\ Y(t) = -A^{\frac{1}{2}}\sin(A^{\frac{1}{2}}t)x + \cos(A^{\frac{1}{2}}t)y - \int_0^t \cos(A^{\frac{1}{2}}(t-s))g(X(s))ds \\ \qquad + \int_0^t \cos(A^{\frac{1}{2}}(t-s))\sigma(X(s))dW(s). \end{cases}
$$

Here $j(r) = \int_0^r g(s)ds, \ \forall r \in \mathbb{R}$.

As regards the global existence in equation (2.2), we have the following result, which is well known (see, e.g., [4], [12]), but we recall it for convenience.

Proposition 2.2 *Let $x \in V,\ y \in H$. Then there is a unique mild solution (X, Y) to* (2.2).

Proof. The proof will be sketched only. First, we prove the existence for the equations

$$
\begin{aligned}
&dX_n = Y_n dt,\ t \in (0, T),\\
&dY_n = -AX_n dt - g_n(X_n)dt + \sigma(X_n)dW,\\
&X_n(0) = x, \quad Y_n(0) = y,
\end{aligned}
\tag{2.3}
$$

where $g_n : V \to H$ is the truncation operator

$$
g_n(u) = \begin{cases} g(u), & \text{if } \|u\|_V \le n,\\ g\left(\dfrac{nu}{\|u\|_V}\right), & \text{if } \|u\|_V > n. \end{cases}
$$

Since, by virtue of (H1), g_n is Lipschitz from V to H, by standard existence results for infinite dimensional stochastic differential equations (see [7]), it follows that there is a unique process

$$
\begin{pmatrix} X_n \\ Y_n \end{pmatrix} \in C([0,T]; L^2(\Omega; V)) \times \mathbb{C}([0,T]; L^2(\Omega; H))
$$

satisfying equation (iii) in Definition 2.1. Moreover, since $e^{t\mathcal{A}}$, $t > 0$, is a semigroup of contractions on $V \times H$, it follows by [7, Proposition 6.11] that this process has $\mathbb{P}$-a.s. continuous sample paths in $V \times H$. In other words, (X_n, Y_n) is a mild solution to (2.3) in the sense of Definition 2.1, of which property (ii) will be proved below.

Now, by (2.3), via Itô's formula applied to the function

$$
\phi(u, v) = \frac{1}{2}(\|u\|_V^2 + |v|_2^2),
$$

we obtain that

$$
\begin{aligned}
&\frac{1}{2}\,\mathbb{E}(\|X_n(t)\|_V^2 + |Y_n(t)|_2^2) + \mathbb{E}\int_0^t \int_{\mathcal{O}} g(X_n(s,\xi))Y_n(s,\xi)d\xi\, ds\\
&\quad = \frac{1}{2}\,(\|x\|_V^2 + |y|_2^2) + \frac{1}{2}\,\mathbb{E}\int_0^t \int_{\mathcal{O}} \sum_{j=1}^{\infty} \mu_j^2 |\sigma(X_n(s))e_j|^2 d\xi\, ds.
\end{aligned}
$$

More precisely, we apply Itô's formula to the square of the Fourier coefficients of $(X_n, Y_n) \in V \times H$ with respect to an orthonormal basis in $V \times H$ and then take the sum.

Recalling that $Y_n = \dfrac{\partial X_n}{\partial t}$ and that $j(X_n(t)) = \displaystyle\int_0^{X_n(t)} g(u)du$, a.e. $t > 0$, we find the estimate

$$
\tag{2.4}
\begin{aligned}
&\frac{1}{2}\,\mathbb{E}(\|X_n(t)\|_V^2 + |\dot{X}_n(t)|_2^2) + \mathbb{E}\int_{\mathcal{O}} j(X_n(t,\xi))d\xi \\
&\le \frac{1}{2}\,(\|x\|_V^2 + |y|_2^2) + C\mathbb{E}\int_0^t |X_n(s)|_2^2 ds + \int_{\mathcal{O}} j(x)d\xi,\ \forall t \in [0,T],
\end{aligned}
$$

which, in particular, implies (ii) in Definition 2.1 for X_n. Hence

$$
\tag{2.5}
\mathbb{E}\|X_n(t)\|_V^2 + \mathbb{E}|\dot{X}_n(t)|_2^2 \le C(1 + \|x\|_V^2 + |x|_{p+2}^{p+2} + |y|_2^2),\ \forall t \in [0,T],
$$

where C is independent of n. By (2.5), it follows that for each n and $t \in [0,T]$

$$
\tag{2.6}
\mathbb{P}[\|X_n(t)\|_V \ge n] \le Cn^{-2}(1 + \|x\|_V^2 + |x|_{p+2}^{p+2} + |y|_2^2).
$$

If $\tau_n = \inf\{t > 0;\ \|X_n(t)\|_V > n\} \wedge T$, we have that $g_n(X_n) = g(X_n)$ on $(0, \tau_n)$ and so $X(t) = X_n(t)$, $Y(t) = Y_n(t)$ for $0 < t \le \tau_n$ is a solution to (2.1). By (2.6), we see that $\lim\limits_{n\to\infty} \tau_n = T$, $\mathbb{P}$-a.s., and so (X, Y) is a solution to (2.1) in the sense of Definition 2.1, as desired, since letting $n \to \infty$ in (2.4), also (ii) in Definition 2.1 holds.

The uniqueness is immediate by the local Lipschitz property of the function g from V to H via Itô's formula. $\square$

3 The main result

If K is a closed subset of $\mathcal{O}$ we denote by $d_K(\xi)$ the distance from $\xi \in \mathcal{O}$ to K, that is,

$$
\inf\{|\xi - \eta|;\ \eta \in K\}
$$

and, for each $r > 0$, we set

$$
K_r = \{\xi \in \mathcal{O};\ d_K(\xi) \le r\}.
$$

For a given function $f : \mathcal{O} \to \mathbb{R}$, let support $\{f\}$ denote the closure of the set $\{\xi \in \mathcal{O};\ f(\xi) \neq 0\}$.

Theorem 3.1 *Let d=1, 2, 3 and let K be a closed subset of $\mathcal{O}$. Let X=$X(t)$ be the solution to* (1.1) *in the sense of Definition* 2.1 *with initial data $x \in V$, $y \in H$. If*

$$\text{support } \{x\} \subset K, \text{ support } \{y\} \subset K, \tag{3.1}$$

then $\mathbb{P}$*-a.s.*

$$\text{support } \{X(t)\} \subset K_t, \quad \forall t \geq 0. \tag{3.2}$$

In other words, $\mathbb{P}$-a.s. the wave front of the solution at time t is in the neighborhood K_t of the set K. This amounts to saying that with probability one the solution $X = X(t)$ to (1.1) propagates with finite velocity ≤ 1 and has its support in the space-time cone $\{(t, \xi) \in (0, \infty) \times \mathcal{O};\ d_K(\xi) \leq t\}$.

Proof. We consider a function $\rho \in C^1(\mathbb{R})$ such that

$$\rho(s) = 0, \ \forall s \leq 0, \quad \rho(s) > 0, \ \forall s > 0, \tag{3.3}$$

$$\rho'(s) \geq 0, \ \forall s \geq 0, \tag{3.4}$$

$$\sup_{s \geq 0} (\rho(s) + \rho'(s)) < \infty. \tag{3.5}$$

Consider the local energy function $\phi : [0, \infty) \times V \times H \to \mathbb{R}$ defined by

$$\phi(t, u, v) = \frac{1}{2} \int_{\mathcal{O}} \rho(d_K(\xi) - t)(|\nabla u(\xi)|^2 + |v(\xi)|^2) d\xi. \tag{3.6}$$

By (3.1), (3.3), we see that

$$\phi(0, x, y) = 0. \tag{3.7}$$

In order to make clear the idea of the proof, we continue with a naive (formal) argument which will be made rigorous later on. (In fact, the formal and not rigorous aspects of this argument presented below consists in the application of Itô's formula in system (2.2) to the Lyapunov function ϕ defined above.) Heuristically, applying Itô's formula in (2.1), we obtain that

$$
\begin{aligned}
&d\phi(t,X(t),Y(t))\\
&\quad= \phi_t(t,X(t),Y(t))dt + \langle \nabla_u\phi(t,X(t),Y(t)),Y(t)\rangle_V\,dt\\
&\quad+\langle \nabla_v\phi(t,X(t),Y(t)),-AX(t)-g(X(t))\rangle_2\,dt\\
&\quad+\frac{1}{2}\sum_{j=1}^{\infty}\mu_j^2\left\langle \nabla_v^2\phi(t,X(t),Y(t))(\sigma(X(t))e_j),\sigma(X(t))e_j\right\rangle_2\,dt\\
&\quad+\langle \nabla_v\phi(t,X(t),Y(t)),\sigma(X(t))dW(t)\rangle_2\,,\ t\ge 0.
\end{aligned}
$$

This yields

$$
\begin{aligned}
&\phi(t,X(t),Y(t))\\
&\quad= -\frac{1}{2}\int_0^t\int_{\mathcal{O}}\rho'(d_K(\xi)-s)(|\nabla X(s,\xi)|^2+|Y(s,\xi)|^2)d\xi\,ds\\
&\quad+\int_0^t\int_{\mathcal{O}}\rho(d_K(\xi)-s)(\nabla X(s,\xi)\cdot\nabla Y(s,\xi)\\
&\qquad\qquad+Y(s,\xi)\Delta X(s,\xi)-Y(s,\xi)g(X(s,\xi)))d\xi\,ds\\
&\quad+\frac{1}{2}\sum_{j=1}^{\infty}\mu_j^2\int_0^t\int_{\mathcal{O}}\rho(d_K(\xi)-s)|\sigma(X(s,\xi))e_j(\xi)|^2d\xi\,ds\\
&\quad+\int_0^t\langle Y(s),\sigma(X(s))dW(s)\rangle_2\,,\ \mathbb{P}\text{-a.s.},\ t\ge 0.
\end{aligned}
\tag{3.8}
$$

because, by (3.7), $\phi(0,X(0),Y(0))=0$.

Taking into account that

$$
|d_K(\xi)-d_K(\bar{\xi})|\le|\xi-\bar{\xi}| \text{ for all } \xi,\bar{\xi}\in\mathbb{R}^d,
$$

we infer that $d_K\in W^{1,\infty}(\mathbb{R}^d)$ and

$$
|\nabla d_K(\xi)|\le 1,\quad \text{a.e. } \xi\in\mathbb{R}^d. \tag{3.9}
$$

On the other hand, by Green's formula we have that

$$
\begin{aligned}
&\int_{\mathcal{O}} \rho(d_K(\xi)-s)Y(s,\xi)\Delta X(s,\xi)d\xi \\
&\qquad = -\int_{\mathcal{O}} \rho(d_K(\xi)-s)\nabla Y(s,\xi)\cdot\nabla X(s,\xi))d\xi \\
&\qquad - \int_{\mathcal{O}} \rho'(d_K(\xi)-s)Y(s,\xi)\nabla d_K(\xi)\cdot\nabla X(s,\xi)d\xi.
\end{aligned}
\tag{3.10}
$$

(The latter is true of course only if $Y(s,\cdot)\in V,\ \forall s\ge 0.$)

Then, by (3.8) and (3.10), we obtain that

$$
\begin{aligned}
\phi(t,X(t),Y(t)) = & -\frac{1}{2}\int_0^t \int_{\mathcal{O}} \rho'(d_K(\xi)-s)|\nabla X(s,\xi)|^2 + |Y(s,\xi)|^2 \\
& +2Y(s,\xi)\nabla d_K(\xi)\cdot\nabla X(s,\xi))d\xi\, ds \\
& +\frac{1}{2}\sum_{j=1}^{\infty}\int_0^t \int_{\mathcal{O}} \rho(d_K(\xi)-s)|\sigma(X(s,\xi))|^2 e_j^2(\xi)d\xi\, ds \\
& -\int_0^t \int_{\mathcal{O}} \rho(d_K(\xi)-s)\frac{\partial}{\partial s}\, j(X(s,\xi))d\xi\, ds \\
& +\int_0^t \langle Y(s),\sigma(X(s))dW(s)\rangle_2\,,\ \forall t\ge 0, \mathbb{P}\text{-a.s.}
\end{aligned}
\tag{3.11}
$$

because $Y(s)=\dfrac{\partial}{\partial s}\, j(X(s,\xi)) = g(X(s,\xi))Y(s,\xi)$, a.e. in $(0,\infty)\times\mathcal{O}$. This yields

$$
\begin{aligned}
\phi(t,X(t),Y(t)) \le & -\frac{1}{2}\int_0^t \int_{\mathcal{O}} \rho'(d_K(\xi)-s)(|\nabla X(s,\xi)|^2 + |Y(s,\xi)|^2 \\
& +2Y(s,\xi)\nabla d_K(\xi)\cdot\nabla X(s,\xi))d\xi\, ds \\
& +\frac{1}{2}\sum_{j=1}^{\infty}\mu_j^2\int_0^t \int_{\mathcal{O}} \rho(d_K(\xi)-s)|\sigma(X(s,\xi))e_j(\xi)|^2 d\xi\, ds \\
& +\int_0^t \langle Y(s),\sigma(X(s))dW(s)\rangle_2\,,\quad \forall t\ge 0,\ \mathbb{P}\text{-a.s.},
\end{aligned}
\tag{3.12}
$$

because $j\ge 0$ on $\mathbb{R}$ and so, by virtue of (3.1), $\rho(d_K(\xi))j(x(\xi))\equiv 0$, $\forall \xi\in\mathcal{O}$.

On the other hand, by (3.9) we see that

$$\frac{1}{2}\left(|\nabla X(s,\xi)|^2 + |Y(s,\xi)|^2 - 2Y(s,\xi)\nabla d_K(\xi)\nabla X(s,\xi)\right) \geq 0,$$
$$\text{a.e. } \xi \in \mathcal{O},\ \ s \geq 0.$$

Then, substituting the latter into (3.12), taking expectation and recalling (1.5), (3.4), (3.5), we obtain that

$$\mathbb{E}\phi(t,X(t),Y(t)) \leq C\int_0^t \mathbb{E}\phi(s,X(s),Y(s))ds,\ \ \forall t \geq 0, \tag{3.13}$$

and hence, since $s \mapsto \phi(s,X(s),Y(s))$ is continuous $\mathbb{P}$-a.s., we obtain that $\mathbb{P}$-a.s.

$$\phi(s,X(s),Y(s)) = 0,\ \ \forall s \geq 0,$$

and, therefore, $\mathbb{P}$-a.s., $\forall t \geq 0$,

$$\rho(d_K(\xi)-t)(|\nabla X(t,\xi)|^2 + |Y(t,\xi)|^2) = 0,\ \text{ for } d\xi\text{- a.e. } \xi \in \mathcal{O}. \tag{3.14}$$

Recalling (3.3) and that $X(t) \in H_0^1(\mathcal{O})$, this yields $X(t,\xi) = 0$ on $\{t < d_K(\xi)\}$, for $d\xi$-a.e. $\xi \in \mathcal{O}$, that is

$$\text{support } \{X(t)\} \subset K_t, \forall t \in [0,T],$$

as claimed.

In order to complete the proof, it remains to give a rigorous proof for the energy formula (3.11). To this purpose, we set

$$X_\varepsilon = (I+\varepsilon A)^{-1}X,\ \ \ Y_\varepsilon = (I+\varepsilon A)^{-1}Y,\ \ \ \varepsilon > 0.$$

Then, by (2.1) we have

$$\begin{aligned} &dX_\varepsilon(t) = Y_\varepsilon(t)dt,\\ &dY_\varepsilon(t) = -AX_\varepsilon(t)dt + (I+\varepsilon A)^{-1}g(X_\varepsilon(t)) + \sigma_\varepsilon(X)dW,\\ &X_\varepsilon(0) = x_\varepsilon = (I+\varepsilon A)^{-1}x,\ Y_\varepsilon(0) = y_\varepsilon = (I+\varepsilon A)^{-1}y, \end{aligned} \tag{3.15}$$

where $\sigma_\varepsilon(X)W = \sum_{j=1}^{\infty} \mu_j(I+\varepsilon A)^{-1}(\sigma(X)e_j)\beta_j$. We note that by (i), (ii) we have

$$X_\varepsilon \in C([0,T];L^2(\Omega,D(A))),\ Y_\varepsilon \in C([0,T];L^2(\Omega;V)). \tag{3.16}$$

Moreover, by (H1) and the Sobolev-Gagliardo-Nirenberg embedding theorem ($H^1(\mathcal{O}) \subset L^6(\mathcal{O})$, for $1 \le d \le 3$), we see that $g(X) \in L^2(0,T;L^2(\Omega\times\mathcal{O}))$ and, therefore,

$$(I+\varepsilon A)^{-1}g(X) \in L^2(0,T;L^2(\Omega,H^2(\mathcal{O})\cap H_0^1(\mathcal{O})),\ \forall\varepsilon>0. \tag{3.17}$$

Then, applying in (3.15) the Itô formula to the function $\phi=\phi(t,u,v)$, we obtain as above (see (3.8))

$$\begin{aligned}
&\phi(t,X_\varepsilon(t),Y_\varepsilon(t))\\
&\quad= -\frac{1}{2}\int_0^t\int_{\mathcal{O}}\rho'(d_K(\xi)-s)(|\nabla X_\varepsilon(s,\xi)|_d^2+|Y_\varepsilon(s,\xi)|^2)d\xi\,ds\\
&\quad+\int_0^t\int_{\mathcal{O}}\rho(d_K(\xi)-s)(\nabla X_\varepsilon(s,\xi)\cdot\nabla Y_\varepsilon(s,\xi)\\
&\quad+Y_\varepsilon(s,\xi)\Delta X_\varepsilon(s,\xi))-Y_\varepsilon(s,\xi)(I+\varepsilon A)^{-1}g(X(s,\xi)))d\xi\,ds\\
&\quad+\frac{1}{2}\sum_{j=1}^\infty\mu_j^2\int_0^t\int_{\mathcal{O}}\rho(d_K(\xi)-s)|(I+\varepsilon A)^{-1}(\sigma(X(s))e_j)|^2d\xi\,ds\\
&\quad+\int_0^t\langle Y_\varepsilon(s),\sigma_\varepsilon(X(s))dW(s)\rangle_2,\ \mathbb{P}\text{-a.s.}
\end{aligned} \tag{3.18}$$

By (3.16) and Greens formula, we have (see (3.10))

$$\begin{aligned}
&\int_{\mathcal{O}}\rho(d_K(\xi)-s)Y_\varepsilon(s,\xi)\Delta X_\varepsilon(s,\xi)d\xi\\
&\qquad=-\int_{\mathcal{O}}\rho(d_K(\xi)-s)\nabla Y_\varepsilon(s,\xi)\cdot\nabla X_\varepsilon(s,\xi))d\xi\\
&\qquad-\int_{\mathcal{O}}\rho'(d_K(\xi)-s)Y_\varepsilon(s,\xi)(\nabla d_K(\xi)\cdot\nabla X_\varepsilon(s,\xi))d\xi.
\end{aligned}$$

Substituting into (3.18), we obtain as above that, for all $\varepsilon>0$,

$$\begin{aligned}
&\mathbb{E}\phi(t,X_\varepsilon(t),Y_\varepsilon(t))\\
&\le\frac{1}{2}\,\mathbb{E}\int_0^t\int_{\mathcal{O}}\sum_{j=1}^\infty\mu_j^2|(I+\varepsilon A)^{-1}(\sigma(X(s))e_j)|^2\rho(d_K(\xi)-s)d\xi ds\\
&-\mathbb{E}\int_0^t\int_{\mathcal{O}}\rho(d_K(\xi)-s)Y_\varepsilon(s,\xi)(I+\varepsilon A)^{-1}g(X(s,\xi))d\xi ds,\ \forall t\ge 0.
\end{aligned} \tag{3.19}$$

On the other hand, we have, for $\varepsilon \to 0$,

$$\begin{array}{rcll} X_\varepsilon & \to & X & \text{in } C([0,T];L^2(\Omega;V)) \\ Y_\varepsilon & \to & Y & \text{in } C([0,T];L^2(\Omega;H)) \\ (I+\varepsilon A)^{-1}g(X) & \to & g(X) & \text{in } L^2(0,T;L^2(\Omega;H)), \end{array}$$

because $X \in L^2(0,T;L^2(\Omega;V))$ and $|g(X)| \le C|X|^3$.

Since $X \in C([0,T];L^2(\Omega;V)$, this implies via the embedding theorem mentioned above that $g(X) \in L^2(0,T;L^2(\Omega;H))$. Then, letting $\varepsilon \to 0$ in (3.19), we get

$$\begin{aligned} \mathbb{E}\phi(t,X(t),Y(t)) &\le C\int_0^t \phi(s,X(s),Y(s))ds \\ &+\mathbb{E}\int_0^t\int_{\mathcal{O}} \rho(d_K(\xi)-s)Y(s,\xi)g(X(s,\xi))ds\,d\xi \\ &\le C\mathbb{E}\int_0^t \phi(s,X(s),Y(s))ds, \quad \forall t \ge 0, \end{aligned}$$

because, as seen earlier, we have

$$\int_{\mathcal{O}} Y_\varepsilon(s,\xi)g(X_\varepsilon(s,\xi))d\xi = \frac{\partial}{\partial s}\int_{\mathcal{O}} j(X_\varepsilon(s,\xi))d\xi, \quad \forall \varepsilon > 0.$$

We have, therefore, obtained (3.13), which, as seen above, completes the proof.

Remark 3.2 As a matter of fact, by (3.16), Theorem 3.1 remains true for the stochastic wave equation (1.1) with nonlinear dissipative damping, that is,

$$\begin{aligned} & d\dot{X} - \Delta X\,dt + g(X)dt + U(\dot{X})dt = \sigma(X)\,dW \text{ in } (0,\infty)\times\mathcal{O}, \\ & X(0) = x,\ \dot{X}(0) = y \text{ in } \mathcal{O}, \\ & X = 0 \text{ on } (0,\infty)\times\partial\mathcal{O}, \end{aligned} \tag{3.20}$$

where g satisfies Hypothesis (H1) and $U : \mathbb{R} \to \mathbb{R}$ is a monotonically non-decreasing C^1 function satisfying a polynomial growth condition. Leaving aside the existence problem for (3.19), which was treated in [2], we note that in this case in the energy equation (3.8) as well as in (3.12) there arises one more term

$$-\int_0^t\int_{\mathcal{O}} \rho(d_K(\xi)-s)Y(s,\xi)U'(Y(s,\xi))ds\,d\xi,$$

which is ≤ 0 and so we conclude the proof as in the previous case. In particular, the equation

$$\varepsilon\, d\dot{X} + \dot{X}\, dt - \Delta X\, dt = \sigma(X)\, dW,\ \varepsilon > 0, \tag{3.21}$$

describes the Cattaneo-Vernotte non Fourier heat conduction model perturbed by Gaussian noise (the relativistic stochastic heat equation). The details are omitted.

4 The wave equation driven by additive noise

Consider here the equation

$$\begin{aligned} & d\dot{X}(t,\xi) - \Delta X(t,\xi)dt + g(X(t,\xi))dt = F(t)dW(t) \text{ in } (0,\infty)\times\mathcal{O}, \\ & X(0,\xi) = x(\xi),\ \xi\in\mathcal{O},\quad \dot{X}(0,\xi) = y(\xi),\ \xi\in\mathcal{O}, \\ & X(t,\xi) = 0 \text{ on } (0,\infty)\times\partial\mathcal{O}, \end{aligned} \tag{4.1}$$

where W is the Wiener process (1.2) , g satisfies Hypothesis (H1) and F satisfies Hypothesis (H2) below.

(H2) $F : [0,\infty)\times\Omega \longrightarrow L^2(\mathcal{O})$ is an adapted process to the filtration $(\mathcal{F}_t)$ and

$$\sum_{j=1}^{\infty} \mu_j^2 \int_0^t \mathbb{E}\int_{\mathcal{O}} (F(s,\xi)^2 e_j^2(\xi) d\xi\, ds \ < \infty.$$

Then, we have

Theorem 4.1 *Let d=1, 2, 3 and let K be a closed subset of $\mathcal{O}$. Let X=$X(t)$ be the solution to* (4.1) *in the sense of Definition* 2.1 *with the initial data $x\in V$, $y\in H$. If*

$$\text{support }\{x\}\subset K,\ \text{support }\{y\}\subset K, \text{support }\{\mathrm{F(t,\cdot)}\subset \mathrm{K_t}, \forall \mathrm{t}>0. \tag{4.2}$$

then $\mathbb{P}$*-a.s.*

$$\text{support }\{X(t)\}\subset K_t,\quad \forall t\geq 0. \tag{4.3}$$

Proof. It is essentially the same as that of Theorem 3.1 and so it will be sketched only. If we use in the system

$$\begin{cases} dX(t) = Y(t)dt, \\ dY(t) = -(AX(t) + g(X(t))dt + F(t)dW(t), \\ X(0) = x, \quad Y(0) = y, \end{cases} \tag{4.4}$$

the Itô formula for the Lyapunov function (3.6), we obtain as above (see (3.12)) that

$$\begin{aligned} \phi(t, X(t), Y(t)) \le & -\frac{1}{2}\int_0^t \int_{\mathcal{O}} \rho'(d_K(\xi) - s)(|\nabla X(s,\xi)|_d^2 + |Y(s,\xi)|^2 \\ & +2Y(s,\xi)\nabla d_K(\xi) \cdot \nabla X(s,\xi))d\xi\, ds \\ & +\frac{1}{2}\sum_{j=1}^{\infty} \mu_j^2 \int_0^t \int_{\mathcal{O}} \rho(d_K(\xi) - s)|)|F(s,\xi)e_j(\xi)|^2 d\xi\, ds \\ & + \int_0^t \langle Y(s), F(s)dW(s)\rangle_2, \quad \forall t \ge 0, \ \mathbb{P}\text{-a.s.}, \end{aligned} \tag{4.5}$$

(As a matter of fact, to make the calculus rigorous, one must proceed as in the previous case by replacing (4.4) by the corresponding system (3.15) and subsequently letting ε go to zero. But the details are omitted.) Taking into account that, by Hypothesis (H2), $\rho(d_K(\xi) - s)F^2(s,\xi) = 0$ a.e on $(0,\infty) \times \mathcal{O}$, we see by (4.5) that (4.3) holds. This completes the proof.

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