# Stochastic Evolution Equations in Weighted $L^2$ Spaces With Jump Noise

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**Summary**

In this thesis, we study two classes of stochastic differential equations (SDEs in short) with jump noise in weighted $L^2$ spaces over $\mathbb{R}^d$. More precisely, the first class of SDEs is a jump-diffusion model in the sense of Merton (see his paper [82] on the theory of option pricing), i.e. the SDE is driven by a Wiener noise and a Poisson noise. The second class consists of SDE's with Lévy noise. We show existence of mild solutions and establish their regularity properties in the case of a drift term consisting of a nonautonomous linear (differential) operator and a non-Lipschitz Nemitskii-type operator. There are two principal issues, that make it impossible to apply the general theory of stochastic evolution equations in Hilbert spaces directly. First, the diffusion coefficients, given by multiplication operators in $L^2$, are not Hilbert-Schmidt and, second, the generating functions of the Nemitskii drift operators are non-Lipschitz and have polynomial growth.

Compared to the framework known for SDEs with Wiener noise, the new situation requires a detailed analysis of the stochastic convolution w.r.t. a compensated Poisson random measure in weighted $L^2$ spaces. To this end, we introduce several regularity conditions on the evolution operator generated by the nonautonomous drift operators, which are additional to those in the Wiener case. Furthermore, we need certain integrability conditions on the Lévy intensity measure associated to the jump process. We prove both the meansquare continuity and, under certain restrictions, the càdlàg property of the stochastic convolution w.r.t. compensated Poisson random measures.

We first show existence (and even uniqueness) of solutions in the case of Lipschitz functions defining the corresponding Nemitskii operators. Then, we prove an infinite-dimensional comparison theorem for jump-type diffusions with different Lipschitz drift coefficients. This allows us to prove the existence of solutions in the case of non-Lipschitz drifts by constructing appropriate Lipschitz approximations and applying the comparison theorem shown before. It should be noted that a sufficient condition for the solvability of the above equations involves an explicit relation between the degree of polynomial growth of the drift coefficients, integrability properties of the Lévy intensity measure and the regularity properties of the evolution operator. Furthermore, in the autonomous case with a Nemitskii drift operator being defined through a maximal monotone function, we even get uniqueness of some of the solutions in the additive case if we restrict our considerations to bounded domains $\Theta \in \mathbb{R}^d$ and cylindrical Wiener processes.

To establish the existence and comparison results in the multiplicative case, we need to analyse approximating equations in Sobolev spaces $W^{m,2}(\Theta)$ of order $m > \frac{d}{2}$ in domains $\Theta \subset \mathbb{R}^d$ obeying the weak cone property. Furthermore, the jump coefficient has to be monotonically increasing and uniformly bounded, whereas the intensity measure corresponding to the jump noise has to be concentrated on the set of only positive functions in $L^2$.

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# Chapter 1

# Introduction

This thesis is concerned with two types of stochastic evolution equations with jump noise in weighted $L^2$-spaces. Such an abstract setting includes a large class of stochastic partial differential equations (SPDEs in short) of parabolic type in (bounded or unbounded) domains $\Theta \subset \mathbb{R}^d$.
The first equation, named (1.1) below, is a stochastic differential equation (SDE in short) consisting of:

- a drift part with a *nonautonomous* operator and a *continuous drift coefficient having polynomial growth*,
- a Wiener part with a *Lipschitz diffusion coefficient*, and
- a Poissonian jump part with a *Lipschitz jump coefficient.*

This is a jump-diffusion model in the sense of option pricing theory (cf. Merton's paper [82], where this terminology was introduced).
The second equation, named (1.2) below, is an SDE consisting of

- a drift part of the same type as in equation (1.1),
- a Lévy jump part with a *Lipschitz jump diffusion coefficient.*

Note that in the whole thesis, we use the term *diffusion coefficient* resp. *jump coefficient* for the coefficient corresponding to the Wiener resp. Poisson noise of an SDE of the form (1.1). Since, in general, a Lévy process $L$ obeys both a diffusion and jumps, we use the term *jump diffusion coefficient* for the coefficient corresponding to the Lévy noise of an SDE of the form (1.2).

The basic aim of our work is to develop a unified theory of infinite-dimensional SDEs driven by jump noises (i.e. Poisson random measures or Lévy processes) in weighted $L^p$-spaces. This includes most of the known results for continuous diffusions in Hilbert spaces driven by a Wiener noise as well as particular results known so far in infinite dimensions for jump diffusions driven by Lévy noise (see the discussion in Section 1.2 below).

In this thesis, we will put particular emphasis on the transition from the standard assumptions of *(globally) Lipschitz drift coefficients* to the case of *continuous coefficients of polynomial growth*. The latter case is of essential interest in various applications. Furthermore, we cover the case of *time-dependent evolution operators* and *non Hilbert-Schmidt coefficients of Nemitskii type*.

The equations will be solved in weighted Lebesgue spaces $L^2_\rho(\Theta)$ resp. $L^{2\nu}_\rho(\Theta)$ over a Borel set $\Theta \subset \mathbb{R}^d$ (for more details on the spaces see Section 1.2 and, in particular, Section 3.1 below). Especially, we will be interseted in the technically more difficult case of *unbounded domains*, e.g. $\Theta = \mathbb{R}^d$.

But before we describe the exact setting for equations (1.1) and (1.2), let us give a general motivation for considering SDEs with jumps.

## 1.1 Motivation for SDEs with jumps

In recent years there has been large interest in SDEs with general, not necessarily continuous, semimartingales as driving noises. This is reflected in a growing number of papers going beyond the well-known framework of SDEs with Wiener noise, e.g. by considering compensated Poisson random measures or Lévy processes as noise.
Monographs considering this topic with a focus on Lévy processes as driving noise are, in finite dimensions, Applebaum [7] and, in infinite dimensions, Peszat and Zabczyk [95].
In the preface of his book [7] on the subject (see p.ix there), Applebaum presents the following list of reasons, why Lévy processes are important in probability theory:

- Lévy processes are analogues of random walks in continuous time;
- Lévy processes form special subclasses of both semimartingales and Markov processes, for which the analysis is, on the one hand, much simpler and, on the other hand, provides valuable guidance for the general case;

- Lévy processes are the simplest examples of random motion whose sample paths are right-continuous and have a number (at most countable) of random jump discontinuities occuring at random times, on each finite interval;
- Lévy processes include a number of very important processes as special cases, among them Brownian motion, Poisson process, stable and self-decomposable processes and subordinated processes.

Concerning the properties of infinite-dimensional Lévy processes, see also Section 2.4.

In general, stochastic evolution equations in infinite dimensions are often used to describe complex models in natural sciences. Numerous examples of SDEs with Wiener noise in infinite dimensions can be found e.g. in the introductory chapter (Chapter 0) in the monograph [26] by DaPrato and Zabczyk.
Stochastic evolution equations with Lévy noise, which constitute a large class of Markov processes, are particularly important. In finite dimensions there is a famous theorem by Courrège (cf. e.g. [23] and [52]), which states that, under rather general assumptions, any Markov semigroup on $\mathbb{R}^d$ is a Lévy-type semigroup, i.e. it can be represented as a transition semigroup corresponding to a certain SDE driven by Lévy noise.
In general, SDEs with compensated Poisson random measures or Lévy processes as driving random forces are candidates to model situations, where the system does not develop in a time-continuous way. Typically, the theory of SDEs with jumps in infinite-dimensional spaces plays a role in modelling critical phenomena. Among areas of application let us mention neurophysiology, environmental pollution and mathematical finance.
A prominent example of an application is the so-called FitzHugh-Nagumo equation in neurophysics. This equation has been treated mathematically e.g. by Bonnacorsi and collaborators in [16].
As an example from mathematical finance, let us mention the Heath-Jarrow-Morton model. This model describing the development of interest rates was originally proposed as an SDE driven by Wiener noise. In recent years, the model has been refined as an SDE with Lévy noise e.g. by Jakubowski and Zabczyk in [55] and by Marinelli in [78]. Furthermore, there have also been attempts to introduce jumps in the Heath-Jarrow-Morton model by considering an SDE with both a Wiener and a jump noise, where the jump noise is usually given by a compensated Poisson random measure. Such models have e.g. been treated by Björk and collaborators (cf. [13]), by Carmona and Tehranchi (cf. [19]) and by Filipovic, Tappe and Teichmann (cf. [38]).

Having motivated the use of SDEs with jumps, we continue with the introduction of the setting we work in. Furthermore, we relate this setting to several important results from the literature.

## 1.2 The basic equations and their relation to the existing literature

Let us describe now in detail the equations (1.1) and (1.2) of our interest. We consider the two SDEs

$$
\begin{aligned}
(1.1)\, dX(t) = \quad & (A(t)X(t) + F(t,\cdot,X(t)))dt + \mathcal{M}_{\Sigma(t,\cdot,X(t))}dW(t) \\
& + \int_{L^2} \mathcal{M}_{\Gamma(t,\cdot,X(t))}x\, \tilde{N}(dt,dx),\ t \in [\,0,T\,],
\end{aligned}
$$

and

$$
\begin{aligned}
(1.2)\, dX(t) = \quad & (A(t)X(t) + E(t,\cdot,X(t)))dt \\
& + \mathcal{M}_{\Sigma(t,\cdot,X(t))}dL(t),\ t \in [\,0,T\,],
\end{aligned}
$$

in weighted $L^p$-spaces over $\mathbb{R}^d$, $d \in \mathbb{N}$, which are defined as follows:

On $\mathbb{R}^d$, we introduce the weight $\alpha$: $\mathbb{R}^d \to [\,1,\infty\,)$ given by

$$\alpha(\theta) = (1 + |\theta|^2)^{\frac{1}{2}},\ \theta \in \mathbb{R}^d.$$

For $\rho \in \mathbb{N} \cup \{0\}$, let $\mu_\rho$ be the (possibly infinite) measure on $\mathbb{R}^d$ given by

$$\mu_\rho(d\theta) := \alpha^{-\rho}(\theta)\, d\theta, \tag{1.3}$$

where $d\theta$ denotes the Lebesgue measure on $\mathbb{R}^d$.
Given a Borel-measurable set $\Theta \subset \mathbb{R}^d$, let $L^{2\nu}_\rho(\Theta)$, $\nu \geq 1$, be the Banach space of $2\nu$-integrable functions w.r.t. the measure $\mu_\rho$ on $\Theta$. In what follows, we will always choose $\rho$ such that $\mu_\rho$ is a finite measure on $\Theta$. Note that this family of weighted $L^p$-spaces is commonly used in the theory of (both deterministic and stochastic) PDEs of parabolic type, see e.g. Appendix B.2 in the monograph [95] by Peszat and Zabczyk. For a closer look at these spaces, see Section 3.1 below.

Concerning the terms in (1.1) and (1.2) we assume that:

- the family $(A(t))_{t\in[\,0,T\,]}$ generates an *almost strong evolution operator* $U = (U(t,s))_{0\leq s\leq t\leq T}$ in $L^2_\rho(\Theta)$ (for its definition see Section 2.1 below),
- $E$, $F$, $\Sigma$ and $\Gamma$ are *Nemitskii-type nonlinear operators* defined through

*predictable* functions $e$, $f$, $\sigma$ and $\gamma$: $[0,T] \times \Omega \times \mathbb{R} \to \mathbb{R}$, e.g. we have

$$E(t,\varphi)(\theta) := e(t,\varphi(\theta)),\ \theta \in \Theta\ ,\ t \in [0,T],\ \varphi \in L^2_\rho(\Theta),$$

- $\mathcal{M}_\Sigma$ resp. $\mathcal{M}_\Gamma$ denotes the *multiplication operator* corresponding to $\Sigma$ resp. $\Gamma$, i.e. $\mathcal{M}_\Sigma$: $L^2(\Theta) \to L^1_\rho(\Theta)$ is defined through

  $\mathcal{M}_{\Sigma(t,\omega,\varphi)}(\psi)(\theta) := \sigma(t,\omega,\varphi(\theta))\psi(\theta)$, $\varphi \in L^2_\rho(\Theta)$, $\theta \in \Theta$ , $t \in [0,T]$, $\varphi \in L^2_\rho(\Theta)$, $\psi \in L^2(\Theta)$

  and $\mathcal{M}_\Gamma$ analogously through $\gamma$.
- $(W(t))_{t\in[0,T]}$ is a $Q$-Wiener process in $L^2(\Theta)$ with the correlation operator $Q$ to be specified in Section 2.3 below,
- $\tilde{N}$: $[0,T] \times \Omega \times L^2(\Theta) \to \mathbb{R}$ is a compensated Poisson random measure (see Section 2.4 below), and
- $(L(t))_{t\in[0,T]}$ is a Lévy process in $L^2(\Theta)$ (see Section 2.5 below).

The solutions to these equations will be constructed in the *mild* sense. More precisely, given an $L^2_\rho(\Theta)$-valued initial condition $\xi$, we look for $L^2_\rho(\Theta)$-valued *predictable* processes $(X(t))_{t\in[0,T]}$ such that for each $t \in [0,T]$ we have, $P$-almost surely,

$$\begin{aligned} X(t) = \quad & U(t,0)\xi + \int_0^t U(t,s)F(s,X(s))\,ds \\ & + \int_0^t U(t,s)\mathcal{M}_{\Sigma(s,X(s))}\,dW(s) \\ & + \int_0^t \int_{L^2(\Theta)} U(t,s)\mathcal{M}_{\Gamma(s,X(s))}x\,\tilde{N}(ds,dx) \end{aligned}$$

resp.

$$\begin{aligned} X(t) = \quad & U(t,0)\xi + \int_0^t U(t,s)E(s,X(s))\,ds \\ & + \int_0^t U(t,s)\mathcal{M}_{\Sigma(s,X(s))}\,dL(s). \end{aligned}$$

Let us stress that equations (1.1) and (1.2) are models of *jump diffusions with Nemitskii-type operators*. In the particular case $\Gamma = 0$, such type of infinite-dimensional diffusion equations was considered e.g. by Manthey and Zausinger in [76]. Compared to [76], our equation (1.1) has an additional *multiplicative* (i.e. solution-dependent) jump noise, which needs a careful

analysis.
In the *autonomous* case, i.e. in the case $A(t) = A$, $t \in [\,0,T\,]$, with $A$ being the generator of a $\mathcal{C}_0$-semigroup, equation (1.2) describes a jump diffusion driven by a Lévy process. In general separable Hilbert spaces, such equations were considered e.g. by Knäble in [59]. As compared to [59], the novelity of our work is that, restricting our consideration to the weighted $L^2$ spaces introduced above, we can cover the case of *time-dependent* operator coefficients $(A(t))_{t\in[\,0,T\,]}$. Furthermore, our *multiplication operators* $\mathcal{M}_\Sigma$ are of *Nemitskii type* and do not preserve the space $L^2(\Theta)$ resp. $L^2_\rho(\Theta)$ and hence are not *Hilbert-Schmidt*. Moreover, the drift coefficients are not necessarily *(local) Lipschitz continuous.*

Before we come to the main results and the structure of our work, let us give a short overview on the results known so far for equations of the above type:

**Equation (1.1):** This equation has two noise sources, a Wiener process and a compensated Poisson random measure.

First, let us give some remarks on the Wiener noise term.

Recall that the standard type of stochastic evolution equations in infinite-dimensional Hilbert spaces, which was considered in numerous papers on S(P)DEs, is

$$dX(t) = (AX(t) + F(t, X(t)))dt + \Sigma(t, X(t))dW(t),\ t \in [\,0,T\,]. \tag{1.4}$$

Given separable Hilbert spaces $G$ and $H$, one usually has the following assumptions:

- $A$ is the generator of a $\mathcal{C}_0$-semigroup $(S(t))_{t\in[\,0,T\,]}$ on $H$,
- $F$: $[\,0,T\,] \times H \to H$ is a drift term fulfilling proper regularity conditions,
- $\Sigma$: $[\,0,T\,] \times H \to \mathcal{L}_2(G,H)$ (with $\mathcal{L}_2(G,H)$ denoting the set of *Hilbert-Schmidt* operators from $G$ to $H$) is an operator diffusion coefficient,
- $(W(t))_{t\in[\,0,T\,]}$ is a Wiener process taking values in $G$.

*Existence* and *uniqueness* of so-called *mild solutions* in the case of *Lipschitz coefficients*, i.e. when we have, uniformly for $t \in [\,0,T\,]$ and $x, y \in H$,

$$||F(t,x) - F(t,y)|| \le C_F||x-y||_H \tag{$L_D$}$$

resp.

$$(L_{HS}) \qquad ||\Sigma(t,x) - \Sigma(t,y)||_{\mathcal{L}_2(G,H)} \le C_\Sigma ||x-y||_H$$

with some positive constant $C_F$ resp. $C_\Sigma$, is a well-known result in the literature (see e.g. the widely cited monograph [26] by DaPrato and Zabczyk). Recall that, given an initial condition $\xi \in H$, a *mild solution* to (1.4) is an $H$-valued pathwise continuous process $(X(t))_{t\in[0,T]}$ such that for each $t \in [0,T]$ we have, $P$-almost surely,

$$X(t) = S(t)\xi + \int_0^t S(t-s)F(s,X(s))\,ds + \int_0^t S(t-s)\Sigma(s,X(s))\,dW(s).$$

Note that *existence* and *uniqueness* in the special case $A = \Delta$ and $H = L^2_\rho(\Theta)$ was shown e.g. in the paper [66] by Kotelenez.

Nevertheless, in many interesting applications we cannot apply this standard framework. There has been a large activity in getting existence results in the case of *non-Lipschitz drift* since the 1980's. A popular example in applications is the situation, where $A$ is the Laplace operator and $F$ is a polynomial.

Without claiming to be complete, the following list of papers contains the main achievements in removing the global Lipschitz condition $(L_D)$ on $F$ in the case of the Laplacian $A = \Delta$ in (1.4):

- In 1971, Marcus in [77] in the case $\Sigma = \mathbf{I}$ ($\mathbf{I}$ denotes the identity operator) and, in 1987, Iwata in [54] in the case of bounded, solution-dependent $\Sigma$ obeying $(L_{HS})$ have showed *existence* and *uniqueness* of *mild* resp. *weak solutions* in the case of $F$ fulfilling the *strong monotonicity* condition

  $$\textbf{(SM)} \qquad < F(x) - F(y), x-y >_H > c_1 ||x-y||^p_H,\ x,y \in H$$

  and the *growth* condition

  $$\textbf{(G)} \qquad ||F(x)||^q_H \le c_2(1 + ||x||^p_H)$$

  with positive constants $c_1$, $c_2$ and $p > 2$, $q = \frac{p}{p-1}$.

- In 1988, in [72] Manthey weakened the assumption **(SM)** on $F$ to the *semi-dissipativity* condition

  $$\textbf{(SD)} \qquad < F(t,x) - F(t,y), x-y >_H \le \beta ||x-y||^2_H,\ x,y \in H,$$

with some $\beta \in \mathbb{R}_+$.

- In 1995, in [27] Da Prato/Zabczyk assumed $F$ to be the *Nemitskii* operator corresponding to a polynomial of the form

$$f(y) = \sum_{k=0}^{2n+1} b_k y^k,\ b_{2n+1} < 0,\ b_k \in \mathbb{R},\ 0 \leq k \leq 2n,\ n \geq 1, \tag{1.5}$$

  and construct so-called *generalized (mild) solutions* in weighted $L^2$-spaces over $\mathbb{R}^d$.
  Similar results for stochastic reaction-diffusion equations with nonlinear terms having polynomial growth and satisfying some dissipativity conditions can also be found in [20], [42] or [57].

- In 1996, in [73] Manthey showed the existence of *mild solutions* in the case of $F$ being the *Nemitskii* operator corresponding to a function $f$, which is *continuous*, has *one-sided linear growth* and is of *at most polynomial growth*. The solutions take their values in weighted $L^2$-spaces with possibly *unbounded domains* (for applications to finance, see e.g. [85]). Similar results are achieved by Gyöngy, Pardoux and Bally in the series of papers [11], [49], [50], [47] for *weak solutions* on *bounded domains*. Here (see also Section 3.2), *one-sided linear growth* means that there is some positive constant $c_f(T)$ such that uniformly in $t \in [0, T]$ we have the estimates

$$\textbf{(LG)} \qquad \begin{aligned} f(t,u) &\geq -c_f(T)(1-u) \text{ if } u \leq 0 \text{ and} \\ f(t,u) &\leq c_f(T)(1+u) \text{ if } u \geq 0, \end{aligned}$$

  whereas the *polynomial growth* property means that there is a $\nu \geq 1$ such that, again uniformly in $t \in [0, T]$,

$$\textbf{(PG)} \qquad |f(t,u)| \leq c_f(1 + |u|^\nu) \text{ if } u \in \mathbb{R}.$$

  Let us note that polynomials of the form (1.5) obey these properties.

Another class of results relates to the case of a *time-dependent* operator family $(A(t))_{t\in[0,T]}$ replacing $A$ in (1.4). In the case of *Lipschitz coefficients* the equation (1.4) has been considered e.g. by Seidler in [103]. Among the contributions extending to the case of a *non-Lipschitz* drift $F$ we mention the following, which both work on weighted $L^2$-spaces with unbounded domains:

- In 1992, in [65] Kotelenez showed *existence* and *uniqueness* of *mild solutions* in the case of $W$ being a *cylindrical* Wiener process (in the sense that $W$ is a $Q$-Wiener process with operator $Q = \mathbf{I}$), $F$ being a polynomial of the form (1.5) and $\Sigma$ obeying the *local Lipschitz property* and having *at most linear growth.*

- In the late 1990's, in [74] and [76], Manthey and his collaborators showed *existence* of solutions in the case of $F$ having *at most polynomial growth* and obeying the *one-sided linear growth* condition. This is a generalization of the previously mentioned paper.

All the papers mentioned above approximate the *non-Lipschitz* drifts by a family of *Lipschitz* ones and then use *comparison theorems* for the solutions to the equations corresponding to the Lipschitz drifts.
Generally speaking, the comparison method compensates the lack of a proper version of Girsanov's formula in the case of *time-dependent* $A(t)$.
Manthey and Zausinger invented a new method to prove their *comparison theorem* in [76], (cf. Theorem 3.3.1 there), simplifying the earlier one from Kotelenez' paper [65]. We intend to adapt and extend the comparison method from [76] to the case of *additive*, i.e. solution-indpendent, jump resp. jump diffusion coefficients in equation (1.1) resp. (1.2).

Given Hilbert spaces $G$ and $H$, the simplest example of an infinite-dimensional SDE with jumps is

$$dX(t) = (AX(t) + F(t, X(t)))dt + \int_G \Gamma(t, X(t))x\tilde{N}(dt, dx),\ t \in [\,0, T\,],$$

where $A$ and $F$ are as above, $\tilde{N}$ is a compensated Poisson random measure and $\Gamma$ is appropriate for stochastic integration (conditions for stochastic integration w.r.t. compensated Poisson random measures are presented in Section 2.5 below).
This equation has been treated e.g. in the following papers:

- in 1998, in [5] Albeverio, Wu and Zhang showed *existence* and *uniqueness* of *càdlàg mild* solutions in the case $A = \Delta$ and $H = L^2_\rho(\Theta)$. It means that, given the initial conditon $\xi$ and the semigroup $S$ generated by $A$, there exists a unique $H$-valued càdlàg process $(X(t))_{t\in[\,0,T\,]}$ satisfying for any $t \in [\,0, T\,]$, $P$-almost surely

$$X(t) = S(t)\xi + \int_0^t S(t-s)F(s, X(s))\, ds + \int_0^t \int_G S(t-s)\Gamma(s, X(s-))\, \tilde{N}(ds, dx).$$

- In 2005, in [60] Knoche showed *existence* and *uniqueness* of *mild càdlàg solutions* in the case of *Lipschitz coefficients* and $A$ being the generator of a $\mathcal{C}_0$-semigroup $(S(t))_{t\geq 0}$ in an abstract Hilbert space.

- In 2008, in [37] Filipovic, Tappe and Teichmann showed *existence* and *uniqueness* of *predictable mild and weak solutions*, taking values in some Hilbert space $H$ and being *meansquare* time-continuous in the case of *Lipschitz* drift and diffusion coefficients and $A$ being the generator of a $\mathcal{C}_0$-semigroup $(S(t))_{t\geq 0}$.

- In 2009, Albeverio, Mandrekar and Rüdiger in [3] showed *existence* and *uniqueness* of $H$-valued *càdlàg mild (but non-Markovian) solutions* in the case of the *Lipschitz* coefficients $F$ and $\Gamma$ depending on the whole solution path $t \mapsto X(t)$, $t \in [\,0,T\,]$.

The next step is to consider SDEs with both Wiener and jump noises (given by compensated Poisson random measures)

$$\begin{aligned} dX(t) = \quad & (AX(t) + F(t, X(t)))dt + \Sigma(t, X(t))dW(t) \\ & + \int\limits_G \Gamma(t, X(t))x\tilde{N}(dt, dx)\,, \quad t \in [\,0, T\,]. \end{aligned}$$

Given an $H$-valued initial condition $\xi$, a *mild solution* to this equation is an $H$-valued *predictable* process $(X(t))_{t\in[\,0,T\,]}$ such that for all $t \in [\,0,T\,]$, $P$-almost surely,

$$\begin{aligned} X(t) = \quad & S(t)\xi + \int\limits_0^t S(t-s)F(s,X(s))\,ds + \int\limits_0^t S(t-s)\Sigma(s,X(s))\,dW(s) \\ & + \int\limits_0^t \int\limits_G S(t-s)\Gamma(s,X(s))x\tilde{N}(ds,dx). \end{aligned}$$

In 2008, in [79] Marinelli, Prévôt and Röckner showed *existence*, *uniqueness* and regular dependence on the initial condition for *mild solutions* to this equation in the case of Lipschitz coefficients. Let us list some further results concerning certain special cases of this equation:

- In 2009, in [14] Bo and his collaborators showed existence of *mild solutions* in the case of $A$ being a *positive self-adjoint* operator on some Hilbert space $H$.
  Under Lipschitz assumptions on the coefficients, the solution takes values in Sobolev spaces $H_\alpha$ constructed by means of the operator $A^\alpha$.

- In 2010, in [81] Marinelli and Röckner proved *existence* and *uniqueness* of *càdlàg weak and mild solutions* to the above equation with *dissipative* drift and *Lipschitz* diffusion and jump coefficients.

- The so-called variational approach to diffusions with jumps in Banach spaces, including those driven by Lévy noise, was developed in [46], [70] and [105].

Compared to our work, these papers impose stronger assumptions on the operator coefficient $\Sigma$, namely that it is *Hilbert-Schmidt*. This is surely not the case for the *Nemitskii-type* operator $\Sigma$ in equation (1.1). In our context, the absence of the *Hilbert-Schmidt* property will be compensated by the smoothing properties of the *evolution operator* $(U(t,s))_{0\leq s\leq t\leq T}$, which are nevertheless not as strong as the smoothing properties of $\mathcal{C}_0$-semigroups $(S(t))_{0\leq t\leq T}$ (see conditions **(A0)**– **(A8)** in Section 3.1 below).

**Equation (1.2):** Recall that the standard class of stochastic evolution equations with Lévy noise is

$$dX(t) = (AX(t) + F(t, X(t)))dt + \Sigma(t, X(t))dL(t),\ t \in [\,0, T\,], \tag{1.6}$$

where

- $A$ is the generator of a $\mathcal{C}_0$-semigroup $(S(t))_{t\in[\,0,T\,]}$ on $H$;
- $F$: $[\,0, T\,] \times H \to H$ is a progressively measurable drift term;
- $\Sigma$: $[\,0, T\,] \times H \to \mathcal{L}_2(G, H)$ is a progressively measurable operator diffusion coefficient;
- $L$: $[\,0, T\,] \times \Omega \to G$ is a Lévy process taking values in some Hilbert space $G$.

Of particular importance for our considerations are *mild solutions* to (1.6). Given an $H$-valued initial condition $\xi$, a *mild solution* to (1.6) is an $H$-valued *predictable* process $X = (X(t))_{t\in[\,0,T\,]}$ obeying for any $t \in [\,0, T\,]$, $P$-almost surely,

$$X(t) = S(t)\xi + \int_0^t S(t-s)F(s, X(s))\,ds + \int_0^t S(t-s)\Sigma(s, X(s))\,dL(s).$$

For example, equations of such kind in infinite dimensions are treated in the recent monograph [95] by Peszat and Zabczyk. For a general theory of Lévy processes in finite dimensions, see e.g. the monographs [12] by Bertoin and [96] by Protter.

Let us give an overview on the results known so far under the standard assumptions, i.e. the conditions on $A$ are the ones from above:

- In 1987, in [21] Chojnowska-Michalik constructed a *weak solution* to (1.6) in the case $F = 0$ and $\Sigma = \mathbf{I}$. The solution process is just the Ornstein-Uhlenbeck process associated with Lévy noise. This is known to be the first paper dealing with a Hilbert space-valued SDE with a Levy process as noise.
- In 2000, in [41] Fuhrmann and Röckner show the existence of weak solutions in the case $F = 0$ and $\Sigma = \mathbf{I}$, which is a generalization to the paper [15], where Bogachev, Röckner and Schmuland treat the Wiener case. Both papers are mainly concerned with the so-called *Mehler semigroups*.
- In 2004, in [7] Applebaum extended the result of Chojnowska-Michalik to the case of $F = 0$ and $\Sigma$ being a bounded, solution-independent operator (not necessarily $\mathbf{I}$).
- In 2005, the result of [7] was improved in [106] by Stolze by allowing for *Lipschitz* drift coefficients $F \neq 0$ and bounded, solution-independent $\Sigma$. Furthermore, *existence* and *uniqueness* of *mild solutions* was shown.
- In 2006, in [59] Knäble showed *existence* and *uniqueness* of *mild solutions* to (1.6) in the case of *Lipschitz* coefficients and the Lévy measure $\eta$ corresponding to $L$ obeying the *square integrability property*

  **(SI)** $$\int_G ||x||_G^2 \eta(dx) < \infty.$$

  (For the definition of a Lévy measure see Section 2.4 below.)
- In 2010, in [17] Brezniak and Hausenblas showed *existence* of a *martingale solution* in the case of a second order uniformly elliptic operator $A$, a *dissipative* drift coefficient $F$ of *polynomial* growth, and a bounded and continuous jump coefficient $\Sigma$.
  A typical example, to which all this applies, is the case, where $A = \Delta$ in $H := L^2(\Theta)$ and $f(u) = -u^3 + u$.

Analogously to the case of equation (1.1), the operator coefficient $\Sigma$ in (1.2) is not *Hilbert-Schmidt* and the absence of this property is compensated by the smoothing property of the *evolution operator* $(U(t,s))_{0\leq s\leq t\leq T}$.
Let us now describe the main results of this thesis and the methods applied to obtain them.

## 1.3 The main results and the structure of the thesis

The thesis is devided into two parts. The first part, including Chapters 2–4, collects technical preliminaries and supporting material.

- In Chapter 2, we recall some general properties of Banach space-valued stochastic processes and *evolution operators*. This includes the definition and properties of Wiener processes, Lévy processes and Poisson random measures in Hilbert spaces. Furthermore, we discuss the stochastic integration w.r.t. Hilbert-space valued Wiener processes and (compensated) Poisson random measures, which will play a crucial role in the second part of the thesis.
  In particular, we recall the *Lévy-Itô decomposition*, which is the key tool to link the results for the two equations (1.1) and (1.2) in the later chapters.

- In Chapter 3, we place ourselves in the framework of the weighted Lebesgue spaces $L^{2\nu}_\rho(\Theta)$, $\nu \geq 1$.
  In Sections 3.1 and 3.2, we specify the conditions on evolution operators and *Nemitskii* operators in the weighted Lebesgue spaces $L^{2\nu}_\rho(\Theta)$, whereas, in Sections 3.3 and 3.4, we discuss the regularity properties of Bochner integrals and stochastic convolutions w.r.t. Wiener processes in the Banach spaces $L^{2\nu}_\rho(\Theta)$.

- In Chapter 4, we study the regularity properties of stochastic convolutions w.r.t. compensated Poisson random measures in the spaces $L^{2\nu}_\rho(\Theta)$.

The main new results in this part (besides that unbounded domains $\Theta$ are allowed) are the continuity properties of the Bochner integrals and of the stochastic convolutions, which we will describe more precisely in the contents of Chapters 3 and 4 below.

In the second part, consisting of Chapters 5–8, we treat the following items:

- In Chapter 5, we prove the general *existence* and *uniqueness* results for equation (1.1) resp. (1.2) in the case of *Lipschitz* coefficients $F$, $\Sigma$ and $\Gamma$ resp. $E$ and $\Sigma$.

- In Chapter 6, we establish *comparison results* for *mild solutions* to equation (1.1) resp. (1.2) in the *Lipschitz case* with functions $f$ resp. $e$ defining $F$ resp. $E$ in the Nemitskii sense being replaced by larger resp. smaller ones and *additive*, i.e. solution-independent, jump resp. jump diffusion coefficients in equation (1.1) resp. (1.2).

- In Chapter 7, we show *existence* and (in some cases) *uniqueness* results for equation (1.1) resp. (1.2) with the drift coefficients $F$ resp. $E$ being of at most *polynomial growth* and the jump resp. jump diffusion coefficients being *additive*.
- In Chapter 8, we extend the results of Chapters 6 and 7 to the case of *multiplicative* jump resp. jump diffusion coefficients in equation (1.1) resp. (1.2).
  In Section 8.2, we establish *comparison results* for solutions to equations (1.1) and (1.2) in the case of *Lipschitz* coefficients with functions $f$ resp. $e$ defining $F$ resp. $E$ in the Nemitskii sense being replaced by larger resp. smaller ones.
  In Sections 8.3 and 8.4, we prove *existence results* for equations (1.1) and (1.2) in the most general setting of *non-Lipschitz* drifts and *multiplicative* diffusion, jump and jump diffusion coefficients.

Mainly, we have two classes of new results in this part. First, we prove *existence* and *uniqueness* results in the case of *nonautonomous*, i.e. time-dependent, generators of the evolution operator and *non-Lipschitz* drift coefficients, both in the case of *additive* (see Theorem 7.1.2 and 7.1.3 below) and *multiplicative* (see Theorems 8.1.1 and 8.1.2) jump resp. jump diffusion coefficients in equation (1.1) resp. (1.2). Second, we have a generalization of the finite-dimensional comparison theory for SDEs with jumps (see e.g. [92], [113], [67]) to the infinite-dimensional case (see Theorem 8.1.1).
In the main results, an explicit relation between the degree of polynomial growth of the drift coefficients, integrability properties of the Lévy intensity measure and the regularity properties of the evolution operator is established.

In the following, we describe the content of our work chapter by chapter.

**The contents of the chapters**

In **Chapter 2**, we collect some technical preliminaries, the main of which we shall briefly describe here.

First, we recall the general definitions of *(almost strong) evolution operators* in Banach spaces $B$ (Section 2.2) and of **$Q$-Wiener processes** (Section 2.3), **Lévy processes** (Section 2.4) and **compensated Poisson random measures** (Section 2.4) in Hilbert spaces $H$.
A crucial issue to have a link between the two equations (1.1) and (1.2) later is the so-called *Lévy-Itô decomposition* (see Lemma 2.4.10 below) and its refinement for *square-integrable* Lévy processes (cf. Lemma 2.4.13 below).
If the *square-integrability property* **(SI)** is fulfilled for the intensity measure $\eta$ corresponding to a Lèvy process $(L(t))_{t\geq 0}$ in a Hilbert space $G$, the *Lévy-*

*Itô decomposition* takes the form

$$L(t) = tm + W(t) + \int_G x\tilde{N}(t, dx)$$

with a drift vector $m \in G$, a $Q$-Wiener process $W$ and a compensated Poisson random measure $\tilde{N}$.

With the help of this decomposition we can rewrite equation (1.2) as an equation of form (1.1) with a new drift term $F := E + \mathcal{M}_\Sigma(m)$. Nevertheless we will treat equation (1.2) separately, because, in our setting, the singular drift term $\mathcal{M}_\Sigma(m)$ is only in $L^1_\rho(\Theta)$, and the general $Q$-Wiener process $(W(t))_{t\in[0,T]}$ does **not** obey the coordinate representation

$$W(t) = \sum_{n\in\mathbb{N}} \sqrt{a_n} w_n(t) e_n,\ t \in [0, T],\ \text{(cf. (2.5) below)}$$

with an orthonormal basis $(e_n)_{n\in\mathbb{N}} \subset L^2(\Theta)$ obeying

$$\sup_{n\in\mathbb{N}} ||e_n||_{L^\infty} < \infty.$$

For technical simplicity only, in Chapters 5–8 below we only consider $L^2(\Theta)$-valued Lévy processes $L$, whose intensity measures $\eta$ obey the *square-integrability property* **(SI)**, but the corresponding results can be extended to general Lévy processes in $L^2(\Theta)$. Note that **(SI)** is equivalent to the finiteness of the second moments of $(L(t))_{t\in[0,T]}$ (see Proposition 2.4.14 below).

Then, in Section 2.5 resp. 2.6, we recall the properties of **stochastic integrals w.r.t. $Q$-Wiener processes resp. compensated Poisson random measures** in Hilbert spaces $H$.

The following analytic tools are of particular importance for the rest of the thesis:

- the *Burkholder-Davis-Gundy inequality* for stochastic integrals w.r.t. $Q$-Wiener processes (cf. Lemma 2.5.4/2.5.6 below);
- the *(infinite-dimensional version of the) Bichteler-Jacod inequality* for the stochastic integration w.r.t. compensated Poisson random measure $\tilde{N}$ from [80], [81](cf. Lemma 2.6.10 below);
- the *Gronwall-Bellman lemma* (see Lemma 2.7.3 below).

In **Chapter 3**, we introduce the special weighted spaces $L^{2\nu}_\rho(\Theta)$, $\nu \geq 1$, where the weight $\mu_\rho$ is defined by (1.3). We introduce some regularity conditions on *almost strong evolution operators* and *Nemitskii* operators in these spaces. Furthermore, we present some preliminary facts on (stochastic) convolutions in $L^2_\rho(\Theta)$.

In Section 3.1, we impose conditions on *almost strong evolution operators* $U = (U(t,s))_{0\le s\le t\le T}$ in $L^2_\rho(\Theta)$ (see **(A0)**–**(A8)** in that section). These conditions later yield the well-definedness and regularity properties of the stochastic convolutions w.r.t. $Q$-Wiener processes resp. compensated Poisson random measures (see $I^W_\varphi$ and $I^{\tilde{N}}_\varphi$ below).

In Section 3.2, we recall the definition of (nonlinear) *Nemitskii* operators and specify the conditions of *Lipschitz continuity*, *linear boundedness*, *one-sided linear growth* and *at most polynomial growth* (see **(LC)**, **(LB)**, **(LG)** and **(PG)** in that section) for their generating functions on $\mathbb{R}$.

In Sections 3.3 and 3.4, we are concerned with the well-definedness of convolution Bochner integrals and the stochastic convolutions w.r.t. $Q$-Wiener processes in $L^2_\rho(\Theta)$. Similar problems have already been treated e.g. by Manthey and Zausinger in [76]. As is typical in the literature, we consider **two basic cases**. In the so-called **nuclear case**, we assume $W = (W(t))_{t\in[0,T]}$ to be a $Q$-Wiener process with a *nuclear* covariance operator $Q$ in $L^2(\Theta)$, which yields a **coordinate representation**

$$W(t) = \sum_{n\in\mathbb{N}} \sqrt{a_n}\, w_n(t) e_n,\ t\in[0,T].$$

Here, $(a_n)_{n\in\mathbb{N}}$ is a summable family of positive numbers, $(e_n)_{n\in\mathbb{N}}$ is a complete orthonormal system in $L^2(\Theta)$, consisting of eigenvectors of $Q$ (corresponding to the eigenvalues $a_n$) such that

$$\sup_{n\in\mathbb{N}} ||e_n||_\infty < \infty,$$

and $(w_n)_{n\in\mathbb{N}}$ is a family of independent real-valued Brownian motions.
The second case (the so-called **cylindrical case**) deals with the *cylindrical* Wiener process, i.e.

$$W(t) = \sum_{n\in\mathbb{N}} w_n(t) e_n,\ t\in[0,T],$$

is a $Q$-Wiener process with $Q = \mathbf{I}$.
Then, the assumptions imposed on the *almost strong evolution operator* $(U(t,s))_{0\le s\le t\le T}$ guarantee the well-definedness of the so-called (Bochner resp. Wiener) convolution processes

$$t \mapsto I_\varphi(t) := \int_0^t U(t,s)\varphi(s)\,ds,$$

$$t \mapsto I_{\varphi_m}(t) := \int_0^t U(t,s)\mathcal{M}_{\varphi(s)} m \, ds, \; m \in L^2(\Theta)$$

and

$$t \mapsto I_\varphi^W(t) := \int_0^t U(t,s)\mathcal{M}_{\varphi(s)} \, dW(s), \; t \in [\,0, T\,],$$

for *predictable processes* $(\varphi(t))_{t \in [\,0,T\,]}$ taking values in $L^2_\rho(\Theta)$ resp. $L^{2\nu}_\rho(\Theta)$.
A crucial role here is played by condition **(A2)**, which assumes that there exists a regularity constant $\zeta \in [\,0, 1\,)$ associated to the evolution operator such that the following estimate for the *Hilbert-Schmidt* norm holds

$$\text{(1.7)} \quad ||U(t,s)\mathcal{M}_\varphi||^2_{\mathcal{L}_2(L^2, L^2_\rho)} \leq c(T)(t-s)^{-\zeta} ||\varphi||^2_{L^2_\rho}, \; 0 \leq s < t \leq T, \; \varphi \in L^2_\rho(\Theta).$$

This allows us to establish the continuity of the above convolutions, which will imply similar properties of the solutions to equations (1.1) and (1.2). Alternatively to **(A2)**, **(A5)** (with $\nu = 1$) from Section 3.1 below implies the same.
The pathwise continuity of $t \mapsto I_\varphi(t)$ and $t \mapsto I^W_\varphi(t)$ was already known e.g. from the paper [76] (see Theorem 3.1.1, p.56 there).
Furthermore, we also show the continuity of the above processes in the Banach spaces

$$L^q(\Omega; L^2_\rho(\Theta)) := \{f : \Omega \to L^2_\rho \mid \int_\Omega ||f(\omega)||^q_{L^2_\rho} \, P(d\omega)\}$$

resp.

$$L^{2\nu}(\Omega; L^{2\nu}_\rho(\Theta)) := \{f : \Omega \to L^{2\nu}_\rho \mid \int_\Omega ||f(\omega)||^{2\nu}_{L^{2\nu}_\rho} \, P(d\omega)\}$$

for $q \geq 2$ and $\nu \geq 1$.

A technical problem is caused here by the fact that $\mathcal{M}_{\varphi(s)}$ is not a *Hilbert-Schmidt* operator in $L^2_\rho(\Theta)$. Well-definedness and continuity properties of the above convolutions are achieved by additional regularity assumptions on $U = (U(t,s))_{0 \leq s \leq t \leq T}$ (see Section 3.1 below).
In particular, the regularity constant $\zeta \in [\,0, 1\,)$ corresponding to $U$ (cf. (1.7) above) plays an important role for the rest of this thesis. It determines the possible choices of the parameter $q$ resp. $\nu$ in the definition of the spaces $L^q(\Omega; L^2_\rho(\Theta))$ resp. $L^{2\nu}(\Omega; L^{2\nu}_\rho(\Theta))$ (Note that, for a given $\zeta$, in Section 3.4 we need $q > \frac{2}{1-\zeta}$ resp. $\nu > \frac{1}{1-\zeta}$.).
In this thesis we will treat both the case of the *Nemitskii* drift operator being *Lipschitz* and obeying a *one-sided linear growth* condition (see **(LG)** in Section 3.1 below). The latter is the most general class of drift coefficients considered so far. In particular, it includes the case of *semi-dissipative* drifts,

i.e. $F$ resp. $E$ in the equation (1.1) resp. (1.2) fulfills condition **(SD)** with $H = L^2_\rho(\Theta)$. The solvability of PDE's similar to equation (1.1) with *dissipative* drifts was established both in an $L^2$-setting (cf. Theorem 13 in [80]) and for general Hilbert spaces (cf. Theorem 3 in [81]) by Marinelli and Röckner.

We note that we have to treat the convolutions in the scale of Banach spaces $L^{2\nu}_\rho(\Theta)$ in view of the later considerations with drifts, which are *non-Lipschitz* but have *at most polynomial growth*.

In **Chapter 4**, we study the properties of stochastic convolutions w.r.t. compensated Poisson random measures. More precisely, given a compensated Poisson random measure $\tilde{N}$ in $[\,0,T\,] \times \Omega \times L^2(\Theta)$, we show the well-definedness and continuity in $L^q(\Omega; L^2_\rho)$ resp. $L^{2\nu}(\Omega; L^{2\nu}_\rho)$ of the stochastic convolution

$$t \mapsto I^{\tilde{N}}_\varphi(t) := \int_0^t \int_{L^2} U(t,s)\mathcal{M}_{\varphi(s)}(x)\, \tilde{N}(ds, dx)$$

for a *predictable* process $\varphi = (\varphi(t))_{t\in[\,0,T\,]}$ obeying

$$\sup_{t\in[\,0,T\,]} \mathbf{E}||\varphi(t)||^q_{L^2_\rho} < \infty \tag{1.8}$$

resp.

$$\sup_{t\in[\,0,T\,]} \mathbf{E}||\varphi(t)||^{2\nu}_{L^{2\nu}_\rho} < \infty. \tag{1.9}$$

Later in Chapter 5, the classes of such processes will be denoted by $\mathcal{H}^q(T)$ resp. $\mathcal{G}_\nu(T)$.
Analogously to the stochastic convolutions w.r.t. $Q$-Wiener processes, we here face the problem that, given any $\varphi \in L^2_\rho(\Theta)$, $\mathcal{M}_\varphi$ is not a *Hilbert-Schmidt* operator from $L^2(\Theta)$ to $L^2_\rho(\Theta)$.
To overcome this problem, in the current chapter we have to impose an additional assumption on $U = (U(t,s))_{0\le s\le t\le T}$ (cf. assumption **(A5)**/(**(A5)*** from Section 3.1), which generalizes **(A2)** for $\nu > 1$. Namely, for a given $\nu \ge 1$ and any $\varphi \in L^{2\nu}_\rho(\Theta)$ with its *multiplication operator* $\mathcal{M}_\varphi$, $U(t,s)\mathcal{M}_\varphi$ should be a *Hilbert-Schmidt* (**(A5)**) resp. bounded (**(A5)***) operator mapping $L^2(\Theta)$ to $L^{2\nu}_\rho(\Theta)$.
This leads to the restriction $q < \frac{2}{\zeta}$ resp. $\nu < \frac{1}{\zeta}$ on the choice of the spaces $\mathcal{H}^q(T)$ resp. $\mathcal{G}_\nu(T)$ in terms of the regularity constant $\zeta \in [\,0,1\,)$.
Let us stress that this condition differs from that needed in the Wiener case ($q > \frac{2}{1-\zeta}$ resp. $\nu > \frac{1}{1-\zeta}$). Note that this is the case, since we apply the *Bichteler-Jacod inequality* (cf. Theorem 2.6.10 below) for compensated Pois-

son random measures instead of the *Burkholder-Davis-Gundy inequality* (cf. Theorem 2.5.4 (nuclear case) resp. Theorem 2.5.6 (cylindrical case) below) in the Wiener case.

Another crucial issue for the well-definedness of the stochastic convolution w.r.t. a compensated Poisson random measure $\tilde{N}$ is the *existence* of higher moments of the intensity measure $\eta$ corresponding to $\tilde{N}$. This is reflected in the assumption

$$\int_{L^2(\Theta)} (||x||_{L^2} \vee 1)^r \, \eta(dx) + \left( \int_{L^2(\Theta)} (||x||_{L^2} \wedge 1)^2 \, \eta(dx) \right)^{\frac{r}{2}} < \infty \tag{1.10}$$

both for $r = q$ and $r = 2\nu$.
It is well-known that the moment estimate (1.10) for the intensity measure $\eta$ implies the *existence* of the corresponding moments of the associated (via the *Lévy-Itô decomposition*) Lévy process $(L(t))_{t\in[0,T]}$, i.e. $\mathbf{E}||L(t)||_{L^2}^r < \infty$ for $r = q$ resp. $r = 2\nu$ (see Proposition 2.4.14 below).

Finally, we show that in the case of $(\varphi(t))_{t\in[0,T]}$ being uniformly bounded in mean in the sense that

$$\sup_{t\in[0,T]} \mathbf{E}||\varphi(t)||_{L^\infty} < \infty \text{ (see Section 3.1 below)}$$

and $U$ being *pseudo contractive*, i.e. there exists a constant $\beta$ such that $||U(t,s)||_{\mathcal{L}(L^2_\rho)} < e^{\beta(t-s)}$ for any $0 \leq s \leq t \leq T$ (cf. condition **(A7)** in Section 3.1 below), there exists a *càdlàg* version of the process $t \mapsto I_\varphi^{\tilde{N}}(t) \in L^2_\rho(\Theta)$ both in the case of $(\varphi(t))_{t\in[0,T]}$ being in $\mathcal{H}^q(T)$ resp. $\mathcal{G}_\nu(T)$ (see (1.8) and (1.9)).

In **Chapter 5**, we prove the main *existence* and *uniqueness* results in the case of *Lipschitz* coefficients.
The solutions will be constructed in the Banach spaces $\mathcal{H}^q(T)$ and $\mathcal{G}_\nu(T)$ of *predictable* $L^2_\rho(\Theta)$ resp. $L^{2\nu}_\rho(\Theta)$-valued processes $(X(t))_{t\in[0,T]}$ obeying (1.8) and (1.9). The parameters $q \geq 2$ and $\nu \geq 1$ will be specified below. We check that *Nemitskii* operators corresponding to functions fulfilling the *Lipschitz property* preserve the spaces $\mathcal{H}^q(T)$ resp. $\mathcal{G}_\nu(T)$.
The spaces $\mathcal{H}^q(T)$ will be used to study equations (1.1) and (1.2) in the case of the drift coefficients having *at most linear growth*, whereas the spaces $\mathcal{G}_\nu$ are needed in the case of the drift coefficients having *at most polynomial growth* of order $\nu > 1$.

We start our study of equations (1.1) and (1.2) with the case of *Lipschitz*

continuous drift and diffusion coefficients. The main results of this chapter are Theorems 5.2.1 and 5.2.2, where we establish the *existence* and *uniqueness* of *mild solutions* to equations (1.1) and (1.2) in the spaces $\mathcal{H}^q(T)$ and $\mathcal{G}_\nu(T)$.
Because of the singularity properties of the *Hilbert-Schmidt* norm $||U(t,s)||_{\mathcal{L}_2(L^2,L^2_\rho)}$ as $t \downarrow s$ resp. $s \uparrow t$ (cf. (1.7) above), we have to overcome some essential technical difficulties to prove solvability even in the *Lipschitz* case.
Namely, to control the convergence of approximations in Banach's fixed point theorem resp. Picard's iteration procedure, we have to apply the *Gronwall-Bellman lemma* (see Lemma 2.7.2 resp. Remark 2.7.3 below) allowing for the time-singularity $(t-s)^{-\zeta}$ as $s \uparrow t$ resp. $t \downarrow s$.
Furthermore, we prove continuity of $t \mapsto X(t)$ in $L^q(\Omega; L^2_\rho(\Theta))$ resp. $L^{2\nu}(\Omega; L^{2\nu}_\rho(\Theta))$ both for the solutions to (1.1) and (1.2) in the case $q \in [\,2, \frac{2}{\zeta}\,)$ resp. $\nu \in [\,1, \frac{1}{\zeta}\,)$ (with the regularity constant $\zeta$ from (1.7)). Crucial for the *existence* of *càdlàg versions* is the restriction to *evolution operators* $U = (U(t,s))_{0\le s\le t\le T}$ with regularity constants $\zeta < \frac{1}{2}$. This assumption excludes the application to second order elliptic operators, but still allows for differential operators of higher order (see Appendix D on this topic). The restriction to the regularity constant $\zeta < \frac{1}{2}$ is necessary, since we need both the (pathwise) theory for Wiener convolutions from Chapter 3 (requiring $q > \frac{2}{1-\zeta}$ resp. $\nu > \frac{1}{1-\zeta}$) and Poisson convolutions from Chapter 4 (requiring $q < \frac{2}{\zeta}$ resp. $\nu < \frac{1}{\zeta}$). To have both conditions, we have to assume that the intervall $(\frac{1}{1-\zeta}, \frac{1}{\zeta})$, where $\nu$ and $\frac{q}{2}$ take their values, is nonempty, which (by setting $\frac{1}{0} := \infty$) gives us the condition $\zeta \in [\,0, \frac{1}{2}\,)$.
Additionally assuming the boundedness of the jump resp. jump diffusion coefficient and the *pseudo contractivity* of the *evolution operator*, we then get the *existence* of càdlàg versions of the solutions.
Finally, let us mention the result of Remark 5.1.11 covering the continuity properties of the Bochner integrals for polynomial drift. In Chapters 7 and 8, this result will allow us to prove time-continuity in $L^q(\Omega; L^2_\rho(\Theta))$ if the drift is polynomial of order 1, whereas for polynomials of order strictly bigger than 1 we have time-continuity in $L^2(\Omega; L^2_\rho(\Theta))$.

In **Chapter 6**, we extend the *comparison theorems* for SDEs with Wiener noise and Lipschitz drift and diffusion coefficients shown by Kotelenez (cf. [65]) resp. Manthey and Zausinger (cf. [76]) to the case of *additive* jump resp. jump diffusions coefficients. The latter means that $\Gamma$ in (1.1) (and later $\Sigma$ in (1.2)) is solution-independent.
Recall that, when considering (1.1), we assume the $Q$-Wiener process $W$ to be as in the nuclear resp. cylindrical case from Chapter 3.
The proof of such *comparison theorems* in infinite dimensions relies on a suitable finite-dimensional approximation. This can be done similar to the comparison theory for SDEs with Wiener noise by Ikeda and Watanabe (cf.

Chapter VI in [53]) resp. Karatzas and Shreve (Chapter 5, Proposition 2.1.8 in [56]). It is crucial to have a family of bounded operators $(A_N(t))_{t\in[0,T]}$, $N \in \mathbb{N}$, approximating $(A(t))_{t\in[0,T]}$ in a proper sense (cf. **(A6)** from Section 3.1). Furthermore, we construct approximations $(X_{N,M}(t))_{t\in[0,T]}$, $N, M \in \mathbb{N}$, of the solutions $(X(t))_{t\in[0,T]}$ to (1.1) resp. (1.2). These approximations are *mild solutions* to equation (1.1) with $A$ being replaced by $A_N$ and $W$ being replaced by the finite dimensional Wiener noises

$$W_M(t) = \sum_{n=1}^{M} \sqrt{a_n} w_n(t) e_n,\ t \in [0,T].$$

Thereafter, showing the convergence in $L^2(\Omega; L^2_\rho(\Theta))$ of the approximating solutions $X_{N,M}$ to the *unique mild solution* $X$ to (1.1) resp. (1.2), we conclude the required comparison for the initial equations.
According to Theorem 6.1.1, which is the main result of Chapter 6, the relation $f^{(1)} \leq f^{(2)}$ resp. $e^{(1)} \leq e^{(2)}$ for the generating function of the *Nemitskii* drift operator $F$ resp. $E$ implies that

$$X^{(1)}(t) \leq X^{(2)}(t) \text{ } P\text{-almost surely for all } t \in [0,T]$$

for the *mild predictable solutions* $(X(t))_{t\in[0,T]}$ to (1.1) resp. (1.2) with solution-independent $\Gamma$ resp. $\Sigma$.

In **Chapter 7**, we show the main *existence* and *uniqueness* results in the case of *non-Lipschitz drift coefficients* and *additive* jump resp. jump diffusion coefficients in equation (1.1) resp. (1.2).

Instead of the *Lipschitz* property **(LC)**, we assume that the function $f$ resp. $e$ corresponding to the *Nemitskii* drift operator $F$ resp. $E$ is *continuous*, satisfies the *one-sided linear growth* condition **(LG)** and is of *at most polynomial growth* satisfying **(PG)** for some $\nu \geq 1$.
The proof of the existence result (cf. Section 7.2 ad 7.3 below) is based on the *comparison theorem* established in Chapter 6. To this end, we adapt the standard scheme of proof used e.g. by Kotelenez in [65] and by Manthey and Zausinger in [76].
More precisely, in the case of equation (1.1) we consider monotone *Lipschitz* approximations $f_N$, $f_{N,M}$, $N, M \in \mathbb{N}$, of the function $f$ obeying $f_{N,M} \uparrow f_N$ as $M \to \infty$ and $f_N \downarrow f$ as $N \to \infty$ (see (7.12) and (7.13) below).
We prove the convergence in $\mathcal{H}^q(T)$ resp. $\mathcal{G}_\nu(T)$ of the corresponding solutions $X_{N,M}$ to a certain process $X$. Thereafter, we check that $X$ is a *mild solution* to equation (1.1). The proof in the case of equation (1.2) works completely analogous with monotone Lipschitz approximations $e_N$, $e_{N,M}$, $N, M \in \mathbb{N}$, of the function $e$.

Recall that whether we consider the equations in $\mathcal{H}^q(T)$ or $\mathcal{G}_\nu(T)$ depends on the drift coefficient. More precisely, the *mild solutions* $X$ to (1.1) resp. (1.2) take their values in $\mathcal{H}^q(T)$ if the function $f$ resp. $e$ corresponding to the *Nemitskii* drift coefficient $F$ resp. $E$ obeys the *polynomial growth* condition **(PG)** with $\nu = 1$, whereas the solutions are in $\mathcal{G}_\nu(T)$ if the function $f$ resp. $e$ corresponding to the *Nemitskii* drift coefficient $F$ resp. $E$ obeys the *polynomial growth* condition **(PG)** with $\nu > 1$.
As in Chapter 5, the possible values of $q$ and $\nu$ depend on the behaviour of the *almost strong evolution operator* $U = (U(t,s))_{0\leq s\leq t\leq T}$ for $s \uparrow t$ resp. $t \downarrow s$. In the case $\nu > 1$, we particularly need the integrability assumption (1.10) on the intensity measure $\eta$ with $r = \nu^2$, i.e.

$$\int\limits_{L^2(\Theta)} (||x||_{L^2} \vee 1)^{2\nu^2}\,\eta(dx) + \left(\int\limits_{L^2} (||x||_{L^2} \wedge 1)^2\,\eta(dx)\right)^{\nu^2} < \infty.$$

This condition seems to be quite natural in the case of higher order polynomials as drifts in SDEs with jumps. In particular, a similar condition was imposed in Marinelli's and Röckner's paper [80] dealing with the well-posedness of stochastic reaction-diffusion equations with Poisson noise (see Section 2.1, p.1531 there).
With the help of Remark 5.1.11 **(i)**, in the case $\nu = 1$ we prove the continuity of $t \mapsto X(t)$ in $L^q(\Omega; L^2_\rho(\Theta))$ analogously to the claims in Chapter 5 (cf. Theorem 5.2.1 there). Furthermore, with the help of Remark 5.1.11 **(ii)** we prove the continuity of $t \mapsto X(t)$ in $L^2(\Omega; L^2_\rho(\Theta))$ in the case $\nu > 1$, i.e., compared to the Lipschitz case, we are no longer able to prove continuity in $L^{2\nu}(\Omega; L^{2\nu}_\rho)$ due to the polynomiality of the drift coefficient. Finally, we show that there are *càdlàg versions* of the solutions if $U$ obeys the *pseudo contractivity* property and $\gamma$ (in the case of equation (1.1)) resp. $\sigma$ (in the case of equation (1.2)) is uniformly bounded.
To this end, we apply the corresponding results on *càdlàg* properties of stochastic convolutions w.r.t. Wiener processes resp. compensated Poisson random measures established in Chapter 3 resp. 4. Thus, as in Chapter 5, we again restrict our consideration to *evolution operators* with regularity constant $\zeta < \frac{1}{2}$. Furthermore in the case of **(PG)** being fulfilled with $\nu = 1$ we impose the condition $q \in (\frac{2}{1-\zeta}, \frac{2}{\zeta})$, whereas in the case of **(PG)** being fulfilled with $\nu > 1$ we need that $\nu \in (\frac{1}{1-\zeta}, \frac{1}{\zeta})$.

Then, in the special case of a $\mathcal{C}_0$-semigroup $(S(t))_{t\in[0,T]}$ replacing the *evolution operator* $(U(t,s))_{0\leq s\leq t\leq T}$, we also show uniqueness of the solutions to (1.1) with the help of Marinelli's and Röckner's uniqueness result (cf. Proposition 7 in [80]) in the additive case. To this end, we restrict to the case of a bounded $\Theta \subset \mathbb{R}^d$, a cylindrical Wiener process $(W(t))_{t\in[0,T]}$ and a *uniformly maximal monotone* drift $f$.

In **Chapter 8**, we show the *existence* of *mild solutions* to equations (1.1) resp. (1.2) in the case of *non-Lipschitz drift* coefficients and *multiplicative* jumps resp. jump diffusions (see Theorems 8.1.1 and 8.1.2 below).

We intend to apply the same scheme of proof as in Chapter 7, which makes it necessary to prove a *comparison theorem* in the case of *Lipschitz* drift and diffusion coefficients (cf. Theorem 8.1.3 below). Therefore, we have to extend the comparison results from Chapter 6 to the case of *multilplicative* jump resp. jump diffusion coefficients.

In Section 8.2, we extend the finite-dimensional comparison theory for SDEs with jumps (see e.g. the papers [92] / [113] resp. [67] by Peng and Zhu resp. Krasin and Melnikov (and also Appendix C below)) to infinite dimensions. Similarly to the scheme in Chapter 6, we first prove a *comparison theorem* for finite-dimensional approximations of equations (1.1) and (1.2). To this end, we consider the approximating equations in the Sobolev spaces $W^{m,2}(\Theta) \subset L^2(\Theta)$. Here, $m \in \mathbb{N}$ is chosen big enough such that $W^{m,2}(\Theta)$ is continuously embedded into the space $C_b(\Theta)$ of continuous bounded functions on $\Theta$. More precisely, we choose a domain $\Theta \subset \mathbb{R}^d$ such that the *weak cone property* is fulfilled. Then, by *Sobolev' s embedding theorem* (cf. also Appendix A, Theorem A.6 below), we can embed $W^{m,2}(\Theta)$ continuously into $C_b(\Theta)$ for suitable $m \in \mathbb{N}$.
We also need to assume that the family of operators $(U_N)_{N\in\mathbb{N}}$ approximating the *almost strong evolution operator* $U$ (recall **(A6)** needed in Chapter 6) obeys certain regularity properties in $W^{m,2}(\Theta)$ (see condition **(A8)** in Section 3.1 below). Finally, the jump coefficient $\Gamma$ resp. the jump diffusion coefficient $\Sigma$ has to be *monotonically increasing* in the sense that it is generated by a monotonically increasing function, and the intensity measure $\eta$ corresponding to the compensated Poisson random measure $\tilde{N}$ resp. the Lévy process $L$ has to be concentrated on the set $L^2_{\geq 0}$ of functions in $L^2(\Theta)$ that are almost everywhere *nonnegative* on $\Theta$. Alternatively, we can also treat the case of a *monotonically decreasing* jump resp. jump diffusion coefficient (in the sense that it is generated by a monotonically decreasing function), and an intensity measure $\eta$ being concentrated on the set $L^2_{\leq 0}$ of functions in $L^2(\Theta)$ that are almost everywhere *nonpositive* on $\Theta$. For fields of application for the latter family of processes see e.g. Chapter 7 in the monograph [12] by Bertoin.
Showing the convergence in $L^2(\Omega; L^2_\rho(\Theta))$ of the approximating solutions $X_{N,M}$ to the *unique mild solution* $X$ to (1.1) resp. (1.2), we conclude the required comparison for the initial equations.

Having achieved a comparison result for *Lipschitz* coefficients, we apply the scheme used in proving the *existence* results in Chapter 7 to get *existence* of *mild solutions* in $\mathcal{H}^q(T)$ resp. $\mathcal{G}_\nu(T)$, which are continuous in $L^q(\Omega; L^2_\rho(\Theta))$ resp. $L^2(\Omega; L^2_\rho(\Theta))$. Again there are *càdlàg versions* of the solutions if $U$ obeys the *pseudo contractivity* property and $\gamma$ (in the case of the equation (1.1)) resp. $\sigma$ (in the case of the equation (1.2)) is uniformly bounded.
By the same reasoning as in Chapter 5, we need to restrict our considerations to evolution operators with regularity constants $\zeta < \frac{1}{2}$.
In comparison to Chapter 7, we are not able to prove *uniqueness* of the solutions in the case of a $\mathcal{C}_0$-semigroup $(S(t))_{t\in[0,T]}$ by directly applying the *uniqueness* condition from [80] (cf. Remark 8.1.4 below and Theorem 13 in [80]). This needs a modification of the assumptions on the compensated Poisson random measure $\tilde{N}$ resp. the Lévy process $(L(t))_{t\in[0,T]}$, which we will not discuss here (see also Remark 8.1.6 **(v)** below).

Finally, this thesis is completed by an Appendix consisting of five chapters.
**Appendix A** deals with *Sobolev spaces* in general and the *Sobolev embedding theorem* (cf. Theorem A.6 below) in particular. This theorem is crucially used in proving the *comparison theorem* in Chapter 8, where we construct approximations of the equations (1.1) and (1.2) in the *Sobolev spaces* $W^{m,2}(\Theta)$ with $\Theta \subset \mathbb{R}^d$ obeying the *weak cone property*.
**Appendix B** recalls the definition of *Bochner integrals* in Banach spaces and is mainly needed to prove the *existence* of the Bochner convolution integrals in Chapter 3.
**Appendix C** collects *comparison theorems* known for finite-dimensional SDEs with jump noise. Such comparison results are crucially used in Chapter 8 to prove the comparison results for the finite-dimensional approximations of the equations (1.1) and (1.2) in $W^{m,2}(\Theta)$.
**Appendix D** presents an example of an *almost strong evolution operator* obeying (most of) the properties **(A0)**–**(A8)** required in Section 3.1. A large class of examples is constituted by all *elliptic differential operators*. The results strongly depend on the space dimension $d \geq 1$ and on the order $m \geq 1$ of the diffusion operators.
Finally, **Appendix E** presents a way of constructing measures, which obey the properties **(QI)** and **(P)** required for the Lévy intensity measures in the main results in Chapter 8.

Concerning the calculations appearing in the thesis, we note that we always add the constants, on which a constant in an estimate depends, in brackets, and that, for simplicity, unessential constants are denoted by the same symbol though they may have different values.

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# Chapter 2

# General definitions and supporting results

In this chapter, we recall some technical preliminaries needed in the thesis. In particular, we present infinite-dimensional analogues of basic probabilistic concepts.
The following results hold for Banach or Hilbert spaces in general, whereas the results in Chapter 3 concern with the special case of the weighted $L^2$-spaces introduced in Section 1.1.

In Section 2.1, we recall some general facts on stochastic processes taking values in Banach spaces.

In Section 2.2, we introduce the notion of an almost strong evolution operator $(A(t))_{t\in[0,T]}$ (with a fixed $0 < T < \infty$) taken from [76].

In Section 2.3, we recall the definition of the $Q$-Wiener process in a separable Hilbert space $G$. In particular, we make use of the representation of a $Q$-Wiener process as an infinite sum, i.e.

$$W(t) := \sum_{n\in\mathbb{N}} \sqrt{a_n} w_n(t) g_n,\ t \in [0, T],$$

where $(a_n)_{n\in\mathbb{N}}$ resp. $(g_n)_{n\in\mathbb{N}}$ is the family of positive eigenvalues resp. eigenvectors of the covariance operator $Q \geq 0$ in $G$ and $(w_n)_{n\in\mathbb{N}}$ is a family of independent real-valued Brownian motions.

In Section 2.4, we recall the definitions of Lévy processes, compensated Poisson random measures and martingale measures in the Hilbert space $G$. The most important facts from this section are the Lévy-Itô decomposition of a Lévy process $L$ (cf. Lemma 2.4.10 below) and its refined form for square-integrable Lévy processes (cf. Lemma 2.4.13 below). Note that the refined form of the decomposition will be used later to rewrite the equation (1.2) as an equation of the form (1.1).

In Section 2.5, we collect basic facts on stochastic integration w.r.t. Wiener processes in Hilbert spaces. In particular, we recall the Burkholder-Davis-Gundy inequality for stochastic integrals w.r.t. Wiener processes (cf. Lemma 2.5.4/2.5.6 below), which will play an important role in the existence (and uniqueness) proofs in Chapters 5, 7, and 8.

In Section 2.6, we recall basic facts on stochastic integration w.r.t. compensated Poisson random measures. The most important result of this section is the Bichteler-Jacod inequality for stochastic integrals w.r.t. compensated Poisson random measures (cf. Lemma 2.6.10 below). This inequality will be the key tool to establish the well-definedness of stochastic convolutions w.r.t. compensated Poisson random measures in Chapter 4. Furthermore, it will also play an important role in the existence (and uniqueness) proofs in Chapters 5, 7, and 8.

In Section 2.7, we collect auxiliary analytic results, among them the Gronwall-Bellman lemma (cf. Lemma 2.7.2 and Remark 2.7.3 below), which will play a crucial role in the existence (and uniqueness) proofs in Chapters 5, 7 and 8.

## 2.1 Some general facts on stochastic processes

In this section, we collect some basic facts on stochastic processes taking values in Banach spaces, which can be found e.g. in the monographs [26], [29], [34], [53] and [96]; see also Sections 1.2 and 1.3 in [61]. To be more precise, we fix the time-intervall $I := \mathbb{R}_+ := [\,0, +\infty\,)$ or $I := [\,0, T\,]$ for some fixed $0 < T < \infty$.

Let $(\Omega, \mathcal{F}, P)$ be a probability space with **complete** filtration $(\mathcal{F}_t)_{t\in I}$. The latter means that $\mathcal{F}_0$ contains all sets of $P$-measure zero. A filtration $(\mathcal{F}_t)_{t\in I}$ is called **right-continuous** if $\mathcal{F}_t = \mathcal{F}_{t+} := \cap_{s>t}\mathcal{F}_s$. Furthermore, let $E$ be a separable Banach space with norm $||\cdot||_E$ and the Borel $\sigma$-algebra $\mathcal{B}(E)$.

**Definition 2.1.1:** *Let $X = (X(t))_{t\in I}$ and $Y = (Y(t))_{t\in I}$ be two $E$-valued stochastic processes.*
*$X$ is called a* ***modification*** *or* ***version*** *of $Y$ if $P[X(t) = Y(t)] = 1$ for each $t \in I$. In this case, we say that $X$ and $Y$ are* ***stochastically equivalent***.

*$X$ and $Y$ are said to be* ***indistinguishable*** *or* ***$P$-equal*** *if there exists a $P$-zero set $N \in \mathcal{F}$ such that we have*

$$X(t,\omega) = Y(t,\omega), \text{ for all } t \in I \text{ and } \omega \in N^c.$$

*We say that a process $X$ is defined* ***$P$-uniquely*** *by certain properties if any further process fulfilling these properties and the process $X$ are $P$-equal.*

**Definition 2.1.2:**
**(i)** *An $E$-valued process $(X(t))_{t \in I}$ is said to have* ***left*** *resp.* ***right limits*** *if, for $P$-almost all $\omega \in \Omega$, the path $I \ni t \mapsto X(t,\omega) \in E$ has left resp. right limits.*

**(ii)** *An $E$-valued process $(X(t))_{t \in I}$ is called* ***continuous*** *(****right-continuous*** *resp.* ***left-continuous****) if, for $P$-almost all $\omega \in \Omega$, $I \ni t \mapsto X(t,\omega) \in E$ is continuous (right-continuous resp. left-continuous).*

**(iii)** *An $E$-valued right-continuous process with paths having left limits is called* ***càdlàg****.*

**(iv)** *An $E$-valued left-continuous process with paths having right limits is called* ***càglàd****.*

**Definition 2.1.3:**
**(i)** *An $E$-valued process $(X(t))_{t \in I}$ is called* ***adapted to the filtration*** *$(\mathcal{F}_t)_{t \in I}$ if the random variable $X(t)$ is $\mathcal{F}_t$-measurable for any $t \in I$.*

**(ii)** *The process $(X(t))_{t \in I}$ is* ***measurable*** *if it is a measurable mapping from $I \times \Omega$ to $E$, where $I \times \Omega$ is equipped with the product $\sigma$-field $\mathcal{B}(I) \otimes \mathcal{F}$.*

**(iii)** *Let $\mathcal{P}_I$ denote the $\sigma$-field of* ***predictable*** *sets, that is, the smallest $\sigma$-field on $I \times \Omega$ containing all sets of the form $\{0\} \times A$, $(\,s,t\,] \cap I \times B$, where $s,t \in I$, $s < t$, $A \in \mathcal{F}_0$ and $B \in \mathcal{F}_t$. Equivalently, $\mathcal{P}_I$ is the minimal $\sigma$-field such that all left-continuous, adapted processes $(X(t))_{t \in I}$ are measurable. In particular, by $\mathcal{P}_T$ we denote the $\sigma$-field of predictable sets on $[\,0,T\,] \times \Omega$. An $E$-valued process $(X(t))_{t \in I}$ is called* ***predictable*** *if it is a measurable mapping from $I \times \Omega$ to $E$, where $I \times \Omega$ is equipped with the $\sigma$-field $\mathcal{P}_I$.*
*A predictable process is not only adapted to the original filtration $(\mathcal{F}_t)_{t \geq 0}$, but also to the smaller filtration $(\mathcal{F}_{t-})_{t \geq 0}$, where $\mathcal{F}_{t-} := \cap_{s<t} \mathcal{F}_s$.*

**(iv)** *Given $p \geq 1$, an $E$-valued process $(X(t))_{t \in I}$ is called* ***p-integrable*** *if*

$$\mathbf{E}||X(t)||_E^p < \infty, \textit{ for all } t \in I.$$

*The space of all such $X$ will be denoted by $L^p(\Omega; E)$. For $X \in L^p(\Omega; E)$, one can define the Bochner integral in $E$*

$$\int_\Omega X(t)\, dP, \textit{ cf. [95], p.24 or Appendix B.}$$

*In the special cases $p = 1$ resp. $p = 2$, we also say that $(X(t))_{t\in I}$ is* ***integrable*** *resp.* ***square-integrable****.*

**Definition 2.1.4:** *Let $(X(t))_{t\in I}$ be an $E$-valued process with left limits. For $0 < t \in I$, we denote $X(t-) := \lim\limits_{\substack{s\uparrow t\\ s\in I}} X(s)$. For $t = 0$, we make the convention $X(0-) := X(0)$.*
*We define a jump at $t$ by $\Delta X(t) := X(t) - X(t-)$.*

Clearly, a càdlàg process $(X(t))_{t\in I}$ can only have jump discontinuities. By Theorem 2.8.1 in [7], for almost all $\omega$, the set of all $t \in I$ such that $\Delta X(t, \omega) \neq 0$ is **at most countable**.
By Proposition 3.17 from [95], if $(X(t))_{t\in I}$ and $(Y(t))_{t\in I}$ are càdlàg processes on $(\Omega, \mathcal{F}, P)$ and $X$ is a modification of $Y$, then $X$ and $Y$ are **indistinguishable**, i.e. there is a set $N \in \mathcal{F}$ of $P$-measure zero such that $X(t, \omega) = Y(t, \omega)$ for all $(t, \omega) \in I \times N^c$.

**Definition 2.1.5:**
**(i)** *An $E$-valued stochastic process $(M(t))_{t\in I}$ is called an $(\mathcal{F}_t)$*-***martingale*** *if it is an integrable $\mathcal{F}_t$-adapted process such that*

$$\mathbf{E}[M(t)|\mathcal{F}_s] = M(s) \; P\textit{-a.s., for all } t \geq s.$$

*For the notion of conditional expectation in Banach spaces, see e.g. p. 24 in [95].*

**(ii)** An $\mathbb{R}$-valued stochastic process $(M(t))_{t\in I}$ is called an $(\mathcal{F}_t)$**-submartingale** if it is an integrable $\mathcal{F}_t$-adapted process such that

$$\mathbf{E}[M(t)|\mathcal{F}_s] \geq M(s) \; P\text{-a.s., for all } t \geq s.$$

**(iii)** *An $\mathbb{R}$-valued stochastic process $(M(t))_{t\in I}$ is called an $(\mathcal{F}_t)$*-***supermartingale*** *if $(-M(t))_{t\in I}$ is a $(\mathcal{F}_t)$-submartingale.*

By Theorem 3.35 from [26], if $(M(t))_{t\in I}$ is an $E$-valued martingale, then $(||M(t)||_E^p)_{t\in I}$ is an $\mathbb{R}_+$-valued submartingale, for each $p \geq 1$.

**Proposition 2.1.6: (Doob's submartingale inequality, see e.g. [34], p.63)**
*Given $T > 0$, any right-continuous $\mathbb{R}_+$-valued $(\mathcal{F}_t)_{t\in[0,T]}$-submartingale $M$ obeys for all $1 < p < \infty$ and $r > 0$*

$$\mathbf{E}\left[\sup_{0\leq t\leq T} M(t)^p\right] \leq \left(\tfrac{p}{p-1}\right)^p \mathbf{E}M(T)^p,$$
$$P\left(\left\{\sup_{0\leq t\leq T} M(t) \geq r\right\}\right) \leq \tfrac{1}{r}\mathbf{E}[M(T)],$$
$$P\left(\left\{\sup_{0\leq t\leq T} M(t) \geq r\right\}\right) \leq \tfrac{1}{r^p}\mathbf{E}[M(T)]^p.$$

As a sequel, we have **Doob's inequality** for $E$-valued càdlàg martingales. Namely, for all $1 < p < \infty$ and $r > 0$

$$\begin{aligned} &\mathbf{E}\left[\sup_{0\leq t\leq T} ||M(t)||_E^p\right] \leq \left(\tfrac{p}{p-1}\right)^p \mathbf{E}||M(T)||_E^p, \\ &P\left(\left\{\sup_{0\leq t\leq T} ||M(t)||_E \geq r\right\}\right) \leq \tfrac{1}{r}\mathbf{E}||M(T)||_E, \\ &P\left(\left\{\sup_{0\leq t\leq T} ||M(t)||_E \geq r\right\}\right) \leq \tfrac{1}{r^p}\mathbf{E}||M(T)||_E^p. \end{aligned} \tag{2.1}$$

**Definition 2.1.7:** ***(i)*** *An $E$-valued process $(X(t))_{t\in I}$ is called **pathwise continuous** if the mapping $I \ni t \mapsto X(t,\omega) \in E$ is continuous for any $\omega \in \Omega$.*
**(ii)** *The process $(X(t))_{t\in I} \subset E$ is called **stochastically continuous** or **continuous in probability** if we have, for any $t \in I$,*

$$\lim_{\substack{s\to t\\ s\in I}} P[|X(t) - X(s)| > \varepsilon] = 0 \text{ for any } \varepsilon > 0.$$

**(iii)** *Given $p \geq 1$, the process $(X(t))_{t\in I} \subset E$ is $L^p$ **continuous** if it is p-integrable and obeys, for any $t \in I$,*

$$\lim_{\substack{s\to t\\ s\in I}} \mathbf{E}||X(s) - X(t)||_E^p = 0.$$

*If $p = 2$, the process $(X(t))_{t\in I}$ is called **meansquare continuous**.*

**Lemma 2.1.8:** (cf. e.g. Proposition 3.21 in [95])
*Any measurable stochastically continuous $(\mathcal{F}_t)$-adapted $E$-valued process $(X(t))_{t\in I}$ has a predictable modification.*

An adapted càdlàg process $(X(t))_{t\in I}$ need not be predictable. But its left limit $(X(t-))_{t\in I}$ will surely be predictable, see definition of the $\sigma$-field $\mathcal{P}_I$.

Suppose that $G$ is a separable Hilbert space. Then, by **Doob' s regularity theorem** (see e.g. Theorem 3.40 in [95]), every stochastically continuous square-integrable martingale $M = (M(t))_{t\in I}$ in $G$ has a càdlàg modification, which will be again denoted by $M$.
Recall that this modification obeys the Doob inequalities (2.1).

Given $T > 0$ and $I := [\,0, T\,]$, the space of all **càdlàg square-integrable $G$-valued martingales** $M = (M(t))_{t\in[\,0,T\,]}$ w.r.t. the filtration $(\mathcal{F}_t)_{t\in[\,0,T\,]}$ is denoted by $\mathcal{M}^2_G(T)$.
As usual, indistinguishable processes $M$: $[\,0, T\,] \times \Omega \to G$ are identified. Endowed with the inner product

$$(M, N) \to \mathbf{E} < M(T), N(T) >_G,$$

$\mathcal{M}^2_G(T)$ is a Hilbert space. We denote by $\mathcal{M}^{2,c}_G(T)$ resp. $\mathcal{M}^{2,0}_G(T)$ the subspace of continuous martingales $M$ resp. those $M$ such that $M_0 = 0$. Both $\mathcal{M}^{2,c}_T(G)$ and $\mathcal{M}^{2,0}_T(G)$ are closed in $\mathcal{M}^2_T(G)$.

Given any $M \in \mathcal{M}^2_T(G)$, by the **Doob-Meyer decomposition** (see e.g. Theorem 3.43 in [95]) there exists a unique increasing predictable process $< M >:= (< M, M >_t)_{t\in[\,0,T\,]}$, called the **predictable quadratic variation** or the **Meyer process** of $M$, such that $< M, M >_0 = 0$ and $||M(t)||^2_G - < M, M >_t$, $t \in [\,0, T\,]$, is a martingale.
The predictable quadratic variation $< M, M >_t$ is used e.g. to construct (via the Itô isometry) the stochastic integrals w.r.t. $dM(t)$.

An alternative process is the so-called **adapted quadratic variation** $[M] := ([M, M]_t)_{t\in[\,0,T\,]}$.
Namely, for any $M \in \mathcal{M}^2_T(G)$, there exists a unique increasing adapted càdlàg process $[M] := ([M, M]_t)_{t\in[\,0,T\,]}$ such that $[M, M]_0 = 0$ and $||M_t||^2_G - [M, M]_t$, $t \in [\,0, T\,]$, is a càdlàg martingale. Note that, for any sequence of partitions of $[\,0, T\,]$

$$\Pi^n := \{0 = t^n_0 < t^n_1 < .... < t^n_{J_n} := T\}$$

such that

$$mesh\ \Pi^n := \sup_j (t_{j+1}^n - t_j^n) \to 0 \text{ as } n \to \infty,$$

one has

$$[M, M]_t = \lim_{n\to\infty} \sum_{j=0}^{J_n - 1} ||M_{t_{j+1}^n} - M_{t_j^n}||_G^2,$$

where the limit is in $L^1(\Omega, \mathcal{F}, P)$. Note that

$$\mathbf{E}[M, M]_t = \mathbf{E}||M(t)||_G^2 - \mathbf{E}||M(0)||_G^2.$$

If $M$ has stationary independent increments, i.e. for all $t \geq s$, $M(t) - M(s)$ is independent of $\mathcal{F}_s$ and has the same distribution as $M(t-s) - M(0)$, then

$$< M, M >_t = t\ \mathbf{E}[||M(1)||_G^2 - ||M(0)||_G^2].$$

Furthermore, if $M \in \mathcal{M}_T^{2,c}(G)$, then $[M, M] =< M, M >$. For discontinuous martingales, $[M, M]_t$ and $< M, M >_t$ need not to coincide.

Note that $\mathbf{E}[M, M]_t = \mathbf{E}||M(t)||_G^2 - \mathbf{E}||M(0)||_G^2$.
The quadratic variation $[M, M]$ is also involved in the following inequality for real-valued martingales $M \in \mathcal{M}_T^{2,0}(\mathbb{R})$ with $G = \mathbb{R}$:

**Theorem 2.1.9:** (**Burkholder-Davis-Gundy**, cf. e.g. Theorem 3.50 in the monograph [95] by Peszat and Zabczyk)
*For every $p \geq 1$, there exists a universal positive constant $C_p$ such that, for any $T > 0$ and any $M \in \mathcal{M}_T^{2,0}(\mathbb{R})$,*

$$\frac{1}{C_p}\mathbf{E}[M, M]_T^{\frac{p}{2}} \leq \mathbf{E}\left(\sup_{t\in[0,T]} |M_t|^p\right) \leq C_p \mathbf{E}[M, M]_T^{\frac{p}{2}}. \tag{2.2}$$

Another important issue, where one needs the quadratic variation $[M]$, is the Itô formula for càdlàg semimartingales (see e.g. [95], Appendix D).

## 2.2 Strong evolution operators in Banach spaces

For this section, let $B$ be a Banach space. Let $\mathcal{L}(B)$ denote the space of linear, bounded operators $A$: $B \to B$ with the usual operator norm $||\cdot||$. We present the definition of an almost strong evolution operator from [111]:

**Definition 2.2.1:** *Let us fix some $T \in (0, \infty)$.*
*A family $U = U(t, s)_{0\leq s\leq t\leq T} \subset \mathcal{L}(B)$ is called an* ***almost strong evolu-***

***tion operator*** *if the following holds:*

***(i)*** $U(t,t) = \mathbf{I}$, $t \in [\,0, T\,]$, *where* $\mathbf{I}$ *denotes the identity operator in* $B$;

***(ii)*** $U(t,r)U(r,s) = U(t,s)$, $0 \leq s \leq r \leq t \leq T$;

***(iii)*** $U$ *is* ***strongly continuous****, i.e. the map* $U(\cdot,\cdot)x$: $\{(t,s)\,|\,0 \leq s \leq t \leq T\} \to B$ *is continuous for any* $x \in B$ *and*

$$\sup_{0\leq s\leq t\leq T} ||U(t,s)|| \leq c(T) < \infty;$$

***(iv)*** *For any* $t \in [\,0, T\,]$, *there exists a closed linear operator* $A(t)$ *on* $B$ *such that* $U(t,s)$: $\mathcal{D}(A(s)) \to \mathcal{D}(A(t))$ *for all* $s < t$ *and*

$$\int_s^t A(r)U(r,s)\varphi\,dr = (U(t,s) - I)\varphi$$

*for any* $\varphi \in \mathcal{D}_{t,s}(A) := \{\varphi \in B \mid U(r,s)\varphi \in \mathcal{D}(A(r))\, for\, all\, r \in [\,s,t\,]\}$.

*Obviously* ***(iv)*** *implies that for every* $\varphi \in \mathcal{D}_{t,s}(A)$

***(v)*** $$\frac{\partial}{\partial t}U(t,s)\varphi = A(t)U(t,s)\varphi$$

*for Lebesgue-almost all* $t \in [\,0, T\,]$, *which justifies the terminology.*

*Analogously to the theory of one-parameter semigroups,* $(A(t))_{t\in[\,0,T\,]}$ *is called the* ***generator*** *of* $U$. *If* ***(v)*** *even holds for all* $t \in [\,0, T\,]$, $U$ *is called a* ***strong evolution operator****.*

**Remark 2.2.2:** **(i)** *For* $U$ *being an almost strong evolution operator in the sense of 2.2.1, by* ***(iii)*** *we have in particular that*

$$[\,s, T\,] \ni t \mapsto U(t,s)x \in B$$

*is a continuous mapping for any fixed* $s \in [\,0, T\,]$ *and* $x \in B$, *and respectively*

$$[\,0, t\,] \ni s \mapsto U(t,s)x \in B$$

*is a continuous mapping for any* $t \in [\,0, T\,]$ *and* $x \in B$.

**(ii)** *It is instructive to compare 2.2.1* ***(iii)*** *with the strong continuity property of operator semigroups.*

*Recall that the $\mathcal{C}_0$-continuity of a semigroup $(S(t))_{t\geq 0}$ in $B$ means that*

$$\lim_{t\downarrow 0} S(t)x = x$$

*for each $x \in B$, which readily implies the continuity of the map*

$$[\,0, T\,] \ni t \mapsto S(t)x \in B$$

*for all $x \in B$.*
*Instead of 2.2.1* ***(iii)****, let us assume the weaker property*

$$\lim_{t\downarrow s} U(t,s)x = x$$

*for any $x \in B$ and any $s \in [\,0, T\,]$ resp.*

$$\lim_{s\uparrow t} U(t,s)x = x$$

*for any $x \in B$ and any $t \in [\,0, T\,]$.*
*Herefrom, by 2.2.1* ***(ii)*** *we get*

$$\lim_{t\downarrow r} U(t,s)x = U(r,s)x$$

*resp.*

$$\lim_{s\uparrow r} U(t,s)x = U(t,r)x$$

*for all $x \in B$ and $0 \leq s \leq r \leq t \leq T$.*

*In contrast to the semigroup $(S(t))_{t\geq 0}$, these properties of right- resp. left-continuity do not imply each other and are weaker than those in 2.2.1* ***(iii)****. Further assumptions on the evolution operator $U(t,s)$, $0 \leq s \leq t \leq T$, will be imposed in Section 3.1.*

## 2.3 $Q$-Wiener processes in Hilbert spaces

The presentation in this subsection is based on [26], Chapter 4 there, and [97], Chapter 2.

Let $(\Omega, \mathcal{F}, P)$ be a probability space, and let $(\mathcal{F}_t)_{t\geq 0}$ be a complete right-continuous filtration in $\mathcal{F}$.
Let $G$ be a separable Hilbert space with inner product $< \cdot, \cdot >_G$ and corre-

sponding Borel $\sigma$-algebra $\mathcal{B}(G)$.

**Definition 2.3.1:** (cf. e.g. [97], Definition 2.1.1, p. 9)
*A probability measure $\mu$ on $(G, \mathcal{B}(G))$ is called **Gaussian** if all bounded linear mappings*

$$\begin{aligned} v: \quad & G \to \mathbb{R} \\ & G \ni g \mapsto < g, v >_G \in \mathbb{R} \end{aligned}$$

*have Gaussian laws, i.e. for any $v \in G$ there exist $m = m(v) \in \mathbb{R}$ and $\sigma := \sigma(v) > 0$ such that*

$$\mu(v \in A) = \frac{1}{\sqrt{2\pi\sigma^2}} \int\limits_A e^{-\frac{(x-m)^2}{2\sigma^2}} \, dx, \; A \in \mathcal{B}(\mathbb{R}),$$

*or*

$$\mu = \delta_g \text{ for some } g \in G, \text{ where } \delta_g \text{ is the Dirac measure placed in } g.$$

**Theorem 2.3.2:** (cf. e.g. [97], Theorem 2.1.2, p. 10)
*A measure $\mu$ on $(G, \mathcal{B}(G))$ is Gaussian if and only if its characteristic functional has the form*

$$\hat{\mu}(g) := \int\limits_G e^{i<g,x>_G} \, \mu(dx) = e^{i<m,g>_G - \frac{1}{2}<Qg,g>_G}, \; g \in G,$$

*where $m \in G$ and $Q \in \mathcal{L}(G)$ is nonnegative, symmetric, with finite trace. The above $\mu$ will be denoted by $N(m, Q)$, where $m$ is called mean and $Q$ is called the covariance operator. Furthermore, $\mu$ is uniquely defined by $m$ and $Q$.*

In what follows, we denote the set of nonnegative, symmetric linear operators $Q \in \mathcal{L}(G)$ with finite trace by $\mathcal{T}^+(G)$.
Recall that $0 \le Q \in \mathcal{L}(G)$ has finite trace if

$$tr\, Q := \sum_{n \in \mathbb{N}} < Qg_n, g_n >_G < \infty \tag{2.3}$$

for some (and thus for any) orthonormal basis $(g_n)_{n \in \mathbb{N}} \subset G$.

**Proposition 2.3.3:** (cf. e.g. [98], Theorem VI.21 and Theorem VI.16)
*For any $Q \in \mathcal{T}^+(G)$, there exists an orthonormal basis $(g_n)_{n \in \mathbb{N}}$ of $G$ such that*

$$Qg_n = a_n g_n, \; a_n \ge 0, \; n \in \mathbb{N}. \tag{2.4}$$

*Hence, by (2.3) we have*

$$tr\, Q = \sum_{n\in\mathbb{N}} a_n < \infty,$$

*and thus* $0$ *is the only accumulation point of the sequence* $(a_n)_{n\in\mathbb{N}}$.

Without loss of generality, we may assume $(a_n)_{n\in\mathbb{N}}$ from 2.3.3 to be in decreasing order.
Then, there is a canonical form of Gaussian random variables.

**Proposition 2.3.4:** (cf. e.g. [97], Proposition 2.1.6, p. 13)
*Let* $m \in G$ *and* $Q \in \mathcal{T}^+(G)$ *obey the eigenvector expansion (2.4).*
*A* $G$*-valued random variable* $X$ *is Gaussian with law*
$P \circ X^{-1} = N(m, Q)$ *if and only if*

$$X = \sum_{n\in\mathbb{N}} \sqrt{a_n} w_n(t) g_n + m,$$

*where* $w_n$ *are independent real-valued random variables with*
$P \circ w_n^{-1} = N(0,1)$ *for all* $n \in \mathbb{N}$ *with* $a_n > 0$*. The series converges in*
$L^2(\Omega, \mathcal{F}, P; G)$*.*

In particular, we have the following statement, which is the inverse to Theorem 2.3.2.

**Corollary 2.3.5:** (cf. e.g. [97], Proposition 2.1.7, p.15)
*Let* $Q \in \mathcal{T}^+(G)$ *and* $m \in G$*.*
*Then, there exists a Gaussian measure* $\mu = N(m, Q)$ *on* $(G, \mathcal{B}(G))$*.*

Now, we will give the definition of a $Q$-Wiener process in the case of
$Q \in \mathcal{T}^+(G)$.

**Definition 2.3.6:** *A* $G$*-valued* $(\mathcal{F}_t)_{t\geq 0}$*-adapted stochastic process* $(W(t))_{t\geq 0}$ *is called a* ***(standard)*** $Q$***-Wiener process*** *(w.r.t. the filtration* $(\mathcal{F}_t)_{t\geq 0}$*) if:*

1. $W(0) = 0$ *(*$P$*-almost surely);*
2. $W$ *has* ***independent increments****, i.e.* $W(t) - W(s)$ *is independent of* $\mathcal{F}_s$ *for all* $0 \leq s < t$*;*
3. $W$ *has* ***stationary increments*** *and for all* $0 \leq s < t$ *the random varaiables* $W(t) - W(s)$ *are normally distributed with mean* $0$ *and covariance operator* $(t - s)Q$*;*
4. $W$ *is a* ***pathwise continuous process****.*

There is a canonical representation of a $Q$-Wiener process given as follows.

**Proposition 2.3.7:** (cf. e.g. [97], Proposition 2.1.10, p. 17)
*Let $(g_n)_{n\in\mathbb{N}}$ be an orthonormal basis of $G$ consisting of eigenvectors of $Q \in \mathcal{T}^+(G)$ with the corresponding eigenvalues $a_n \geq 0$, $n \in \mathbb{N}$.*
*Then, a $G$-valued stochastic process $(W(t))_{t\geq 0}$ is a $Q$-Wiener process if and only if*

$$W(t) := \sum_{n\in\mathbb{N}} \sqrt{a_n} w_n(t) g_n,\ t \geq 0, \tag{2.5}$$

*where $w_n$, $n \in \{n \in \mathbb{N} \,|\, a_n > 0\}$, are independent real-valued Brownian motions.*
*The series is convergent in $L^2(\Omega, \mathcal{F}, P; C([0,T], G))$ and thus has a $P$-almost surely time-continuous modification.*

So, by Corollary 2.3.5 and Proposition 2.3.7, for any operator $Q \in \mathcal{T}^+(G)$ there is a $Q$-Wiener process and it is of the form (2.5).

There is an equivalent coordinate representation of $W$ using an orthonormal basis in the so-called **reproducing kernel Hilbert space**(in short RKHS, see e.g. Section 7.1, Definition 7.2 in [95]).
Given $Q \in \mathcal{T}^+(G)$ obeying (2.4), the corresponding RKHS is given by $\mathcal{G} := Q^{\frac{1}{2}} G$. This is a Hilbert space with the inner product

$$< \varphi, \psi >_{\mathcal{G}} := \sum_{\substack{n\in\mathbb{N}\\ a_n \neq 0}} a_n^{-1} < \varphi, g_n >_G < \psi, g_n >_G,\ \varphi,\ \psi \in \mathcal{G},$$

where $< \cdot, \cdot >_G$ denotes the inner product in $G$. Since this space is generated by the orthonormal basis $(\tilde{g}_n)_{n\in\mathbb{N}}$ defined by

$$\tilde{g}_n := Q^{\frac{1}{2}} g_n = a_n^{\frac{1}{2}} g_n,\ n \in \mathbb{N},$$

we have the Hilbert-Schmidt embedding $\mathcal{G} \underset{\to}{\subseteq} G$.
This leads to the representation

$$W(t) := \sum_{n\in\mathbb{N}} w_n(t) \tilde{g}_n,\ t \in [0,T], \tag{2.6}$$

whereby the series is convergent in $L^2(\Omega, \mathcal{F}, P; C([0,T], G))$ by 2.3.7 (cf. equation (2.5) there).

In general, an operator $Q \geq 0$ need not to have finite trace. Given such situation, we write $Q \notin \mathcal{T}^+(G)$. This leads to the class of so-called **cylindrical** Wiener processes.

In the following, we construct such a Wiener process in the important case $Q = \mathbf{I} \notin \mathcal{T}^+(G)$. This is done by establishing a representation similar to (2.6), but converging in an appropriately large space.
More precisely, a cylindrical Wiener process can be defined in the following way (cf. [97], Section 2.5.1, pp 39–41):
We need a further Hilbert space $(G_1, < \cdot, \cdot >_{G_1})$, $G \subset G_1$, such that $G = Q^{\frac{1}{2}}(G_1)$ with $Q \in \mathcal{T}^+(G_1)$. By Remark 2.5.1 from [97], such $G_1$ and $Q$ always exist. To this end, we just take a sequence $(b_n)_{n\in\mathbb{N}}$ of nonnegative real numbers such that

$$\sum_{n\in\mathbb{N}} b_n^2 < \infty.$$

Let $(g_n)_{n\in\mathbb{N}}$ be an orthonormal basis of $G$. Then, we define $G_1$ as a completion of $G$ w.r.t. the inner product

$$< g, h, >_{G_1} := \sum_{n\in\mathbb{N}} b_n^2 < g_n, g >_G < g_n, h >_G .$$

Obviously, the orthonormal basis in $G_1$ consists of the vectors $\tilde{g}_n := b_n^{-1} g_n$, $n \in \mathbb{N}$.
Let $J$: $G \underset{\rightarrow}{\subseteq} G_1$ be the embedding operator and $J^*$: $G_1 \to G$ its adjoint. Then, $Q_1 := JJ^* \in \mathcal{T}^+(G_1)$ with

$$tr(Q_1) = \sum_{n\in\mathbb{N}} b_n^2 < \infty.$$

Furthermore, $Q_1^{\frac{1}{2}}(G_1) = G$ and $Q_1^{\frac{1}{2}}$: $G_1 \to G$ is an isometry. Now, the previous scheme runs with the RKHS $G$ and the covariance operator $Q_1 \in \mathcal{T}^+(G_1)$. We get (see also [97], Proposition 2.5.2, p. 47):

**Theorem 2.3.8:** *Given an orthonormal basis $(g_n)_{n\in\mathbb{N}}$ of $G$ and a family $(w_n)_{n\in\mathbb{N}}$ of independent real-valued Brownian motions, define $Q_1 := JJ^* \in \mathcal{T}^+(G_1)$ with $J$ as above.*
*Then, the series*

$$W(t) := \sum_{n\in\mathbb{N}} w_n(t) J g_n,\ t \geq 0, \tag{2.7}$$

*converges in $L^2(\Omega, \mathcal{F}, P; G_1)$ and defines a continuous, square-integrable martingale. Moreover, $(W(t))_{t\geq 0}$ is a $Q_1$-Wiener process in $G_1$.*

When we talk about cylindrical **I**-Wiener process in the following, we always mean the process (2.7) from the theorem above.

## 2.4 Lévy processes, compensated Poisson random measures and martingale measures

In this section, we recall the definition of Hilbert space valued Lévy processes, compensated Poisson random measures and martingale measures.

**Lévy processes and their path properties**

First, we introduce the notion of Lévy processes in Hilbert spaces and discuss some basic results. This topic is treated e.g. in the monograph [95] by Peszat and Zabczyk as well as in the monograph [7] by Applebaum and the paper [8] by the same author. We also refer to the papers of Albeverio, Mandrekar, Rüdiger and Ziglio ([3], [4], and [102]), which cover the more general case of Lévy processes in Banach spaces.
For a concise review, see also the manuscripts [58] and [60].

We assume $G$ to be a separable Hilbert space (again denoting its inner product by $< \cdot, \cdot, >_G$ and the corresponding $\sigma$-algebra by $\mathcal{B}(G)$).
Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $(\mathcal{F}_t)_{t\geq 0}$ be a complete right-continuous filtration.

**Definition 2.4.1:** *A $G$-valued, $(\mathcal{F}_t)_{t\geq 0}$ adapted process $(L(t))_{t\geq 0}$ is called a **Lévy process** if:*

1. *$L(0) = 0$ ($P$-almost surely);*
2. *$L$ has **independent increments**, i.e. $L(t) - L(s)$ is independent of $\mathcal{F}_s$ for all $0 \leq s < t$;*
3. *$L$ has **stationary increments**, i.e. for all $0 \leq s < t$ the random variables $L(t) - L(s)$ and $L(t - s)$ have the same distribution;*
4. *$L$ is **stochastically continuous**.*

A principal difference between Lévy processes and $Q$-Wiener processes is that Lévy processes in general **do not have continuous paths**.

**Remark 2.4.2:** ***(i)*** *By Theorem 2.17 from [8] every Lévy process has a **càdlàg modification**, which is itself a Lévy process.*
*Thus, concerning the path structure of Lévy processes, it is clear that all possible discontinuities could only be of jump type.*
*In what follows, we always assume a Lévy process $L$ to be càdlàg.*

***(ii)*** *Given a $G$-valued Lévy process $(L(t))_{t\geq 0}$, for any $\omega \in \Omega$, the path*

*$(L(t,\omega))_{t\geq 0}$ only has **finitely many jumps of norm** $\geq 1$. Otherwise we could find an accumulation point $\bar{t}\geq 0$ such that $t\mapsto L(t,\omega)$ does not have a left limit at $\bar{t}$. This would contradict the càdlàg property!*

***(iii)*** *Given a G-valued Lévy process $(L(t))_{t\geq 0}$, the process $(< L(t), g >_G)_{t\geq 0}$ is a real-valued càdlàg Lévy process for any fixed $g \in G$ (cf. Lemma 2.7 from [58]). Let us take an orthonormal basis $(g_n)_{n\in\mathbb{N}}$ in $G$ and set $l_n(t) :=< L(t), g_n >_G$, $t \geq 0$.*
*Then, by Theorem 4.39 in [95], we have the expansion*

$$L(t) = \sum_{n\in\mathbb{N}} l_n(t) g_n, \tag{2.8}$$

*where the series is convergent in probability uniformly on any compact time-intervall.*
*In particular, if $L$ is square-integrable, i.e.*

$$\mathbf{E}||L(t)||_G^2 < \infty \text{ for any } t \geq 0,$$

*the right hand side in (2.8) converges in $L^2(\Omega, \mathcal{F}, P; G)$ (cf. e.g. [88], Proposition 2.1).*

***(iv)*** *Basic examples of Lévy processes are Poisson processes and Q-Wiener processes.*
*In fact, any Lévy process can be built from Poisson processes and Q-Wiener processes in a constructive way by the Lévy-Itô decomposition, see Theorem 2.4.9 below.*

***(v)*** *In this manuscript, we do not consider the so-called **cylindrical** Lévy or Poisson processes with the corresponding RKHS equal to G.*
*Such processes are represented as*

$$L(t) = \sum_{n\in\mathbb{N}} l_n(t) g_n,\ t \geq 0,$$

*with $(g_n)_{n\in\mathbb{N}}$ being an orthonormal basis in $G$ and $(l_n)_{n\in\mathbb{N}}$ a family of independent, identically distributed real-valued Lévy processes on $(\Omega, \mathcal{F}, P)$. The above series converges $P$-a.s. uniformly on compact time-intervalls, but in a larger Hilbert space $G_1$ with the Hilbert-Schmidt embedding $G\subset G_1$. If the Lévy processes $l_n$, $n \in \mathbb{N}$, are themselves square-integrable, then the convergence is also in $L^2(\Omega, \mathcal{F}, P; G_1)$ (see Section 4.8 in [95] or the paper [99]).*

Next, we are going to describe the **discontinuities** of a Lévy process. So let $(L(t))_{t\geq 0}$ be a Lévy process as in 2.4.1:

**Definition 2.4.3:** ***(i)*** *Set*

$$\Delta L(t,\omega) := L(t,\omega) - L(t-,\omega),\ t \geq 0,\ \omega \in \Omega.$$

*By definition, $\Delta L(t,\omega)$ is the jump size of the path $L(\omega)$ at time $t$.*
*Note that by Lemma 2.3.2 in [8], for a fixed $t \geq 0$, $\Delta L(t) = 0$, $P$-a.s.*

***(ii)*** *Introduce a random variable*

(2.9) $N(t,A,\omega) := card(\{0 < s \leq t : \Delta L(s,\omega) \in A\}) = \sum\limits_{0<s\leq t} \mathbf{1}_A(\Delta L(s,\omega)),$
$t \in [\,0,T\,]$, $A \in \mathcal{B}(G)$, $\omega \in \Omega$,

*with $card(\cdot)$ denoting the set cardinality. In other words,*
*$N(t,A,\omega) \in \mathbb{Z}_+ \cup \{\infty\}$ counts the jumps of a path $L(\omega)$ that take values in $A \in \mathcal{B}(G)$.*
*Concerning well-definedness of $N(t,A,\omega)$, see Lemma 2.4.4 below.*

***(iii)*** *A (possibly infinite) measure $\eta$ on $(G, \mathcal{B}(G))$ is called a* ***Lévy measure*** *if*

$$\int\limits_{G\setminus\{0\}} (||x||_G^2 \wedge 1)\eta(dx) < \infty. \tag{2.10}$$

*We will see below that for each Lévy process $L$ there is a Lévy measure $\eta$ giving information about size and likelihood of jumps of $L$.*

Let us denote by $\mathcal{A}_0$ the family of all $A \in \mathcal{B}(G)$ such that $0 \notin \bar{A}$.
Such sets are called **bounded below**, whereas 0 is called the **forbidden point**.
It is easy to see (cf. e.g. Lemma 2.18 in [58]) that $\mathcal{A}_0$ is a **ring** in $G\setminus\{0\}$, i.e, (i) $\emptyset \in \mathcal{A}_0$, (ii) $A, B \in \mathcal{A}_0 \Rightarrow A\setminus B \in \mathcal{A}_0$ and (iii) $A, B \in \mathcal{A}_0 \Rightarrow A \cup B \in \mathcal{A}_0$.

Now, let us qoute some results on the notations introduced above.

**Lemma 2.4.4:** (cf. Theorem 2.7 in [4])
*Suppose that $A \in \mathcal{A}_0$.*
*Then, $N(t,A)$ is finite for each $t \geq 0$ $P$-a.s..*

Let us introduce

$$\mathcal{B}(G \setminus \{0\}) := \sigma(\mathcal{A}_0) = \{A \in \mathcal{B}(G) \,|\, 0 \notin A\},$$

which is the minimal $\sigma$-algebra containing the ring $\mathcal{A}_0$. It is easy to check that $\sigma(\mathcal{A}_0) = \{A := B \setminus \{0\} \,|\, B \in \mathcal{B}(G)\}$.

**Lemma 2.4.5:** ***(i)*** *(cf. Proposition 2.3.5 from [106] resp. Theorem 2.13 from [4])*
*For each $0 \leq t \leq T$ and $\omega \in \Omega$, we get a mapping*

$$\begin{aligned} N(t,\cdot,\omega)\colon \mathcal{A}_0 &\mapsto \mathbb{R}_+ \cup \{0\} \\ A &\mapsto N(t,A,\omega). \end{aligned}$$

*For all $0 \leq t \leq T$ and $P$-almost all $\omega \in \Omega$ (cf. Proposition 2.4.4), this is a $\sigma$-finite pre-measure, i.e.:*

- $N(t,\emptyset,\omega) = 0$;
- *For any family $(A_n)_{n\in\mathbb{N}}$ of pairwise disjoint sets from $\mathcal{A}_0$, we have*

$$N\left(t, \bigsqcup_{n=1}^{\infty} A_n, \omega\right) = \sum_{n=1}^{\infty} N(t,A_n,\omega);$$

- *There is a sequence $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A}_0$ exhausting $G \setminus \{0\}$ such that*

$$N(t,A_n,\omega) < \infty \text{ for any } n \in \mathbb{N}.$$

*In particular, one can take here $A_n := \{x \in G \mid \frac{1}{n} < ||x||_G\}$, $n \in \mathbb{N}$.*

***(ii)*** *(cf. Corollary 2.14 from [4])*
*Given $0 \leq t \leq T$ and $\omega \in \Omega$, there is a unique $\sigma$-finite measure $N(t,dx,\omega)$ on $\mathcal{B}(G \setminus \{0\})$ extending $N(t,\cdot,\omega)$ from part* ***(i)***.

***(iii)*** *(cf. Theorem 2.17 from [4])*
*The mapping $\eta$ given by*

$$\eta(A) := \mathbf{E}[N(1,A)],\ A \in \mathcal{A}_0,$$

*is a $\sigma$-finite pre-measure on $\mathcal{A}_0$.*

***(iv)*** *(cf. Corollary 2.18 from [4])*
*There is a unique $\sigma$-finite measure $\eta$ on $\mathcal{B}(G \setminus \{0\})$ extending $\eta$ from part* ***(iii)***.
*By Theorem 2.21 from [4], $\eta$ is a Lévy measure, i.e. it fulfills (2.10).*
*In the following we call $\eta$ the* ***intensity measure*** *corresponding to $L$.*

To describe the properties of the random variables $N(t,\cdot)$, let us recall the following general definition.

**Definition 2.4.6:** *Given a measurable space $(S,\mathcal{S})$, a family $(N(S))_{S\in\mathcal{S}}$ of random variables on a common probability space $(\Omega,\mathcal{F},P)$ is called a* ***Poisson random measure*** *if:*

- *For almost all $\omega$, $N(\cdot,\omega)$ is a measure on $(S,\mathcal{S})$;*
- *The random variables $N(A_1)$, $N(A_2)$, ..., $N(A_n)$ are mutually independent for any finite family of mutually disjoint $A_1$, $A_2$, ..., $A_n \in \mathcal{S}$ and $n \in \mathbb{N}$;*
- *Each $N(A)$ has a Poisson distribution whenever $\mathbf{E}N(A) < \infty$, i.e. there is a $\lambda_A > 0$ such that*

$$P(N(A)=k) = \frac{\lambda_A^k exp\{-\lambda_A\}}{k!},\ k \in \mathbb{N}\cup\{0\}.$$

*Indeed, here $\lambda_A = \mathbf{E}N(A)$.*

**Proposition 2.4.7:** *Setting $S := G\setminus\{0\}$ and $\mathcal{S} := \mathcal{B}(G\setminus\{0\})$ as above and fixing $t \geq 0$, $N(t,\cdot)$ from Definition 2.4.3* ***(ii)*** *is a Poisson random measure.*

**Proof:** Note that $N(t,\omega,\cdot)$ is a measure on $(S,\mathcal{S})$ for all $t \geq 0$ and almost all $\omega \in \Omega$ by 2.4.5 **(ii)**.
The two remaining properties from Definition 2.4.6 follow from Theorem 2.7 from [4] and its proof. ■

The so-called **compensated Poisson random measure** $\tilde{N}(t,\cdot)$ corresponding to $N(t,\cdot)$ is defined on $(G\setminus\{0\},\mathcal{B}(G\setminus\{0\}))$ by

$$\tilde{N}(t,dx) := N(t,dx) - t\eta(dx),\ t \geq 0, \tag{2.11}$$

where $\eta$ is the same as in Lemma 2.4.5 **(iv)**.

Actually, one starts from the definition $\tilde{N}(t,A) = N(t,A) - t\eta(A) \in \mathbb{R}$ for all $A \in \mathcal{A}_0$.

By Lemma 2.4.5, $\tilde{N}(t,dx)$ then extends to a random measure on $G$ with the forbidden set $\{0\}$.

**Lemma 2.4.8:** (cf. e.g. Example 2.3.7(3) from [8])

*For any $A \in \mathcal{A}_0$, $t \mapsto \tilde{N}(t, A)$ is an $(\mathcal{F}_t)_{t\geq 0}$-martingale.*

**Remark 2.4.9:** *Recall from Remark 2.4.2 that, given a $G$-valued Lévy process $(L(t))_{t\in[0,T]}$ and a vector $g \in G$, the process $L_g(t) :=< L(t), g >_G$, $t \in [0, T]$, is a real-valued Lévy process.*
*Analogously to Definition 2.4.3, Lemma 2.4.5 **(iii)** and (2.11), we define*

$$\begin{aligned} N_g(t, A, \omega) :=\; & card(\{0 < s \leq t : \Delta L_g(s, \omega) \in A\}) \\ =\; & \sum_{0<s\leq t} \mathbf{1}_A(\Delta L_g(s, \omega)),\ t \in [0, T],\ A \in \mathcal{B}(\mathbb{R} \setminus \{0\}),\ \omega \in \Omega, \end{aligned}$$

$$\eta_g(A) := \mathbf{E}[N_g(1, A)],\ A \in \mathcal{B}(\mathbb{R} \setminus \{0\})$$

*and*

$$\tilde{N}_g(t, dx) := N_g(t, dx) - t\eta_g(dx),\ t \geq 0.$$

*Then, $N_g$ is a Poisson random measure, $\eta_g$ a $\sigma$-finite premeasure and $\tilde{N}_g$ a compensated Poisson random measure with compensator $\eta_g \otimes dt$. Obviously, for a given $g \in G$ $N_g$ can be seen as a projection of $N$ to the one-dimensional subspace*

$$G_g := \{< x, g >_G g \,|\, x \in G\} \subset G,$$

*whereas $\eta_g$ and $\tilde{N}_g$ are the projections of $\eta$ resp. $\tilde{N}$ to $G_g$.*

### The Lévy-Itô decomposition

There is a canonical representation for Lévy processes, which is given by the celebrated **Lévy-Itô decomposition**.

**Theorem 2.4.10:** (cf. e.g. [7], Theorem 2.4.16 or [8], Theorem 1)
*For any $G$-valued Lévy process $(L(t))_{t\geq 0}$, there exist a drift vector $b \in G$ and a $Q$-Wiener process $W$ with $Q \in \mathcal{T}^+(G)$ such that, for all $t \geq 0$,*

$$L(t) = tb + W(t) + \int_{\{||x||_G<1\}} x\tilde{N}(t, dx) + \int_{\{||x||_G\geq 1\}} xN(t, dx). \tag{2.12}$$

*Furthermore, $W$ is independent of $N(\cdot, A)$ for all $A \in \mathcal{A}_0$, where $N(t, dx)$, $t \geq 0$, are Poisson random measures defined by (2.9).*
*The compensated Poisson random measure $\tilde{N}(t, dx)$ is defined by (2.11) and the intensity measure $\eta$ respectively by Lemma 2.4.5.*
*The triple $(b, Q, \eta)$ is called the **characteristics** of $L$. It is uniquely determined by the Lévy process $L$.*

The integrals in (2.12) are understood in the Bochner sense (cf. Appendix B for the general definition of Bochner integrals in Banach spaces).
Concerning the **random Bochner integral** w.r.t. $N$, we note the following:

**Remark 2.4.11:** *Since, by Remark 2.4.2* ***(ii)****, for each $\omega \in \Omega$, there is only a finite number of jumps obeying $||\Delta L(s,\omega)||_G \geq 1$, the Bochner integral w.r.t. $N$ in (2.12) can be calculated as a random finite sum*

$$\int\limits_{\{||x||_G \geq 1\}} xN(t,dx) = \sum_{0<s\leq t} \Delta L(s)\mathbf{1}_{\{||x||_G \geq 1\}}(\Delta L(s)).$$

Clearly,

$$\int\limits_{\{||x|| \geq 1\}} xN(t,dx)$$

gives rise to a càdlàg stochastic process in $G$ as we vary $t \geq 0$.
Next, we define the compensated sum of small jumps

$$\int\limits_{\{||x||_G < 1\}} x\tilde{N}(t,dx).$$

For **deterministic integrands**, the construction of the **compensated Poisson integrals** is described in Chapter 3 of [4] in the most general case of Banach spaces and in Section 2.3 of [8] in the case of Hilbert spaces.

To define the above integral for $G$-valued functions w.r.t. the compensated Poisson random measure $\tilde{N}(t,dx)$, we first need the Bochner integral w.r.t. the Lévy intensity measure $\eta$ from 2.4.5 **(iv)**.
We call a mapping $f\colon G \setminus \{0\} \to G$ **elementary** if it can be written as

$$f(x) = \sum_{k=1}^{K} a_k \mathbf{1}_{A_k}(x) \tag{2.13}$$

with some $a_k \in G$, $A_k \in \mathcal{A}_0$, $k \in \{1,2,\ldots,K\}$, and $K \in \mathbb{N}$.

Given $A \in \sigma(\mathcal{A}_0) = \mathcal{B}(G \setminus \{0\})$ and an elementary $f$ as in (2.13), we define the Bochner integral of $f$ w.r.t. $\eta$ on $A$ by

$$\int\limits_A f(x)\,\eta(dx) := \sum_{k=1}^{K} a_k \eta(A_k \cap A).$$

The notion of integral will be extended to general $f$ by a limit procedure.
Given $p \geq 1$, a $\mathcal{B}(G \setminus \{0\})/\mathcal{B}(G)$-measurable $f\colon G \setminus \{0\} \to G$ is said to be **Bochner $p$-integrable** w.r.t. $\eta$ on $A \in \mathcal{B}(G \setminus \{0\})$ if there exists a sequence

$(f_n)_{n\in\mathbb{N}}$ of elementary functions such that $f_n \to f$ $\eta$-almost surely on $A$ and

$$\lim_{n\to\infty} \int_A ||f_n(x) - f(x)||_G^p \, \eta(dx) = 0. \tag{2.14}$$

Such sequences will be called $L^p$-approximating $f$ on $A$.
Then, the Bochner $p$-integral of $f$ on $A \in \mathcal{B}(G \setminus \{0\})$ is defined as

$$\int_A f(x)\, \eta(dx) = \lim_{n\to\infty} \int_A f_n(x)\, \eta(dx),$$

and this definition is independent of the $L^p$-approximating sequence $f_n \to f$ we choose in (2.14).

By Remark 3.7 from [4], a $\mathcal{B}(G \setminus \{0\})/\mathcal{B}(G)$-measurable mapping $f\colon G \setminus \{0\} \to G$ is Bochner $p$-integrable w.r.t. $\eta$ on $G \setminus \{0\}$ if and only if it fulfills

$$\int_{G\setminus\{0\}} ||f(x)||_G^p \, \eta(dx) < \infty. \tag{2.15}$$

In this case, $f$ is also Bochner $p$-integrable w.r.t. $\eta$ on any $A \in \mathcal{B}(G\setminus\{0\})$ and

$$\int_A f(x)\, \eta(dx) = \int_{G\setminus\{0\}} \mathbf{1}_A(x) f(x)\, \eta(dx).$$

We assume that $t \geq 0$, $p \geq 1$ and $f\colon G \setminus \{0\} \to G$ fulfills (2.15).
Our aim is to define the integral w.r.t. the compensated Poisson random measure $\tilde{N}(t, dx)$ from Lemma 2.4.8.
For elementary $f$ of the form (2.13) and $A \in \mathcal{B}(G \setminus \{0\})$, we define

$$\int_A f(x)\, \tilde{N}(t, dx) := \sum_{k=1}^{K} a_k \tilde{N}(t, A_k \cap A).$$

We say that a $\mathcal{B}(G \setminus \{0\})/\mathcal{B}(G)$-measurable $f\colon G \setminus \{0\} \to G$ is **strong $p$-integrable on $A \in \mathcal{B}(G \setminus \{0\})$ w.r.t. $\tilde{N}(t, dx)$** if

$$\int_A f_n(x)\, \tilde{N}(t, dx)$$

converges in $L^p(\Omega, \mathcal{F}, P; G)$ for any sequence $(f_n)_{n\in\mathbb{N}}$ of simple functions $L^p$-approximating $f$ on $A$ in the sense of (2.14), and the limit does not depend on the choice of such sequence.
The integral $\int_A f(x)\tilde{N}(t, dx)$ is called the **strong 2-integral** of $f$ w.r.t. $\tilde{N}(t, dx)$ on $A$.

In [4], cf. Theorems 3.21 and 3.24 there, it was proven that the integrability condition (2.15) with $p = 1$ resp. $p = 2$ implies the strong 1-integrability resp. the strong 2-integrability of $f$ w.r.t. $\tilde{N}(t, dx)$.

In this case, for any $t \geq 0$ and $A \in \mathcal{B}(G \setminus \{0\})$,

$$\int_A f(x)\,\tilde{N}(t,dx) = \int_{G\setminus\{0\}} \mathbf{1}_A(x) f(x)\,\tilde{N}(t,dx), \text{ } P\text{-almost surely.}$$

If not pointed out explicitly, below we restrict ourselves to the case $p = 2$.

**Proposition 2.4.12:** (cf. Proposition 3.26 from [4])
*Let $f$ fulfill (2.15) with $p = 2$.*
*For all $A \in \mathcal{A}_0$, the strong 2-integral of $f$ coincides with the natural integral of $f$, i.e.*

$$\int_A f(x)\,\tilde{N}(t,dx) = \sum_{0<s\leq t\leq T} f(\Delta L(s))\mathbf{1}_A(\Delta(L(s))) - t\int_A f(x)\,\eta(dx), \text{ } P\text{-}a.s.$$

By standard arguments (see e.g. [7], Chapter 2 or [8]), we can see that for each $A \in \mathcal{B}(G \setminus \{0\})$

$$\int_A f(x)\,\tilde{N}(t,dx),\ t \geq 0,$$

is a centered, square-integrable martingale with

$$\mathbf{E}\left|\left|\int_A f(x)\,\tilde{N}(t,dx)\right|\right|_G^2 = t\int_A ||f(x)||_G^2\,\eta(dx),$$

provided $f$ satisfies (2.15) with $p = 2$.
In particular, since

$$\int_{G\setminus\{0\}} (||x||_G^2 \wedge 1)\,\eta(dx) < \infty,$$

the strong 2-integral

$$\int_{\{0<||x||_G<1\}} x\,\tilde{N}(t,dx)$$

is correctly defined. For notational simplicity, we will denote it just by

$$\int_{||x||_G<1} x\,\tilde{N}(t,dx).$$

Furthermore, see e.g. [8], Section 2.3., p.179,

$$\int_{\{||x||_G<1\}} x\,\tilde{N}(t,dx) = \lim_{n\to\infty} \int_{\{\frac{1}{n}<||x||_G<1\}} x\,\tilde{N}(t,dx),$$

where the limit is taken in $L^2(\Omega,\mathcal{F},P;G)$.
Having explained the terms in the decomposition 2.4.10, we continue with a

special form of the Lévy-Itô decomposition needed in the subsequent chapters. To this end, we impose the square-integrability assumption **(SI)** already mentioned in the Introduction.

**Theorem 2.4.13:** (cf. Lemma 1.1 from [59] )
*Let $L$ be a Lévy process with characteristics $(b, Q, \eta)$ such that $\eta$ fulfills **(SI)**, i.e.*

$$\int_G ||x||_G^2 \eta(dx) < \infty.$$

*Then, the Lévy-Itô decomposition can be written as*

$$L(t) = tm + W(t) + \int_G x\tilde{N}(t, dx) \tag{2.16}$$

*with a drift vector $m \in G$ given by $m = b + \int_{\{||x||_G \geq 1\}} x\eta(dx)$.*

Let us note the following equivalence relation between a Lévy process and its intensity measure.

**Proposition 2.4.14:** (cf. [95], Theorem 4.47, p.67)
*A Lévy process $L$ on a Hilbert space $G$ is square-integrable, i.e.*

$$\mathbf{E}||L(t)||_G^2 < \infty \text{ for any } t \geq 0,$$

*if and only if its intensity measure satisfies **(SI)**.*
*Furthermore the assumption*

$$\textbf{(PI)} \qquad \int_{G\backslash\{0\}} ||x||_G^p \eta(dx) < \infty \text{ for some } p \geq 1$$

*implies by Proposition 6.9 in [95] that*

$$\mathbf{E}||L(t)||_G^p < \infty \text{ for all } t \geq 0.$$

**Martingale measures**

It is an important observation that the third term on the right hand side of the Lévy-Itô decomposition (2.16) is a **martingale measure**. The notion of the martingale measure combines the two important concepts of random measure and martingale.
In particular, there is a well-developed $L^2$-theory of stochastic integration w.r.t. martingale measures in Hilbert spaces, which is presented e.g. in

the monograph [7] by Applebaum (for some basic facts see also Section 2.6 below).

The notion of martingale measure for $G = \mathbb{R}$ was first introduced by Walsh in Chapter 2 of [110]. For the corresponding extension to the case of Hilbert spaces, see e.g. [7] or [8], Section 2.2. Below, we give a short account adapted to our purposes.
So, let $G$ be a separable Hilbert space with the forbidden point 0. Let us define the family $\mathcal{A}_0$ of sets $A \in \mathcal{B}(G)$ bounded below (i.e. such that $0 \notin \bar{A}$). Next, let $S := \{x \in G \,|\, 0 < ||x||_G < 1\}$, $S_n := \{x \in G \,|\, \frac{1}{n} < ||x||_G < 1\}$, $\mathcal{S}_0 := \mathcal{A}_0 \cap S$, $\mathcal{B}(S) = \sigma(\mathcal{S}_0) = S \cap \mathcal{B}(G)$, and $\mathcal{B}(S_n) = S_n \cap \mathcal{B}(G)$, $n \in \mathbb{N}$. Obvously, $S = \bigcup\limits_{n\in\mathbb{N}} S_n$ and $\mathcal{S}_0$ is a ring in $S$.

**Definition 2.4.15:** *A family of $G$-valued random variables $M(t, A)$, indexed by $t \in \mathbb{R}_+$ and $A \in \mathcal{S}_0$, is called a* ***martingale-valued measure*** *if it has the following properties:*

1. *$M(0, A) = M(t, \emptyset) = 0$ $P$-a.s. for all $t \geq 0$, $A \in \mathcal{S}_0$;*
2. ***Finite additivity:*** *$M(t, A \sqcup B) = M(t, A) + M(t, B)$ $P$-a.s. for all disjoint $A, B \in \mathcal{A}$ and $t \in [0, T]$;*
3. ***$\sigma$-finedness:*** *$\sup\{\mathbf{E}||M(t, A)||_G^2 \,|\, A \in \mathcal{B}(S_n)\} < \infty$ for all $n \in \mathbb{N}$, $t \in [0, T]$;*
4. ***$\sigma$-additivity*** *on each $\mathcal{B}(S_n)$, $n \in \mathbb{N}$:*
   *$\lim\limits_{j\to\infty} \mathbf{E}||M(t, A_j)||_G^2 = 0$ for any sequence $(A_j)_{j\in\mathbb{N}} \subset \mathcal{B}(S_n)$ decreasing to the empty set;*
5. *$\mathbb{R}_+ \ni t \mapsto M(t, A)$ is a* ***square integrable martingale*** *for each $A \in \mathcal{S}_0$.*

*$M$ is called* ***orthogonal*** *if*

$$[0, T] \ni t \mapsto (M(t, A), g_n >_G < M(t, B), g_m >_G),$$

*is a martingale for all disjoint $A, B \in \mathcal{A}$, all $n, m \in \mathbb{N}$ and an orthonormal basis $(g_n)_{n\in\mathbb{N}}$ of $G$.*

To continue, let us recall that a family $\mathbf{T} = \{\mathbf{T}_A \,|\, A \in \mathcal{S}_0\}$ of nonnegative bounded symmetric operators in $G$ is called a **positive operator-valued measure** if:

- $\mathbf{T}_\emptyset = 0$;

- $\mathbf{T}_{A \sqcup B} = \mathbf{T}_A + \mathbf{T}_B$ for all disjoint $A, B \in \mathcal{S}_0$.

$\mathbf{T}$ is said to be of **trace class** if $\mathbf{T}_A \in \mathcal{T}^+(G)$ for every $A \in \mathcal{S}_0$.

Let $\mathbf{T}$ be a trace class positive operator-valued measure and let $\rho$ be a Radon measure on $(0, \infty)$.

The martingale measure $M$ is called **nuclear with** $(\mathbf{T}, \rho)$ if

$$\mathbf{E}[< M((s,t], A), g >_G < M((s,t], A), h >_G] =< g, \mathbf{T}_A h >_G \rho((s,t])$$

for all $0 \leq s \leq t < \infty$, $A \in \mathcal{S}_0$ and $g, h, \in G$.

$M$ is called **decomposable** if $\mathbf{T}$ is decomposable, i.e. there exist a $\sigma$-finite measure $\eta$ on $\mathcal{B}(S)$ and a family $(\mathbf{T}_x)_{x \in S}$ of nonnegative bounded symmetric operators in $G$ such that $S \ni x \mapsto \mathbf{T}_x g \in G$ is measurable for all $g \in G$ and

$$\mathbf{T}_A g = \int_A \mathbf{T}_x g \, \eta(dx)$$

for all $A \in \mathcal{A}$ and $g \in G$.
Again, the integral is understood in the Bochner-sense in $G$ (with $p = 1$).

Next, we note the following relation between martingale measures and compensated Poisson random measures.

**Theorem 2.4.16:** (cf. e.g. Theorem 2 in [8] or Theorem 2.5.2 and Proposition 2.5.4 in [106])
*Let $\tilde{N}(t, dx)$ be a compensated Poisson random measure corresponding to a Lévy process $L(t)$, $t \geq 0$, in $G$.*
*Then,*

$$M(t, A) := \int_A x \, \tilde{N}(t, dx), \; t \geq 0, \; A \in \mathcal{B}(S),$$

*is an orthogonal martingale-valued measure with independent increments. It is nuclear with $(\mathbf{T}, dt)$, where $dt$ denotes the Lebesgue measure on $\mathbb{R}_+$ and $\mathbf{T} = \{\mathbf{T}_A \,|\, A \in \mathcal{S}_0\}$ is given by*

$$\mathbf{T}_A g := \int_A (x, g)_G x \, \eta(dx), \text{ for } A \in \mathcal{S}_0, \; g \in G.$$

*Here, $\eta$ is the Lévy intensity measure defined in Lemma 2.4.5 **(iv)**. Each $\mathbf{T}_A \in \mathcal{T}^+(G)$ and*

$$tr(\mathbf{T}_A) = \int_A ||x||_G^2 \, \eta(dx) < \infty, \; A \in \mathcal{S}_0.$$

*Furthermore,* $\mathbf{T}$ *is decomposable with the measure* $\eta$ *and the operator family*

$$G \ni g \mapsto \mathbf{T}_x g := < x, g >_G x \in G. \tag{2.17}$$

We close this section by taking a closer look at the operators $\mathbf{T}_x$, $x \in S$. Let us recall the following definition:

**Definition 2.4.17:** *Given two separable Hilbert spaces* $G$ *and* $H$*, a bounded linear operator* $T$*:* $G \to H$ *is said to be a* ***Hilbert-Schmidt operator*** *if*

$$||T||^2_{\mathcal{L}_2(G,H)} := tr_G(T^*T) = \sum_{n\in\mathbb{N}} ||Tg_n||^2_H < \infty,$$

*where* $(g_n)_{n\in\mathbb{N}}$ *is an orthonormal basis of* $G$*.*

*We denote the space of all Hilbert-Schmidt operators from* $G$ *to* $H$ *by* $\mathcal{L}_2(G,H)$*.*
*It is a separable Hilbert space with inner product*

$$< T_1, T_2 >_{\mathcal{L}_2(G,H)} = tr_G(T_1^* T_2)$$

*and induced norm* $||T||_{\mathcal{L}_2(G,H)}$*. Actually the above definition does not depend on the choice of the orthonormal basis* $(g_n)_{n\in\mathbb{N}}$*. Furthermore,* $\mathcal{L}_2(G,H)$ *is a two-sided ideal in the Banach space* $\mathcal{L}(G,H)$ *of all bounded linear operators from* $G$ *to* $H$*, i.e., for any* $T \in \mathcal{L}_2(G,H)$ *and* $R_1 \in \mathcal{L}(G)$*,* $R_2 \in \mathcal{L}(H)$*, we have* $R_2TR_1 \in \mathcal{L}_2(G,H)$ *and*

$$||R_2TR_1||_{\mathcal{L}(G,H)} \le ||R_2TR_1||_{\mathcal{L}_2(G,H)} \le ||R_2||_{\mathcal{L}(H)}||T||_{\mathcal{L}_2(G,H)}||R_1||_{\mathcal{L}(G)}. \tag{2.18}$$

*The spaces* $\mathcal{L}(G,H)$ *and* $\mathcal{L}_2(G,H)$ *will be equipped by the corresponding* $\sigma$*-algebras* $\mathcal{B}(\mathcal{L}(G,H))$ *and* $\mathcal{B}(\mathcal{L}_2(G,H))$*.*
*Because of the continuous embedding* $\mathcal{L}_2(G,H) \underset{\rightarrow}{\subseteq} \mathcal{L}(G,H)$*, a general argument based on* ***Kuratowski's theorem*** *(see Theorem 3.9, p.21 in [91]) yields that* $\mathcal{L}_2(G,H)$ *is a measurable subset of* $\mathcal{L}(G,H)$ *and*

$$\mathcal{B}(\mathcal{L}_2(G,H)) = \mathcal{L}_2(G,H) \bigcap \mathcal{B}(\mathcal{L}(G,H)).$$

**Remark 2.4.18:** *It is easy to check that each* $\mathbf{T}_x$*,* $x \in S$*, defined in (2.17) is a bounded, nonnegative and symmetric operator in* $G$*. Its square-root operator has an explicit form as*

$$G \ni g \mapsto \mathbf{T}_x^{\frac{1}{2}} g := \frac{<x,g>_G}{||x||_G} x \in G,\ x \in S.$$

*Furthermore,* $\mathbf{T}_x \in \mathcal{T}^+(G)$, $\mathbf{T}_x^{\frac{1}{2}} \in \mathcal{L}_2(G)$ *and*

$$||\mathbf{T}_x^{\frac{1}{2}}||^2_{\mathcal{L}_2(G)} = tr(T_x) = ||\mathbf{T}_x|| = ||x||^2_G.$$

Assuming the global integrability condition **(SI)**, one can prove the same results for $S := G \setminus \{0\}$ and $\mathcal{S}_0 = \mathcal{A}_0$.

## 2.5 Stochastic integration w.r.t. Wiener processes

In this section, we briefly recall the standard construction of stochastic integrals w.r.t. Wiener processes in Hilbert spaces.
For more details, see e.g. the monographs [26] and [97].

Let $G$ and $H$ be separable Hilbert spaces.
We have to distinguish two main cases:

- **Nuclear case:** stochastic integration w.r.t. $Q$-Wiener processes with $Q \in \mathcal{T}^+(G)$.

- **Cylindrical case:** stochastic integration w.r.t. cylindrical **I**-Wiener processes.

As in the previous sections, let $(\Omega, \mathcal{F}, P)$ be a probability space and let $(\mathcal{F}_t)_{t\in[0,T]}$ be a complete right-continuous filtration in $\mathcal{F}$.

**Nuclear case**
Given $Q \in \mathcal{T}^+(G)$, the integration w.r.t. $Q$-Wiener process $(W(t))_{t\geq 0}$ taking values in $G$ is defined as follows.

Recall that we write $\mathcal{G}$ for the Hilbert space $Q^{\frac{1}{2}}(G)$ defined in Section 2.3:

For the rest of this section, we fix $0 < T < \infty$. First, we introduce the class of elementary processes.

**Definition 2.5.1:** (cf.[97], Section 2.3, Definition 2.3.1 resp. Propositions 2.3.5 and 2.3.6)
***(i)*** *An* $\mathcal{L}(G, H)$*-valued process* $(\Phi(t))_{t\in[0,T]}$ *is called an* ***elementary*** *(or simple)* ***process*** *if there exists a partition of* $[0, T]$,

$$0 =: t_0 < t_1 < \ldots < t_M := T,\ M \in \mathbb{N},$$

*such that*

$$\Phi(t) := \sum_{m=0}^{M-1} \Phi_m \mathbf{1}_{(\, t_m, t_{m+1}\,]}(t),$$

*where for each* $0 \leq m \leq M-1$*:*

- $\Phi_m$ *is* $\mathcal{F}_{t_m}$*-measurable;*
- $\Phi_m$ *only takes a finite number of values in* $\mathcal{L}(G, H)$*.*

***(ii)*** *Given such a process, for each* $t \in [\,0, T\,]$ *the* ***stochastic integral*** *is defined by*

$$\left(\int_0^t \Phi(s)\, dW(s)\right)(\omega) := \sum_{m=0}^{M-1} \Phi_m(\omega)(W(t_{m+1} \wedge t)(\omega) - W(t_m \wedge t)(\omega)),\ \omega \in \Omega.$$

*Furthermore, we have the* ***Itô-isometry***

$$\mathbf{E}\left|\left|\int_0^t \Phi(s)\, dW(s)\right|\right|_H^2 = \mathbf{E}\int_0^t ||\Phi(s)||^2_{\mathcal{L}_2(\mathcal{G},H)}\, ds \tag{2.19}$$

*and the mapping*

$$[\,0, T\,] \ni t \mapsto \int_0^t \Phi(s)\, dW(s) \in H$$

*is a continuous, square-integrable martingale w.r.t.* $(\mathcal{F}_t)_{t\in[\,0,T\,]}$*.*

Note that the Hilbert-Schmidt norm in the right hand side of (2.19) is finite due to the elementary estimate

$$||\Phi||_{\mathcal{L}_2(\mathcal{G},H)} \leq (tr\, Q)^{\frac{1}{2}} ||\Phi||_{\mathcal{L}(G,H)}$$

valid for any operator $\Phi \in \mathcal{L}(G, H)$.

By the Itô-isometry, the notion of stochastic integrals is extended to a larger class of integrands $\Phi$:

**Proposition 2.5.2:** (cf. [97], Section 2.3, Proposition 2.3.8)
*For any* $\mathcal{L}_2(\mathcal{G}, H)$*-valued, predictable process* $\Phi$ *obeying*

$$\mathbf{E}\int_0^T ||\Phi(s)||^2_{\mathcal{L}_2(\mathcal{G},H)}\,ds < \infty, \tag{2.20}$$

*there exists a sequence of elementary processes $(\Phi_n)_{n\in\mathbb{N}}$ such that*

$$\lim_{n\to\infty} \mathbf{E}\int_0^T ||\Phi(s)-\Phi_n(s)||^2_{\mathcal{L}_2(\mathcal{G},H)}\,ds = 0.$$

*Then, one defines the stochastic integral for $\Phi$ as the $L^2$-limit of the stochastic integrals corresponding to $\Phi_n$ that were constructed by 2.5.1, i.e.*

$$\int_0^t \Phi(s)\,dW(s) := \lim_{n\to\infty}\int_0^t \Phi_n(s)\,dW(s),\ t\in[\,0,T\,],$$

*in $L^2(\Omega,\mathcal{F},P;H)$.*
*Obviously, the limit does not depend on the choice of the approximating sequence $(\Phi_n)_{n\in\mathbb{N}}$. Furthermore, Itô's isometry (2.19) holds true for this stochastic integral and*

$$[\,0,T\,]\ni t\mapsto \int_0^t \Phi(s)\,dW(s) =: M(t)\in H$$

*is a continuous square integrable martingale as well. Its quadratic variation equals*

$$< M,M>_t = [M,M]_t = \int_0^t ||\Phi(s)||^2_{\mathcal{L}_2(\mathcal{G},H)}\,ds,\ t\in[\,0,T\,].$$

The Wiener stochastic integral has the following continuity property.

**Proposition 2.5.3:** (cf. [97], Lemma 2.4.1, p.35)
*Let $\Phi$ be as in 2.5.2, let $L\in\mathcal{L}(H,\tilde{H})$, where $\tilde{H}$ is another separable Hilbert space. Then, $(L(\Phi(t)))_{t\in[\,0,T\,]}\subset\mathcal{L}_2(\mathcal{G},\tilde{H})$ and, for any $t\in[\,0,T\,]$,*

$$L\left(\int_0^t \Phi(s)\,dW(s)\right) = \int_0^t L(\Phi(t))\,dW(s),\ P\text{-a.s.}.$$

We finish our consideration of the nuclear case with the following fundamental inequality due to **Burkholder, Davies and Gundy**.

**Proposition 2.5.4:** (cf. [26], Chapter 7, Lemma 7.2)
*Let $r\geq 1$ and define positive constants*

$$c_r := \left(\tfrac{2r}{2r-1}\right)^{2r}$$

*and*

$$C_r := (r(2r-1))^r \left(\frac{2r}{2r-1}\right)^{2r^2}.$$

*Then, for any $\mathcal{L}_2(\mathcal{G}, H)$-valued, predictable process $(\Phi(t))_{t\in[0,T]}$ obeying (2.20), we have*

$$\begin{aligned}\mathbf{E}\left(\sup_{s\in[0,t]}\left\|\int_0^s \Phi(u)\,dW(u)\right\|_H^{2r}\right) &\leq c_r \sup_{s\in[0,t]} \mathbf{E}\left(\left\|\int_0^s \Phi(u)\,dW(u)\right\|_H^{2r}\right)\\ &\leq C_r \mathbf{E}\left(\int_0^t \|\Phi(s)\|^2_{\mathcal{L}_2(\mathcal{G},H)}\,ds\right)^r,\ t\in[0,T].\end{aligned}$$

**Cylindrical case**
Let $Q = \mathbf{I}$ and $W$ be the corresponding cylindrical Wiener process defined in Section 2.3.
Recall that to construct this process, we need an auxiliary Hilbert space $G_1$ with the Hilbert-Schmidt embedding $J$: $G \underset{\to}{\subseteq} G_1$.
To this end, let $(b_n)_{n\in\mathbb{N}}$ be a sequence of nonnegative real numbers such that

$$\sum_{n\in\mathbb{N}} b_n^2 < \infty,$$

and let $(g_n)_{n\in\mathbb{N}}$ be an orthonormal basis of $G$.
Then, $G_1$ is defined as a completion of $G$ w.r.t. the inner product

$$< g, h >_{G_1} := \sum_{n\in\mathbb{N}} b_n^2 < g_n, g >_G < g_n, h >_G.$$

By the above construction, we get the Hilbert-Schmidt embedding operator $J \in \mathcal{L}_2(G, G_1)$

$$G \ni g \mapsto Jg := \sum_{n\in\mathbb{N}} b_n < g, g_n >_G g_n \in G_1.$$

Now, the idea is to define the stochastic integral w.r.t. the cylindrical Wiener process as the stochastic integral w.r.t. to the $Q_1$-Wiener process introduced in 2.3.8. Its correlation operator is $Q_1 := JJ^* \in \mathcal{T}^+(G_1)$ with $G = Q_1^{\frac{1}{2}}(G_1)$.
Now the previuos construction runs for $G := G_1$ and $\mathcal{G} := Q_1^{\frac{1}{2}}(G_1) = G$.
By Theorem 2.5.2, the stochastic integral w.r.t. the cylindrical Wiener process $W$ exists for all predictable $\mathcal{L}_2(Q_1^{\frac{1}{2}}(G_1), H)$-valued processes $(\Psi(t))_{t\in[0,T]}$ fulfilling

$$\mathbf{E}\int_0^T \|\Psi(t)\|^2_{\mathcal{L}_2(Q_1^{\frac{1}{2}}G_1,H)}\,dt < \infty.$$

It is clear that a predictable process $(\Phi(t))_{t\in[0,T]}$ is $\mathcal{L}_2(G,H)$-valued if and only if $(\Psi(t) := \Phi(t)\circ J^{-1})_{t\in[0,T]}$ is $\mathcal{L}_2(Q_1^{\frac{1}{2}}G_1, H)$-valued. This leads to the following:

**Definition 2.5.5:** *Let $W$ be a cylindrical $\mathbf{I}$-Wiener process in the sense of Section 2.3. Let $J$ be as defined above.*

*Given a predictable $\mathcal{L}_2(G,H)$-valued process $(\Phi(t))_{t\in[0,T]}$ such that*

$$\mathbf{E}\int_0^T ||\Phi(s)||^2_{\mathcal{L}_2(G,H)}\,ds < \infty,$$

*its stochastic integral w.r.t. $W$ is defined by*

$$\int_0^t \Phi(s)\,dW(s) := \int_0^t \Phi(s)\circ J^{-1}\,dW(s) \in H,\ t\in[0,T]. \tag{2.21}$$

*Here, Itô's isometry takes the form*

$$\mathbf{E}\left|\left|\int_0^t \Phi(s)\,dW(s)\right|\right|_H^2 = \mathbf{E}\int_0^T ||\Phi(s)||^2_{\mathcal{L}_2(G,H)}\,ds,\ t\in[0,T].$$

Let us note (cf. [97], Remark 2.5.3, p.42) that the integral in (2.21) is independent of the choice of $(b_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$.

Furthermore, we have the following version of the **Burkholder-Davis-Gundy** inequality for the cylindrical **I**-Wiener process:

**Proposition 2.5.6:** (cf. [26], Chapter 7, Lemma 7.7)
*Given an $\mathcal{L}_2(G,H)$-valued, predictable process $(\Phi(t))_{t\in[0,T]}$, for each $r\geq 1$ and $t\in[0,T]$, the following estimate holds*

$$\sup_{s\in[0,t]}\mathbf{E}\left|\left|\int_0^s \Phi(u)\,dW(u)\right|\right|_H^{2r} \leq (r(2r-1))^r\left(\int_0^t (\mathbf{E}||\Phi(s)||^{2r}_{\mathcal{L}_2(G,H)})^{\frac{1}{r}}\,ds\right)^r.$$

We finish this section with the following notation convention:

**Definition 2.5.7:** *If it does not lead to misunderstandings, we use the notation $\mathcal{L}_2$ for $\mathcal{L}_2(Q^{\frac{1}{2}}L^2, L^2_\rho)$ both in the case of $Q\in\mathcal{T}^+(G)$ and in the case of $Q=\mathbf{I}\notin\mathcal{T}^+(G)$.*

**Definition 2.5.8:** *For a given $T>0$, we denote by $\mathcal{S}_W(T)$ the set of elementary (or simple) processes $(\Phi(t))_{t\in[0,T]}$ with values in $\mathcal{L}_2$ and by $\mathcal{N}_W(T)$*

*the set of all predictable processes* $(\Phi(t))_{t\in[0,T]}$ *with values in* $\mathcal{L}_2$ *such that*

$$\mathbf{E}\int_0^T ||\Phi(s)||_{\mathcal{L}_2}^2\, ds < \infty.$$

**Remark 2.5.9:** *Actually, the predictability of the integrand process* $\Phi$ *is not necessary to define*

$$\int_0^t \Phi(s)\, dW(s).$$

*Using Itô's isometry, by the previous scheme one can extend the definition of stochastic integral to all* ***measurable, adapted*** *processes* $\Phi$ *such that*

$$||\Phi||_{L^2([0,T]\times\Omega;\mathcal{L}_2)}^2 = \int_{[0,T]\times\Omega} ||\Phi(t,\omega)||_{\mathcal{L}_2}^2\, dt\, dP < \infty.$$

## 2.6 Stochastic integration w.r.t. compensated Poisson random measures

The main ingredient of equations (1.1) and (1.2) is the stochastic integral w.r.t. compensated Poisson random measures and orthogonal martingale measures.

Here, we briefly present the $L^p$-theory of stochastic integration w.r.t. that kind of measures. For a more detailed exposition for $p = 2$ resp. $p = 1$ see [7] resp. [102].
Furthermore, we discuss the so-called Bichteler-Jacod inequality and path properties of stochastic integrals w.r.t. compensated Poisson random measures.

Analogously to Section 2.5, we fix some $0 < T < \infty$.

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $(\mathcal{F}_t)_{t\in[0,T]}$ be a complete right-continuous filtration in $\mathcal{F}$.

**Stochastic integration w.r.t. a compensated Poisson random measure**

Let $G$, $H$ be two separable Hilbert spaces as in Section 2.5. As in Section 2.4, let $\tilde{N}$ be a compensated Poisson random measure on $[0,T]\times G$

with compensator $\eta(dx) \otimes dt$, where $\eta$ is a Lévy measure.

The following presentation is based on Section 3 of [102], where Rüdiger and Ziglio elaborate an $L^p$-theory (with $p \geq 1$) of stochastic integration w.r.t. Poisson random measures on separable Banach spaces. We restrict ourselves to the case of separable Hilbert spaces, which will be sufficient for our purposes.

Recall from Section 2.4 that $N(t, A)$ is $P$-a.s. finite for any $t \in [0, T]$ and $A \in \mathcal{A}_0$. Here,

$$\mathcal{A}_0 := \{A \in \mathcal{B}(G) \mid 0 \notin \bar{A}\}$$

is the ring of the so-called bounded below sets in $G$. Furthermore (cf. (7), (8) from [102], p. 5 there), for any $0 \leq t_1 < t_2 \leq T$,

$$N((t_1, t_2] \times A)(\omega) = \sum_{t_1 < s \leq t_2} \mathbf{1}_A(\Delta L(s))(\omega),\ \omega \in \Omega. \tag{2.22}$$

Herefrom, by the definition (2.11) of $\tilde{N}(t, dx)$,

$$\tilde{N}((t_1, t_2] \times A)(\omega) = N((t_1, t_2] \times A)(\omega) - (t_2 - t_1)\eta(A),\ \omega \in \Omega. \tag{2.23}$$

We define the stochastic integration w.r.t. the compensated Poisson random measure $\tilde{N}$ for vector-valued integrands $f\colon [0, T] \times \Omega \times G \setminus \{0\} \to H$.

First, we need the proper notion of measurability for the integrand functions.
By $\mathcal{P}_{T,\mathcal{A}_0}$ we denote the $\sigma$-algebra on $[0, T] \times \Omega \times G \setminus \{0\}$ generated by product sets of the form $\{0\} \times B \times A$ and $(s, t] \times C \times A$ with $0 < s < t \leq T$, $A \in \mathcal{A}_0$, $B \in \mathcal{F}_0$ and $C \in \mathcal{F}_s$.
The integrand functions $f$ are assumed to be $\mathcal{P}_{T,\mathcal{A}_0}/\mathcal{B}(H)$-measurable and hence such that $t \mapsto f(t, \omega, x)$ is $\mathcal{F}_t$-adapted for any fixed $x \in G \setminus \{0\}$. We will call such $f$ **predictable** and denote their set by $\mathcal{N}_{G/H}(T)$.

As in the Wiener case, we start with the definition of the integral for simple functions.

**Definition 2.6.1:** (cf. Definition 3.3 from [102])
***(i)*** *A function $f \in \mathcal{N}_{G/H}(T)$ belongs to the set $\mathcal{S}_{G/H}(T)$ of* ***elementary (or simple) functions*** *if it can be represented as*

$$(2.24)\ f(t,\omega,x) = \sum_{m=1}^{M} \mathbf{1}_{\{0\}}(t)\mathbf{1}_{A_{0,m}}(x)\mathbf{1}_{B_{0,m}}(\omega)a_{0,m} + \sum_{k=1}^{K-1}\sum_{m=1}^{M} \mathbf{1}_{(t_k,t_{k+1}]}(t)\mathbf{1}_{A_{k,m}}(x)\mathbf{1}_{B_{k,m}}(\omega)a_{k,m},$$

*where $K, M \in \mathbb{N}$, $A_{k,m} \in \mathcal{A}_0$, $B_{k,m} \in \mathcal{F}_{t_k}$, $a_{k,m} \in H$ and $0 =: t_0 < t_1 < ... < t_M = T$. For each $k \in \{1, 2, ..., K\}$ fixed, we assume*

$$(A_{k,m_1} \times B_{k,m_1}) \cap (A_{k,m_2} \times B_{k,m_2}) = \emptyset \text{ if } m_1 \neq m_2.$$

***(ii)*** *Given $t \in [0,T]$, $A \in \mathcal{B}(G \setminus \{0\})$ and $f \in \mathcal{S}_{G/H}(T)$, we define the Poisson stochastic integral as a random variable*

$$(2.25)\ \int_0^t \int_A f(t,\omega,x)\,\tilde{N}(ds,dx)(\omega)$$
$$:= \int_{(0,t]} \int_A f(t,\omega,x)\,\tilde{N}(ds,dx)(\omega)$$
$$:= \sum_{k=0}^{K-1}\sum_{m=0}^{M} a_{k,m}\mathbf{1}_{B_{k,m}}(\omega)\tilde{N}((t_k,t_{k+1}] \cap (0,t] \times A_{k,m} \cap A)(\omega),\ \omega \in \Omega.$$

*Moreover, for $0 \le t_1 \le t_2 \le T$, we set*

$$\int_{t_1}^{t_2} \int_A f(s,\omega,x)\,\tilde{N}(ds,dx)(\omega)$$
$$:= \int_{(t_1,t_2]} \int_A f(s,\omega,x)\,\tilde{N}(ds,dx)(\omega)$$
$$:= \int_{(0,t_2]} \int_A f(s,\omega,x)\,\tilde{N}(ds,dx)(\omega) - \int_{(0,t_1]} \int_A f(s,\omega,x)\,\tilde{N}(ds,dx)(\omega)$$
$$= \int_0^{t_2} \int_A f(s,\omega,x)\,\tilde{N}(ds,dx)(\omega) - \int_0^{t_1} \int_A f(s,\omega,x)\,\tilde{N}(ds,dx)(\omega).$$

**Remark 2.6.2:** *We set $t_0 = t_1 = 0$ in (2.24) in order to include the value of $f \in \mathcal{S}_{G/H}(T)$ at $t_0 = 0$ and hence to define the integrand function on the whole intervall $[0,T]$. However, (2.25) shows that the concrete value of $f$ at $t = 0$ does not influence the integral. For this reason, it is more accurate to use the notation*

$$\int_{(0,t]} \int_A f(s,x)\,\tilde{N}(ds,dx).$$

For $p \ge 1$, we denote by $N^{p,\eta}_{G/H}(T)$ the set of functions $f \in \mathcal{N}_{G/H}(T)$ such that

$$\int_0^T \int_G \mathbf{E}||f(t,x)||_H^p \, \eta(dx)\, dt < \infty.$$

**Definition 2.6.3:** (cf. Definition 3.5 from [102])
*Let $p \geq 1$, $A \in \mathcal{B}(G \setminus \{0\})$ and $f \in \mathcal{N}_{G/H}(T)$.*
*A sequence $(f_n)_{n\in\mathbb{N}} \subset \mathcal{S}_{G/H}(T)$ is called* ***$L^p$-approximating*** *$f$ on $(0,T] \times \Omega \times A$ w.r.t. $dt \times P \times \eta$ if $f_n$ is $dt \times P \times \eta$-almost surely converging to $f$ as $n \to \infty$, and*

$$\lim_{n\to\infty} \int_0^T \int_A \mathbf{E}[||f_n(t,x) - f(t,x)||_H^p] \, \eta(dx)\, dt = 0.$$

**Theorem 2.6.4:** (cf. Theorem 3.6 from [102])
*Let $p \geq 1$, $T > 0$. Then, for each $f \in N_{G/H}^{p,\eta}(T)$, there is a sequence $(f_n)_{n\in\mathbb{N}} \subset S_{G/H}^{p,\eta}(T)$, which is $L^p$-approximating $f$ on $(0,T] \times \Omega \times A$ for all $A \in \mathcal{B}(G \setminus \{0\})$.*

Now, we describe the class of admissible integrands.

**Definition 2.6.5:** (cf. Definition 3.9 from [102])
***(i)*** *Let $p \geq 1$, $T > 0$ and $A \in \mathcal{B}(G \setminus \{0\})$. We say that $f \in \mathcal{N}_{G/H}(T)$ is* ***strong-p-integrable*** *on $(0,T] \times \Omega \times A$ w.r.t. $\tilde{N}$ if there exists a sequence $(f_n)_{n\in\mathbb{N}}$ of simple functions $L^p$-approximating $f$ on $(0,T] \times \Omega \times A$.*
***(ii)*** *For any such sequence and any $t \in [0,T]$, the limit of the integrals of $f_n$ w.r.t. $\tilde{N}$ exists, i.e.*

$$\begin{aligned}(2.26)\quad \int_0^t \int_A f(s,\omega,x)\, \tilde{N}(ds,dx)(\omega) :=& \int_0^T \int_A \mathbf{1}_{[0,t]}(s) f(s,\omega,x)\, \tilde{N}(ds,dx)(\omega) \\ :=& \lim_{n\to\infty} \int_0^T \int_A \mathbf{1}_{[0,t]}(s) f_n(s,\omega,x)\, \tilde{N}(ds,dx)(\omega) \\ =& \lim_{n\to\infty} \int_0^t \int_A f_n(s,\omega,x)\, \tilde{N}(ds,dx)(\omega),\end{aligned}$$

*and the limit does not depend on the choice of the $L^p$-approximating sequence $(f_n)_{n\in\mathbb{N}}$.*
*The limit in (2.26) will be called the* ***strong-p-integral of*** *$f$ w.r.t. $\tilde{N}$ on $(0,t] \times A$.*
*Moreover, given $0 \leq t_1 \leq t_2 \leq T$, we set*

$$\begin{aligned}\int_{t_1}^{t_2} \int_A f(s,\omega,x)\, \tilde{N}(ds,dx)(\omega) :=& \int_0^{t_2} \int_A f(s,\omega,x)\, \tilde{N}(ds,dx)(\omega) \\ &- \int_0^{t_2} \int_A f(s,\omega,x)\, \tilde{N}(ds,dx)(\omega).\end{aligned}$$

In the special case $p = 2$, there is a constructive description of the possible integrands.

**Theorem 2.6.6:** (cf. Theorem 3.13 from [102], Proposition 3.15 from [71] and Proposition 3.6. in [61])
*Let $f \in N^{2,\eta}_{G/H}(T)$. Then, $f$ is strong-2-integrable w.r.t. $\tilde{N}(dt,dx)$ on $(\,0,t\,] \times A$ for any $0 \le t \le T$, $A \in \mathcal{B}(G \setminus \{0\})$ and*

$$\int_0^t \int_A f(s,x)\, \tilde{N}(ds,dx) = \int_0^T \int_{G\setminus\{0\}} \mathbf{1}_{[\,0,t\,]}(s)\mathbf{1}_A(x) f(s,x)\, \tilde{N}(ds,dx).$$

*Moreover, we have the isometry*

$$\mathbf{E}\left[\left|\left|\int_0^t \int_A f(s,x)\, \tilde{N}(ds,dx)\right|\right|_H^2\right] = \int_0^t \int_A \mathbf{E}||f(s,x)||_H^2\, \eta(dx)\, ds. \tag{2.27}$$

*The process*

$$M(t) := \int_0^t \int_{G\setminus\{0\}} f(s,x)\, \tilde{N}(ds,dx),\; t \in [\,0,T\,] \tag{2.28}$$

*is a square-integrable $\mathcal{F}_t$-martingale with mean zero.*
*Its predictable quadratic variation (i.e. the Meyer process) is given by*

$$< M, M >_t = \int_0^t \int_{G\setminus\{0\}} ||f(s,x)||_H^2\, \eta(dx)\, ds,$$

*whereas for its adpted quadratic variation we have*

$$[M,M]_t = \int_0^t \int_{G\setminus\{0\}} ||f(s,x)||_H^2\, N(ds,dx).$$

*Furthermore, $M$ is càdlàg and $M(t) = M(t-)$, $P$-a.s. for all $t \in [\,0,T\,]$.*

**Remark 2.6.7:** *From Itô's isometry we see that*

$$\int_0^t \int_{G\setminus\{0\}} f_1(s,x)\, \tilde{N}(ds,dx) = \int_0^t \int_{G\setminus\{0\}} f_2(s,x)\, \tilde{N}(ds,dx)$$

*for any two predictable processes $f_1$, $f_2 \in \mathcal{N}^{2,\eta}_{G/H}$ satisfying*

$$\int_0^t \int_{G\setminus\{0\}} \mathbf{E}||f_1(s,x) - f_2(s,x)||_H^2\, \eta(dx)\, ds = 0.$$

*In particular, except for a zeroset, the value of $f$ at $t_0 = 0$ does not influence the integral*

$$\int_0^t \int_A f(s,x)\,\tilde{N}(ds,dx),$$

*since the process $\mathbf{1}_{\{0\}}(s)f(s,x)$ is $P\otimes dt\otimes\eta$-equivalent to the indentity zero process. Again it would be more accurate to use the notation*

$$\int_{(0,t\,]} \int_A f(s,x)\,\tilde{N}(ds,dx).$$

*Recall that by the definition*

$$M(t) := \int_0^t \int_{G\backslash\{0\}} f(s,x)\,\tilde{N}(ds,dx),\; t\in[\,0,T\,],$$

*is a càdlàg process, and hence it obeys a predictable modification $M(t-)$, $t\in[\,0,T\,]$. To distinguish between the càdlàg and the predictable versions some authors use respectively the notation*

$$\int_0^{t+} \int_{G\backslash\{0\}} f(s,x)\,\tilde{N}(ds,dx)$$

*and*

$$\int_0^{t-} \int_{G\backslash\{0\}} f(s,x)\,\tilde{N}(ds,dx).$$

We have an anlogous proposition to 2.5.3.

**Proposition 2.6.8:** (cf. [60], Proposition 3.7, p.58)
*Let $f\in\mathcal{N}^{2,\eta}_{G/H}(T)$ and let $L\in\mathcal{L}(H,\tilde{H})$, where $\tilde{H}$ is another separable Hilbert space. Then, $Lf\in\mathcal{N}^{2,\eta}_{G/\tilde{H}}(T)$ and, for each $t\in[\,0,T\,]$,*

$$L\left(\int_0^t \int_{G\backslash\{0\}} f(s,x)\,\tilde{N}(ds,dx)\right) = \int_0^t Lf(s,x)\,\tilde{N}(ds,dx),\; P\text{-a.s.}.$$

**Remark 2.6.9:** *Let us apply the Theorem 2.6.6 to the concrete function $f\in N^{2,\eta}_{G/H}(T)$ given by*

$$f(t,\omega,x)=\mathbf{1}_S(x)g(t,\omega)x,\; (t,\omega,x)\in[\,0,T\,]\times\Omega\times G, \tag{2.29}$$

*with $S=\{x\in G\,|\,0<||x||_G<1\}$. Under the assumption (2.10), i.e.*

$$\int_G (||x||_G^2\wedge 1)\,\eta(dx)<\infty,$$

*a sufficient condition for the above $f$ to belong to $\mathcal{N}^{2,\eta}_{G/H}(T)$ is that*

$$[0,T] \times \Omega \ni (t,\omega) \mapsto g(t,\omega) \in \mathcal{L}(G,H)$$

*is predictable and obeys*

$$\mathbf{E}\int_0^T ||g(t,\omega)||^2_{\mathcal{L}(G,H)}\, ds < \infty.$$

*Note that if $g_1,\ g_2 \in L^2([0,T] \times \Omega; \mathcal{L}(G,H))$ are predictable and stochastically equivalent in the sense of Definition 2.1.1, then by Itô's isometry*

$$\mathbf{E}\left[\left|\left|\int_0^t\int_G f_1(s,x)\,\tilde{N}(ds,dx) - \int_0^t\int_G f_2(s,x)\,\tilde{N}(ds,dx)\right|\right|^2_H\right]$$
$$\leq \left(\int_G ||x||^2_G\,\eta(dx)\right)\mathbf{E}\int_0^T ||g_1(s)-g_2(s)||^2_{\mathcal{L}_2(G,H)}\, ds = 0,\ 0 \leq t \leq T,$$

*i.e. the integrals are also stochastically equivalent.*

*The integrands of such form naturally appear in the theory of SDEs driven by Lévy processes.*
*For notational simplicity, for $f \in \mathcal{N}^{2,\eta}_{G/H}(T)$ such that $f(x) = 0$ if $x = 0$, we shall write*

$$\int_0^t\int_G f(s,x)\,\tilde{N}(ds,dx) = \int_0^t\int_{G\setminus\{0\}} f(s,x)\,\tilde{N}(ds,dx).$$

We finish this section by recalling $L^p$-properties of the Poisson stochastic integral.

We state the important **Bichteler-Jacod inequality**, which e.g. can be found in [18] (cf. Lemma 3.1 there) and [79] (cf. Lemma 3.1 , p.7 there).

To this end, for $p \geq 2$, we consider the space of integrands $f \in N^{p,\eta}_{G/H}(T)$, where the functions $f$ are predictable, i.e. $f \in \mathcal{N}_{G/H}(T)$, and fulfill

$$\mathbf{E}\int_0^T\left[\int_G ||f(s,x)||^p_H\,\eta(dx) + \left(\int_G ||f(s,x)||^2_H\,\eta(dx)\right)^{\frac{p}{2}}\right] ds < \infty. \tag{2.30}$$

Obviously $N^{p,\eta}_{G/H}(T) \subset N^{2,\eta}_{G/H}(T)$ and hence the integral (2.28) is well-defined for such $f$.

**Theorem 2.6.10:** *For $p \geq 2$ and $f \in N^{p,\eta}_{G/H}(T)$, the process*

$$M(t) := \int_0^t \int_G f(s,x)\, \tilde{N}(ds,dx),\ t \in [0,T],$$

*obeys the supremum bound*

$$\begin{aligned}(2.31)\quad \sup_{t\in[0,T]} \mathbf{E}||M(t)||_H^p \leq\ & \mathbf{E}\left(\sup_{t\in[0,T]} ||M(t)||_H^p\right) \\ \leq\ & K_{p,T}\mathbf{E}\int_0^T \left[\left(\int_{G\setminus\{0\}} ||f(s,x)||_H^p\, \eta(dx)\right)\right. \\ & \left.+\mathbf{E}\left(\int_{G\setminus\{0\}} ||f(s,x)||_H^2\, \eta(dx)\right)^{\frac{p}{2}}\right] ds,\end{aligned}$$

*where* $(p,T) \mapsto K_{p,T} \in \mathbb{R}_+$ *is continuous.*

An advantage of the Bichteler-Jacod inequality (2.31) (as compared with the Doob-Meyer decomposition and Burkholder-Davies-Gundy inequality) is that we do not need to calculate the corresponding quadratic variation processes.
A lower bound for the left hand side in (2.31) was established in the recent work [31] by Dirksen, see also Remark 4.5 below.

**Remark 2.6.11:** *Note that, for* $f \in N^{p,\eta}_{G/H}(T)$ *of the form (2.22), to have (2.31) it is sufficient to assume that*

$$\int_0^T \mathbf{E}||g(t,\omega)||^p_{\mathcal{L}(G,H)}\, ds < \infty.$$

**Stochastic integration w.r.t. martingale measures**

In Hilbert spaces there is a unified $L^2$-theory for stochastic integration, which includes both integration w.r.t. Wiener and Poisson processes.
This theory, which is based on the concept of a martingale-valued measure (see Section 2.4 above), was developed by Walsh in finite dimensions (cf. the monograph [110]) and by Applebaum (cf. [7]) in general Hilbert spaces. We will briefly explain the basic issues of this theory adapted to our framework.
Let again $G$ and $H$ be two separable Hilbert spaces. As before, let

$$S = \{x \in G \,|\, 0 < ||x||_G < 1\}$$

and

$$\mathcal{S}_0 = \{A \in \mathcal{B}(G) \,|\, A \subset S, 0 \notin \bar{A}\}.$$

Furthermore, let $N(t, dx)$ be a $G$-valued orthogonal martingale measure, which is nuclear with $(\mathbf{T}, dt)$ and decomposable with intensity measure $\eta$.

**Definition 2.6.12 :** *We denote by $\mathcal{N}^2_M(T)$ the set of all mappings $F : [\,0, T\,] \times S \times \Omega \mapsto \mathcal{L}(G, H)$ obeying the following properties:*

- ***Predictability:*** *$(t, x) \mapsto F(t, x)g$ is $\mathcal{P}_T \otimes \mathcal{B}(S)$-measurable for each $g \in G$;*
- *For any $(t, x, \omega) \in [\,0, T\,] \times S \times \Omega$, $F(t, x)(\omega)T_x^{\frac{1}{2}} : G \to H$ is a* ***Hilbert-Schmidt*** *operator, i.e. it belongs to $\mathcal{L}_2(G, H)$, and we have*

$$||F||_{\mathcal{N}^2_M(T)} := \left(\mathbf{E} \int_0^T \int_S ||F(s, x)T_x^{\frac{1}{2}}||^2_{\mathcal{L}_2(G,H)}\, \eta(dx)\, ds\right)^{\frac{1}{2}} < \infty. \tag{2.32}$$

As usual, the construction of the integral is started by considering simple integrands.

**Definition 2.6.13:** *The subspace $\mathcal{S}^2_M(T) \in \mathcal{N}^2_M(T)$ consists of all* ***elementary processes*** *$F$: $[\,0, T\,] \times S \times \Omega \mapsto \mathcal{L}(G, H)$ having the form*

$$\begin{aligned} F = \; & \sum_{m=1}^{M} \mathbf{1}_{\{0\}} \mathbf{1}_{A_m} F_{0m} \\ & + \sum_{k=0}^{K-1} \sum_{m=1}^{M} \mathbf{1}_{(\,t_k, t_{k+1}\,]} \mathbf{1}_{A_m} F_{km} \end{aligned}$$

*with $K, M \in \mathbb{N}$, $0 := t_0 = t_1 < t_2 < \ldots < t_K := T$, pairwise disjoint $A_m \in \mathcal{S}_0$, and random variables $F_{km} \in \mathcal{L}(G, H)$ such that $F_{km}g \in H$ is $\mathcal{F}_{t_k}$-measurable for all $g \in G$.*

It can be checked that $\mathcal{N}^2_M(T)$ is a Banach space and $\mathcal{S}^2_M(T)$ is dense in $\mathcal{N}^2_M(T)$ w.r.t. the norm (2.32) (cf. e.g. Lemma 3.1.2 in [106]).
For each $F \in \mathcal{S}^2_M(T)$, we define

$$\begin{aligned} \int_0^t \int_S F(s, x)\, M(ds, dx) := & \int_{(0,t]} \int_S F(s, x)\, M(ds, dx) \\ := & \sum_{k=0}^{K-1} \sum_{m=1}^{M} F_{km} M((t \wedge t_k, t \wedge t_{k+1}], A_m). \end{aligned}$$

Analogously to Definition 2.6.5, given $0 \le t_1 \le t_2 \le T$, we set

$$\begin{aligned}\int_{t_1}^{t_2}\int_S F(s,x)\,M(ds,dx) &:= \int_{(t_1,t_2]}\int_S F(s,x)\,M(ds,dx)\\ &:= \int_{(0,t_2]}\int_S F(s,x)\,M(ds,dx) - \int_{(0,t_1]}\int_S F(s,x)\,M(ds,dx)\\ &= \int_0^{t_2}\int_S F(s,x)\,M(ds,dx) - \int_0^{t_1}\int_S F(s,x)\,M(ds,dx).\end{aligned}$$

A crucial issue is the Itô-isometry

$$\mathbf{E}\left|\left|\int_0^t\int_S F(s,x)\,M(ds,dx)\right|\right|_H^2 = \mathbf{E}\int_0^t\int_S ||F(s,x)T_x^{\frac{1}{2}}||_{\mathcal{L}_2(G,H)}\,\eta(dx)\,ds. \tag{2.33}$$

**Lemma 2.6.14 :** *For each $F \in \mathcal{N}_M^2(T)$, there exists a sequence $(F_n)_{n\in\mathbb{N}} \subset \mathcal{S}_M^2(T)$ such that*

$$||F_n - F||^2_{\mathcal{N}_M^2(T)} = \mathbf{E}\int_0^T\int_S ||(F_n(t,x) - F(t,x))T_x^{\frac{1}{2}}||^2_{\mathcal{L}_2(G,H)}\eta(dx)\,dt \to 0.$$

*Then, $\int_0^t\int_S F(s,x)\,M(ds,dx)$ is defined as the $L^2(\Omega,\mathcal{F},P;H)$-limit of the integrals corresponding to any approximating sequence $(F_n)_{n\in\mathbb{N}} \subset \mathcal{S}_M^2(T)$. In particular, by Itô's isometry the limit does not depend on the concrete choice of such sequence.*
*Respectively for $A \in \mathcal{B}(S)$, we call*

$$\begin{aligned}\int_0^t\int_A F(s,x)\,M(ds,dx) &:= \int_0^t\int_S \mathbf{1}_A(x)F(s,x)\,M(ds,dx)\\ &:= \int_{(0,t]}\int_A F(s,x)\,M(ds,dx)\end{aligned}$$

*the **strong integral** w.r.t the martingale measure.*

The properties of the integral defined above are described by

**Theorem 2.6.15:**(cf. f.e. [106],Theorem 3.1.5)
*The process $(\int_0^t\int_S F(s,x)M(ds,dx))_{t\in[0,T]}$ is an $H$-valued square integrable martingale with càdlàg paths. Furthermore, for all $t \in [0,T]$,*

$$\mathbf{E}\left|\left|\int_0^t\int_S F(s,x)\,M(ds,dx)\right|\right|_H^2 = \mathbf{E}\int_0^t\int_S ||F(s,x)T_x^{\frac{1}{2}}||^2_{\mathcal{L}_2(G,H)}\,\eta(dx)\,ds.$$

To complete the discussion, let us compare the definitions of stochastic integrals given in this section. Having in mind applications to SDEs with Lévy processes, we are interested in the stochastic integral

$$\int_0^t \int_{\{0<||x||<1\}} g(s)x\,\tilde{N}(ds,dx) \tag{2.34}$$

with predictable

$$[\,0,T\,]\times\Omega \ni (t,\omega)\mapsto g(t,\omega)\in\mathcal{L}(G,H)$$

obeying

$$\mathbf{E}\int_0^T ||g(t)||^2_{\mathcal{L}(G,H)}\,dt<\infty. \tag{2.35}$$

(2.34) is correctly defined in the $L^2$-sense, see Remark 2.6.6. On the other hand, (2.34) can be seen as the stochastic integral w.r.t. the Lévy martingale measure

$$M(t,A):=\int_A x\,\tilde{N}(ds,dx),\ A\in\mathcal{S}_0.$$

By Theorem 2.4.15, $M$ is decomposable with $(\mathbf{T}_x)_{x\in S}\subset\mathcal{T}^+(G)$ given by

$$G\ni g\mapsto\mathbf{T}_x g:=<x,g>_G x\in G.$$

Obviously, for $x\in S:=\{x\in G\,|\,0<||x||_G<1\}$ we have

$$\mathbf{T}_x^{\frac{1}{2}}g:=\frac{<x,g>_G}{||x||_G}x,\ g\in G.$$

Then, by considering elementary processes $g(t)$, $t\in[\,0,T\,]$, and using Itô's isometry, one can check that these two integrals coincide, i.e.

$$\int_0^t\int_{\{0<||x||<1\}} g(s)x\,\tilde{N}(ds,dx)=\int_0^t\int_{\{0<||x||<1\}} g(s)\,M(ds,dx).$$

Moreover, under the assumption (2.35)

$$\begin{aligned}
||g||^2_{\mathcal{N}^2_M(T)} &= \mathbf{E}\int_0^T\int_S ||g(t)\mathbf{T}_x^{\frac{1}{2}}||^2_{\mathcal{L}_2(G,H)}\,\eta(dx)\,dt \\
&= \mathbf{E}\int_0^T\int_S \sum_{n\in\mathbb{N}} ||g(t)\mathbf{T}_x^{\frac{1}{2}}g_n||^2_H\,\eta(dx)\,dt \\
&= \mathbf{E}\int_0^T\int_S \sum_{n\in\mathbb{N}} \left|\left|g(t)\frac{<x,g_n>_G x}{||x||_G}x\right|\right|^2_H\,\eta(dx)\,dt \\
&= \mathbf{E}\int_0^T\int_S ||g(t)x||^2_H\,\eta(dx)\,dt \\
&\leq \left(\mathbf{E}\int_0^T ||g(t)||^2_{\mathcal{L}(G,H)}\,dt\right)\left(\int_{\{0<||x||<1\}} ||x||^2_H\,\eta(dx)\right) \\
&< \infty,
\end{aligned}$$

where $(g_n)_{n\in\mathbb{N}}$ is an orthonormal basis in $G$.

## 2.7 Supporting analytical results

We finish this chapter by collecting some lemmata, which shall be needed in the manuscript. All these results are more or less known in the literature, so we restrict ourselves to just quoting and giving references.

Our first result is a discrete analogon of Lebesgue's dominated convergence theorem.

**Lemma 2.7.1:** (cf. Lemma 2.5 from [59])
*Let $(x_{n,m})_{m\in\mathbb{N}}$, $n\in\mathbb{N}$, be sequences of real numbers such that, for each $n\in\mathbb{N}$, there exists*

$$\lim_{m\to\infty} x_{n,m} =: x_n \in \mathbb{R}.$$

*If there exists a majorizing sequence $(y_n)_{n\in\mathbb{N}}\subset\mathbb{R}_+$ such that $|x_{n,m}|\leq y_n$ for all $m\in\mathbb{N}$ and $\sum_{n\in\mathbb{N}} y_n<\infty$, then*

$$\lim_{m\to\infty}\sum_{n\in\mathbb{N}} x_{n,m} = \sum_{n\in\mathbb{N}} x_n.$$

Next, we give a generalization of Gronwall's lemma, which will be used to prove the existence and uniqueness result for the stochastic convolutions under consideration. Generalized versions of the Gronwall inequality have been given e.g. in [110] (cf. Theorem 3.3 there) or in [66]. We will use the following version of it, the so-called **Gronwall-Bellman lemma**:

**Lemma 2.7.2:** (cf. [75], Appendix, Lemma A1 or [112])

*Let $(g_n)_{n\in\mathbb{N}}$ be a sequence of measurable functions $g_n\colon \mathbb{R}_+ \to \mathbb{R}_+$ obeying*

$$g_n(t) \le q + b\int_0^t (t-s)^{-\delta} g_{n-1}(s)\,ds,\ t\in[\,0,T\,],\ n\in\mathbb{N},$$

*with some $\delta\in[\,0,1\,)$, $b>0$, $q\ge 0$. Then,*

$$g_n(t) \le q\sum_{k=0}^{n-1} q_k\, t^{k(1-\delta)} + q_n\, t^{n(1-\delta)} \sup_{s\in[\,0,T\,]} g_0(s)$$

*with*

$$q_0 = 1,\ q_1 = \tfrac{b}{1-\delta},\ q_k = \tfrac{c^k(b,\delta)}{\Gamma(k(1-\delta)+1)} \text{ for } k>1,$$

*where $\Gamma$ is the Gamma-function defined by*

$$\Gamma(t) := \int_0^\infty x^{t-1}e^{-x}\,dx,\ t>0.$$

*Furthermore, one has the following summability property*

$$\sum_{k=0}^{\infty} q_k\, T^{k(1-\delta)} < \infty,$$

*and in particular*

$$\lim_{k\to\infty} q_k = 0.$$

**Remark 2.7.3:** (cf. [76], Remark 3.2.5, p. 59)
*In the special case of $g_n = g$ for all $n\in\mathbb{N}$ with a bounded $g\colon \mathbb{R}_+\to\mathbb{R}_+$, 2.6.3 implies*

$$\begin{aligned} g(t) \le\ & q\lim_{n\to\infty}\left(\sum_{k=0}^{n-1} q_k\, T^{k(1-\delta)}\right) + \left(\lim_{n\to\infty} q_n\, T^{n(1-\delta)}\right)\sup_{r\in[\,0,T\,]} g(r) \\ =\ & q\sum_{k=0}^{\infty} q_k T^{k(1-\delta)} =: q\,c(T,b,\delta) \end{aligned}$$

*with a proper constant $c(T,b,\delta)>0$.*
*In particular, the case $\delta=0$ here corresponds to the usual Gronwall's lemma widely used in the literature on SDEs.*

# Chapter 3

# Introduction to Stochastic Analysis in weighted $L^2$-spaces

In this chapter, we concentrate on the case the weighted Lebesgue spaces $L^{2\nu}_{\rho}(\Theta)$, $\nu \geq 1$, as underlying Banach spaces for equation (1.1) and equation (1.2). The weight $\mu_\rho$ is the same as in the Introduction (cf. equation (1.3), p. 7 there). We start with elements of functional analysis in these spaces, namely we introduce some conditions on almost strong evolution operators and Nemitskii operators in $L^{2\nu}_{\rho}(\Theta)$.
The main issue of this chapter is to define Bochner integrals and stochastic convolutions w.r.t. Wiener processes in $L^2_\rho(\Theta)$ resp. $L^{2\nu}_{\rho}(\Theta)$ needed for the existence of mild solutions to our basic equations (1.1) and (1.2).
We stress that in this chapter we have time-continuity of the Bochner integrals and the stochastic convolutions w.r.t. Wiener processes not only in the pathwise sense (a result, which is already well-known) but also in the meansquare sense, i.e. in the spaces

$$L^q(\Omega; L^2_\rho(\Theta)) := \{f : \Omega \to L^2_\rho(\Theta) \mid \int\limits_{\Omega} ||f(\omega)||^q_{L^2_\rho} P(d\omega) < \infty\}$$

resp.

$$L^{2\nu}(\Omega; L^{2\nu}_\rho(\Theta)) := \{f : \Omega \to L^{2\nu}_\rho(\Theta) \mid \int\limits_{\Omega} ||f(\omega)||^{2\nu}_{L^{2\nu}_\rho} P(d\omega) < \infty\}$$

(for the precise conditions on $q \geq 2$ and $\nu \geq 1$ see Sections 3.3 and 3.4 below) .

Recall that, for a given Borel domain $\Theta \subset \mathbb{R}^d$, we assume $\rho$ to be such that $\mu_\rho(\Theta) < \infty$. Respectively, we consider the two basic cases:

- $\rho > d$ for unbounded $\Theta$ and
- $\rho = 0$ for bounded $\Theta$.

Under these assumptions, our results hold true for arbitrary $\Theta \in \mathcal{B}(\mathbb{R}^d)$. Thus, to shorten notation, we write $L^2_\rho$ resp. $L^{2\nu}_\rho$ instead of $L^2_\rho(\Theta)$ resp. $L^{2\nu}_\rho(\Theta)$.

Let us briefly describe the content of this chapter.
First, in Section 3.1 we take a closer look at the Banach spaces $L^{2\nu}_\rho$ and impose conditions on the evolution operators $U(t,s)$, $0 \leq s \leq t \leq T$, in $L^2_\rho$ (see **(A0)**–**(A8)** there). Most of the conditions are taken from [76], but there are some additional conditions caused by the jump property of the noise in equation (1.1) resp. equation (1.2) (see Remark 3.1.1 below). These conditions later yield the well-definedness and regularity properties of the stochastic convolutions w.r.t. $Q$-Wiener processes and compensated Poisson random measures.
More precisely, we need these assumptions on the evolution operator in order to overcome the problem that the multiplication operators $\mathcal{M}_\varphi$ corresponding to $L^2_\rho$-valued functions $\varphi$ are in general no Hilbert-Schmidt operators. By these conditions (cf. e.g. **(A2)** in Section 3.1 below) there is a constant $\xi \in [\,0,1\,)$ associated to the evolution operator describing singularity behaviour allowed for $||U(t,s)||_{\mathcal{L}_2(L^2,L^2_\rho)}$ at the diagonal $t = s$, which plays a crucial role for the regularity properties of the stochastic convolution w.r.t. $Q$-Wiener processes and compensated Poisson random measures.
In Section 3.2 we take a closer look at the Nemitskii-type operators and recall some notation from [76].
After these preparations, in Section 3.3 we consider the well-definedness, moment estimates and regularity properties of Bochner integrals both in $L^2_\rho$ and $L^{2\nu}_\rho$. The integrands will be convolutions of an evolution operator and a predictable process $(\varphi(t))_{t\in[\,0,T\,]} \subset L^2_\rho$ resp. $\subset L^{2\nu}_\rho$ for some fixed $T > 0$.
Finally, in Section 3.4 we consider the well-definedness, moment estimates and regularity properties of stochastic convolutions w.r.t. Wiener processes. A part of these results has already been proven by Manthey and Zausinger in [76], but for convenience of the reader we will present all necessary details.
A crucial fact for the theory of SPDEs in $L^{2\nu}_\rho$ is that there exists a special orthonormal basis $(e_n)_{n\in\mathbb{N}} \subset L^2$, which additionally is uniformly bounded in the supremum norm (see Section 3.1). Thus, we have

$$\sup_{n\in\mathbb{N}} ||e_n||_\infty < \infty. \tag{3.1}$$

For the $Q$-Wiener process $W$, we consider the following two cases:

- **Nuclear case:** $W$ is a $Q$-Wiener process with the covariance operator $Q \in \mathcal{T}^+(L^2)$, i.e. $Q$ is a nonnegative, symmetric operator with finite trace.
  Furthermore, the elements of the above orthonormal basis $(e_n)_{n\in\mathbb{N}}$ constitute a complete system of eigenvectors of the operator $Q$, i.e. $Qe_n = a_n e_n$ with $a_n \geq 0$, (3.1) and
  $$tr\, Q = \sum_{n\in\mathbb{N}} a_n < \infty.$$
  The $Q$-Wiener process is represented by the convergent series in $L^2$
  $$W(t) = \sum_{n\in\mathbb{N}} \sqrt{a_n} w_n(t) e_n,\ t \geq 0,$$
  where $(w_n)_{n\in\mathbb{N}}$ are independent Brownian motions in $\mathbb{R}$.

- **Cylindrical case:** $W$ is a cylindrical **I**-Wiener process.
  In other words, it obeys the coordinate representation
  $$W(t) = \sum_{n\in\mathbb{N}} w_n(t) e_n,\ t \geq 0,$$
  with some (not necessarily uniformly bounded) basis $(e_n)_{n\in\mathbb{N}}$ in $L^2$ and a system $(w_n)_{n\in\mathbb{N}}$ of independent scalar Brownian motions.

**Remark 3.1:** *Another two possible cases, which however will be briefly touched upon in this manuscript, are the following:*

- ***General nuclear case:*** *The covariance operator $Q$ is of trace class, i.e. $Q \in \mathcal{T}^+(L^2)$, but it does not need to posess a complete system of eigenvectors $(e_n)_{n\in\mathbb{N}}$ that is uniformly bounded in the supremum norm, i.e. (3.1) possibly fails.*

- ***General cylindrical case:*** *$Q$ is a nonnegative, symmetric bounded operator in $L^2$, but it need not to have finite trace, i.e. $Q \notin \mathcal{T}^+(L^2)$.*

*Note that the general nuclear case typically occurs in the Wiener term of the Lévy-Itô decomposition (2.16). The general cylindrical case is not relevant for our work. The nuclear and the cylindrical case have been treated in [76].*

## 3.1 Some facts on the spaces $L^{2\nu}_\rho$ and conditions for evolution operators in $L^2_\rho$

### 3.1.1 The spaces $L^{2\nu}_\rho$

Consider $\mathbb{R}^d$, $d \in \mathbb{N}$, with Euclidean norm $|\cdot|$, Borel $\sigma$-algebra $\mathcal{B}(\mathbb{R}^d)$ and Lebesgue measure $d\theta$. Let $\alpha$: $\mathbb{R}^d \to [\,1, \infty\,)$ be a weight function given by

$$\alpha(\theta) = (1 + |\theta|^2)^{\frac{1}{2}},\ \theta \in \mathbb{R}^d.$$

For $\rho \in \mathbb{N} \cup \{0\}$, let us define a measure $\mu_\rho$ on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)$ by

$$\mu_\rho(d\theta) := \alpha^{-\rho}(\theta)\, d\theta.$$

Note that, by definition, we have $\mu_0(d\theta) = d\theta$.

Let us fix some (possibly unbounded) Borel subset $\Theta$ of $\mathbb{R}^d$.

As already said in the beginning of the chapter, we choose $\rho \in \mathbb{N} \cup \{0\}$ in such a way that $\mu_\rho(\Theta) < \infty$. This allows us to consider the two cases of $\Theta = \mathbb{R}^d$ and $\Theta \subset \mathbb{R}^d$ bounded simultaneously (in a similar way as Manthey and Zausinger did in [76]).

Note that there is the following relation between the Borel $\sigma$-algebras:

$$\mathcal{B}(\Theta) = \mathcal{B}(\mathbb{R}^d) \cap \Theta.$$

For $\nu \geq 1$ and $\rho \in \mathbb{N} \cup \{0\}$, by $L^{2\nu}_\rho(\Theta)$ we denote the set of all Borel-measurable mappings $\varphi :\ \Theta \to \mathbb{R}$ such that

$$\int\limits_\Theta |\varphi|^{2\nu}(\theta)\mu_\rho(d\theta) < \infty.$$

$L^{2\nu}_\rho(\Theta)$ is a Banach space with norm

$$||\varphi||_{L^{2\nu}_\rho} := \left(\int\limits_\Theta |\varphi|^{2\nu}(\theta)\mu_\rho(d\theta)\right)^{\frac{1}{2\nu}}.$$

In the special case $\nu = 1$, we get a Hilbert space $L^2_\rho(\Theta)$ with inner product

$$< \varphi_1, \varphi_2 >_{L^2_\rho} = \int\limits_\Theta \varphi_1(\theta)\varphi_2(\theta)\mu_\rho(d\theta)$$

and norm

$$||\varphi||_{L^2_\rho} = \left( \int\limits_\Theta \varphi^2(\theta)\mu_\rho(d\theta) \right)^{\frac{1}{2}}.$$

Denoting by $L^p(\Theta)$ the usual $L^p$-space on $(\Theta, \mathcal{B}(\Theta))$, we obviously have $L^{2\nu}_0(\Theta) = L^{2\nu}(\Theta)$.

A crucial fact (pointed out in [76]) is the existence of an orthonormal basis $(e_n)_{n\in\mathbb{N}}$ of $L^2(\Theta)$ obeying (3.1).
Manthey and Zausinger prove the existence of such an orthonormal basis with the help of general arguments from [87] (cf. Section 2, p.41 in [76]). We will often use this fact in the following.

### 3.1.2 Conditions for evolution operators in $L^2_\rho$

Let $U = \{U(t,s) \,|\, (t,s) \in \mathbb{R}^2_+, 0 \le s \le t \le T\}$ be an almost strong evolution operator in the sense of Definition 2.2.1 with $B := L^2_\rho$ there.
Recall that each $U(t,s)$ is a closed linear operator in $L^2_\rho$, see item **(iv)** in Definition 2.2.1. The generator of $U$ is denoted by $(A(t))_{t\in[0,T]}$.
Depending on the problems under consideration, for the evolution operator $U$ in $L^2_\rho$ we introduce the following additional conditions:

**(A0)** The domain

$$\mathcal{D}(A) := \bigcap_{0\le t\le T} \mathcal{D}(A(t))$$

is **dense in** $L^2_\rho$.

**(A1)** The evolution operator $U$ on $L^2_\rho$ is **positivity preserving**, i.e., for all $\varphi \in L^2_\rho$ and $0 \le s \le t \le T$,

$$\varphi \ge 0 \Rightarrow U(t,s)\varphi \ge 0 \ (d\theta\text{-a.s.}).$$

**(A2)** For a given $\varphi \in L^2_\rho$, let us define the **multiplication operator**

$$L^2 \ni \psi \mapsto \mathcal{M}_\varphi(\psi) := \varphi\psi \in L^1_\rho.$$

We suppose that, for $0 \le s < t \le T$, there exists an **extension** of $U(t,s)$ to the domain

$$\mathcal{M} := \{h \in L^1_\rho \mid h = \mathcal{M}_\varphi(\psi), \varphi \in L^2_\rho, \psi \in L^2\}$$

(again denoted by $U(t,s)$) such that $U(t,s)\mathcal{M}_\varphi \in \mathcal{L}_2(L^2, L^2_\rho)$ for any $\varphi \in L^2_\rho$. Furthermore, there exist $\zeta \in [\,0,1\,)$ and $c(T) > 0$ such that

$$||U(t,s)\mathcal{M}_\varphi||^2_{\mathcal{L}_2(L^2,L^2_\rho)} \leq c(T)(t-s)^{-\zeta}||\varphi||^2_{L^2_\rho} \tag{3.2}$$

is fulfilled for any $\varphi \in L^2_\rho$ and $0 \leq s < t \leq T$.
Taking in particular $\varphi \equiv 1$, we get

$$||U(t,s)||^2_{\mathcal{L}_2(L^2,L^2_\rho)} \leq c(T)(t-s)^{-\zeta}||1||^2_{L^2_\rho},$$

i.e. $U(t,s)$: $L^2 \to L^2_\rho$ is Hilbert-Schmidt whenever $s < t$.

**(A3)** For a given $\nu \geq 1$, there exists a constant $c(\nu,T) > 0$ such that for any $\varphi \in L^{2\nu}_\rho$ and $0 \leq s \leq t \leq T$

$$(U(t,s)|\varphi|)^\nu \leq c(\nu,T)U(t,s)|\varphi|^\nu \ (d\theta\text{-a.s.}).$$

This implies that $U(t,s)|\varphi| \in L^{2\nu}_\rho$ and

$$||U(t,s)|\varphi|||^{2\nu}_{L^{2\nu}_\rho} \leq c(\nu,T,c(T))||\varphi||^{2\nu}_{L^{2\nu}_\rho}. \tag{3.3}$$

Furthermore, we assume that $U$ is strongly continuous in $L^{2\nu}_\rho$, i.e. the mapping $U(\cdot,\cdot)\varphi$: $\{(t,s)\,|\,0 \leq s \leq t \leq T\} \to L^{2\nu}_\rho$ is continuous for each $\varphi \in L^{2\nu}_\rho$.
Since $U$ is positivity preserving (cf. **(A1)**), a sufficient condition for the strong continuity in $L^{2\nu}_\rho$ is that $U(t,s)1 = 1$ for each $(t,s)$.
By **(A1)** and (3.3), we have $U(t,s) \in \mathcal{L}(L^{2\nu}_\rho)$ with

$$||U(t,s)||_{\mathcal{L}(L^{2\nu}_\rho)} \leq [c(\nu,T,c(T))]^{\frac{1}{2\nu}}. \tag{3.4}$$

**(A4)** For a given $\nu \geq 1$, there exist $\zeta \in [\,0,1\,)$ and a constant $c(\nu,T) > 0$ such that for each $L^{2\nu}_\rho$-valued predictable process $\varphi = (\varphi(t))_{t\in[\,0,T\,]}$

$$\mathbf{E}\int_\Theta \left[\int_0^t \sum_{n\in\mathbb{N}} [U(t,s)\mathcal{M}_{\varphi(s)}(Q^{\frac{1}{2}}e_n)]^2(\theta)\,ds\right]^\nu \mu_\rho(d\theta)$$
$$\leq c(\nu,T)\int_0^t (t-s)^{-\zeta}\mathbf{E}||\varphi(s)||^{2\nu}_{L^{2\nu}_\rho}\,ds.$$

In the nuclear case, i.e. if $Q \in \mathcal{T}^+(L^2)$, we have $\zeta = 0$, whereas in the cylindrical case, i.e. if $Q = \mathbf{I}$, we have $\zeta \in [\,0,1\,)$ as in **(A2)**. Here, $(e_n)_{n\in\mathbb{N}}$ denotes an orthonormal basis in $L^2$ consisting of the

eigenvectors of $Q$ and obeying (3.1).
The integral and the infinite sum in the left hand side are understood in the Bochner sense in $L^1_\rho$, see Remark 3.1.1 **(ii)**.

Note that **(A2)** is equivalent to **(A4)** with $Q = \mathbf{I}$ and $\nu = 1$.
At first sight, condition **(A4)** seems to be not transparent enough. Below we formulate the next condition **(A5)**, which generalizes **(A2)** to all $\nu \geq 1$ and readily implies **(A4)**, see Remark 3.1.1 **(iv)**.

**(A5)** For a given $\nu \geq 1$ and any $\varphi \in L^{2\nu}_\rho$, let $\mathcal{M}_\varphi$ be the multiplication operator as defined in **(A2)**. We suppose that, for $0 \leq s < t \leq T$, the operator $U(t,s)$ extends to the domain

$$\mathcal{M}_\nu := \{h \in L^\nu_\rho \mid h = \mathcal{M}_\varphi(\psi), \varphi \in L^{2\nu}_\rho, \psi \in L^2\}.$$

Furthermore, for any $\varphi \in L^{2\nu}_\rho$ we have $U(t,s)\mathcal{M}_\varphi \in \mathcal{L}_2(L^2, L^{2\nu}_\rho)$, and there exist $\zeta \in [\,0,1\,)$ and $c(T) > 0$ such that

$$||U(t,s)\mathcal{M}_\varphi||^2_{\mathcal{L}_2(L^2,L^{2\nu}_\rho)} \leq c(\nu,T)(t-s)^{-\zeta}||\varphi||^2_{L^{2\nu}_\rho}. \tag{3.5}$$

In some cases it will be enough to assume the following version of **(A5)**:

**(A5)*** For a given $\nu \geq 1$, the estimate (3.5) in **(A5)** holds with the usual operator norm in $\mathcal{L}(L^2, L^2_\rho)$, i.e. there exist $\zeta \in [\,0,1\,)$ and $c(T) > 0$ such that

$$||U(t,s)\mathcal{M}_\varphi\psi||^2_{L^{2\nu}_\rho} \leq c(\nu,T)(t-s)^{-\zeta}||\psi||^2_{L^2}||\varphi||^2_{L^{2\nu}_\rho}, \tag{3.6}$$

for any $\varphi \in L^{2\nu}_\rho$, $\psi \in L^2$ snd $0 \leq s < t \leq T$.

**(A6)** There exists a family of bounded operators $((A_N(t))_{t\in[\,0,T\,]})_{N\in\mathbb{N}} \subset \mathcal{L}(L^2_\rho)$ with the following properties:

**(i)** Denoting the operator norm in $L^2_\rho$ by $||\cdot||$, we have

$$\sup_{t\in[\,0,T\,]} ||A_N(t)|| \leq c(N),\ N \in \mathbb{N}.$$

**(ii)** For each $N \in \mathbb{N}$, the family $(A_N(t))_{t\in[\,0,T\,]}$ generates an almost strong evolution operator $U_N$ in $L^2_\rho$, which is positivity preserving and fulfills

$$\sup_{0\leq s\leq t\leq T} ||(U_N(t,s) - U(t,s))\varphi||^2_{L^2_\rho} \to 0,\ N \to \infty,$$

for any $\varphi \in L^2_\rho$. Furthermore, there is a uniform bound

$$K(T) := \sup_{N\in\mathbb{N}} \sup_{0\le s\le t\le T} ||U_N(t,s)|| < \infty.$$

**(A7)** $U$ is **pseudo contractive in** $L^2_\rho$, i.e. there is a nonnegative constant $\beta$ such that

$$||U(t,s)||_{\mathcal{L}(L^2_\rho)} \le e^{\beta(t-s)},\ 0 \le s \le t \le T.$$

**(A8)** The family $(A_N(t))_{t\in[0,T]}$ from **(A6)** is such that, for any $N \in \mathbb{N}$, we have $(A_N(t))_{t\in[0,T]} \subset \mathcal{L}(W^{m,2}(\Theta))$ and, for the corresponding evolution operators $U_N$,

$$||U_N(t,s)||_{\mathcal{L}(W^{m,2})} < \bar{c}_N(T),\ 0 \le s \le t \le T.$$

Here, for a given $m > \frac{d}{2}$, $W^{m,2}(\Theta)$ is the Sobolev space of order $m$ defined in Appendix A

**Remark 3.1.2.1:** *Conditions **(A0)**–**(A4)** and **(A6)** have been introduced in the paper [76], dealing with SPDEs driven by a Wiener noise.*
*The rest of the conditions is new and appears first in the context of Poisson and Lévy integration.*
*Let us comment in more detail on the above assumptions.*

***(i)*** *Conditions **(A0)**–**(A2)** are needed to study the stochastic convolution w.r.t. Q-Wiener processes in $L^2_\rho$. Note that we make use of **(A2)** only in the case $Q = \boldsymbol{I}$. In the case of a nuclear Wiener process, it suffices to assume just **(A0)** and **(A1)** (cf. the discussion in [76], Section 2).*
*For the corresponding $Q \in \mathcal{T}^+(L^2)$ we always have the following modification of **(A2)** with $\zeta = 0$*

$$\text{(3.7)}\quad \begin{aligned} ||U(t,s)\mathcal{M}_\varphi||^2_{\mathcal{L}_2(Q^{\frac{1}{2}}L^2,L^2_\rho)} &\le ||U(t,s)||^2_{\mathcal{L}(L^2_\rho)}||\mathcal{M}_\varphi||^2_{\mathcal{L}_2(Q^{\frac{1}{2}}L^2,L^2_\rho)} \\ &\le c(T)\, tr\, Q \left(\sup_{n\in\mathbb{N}} ||e_n||^2_\infty\right) ||\varphi||^2_{L^2_\rho}. \end{aligned}$$

***(ii)*** *Conditions **(A3)** and **(A4)** are needed to show that taking the stochastic convolution w.r.t. the Wiener process preserves the space of $L^{2\nu}_\rho$-valued predictable processes with $\nu \ge 1$ (see Proposition 3.4.3 below).*
*In **(A4)**, we understand*

$$\int_0^t \sum_{n\in\mathbb{N}} (U(t,s)\mathcal{M}_{\varphi(s)}Q^{\frac{1}{2}}e_n)^2\, ds$$

*as a Bochner integral in $L^1_\rho$. Concerning well-definedness of the above integral, see Section 3.4, proof of Lemma 3.4.3, Step 1, below.*

***(iii)*** *In the nuclear case, we get* ***(A4)*** *with $\zeta = 0$ directly from* ***(A1)*** *and* ***(A3)***\(*see also Remark 2.3* ***(ii)*** *in [76]).*

***(iv)*** *In the general nuclear case* ***(A5)**** *implies* ***(A4)*** *with the same $\zeta \in [0,1)$.*
*Indeed, considering* ***any*** *orthonormal basis $(e_n)_{n\in\mathbb{N}}$ of $L^2$ consisting of eigenvectors of $Q$, assuming* ***(A5)**** *we can estimate the left hand side of* ***(A4)*** *as follows*

$$
\begin{aligned}
(3.8)\ &\mathbf{E}\int_\Theta \left[\int_0^t \sum_{n\in\mathbb{N}} [U(t,s)\mathcal{M}_{\varphi(s)}Q^{\frac12}e_n]^2(\theta)\,ds\right]^\nu \mu_\rho(d\theta)\\
&\le \mathbf{E}\left[\int_0^t \sum_{n\in\mathbb{N}} \left[\int_\Theta [U(t,s)\mathcal{M}_{\varphi(s)}Q^{\frac12}e_n]^{2\nu}(\theta)\,\mu_\rho(d\theta)\right]^{\frac{1}{\nu}} ds\right]^\nu\\
&= \mathbf{E}\left[\int_0^t \sum_{n\in\mathbb{N}} ||U(t,s)\mathcal{M}_{\varphi(s)}Q^{\frac12}e_n||^2_{L^{2\nu}_\rho}\,ds\right]^\nu\\
&= \mathbf{E}\left[\int_0^t ||U(t,s)\mathcal{M}_{\varphi(s)}Q^{\frac12}||^2_{\mathcal{L}_2(L^2,L^{2\nu}_\rho)}\,ds\right]^\nu\\
&\le ||Q^{\frac12}||^{2\nu}_{\mathcal{L}_2(L^2)}\mathbf{E}\left[\int_0^t ||U(t,s)\mathcal{M}_{\varphi(s)}||^2_{\mathcal{L}(L^2,L^{2\nu}_\rho)}\,ds\right]^\nu\\
&\le (tr\,Q)^\nu c(\nu,T)\mathbf{E}\left[\int_0^t (t-s)^{-\zeta}||\varphi(s)||^2_{L^{2\nu}_\rho}\,ds\right]^\nu\\
&= (tr\,Q)^\nu c(\nu,T,c(T))\mathbf{E}\left[\int_0^t (t-s)^{-\frac{\zeta(\nu-1)}{\nu}}(t-s)^{-\frac{\zeta}{\nu}}||\varphi(s)||^2_{L^{2\nu}_\rho}\,ds\right]^\nu\\
&\le (tr\,Q)^\nu c(\nu,T)\left(\int_0^T s^{-\zeta}\,ds\right)^{\nu-1}\mathbf{E}\int_0^t (t-s)^{-\zeta}||\varphi(s)||^{2\nu}_{L^{2\nu}_\rho}\,ds.
\end{aligned}
$$

*Here, in the first step, we used Minkowski's inequality (4.25). To be rigorous, one has to take here $\mathcal{B}([0,T])\otimes\mathcal{F}\otimes\mathcal{B}(\Theta)$-measurable realizations of $U(t,s)\mathcal{M}_{\varphi(s)}Q^{\frac12}e_n$ (see Section 3.4).*

***(v)*** *Respectively in the cylindrical case, i.e. $Q = \boldsymbol{I}$,* ***(A4)*** *is always implied by* ***(A5)***. *This is obvious from the following modification of (3.8) (with $Q = \mathbf{I}$)*

$$
\begin{aligned}
&\mathbf{E}\int_\Theta \left[\int_0^t \sum_{n\in\mathbb{N}} [U(t,s)\mathcal{M}_{\varphi(s)}e_n]^2(\theta)\,ds\right]^\nu \mu_\rho(d\theta)\\
&\le \mathbf{E}\left[\int_0^t ||U(t,s)\mathcal{M}_{\varphi(s)}||^2_{\mathcal{L}_2(L^2,L^{2\nu}_\rho)}\,ds\right]^\nu
\end{aligned}
$$

$$\leq c(\nu,T)\mathbf{E}\left[\int_0^t (t-s)^{-\zeta}||\varphi(s)||^2_{L^{2\nu}_\rho}\, ds\right]^\nu$$
$$\leq c(\nu,T)\left(\int_0^T s^{-\zeta}\, ds\right)^{\nu-1}\mathbf{E}\int_0^t (t-s)^{-\zeta}||\varphi(s)||^{2\nu}_{L^{2\nu}_\rho}\, ds.$$

***(vi) (A6)*** *is needed to prove the comparison result by the approximation method of Manthey and Zausinger (see proof of Theorem 3.3.1 in [76]).*

***(vii) (A7)*** *is the generalization of the contraction property* $||U(t,s)||_{\mathcal{L}(L^2_\rho)} \leq 1$ *resp.* $||U(t,s)||_{\mathcal{L}(L^2)} \leq 1$, $0 \leq s \leq t \leq T$. *In the case of a semigroup* $U(t,s) := e^{(t-s)A}$, $0 \leq s \leq t < \infty$, *a sufficient condition of pseudo-contraction is that* $(A + \beta\mathbf{I})$ *is a nonnegative self-adjoint operator in* $L^2_\rho$ *resp.* $L^2$. *On the other hand, by the Hille-Yosida theorem, any* $\mathcal{C}_0$-*semigroup* $e^{-tA}$, $t \geq 0$, *obeys the bound*

$$||e^{tA}||_{\mathcal{L}(L^2_\rho)} \leq Ce^{\beta t},\ t \geq 0,$$

*with proper constants* $C$, $\beta \in \mathbb{R}_+$. *In the case of pseudo-contractivity, we have* $C = 1$ *in the later estimate.*
*A big class of elliptic differential operators satisfying* ***(A7)*** *in* $L^2$ *will be constructed in Appendix D.*

The above conditions **(A0)**–**(A8)** are satisfied for a large class of elliptic differential operators $A(t)$ in $L^2_\rho$, see Appendix D.
The constant $\zeta$ depends on the dimension of the underlying space $\mathbb{R}^d$ and on the order of the differential operators $A(t)$.
Depending on the problems under consideration, we will assume that some or even all of the above conditions **(A0)**–**(A8)** are satisfied.

## 3.2 Nemitskii operators

As already discussed in the Introduction, our coefficients $F$ and $E$ in (1.1) resp. (1.2) will be nonlinear operators of **Nemitskii-type**.
Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\lambda$: $[\,0,T\,] \times \Omega \times \mathbb{R} \to \mathbb{R}$ be a measurable mapping.
Pointwise, for $\varphi \in L^2_\rho$ we define a Nemitskii-type-operator $\Lambda$ by

**(NEM)** $$\Lambda(t,\omega,\varphi)(\theta) := \lambda(t,\omega,\varphi(\theta)),\ \theta \in \Theta\ ,\ (t,\omega) \in [\,0,T\,] \times \Omega.$$

Below, we discuss the conditions needed to make $\Lambda$ a mapping in $L^2_\rho$.
We recall some standard notation from [76] to describe the regularity prop-

erties of $\lambda$: $[0,T] \times \Omega \times \mathbb{R} \to \mathbb{R}$:

**(LC)** (**Lipschitz continuity**) There is an $L(T) > 0$ such that

$$|\lambda(t,\omega,u) - \lambda(t,\omega,v)| \leq L(T)|u-v|,$$

for all $(t,\omega) \in [0,T] \times \Omega$ and $u,v \in \mathbb{R} \times \mathbb{R}$.

**(LB)** (**Local boundedness**) There is an $L(T) > 0$ such that

$$|\lambda(t,\omega,0)| \leq L(T),$$

for all $(t,\omega) \in [0,T] \times \Omega$.

**(PG)** (**Polynomial growth**) There exist $\nu \in \mathbb{N}$ and $L(T) > 0$ such that

$$|\lambda(t,\omega,u)| \leq L(T)(1+|u|^\nu)$$

for all $(t,\omega) \in [0,T] \times \Omega$ and $u \in \mathbb{R}$.

**(LG)** (**One-sided linear growth**) There exists $L(T) > 0$ such that

$$\begin{aligned} \lambda(t,\omega,u) &\geq -L(T)(1-u) \text{ if } u \leq 0, \\ \lambda(t,\omega,u) &\leq L(T)(1+u) \text{ if } u \geq 0, \end{aligned}$$

for all $(t,\omega) \in [0,T] \times \Omega$.

**Remark 3.2.1:** *Obviously, each $\lambda$ fulfilling **(LC)** and **(LB)** also fulfills **(PG)** with exponent $\nu = 1$. It is easy to see that **(LG)** is equivalent to claiming that*

$$\lambda(t,\omega,u)u \leq L(T)(1+u^2),\ u \in \mathbb{R}.$$

*In particular, the class of functions with one-sided linear growth includes all **semi-dissipative** functions, i.e. those $\lambda$, which obey*

$$(\lambda(t,\omega,u) - \lambda(t,\omega,v))(u-v) \leq L(T)(u-v)^2$$

*with some $c(T) \geq 0$ that is uniform for all $(t,\omega) \in [0,T] \times \Omega$ and $u,v \in \mathbb{R}$.*

The following simple lemma is crucial for our further considerations.

**Lemma 3.2.2:** *Suppose $\lambda$ obeys **(PG)** with exponent $\nu \in \mathbb{N}$.*
*Then, the corresponding Nemitskii-operator $\Lambda$ defined by **(NEM)** maps $L^{2\nu}_\rho$ into $L^2_\rho$. Furthermore, $\Lambda$ is locally bounded.*

**Proof:** By direct calculations

$$\begin{aligned}
\int_\Theta (\Lambda(t,\omega,\varphi))^2(\theta)\,\mu_\rho(d\theta) &= \int_\Theta (\lambda(t,\omega,\varphi(\theta)))^2\,\mu_\rho(d\theta)\\
&\leq \int_\Theta (L(T)(1+|\varphi(\theta)|^\nu))^2\,\mu_\rho(d\theta)\\
&\leq 2L(T)^2\mu_\rho(\Theta) + 2L(T)^2\int_\Theta (|\varphi(\theta)|^\nu)^2\,\mu_\rho(d\theta)\\
&= 2L(T)^2\mu_\rho(\Theta) + 2L(T)^2||\varphi||^{2\nu}_{L^{2\nu}_\rho}\\
&< \infty.
\end{aligned}$$

Here, we used **(PG)** with exponent $\nu$ in the second step and the assumption on $\rho$ that

$$\mu_\rho(\Theta) = \int_\Theta \mu_\rho(d\theta) < \infty$$

in the last step, respectively.
From the above calculation, it is obvious that $\Lambda$ maps $L^{2\nu}_\rho$ into $L^2_\rho$ and is locally bounded there, which completes the proof. ■

In Chapters 5–7, given measurable $e$, $f$: $[0,T]\times\Omega\times\mathbb{R}\to\mathbb{R}$, the drifts $E$ and $F$ in (1.2) and (1.1) will be defined through $e$ and $f$ by **(NEM)**.
Furthermore, $\Sigma$ and $\Gamma$ in the multiplication operators $\mathcal{M}_\Sigma$ and $\mathcal{M}_\Gamma$ in (1.2) and (1.1) will be defined through measurable $\sigma$, $\gamma$: $[0,T]\times\Omega\times\mathbb{R}\to\mathbb{R}$ by **(NEM)**.

## 3.3 Bochner integrals depending on parameters

In this section we consider the Bochner integrals

$$I_\varphi(t) := \int_0^t U(t,s)\varphi(s)\,ds,\; t\in[0,T], \tag{3.9}$$

and

$$I_{\varphi m}(t) := \int_0^t U(t,s)\mathcal{M}_{\varphi(s)}m\,ds,\; t\in[0,T], \tag{3.10}$$

in $L^2_\rho$ resp. $L^{2\nu}_\rho$, where $\varphi = (\varphi(t))_{t\in[0,T]}$ is a predictable process in $L^2_\rho$ resp. $L^{2\nu}_\rho$ and $m\in L^2$.
First, we consider the case $\nu=1$, i.e. $L^2_\rho$, which only requires the assumptions **(A0)**–**(A2)**. The above integrals will be defined pathwise, i.e. for $P$-almost all $\omega\in\Omega$. Especially, we will be interested in the meansquare and time-continuity properties of the Bochner convolution processes (3.9) and

(3.10). It is convinient to fix the concrete representation for the integrand mappings as

$$[0,T] \times \Omega \ni (s,\omega) \mapsto \mathbf{1}_{[0,t)}(s)U(t,s)\varphi(s) \in L^2_\rho$$

resp.

$$[0,T] \times \Omega \ni (s,\omega) \mapsto \mathbf{1}_{[0,t)}(s)U(t,s)\mathcal{M}_{\varphi(s)}m \in L^2_\rho.$$

For each fixed $t \in [0,T]$, the integral is well-defined $P$-a.s. if

$$\int_0^t ||U(t,s)\varphi(s)||_{L^2_\rho}\, ds < \infty, \ \int_0^t ||U(t,s)\mathcal{M}_{\varphi(s)}m||_{L^2_\rho}\, ds < \infty.$$

To proceed, we need the following general measurability result, which will also be used later for constructing Wiener and Poisson stochastic convolutions.

**Lemma 3.3.1:** *For any fixed $t \in [0,T]$ and any $\mathcal{P}_T$-measurable process $(\varphi(s))_{s\in[0,T]} \in L^2_\rho$, the mapping*

$$[0,T] \times \Omega \ni (s,\omega) \mapsto \mathbf{1}_{[0,t)}(s)U(t,s)\varphi(s) \in L^2_\rho \tag{3.11}$$

*is $\mathcal{P}_T/\mathcal{B}(L^2_\rho)$-measurable, i.e. predictable.*

**Proof:** We extend the method used in [62] for proving Lemma 3.5 there. The proof involves the following two steps:

- **(i)** We show $\mathcal{P}_T$-measurability of (3.11) for simple (elementary) predictable processes $(\varphi(s))_{s\in[0,T]} \in L^2_\rho$ having the form

$$\varphi(s) := \sum_{k=1}^{K} \varphi_k \mathbf{1}_{A_k}(s),\ s \in [0,T], \tag{3.12}$$

  where $\varphi_k \in L^2_\rho$, $A_k \in \mathcal{P}_T$, $1 \le k \le K \in \mathbb{N}$.

- **(ii)** We show $\mathcal{P}_T$-measurability of (3.11) for general predictable processes $(\varphi(s))_{s\in[0,T]} \subset L^2_\rho$ by approximating them by simple processes.

Concerning **(i)**, note that for any simple predictable process of form (3.12) and any $B \in \mathcal{B}(L^2_\rho)$

$$(\mathbf{1}_{[0,t)}(s)U(t,s)\varphi(s))^{-1}(B) = \bigcup_{k=1}^{K} \underbrace{\underbrace{(\{s \in [0,t) \,|\, \mathbf{1}_{[0,t)}(s)U(t,s)\varphi_k \in B\})}_{\in\mathcal{B}([0,t))} \times \Omega \bigcap A_k}_{\in\mathcal{P}_T},$$

because of the strong continuity of $s \mapsto U(t,s)$ in $L^2_\rho$.

Note that for any predictable process $(\varphi(s))_{s\in[0,T]}$ there exists a sequence of simple predictable processes of form (3.12) such that in $L^2_\rho$

$$\varphi_N(s,\omega) \to \varphi(s,\omega), \text{ as } N \to \infty$$

for all $(s,\omega) \in [0,T] \times \Omega$ (see e.g. Lemma A.4 in [62]). Since $U(t,s) \in \mathcal{L}(L^2_\rho)$, we have $U(t,s)\varphi_N(s) \to U(t,s)\varphi(s)$ in $L^2_\rho$ as $N \to \infty$.
Thus, $\mathbf{1}_{[0,t)}(s)U(t,s)\varphi(s)$ is predictable as a pointwise limit of predictable processes, which shows **(ii)**. ■

The main results of this section are Propositions 3.3.2/3.3.3 resp. Propositions 3.3.4/3.3.5, which state the time-continuity of the Bochner integrals (3.9) and (3.10) both pathwise and in $L^q(\Omega; L^2_\rho)$ resp. $L^{2\nu}(\Omega; L^{2\nu}_\rho)$.
Actually, instead of the predictability of $\varphi$ it would be enough to assume $\mathcal{B}([0,T]) \otimes \mathcal{F}$-measurability.

**Proposition 3.3.2:** *Let $U$ be an almost strong evolution operator in the sense of Definition 2.2.1. Let $\varphi = (\varphi(t))_{t\in[0,T]}$ be an $L^2_\rho$-valued predictable process obeying*

$$\int_0^T \mathbf{E}||\varphi(t)||^q_{L^2_\rho}\, dt < \infty \tag{3.13}$$

*for some $q \geq 2$. Then, for each $t \in [0,T]$, the convolution $I_\varphi(t)$ is well-defined in $L^2_\rho$. Furthermore,*

$$\mathbf{E}||I_\varphi(t)||^q_{L^2_\rho}\, ds \leq c(q,c(T)) \int_0^t \mathbf{E}||\varphi(s)||^q_{L^2_\rho}\, ds \tag{3.14}$$

*and hence*

$$\sup_{t\in[0,T]} \mathbf{E}||I_\varphi(t)||^q_{L^2_\rho} < \infty. \tag{3.15}$$

*The process $[0,T] \ni t \mapsto I_\varphi(t) \in L^2_\rho$ is pathwise continuous, as well as continuous in $L^q(\Omega; L^2_\rho)$.*

**Proof:** The integral (3.9) will be defined, for $P$-almost all $\omega \in \Omega$, as a Bochner integral in $L^2_\rho$.
For each $t \in [0,T]$, the integrand function
$[0,T] \ni s \mapsto \mathbf{1}_{[0,t)}(s)U(t,s)\varphi(s) \in L^2_\rho$ is $\mathcal{P}_T$-measurable by Lemma 3.3.1.
By (3.13) we have

$$\mathbf{E}\int_0^T ||\varphi(t)||_{L^2_\rho}\,dt < \infty,$$

and hence there exists a subset $\Omega_0 \subset \Omega$ of full $P$-measure such that

$$\int_0^t ||U(t,s)\varphi(s)||_{L^2_\rho}\,ds \le c(T)\int_0^t ||\varphi(s)||_{L^2_\rho}\,ds < \infty \tag{3.16}$$

for all $t \in [\,0,T\,]$ and $\omega \in \Omega_0$. Therefore, $I_\varphi(t)$ is well-defined for all $t \in [\,0,T\,]$ and $\omega \in \Omega_0$.
Let us assume that $q \ge 2$. By Bochner's and Hölder's inequalities we get the following chain of estimates

$$\begin{aligned}\mathbf{E}\left|\left|\int_0^t U(t,s)\varphi(s)\,ds\right|\right|^q_{L^2_\rho} &\le \mathbf{E}\left[\int_0^t ||U(t,s)\varphi(s)||_{L^2_\rho}\,ds\right]^q \\ &\le c^q(T)T^{q-1}\int_0^t \mathbf{E}||\varphi(s)||^q_{L^2_\rho}\,ds,\end{aligned}$$

which is just (3.14), (3.15).
To prove the continuity result, let us consider $0 \le r \le t \le T$ and $\omega \in \Omega_0$. In this case, we have

$$\begin{aligned}||I_\varphi(t) - I_\varphi(r)||_{L^2_\rho} \le &\int_0^r ||[U(t,s)-U(r,s)]\varphi(s)||_{L^2_\rho}\,ds \\ &+\int_r^t ||U(t,s)\varphi(s)||_{L^2_\rho}\,ds.\end{aligned} \tag{3.17}$$

By Lebesgue's dominated convergence theorem, the first integral on the right hand side tends to 0 as $t \downarrow r$ resp. $r \uparrow t$ due to the strong continuity of $U(t,s)$ and the uniform bound

$$\sup_{0\le r\le t\le T} ||[U(t,s)-U(r,s)]\varphi(s)||_{L^2_\rho} \le 2c(T)||\varphi(s)||_{L^2_\rho},$$

whereby by (3.16)

$$\int_0^T ||\varphi(s)||_{L^2_\rho}\,ds < \infty \text{ for all } \omega \in \Omega_0.$$

Since the second integral on the right hand side in (3.17) obviously tends to 0 as $r \uparrow t$ resp. $t \downarrow r$ due to the uniform bound

$$\sup_{0\le s\le t\le T} ||U(t,s)\varphi(s)||_{L^2_\rho} \le c(T)||\varphi(s)||_{L^2_\rho},$$

we get the continuity of the map $t \mapsto I_\varphi(t)$ for $P$-almost all $\omega \in \Omega$.

By the same arguments based on Lebesgue's theorem and (3.13), the pathwise continuity of $t \mapsto I_\varphi(t)$ implies the $L^q(\Omega; L^2_\rho)$-continuity, which finishes the proof. ■

**Proposition 3.3.3:** *Suppose **(A2)** (or the weaker assumption **(A5)*** with $\nu = 1$) holds for the almost strong evolution operator $U$.*
*Let $\varphi = (\varphi(t))_{t \in [0,T]}$ be an $L^2_\rho$-valued predictable process obeying (3.13) for some $q \geq 2$.*

*Then, for each $t \in [0,T]$, the convolution $I_{\varphi,m}$ is well-defined in $L^2_\rho$.*
*Furthermore,*

$$\mathbf{E}||I_{\varphi,m}(t)||^q_{L^2_\rho}\, ds \leq c(q,\zeta,m,T) \int_0^t \mathbf{E}||\varphi(s)||^q_{L^2_\rho}\, ds, \tag{3.18}$$

*where $\zeta \in [0,1)$ is the same as in **(A2)**.*
*Furthermore, the process $t \mapsto I_{\varphi,m}(t)$ is, on the one hand, pathwise continuous in $L^2_\rho$ and, on the other hand, continuous in $L^q(\Omega; L^2_\rho)$.*

**Proof:** A technical problem is that, for general $m \in L^2$, $\mathcal{M}_{\varphi(t)}m$ does not belong to $L^2_\rho$. Thus, a proper approximation is needed.
Let us first consider $m \in L^\infty$. Then, $\mathcal{Y}(t) := \mathcal{M}_{\varphi(t)}m$, $t \in [0,T]$, is $\mathcal{P}_T/\mathcal{B}(L^2_\rho)$-measurable and, by (3.13), is surely such that

$$\int_0^T \mathbf{E}||\mathcal{Y}(t)||^q_{L^2_\rho}\, dt < ||m||^q_{L^\infty} \int_0^T \mathbf{E}||\varphi(t)||^q_{L^2_\rho}\, dt < \infty.$$

Hence, Proposition 3.3.2 applies, which yields the well-definedness of $I_{\varphi,m}(t) = I_{\mathcal{Y}}(t)$, the moment estimates and the required continuity properties in this case.

In the general case $m \in L^2$, let us take a sequence $(m_N)_{N \in \mathbb{N}} \subset L^\infty$ such that

$$||m_N - m||_{L^2} \to 0 \text{ as } n \to \infty.$$

Then, by **(A2)** and Hölder's inequality, for each $s \in [0,t)$ and $\omega \in \Omega$, we have

$$\begin{aligned} ||U(t,s)\mathcal{M}_{\varphi(s)}(m_N - m)||^2_{L^2_\rho} \leq &\quad c(T)(t-s)^{-\zeta}||m_N - m||^2_{L^2}||\varphi(s)||^2_{L^2_\rho} \\ \longrightarrow &\quad 0 \text{ as } N \to \infty. \end{aligned}$$

Thus, $[0,T] \ni s \mapsto \mathbf{1}_{[0,t)}(s)U(t,s)\mathcal{M}_{\varphi(s)}m \in L^2_\rho$ is $\mathcal{P}_T/\mathcal{B}(L^2_\rho)$-measurable as a pointwise limit of predictable functions (see Lemma 3.3.1).

Therefore, $I_{\varphi,m}(t)$ is well-defined as a Bochner integral in $L^2_\rho$ provided

$$(3.19)\ \int_0^t ||U(t,s)\mathcal{M}_{\varphi(s)}m||_{L^2_\rho}\,ds$$
$$\leq c(T)||m||_{L^2}\int_0^t (t-s)^{-\frac{\zeta}{2}}||\varphi(s)||_{L^2_\rho}\,ds$$
$$\leq c(T)||m||_{L^2}\left(\int_0^T s^{-\frac{\zeta q}{2(q-1)}}\,ds\right)^{\frac{q-1}{q}}\left(\int_0^T ||\varphi(s)||^q_{L^2_\rho}\,ds\right)^{\frac{1}{q}} < \infty,\ P\text{-a.s.},$$

which holds by assumption (3.13) and the relation

$$\tfrac{\zeta q}{2(q-1)} < 1 \text{ for } q \geq 2.$$

Here, we used **(A2)** with $\zeta \in [0,1)$.
Similarly, by Bochner's inequality (cf. Appendix B) we have

$$(3.20)\ \mathbf{E}\left|\left|\int_0^t U(t,s)\mathcal{M}_{\varphi(s)}m\,ds\right|\right|^q_{L^2_\rho}$$
$$\leq c^{\frac{q}{2}}(T)||m||^q_{L^2}c(q,\zeta,T)\int_0^T \mathbf{E}||\varphi(s)||^q_{L^2_\rho}\,ds,$$

which proves (3.18).
Let $\Omega_0 \in \mathcal{B}(\Omega)$ be a subset of full $P$-measure such that

$$\int_0^T ||\varphi(t,\omega)||^2_{L^2_\rho}\,dt < \infty,\ \omega \in \Omega_0.$$

Such a subset exists by (3.19).
To check the continuity of $[0,T] \ni t \mapsto I_{\varphi,m}(t) \in L^2_\rho$, we again use the approximation of $m$ by $(m_N)_{N\in\mathbb{N}} \subset L^\infty$. Thus (see the proof of Proposition 3.3.2), all $I_{\varphi,m_N}$, $N \in \mathbb{N}$, $\omega \in \Omega_0$, are well-defined and time-continuous on $[0,T]$. But, for each $\omega \in \Omega_0$,

$$\sup_{t\in[0,T]} ||I_{\varphi,m_N}(t,\omega) - I_{\varphi,m}(t,\omega)||_{L^2_\rho}$$
$$= \sup_{t\in[0,T]}\left|\left|\int_0^t U(t,s)\mathcal{M}_{\varphi(s,\omega)}(m_N - m)\right|\right|_{L^2_\rho}$$
$$\leq \sup_{t\in[0,T]}\int_0^t ||U(t,s)\mathcal{M}_{\varphi(s,\omega)}(m_N-m)||_{L^2_\rho}\,ds$$
$$\leq ||m_N - m||_{L^2}\sup_{t\in[0,T]}\int_0^t ||U(t,s)\mathcal{M}_{\varphi(s,\omega)}||_{\mathcal{L}(L^2,L^2_\rho)}\,ds$$
$$\leq ||m_N - m||_{L^2}c^{\frac{1}{2}}(T)\sup_{t\in[0,T]}\int_0^t (t-s)^{-\frac{\zeta}{2}}||\varphi(s,\omega)||_{L^2_\rho}\,ds$$

$$\leq ||m_N - m||_{L^2} c^{\frac{1}{2}}(T) \left( \int_0^T s^{-\zeta}\, ds \right)^{\frac{1}{2}} \left( \int_0^T ||\varphi(s,\omega)||^2_{L^2_\rho}\, ds \right)^{\frac{1}{2}}$$
$$\to 0 \text{ as } N \to \infty,$$

where we need **(A2)** (or the weaker assumption **(A5)*** with $\nu = 1$).
Thus, for each $\omega \in \Omega_0$, $I_{\varphi,m}(t,\omega)$ is continuous in $L^2_\rho$ as a uniform limit of continuous functions. Herefrom, by Lebesgue's dominated convergence theorem, we also get

$$\sup_{t\in[0,T]} \mathbf{E}||I_{\varphi,m_N}(t) - I_{\varphi,m}(t)||^q_{L^2_\rho} \to 0 \text{ as } N \to \infty,$$

which in turn implies the continuity of $I_{\varphi,m}$ in $L^q(\Omega; L^2_\rho)$. ■

To control the properties of the (stochastic) Bochner convolutions (3.9)/(3.10) in the Banach spaces $L^{2\nu}_\rho$, $\nu \geq 1$, one needs regularity properties of the evolution family $U = (U(t,s))_{0\leq s\leq t\leq T}$ in these spaces. To this end, we have to additionally assume **(A3)** and **(A4)**.
The properties of the convolutions in $L^{2\nu}_\rho$ are described by the following two propositions (generalizing Propositions 3.3.2 and 3.3.3).

**Proposition 3.3.4:** *Let $\nu \geq 1$ and suppose that **(A3)** holds.*
*Let $\varphi = (\varphi(t))_{t\in[0,T]}$ be an $L^{2\nu}_\rho$-valued predictable process obeying*

$$\int_0^T \mathbf{E}||\varphi(t)||^{2\nu}_{L^{2\nu}_\rho} < \infty. \tag{3.21}$$

*Then, for each $t \in [0,T]$, one has*

$$I_\varphi(t) = \int_0^t U(t,s)\varphi(s)\, ds \in L^{2\nu}_\rho \ (P\text{-a.s.}).$$

*Furthermore, there exists a positive constant $c(\nu,T)$ such that*

$$\mathbf{E}||I_\varphi(t)||^{2\nu}_{L^{2\nu}_\rho} \leq c(\nu,T) \int_0^t \mathbf{E}||\varphi(s)||^{2\nu}_{L^{2\nu}_\rho}\, ds < \infty, \tag{3.22}$$

*and hence*

$$\sup_{t\in[0,T]} \mathbf{E}||I_\varphi(t)||^{2\nu}_{L^{2\nu}_\rho} < \infty. \tag{3.23}$$

*Finally, the mapping $t \mapsto I_\varphi(t)$ is continuous both pathwise in $L^{2\nu}_\rho$ and in $L^{2\nu}(\Omega; L^{2\nu}_\rho)$.*

**Proof:** For any $t \in [0,T]$, the integrand function

$[0,T] \ni s \mapsto \mathbf{1}_{[0,t)}(s)U(t,s)\varphi(s) \in L_\rho^{2\nu}$ is $\mathcal{P}_T$-measurable by Lemma 3.3.1. Furthermore, by (3.21) there exists a subset $\Omega_0 \in \mathcal{B}(\Omega)$ with $P(\Omega_0) = 1$ such that for all $\omega \in \Omega_0$

$$\int_0^T ||\varphi(t,\omega)||_{L_\rho^{2\nu}}\, dt < \infty.$$

Since by **(A3)**

$$\int_0^t ||U(t,s)\varphi(s,\omega)||_{L_\rho^{2\nu}}\, ds \le (c(\kappa,T))^{\frac{1}{2\nu}} \int_0^t ||\varphi(s,\omega)||_{L_\rho^{2\nu}}\, ds,$$

$I_\varphi(t,\omega)$ is well-defined as a Bochner integral in $L_\rho^{2\nu}$ for all $t \in [0,T]$ and $\omega \in \Omega_0$.
As $L_\rho^{2\nu}$ is continuously embedded in $L_\rho^2$, Proposition B.2.2 says that $I_\varphi(t)$ coincides with the Bochner integral in $L_\rho^2$ defined by Proposition 3.3.3.
By **(A3)**, together with Bochner's and Hölder's inequalities, we have

$$\begin{aligned}
\mathbf{E}||I_\varphi(t)||_{L_\rho^{2\nu}}^{2\nu} \le \quad & \mathbf{E}\left[\int_0^t ||U(t,s)\varphi(s)||_{L_\rho^{2\nu}}\, ds\right]^{2\nu} \\
\le \quad & c(\nu,T)\int_0^t \mathbf{E}||\varphi(s)||_{L_\rho^{2\nu}}^{2\nu}\, ds,
\end{aligned}$$

which proves (3.22) and (3.23).

To check the continuty properties, we proceed analogously to the proof of Proposition 3.3.2. We have for $0 \le r \le t \le T$ and $\omega \in \Omega_0$

$$\begin{aligned}
||I_\varphi(t,\omega) - I_\varphi(r,\omega)||_{L_\rho^{2\nu}} \le \quad & \int_0^r ||[U(t,s) - U(r,s)]\varphi(s,\omega)||_{L_\rho^{2\nu}}\, ds \\
& + \int_r^t ||U(t,s)\varphi(s,\omega)||_{L_\rho^{2\nu}}\, ds.
\end{aligned}$$

By the strong continuity of $U$ in $L_\rho^{2\nu}$ (cf. **(A3)**), we can literally repeat the previous arguments to get the pathwise continuity of

$$[0,T] \ni t \mapsto I_\varphi(t) \in L_\rho^{2\nu}.$$

Finally, by (3.23) and the pathwise continuity shown before, Lebesgue's convergence theorem yields the time-continuity of $I_\varphi$ in $L^{2\nu}(\Omega; L_\rho^{2\nu})$. ■

**Proposition 3.3.5:** *Suppose that **(A3)** and **(A5)** (or even the weaker assumption **(A5)***) with $\nu \ge 1$ and $\zeta \in [0,1)$ hold.*
*Let $\varphi = (\varphi(t))_{t\in[0,T]}$ be an $L_\rho^{2\nu}$-valued predictable process obeying*

$$\int_0^T \mathbf{E}||\varphi(t)||_{L^{2\nu}_\rho}^{2\nu}\, dt < \infty. \tag{3.24}$$

*Then, for each $m \in L^2$ and $t \in [\,0, T\,]$, one has*

$$I_{\varphi,m}(t) = \int_0^t U(t,s)\mathcal{M}_{\varphi(s)} m\, ds \in L^{2\nu}_\rho \ (P\text{-a.s.}).$$

*Furthermore, there exists a positive constant $c(\nu, T, c(T))$ such that*

$$\mathbf{E}||I_{\varphi,m}(t)||_{L^{2\nu}_\rho}^{2\nu} \le c(\nu, m, T, c(T)) \int_0^t (t-s)^{-\zeta\nu} \mathbf{E}||\varphi(s)||_{L^{2\nu}_\rho}^{2\nu}\, ds < \infty. \tag{3.25}$$

*Finally, the mapping $t \mapsto I_{\varphi,m}(t)$ is continuous both pathwise and in $L^{2\nu}(\Omega; L^{2\nu}_\rho)$.*

**Proof:** By **(A5)*** (cf. (3.6)) and Hölder's inequality we have the following chain of estimates

$$\begin{aligned}
\mathbf{E}||I_{\varphi,m}(t)||_{L^{2\nu}_\rho}^{2\nu} &\le \left[\int_0^t \mathbf{E}||U(t,s)\mathcal{M}_{\varphi(s)} m||_{L^{2\nu}_\rho}\, ds\right]^{2\nu} \\
&\le c(\nu,T)||m||_{L^2}^{2\nu} \left[\int_0^t \sqrt{c(T)}(t-s)^{-\frac{\zeta}{2}} \mathbf{E}||\varphi(s)||_{L^{2\nu}_\rho}\, ds\right]^{2\nu} \\
&\le c(\nu,T)c(T)^\nu ||m||_{L^2}^{2\nu} \left[\int_0^t (t-s)^{-\frac{\zeta}{2}} \mathbf{E}||\varphi(s)||_{L^{2\nu}_\rho}\, ds\right]^{2\nu} \\
&\le c(\nu,T)c(T)^\nu ||m||_{L^2}^{2\nu} T^{2\nu-1} \int_0^t (t-s)^{-\zeta\nu} \mathbf{E}||\varphi(s)||_{L^{2\nu}_\rho}^{2\nu}\, ds \\
&=: c(\nu, m, T, c(T)) \int_0^t (t-s)^{-\zeta\nu} \mathbf{E}||\varphi(s)||_{L^{2\nu}_\rho}^{2\nu}\, ds,
\end{aligned}$$

which yields (3.25).
Concerning the required continuity property, analogously to the proof of Proposition 3.3.3, let us start with the special case $m \in L^\infty$. This gives us a predictable mapping $[\,0, T\,] \ni t \mapsto \mathcal{M}_{\varphi(t)} m \in L^{2\nu}_\rho$, for which, by Proposition 3.3.4, we have the well-definedness and continuity of $I_{\varphi,m}(t,\omega) \in L^{2\nu}_\rho$ for all $t \in [\,0, T\,]$ and $\omega \in \Omega_0$.
Here, $\Omega_0$ is the set of full $P$-measure such that

$$\int_0^T ||\varphi(t,\omega)||_{L^{2\nu}_\rho}\, dt < \infty \text{ for all } \omega \in \Omega_0.$$

Such $\Omega_0$ exists by (3.24).
Next, we consider the general case $m \in L^2$. There exists a sequence $(m_N)_{N\in\mathbb{N}} \subset L^\infty$ such that

$$\lim_{N\to\infty} ||m_N - m||_{L^2} = 0.$$

Now, by **(A5)**/ **(A5)*** we have, for each $\omega \in \Omega_0$,

$$\begin{aligned}
&\sup_{t\in[0,T]} ||I_{\varphi,m_N}(t,\omega) - I_{\varphi,m}(t,\omega)||_{L^{2\nu}_\rho} \\
&= \sup_{t\in[0,T]} \left|\left| \int_0^t U(t,s)\mathcal{M}_{\varphi(s,\omega)}(m_N - m) \right|\right|_{L^{2\nu}_\rho} \\
&\le \sup_{t\in[0,T]} \int_0^t ||U(t,s)\mathcal{M}_{\varphi(s,\omega)}(m_N - m)||_{L^{2\nu}_\rho}\, ds \\
&\le ||m_N - m||_{L^2} \sup_{t\in[0,T]} \int_0^t ||U(t,s)\mathcal{M}_{\varphi(s,\omega)}||_{\mathcal{L}(L^2,L^{2\nu}_\rho)}\, ds \\
&\le ||m_N - m||_{L^2} (c(\nu,T))^{\frac{1}{2\nu}} \sup_{t\in[0,T]} \int_0^t (t-s)^{-\frac{\zeta}{2}} ||\varphi(s,\omega)||_{L^{2\nu}_\rho}\, ds \\
&\le ||m_N - m||_{L^2} (c(\nu,T))^{\frac{1}{2\nu}} \left( \int_0^T s^{-\frac{\zeta(2\nu)}{2(2\nu-1)}}\, ds \right)^{\frac{2\nu-1}{2\nu}} \left( \int_0^T ||\varphi(s,\omega)||^{2\nu}_{L^{2\nu}_\rho}\, ds \right)^{\frac{1}{2\nu}} \\
&\to 0 \text{ as } N \to \infty.
\end{aligned}$$

Thus, $t \mapsto I_{\varphi,m}(t,\omega)$ is continuous as the uniform limit of continuous mappings $t \mapsto I_{\varphi,m_N}(t,\omega)$.
Finally, by the finiteness of the right hand side in (3.25) and the pathwise continuity shown before, Lebesgue's dominated convergence theorem gives us the time-continuity of $I_{\varphi,m}$ in $L^{2\nu}(\Omega; L^{2\nu}_\rho)$. ■

## 3.4 Stochastic convolution w.r.t. $Q$-Wiener process in weighted $L^2$-spaces

In this section, we present results on the stochastic convolution w.r.t. Wiener processes in $L^2_\rho$ resp. $L^{2\nu}_\rho$ (see e.g. [76]).
An emphasis is put on the well-definedness (Proposition 3.4.1 resp. Proposition 3.4.3) and on the (pathwise (see Proposition 3.4.4) and meansquare (see Proposition 3.4.5–3.4.7)) continuity properties of Wiener stochastic convolutions in $L^2_\rho$ resp. $L^{2\nu}_\rho$, $\nu \ge 1$. A new result of this section is Proposition 3.4.7, where we prove the time-continuity in $L^{2\nu}(\Omega; L^{2\nu}_\rho)$. This will be important later for establishing similar continuity properties of the mild solutions to (1.1) resp. (1.2). Furthermore, unlike Manthey and Zausinger in [76], we also consider the general nuclear case, which is of importance for the later considerations of equation (1.2), since the $Q$-Wiener process appearing in the Lévy-Itô decomposition of a Lévy process in general does not fit to the nuclear case introduced above. We emphasize that, in the general nuclear case

of this section, we apply the assumption **(A5)*** to get the well-definedness of the stochastic convolution in $L^{2\nu}_\rho$.

The other results mentioned before are more or less known. Nevertheless, we include detailed proofs of them, especially of Proposition 3.4.3, since later we will adapt them to the case of compensated Poisson random measures in Chapter 4. In doing so, we will fill some gaps in the original proof in [76].

In the whole section, we assume that **(A0)**–**(A2)** hold.

First, we consider the case $\nu = 1$.
Given a predictable process $(\varphi(t))_{t\in[0,T]}$ taking values in $L^2_\rho$, we consider the following stochastic integral, which is called **Wiener stochastic convolution**,

$$I^W_\varphi(t) := \int_0^t U(t,s)\mathcal{M}_{\varphi(s)}\, dW(s). \tag{3.26}$$

A technical problem is caused by the singularity of the integrand function at $s = t$.
In particular, we will show that the simplest condition for (3.26) to be well-defined is

$$\sup_{t\in[0,T]} \mathbf{E}||\varphi(t)||^2_{L^2_\rho} < \infty. \tag{3.27}$$

The Banach space of all predictable processes $\varphi$ obeying this property will be denoted by $\mathcal{H}^2(T)$ (for an exact definition of this space, see in Section 5.1).

**Proposition 3.4.1:** *Let $\varphi = (\varphi(t))_{t\in[0,T]}$ be an $L^2_\rho$-valued predictable process obeying*

$$\sup_{t\in[0,T]} \int_0^t (t-s)^{-\zeta}\mathbf{E}||\varphi(s)||^2_{L^2_\rho}\, ds < \infty, \tag{3.28}$$

*where $\zeta = 0$ in the nuclear case and $\zeta \in [0,1)$ as in **(A2)** in the general nuclear and in the cylindrical case.*
*Then, for each $t \in [0,T]$, the stochastic convolution $I^W_\varphi$ is well-defined in $L^2_\rho$ in both the nuclear and the cylindrical case.*
*In particular, (3.28) is fulfilled in case of (3.27) being fulfilled. In this case, we even have well-definedness of the stochastic convolution in the general nuclear case.*

*Furthermore, if for some $q \geq 2$*

(3.29)
$$\sup_{t\in[0,T]} \int_0^t (t-s)^{-\zeta} \mathbf{E}||\varphi(s)||^q_{L^2_\rho}\, ds < \infty,$$

*then also*

(3.30)
$$\mathbf{E}||I^W_\varphi(t)||^q_{L^2_\rho}\, ds \le c(q,\zeta,T) \int_0^t (t-s)^{-\zeta} \mathbf{E}||\varphi(s)||^q_{L^2_\rho}\, ds < \infty.$$

*In particular,*

$$\sup_{t\in[0,T]} \mathbf{E}||\varphi(t)||^q_{L^2_\rho} < \infty$$

*implies*

$$\sup_{t\in[0,T]} \mathbf{E}||I^W_\varphi(t)||^q_{L^2_\rho} < \infty.$$

**Proof:** As in the Bochner case, let us first fix some $t \in [0,T]$.
We fix the representative of the integrand process as

(3.31)
$$[0,T]\times\Omega \ni (s,\omega) \mapsto \chi(s,\omega) := \mathbf{1}_{[0,t)}(s)U(t,s)\mathcal{M}_{\varphi(s,\omega)} \in \mathcal{L}_2.$$

First, we show that it is $\mathcal{P}_T/\mathcal{B}(\mathcal{L}_2)$-measurable, where $\mathcal{L}_2$ stands for $\mathcal{L}_2(Q^{\frac{1}{2}}L^2, L^2_\rho)$. By the arguments used in proving Lemma 3.6 in [62], this is equivalent to the $\mathcal{P}_T/L^2_\rho$-measurability of the mappings

$$[0,T]\times\Omega \ni (s,\omega) \mapsto \chi(s,\omega)Q^{\frac{1}{2}}e_n \in L^2_\rho,\ n\in\mathbb{N},$$

where $(e_n)_{n\in\mathbb{N}} \subset L^2$ is an orthonormal basis in $L^2$ of eigenvectors of $Q$, which always exists by 2.3.3.

The statement will be a corollary of the general fact that $[0,T]\times\Omega \ni (s,\omega) \mapsto \mathbf{1}_{[0,t)}(s)U(t,s)\mathcal{Y}(s)$ is $\mathcal{P}_T$-measurable for each $\mathcal{P}_T$-measurable process $(\mathcal{Y}(t))_{t\in[0,T]} \subset L^2_\rho$, which has already been proved in Proposition 3.3.1.

In both the nuclear and the cylindrical case we take $\mathcal{Y}(s) := \varphi(s)Q^{\frac{1}{2}}e_n \in L^2_\rho$ (recall that $Q^{\frac{1}{2}}e_n \in L^\infty$ for any $n\in\mathbb{N}$ in both cases).
In the general nuclear case, given any $n\in\mathbb{N}$, we set $\mathcal{Y}(s) := \varphi(s)h_{n,M} \in L^2_\rho$ for any $s\in[0,T]$ and a sequence $(h_{n,M})_{M\in\mathbb{N}} \subset L^\infty$ approximating $Q^{\frac{1}{2}}e_n$ in the $L^2$-norm. Then, by **(A2)** we have, for each $(s,\omega)\in[0,T]\times\Omega$,

$$\lim_{M\to\infty} ||\chi(s,\omega)h_{n,M} - \chi(s,\omega)Q^{\frac{1}{2}}e_n||^2_{L^2_\rho} = 0,$$

which implies the required measurability in the general nuclear case.

Now, as was discussed in Section 2.5 (cf. Definitions 2.5.2 and 2.5.5 there), the stochastic integral

$$\begin{aligned} I_\varphi^W(t) := & \int_0^t U(t,s)\mathcal{M}_{\varphi(s)}\, dW(s) \\ := & \int_0^t \chi(s)\, dW(s) \end{aligned}$$

is well-defined in all three cases provided

$$\mathbf{E}\int_0^t ||U(t,s)\mathcal{M}_{\varphi(s)}||^2_{\mathcal{L}_2}\, ds < \infty,\ t \in [0,T]. \tag{3.33}$$

It is easy to see that, combined with (3.1) and **(A2)**, (3.33) follows from (3.28). Indeed, we have

$$\begin{aligned} \mathbf{E}\int_0^t ||U(t,s)\mathcal{M}_{\varphi(s)}||^2_{\mathcal{L}_2}\, ds = & \ \mathbf{E}\int_0^t \sum_{n\in\mathbb{N}} ||U(t,s)\mathcal{M}_{\varphi(s)}(Q^{\frac{1}{2}}e_n)||^2_{L^2_\rho}\, ds \\ \le & \ c(T)\left(\sum_{n\in\mathbb{N}} a_n\right)\left(\sup_{n\in\mathbb{N}} ||e_n||^2_\infty\right)\int_0^t \mathbf{E}||\varphi(s)||^2_{L^2_\rho}\, ds \\ < & \ \infty \text{ (by (3.28) with } \zeta = 0 \text{ and (3.1))} \end{aligned} \tag{3.34}$$

in the nuclear case, respectively,

$$\begin{aligned} \mathbf{E}\int_0^t ||U(t,s)\mathcal{M}_{\varphi(s)}||^2_{\mathcal{L}_2(L^2,L^2_\rho)}\, ds \le & \ c(T)\int_0^t (t-s)^{-\zeta}\mathbf{E}||\varphi(s)||^2_{L^2_\rho}\, ds \\ < & \ \infty \text{ (by (3.28) and } \mathbf{(A2)}) \end{aligned} \tag{3.35}$$

in the cylindrical case, respectively

$$\begin{aligned} \mathbf{E}\int_0^t ||U(t,s)\mathcal{M}_{\varphi(s)}||^2_{\mathcal{L}_2}\, ds \le & \ tr\, Q\mathbf{E}\int_0^t ||U(t,s)\mathcal{M}_{\varphi(s)}||^2_{\mathcal{L}_2(L^2,L^2_\rho)}\, ds \\ \le & \ c(T) tr\, Q\mathbf{E}\int_0^t (t-s)^{-\zeta}||\varphi(s)||^2_{L^2_\rho}\, ds \\ < & \ \infty \text{ (by (3.28) and } \mathbf{(A2)}) \end{aligned} \tag{3.36}$$

in the general nuclear case.
It remains to prove the estimate (3.30) for $q \ge 2$.
By the Burkholder-Davis-Gundy inequalities 2.5.4/2.5.6, estimate (3.2) from **(A2)** and Hölder's inequality, we have

$$\begin{aligned}
\mathbf{E}||I_{\varphi}^{W}(t)||_{L_\rho^2}^q \leq\ & c(q,T)\mathbf{E}\left(\int_0^t ||U(t,s)\mathcal{M}_{\varphi(s)}||_{\mathcal{L}_2}^2\, ds\right)^{\frac{q}{2}} \\
\leq\ & c(q,T)\mathbf{E}\left(\int_0^t (t-s)^{-\zeta}||\varphi(s)||_{L_\rho^2}^2\, ds\right)^{\frac{q}{2}} \\
=\ & c(q,T)\mathbf{E}\left(\int_0^t (t-s)^{-\frac{(q-2)\zeta}{q}}(t-s)^{-\frac{2\zeta}{q}}||\varphi(s)||_{L_\rho^2}^2\, ds\right)^{\frac{q}{2}} \\
\leq\ & c(q,T)\left(\int_0^T s^{-\zeta}\, ds\right)^{\frac{q-2}{2}} \int_0^t (t-s)^{-\zeta}\mathbf{E}||\varphi(s)||_{L_\rho^2}^q\, ds \\
=:\ & c(q,\zeta,T)\int_0^t (t-s)^{-\zeta}\mathbf{E}||\varphi(s)||_{L_\rho^2}^q\, ds,
\end{aligned}$$

which proves the claim. ■

**Remark 3.4.2:** ***(i)*** *As one can see from (3.34)–(3.36), the stochastic convolution is well-defined in* $L_\rho^2$ *even under the sufficient conditions*

$$\int_0^T \mathbf{E}||\varphi(s)||_{L_\rho^2}^2\, ds < \infty \tag{3.37}$$

*in the nuclear case and, respectively,*

$$\int_0^T (\mathbf{E}||\varphi(t)||_{L_\rho^2}^2)^r\, dt < \infty \text{ for some } r > \tfrac{1}{1-\zeta} \tag{3.38}$$

*in the general nuclear and the cylindrical case. The latter condition comes from the following estimate of the integral, appearing on the right hand side of (3.35) and (3.36),*

$$\begin{aligned}
&\int_0^t (t-s)^{-\zeta}\mathbf{E}||\varphi(s)||_{L_\rho^2}^2\, ds \\
&\leq \left(\int_0^t s^{-\zeta(1+\delta)}\, ds\right)^{\frac{1}{1+\delta}} \left(\int_0^t (\mathbf{E}||\varphi(s)||_{L_\rho^2}^2)^{\frac{\delta+1}{\delta}}\, ds\right)^{\frac{\delta}{\delta+1}},
\end{aligned}$$

*where we used Hölder's inequality and choose some* $\delta > 0$ *such that* $\zeta(1+\delta) < 1$.
*The last integral is finite for* $\delta = \frac{1}{r-1} > 0$ *under the assumption (3.38). Besides the process* $\varphi$ *being in* $\mathcal{H}^2(T)$*, a sufficient condition for (3.38) is also*

$$\int_0^T \mathbf{E}||\varphi(t)||_{L_\rho^{2r}}^{2r}\, dt < \infty \text{ for some } r > \tfrac{1}{1-\zeta}.$$

***(ii)*** *As we have seen,* ***(A2)*** *allows to consider all three cases from 3.4.1 simultaneously. In the cylindrical case, we could also apply* ***(A5)*** *with* $\nu = 1$, *whereas in the general nuclear case we could assume **(A5)*** with $\nu = 1$ instead of **(A2)**.*

***(iii)** Actually, the results of this section remain true if we just assume that the integrand process $\varphi$ is measurable and $(\mathcal{F}_t)_{t\in[0,T]}$-adapted.*

Let us stress that the stochastic convolution process (3.26) is not a martingale, and thus one needs more comprehensive methods to study its regularity properties.

Now, we consider the general case $\nu \geq 1$.
A key idea is to control the well-definedness of the stochastic convolution (3.26) in the Banach spaces $L^{2\nu}_\rho$, $\nu \geq 1$, by additional regularity properties of the evolution family $U = (U(t,s))_{0\leq s\leq t\leq T}$ in these spaces.
The properties of the stochastic convolution in $L^{2\nu}_\rho$ are described by

**Proposition 3.4.3:** (cf. [76], Chapter 2, Remark 2.3(iii))
*Let $\nu \geq 1$. Suppose that, additionally to the previous assumptions, $U$ obeys **(A3)** and **(A4)** (In the nuclear case, **(A4)** certainly holds with $\zeta = 0$, cf. Remark 3.1.2.1 **(iii)**, in the cylindrical case **(A4)** is implied by **(A5)** with the same $\zeta \in [0,1)$, cf. Remark 3.1.2.1 **(v)**, whereas in the general nuclear case, **(A4)** is implied by **(A5)*** with the same $\zeta \in [0,1)$, cf. Remark 3.1.2.1 **(iv)**).*
*Let $\varphi = (\varphi(t))_{t\in[0,T]}$ be an $L^{2\nu}_\rho$-valued, predictable process obeying*

$$\text{(3.39)} \qquad \sup_{t\in[0,T]} \int_0^t (t-s)^{-\zeta}\mathbf{E}||\varphi(s)||^{2\nu}_{L^{2\nu}_\rho}\, ds < \infty,$$

*where $\zeta = 0$ in the nuclear case and $\zeta \in [0,1)$ as in **(A2)** in the general nuclear and the cylindrical case. Then, in all cases we have, for each $t \in [0,T]$,*

$$I^W_\varphi(t) = \int_0^t U(t,s)\mathcal{M}_{\varphi(s)}\, dW(s) \in L^{2\nu}_\rho \ (P\text{-a.s.}).$$

*Furthermore, there exists a positive constant $c(\nu,T)$ such that*

$$\text{(3.40)}\ \mathbf{E}||I^W_\varphi(t)||^{2\nu}_{L^{2\nu}_\rho} \leq c(\nu,T)\int_0^t (t-s)^{-\zeta}\mathbf{E}||\varphi(s)||^{2\nu}_{L^{2\nu}_\rho}\, ds < \infty.$$

*In particular,*

$$\sup_{t\in[0,T]} \mathbf{E}||\varphi(t)||^{2\nu}_{L^{2\nu}_\rho} < \infty$$

*is sufficient for (3.39) and implies*

$$\sup_{t\in[\,0,T\,]} \mathbf{E}||I_\varphi^W(t)||^{2\nu}_{L^{2\nu}_\rho} < \infty.$$

**Proof:** We follow the original proof from [76], but give more explanation to some of the key steps.

Let us fix an arbitrary $t \in [\,0,T\,]$ and define the predictable integrand process, cf. (3.31),

$$[\,0,T\,] \ni s \mapsto \chi(s) := \mathbf{1}_{[\,0,t\,)}(s)U(t,s)\mathcal{M}_{\varphi(s)} \in \mathcal{L}_2.$$

Note that we have already shown the well-definedness in $L^2_\rho$ of

$$I_\varphi^W(t) := \int_0^t \chi(s)\, dW(s).$$

Recalling the coordinate structure of the $Q$-Wiener process from Section 2.3, heuristically we could also consider the infinite series of stochastic integrals

$$\sum_{n\in\mathbb{N}} \int_0^t \chi(s)(Q^{\frac{1}{2}}e_n)\, dw_n(s) \tag{3.41}$$

constructed by means of the family $(w_n)_{n\in\mathbb{N}}$ of independent scalar Brownian motions as in Section 2.3. Our aim is to identify (3.41) with the $L^2_\rho$-valued stochastic integral (3.26). Then, we will examine the $L^{2\nu}_\rho$-properties of each term and establish the convergence of the above expansion in $L^{2\nu}(\Omega; L^{2\nu}_\rho)$.

The proof works in the following way:

**Step 1** We find a family $(\psi^{(n)})_{n\in\mathbb{N}}$ of $\mathcal{P}_T\otimes\mathcal{B}(\Theta)-\mathcal{B}(\mathbb{R})$-measurable **representatives** for the $L^2_\rho$-valued functions $(\chi Q^{\frac{1}{2}}e_n)_{n\in\mathbb{N}}$ (for their definition see below).

**Step 2** In $L^2(\Omega; L^2_\rho)$ we check the identity

$$\int_0^t \chi(s)\, dW(s) = \sum_{n\in\mathbb{N}} \int_0^t \psi^{(n)}(s,\cdot)\, dw_n(s)$$

with $\chi$ as in (3.31).

**Step 3** We show the required inclusion

$$\sum_{n\in\mathbb{N}} \int_0^t \psi^{(n)}(s,\cdot)\, dw_n(s) \in L^{2\nu}_\rho \;\; (P\text{-a.s.}).$$

**Step 1** is done by the following claim.

**Claim 1:** *There are* ***representatives*** $\psi^{(n)}$ *for the functions*

$$[0,T] \ni s \mapsto \mathbf{1}_{[0,t)}(s)U(t,s)\mathcal{M}_{\varphi(s)}Q^{\frac{1}{2}}e_n \in L^2_\rho,$$

*i.e.* $\mathcal{P}_T \otimes \mathcal{B}(\Theta) - \mathcal{B}(\mathbb{R})$*-measurable functions*
$\psi^{(n)}\colon \Omega \times [0,T] \times \Theta \to \mathbb{R}$ *such that*

$$(3.42)\quad \mathbf{E} \sum_{n\in\mathbb{N}} \int_0^T ||U(t,s)\mathcal{M}_{\varphi(s)}Q^{\frac{1}{2}}e_n - \psi^{(n)}(s,\cdot)||^2_{L^2_\rho}\, ds = 0.$$

**Proof:** In order to find such measurable representatives, we need to "evaluate" $U(t,s)\mathcal{M}_{\varphi(s)}Q^{\frac{1}{2}}e_n$ at any point $\theta \in \Theta$.
To this end, we exploit the smoothing properties of a standard convolution operator for real-valued functions.

Let us first recall the following:

**Definition:**
***(i)*** *A sequence* $(\delta_k)_{k\in\mathbb{N}} \subset L^1(\mathbb{R}^d)$ *is called a* ***(general-) Dirac sequence*** *if*

$$(3.43)\quad \begin{aligned} &\delta_k \geq 0 \ (dx\text{-a.s.}),\ \int_{\mathbb{R}^d} \delta_k(\theta)\, d\theta = 1 \text{ and} \\ &\lim_{k\to\infty} \int_{\mathbb{R}^d\setminus B_\rho(0)} \delta_k(\theta)\, d\theta = 0 \text{ for any } \rho > 0, \end{aligned}$$

*where* $B_\rho(0)$ *is the ball of radius* $\rho > 0$ *around* $0$.

***(ii)*** *If* $\psi_1\colon \mathbb{R}^d \to \mathbb{R}$ *is measurable and* $\psi_2 \in L^p(\mathbb{R}^d)$ *for some* $1 \leq p \leq \infty$*, the* ***standard convolution mapping*** *conv is given by*

$$(conv(\psi_1,\psi_2))(\theta) := \int_{\mathbb{R}^d} \psi_1(\xi-\theta)\psi_2(\xi)\, d\xi,\ \theta \in \mathbb{R}^d,$$

*provided the integral in the right hand side exists.*

It is well-known that:

- For any function $\varphi \in C_0^\infty(\mathbb{R}^d)$, the sequence $(\delta_k)_{k\in\mathbb{N}}$ given by

  $$(3.44)\qquad \delta_k(\theta) := \left(\tfrac{1}{k}\right)^{-d}\varphi(k\theta),\ \theta \in \mathbb{R}^d,$$

  is a Dirac sequence (cf. 2.13 2 in [6]).
- The Dirac sequence from (3.44) fulfills

$$(3.45) \qquad conv(\delta_k, \psi) \in C^\infty(\mathbb{R}^d)$$

for any $\psi \in L^2(\mathbb{R}^d)$ (cf. 2.12.4 from [6]).

- For any Dirac sequence $(\delta_k)_k \subset L^1(\mathbb{R}^d)$ and any $\psi \in L^2(\mathbb{R}^d)$, we have (cf. 2.14.2 from [6]) that

$$(3.46) \qquad conv(\delta_k, \psi) \underset{L^2}{\longrightarrow} \psi \text{ as } k \to \infty.$$

- For any $\varphi \in L^1(\mathbb{R}^d)$ and any $\psi \in L^2(\mathbb{R}^d)$, the convolution exists and obeys the bound (cf. 2.12 2. from [6])

$$(3.47) \qquad ||conv(\varphi, \psi)||_{L^2} \leq ||\varphi||_{L^1} ||\psi||_{L^2}.$$

We show Claim 1 with the help of the above definitions and properties.

Let $(e_n)_{n\in\mathbb{N}} \subset L^2$ be a complete orthonormal system of eigenvectors of the operator $Q \in \mathcal{T}(L^2)$.
Given the weight function $\mu_\rho$ as in the Introduction, for $n \in \mathbb{N}$ and almost all $(s, \omega) \in [\,0, T\,] \times \Omega$, we have (cf. (3.34)–(3.36))

$$\mu_\rho^{\frac{1}{2}} \chi(s) Q^{\frac{1}{2}} e_n = \mathbf{1}_{[\,0,t\,)}(s) \mu_\rho^{\frac{1}{2}} U(t,s) \mathcal{M}_{\varphi(s)} Q^{\frac{1}{2}} e_n \in L^2(\Theta).$$

Outside $\Theta$ we trivially continue this function by 0.
Thus, by (3.46) we have, for any $n \in \mathbb{N}$,

$$(3.48) \qquad conv(\delta_k, \mu_\rho^{\frac{1}{2}} \chi(s) Q^{\frac{1}{2}} e_n) \underset{L^2}{\longrightarrow} \mu_\rho^{\frac{1}{2}} \chi(s) Q^{\frac{1}{2}} e_n \text{ as } k \to \infty,$$

where $(\delta_k)_{k\in\mathbb{N}}$ is the Dirac sequence from (3.44).
Furthermore, by (3.45) we get

$$conv(\delta_k, \mu_\rho^{\frac{1}{2}} \chi(s) Q^{\frac{1}{2}} e_n) \in C^\infty(\mathbb{R}^d).$$

Thus, for any $n \in \mathbb{N}$ and almost all $(s, \omega) \in [\,0, T\,] \times \Omega$, we can calculate $(conv(\delta_k, \mu_\rho^{\frac{1}{2}} \chi(s) Q^{\frac{1}{2}} e_n))(\theta)$ for any fixed $\theta \in \Theta$.

Given $n, k \in \mathbb{N}$, we define $\psi_k^{(n)}\colon [\,0, T\,] \times \Omega \times \Theta \to \mathbb{R}$ by

$$(3.49) \qquad \begin{aligned} \psi_k^{(n)}(s, \omega, \theta) := {} & (\mu_\rho^{-\frac{1}{2}} conv(\delta_k, \mu_\rho^{\frac{1}{2}} \chi(s) Q^{\frac{1}{2}} e_n))(\theta) \\ = {} & \mathbf{1}_{[\,0,t\,)}(s) (\mu_\rho^{-\frac{1}{2}} conv(\delta_k, \mu_\rho^{\frac{1}{2}} U(t,s) \mathcal{M}_{\varphi(s)} Q^{\frac{1}{2}} e_n))(\theta) \end{aligned}$$

Obviously, $\mu_\rho^{-\frac{1}{2}}\psi \in L^2_\rho$ for any $\psi \in L^2$. Thus, for $\psi_k^{(n)}$ given by (3.49) and for almost all $(s,\omega) \in [\,0,T\,] \times \Omega$, we get

$$\psi_k^{(n)}(s,\omega,\cdot) \in L^2_\rho$$

and

$$\lim_{k\in\mathbb{N}} ||\psi_k^{(n)}(s,\omega,\cdot) - U(t,s)\mathcal{M}_{\varphi(s)}Q^{\frac{1}{2}}e_n||_{L^2_\rho} = 0. \tag{3.50}$$

We next show $\mathcal{P}_T \otimes \mathcal{B}(\Theta) - \mathcal{B}(\mathbb{R})$-measurability of each $\psi_k^{(n)}$, $n,k \in \mathbb{N}$.

To this end, we use Theorem 6.1 from [51]. To apply it, we need the following two properties to be fulfilled:

- continuity of the mapping

$$\Theta \ni \theta \mapsto \psi_k^{(n)}(s,\omega,\theta)$$

  for any $(s,\omega) \in [\,0,T\,] \times \Omega$, and

- $\mathcal{P}_T$-measurability of

$$(s,\omega) \mapsto \psi_k^{(n)}(s,\omega,\theta)$$

  for any fixed $\theta \in \Theta$.

By (3.45)/(3.49) there is a version of $\psi_k^{(n)}$ obeying the required continuity property.

Concerning the $\mathcal{P}_T$-measurability required for any fixed $\theta \in \Theta$, note that

$$\begin{aligned}\psi_k^{(n)}(s,\omega,\theta) = \; & \mu_\rho^{-\frac{1}{2}}(\theta) \int_\Theta \delta_k(\xi-\theta)(\mu_\rho^{\frac{1}{2}}\chi(s,\omega)Q^{\frac{1}{2}}e_n)(\theta)\,d\xi \\ = \; & \mu_\rho^{-\frac{1}{2}}(\theta) < \delta_k(\cdot-\theta), \mu_\rho^{\frac{1}{2}}\chi(s,\omega)Q^{\frac{1}{2}}e_n >_{L^2}.\end{aligned}$$

Since $\chi$ is predictable, we get the $\mathcal{P}_T$-measurability of $(s,\omega) \mapsto \psi_k^{(n)}(s,\omega,\theta)$ by Fubini's Theorem.

Thus, for each $n,k \in \mathbb{N}$, the assumptions of Theorem 6.1 from [51] are fulfilled. This gives us $\mathcal{P}_T \otimes \mathcal{B}(\Theta)$-measurability of

$$(s,\omega,\theta) \mapsto \psi_k^{(n)}(s,\omega,\theta).$$

By construction, see (3.46),

$$(\psi_k^{(n)}(s,\omega,\cdot))_{k\in\mathbb{N}}$$

is a Cauchy sequence in $L^2_\rho$ for each $n \in \mathbb{N}$ and almost all $(s,\omega) \in [0,T] \times \Omega$.

Due to the uniform bound (3.45), Lebesgue's theorem is applicable, which yields that

$$(s,\omega) \mapsto \psi_k^{(n)}(s,\omega,\cdot),\ k \in \mathbb{N},$$

is a Cauchy sequence in $L^2(\Omega \times [0,T], \mathcal{P}_T, P \otimes dt; L^2_\rho)$ for each $n \in \mathbb{N}$.
Furthermore, by Fubini's theorem we have for any $k$, $\bar{k} \in \mathbb{N}$

$$\sum_{n\in\mathbb{N}} \mathbf{E} \int_0^T \int_\Theta |\psi_k^{(n)}(s,\theta) - \psi_{\bar{k}}^{(n)}(s,\theta)|^2 \, \mu_\rho(d\theta)\, ds = \mathbf{E} \int_0^T \sum_{n\in\mathbb{N}} ||\psi_k^{(n)}(s,\cdot) - \psi_{\bar{k}}^{(n)}(s,\cdot)||^2_{L^2_\rho} \, ds.$$

This implies that, for a fixed $n \in \mathbb{N}$,

$$(s,\omega,\theta) \mapsto \psi_k^{(n)}(s,\omega,\theta),\ k \in \mathbb{N},$$

is a Cauchy sequence in

$$L^2([0,T] \times \Omega \times \Theta) := L^2([0,T] \times \Omega \times \Theta, \mathcal{P}_T \otimes \mathcal{B}(\Theta), P \otimes ds \otimes \mu_\rho).$$

Moreover, by Lebesgue's theorem and (3.46) $(\psi_k^{(n)})_{k\in\mathbb{N}}$ is a Cauchy sequence in $L^2([0,T] \times \Omega \times \Theta; l^2)$, which is the space of all sequences $(\psi^{(n)})_{n\in\mathbb{N}}$ in $L^2([0,T] \times \Omega \times \Theta)$ such that

$$\sum_{n\in\mathbb{N}} \mathbf{E} \int_0^T \int_\Theta |\psi^{(n)}(s,\theta)|^2 \, \mu_\rho(d\theta)\, ds < \infty.$$

Since $L^2([0,T] \times \Omega \times \Theta; l^2)$ is a Hilbert space, there exists a limit sequence $(\psi^{(n)})_{n\in\mathbb{N}} \in L^2([0,T] \times \Omega \times \Theta; l^2)$ such that each $\psi^{(n)}$ is $\mathcal{P}_T \times \mathcal{B}(\Theta)$-measurable and

$$\sum_{n\in\mathbb{N}} \mathbf{E} \int_0^T \int_\Theta |\psi_k^{(n)}(s,\theta) - \psi^{(n)}(s,\theta)|^2 \, \mu_\rho(d\theta)\, ds \to 0 \text{ as } k \to \infty. \tag{3.51}$$

Obviously, this implies that each component $\psi^{(n)}(s,\omega,\cdot) \in L^2_\rho$ for $dt \otimes P$-almost all $(s,\omega) \in [0,T] \times \Omega$.
On the other hand, by the definition of $\psi_k^{(n)}$ we have, for each $n \in \mathbb{N}$ and almost all
$(s,\omega) \in [0,T] \times \Omega$,

$$\lim_{k\to\infty} ||\psi_k^{(n)}(s,\omega,\cdot) - \chi(s,\omega)Q^{\frac{1}{2}}e_n||_{L^2_\rho} = 0$$

and thus

$$\lim_{k\to\infty} ||\psi_k^{(n)}(s,\omega,\cdot) - \psi^{(n)}(s,\omega,\cdot)||^2_{L^2_\rho} = ||\chi(s,\omega)Q^{\frac{1}{2}}e_n - \psi^{(n)}(s,\omega,\cdot)||_{L^2_\rho}.$$

This implies

$$\begin{aligned}
&(3.52)\ \mathbf{E}\int_0^T \sum_{n\in\mathbf{N}} ||\chi(s,\omega)Q^{\frac{1}{2}}e_n - \psi^{(n)}(s,\cdot)||^2_{L^2_\rho}\,ds \\
&= \mathbf{E}\int_0^T \sum_{n\in\mathbf{N}} \lim_{k\to\infty} ||\psi_k^{(n)}(s,\omega,\cdot) - \psi^{(n)}(s,\omega,\cdot)||^2_{L^2_\rho}\,ds \\
&= \lim_{k\to\infty} \mathbf{E}\int_0^T \sum_{n\in\mathbf{N}} ||\psi_k^{(n)}(s,\omega,\cdot) - \psi^{(n)}(s,\omega,\cdot)||^2_{L^2_\rho}\,ds \\
&= \lim_{k\to\infty} \sum_{n\in\mathbb{N}} \mathbf{E}\int_0^T\int_\Theta |\psi_k^{(n)}(s,\theta) - \psi^{(n)}(s,\theta)|^2\,\mu_\rho(d\theta)\,ds \\
&= 0,
\end{aligned}$$

which yields (3.42). $\square$

Thus, the proof of Step 1 is finished. $\triangle$

**Step 2:** This step is proven by the following claim.

**Claim 2:** *In $L^2(\Omega; L^2_\rho)$, for $\chi$ given by (3.31) we have*

$$I_\varphi^W(t) = \int_0^t \chi(s)\,dW(s) = \sum_{n\in\mathbb{N}}\int_0^t \chi(s)Q^{\frac{1}{2}}e_n\,dw_n(s) = \sum_{n\in\mathbb{N}}\int_0^t \psi^{(n)}(s,\cdot)\,dw_n(s),$$

*where for each $n\in\mathbb{N}$ (by (3.49))*

$$I_n(t) := \int_0^t \psi^{(n)}(s,\cdot)\,dw_n(s) \tag{3.53}$$

*is the usual $L^2_\rho$-valued stochastic integral of the predictable process $[\,0,T\,]\ni s\mapsto \psi^{(n)}(s,\cdot)\in L^2_\rho$ (see e.g. [97]).*

**Proof:** By the definition of $\psi^{(n)}$ as a measurable modification of $\chi(s)Q^{\frac{1}{2}}e_n$ (see (3.49)), the claim is equivalent to showing that (in any case) the sum

$$\sum_{n\in\mathbb{N}}\int_0^t \chi(s)Q^{\frac{1}{2}}e_n\,dw_n(s) \tag{3.54}$$

converges in $L^2(\Omega; L^2_\rho)$ and that

$$\int_0^t \chi(s)\,dW(s) = \sum_{n\in\mathbb{N}} \int_0^t \chi(s)Q^{\frac{1}{2}}e_n\,dw_n(s) = \sum_{n\in\mathbb{N}} I_n(t), \tag{3.55}$$

where the $I_n(t)$ are defined by (3.53). To prove the convergence of (3.54), we use a proper version of Itô's isometry. Recall that, for a real-valued Brownian motion $(w(t))_{t\in[0,T]}$ and an $L^2_\rho$-valued, predictable process $(\varphi(t))_{t\in[0,T]}$, we have (cf. e.g. [97])

$$\mathbf{E}\left|\left|\int_0^t \varphi(s)\,dw(s)\right|\right|^2_{L^2_\rho} = \mathbf{E}\int_0^t ||\varphi(s)||^2_{L^2_\rho}\,ds,\ t\in[0,T],$$

provided the right hand side is finite.

Due to the mutual independence of $(w_n)_{n\in\mathbb{N}}$, we have by (3.15)

$$\mathbf{E}\left|\left|\sum_{n\in\mathbb{N}}\int_0^t \chi(s)Q^{\frac{1}{2}}e_n\,dw_n(s)\right|\right|^2_{L^2_\rho}$$
$$= \sum_{n\in\mathbb{N}} \mathbf{E}\int_0^t ||\chi(s)Q^{\frac{1}{2}}e_n||^2_{L^2_\rho}\,ds < \infty.$$

Hence, the series (3.54) is convergent in $L^2(\Omega; L^2_\rho)$.

To check the identity (3.55), let us first proceed in the nuclear and general nuclear case, i.e. we suppose $W$ is a $Q$-Wiener process with the covariance operator $Q\in\mathcal{T}^+(L^2)$.

Recall that by 2.5.2 there is a family of elementary processes $(\chi_m)_{m\in\mathbb{N}}$ such that

$$\lim_{m\to\infty} \mathbf{E}\int_0^T ||\chi_m(s)-\chi(s)||^2_{\mathcal{L}_2}\,ds = 0, \tag{3.56}$$

where $\mathcal{L}_2$ denotes the set $\mathcal{L}_2(Q^{\frac{1}{2}}L^2, L^2_\rho)$ of Hilbert-Schmidt operators from $Q^{\frac{1}{2}}L^2$ to $L^2_\rho$ (see also Definition 2.5.7 above).
We first show (3.55) for the elementary processes $\chi_m\in\mathcal{S}_W(T)$.
Any such elementary process can be written in the form

$$\chi_m(t) = \sum_{j=0}^{j_m-1} \chi_m^j \mathbf{1}_{[t_j^m, t_{j+1}^m)}(s),$$

with $0 := t_0^m < t_1^m < ... < t_{j_m}^m := T$ and $\chi_m^j \in \mathcal{L}(Q^{\frac{1}{2}}L^2, L^2_\rho)$, $0\le j\le j_m-1$.

For this $\chi_m$, we have the following chain of equations in $L^2(\Omega,\mathcal{F},P;L^2_\rho)$.

$$\sum_{n\in\mathbf{N}}\int_0^t \chi_m(s)Q^{\frac{1}{2}}e_n\,dw_n(s)$$

$$= \sum_{n\in\mathbf{N}} \left( \sum_{i=0}^{j_m-1} \chi_m^j \mathbf{1}_{[t_j^m, t_{j+1}^m)}(s) \right) Q^{\frac{1}{2}} e_n (w_n(t_{j+1}^m \wedge t) - w_n(t_j^m \wedge t))$$
$$= \sum_{j=0}^{j_m-1} \chi_m^j \mathbf{1}_{[t_j^m, t_{j+1}^m)}(s) \left( \sum_{n\in\mathbb{N}} Q^{\frac{1}{2}} e_n (w_n(t_{j+1}^m \wedge t) - w_n(t_j^m \wedge t)) \right)$$
$$= \sum_{j=0}^{j_m-1} \chi_m^j \mathbf{1}_{[t_j^m, t_{j+1}^m)}(s) (W(t_{j+1}^m \wedge t) - W(t_j^m \wedge t)) = \int_0^t \chi_m \, dW(s),$$

where we used (2.5) (with $G := L^2$ and $(e_n)_{n\in\mathbb{N}}$) for $W$ in the last line. Thus, the claim holds true for any elementary $\chi_m \in \mathcal{S}_W(T)$.
On the other hand, we have

$$(3.57)\ \mathbf{E} \left|\left| \int_0^t (\chi_m(s) - \chi(s)) \, dW(s) \right|\right|_{L^2_\rho}^2$$
$$= \mathbf{E} \int_0^t ||\chi_m(s) - \chi(s)||^2_{\mathcal{L}_2} \, ds$$
$$\leq \mathbf{E} \int_0^T ||\chi_m(s) - \chi(s)||^2_{\mathcal{L}_2} \, ds \to 0, \text{ as } m \to \infty$$

by Itô's isometry and (3.56). Taking into account (3.15), (3.56) and the mutual independence of $w_n$ for different $n \in \mathbb{N}$, we also have

$$(3.58)\ \mathbf{E} \left|\left| \sum_{n\in\mathbb{N}} \int_0^t (\chi_m(s) - \chi(s)) Q^{\frac{1}{2}} e_n \, dw_n(s) \right|\right|_{L^2_\rho}^2$$
$$= \sum_{n\in\mathbb{N}} \mathbf{E} \int_0^t ||(\chi_m(s) - \chi(s)) Q^{\frac{1}{2}} e_n||^2_{L^2_\rho} \, ds$$
$$\leq \mathbf{E} \int_0^T ||\chi_m(s) - \chi(s)||^2_{\mathcal{L}_2} \, ds \to 0 \text{ as } m \to \infty.$$

Combining (3.57) and (3.58), we get (3.55) both in the nuclear and the general nuclear case.

Concerning the cylindrical case, note that the stochastic integration w.r.t. the cylindrical Wiener process is defined via an auxiliary $Q_1$-Wiener process for some $Q_1 \in \mathcal{T}^+(L^2)$, see Section 2.5. Thus, (3.55) in the cylindrical case readily follows from the (general) nuclear case. □

Thus, we have finished Step 2. △

Note that so far we did not use the measurability properties of the family $(\psi^{(n)})_{n\in\mathbb{N}}$. This is needed in

**Step 3:** *$P$-almost surely, each*

$$I_n(t) := \int_0^t \psi^{(n)}(s, \cdot)\, dw_n(s)$$

*belongs to* $L^{2\nu}_\rho$. *Furthermore, their sum*

$$\sum_{n\in\mathbb{N}} \int_0^t \psi^{(n)}(s, \cdot)\, dw_n(s)$$

*converges* $P$*-almost surely in* $L^{2\nu}_\rho$.

To prove this step, we first need the pointwise representation for the above integrals:

**Claim 3:** *For* $n \in \mathbb{N}$, *let us consider the stochastic integral*

$$\tilde{I}_n(t, \theta) := \int_0^t \psi^{(n)}(s, \theta) dw_n(s) \in \mathbb{R} \tag{3.59}$$

*depending on the parameter* $\theta \in \Theta$.
*Then, there exists an* $\mathcal{F}_t \otimes \mathcal{B}(\Theta)$*-measurable realization of (3.59), which we again denote by* $\tilde{I}_n(t, \theta)$.
*Furthermore,* $\tilde{I}_n(t)$ *coincides* $P$*-almost surely with the* $L^2_\rho$*-valued integral (cf. (3.53))*

$$I_n(t) := \int_0^t \psi^{(n)}(s)\, dw_n(s),$$

*i.e.*

$$\mathbf{E}||\tilde{I}_n(t) - I_n(t)||^2_{L^2_\rho} = \mathbf{E}\left|\int_\Theta |\tilde{I}_n(t, \theta) - (I_n(t))(\theta)|^2\, \mu_\rho(d\theta)\right| = 0.$$

**Proof of Claim 3:** Let us first check that $\tilde{I}_n(t)$ is well-defined for $\mu_\rho$-almost all $\theta \in \Theta$.
Indeed, by the construction (see (3.49) and (3.51)),
$(\psi^{(n)})_{n\in\mathbb{N}} \in L^2(\Omega \times [\,0, T\,] \times \Theta, P \otimes dt \otimes \mu_\rho; l_2)$. Thus, we have

$$\mathbf{E}\int_0^T \int_\Theta \sum_{n\in\mathbb{N}} |\psi^{(n)}(s, \theta)|^2\, \mu_\rho(d\theta)\, ds < \infty.$$

Then, Fubini's theorem gives us

$$\int_\Theta \int_0^T \mathbf{E}|\psi^{(n)}(s, \omega, \theta)|^2\, ds\, \mu_\rho(d\theta) = \mathbf{E}\int_0^T \int_\Theta |\psi^{(n)}(s, \omega, \theta)|^2\, \mu_\rho(d\theta)\, ds < \infty \tag{3.60}$$

and hence

$$\int_0^T \mathbf{E}(\psi^{(n)}(s,\theta))^2\,ds < \infty \tag{3.61}$$

for $\mu_\rho$-almost all $\theta \in \Theta$.
By Itô's isometry, this implies the well-definedness of $\tilde{I}_n(t,\theta)$, $n \in \mathbb{N}$, $t \in [0,T]$, for $\mu_\rho$-almost all $\theta \in \Theta$.

In the following, we will crucially use the measurability of $\tilde{I}_n(t,\theta)$.

We claim that $\tilde{I}_n(t)$ allows an $\mathcal{F}_t \otimes \mathcal{B}(\Theta)$-measurable realization for all $n \in \mathbb{N}$. A general measurability result A.1(a) from [13] states that, under the sufficient condition (3.61), for each $t \in [0,T]$, there exists an $\mathcal{F}_t \otimes \mathcal{B}(\Theta)$-measurable realization of the stochastic integral (3.59). More precisely, one can find an $\mathcal{F}_t \otimes \mathcal{B}(\Theta)$-measurable function $\bar{I}_t^n$: $\Omega \times \Theta \to \mathbb{R}$ and a set $\Theta_0 \in \mathcal{B}(\Theta)$ of full $\mu_\rho$-measure such that $\bar{I}_t^n(\omega,\theta) = \tilde{I}_n(t,\omega,\theta)$ for all $(\omega,\theta) \in \Omega \times \Theta_0$.
Below, we identify $\tilde{I}_n(t)$ with its measurable representative $\bar{I}_t^n$.

So, we want to identify the map $\tilde{I}_n(t)$: $\Omega \times \Theta \to \mathbb{R}$ with the $L^2_\rho$-valued random variable

$$I_n(t) := \int_0^t \psi^{(n)}(s,\cdot)dw_n(s).$$

To this end, we consider cylinder functions $F$ of the form

$$F = F_1 \cdot F_2,\ F_1 \in L^2(\Omega,\mathcal{F},P),\ F_2 \in L^2(\Theta,\mathcal{B}(\Theta),\mu_\rho). \tag{3.62}$$

Obviously, such $F$ belong both to $L^2(\Omega;L^2_\rho) := L^2(\Omega,\mathcal{F},P;L^2_\rho)$ and

$$L^2(\Omega\times\Theta) := L^2(\Omega\times\Theta,\mathcal{F}\otimes\mathcal{B}(\Theta),P\otimes\mu_\rho).$$

We will show that the pairings of $\tilde{I}_n$ and $I_n$ with such functions coincide. Since functions $F$ of the form (3.62) constitute a total set in $L^2(\Omega;L^2_\rho)$, this would imply the identity $\tilde{I}_n = I_n$ as elements of $L^2(\Omega;L^2_\rho)$.

So, let $F \in L^2(\Omega;L^2_\rho)$ be of the form (3.62).

An important observation is that $P$-a.s.

$$\int_0^t \left[\int_\Theta F_2(x)\psi^{(n)}(s,\theta)\,\mu_\rho(d\theta)\right] dw_n(s) = \int_\Theta F_2(\theta)I_n(t,\theta)\,\mu_\rho(d\theta). \tag{3.63}$$

This follows by the stochastic Fubini theorem 4.18 from [26]. A sufficient condition to apply this theorem is that

$F_2\psi \in L^1(\Omega, \mu_\rho; L^2(\Omega \times [0,T], P \otimes ds))$.
Indeed, by the Cauchy inequality we have

$$\int_\Theta \left[ \mathbf{E} \int_0^T (F_2(\theta)\psi^{(n)}(s,\theta))^2\, ds \right]^{\frac{1}{2}} \mu_\rho(d\theta)$$
$$= \int_\Theta F_2(\theta) \left[ \mathbf{E} \int_0^T (\psi^{(n)}(s,\theta))^2\, ds \right]^{\frac{1}{2}} \mu_\rho(d\theta)$$
$$\leq ||F_2||_{L^2_\rho} \left[ \mathbf{E} \int_0^T \int_\Theta (\psi^{(n)}(s,\theta))^2\, \mu_\rho(d\theta)\, ds \right]^{\frac{1}{2}}$$
$$< \infty,$$

where we used (3.61) in the last step.

Thus, we can rewrite the inner product as

$$< I_n(t), F >_{L^2(\Omega; L^2_\rho)}$$
$$= \int_\Omega \left[ \int_\Theta F(\omega,\theta) \left( \int_0^t \psi^{(n)}(s,\cdot)\, dw_n(s) \right) (\omega,\theta)\, \mu_\rho(d\theta) \right] P(d\omega)$$
$$= \int_\Omega F_1(\omega) \left( \int_\Theta F_2(\theta) \left( \int_0^t \psi^{(n)}(s,\cdot)\, dw_n(s) \right) (\omega,\theta)\, \mu_\rho(d\theta) \right) P(d\omega)$$
$$= \mathbf{E} \left[ F_1 \left\langle F_2, \int_0^t \psi^{(n)}(s,\cdot)\, dw_n(s) \right\rangle_{L^2_\rho} \right]$$
$$= \mathbf{E} \left[ F_1 \int_0^t \left\langle F_2, \psi^{(n)}(s,\cdot) \right\rangle_{L^2_\rho} dw_n(s) \right]$$
$$= \mathbf{E} \left[ F_1 \left( \int_0^t \int_\Theta F_2(\theta)\psi^{(n)}(s,\theta)\, \mu_\rho(d\theta)\, dw_n(s) \right) \right]$$
$$= \mathbf{E} \left[ F_1 \left( \int_\Theta F_2(\theta)\tilde{I}_n(t,\theta)\, \mu_\rho(d\theta) \right) \right]$$
$$= \int_\Omega \left[ \int_\Theta F(\omega,\theta)\tilde{I}_n(t,\omega,\theta)\, \mu_\rho(d\theta) \right] P(d\omega)$$
$$=< \tilde{I}_n(t), F >_{L^2(\Omega; L^2_\rho)}.$$

Here, we simply used the definition of the inner product in $L^2_\rho$ in the third and the fifth, Proposition 2.5.3 in the fourth, (3.56) in the sixth, and (3.62) in the second and the second last step.
So, the inner products of $\tilde{I}_n(t)$ and $I_n(t)$ with $F$ of the form (3.62) coincide, which proves Claim 3. $\square$

We finish Step 3 by the following claim:

**Claim 4:** *For any $n \in \mathbb{N}$, we have $I_n(t) \in L^{2\nu}_\rho$, $P$-a.s., and the series*

$$\sum_{n\in\mathbb{N}} I_n(t)$$

*converges in $L^{2\nu}(\Omega, \mathcal{F}, P; L^{2\nu}_\rho)$.*

**Proof:** By the above construction, see Claim 3, there is an $\mathcal{F}_t \otimes \mathcal{B}(\Theta)$-measurable version

$$\tilde{I}_n(t,\theta) = \int_0^t \psi^{(n)}(s,\theta)\, dw_n(s)$$

of $I_n(t)$ for any $n \in \mathbb{N}$.
Since, by Claim 2, the infinite sum

$$\sum_{n\in\mathbb{N}} I_n(t) = \sum_{n\in\mathbb{N}} \int_0^t \psi^{(n)}(s,\cdot)\, dw_n(s)$$

is convergent in $L^2(\Omega, \mathcal{F}, P; L^2_\rho)$, by Claim 3 we get the convergence of

$$\sum_{n\in\mathbb{N}} \tilde{I}_n(t,\theta) = \sum_{n\in\mathbb{N}} \int_0^t \psi^{(n)}(s,\theta)\, dw_n(s)$$

in $L^2(\Omega \times \Theta, \mathcal{F} \otimes \mathcal{B}(\Theta), P \otimes \mu_\rho)$.
Thus, we can apply Fubini's theorem yielding

$$\begin{aligned}
&\mathbf{E}\left\|\int_0^t \chi(s)\, dW(s)\right\|^{2\nu}_{L^{2\nu}_\rho} \\
&= \mathbf{E}\left\|\sum_{n\in\mathbb{N}} I_n(t)\right\|^{2\nu}_{L^{2\nu}_\rho} \\
&= \mathbf{E}\int_\Theta \left(\sum_{n\in\mathbb{N}} \int_0^t \psi^{(n)}(s,\theta)\, dw_n(s)\right)^{2\nu} \mu_\rho(d\theta) \qquad (3.64)\\
&= \int_\Theta \mathbf{E}\left(\sum_{n\in\mathbb{N}} \int_0^t \psi^{(n)}(s,\theta)\, dw_n(s)\right)^{2\nu} \mu_\rho(d\theta) \\
&= \int_\Theta \mathbf{E}\left[\sum_{n\in\mathbb{N}} \tilde{I}_n(t,\theta)\right]^{2\nu} \mu_\rho(d\theta),
\end{aligned}$$

provided the right hand side is finite.
Now, we will show that the partial sums

$$\sum_{n=1}^N \tilde{I}_n(t),\ N \in \mathbb{N},$$

constitute a Cauchy sequence in $L^{2\nu}(\Omega; L^{2\nu}_\rho) := L^{2\nu}(\Omega, \mathcal{F}, P; L^{2\nu}_\rho)$.

Indeed, for any $N, K \in \mathbb{N}$,

$$\text{(3.65)}\quad \mathbf{E}\left|\left|\sum_{n=N}^{N+K-1} \tilde{I}_n(t)\right|\right|_{L_\rho^{2\nu}}^{2\nu} = \mathbf{E}\int_\Theta \left(\sum_{n=N}^{N+K-1} \tilde{I}_n(t,\theta)\right)^{2\nu} \mu_\rho(d\theta)$$
$$= \int_\Theta \mathbf{E}\left(\sum_{n=N}^{N+K-1} \int_0^t \psi^{(n)}(s,\omega,\theta)\, dw_n(s,\omega)\right)^{2\nu} \mu_\rho(d\theta).$$

Setting

$$\psi^K(s,\theta) := \left(\psi^{(n)}(s,\theta)\right)_{n=N}^{N+K-1}$$

and

$$W^K(s) := (w_n(s))_{n=N}^{N+K-1},$$

by the Burkholder-Gundy inequality for multi-dimensional Wiener processes we have, for a fixed $\theta \in \Theta$,

$$\begin{aligned}\mathbf{E}\left(\sum_{n=N}^{N+K-1} \int_0^t \psi^{(n)}(s,\theta)\, dw_n(s)\right)^{2\nu} &= \mathbf{E}\left(\left\langle \int_0^t \psi^K(s,\cdot,\theta), dW^K\right\rangle_{\mathbb{R}^K}\right)^{2\nu} \\ &\le c(\nu)\mathbf{E}\left(\int_0^t ||\psi^K(s,\cdot,\theta)||_{\mathbb{R}^K}^2\, ds\right)^{\nu} \\ &= c(\nu)\mathbf{E}\left[\int_0^t \sum_{n=N}^{N+K-1} \psi^{(n)}(s,\theta)^2\, ds\right]^{\nu}.\end{aligned}$$

Thus, we can continue (3.65) as

$$\text{(3.66)}\quad \mathbf{E}\left|\left|\sum_{n=N}^{N+K-1} \tilde{I}_n(t)\right|\right|_{L_\rho^{2\nu}}^{2\nu}$$
$$\le c(\nu)\int_\Theta \mathbf{E}\left[\int_0^t \sum_{n=N}^{N+K-1} (\psi^{(n)}(s,\theta))^2\, ds\right]^{\nu} \mu_\rho(d\theta)$$
$$= c(\nu)\mathbf{E}\int_\Theta \left[\int_0^t \sum_{n=N}^{N+K-1} (\psi^{(n)}(s))^2\, ds\right]^{\nu}(\theta)\, \mu_\rho(d\theta)$$
$$= c(\nu)\mathbf{E}\int_\Theta \left(\int_0^t \sum_{n=N}^{N+K-1} (\chi(s)Q^{\frac{1}{2}}e_n)^2\, ds\right)^{\nu}(\theta)\, \mu_\rho(d\theta)$$
$$= c(\nu)\mathbf{E}\int_\Theta \left(\int_0^t \sum_{n=N}^{N+K-1} (U(t,s)\mathcal{M}_{\varphi(s)}Q^{\frac{1}{2}}e_n)^2\, ds\right)^{\nu}(\theta)\, \mu_\rho(d\theta),$$

where in the last three lines we passed to the Bochner integral over $[0, T]$ in $L_\rho^1$ (cf. Remark 3.1.2.1 **(ii)** above) and used Claim 1.
Since by assumption **(A4)**

$$\begin{aligned}
&\mathbf{E}\int_{\Theta}\left(\int_0^t \sum_{n\in\mathbb{N}} (U(t,s)\mathcal{M}_{\varphi(s)}Q^{\frac{1}{2}}e_n)^2\, ds\right)^{\nu} d\mu_\rho \\
&\le c(\nu,T)\int_0^t (t-s)^{-\zeta}\mathbf{E}||\varphi(s)||^{2\nu}_{L^{2\nu}_\rho}\, ds \\
&< \infty,
\end{aligned}$$

by Lebesgue's theorem we can conclude that the last line in (3.66) tends to 0 as $N, K \to \infty$.
So, we have proven that $\sum_n \tilde{I}_n(t)$ converges in $L^{2\nu}(\Omega; L^{2\nu}_\rho)$. $\triangle$

To complete the proof of Proposition 3.3.1, let us recall that $\sum_n \tilde{I}_n(t)$ converges to $I^W_\varphi(t)$ in $L^2(\Omega; L^2_\rho)$ by Claims 2 and 3.
Thus, $P$-a.s., we get the required inclusion

$$I^W_\varphi(t) = \int_0^t \chi(s)\, dW(s) \in L^{2\nu}_\rho.$$

It remains to show estimate (3.40). But this follows immediately by combining (3.66) with (3.64) and (3.65). ■

**Remark 3.4.4:** *Actually, Proposition 3.4.3 extends to any predictable $\mathcal{L}_2$-valued process $(\chi(t))_{t\in[0,T]}$ such that*

$$\mathbf{E}\int_0^T ||\chi(s)||^2_{\mathcal{L}_2}\, dt < \infty.$$

*Namely, we can prove that*

$$\int_0^t \chi(s)\, dW(s) \in L^{2\nu}_\rho \;\; (P\text{-a.s.})$$

*and*

$$\begin{aligned}
(3.67)\ &\mathbf{E}\left|\left|\int_0^t \chi(s)\, dW(s)\right|\right|^{2\nu}_{L^{2\nu}_\rho} \\
&\le c(\nu)\mathbf{E}\left[\int_0^t ||\chi(s)Q^{\frac{1}{2}}||_{\mathcal{L}_2(L^2,L^{2\nu}_\rho)}\, ds\right]^{\nu} \\
&\le c(\nu,T)\int_0^t \mathbf{E}||\chi(s)Q^{\frac{1}{2}}||^{2\nu}_{\mathcal{L}_2(L^2,L^{2\nu}_\rho)}\, ds,
\end{aligned}$$

*provided the right hand side in (3.67) is finite.*
*The proof of (3.67) follows by Minkowski's inequality applied to (3.66) (see Remark 3.1.2.1 **(iv)**).*

To finish this section, we discuss the continuity property of the stochastic convolution (3.26).

First, we recall the following proposition from [76], the proof of which is based on the so-called factorization method for Wiener-type convolutions (cf. e.g. [25] or the proof of Theorem 5.9 in [26]) and the Burkholder-Davis-Gundy inequality (see Section 2.5).

**Theorem 3.4.5:** (cf. [76], Theorem 3.1.1 there)
*Given a predictable process $\varphi$: $[0,T] \times \Omega \to L^2_\rho$, suppose that*

$$\mathbf{E}\int_0^T ||\varphi(t)||^q_{L^2_\rho}\, dt < \infty \tag{3.68}$$

*for some $q > 2$ in the nuclear case and $q > \frac{2}{1-\zeta}$ with $\zeta \in [0,1)$ as in* ***(A2)*** *in the general nuclear and in the cylindrical case.*

*Then, there exists a continuous modification of the process*

$$[0,T] \ni t \mapsto \int_0^t U(t,s)\mathcal{M}_{\varphi(s)}\, dW(s) \in L^2_\rho.$$

Note that in Theorem 3.4.5 it is really necessary to assume $q > 2$ in the nuclear case resp. $q > \frac{2}{1-\zeta}$ in the general nuclear and in the cylindrical case. Otherwise the factorization method for Wiener-type convolutions would not be applicable.

Since, in later chapters, the presence of jump terms in the equations (1.1) and (1.2) causes us to consider other continuity properties, we finish this section by the following propositions, which seem to be new for evolution operators $U(t,s)$, $0 \le s \le t \le T$.

**Proposition 3.4.6:** *Suppose that the conditions of Proposition 3.4.1 hold for the evolution operator $U$.*
*Let $(\varphi(t))_{t\in[0,T]}$ be an $L^2_\rho$-valued predictable process obeying the uniform moment bound*

$$\sup_{t\in[0,T]} \mathbf{E}||\varphi(t)||^2_{L^2_\rho} < \infty. \tag{3.69}$$

*Then,*

$$t \mapsto I^W_\varphi(t) := \int_0^t U(t,s)\mathcal{M}_{\varphi(s)}\, dW(s)$$

*is continuous in* $L^2(\Omega; L^2_\rho)$ *in the nuclear, general nuclear and cylindrical case.*

**Proof:** We extend here a method of proving meansquare continuity, which is used e.g. by Knoche in [60] and by Knäble in [59], to the case of non-Hilbert-Schmidt operator valued coefficients $\mathcal{M}_{\varphi(t)}$.

For $\alpha > 1$, consider the process

$$\Phi^\alpha(t) := \int_0^{\frac{t}{\alpha}} U(t,s)\mathcal{M}_{\varphi(s)}\, dW(s) \in L^2_\rho,\ 0 \le t \le T, \tag{3.70}$$

which is well-defined in all three cases by **(A2)** and (3.69).
We claim that $\Phi^\alpha(t)$, $0 \le t \le T$, is meansquare continuous. Indeed, for any $0 \le r \le t \le T$, we have by Itô's isometry

$$\begin{aligned}\mathbf{E}||\Phi^\alpha(t) - \Phi^\alpha(r)||^2_{L^2_\rho} \le\ & 2\left(\int_0^{\frac{r}{\alpha}} \mathbf{E}||[U(t,s) - U(r,s)]\mathcal{M}_{\varphi(s)}||^2_{\mathcal{L}_2}\, ds\right.\\ &\left. + \int_{\frac{r}{\alpha}}^{\frac{t}{\alpha}} \mathbf{E}||U(t,s)\mathcal{M}_{\varphi(s)}||^2_{\mathcal{L}_2}\, ds\right).\end{aligned} \tag{3.71}$$

Let us start with the first integral on the right hand side. We consider simultaneously the nuclear and the cylindrical case. Note that

$$\begin{aligned}&\int_0^{\frac{r}{\alpha}} \mathbf{E}||[U(t,s) - U(r,s)]\mathcal{M}_{\varphi(s)}||^2_{\mathcal{L}_2}\, ds\\ &= \int_0^{\frac{r}{\alpha}} \mathbf{E} \sum_{n\in\mathbb{N}} ||[U(t,s) - U(r,s)]\mathcal{M}_{\varphi(s)}Q^{\frac{1}{2}}e_n||^2_{L^2_\rho}\, ds\\ &= \int_0^{\frac{r}{\alpha}} \mathbf{E} \sum_{n\in\mathbb{N}} ||[U(t,\alpha s) - U(r,\alpha s)]U(\alpha s,s)\mathcal{M}_{\varphi(s)}Q^{\frac{1}{2}}e_n||^2_{L^2_\rho}\, ds,\end{aligned}$$

where we have $Q = \mathbf{I}$ in the cylindrical case.
Note that both in the nuclear and in the cylindrical case the orthonormal basis $(e_n)_{n\in\mathbb{N}} \subset L^2$ of eigenvectors of $Q$ especially fulfills $(e_n)_{n\in\mathbb{N}} \subset L^\infty$.
By the strong continuity of $U$ in $L^2_\rho$ we have, for any $s \in [\,0,T\,]$

(3.72) $\mathbf{1}_{[\,0,\frac{r}{\alpha}]}(s)||[U(t,\alpha s) - U(r,\alpha s)]U(\alpha s,s)\mathcal{M}_{\varphi(s)}Q^{\frac{1}{2}}e_n||^2_{L^2_\rho} \to 0$ $P$-a.s. as $r \uparrow t$ resp. $t \downarrow r$.

On the other hand, uniformly for all $0 \le r \le t \le T$,

(3.73) $\mathbf{1}_{[\,0,\frac{r}{\alpha}]}(s)||[U(t,\alpha s) - U(r,\alpha s)]U(\alpha s,s)\mathcal{M}_{\varphi(s)}Q^{\frac{1}{2}}e_n||^2_{L^2_\rho}$

$$\le 2c(T)||U(\alpha s,s)\mathcal{M}_{\varphi(s)}Q^{\frac{1}{2}}e_n||^2_{L^2_\rho}.$$

In the nuclear case we have

$$(3.74)\quad \int_0^T \mathbf{E}||U(\alpha s,s)\mathcal{M}_{\varphi(s)}||^2_{\mathcal{L}_2}$$
$$\le c(T)\, tr\, Q\, T\left(\sup_{t\in[0,T]}\mathbf{E}||\varphi(t)||^2_{L^2_\rho}\right) < \infty,$$

whereas in the cylindrical case we have, by **(A2)**,

$$(3.75)\quad \int_0^T \mathbf{E}||U(\alpha s,s)\mathcal{M}_{\varphi(s)}||^2_{\mathcal{L}_2}$$
$$\le c(T)\int_0^T((\alpha-1)s)^{-\zeta}\mathbf{E}||\varphi(s)||^2_{L^2_\rho}\,ds$$
$$\le c(T)\left(\sup_{t\in[0,T]}\mathbf{E}||\varphi(t)||^2_{L^2_\rho}\right)(\alpha-1)^{-\zeta}\int_0^T s^{-\zeta}\,ds < \infty.$$

Thus, Lebesgue's theorem is applicable, which gives us the convergence to 0 as $r\uparrow t$ resp. $t\downarrow r$ of the first integral in (3.71).
The second integral in (3.71) can be estimated as follows. In the nuclear case, analogously to (3.74), we get

$$\int_{\frac{r}{\alpha}}^{\frac{t}{\alpha}} \mathbf{E}||U(t,s)\mathcal{M}_{\varphi(s)}||^2_{\mathcal{L}_2}\,ds$$
$$\le c(T) tr\, Q\left(\sup_{t\in[0,T]}\mathbf{E}||\varphi(t)||^2_{L^2_\rho}\right)\left(\tfrac{t-r}{\alpha}\right),$$

whereas in the cylindrical case, analogously to (3.75), we get

$$\int_{\frac{r}{\alpha}}^{\frac{t}{\alpha}} \mathbf{E}||U(t,s)\mathcal{M}_{\varphi(s)}||^2_{\mathcal{L}_2}\,ds$$
$$\le c(T)\int_{\frac{r}{\alpha}}^{\frac{t}{\alpha}}(t-s)^{-\zeta}\mathbf{E}||\varphi(s)||^2_{L^2_\rho}\,ds$$
$$\le c(T)\left(\sup_{t\in[0,T]}\mathbf{E}||\varphi(t)||^2_{L^2_\rho}\right)\int_{\frac{r}{\alpha}}^{\frac{t}{\alpha}} s^{-\zeta}\,ds \to 0 \text{ as } r\uparrow t \text{ resp. } t\downarrow r.$$

Thus, it remains to consider the general nuclear case.
Similar to the previous considerations, for the first integral in the right hand side of (3.71) we have

$$\int_0^{\frac{r}{\alpha}} \mathbf{E}||[U(t,s)-U(r,s)]\mathcal{M}_{\varphi(s)}||^2_{\mathcal{L}_2}\,ds$$
$$= \int_0^{\frac{r}{\alpha}} \mathbf{E}\sum_{n\in\mathbb{N}} ||[U(t,s)-U(r,s)]\mathcal{M}_{\varphi(s)}Q^{\frac{1}{2}}e_n||^2_{L^2_\rho}\,ds$$
$$= \int_0^{\frac{r}{\alpha}} \mathbf{E}\sum_{n\in\mathbb{N}} ||[U(t,\alpha s)-U(r,\alpha s)]U(\alpha s,s)\mathcal{M}_{\varphi(s)}Q^{\frac{1}{2}}e_n||^2_{L^2_\rho}\,ds.$$

In contrast to the nuclear case, we do not assume that $(e_n)_{n\in\mathbb{N}} \subset L^\infty$ obeying (3.1). We have

$$\int_0^{\frac{r}{\alpha}} \mathbf{E}||[U(t,s)-U(r,s)]\mathcal{M}_{\varphi(s)}||^2_{\mathcal{L}_2}\,ds = \int_0^T \sum_{n\in\mathbb{N}} \mathbf{1}_{[0,\frac{r}{\alpha}]}(s)\mathbf{E}||[U(t,s)-U(r,s)]\mathcal{M}_{\varphi(s)}Q^{\frac{1}{2}}e_n||^2_{L^2_\rho}\,ds,$$

whereby uniformly for all $n \in \mathbb{N}$ and all $0 \le r \le t \le T$,

$$\mathbf{1}_{[0,\frac{r}{\alpha}]}(s)||[U(t,s)-U(r,s)]\mathcal{M}_{\varphi(s)}Q^{\frac{1}{2}}e_n||^2_{L^2_\rho}$$
$$= \mathbf{1}_{[0,\frac{r}{\alpha}]}(s)||[U(t,\alpha s)-U(r,\alpha s)]U(\alpha s,s)\mathcal{M}_{\varphi(s)}Q^{\frac{1}{2}}e_n||^2_{L^2_\rho}$$
$$\le 2c(T)\mathbf{1}_{[0,\frac{r}{\alpha}]}(s)||U(\alpha s,s)\mathcal{M}_{\varphi(s)}Q^{\frac{1}{2}}e_n||^2_{L^2_\rho}\,ds.$$

Since, by **(A2)**$U(\alpha s,s)\mathcal{M}_{\varphi(s)}Q^{\frac{1}{2}}e_n \in L^2_\rho$ $P$-a.s., for any $s \in [0,T]$ and any $n \in \mathbb{N}$, we get

$$\mathbf{1}_{[0,\frac{r}{\alpha}]}(s)||[U(t,\alpha s)-U(r,\alpha s)]U(\alpha s,s)\mathcal{M}_{\varphi(s)}Q^{\frac{1}{2}}e_n||^2_{L^2_\rho} \to 0$$

$P$-a.s. as $r \uparrow t$ resp. $t \downarrow r$. Furthermore,

$$\int_0^T \mathbf{E}\sum_{n\in\mathbb{N}} 2c(T)\mathbf{1}_{[0,\frac{r}{\alpha}]}(s)||U(\alpha s,s)\mathcal{M}_{\varphi(s)}Q^{\frac{1}{2}}e_n||^2_{L^2_\rho}\,ds$$
$$\le 2c(T)\,tr\,Q\int_0^T \mathbf{E}||\varphi(s)||^2_{L^2_\rho}\,ds\,\underbrace{\sup_{n\in\mathbb{N}}||e_n||_{L^2}}_{=1}.$$

Thus, Lebesgue's theorem gives the convergence to 0 of the first integral in (3.71).
The second integral can be estimated by

$$\int_{\frac{r}{\alpha}}^{\frac{t}{\alpha}} \mathbf{E}||U(t,s)\mathcal{M}_{\varphi(s)}||^2_{\mathcal{L}_2}\,ds \le c(T)T\,tr\,Q\underbrace{\sup_{n\in\mathbb{N}}||e_n||_{L^2}}_{=1}\left(\sup_{t\in[0,T]}\mathbf{E}||\varphi(t)||^2_{L^2_\rho}\right)\left(\tfrac{t-r}{\alpha}\right)$$

$\to 0$ as $r \uparrow t$ resp. $t \downarrow r$.

Thus, in all three cases, the mapping $[0,T] \ni t \mapsto \Phi^\alpha(t) \in L^2_\rho$ is mean-square continuous. Now we observe that, for any $\alpha > 1$,

$$\sup_{t\in[0,T]} \mathbf{E}||I_\varphi^W(t) - \Phi^\alpha(t)||^2_{L^2_\rho} = \sup_{t\in[0,T]} \int_{\frac{t}{\alpha}}^{t} \mathbf{E}||U(t,s)\mathcal{M}_{\varphi(s)}||^2_{\mathcal{L}_2}\, ds$$

in all three cases.

Since in the nuclear and in the general nuclear case we have

$$||U(t,s)\mathcal{M}_{\varphi(s)}||^2_{\mathcal{L}_2} \le trQ\, ||U(t,s)\mathcal{M}_{\varphi(s)}||^2_{\mathcal{L}_2(L^2,L^2_\rho)}$$

$P$-a.s. for any $s \in [0,T]$, it suffices to consider only the cylindrical case.

In the cylindrical case, we have

$$\begin{aligned}
\sup_{t\in[0,T]} \mathbf{E}||I_\varphi^W(t) - \Phi^\alpha(t)||^2_{L^2_\rho} &= \sup_{t\in[0,T]} \int_{\frac{t}{\alpha}}^{t} \mathbf{E}||U(t,s)\mathcal{M}_{\varphi(s)}||^2_{\mathcal{L}_2}\, ds \\
&\le c(T) \sup_{t\in[0,T]} \int_{\frac{t}{\alpha}}^{t} (t-s)^{-\zeta}\mathbf{E}||\varphi(s)||^2_{L^2_\rho}\, ds \\
&\le c(T) \left( \sup_{t\in[0,T]} \mathbf{E}||\varphi(t)||^2_{L^2_\rho} \right) \sup_{t\in[0,T]} \int_{\frac{t}{\alpha}}^{t} s^{-\zeta}\, ds \\
&\le c(T) \left( \sup_{t\in[0,T]} \mathbf{E}||\varphi(t)||^2_{L^2_\rho} \right) \frac{T^{1-\zeta} - \left(\frac{T}{\alpha}\right)^{1-\zeta}}{1-\zeta},
\end{aligned}$$

which tends to 0 as $\alpha \downarrow 1$.
Thus, $I_\varphi^W$ is also meansquare continuous as a uniform limit in $C([0,T], L^2(\Omega; L^2_\rho))$ of $I^\alpha$ as $\alpha \downarrow 1$. ■

A generalization of Proposition 3.4.6 to $q \ge 2$ is the following:

**Proposition 3.4.7:** *Let the assumptions of Proposition 3.4.5 hold. Suppose additionally that*

$$\sup_{t\in[0,T]} \mathbf{E}||\varphi(t)||^q_{L^2_\rho} < \infty$$

*for some $q \in [2, \frac{2}{\zeta})$ with $\zeta = 0$ in the nuclear case and $\zeta$ as in **(A2)** (or **(A5)** with $\nu = 1$) in the general nuclear and in the cylindrical case. Then, the mapping $t \mapsto I_\varphi^W(t)$ is continuous in $L^q(\Omega; L^2_\rho)$.*

**Proof:** We keep the same notation and repeat the arguments used in proving Proposition 3.4.6.
Let us start again with the cylindrical case. Using the Burkholder-Davis-Gundy inequality and Hölder's inequality (similarly to the proof of Propo-

sition 3.4.1), we arrive at the following estimate for $0 \le r \le t \le T$:

$$(3.76)\quad \mathbf{E}||\Phi^{\alpha}(t) - \Phi^{\alpha}(r)||^q_{L^2_\rho} \le c(q,T)\left[\int_0^{\frac{r}{\alpha}} \mathbf{E}||[U(t,s) - U(r,s)]\mathcal{M}_{\varphi(s)}||^q_{\mathcal{L}_2}\,ds + \int_{\frac{r}{\alpha}}^{\frac{t}{\alpha}} \mathbf{E}||U(t,s)\mathcal{M}_{\varphi(s)}||^q_{\mathcal{L}_2}\,ds\right].$$

The first integral tends to 0 as $r \uparrow t$ resp. $t \downarrow r$ by Lebesgue's theorem, where we use (3.72), (3.73) and the uniform bound

$$\int_0^T \mathbf{E}||U(\alpha s,s)\mathcal{M}_{\varphi(s)}||^q_{\mathcal{L}_2}\,ds \le c(q,T)\int_0^T ((\alpha-1)s)^{-\frac{q\zeta}{2}}\mathbf{E}||\varphi(s)||^q_{L^2_\rho}\,ds.$$

The second integral in (3.76) tends to 0 as $r \uparrow t$ resp. $t \downarrow r$, since

$$\begin{aligned}\int_{\frac{r}{\alpha}}^{\frac{t}{\alpha}} \mathbf{E}||U(t,s)\mathcal{M}_{\varphi(s)}||^q_{\mathcal{L}_2}\,ds &\le \int_{\frac{r}{\alpha}}^{\frac{t}{\alpha}} (t-s)^{-\frac{q\zeta}{2}}\mathbf{E}||\varphi(s)||^q_{L^2_\rho}\,ds \\ &\le \left(\sup_{t\in[0,T]}\mathbf{E}||\varphi(t)||^q_{L^2_\rho}\right)\int_{\frac{r}{\alpha}}^{\frac{t}{\alpha}} s^{-\frac{q\zeta}{2}}\,ds \\ &\le \left(\sup_{t\in[0,T]}\mathbf{E}||\varphi(t)||^q_{L^2_\rho}\right)\left(\frac{\left(\frac{t}{\alpha}\right)^{1-\frac{q\zeta}{2}} - \left(\frac{r}{\alpha}\right)^{1-\frac{q\zeta}{2}}}{1-\frac{q\zeta}{2}}\right) \\ &\longrightarrow 0 \text{ as } r\uparrow t \text{ esp. } t\downarrow r.\end{aligned}$$

In the nuclear and general nuclear case, the integrals in the right hand side of (3.76) are estimated by the ones from the cylindrical case similary to the proof of Proposition 3.4.6. ■

Moreover, we have an extension of Proposition 3.4.4 to the spaces $L^{2\nu}_\rho$ with $\nu > 1$.

**Proposition 3.4.8:** *Suppose that $U$ obeys the assumptions from Proposition 3.4.5 and additionally **(A5)** in the cylindrical and **(A5)*** in the general nuclear case (see also the remark about these conditions in Section 3.1 (cf. Remark 3.1.2.1) and in the formulation of Proposition 3.4.3).*
*Furthermore, let $(\varphi(t))_{t\in[0,T]}$ be an $L^{2\nu}_\rho$-valued predictable process as in Proposition 3.4.3 but additionally obeying the uniform moment bound*

$$(3.77)\quad \sup_{t\in[0,T]}\mathbf{E}||\varphi(t)||^{2\nu}_{L^{2\nu}_\rho} < \infty.$$

*Then, assuming that $\nu \in [1\frac{1}{\zeta})$ with $\zeta = 0$ in the nuclear case and $\zeta \in [0,1)$ as in **(A5)** resp. **(A5)*** in the cylindrical resp. in the general nuclear case,*

$$t \mapsto I_\varphi^W(t) := \int_0^t U(t,s)\mathcal{M}_{\varphi(s)}\, dW(s)$$

*is continuous in* $L^{2\nu}(\Omega; L_\rho^{2\nu})$.

**Proof:** As was shown in Proposition 3.4.3, $I_\varphi^W \in L_\rho^{2\nu}$, $P$-a.s., for each $t \in [\,0, T\,]$.
Now, the previous scheme of proving Proposition 3.4.6 and 3.4.7 runs with $\nu > 1$ if we use the strong continuity of $U(t,s)$ in $L_\rho^{2\nu}$ (see assumption **(A3)**). Indeed, for $\Phi^\alpha(t) \in L_\rho^{2\nu}$ defined by (3.70) and $0 \le r \le t \le T$, we have

$$(3.78)\ \mathbf{E}||\Phi^\alpha(t) - \Phi^\alpha(r)||_{L_\rho^{2\nu}}^{2\nu}$$
$$\le c(\nu)\left[\mathbf{E}\left(\int_0^{\frac{r}{\alpha}} ||[U(t,\alpha s) - U(r,\alpha s)]U(\alpha s,s)\mathcal{M}_{\varphi(s)}Q^{\frac{1}{2}}||_{\mathcal{L}_2(L^2,L_\rho^{2\nu})}^2\, ds\right)^\nu\right.$$
$$\left.+\mathbf{E}\left(\int_{\frac{r}{\alpha}}^{\frac{r}{\alpha}} ||U(t,s)\mathcal{M}_{\varphi(s)}Q^{\frac{1}{2}}||_{\mathcal{L}_2(L^2,L_\rho^{2\nu})}^2\, ds\right)^\nu\right],$$

where we use the moment estimate (3.67) (cf. Remark 3.4.4).
Concerning the first integral on the right hand side of (3.78) we note that, by **(A5)**/ **(A5)***, in all three cases we have, $P$-a.s., $U(\alpha s,s)\mathcal{M}_{\varphi(s)}Q^{\frac{1}{2}}e_n \in L_\rho^{2\nu}$ for any $n \in \mathbb{N}$, where $(e_n)_{n\in\mathbb{N}} \subset L^2$ denotes an orthonormal basis of $L^2$ consisting of eigenvectors of the operator $Q$ in the nuclear and general nuclear case.
Therefore, by the continuity assumption from **(A3)**, we have, for any $n \in \mathbb{N}$ and $s \in [\,0, T\,]$,

$\mathbf{1}_{[\,0,\frac{r}{\alpha}\,]}(s)[U(t,\alpha s) - U(r,\alpha s)]U(\alpha s,s)\mathcal{M}_{\varphi(s)}Q^{\frac{1}{2}}e_n \to 0 \in L_\rho^{2\nu}$ as $r \uparrow t$ resp. $t \downarrow r$, $P$-a.s.,

which implies that, for any $s \in [\,0, T\,]$,

$\mathbf{1}_{[\,0,\frac{r}{\alpha}\,]}(s)[U(t,\alpha s) - U(r,\alpha s)]U(\alpha s,s)\mathcal{M}_{\varphi(s)}Q^{\frac{1}{2}} \to 0 \in \mathcal{L}_2(L^2, L_\rho^{2\nu})$ as $r \uparrow t$ resp. $t \downarrow r$, $P$-a.s..

Note that so far there were no restrictions on $Q$.
Furthermore, by **(A3)**, **(A5)***, Hölder's inequality and the fact that

$$\nu \in [\,1, \tfrac{1}{\zeta}\,) \Rightarrow \zeta\nu < 1,$$

in the nuclear and general nuclear case we have the estimate

$$\mathbf{E}\left(\int_0^{\frac{r}{\alpha}} ||[U(t,\alpha s) - U(r,\alpha s)]U(\alpha s,s)\mathcal{M}_{\varphi(s)}Q^{\frac{1}{2}}||_{\mathcal{L}_2(L^2,L_\rho^{2\nu})}^2\, ds\right)^\nu$$

$$\le (tr\, Q)^\nu \, \mathbf{E} \left( \int_0^{\frac{r}{\alpha}} ||U(t,\alpha s) - U(r,\alpha s)||^2_{\mathcal{L}(L^{2\nu}_\rho)} ||U(\alpha s,s)\mathcal{M}_{\varphi(s)}||^2_{\mathcal{L}(L^2,L^{2\nu}_\rho)}\, ds \right)^\nu$$
$$\le c(\nu,T,c(T))\,(tr\, Q)^\nu \, \mathbf{E} \left( \int_0^{\frac{r}{\alpha}} ((\alpha-1)s)^{-\zeta} ||\varphi(s)||^2_{L^{2\nu}_\rho}\, ds \right)^\nu$$
$$\le c(\nu,T,c(T))\,(tr\, Q)^\nu \, \mathbf{E} \int_0^{\frac{r}{\alpha}} ((\alpha-1)s)^{-\zeta\nu} ||\varphi(s)||^{2\nu}_{L^{2\nu}_\rho}\, ds$$
$$\le c(\nu,T,c(T))\,(tr\, Q)^\nu \left( \sup_{t\in[0,T]} \mathbf{E}||\varphi(s)||^{2\nu}_{L^{2\nu}_\rho} \right) (\alpha-1)^{\zeta\nu} \left( \tfrac{s^{1-\zeta\nu}}{1-\zeta\nu} \right)$$
$$< \infty.$$

Concerning the cylindrical case, we note that just applying **(A5)** instead of **(A5)*** the previous chain of arguments changes to

$$\mathbf{E} \left( \int_0^{\frac{r}{\alpha}} ||[U(t,\alpha s) - U(r,\alpha s)]U(\alpha s,s)\mathcal{M}_{\varphi(s)}Q^{\frac{1}{2}}||^2_{\mathcal{L}_2(L^2,L^{2\nu}_\rho)}\, ds \right)^\nu$$
$$\le (tr\, Q)^\nu \, \mathbf{E} \left( \int_0^{\frac{r}{\alpha}} ||U(t,\alpha s) - U(r,\alpha s)||^2_{\mathcal{L}(L^{2\nu}_\rho)} ||U(\alpha s,s)\mathcal{M}_{\varphi(s)}||^2_{\mathcal{L}_2(L^2,L^{2\nu}_\rho)}\, ds \right)^\nu$$
$$\le c(\nu,T,c(T))\,(tr\, Q)^\nu \, \mathbf{E} \left( \int_0^{\frac{r}{\alpha}} ((\alpha-1)s)^{-\zeta} ||\varphi(s)||^2_{L^{2\nu}_\rho}\, ds \right)^\nu$$
$$\le c(\nu,T,c(T))\,(tr\, Q)^\nu \, \mathbf{E} \int_0^{\frac{r}{\alpha}} ((\alpha-1)s)^{-\zeta\nu} ||\varphi(s)||^{2\nu}_{L^{2\nu}_\rho}\, ds$$
$$\le c(\nu,T,c(T))\,(tr\, Q)^\nu \left( \sup_{t\in[0,T]} \mathbf{E}||\varphi(s)||^{2\nu}_{L^{2\nu}_\rho} \right) (\alpha-1)^{\zeta\nu} \left( \tfrac{s^{1-\zeta\nu}}{1-\zeta\nu} \right)$$
$$< \infty.$$

Thus, we can apply Lebesgue's dominated convergence theorem to get convergence to 0 as $r \uparrow t$ resp. $t \downarrow r$ for the first integral in estimate (3.78) in all three cases.
Concerning the second integral on the right hand side of (3.78) note that, in the nuclear and general nuclear case, we have

$$\begin{aligned}
\mathbf{E} \left( \int_{\frac{r}{\alpha}}^{\frac{r}{\alpha}} ||U(t,s)\mathcal{M}_{\varphi(s)}Q^{\frac{1}{2}}||^2_{\mathcal{L}_2(L^2,L^{2\nu}_\rho)}\, ds \right)^\nu &\le (tr\, Q)^\nu \mathbf{E} \left( \int_{\frac{r}{\alpha}}^{\frac{r}{\alpha}} (t-s)^{-\zeta} ||\varphi(s)||^2_{L^{2\nu}_\rho}\, ds \right)^\nu \\
&\le (tr\, Q)^\nu \left( \sup_{t\in[0,T]} \mathbf{E}||\varphi(s)||^{2\nu}_{L^{2\nu}_\rho} \right) \frac{(\frac{t-r}{\alpha})^{1-\zeta\nu}}{1-\zeta\nu},
\end{aligned}$$

whereas in the cylindrical case we have

$$\mathbf{E}\left(\int_{\frac{r}{\alpha}}^{\frac{r}{\alpha}} ||U(t,s)\mathcal{M}_{\varphi(s)}||^2_{\mathcal{L}_2(L^2,L^2_\rho)}\,ds\right)^\nu \le \mathbf{E}\left(\int_{\frac{r}{\alpha}}^{\frac{r}{\alpha}} (t-s)^{-\zeta}||\varphi(s)||^2_{L^{2\nu}_\rho}\,ds\right)^\nu$$
$$\le \left(\sup_{t\in[\,0,T\,]} \mathbf{E}||\varphi(s)||^{2\nu}_{L^{2\nu}_\rho}\right) \frac{(\frac{t-r}{\alpha})^{1-\zeta\nu}}{1-\zeta\nu}.$$

Obviously, both estimates tend to 0 as $r \uparrow t$ resp. $t \downarrow r$. ■

Next, we give an alternative and very short proof of Proposition 3.4.6 by using the result of Theorem 3.4.4.

**Alternative proof of Proposition 3.4.8:** First, assume that the integrand process $\varphi(t)$, $t \in [\,0,T\,]$, satisfies conditions of Theorem 3.4.4, i.e.

$$\sup_{t\in[\,0,T\,]} \mathbf{E}||\varphi(t)||^{2q}_{L^{2q}_\rho} < \infty \tag{3.79}$$

for a large enough $q > \nu$.
We have to prove that for each sequence $(t_n)_{n\in\mathbb{N}} \subset [\,0,T\,]$, $t_n \to t$ as $n \to \infty$,

$$\mathbf{E}||I^W_\varphi(t_n) - I^W_\varphi(t)||^{2\nu}_{L^{2\nu}_\rho} \to 0 \text{ as } n \to \infty.$$

To this end, we use an $\mathcal{F}_t \otimes \mathcal{B}(\Theta)$-measurable modification of $I^W_\varphi(t)$, $t \in [\,0,T\,]$, which exists by Step 3 from the proof of Proposition 3.3.1. We have

$$\sup_{t\in[\,0,T\,]} \mathbf{E}\int_\Theta |I^W_\varphi(t,\theta)|^{2q}\,\mu_\rho(d\theta) = \sup_{t\in[\,0,T\,]} \mathbf{E}||I^W_\varphi(t)||^{2q}_{L^{2q}_\rho} < \infty. \tag{3.80}$$

We claim that the pathwise $L^2_\rho$-continuity of $t \mapsto I^W_\varphi(t)$ together with the uniform bound (3.80) imply the $P \otimes \mu_\rho$-continuity of

$$[\,0,T\,] \ni t \mapsto I^W_\varphi(t,\theta), \theta \in \Theta.$$

Indeed, by Lebesgue's dominated covergence theorem for any $\varepsilon > 0$ we have

$$P \otimes \mu_\rho(\{(\omega,\theta)\,|\,|I^W_\varphi(t_n,\omega,\theta) - I^W_\varphi(t,\omega,\theta)| > \varepsilon\})$$
$$= \int_\Omega \left(\int_\Theta \mathbf{1}_{\{(\omega,\theta)\,|\,|I^W_\varphi(t_n,\omega,\theta)-I^W_\varphi(t,\omega,\theta)|>\varepsilon\}}\,d\mu_\rho\right) dP \to 0 \text{ as } n \to \infty.$$

Due to the $L^2_\rho$-continuity of $I^W_\varphi$, the inner integral in the previous line converges to 0 as $n \to \infty$ for almost all $\omega \in \Omega$. Thus, by (3.73) and (3.74), we can apply the de la Vallée-Poussin theorem with any $q > \nu$. This yields

$$\lim_{n\to\infty} \int_\Omega \int_\Theta |I^W_\varphi(t_n,\theta) - I^W_\varphi(t,\theta)|^{2\nu}\,\mu_\rho(d\theta)\,dP = 0.$$

Obviously, the left hand side of the above equation is just

$$\lim_{n\to\infty} \mathbf{E}||I_\varphi^W(t_n) - I_\varphi^W(t)||_{L_\rho^{2\nu}}^{2\nu},$$

which proves the required continuity under the assumption (3.79).

Now, let us take any general $\varphi$ satisfying (3.77).
For such $\varphi$ one can always find a sequence of regular $\varphi_n$, $n \in \mathbb{N}$, satisfying (3.79) and approximating $\varphi$ in the sense that, given $\delta > 0$ as in Remark 3.3.1 **(i)**,

$$\int_0^T (\mathbf{E}||\varphi_n(t) - \varphi(t)||_{L_\rho^{2\nu}}^{2\nu})^{\frac{\delta+1}{\delta}}\, ds \to 0 \text{ as } n \to \infty.$$

Actually, for $\varphi_n$ one can take simple processes from $\mathcal{N}_W(T)$, see e.g. Proposition 2.24 in [61].
Then, we have by **(A4)** (or **(A5)**) and Hölder's inequality

$$\begin{aligned}
&\sup_{t\in[0,T]} \mathbf{E}||I_{\varphi_n}^W - I_\varphi^W||_{L_\rho^{2\nu}}^{2\nu} \\
&\le c(\nu,T) \sup_{t\in[0,T]} \int_0^t (t-s)^{-\zeta}\mathbf{E}||\varphi_n(t) - \varphi(t)||_{L_\rho^{2\nu}}^{2\nu}\, ds \\
&\le c(\nu,T) \left(\int_0^T s^{-\zeta(1+\delta)}\, ds\right)^{\frac{1}{1+\delta}} \left(\int_0^T (\mathbf{E}||\varphi_n(s) - \varphi(s)||_{L_\rho^{2\nu}}^{2\nu})^{\frac{\delta+1}{\delta}}\, ds\right)^{\frac{\delta}{\delta+1}}.
\end{aligned}$$

Thus, $t \mapsto I_\varphi^W(t)$ is continuous in $L^{2\nu}(\Omega; L_\rho^{2\nu})$ as a uniform limit of $L^{2\nu}(\Omega; L_\rho^{2\nu})$-continuous mappings. ■

To finish this section, we give a remark on the special case of a bounded integrator $\varphi$, which will be relevant for the equation (1.2) driven by Lévy noise.

**Remark 3.4.9:** *If $(\varphi(t))_{t\in[0,T]}$ is bounded in the sense that*

$$\sup_{t\in[0,T]} \mathbf{E}||\varphi(t)||_{L^\infty} < \infty, \tag{3.81}$$

*then the statements of Propositions 3.4.1, 3.4.3, and 3.4.5–3.4.7 remain valid under the weaker assumption that **(A5)** resp. **(A5)*** holds only for $\varphi \equiv 1 \in L_\rho^{2\nu}$, i.e., instead of (3.5) resp. (3.6), it is enough to suppose*

$$||U(t,s)||_{\mathcal{L}_2(L_\rho^{2\nu})}^2 \le c(T)(t-s)^{-\zeta}$$

*resp.*

$$||U(t,s)||_{\mathcal{L}(L_\rho^{2\nu})}^2 \le c(T)(t-s)^{-\zeta}.$$

# Chapter 4

# Stochastic convolution w.r.t. compensated Poisson random measures in weighted $L^2$-spaces

In this chapter, we are concerned with stochastic convolutions w.r.t. compensated Poisson random measures on the weighted $L^2$-spaces from Section 1.1.
The main results of this chapter establish the existence and moment bounds of the stochastic convolution w.r.t. compensated Poisson random measures in $L^2_\rho(\Theta)$ resp. $L^{2\nu}_\rho(\Theta)$ and its time-continuity in $L^q(\Omega; L^2_\rho(\Theta))$ resp. $L^{2\nu}(\Omega; L^{2\nu}_\rho(\Theta))$ with $q \geq 2$ resp. $\nu \geq 1$.
Analogously to the stochastic convolution w.r.t. $Q$-Wiener processes, we face the problem that, given any $\varphi \in L^2_\rho(\Theta)$, the multiplication operator $\mathcal{M}_\varphi$ is not a Hilbert-Schmidt operator from $L^2(\Theta)$ to $L^2_\rho(\Theta)$. Thus, we have to impose additional conditions on the Poisson random measure and the evolution operator.
The key assumption on the compensated Poisson random measure is that the corresponding Lévy intensity measure $\eta$ obeys

**(QI)** $$\int_{\{0<||x||_{L^2}<1\}} ||x||^2_{L^2}\,\eta(dx) + \int_{\{||x||_{L^2}\geq 1\}} ||x||^q_{L^2}\,\eta(dx) < \infty$$

with $q \geq 2$. The necessity of this assumptions follows from the Bichteler-Jacod inequality for Poisson integrals, cf. Lemma 2.6.10 in Section 2.6. Respectively, the key (and, compared to the Wiener case, new) assumption on the almost strong evolution operator $U$ is **(A5)** resp. **(A5)*** from Section 3.1. These assumptions allow us to control the Poisson stochastic convolution in the spaces $L^q(\Omega; L^2_\rho(\Theta))$ resp. $L^{2\nu}(\Omega; L^{2\nu}_\rho(\Theta))$.
Assuming pseudo contractivity of the evolution operator (cf. condition **(A7)**

from Section 3.1) and uniform boundedness of the multiplicator function, we can also show existence of a càdlàg version of the stochastic convolution (see Proposition 4.11 below).
Similar to the case of stochastic convolutions w.r.t. Wiener processes, a crucial role is played by the regularity constant $\zeta \in [0,1)$ associated to the evolution operator via condition **(A2)**.

As in Chapter 3, we assume $\rho \in \mathbb{N} \cup \{0\}$ to be such that $\mu_\rho(\Theta) < \infty$. Under this assumption, the following considerations do not depend on the choice of $\Theta$. Thus, we use the shortened notations for the $L^p_\rho$-spaces on $\Theta$.

More precisely, in this chapter we are focused on establishing analogons to Propositions 3.4.1 and 3.4.3 by extending their methods of proof to the case of compensated Poisson random measures. After that we also establish continuity properties similar to that from Chapter 3. In that respect, recall the shortened notations $L^q(\Omega; L^2_\rho) := L^q(\Omega, \mathcal{F}, P; L^2_\rho)$ and $L^{2\nu}(\Omega, \mathcal{F}, P; L^{2\nu}_\rho)$.

Let $(\tilde{N}(t,\cdot))_{t\in[0,T]}$ be a family of compensated Poisson random measures with commutator $dt \otimes \eta$ in the sense of Section 2.4.
Given an almost strong evolution operator $U$ on $L^2_\rho$ with properties **(A0)**, **(A1)** and a predictable process $(\varphi(t))_{t\in[0,T]}$ taking values in $L^2_\rho$, we consider the stochastic integral

$$I^{\tilde{N}}_\varphi(t) := \int_0^t \int_{L^2} U(t,s)\mathcal{M}_{\varphi(s)}(x)\, \tilde{N}(ds,dx). \tag{4.1}$$

Let us check the well-definedness of the Poisson stochastic convolution (4.1) in $L^2(\Omega; L^2_\rho)$ and, more generally, in the spaces $L^q(\Omega; L^2_\rho)$ and $L^{2\nu}(\Omega; L^{2\nu}_\rho)$ with $q \geq 2$ and $\nu \geq 1$.

To this end, we introduce the notation **(QI)** for both integrability conditions **(SI)** and **(PI)** (with $p := q$) being fulfilled for the corresponding Lévy measure $\eta$ for some $q \geq 2$, i.e.

$$\textbf{(QI)} \qquad \int_{\{0<||x||_{L^2}<1\}} ||x||^2_{L^2}\, \eta(dx) + \int_{\{||x||_{L^2}\geq 1\}} ||x||^q_{L^2}\, \eta(dx) < \infty.$$

Similar to the case of the Wiener stochastic convolution, the simplest assumption guaranteeing well-definedness of the Poisson stochastic convolution (4.1) is

$$\sup_{t\in[0,T]} \mathbf{E}||\varphi(t)||^2_{L^2_\rho} < \infty.$$

The following proposition is the analogon of Proposition 3.4.2 for the case of Wiener stochastic convolution.

**Proposition 4.1:** *Suppose that **(A0)**– **(A2)** (or even the weaker assumption **(A5)*** with $\nu = 1$ instead of **(A2)**) hold. Furthermore, assume **(QI)** with*

$$C_{q,\eta} := \int\limits_{L^2} ||x||^q_{L^2}\,\eta(dx) + \left(\int\limits_{L^2} ||x||^2_{L^2}\,\eta(dx)\right)^{\frac{q}{2}} < \infty. \tag{4.2}$$

*Let $\varphi = (\varphi(t))_{t\in[0,T]}$ be an $L^2_\rho$-valued predictable process obeying*

$$\sup_{t\in[0,T]} \int\limits_0^t (t-s)^{-\zeta}\mathbf{E}||\varphi(s)||^2_{L^2_\rho}\,ds < \infty, \tag{4.3}$$

*where $\zeta \in [0,1)$ as in **(A2)**. Then, for each $t \in [0,T]$, the convolution $I^{\tilde{N}}_\varphi$ is well-defined in $L^2_\rho$.*
*Furthermore, for any $q \in [2, \frac{2}{\zeta})$ such that $\eta$ obeys **(QI)** and*

$$\sup_{t\in[0,T]} \int\limits_0^t (t-s)^{-\zeta}\mathbf{E}||\varphi(s)||^q_{L^2_\rho}\,ds < \infty, \tag{4.4}$$

*we have the moment bound*

$$\mathbf{E}||I^{\tilde{N}}_\varphi(t)||^q_{L^2_\rho}\,ds \le c(q,T,C_{q,\eta}) \int\limits_0^t (t-s)^{-\frac{q\zeta}{2}}\mathbf{E}||\varphi(s)||^q_{L^2_\rho}\,ds. \tag{4.5}$$

*In particular,*

$$\sup_{t\in[0,T]} \mathbf{E}||\varphi(t)||^q_{L^2_\rho} < \infty \tag{4.6}$$

*is sufficient for (4.4) and implies*

$$\sup_{t\in[0,T]} \mathbf{E}||I^{\tilde{N}}_\varphi(t)||^q_{L^2_\rho} < \infty.$$

**Proof:** Analogously to the proof of Proposition 3.4.3, we fix $t \in [0,T]$. Setting (cf. (3.31))

$$\chi(s) := \mathbf{1}_{[0,t)}(s)U(t,s)\mathcal{M}_{\varphi(s)},\ 0 \le s \le T,$$

we need to establish the $\mathcal{P}_T \otimes \mathcal{B}(L^2)/\mathcal{B}(L^2_\rho)$-measurability of the integrand function

$$[0,T] \times \Omega \times L^2 \ni (s,\omega,x) \mapsto \chi(s,\omega)x \in L^2_\rho.$$

To this end, we will use Theorem 6.1 from [51], which requires the following properties to be fulfilled:

**(i)** The mapping

$$L^2 \ni x \mapsto \chi(s,\omega)x \in L^2_\rho$$

is continuous for almost any fixed $(s,\omega) \in [\,0,T\,] \times \Omega$.

**(ii)** The mapping

$$[\,0,T\,] \times \Omega \ni (s,\omega) \mapsto \chi(s,\omega)x \in L^2_\rho$$

is $\mathcal{P}_T$-measurable for any fixed $x \in L^2$.

Note that **(i)** immediately follows from **(A2)** (or **(A5)***). To show **(ii)**, let us note that, for each $x \in L^2$, the function $\chi(s,\omega)x \in L^2_\rho$ can be pointwise approximated on $[\,0,T\,]\times\Omega$ by linear combinations of $\chi(s,\omega)e_n \in L^2_\rho$, $n \in \mathbb{N}$. Under the same assumptions, each $\chi e_n$ is predictable as we showed in the proof of Proposition 3.4.1, cf. Step 1 there. This implies the predictability of $\chi x$ required in **(ii)**.

Thus, Theorem 6.1 from [51] is applicable and gives us existence of a $\mathcal{P}_T \otimes \mathcal{B}(L^2_\rho)$-measurable modification of the integrand function, which we again denote by $\chi x$.
Being restricted to the set $[\,0,T\,] \times \Omega \times L^2 \setminus \{0\}$, the integrand $\chi(s,\omega)x$ is obviously $\mathcal{P}_{T,\mathcal{A}_0}$-predictable as required to define Poisson integrals (see Section 2.6, p.57). In our context $\mathcal{A}_0 = \{B \in \mathcal{B}(L^2) \,|\, 0 \notin \bar{B}\}$.

For the well-definedness of (4.1), we have to check that

$$\mathbf{E}\int_0^T \int_{L^2} ||\chi(s)x||^2_{L^2_\rho}\, \eta(dx)\, ds < \infty, \tag{4.7}$$

i.e. $\chi x \in L^2([\,0,T\,] \times \Omega \times L^2; L^2_\rho)$.
Indeed, we have

$$\begin{aligned}
&\mathbf{E}\int_0^T \int_{L^2} ||\chi(s)x||^2_{L^2_\rho}\, \eta(dx)\, ds \\
&= \mathbf{E}\int_0^t \int_{L^2} ||U(t,s)\mathcal{M}_{\varphi(s)}(x)||^2_{L^2_\rho}\, \eta(dx)\, ds \\
&\le \left(\int_{L^2} ||x||^2_{L^2}\, \eta(dx)\right) \mathbf{E}\int_0^t ||U(t,s)\mathcal{M}_{\varphi(s)}||^2_{\mathcal{L}_2(L^2,L^2_\rho)}\, ds \\
&\le C_{q,\eta}c(T)\int_0^t (t-s)^{-\zeta}\mathbf{E}||\varphi(s)||^2_{L^2_\rho}\, ds
\end{aligned}$$

$< \infty$.

Here, we used (3.2) from **(A2)** in the third step, respectively (4.2) and (4.3) in the last step.
Thus, $f(s,\omega,x) := \chi(s,\omega)x \in N^{2,\eta}_{L^2/L^2_\rho}(T)$ such that Definition 2.6.6 gives us the well-definedness of $I^{\tilde{N}}_\varphi(t)$ as an element of $L^2(\Omega; L^2_\rho)$.

It remains to prove the estimate (4.5) for $q \geq 2$.
By the Bichteler-Jacod type inequality 2.6.10, (4.2), estimate (3.2) from **(A2)** and Hölder's inequality, we have

$$\begin{aligned}
\mathbf{E}||I^{\tilde{N}}_\varphi(t)||^q_{L^2_\rho} \leq \quad & c(q,T)\int_0^T \Bigg[\mathbf{E}\int_{L^2} ||\chi(s)x||^q_{L^2_\rho}\,\eta(dx) \\
& +\mathbf{E}\left(\int_{L^2} ||\chi(s)x||^2_{L^2_\rho}\,\eta(dx)\right)^{\frac{q}{2}}\Bigg]\,ds \\
= \quad & c(q,T)\left[\int_{L^2} ||x||^q_{L^2}\,\eta(dx) + \left(\int_{L^2} ||x||^2_{L^2}\,\eta(dx)\right)^{\frac{q}{2}}\right]\mathbf{E}\int_0^T ||\chi(s)||^q_{\mathcal{L}_2}\,ds \\
= \quad & c(q,T,C_{q,\eta})\mathbf{E}\int_0^t ||U(t,s)\mathcal{M}_{\varphi(s)}||^q_{\mathcal{L}_2}\,ds \\
\leq \quad & c(q,T,c(T),C_{q,\eta})\int_0^t (t-s)^{-\frac{q\zeta}{2}}\mathbf{E}||\varphi(s)||^q_{L^2_\rho}\,ds,
\end{aligned}$$

which finishes the proof. ■

**Remark 4.2:** *Analogously to Remark 3.4.2 in the Wiener case, the Poisson stochastic convolution is well-defined even under the weaker assumptions*

$$\text{(4.8)} \qquad \int_0^T (\mathbf{E}||\varphi(t)||^2_{L^2_\rho})^r\,dt < \infty \textit{ for some } r > \tfrac{1}{1-\zeta}.$$

*Indeed, the sufficient condition (4.3) is implied by the following estimate with* $\delta = \frac{1}{r-1} > 0$

$$\begin{aligned}
&\int_0^t (t-s)^{-\zeta}\mathbf{E}||\varphi(s)||^2_{L^2_\rho}\,ds \\
&\leq \left(\int_0^t s^{-\zeta(1+\delta)}\,ds\right)^{\frac{1}{1+\delta}} \left(\int_0^t (\mathbf{E}||\varphi(s)||^2_{L^2_\rho})^{\frac{\delta+1}{\delta}}\,ds\right)^{\frac{\delta}{\delta+1}}
\end{aligned}$$

*already appearing in Remark 3.4.2* ***(i)****.*
*In turn, (4.8) is implied by*

$$\int_0^T \mathbf{E}||\varphi(t)||_{L^r_\rho}^{2r}\, dt < \infty \textit{ for some } r > \tfrac{1}{1-\zeta}.$$

To describe admissible integrands in (4.1), for $\nu \geq 1$ we introduce the Banach spaces $\mathcal{G}_\nu(T)$ of all predictable $L^{2\nu}_\rho$-valued processes $(\varphi(t))_{t\in[0,T]}$ such that

$$||\varphi||_{\mathcal{G}_\nu(T)} := \left(\sup_{t\in[0,T]} \mathbf{E}||\varphi(t)||^{2\nu}_{L^{2\nu}_\rho}\right)^{\frac{1}{2\nu}} < \infty. \tag{4.9}$$

Since $L^{2\nu}_\rho \in \mathcal{B}(L^\nu_\rho)$ and $\mathcal{B}(L^{2\nu}_\rho) = \mathcal{B}(L^2_\rho) \bigcap L^{2\nu}_\rho$, any $\varphi \in \mathcal{G}_\nu(T)$ is even predictable as a process with values in $L^{2\nu}_\rho$. Note that the spaces $\mathcal{G}_\nu(T)$ will be treated more detailed in Section 5.1.

Let us start with the following proposition, which is a **Bichteler-Jacod** type inequality for Poisson convolutions in $L^{2\nu}_\rho$.

**Proposition 4.3:** *Let $\nu \geq 1$ and suppose that **(A0)**–**(A2)** (Here, **(A2)** can be replaced by **(A5)*** with $\nu = 1$) hold. Furthermore, let the integrability condition **(QI)** on the Lévy measure $\eta$ be fulfilled with $q = 2\nu$.*
*Given an $L^2_\rho$-valued predictable process obeying (4.3), we have*

$$I^{\tilde N}_\varphi := \int_0^t \int_{L^2} U(t,s)\mathcal{M}_{\varphi(s)}(x)\,\tilde N(ds,dx) \in L^{2\nu}_\rho,\ P\text{-a.s.},$$

*for all $t \in [0,T]$, provided*

$$\mathbf{E}\int_0^t \left[\int_{L^2} ||U(t,s)\mathcal{M}_{\varphi(s)}(x)||^{2\nu}_{L^{2\nu}_\rho}\,\eta(dx) + \left(\int_{L^2} ||U(t,s)\mathcal{M}_{\varphi(s)}(x)||^2_{L^{2\nu}_\rho}\,\eta(dx)\right)^\nu\right] ds < \infty. \tag{4.10}$$

*Furthermore, for any fixed $t \in [0,T]$, we have the moment estimate*

$$\mathbf{E}\left|\left|\int_0^t \int_{L^2} U(t,s)\mathcal{M}_{\varphi(s)}(x)\tilde N(ds,dx)\right|\right|^{2\nu}_{L^{2\nu}_\rho} \tag{4.11}$$
$$\leq c(\nu,T)\mathbf{E}\int_0^t \left[\int_{L^2} ||U(t,s)\mathcal{M}_{\varphi(s)}(x)||^{2\nu}_{L^{2\nu}_\rho}\,\eta(dx) + \left(\int_{L^2} ||U(t,s)\mathcal{M}_{\varphi(s)}(x)||^2_{L^{2\nu}_\rho}\,\eta(dx)\right)^\nu\right] ds.$$

The main result of this section is the following proposition, which will be relevant for later considerations of the SDEs (1.1) and (1.2). It is an immediate corollary of the previous proposition (cf. (4.10)/(4.11)).

**Proposition 4.4:** *Let $\nu \geq 1$ and assume additionally to the assumptions of 4.3 that **(A5)** (or even the weaker assumption **(A5)***) holds.*

*If*

$$\sup_{t\in[0,T]} \int_0^t (t-s)^{-\zeta\nu} \mathbf{E}||\varphi(s)||_{L^{2\nu}_\rho}^{2\nu}\, ds < \infty, \tag{4.12}$$

*then, for all* $t \in [0,T]$,

$$\int_0^t \int_{L^2} U(t,s)\mathcal{M}_{\varphi(s)}(x)\, \tilde{N}(ds,dx) \in L^{2\nu}_\rho,\ P\text{-a.s..}$$

*Furthermore, we have the moment estimate*

$$\begin{aligned} &\mathbf{E}\left|\left|\int_0^t \int_{L^2} U(t,s)\mathcal{M}_{\varphi(s)}(x)\, \tilde{N}(ds,dx)\right|\right|_{L^{2\nu}_\rho}^{2\nu} \\ &\le c(\nu,T,c(T),C_{2\nu,\eta}) \int_0^t (t-s)^{-\zeta\nu} \mathbf{E}||\varphi(s)||_{L^{2\nu}_\rho}^{2\nu}\, ds \end{aligned} \tag{4.13}$$

*with a positive constant only depending on* $\nu$, $T$, $U$ *and* $\eta$.
*In particular, the right hand side is finite if* $\nu < \frac{1}{\zeta}$ *(of course, this condition is only needed in case* $\zeta \neq 0$*) and* $\varphi$ *obeys (4.9), i.e.* $\varphi \in \mathcal{G}_\nu(T)$.

**Remark 4.5:** *The estimate (4.13) (aswell as estimate (4.27) in Remark 4.6* ***(i)*** *below) extends the Bichteler-Jacod inequality for Hilbert spaces (cf. Section 2.6) to the Banach spaces* $L^{2\nu}_\rho$ *with* $\nu \ge 1$. *A similar result in* $L^p$*-spaces in bounded domains was established by Marinelli and Röckner in [80] (see Lemma 4, p.1532 there).*
*There is also a general theory on stochastic integration w.r.t. compensated Poisson random measures in a special family of Banach spaces, the so-called UMD-Banach spaces, which is based on the theory of stochastic integration w.r.t. Wiener processes developed in 2007 in [109] by van Neerven and collaborators. After this thesis was almost completed, we got to know about the recent PhD-thesis [31] by Dirksen, where the Bichteler-Jacod inequality for stochastic integrals w.r.t. compensated Poisson random measures is proven. There an upper and a lower estimate on the stochastic integral w.r.t. compensated Poisson random measures is given.*

**Proof of 4.3:** Let us fix an arbitrary $t \in [0,T]$.
Setting (cf. (3.31) in the proof of Proposition 3.4.3)

$$\chi(s,\omega) := \mathbf{1}_{[0,t]}(s)U(t,s)\mathcal{M}_{\varphi(s,\omega)} \in \mathcal{L}_2(L^2, L^2_\rho),\ (s,\omega) \in [0,T]\times\Omega,$$

we are interested in the properties of the Poisson integral

$$\begin{aligned} I_\varphi^{\tilde N}(t) &= \int_0^t \int_{L^2} U(t,s)\mathcal{M}_{\varphi(s)}x\, \tilde N(ds,dx) \\ &= \int_0^t \int_{L^2} \chi(s)x\, \tilde N(ds,dx). \end{aligned}$$

Note that we have the expansion in $L^2$

$$x = \sum_{n\in\mathbb{N}} (x, e_n)_{L^2} e_n,$$

which implies the corresponding expansion in $L^2_\rho$

$$\chi(t)x = \sum_{n\in\mathbb{N}} (x, e_n)_{L^2} \chi(t)e_n, \ P\text{-a.s.} \tag{4.14}$$

Analogously to the proof of Theorem 3.4.3, we will proceed in three steps.

**Step 1** Making use of the expansion (4.14), we find a measurable representative $\psi$ of the mapping

$$[\,0,T\,] \times \Omega \times L^2 \ni (s,\omega,x) \mapsto \chi(s,\omega)x \in L^2_\rho.$$

By definition, this is a $\mathcal{P}_T \otimes \mathcal{B}(L^2 \setminus \{0\}) \otimes \mathcal{B}(\Theta)/\mathcal{B}(\mathbb{R})$-measurable function $\psi$: $\Omega \times [\,0,T\,] \times L^2 \setminus \{0\} \times \Theta \to \mathbb{R}$ such that

$$\mathbf{E}\int_0^T \int_{L^2\setminus\{0\}} ||\chi(s)x - \psi(s,x,\cdot)||^2_{L^2_\rho} \eta(dx)\, ds = 0. \tag{4.15}$$

**Step 2** In $L^2(\Omega; L^2_\rho)$, we check the identity

$$I_\varphi^{\tilde N}(t) := \int_0^t \int_{L^2} \chi(s)x\, \tilde N(ds,dx) = \int_0^t \int_{L^2\setminus\{0\}} \psi(s,x,\cdot)\, \tilde N(ds,dx).$$

**Step 3** We show that

$$\int_0^t \int_{L^2\setminus\{0\}} \psi(s,x,\cdot)\, \tilde N(ds,dx) \in L^{2\nu}_\rho,\ P\text{-a.s.},$$

and satisfies the estimate (4.11).

**Step 1:** To this end, everything was mainly prepared in Step 1 in the proof of Proposition 3.4.3.
Indeed, there we have constructed a family $(\psi^{(n)})_{n\in\mathbb{N}}$ of $\mathcal{P}_T \otimes \mathcal{B}(\Theta)/\mathcal{B}(\mathbb{R})$-measurable representatives for the functions $\chi e_n$, $n \in \mathbb{N}$, such that

$$\mathbf{E}\sum_{n\in\mathbb{N}}\int_0^T ||\chi(s)e_n - \psi^{(n)}(s,\cdot)||^2_{L^2_\rho}\, ds = 0. \tag{4.16}$$

We claim that the series

$$\psi(s,x,\theta) := \sum_{n\in\mathbb{N}}(x,e_n)_{L^2}\psi^{(n)}(s,\theta),\ (s,x,\theta)\in[\,0,T\,]\times L^2\times\Theta \tag{4.17}$$

is convergent in

$L^2(\Omega\times[\,0,T\,]\times L^2\backslash\{0\}\times\Theta) := L^2(\Omega\times[\,0,T\,]\times L^2\backslash\{0\}\times\Theta, dP\otimes ds\otimes d\eta\otimes d\mu_\rho)$.
Indeed, for any $N,K\in\mathbb{N}$ by Fubini's theorem and (4.16)

$$\begin{aligned}
&\int_\Omega\int_0^T\int_{L^2\backslash\{0\}}\int_\Theta\left(\sum_{n=N}^{N+K-1}(x,e_n)_{L^2}\psi^{(n)}(s,\omega,\theta)\right)^2\mu_\rho(d\theta)\,\eta(dx)\,ds\,P(d\omega) \qquad (4.18)\\
&=\int_\Omega\int_0^T\int_{L^2\backslash\{0\}}\int_\Theta\left(\sum_{n,m=N}^{N+K-1}(x,e_n)_{L^2}(x,e_m)_{L^2}\psi^{(n)}(s,\omega,\theta)\psi^{(m)}(s,\omega,\theta)\right)\mu_\rho(d\theta)\,\eta(dx)\,ds\,P(d\omega)\\
&=\int_{L^2\backslash\{0\}}\sum_{n,m=N}^{N+K-1}(x,e_n)_{L^2}(x,e_m)_{L^2}\left[\int_\Omega\int_0^T\int_\Theta\psi^{(n)}(s,\omega,\theta)\psi^{(m)}(s,\omega,\theta)\mu_\rho(d\theta)\,ds\,P(d\omega)\right]\eta(dx)\\
&=\int_{L^2\backslash\{0\}}\sum_{n,m=N}^{N+K-1}(x,e_n)_{L^2}(x,e_m)_{L^2}\left[\mathbf{E}\int_0^T <\chi(s)e_n,\chi(s)e_m>_{L^2_\rho}\ ds\right]\eta(dx)\\
&=\mathbf{E}\int_0^T\int_{L^2}\left[\sum_{n,m=N}^{N+K-1}(x,e_n)_{L^2}(x,e_m)_{L^2}<\chi(s)e_n,\chi(s)e_m>_{L^2_\rho}\right]\eta(dx)\,ds\\
&=\mathbf{E}\int_0^T\int_{L^2}\left|\left|\sum_{n,m=N}^{N+K-1}(x,e_n)_{L^2}\chi(s)e_n\right|\right|^2_{L^2_\rho}\eta(dx)\,ds\\
&=\mathbf{E}\int_0^T\int_{L^2}||\chi(s)P_{N,N+K-1}x||^2_{L^2_\rho}\,\eta(dx)\,ds\\
&\le\mathbf{E}\int_0^T\int_{L^2}||\chi(s)x||^2_{L^2_\rho}\,\eta(dx)\,ds\\
&=\mathbf{E}\int_0^t\int_{L^2}||U(t,s)\mathcal{M}_{\varphi(s)}x||^2_{L^2_\rho}\,\eta(dx)\,ds.
\end{aligned}$$

Here,

$$P_{N,N+K-1}x := \sum_{n=N}^{N+K-1}(x,e_n)_{L^2}e_n$$

is the projection of $x\in L^2$ on the linear subspace generated by the vectors $e_N$, $e_{N+1}$, ..., $e_{N+K-1}$. Obviously $||P_{N,N+K-1}x||_{L^2}\to 0$ as $N$, $K\to\infty$. The last integral is finite by the integrability assumption (4.10) on $\eta$. Therefore, by Lebesgue's dominated convergence theorem the left hand side in (4.18)

tends to 0 as $N,\, K\to\infty$.
Thus, (4.17) defines $\psi\in L^2(\Omega\times[\,0,T\,]\times L^2\setminus\{0\}\times\Theta)$.
The above reasoning also shows that

$$\sum_{n\in\mathbb{N}}(x,e_n)_{L^2}\chi(s)e_n \tag{4.19}$$

is convergent in $L^2(\Omega\times[\,0,T\,]\times L^2;L^2_\rho)$ and by (4.14) its limit coincides with $\chi(s)x$.
Thus, by (4.16),

$$\begin{aligned}&\mathbf{E}\int_0^T\int_{L^2\setminus\{0\}}\|\chi(s)x-\psi(s,x,\cdot)\|^2_{L^2_\rho}\,\eta(dx)\,ds\\ &=\lim_{N\to\infty}\mathbf{E}\int_0^T\int_{L^2\setminus\{0\}}\left\|\sum_{n=1}^N(x,e_n)_{L^2}\chi(s)e_n-(x,e_n)_{L^2}\psi^{(n)}(s,x,\cdot)\right\|^2_{L^2_\rho}\\ &=0,\end{aligned}$$

which we needed to prove. $\triangle$

**Step 2:** This step is proven by the following claim:

**Claim 2:** *In $L^2(\Omega;L^2_\rho)$, we have*

$$I^{\tilde N}_\varphi(t)=\int_0^t\int_{L^2}\chi(s,x)\,\tilde N(ds,dx)=\int_0^t\int_{L^2\setminus\{0\}}\psi(s,x,\cdot)\,\tilde N(ds,dx).$$

**Proof:** The well-definedness of the integral on the left hand side has already been shown in the beginning of the section. Thus, we get the claim by Itô's isometry and (4.15) from Claim 1. $\square$ $\triangle$

**Step 3:** $\int_0^t\int_{L^2\setminus\{0\}}\psi(s,x,\cdot)\,\tilde N(ds,dx)\in L^{2\nu}_\rho$, *$P$-a.s..*

This step will be shown by the following two claims.

**Claim 3:** *Let us consider the stochastic integral*

$$\tilde I(t,\theta):=\int_0^t\int_{L^2\setminus\{0\}}\psi(s,x,\theta)\,\tilde N(ds,dx)\in\mathbb{R} \tag{4.20}$$

*depending on the parameter $\theta\in\Theta$.*
*Then, (4.20) allows an $\mathcal{F}\otimes\mathcal{B}(\Theta)$- measurable modification, which we again denote by $\tilde I(t)$.*
*Furthermore, $P$-almost surely, $\tilde I(t)$ coincides with the $L^2_\rho$-valued random variable*

$$I(t) := \int_0^t \int_{L^2\setminus\{0\}} \psi(s,x,\cdot)\,\tilde{N}(ds,dx),$$

*i.e.*

$$\mathbf{E}||\tilde{I}(t) - I(t)||^2_{L^2_\rho} = \mathbf{E}\left(\int_\Theta |\tilde{I}(t,\theta) - (I(t))(\theta)|^2\,\mu_\rho(d\theta)\right) = 0.$$

**Proof:** First of all, we note that $\tilde{I}(t,\theta)$ is well-defined for $\mu_\rho$-almost all $\theta \in \Theta$. Indeed, by Step 1 $\psi \in L^2(\Omega \times [0,T] \times L^2 \times \Theta)$, which means

$$\mathbf{E}\int_0^t \int_{L^2\setminus\{0\}} \int_\Theta (\psi(s,x,\theta))^2\,\mu_\rho(d\theta)\,\eta(dx)\,ds < \infty,$$

and hence, by Fubini's theorem,

$$\mathbf{E}\int_0^t \int_{L^2\setminus\{0\}} (\psi(s,x,\theta))^2\,\eta(dx)\,ds < \infty \tag{4.21}$$

for $\mu_\rho$-almost all $\theta \in \Theta$.
By Itô's isometry this implies the well-definedness of $\tilde{I}(t,\theta)$ for $\mu_\rho$-almost all $\theta \in \Theta$.

Next, we are going to show the measurability property of $\tilde{I}$.
To this end, we apply a general measurability result A.1.(b) from [13]. It says that there exists an $\mathcal{F}_t \otimes \mathcal{B}(\Theta)$-measurable mapping $\bar{I}$: $\Omega \times \Theta \to \mathbb{R}$ and a Borel subset $\Theta_0 \subset \Theta$ of full $\mu_\rho$-measure such that for each $\theta \in \Theta_0$ $\bar{I}(\theta) = \tilde{I}(t,\theta)$.
Below, we will always identify $\tilde{I}(t)$ with its measurable realization $\bar{I}$.

Next, we show that $\tilde{I}(t)$ can be identified with the $L^2_\rho$-valued random variable $I(t)$.

To this end, we take the inner product of $\tilde{I}$ and $I$ with cylinder functions $F \in L^2(\Omega \times \Theta)$ of the form

$$F(\omega,\theta) = F_1(\omega)\left(\sum_{j=0}^{J-1} d_j \mathbf{1}_{B_j}(\theta)\right),\ (\omega,\theta) \in \Omega \times \Theta$$

with $F_1 \in L^2(\Omega,\mathcal{F},P)$, $d_j \in \mathbb{R}$, pairwise disjoint $B_j \in \mathcal{B}(\Theta)$ for $1 \le j \le J$ and $J \in \mathbb{N}$. Since such $F$ constitute a total set in $L^2(\Omega; L^2_\rho)$, it suffices to show that the inner products coincide.

An important observation is that, $P$-a.s.,

$$(4.22)\quad \int_0^t \int_{L^2\setminus\{0\}} \int_\Theta F_2(\theta)\psi(s,x,\theta)\,\mu_\rho(d\theta)\,\tilde{N}(ds,dx) = \int_\Theta F_2(\theta)\tilde{I}(t,\theta)\,\mu_\rho(d\theta).$$

This follows by the stochastic Fubini theorem for Poisson processes, which can be found e.g. in [106] (cf. Proposition 3.3.4 there).
A sufficient condition to apply this theorem is that

$$\int_\Theta \mathbf{E}\int_0^T \int_{L^2\setminus\{0\}} (F_2(\theta)\psi(s,x,\theta))^2\,\eta(dx)\,ds\,\mu_\rho(d\theta) < \infty,$$

which is checked by the following estimates

$$\begin{aligned}
&\int_\Theta \mathbf{E}\int_0^T \int_{L^2\setminus\{0\}} (F_2(\theta)\psi(s,x,\theta))^2\,\eta(dx)\,ds\,\mu_\rho(d\theta)\\
&= \int_\Theta F_2^2(\theta)\left(\mathbf{E}\int_0^T \int_{L^2\setminus\{0\}} (\psi(s,x,\theta))^2\,\eta(dx)\,ds\right)\mu_\rho(d\theta)\\
&= \sum_{j=0}^M \int_{B_j} d_j^2\left(\mathbf{E}\int_0^T \int_{L^2\setminus\{0\}} (\psi(s,x,\theta))^2\,\eta(dx)\,ds\right)\mu_\rho(d\theta)\\
&\le \left(\max_j |d_j|\right)^2 \sum_{j=0}^M \int_{B_j} \mathbf{E}\int_0^T \int_{L^2\setminus\{0\}} (\psi(s,x,\theta))^2\,\eta(dx)\,ds\,\mu_\rho(d\theta)\\
&= \left(\max_j |d_j|\right)^2 \int_\Theta \mathbf{E}\int_0^T \int_{L^2\setminus\{0\}} (\psi(s,x,\theta))^2\,\eta(dx)\,ds\,\mu_\rho(d\theta)\\
&< \infty.
\end{aligned}$$

Now let us show that the inner products are equal. We get

$$\begin{aligned}
&< I(t), F >_{L^2(\Omega;L^2_\rho)}\\
&= \mathbf{E}\left[\int_\Theta F(\cdot,\theta)\left(\int_0^t \int_{L^2\setminus\{0\}} \psi(s,x,\cdot)\,\tilde{N}(ds,dx)\right)(\theta)\,\mu_\rho(d\theta)\right]\\
&= \mathbf{E}\left[F_1\left\langle F_2, \int_0^t \int_{L^2\setminus\{0\}} \psi(s,x,\cdot)\,\tilde{N}(ds,dx)\right\rangle_{L^2_\rho}\right]\\
&= \mathbf{E}\left[F_1 \int_0^t \int_{L^2\setminus\{0\}} \langle F_2, \psi(s,x,\cdot)\rangle_{L^2_\rho}\,\tilde{N}(ds,dx)\right]\\
&= \mathbf{E}\left[F_1 \int_0^t \int_{L^2\setminus\{0\}} \int_\Theta F_2(\theta)\psi(s,x,\theta)\,\mu_\rho(d\theta)\,\tilde{N}(ds,dx)\right]\\
&= \mathbf{E}\left[F_1 \int_\Theta F_2(\theta)\left(\int_0^t \int_{L^2\setminus\{0\}} \psi(s,x,\theta)\,\tilde{N}(ds,dx)\right)\mu_\rho(d\theta)\right]
\end{aligned}$$

$$= \mathbf{E}\left[\int\limits_{\Theta} F(\cdot,\theta)\tilde{I}(t,\theta)\,\mu_\rho(d\theta)\right]$$
$$=< \tilde{I}(t), F >_{L^2(\Omega;L^2_\rho)}.$$

Here, we used the Fubini theorem (4.22) in the sixth, the definition of the inner product in $L^2_\rho$ in the second and the fourth, Proposition 2.6.8 in the third and the form of $F$ in the second and the second last step.
By the previous chain of equations, the inner products of $I(t)$ and $\tilde{I}(t)$ with $F$ coincide as elements of
$L^2(\Omega; L^2_\rho)$, which proves Claim 3. □

Now, we can finish Step 3 with the following claim.

**Claim 4:** *The integral*

$$\tilde{I}(t,\theta) := \int\limits_0^t \int\limits_{L^2\setminus\{0\}} \psi(s,x,\theta)\,\tilde{N}(ds,dx),$$

*as a function of* $\theta \in \Theta$, *belongs* $P$*-almost surely to* $L^{2\nu}_\rho$.

**Proof:** Applying the Bichteler-Jacod inequality 2.6.10 with $H = \mathbb{R}$, we get

$$(4.23)\ \int\limits_{\Theta} \mathbf{E}\left(\int\limits_0^t \int\limits_{L^2\setminus\{0\}} \psi(s,x,\theta)\,\tilde{N}(ds,dx)\right)^{2\nu} \mu_\rho(d\theta)$$
$$\le \int\limits_{\Theta} \mathbf{E}\left(\sup_{0\le r\le t}\int\limits_0^r \int\limits_{L^2\setminus\{0\}} \psi(s,x,\theta)\,\tilde{N}(ds,dx)\right)^{2\nu} \mu_\rho(d\theta)$$
$$\le c(\kappa,T)\int\limits_{\Theta}\int\limits_0^t \left(\mathbf{E}\left(\int\limits_{L^2\setminus\{0\}} |\psi(s,x,\theta)|^{2\nu}\,\eta(dx)\right)\right.$$
$$\left.+\mathbf{E}\left[\left(\int\limits_{L^2\setminus\{0\}} |\psi(s,x,\theta)|^2\,\eta(dx)\right)^{\nu}\right]\right) ds\mu_\rho(d\theta).$$

It remains to show that both integrals in the right hand side are finite.

By the measurabilty property of $\psi$ and Fubini's theorem, we immediately get

$$(4.24)\ \int\limits_{\Theta}\int\limits_0^T \mathbf{E}\int\limits_{L^2\setminus\{0\}} |\psi(s,x,\theta)|^{2\nu}\,\eta(dx)\,ds\mu_\rho(d\theta)$$
$$= \int\limits_0^T \int\limits_{L^2\setminus\{0\}} \mathbf{E}\int\limits_{\Theta} |\psi(s,x,\theta)|^{2\nu}\mu_\rho(d\theta)\,\eta(dx)\,ds$$
$$= \int\limits_0^t \int\limits_{L^2\setminus\{0\}} \mathbf{E}\int\limits_{\Theta} |(U(t,s)\mathcal{M}_{\varphi(s)}(x))|^{2\nu}\mu_\rho(d\theta)\,\eta(dx)\,ds$$

$$= \int_0^t \int_{L^2} \mathbf{E}||U(t,s)\mathcal{M}_{\varphi(s)}(x)||^{2\nu}_{L^{2\nu}_\rho}\,\eta(dx)\,ds,$$

which is finite, since by the assumption of 4.1 the right hand side is finite.

To estimate the second integral in (4.24), we use the following Minkowski inequality (see Theorem 2.4 in [69]) with $p \geq 1$

$$(4.25)\quad \left(\int_X \left[\int_Y f(x,y)\,\nu(dy)\right]^p \mu(dx)\right)^{\frac{1}{p}} \leq \int_Y \left[\int_X f^p(x,y)\,\mu(dx)\right]^{\frac{1}{p}} \nu(dy).$$

This inequality is valid for any measurable spaces $X$ and $Y$, $\sigma$-finite measures $\mu$ and $\eta$ on $X$ and $Y$ and any nonnegative measurable function $f\colon X \times Y \to \mathbb{R}_+$.

Hence, by (4.25) we get

$$(4.26)\quad \int_\Theta \int_0^T \mathbf{E}\left[\left(\int_{L^2\setminus\{0\}} |\psi(s,x,\theta)|^2\,\eta(dx)\right)^\nu\right] ds\,\mu_\rho(d\theta)$$

$$= \mathbf{E}\int_\Theta \int_0^T \left(\int_{L^2\setminus\{0\}} |\psi(s,x,\theta)|^2\,\eta(dx)\right)^\nu ds\,\mu_\rho(d\theta)$$

$$\leq \mathbf{E}\int_0^T \left(\int_{L^2\setminus\{0\}} \left(\int_\Theta |\psi(s,x,\theta)|^{2\nu}\,\mu_\rho(d\theta)\right)^{\frac{1}{\nu}} \eta(dx)\right)^\nu ds$$

$$= \mathbf{E}\int_0^t \left(\int_{L^2} \left(\int_\Theta |(U(t,s)\mathcal{M}_{\varphi(s)}(x))|^{2\nu}\,d\mu_\rho\right)^{\frac{1}{\nu}} \eta(dx)\right)^\nu ds$$

$$= \mathbf{E}\int_0^t \left(\int_{L^2} ||U(t,s)\mathcal{M}_{\varphi(s)}(x)||^2_{L^{2\nu}_\rho}\,\eta(dx)\right)^\nu ds,$$

which is finite by the assumption (4.10).

Now, combining (4.23), (4.24) and (4.26) gives us Claim 4. □

It is easy to see that the stochastic integral $I^{\tilde{N}}_\varphi(t)$, which was initially defined in $L^2_\rho$, actually belongs to $L^{2\nu}_\rho$.

By Claim 2 and 3, we have the identity $I^{\tilde{N}}_\varphi(t) = \tilde{I}(t)$ in $L^2(\Omega; L^2_\rho)$.
But in Claim 4, we have just shown that $\tilde{I}(t)$ belongs to $L^{2\nu}_\rho$, $P$-a.s., as a function of $\theta$.

To finish Step 3 and the proof of 4.3, it remains to show the estimate (4.11).

With the help of Step 2 and Claim 3, we immediately get

$$\mathbf{E}\left|\left|\int_0^t\int_{L^2} U(t,s)\mathcal{M}_{\varphi(s)}(x)\,\tilde{N}(ds,dx)\right|\right|^{2\nu}_{L^{2\nu}_\rho}$$
$$=\mathbf{E}\left|\left|\int_0^T\int_{L^2\setminus\{0\}} \psi(s,x,\cdot)\,\tilde{N}(ds,dx)\right|\right|^{2\nu}_{L^{2\nu}_\rho}$$
$$=\mathbf{E}\int_\Theta\left(\int_0^T\int_{L^2\setminus\{0\}} \psi(s,x,\cdot)\,\tilde{N}(ds,dx)\right)^{2\nu}(\theta)\,\mu_\rho(d\theta)$$
$$=\int_\Theta\mathbf{E}\left(\int_0^T\int_{L^2\setminus\{0\}} \psi(s,x,\theta)\,\tilde{N}(ds,dx)\right)^{2\nu}\mu_\rho(d\theta)$$
$$\le c(\kappa,T)\int_\Theta\int_0^T\left(\mathbf{E}\left(\int_{L^2\setminus\{0\}}|\psi(s,x,\theta)|^{2\nu}\,\eta(dx)\right)\right.$$
$$\left.+\mathbf{E}\left[\left(\int_{L^2\setminus\{0\}}|\psi(s,x,\theta)|^2\,\eta(dx)\right)^\nu\right]\right)ds\mu_\rho(d\theta).$$

The last integral was aready estimated by (4.23)–(4.26). This gives us the required estimate (4.11). △

Thus, we are finished with the proof. ■

**Proof of 4.4:** By **(A5)**/ **(A5)*** for $U$ and **(QI)** with $q=2\nu$ for $\eta$ and (4.12), we get

$$\mathbf{E}\int_0^t\left[\int_{L^2}||U(t,s)\mathcal{M}_{\varphi(s)}(x)||^{2\nu}_{L^{2\nu}_\rho}\,\eta(dx)+\left(\int_{L^2}||U(t,s)\mathcal{M}_{\varphi(s)}(x)||^2_{L^{2\nu}_\rho}\,\eta(dx)\right)^\nu\right]ds<\infty.$$

Thus, by 4.3, we have for any fixed $t\in[\,0,T\,]$

$$\int_0^t\int_{L^2}U(t,s)\mathcal{M}_{\varphi(s)}(x)\,\tilde{N}(ds,dx)\in L^{2\nu}_\rho,\ P\text{-a.s.}.$$

Furthermore, the estimate (4.11) together with **(A5)**/**(A5)*** for $U$ resp. **(QI)** for $\eta$ with $q=2\nu$ gives us

$$\mathbf{E}\left|\left|\int_0^t\int_{L^2} U(t,s)\mathcal{M}_{\varphi(s)}(x)\,\tilde{N}(ds,dx)\right|\right|^{2\nu}_{L^{2\kappa}_\rho}$$
$$\le c(\nu,T)\left(\mathbf{E}\int_0^t\left[\int_{L^2}||U(t,s)\mathcal{M}_{\varphi(s)}(x)||^{2\nu}_{L^{2\nu}_\rho}\,\eta(dx)+\left(\int_{L^2}||U(t,s)\mathcal{M}_{\varphi(s)}(x)||^2_{L^{2\nu}_\rho}\,\eta(dx)\right)^\nu\right]ds\right)$$

$$\leq c(\nu, T, c(T), C_{2\nu,\eta}) \int_0^t (t-s)^{-\zeta\nu} \mathbf{E}||\varphi(s)||_{L_\rho^{2\nu}}^{2\nu}\, ds,$$

which is just (4.13). ■

**Remark 4.6:** ***(i)*** *Actually, analogously to Proposition 3.4.3 and Remark 3.4.4, Proposition 4.3 extends to any predictable process* $\chi\colon [0,T] \times \Omega \times L^2 \setminus \{0\} \to L_\rho^{2\nu}$ *such that* $\chi \in \mathcal{N}^{2,\eta}_{L^2/L_\rho^2}(T)$.
*Namely, we can prove that (cf. (4.23)–(4.26))*

$$\int_0^t \int_{L^2} \chi(s,x)\, \tilde{N}(ds,dx) \in L_\rho^{2\nu} \ (P\text{-}a.s.)$$

*and*

$$(4.27)\ \mathbf{E}\left|\left|\int_0^t \int_{L^2} \chi(s,x)\, \tilde{N}(ds,dx)\right|\right|^{2\nu}_{L_\rho^{2\nu}}$$
$$\leq c(\nu,T)\left(\mathbf{E}\int_0^t \left[\int_{L^2} ||\chi(s,x)||^{2\nu}_{L_\rho^{2\nu}}\, \eta(dx) + \left(\int_{L^2} ||\chi(s,x)||^2_{L_\rho^{2\nu}}\, \eta(dx)\right)^{\nu}\right] ds\right),$$

*provided the right hand side is finite.*

***(ii)*** *Let us give a direct proof of Step 1 in the proof of Proposition 4.3 without referring to Step 1 in the proof of Proposition 3.4.3.*
*So, a measurable realization* $\psi(s,x,\theta)$ *of* $\chi(s)x$ *can be constructed as follows. By arguments analogous to that in Step 1 in the proof of Proposition 3.4.3, we can approximate* $\chi(s)x \in L_\rho^2$ *by standard convolutions*

$$(4.28)\qquad \psi_m(s,x) := \mu_\rho^{-1} conv(\delta_m, \mu_\rho \chi(s)x) \in L_\rho^2 \bigcap C(\mathbb{R}^d)$$

*in such a way that, for all* $(s,\omega,x) \in [0,T] \times \Omega \times L^2 \setminus \{0\}$,

$$(4.29)\qquad \lim_{m\to\infty} ||\psi_m(s,x) - \chi(s)x||^2_{L_\rho^2} = 0$$

*and*

$$(4.30)\qquad \sup_{m\in\mathbb{N}} ||\psi_m(s,x)||_{L_\rho^2} \leq ||\chi(s)x||_{L_\rho^2}.$$

*Since* $\psi_m(s,x) \in C(\mathbb{R}^d)$, *we can evaluate* $\psi_m(s,x,\theta)$ *for any* $\theta \in \Theta$. *Furthermore, Theorem 6.1 from [51] guarantees that there exists a* $\mathcal{P}_T \otimes \mathcal{B}(L^2 \setminus \{0\}) \otimes \mathcal{B}(\Theta)/\mathcal{B}(\mathbb{R})$*-measurable realization of*

$$(s,\omega,x,\theta) \mapsto \psi_m(s,\omega,x,\theta).$$

*Sufficient conditions to apply this theorem are*

***(i)*** *continuity of the mapping*

$$\Theta \ni \theta \mapsto \psi_m(s, \omega, x, \theta) \in \mathbb{R}$$

*for almost any fixed* $(s, \omega, x) \in [0, T] \times \Omega \times L^2 \setminus \{0\}$, *which holds by (3.45);*

***(ii)*** $\mathcal{P}_T \otimes \mathcal{B}(L^2 \setminus \{0\})$*-measurability of*

$$[0, T] \times \Omega \times L^2 \setminus \{0\} \ni (s, \omega, x) \mapsto \psi_m(s, \omega, x, \theta) \in \mathbb{R}$$

*for any fixed* $\theta \in \Theta$.

*The latter readily follows by Fubini's theorem from the definition (4.28) of* $\psi_m$, *i.e.*

$$\psi_m(s, \omega, x, \theta) = \mu_\rho^{-\frac{1}{2}}(\theta) < \delta_m(\cdot - \theta), \mu_\rho^{\frac{1}{2}} \chi(s, \omega) x >_{L^2}.$$

*Next, by (4.29), (4.30) and Lebesgue's theorem, we observe that the corresponding sequence of measurable realizations* $\psi_m$, $m \in \mathbb{N}$, *is a Cauchy sequence in the Hilbert space*
$L^2(\Omega \times [0, T] \times L^2 \setminus \{0\} \times \Theta, \mathcal{P}_T \otimes \mathcal{B}(L^2 \setminus \{0\}) \otimes \mathcal{B}(\Theta), P \otimes dt \otimes \eta \otimes \mu_\rho; \mathbb{R})$.
*Indeed, for all* $n, m, \in \mathbb{N}$

$$\begin{aligned}&\mathbf{E} \int_0^T \int_{L^2\setminus\{0\}} \int_\Theta |\psi_m(s, x, \theta) - \psi_n(s, x, \theta)|^2 \, \mu_\rho(d\theta) \, \eta(dx) \, ds \\ &= \mathbf{E} \int_0^T \int_{L^2\setminus\{0\}} ||\psi_m(s, x) - \psi_n(s, x)||^2_{L^2_\rho} \, \eta(dx) \, ds\end{aligned}$$

*with the uniform bound*

$$\sup_{m\in\mathbb{N}} ||\psi_m(s, x)||_{L^2_\rho} \le ||\chi(s)x||_{L^2_\rho} \in L^2(\Omega \times [0, t] \times L^2).$$

*Thus, there exists a limit function*
$\psi \in L^2(\Omega \times [0, T] \times L^2 \setminus \{0\} \times \Theta, \mathcal{P}_T \otimes \mathcal{B}(L^2 \setminus \{0\}) \otimes \mathcal{B}(\Theta), P \otimes dt \otimes \eta \otimes \mu_\rho; \mathbb{R})$
*such that*

$$\begin{aligned}&\lim_{m\to\infty} \mathbf{E} \int_0^T \int_{L^2\setminus\{0\}} \int_\Theta |\psi_m(s, x, , \theta) - \psi(s, x, \theta)|^2 \, \mu_\rho(d\theta) \, \eta(dx) \, ds \\ &= \lim_{m\to\infty} \mathbf{E} \int_0^T \int_{L^2\setminus\{0\}} ||\psi_m(s, x) - \psi(s, x)||^2_{L^2_\rho} \, \eta(dx) \, ds\end{aligned} \tag{4.31}$$

$= 0.$

*Combining (4.29)–(4.31), we conclude that*

$$\mathbf{E}\int_0^T \int_{L^2\setminus\{0\}} ||\psi(s,x) - \chi(s)x||^2_{L^2_\rho}\,\eta(dx)\,ds = 0,$$

*i.e.* $\psi(s,x,\theta)$ *is a measurable realization of* $\chi(s)x$ *obeying (4.15).* ■

Recall that in Section 3.4 we showed pathwise continuity for the stochastic convolution w.r.t. the Wiener process.
But such continuity is surely not the case for Poisson processes (even in the finite-dimensional case). Instead of this, one may expect to have meansquare continuity. We finish this section by the following propositions, which are the analogons of Propositions 3.4.4–3.4.6 in the case of Poisson stochastic convolutions:

**Proposition 4.7:** *Suppose, we have **(A0)**–**(A2)** (or **(A5)*** with* $\nu = 1$ *instead of **(A2)**) for an almost strong evolution operator* $U$ *and*

$$\int_{L^2} ||x||^2_{L^2}\,\eta(dx) < \infty$$

*for the Lévy measure* $\eta$ *corresponding to the compensated Poisson random measure*$\tilde{N}$*.*
*Furthermore, let* $(\varphi(t))_{t\in[0,T]}$ *be an* $L^2_\rho$*-valued predictable process obeying the uniform moment bound (3.27) from Section 3.4, i.e.*

$$\sup_{t\in[0,T]} \mathbf{E}||\varphi(t)||^2_{L^2_\rho} < \infty. \tag{4.32}$$

*Then,*

$$t \mapsto I^{\tilde{N}}_\varphi(t) := \int_0^t \int_{L^2} U(t,s)\mathcal{M}_{\varphi(s)}x\,\tilde{N}(ds,dx)$$

*is continuous in* $L^2(\Omega; L^2_\rho)$*.*

**Proof:** Again, we extend here a method of proving meansquare continuity, which is used e.g. in [60] and [59], to the case of non-Hilbert-Schmidt operator valued coefficients $\mathcal{M}_{\varphi(t)}$ and two-parameter evolution operators $U(t,s)$.

For $\alpha > 1$, consider the process

$$\Phi^\alpha(t) := \int_0^{\frac{t}{\alpha}} \int_{L^2} U(t,s)\mathcal{M}_{\varphi(s)} x \, \tilde{N}(ds,dx) \in L^2_\rho, \; 0 \le t \le T, \tag{4.33}$$

which is well-defined due to (4.32).
We claim that $\Phi^\alpha(t)$, $0 \le t \le T$, is meansquare continuous. Indeed, for any $0 \le r \le t \le T$, we have by Itô's isometry for compensated Poisson random measures and **(QI)**

$$\begin{aligned} \mathbf{E}||\Phi^\alpha(t) - \Phi^\alpha(r)||^2_{L^2_\rho} \le \;& 2\Bigg( \int_0^{\frac{r}{\alpha}} \int_{L^2} \mathbf{E}||[U(t,s) - U(r,s)]\mathcal{M}_{\varphi(s)}x||^2_{L^2_\rho} \, \eta(dx)\, ds \\ & + \int_{\frac{r}{\alpha}}^{\frac{t}{\alpha}} \int_{L^2} \mathbf{E}||U(t,s)\mathcal{M}_{\varphi(s)}x||^2_{L^2_\rho} \, \eta(dx)\, ds \Bigg) \\ \le \;& 2 \underbrace{\int_{L^2} ||x||^2_{L^2} \, \eta(dx)}_{<\infty} \Bigg[ \int_0^{\frac{r}{\alpha}} \mathbf{E}||[U(t,s) - U(r,s)]\mathcal{M}_{\varphi(s)}||^2_{\mathcal{L}_2(L^2, L^2_\rho)} \, ds \\ & + \int_{\frac{r}{\alpha}}^{\frac{t}{\alpha}} \mathbf{E}||U(t,s)\mathcal{M}_{\varphi(s)}||^2_{\mathcal{L}_2(L^2, L^2_\rho)} \, ds \Bigg]. \end{aligned} \tag{4.34}$$

Thus, see inequality (3.71), by the same arguments as in the cylindrical case of Proposition 3.4.6, we get the convergence to 0 as $r \uparrow t$ resp. $t \downarrow r$.

Now, we observe that, for any $\alpha > 1$,

$$\begin{aligned} \sup_{t \in [0,T]} \mathbf{E}||I^{\tilde{N}}_\varphi(t) - \Phi^\alpha(t)||^2_{L^2_\rho} = \;& \sup_{t \in [0,T]} \int_{\frac{t}{\alpha}}^{t} \int_{L^2} \mathbf{E}||U(t,s)\mathcal{M}_{\varphi(s)}x||^2_{L^2_\rho} \, \eta(dx)\, ds \\ \le \;& \left( \int_{L^2} ||x||^2_{L^2} \, \eta(dx) \right) \int_{\frac{t}{\alpha}}^{t} \mathbf{E}||U(t,s)\mathcal{M}_{\varphi(s)}||^2_{\mathcal{L}_2(L^2, L^2_\rho)} \, ds. \end{aligned}$$

Since the first term on the right hand side is finite and the second term tends to 0 as $\alpha \downarrow 1$ uniformly in $[0,T]$ (cf. the cylindrical case in the proof of Proposition 3.4.5), the term from the left hand side above tends to 0 as $\alpha \downarrow 1$.
Thus, $I^{\tilde{N}}_\varphi$ is also meansquare continuous as a uniform limit in $C([0,T], L^2(\Omega; L^2_\rho))$ of $I^\alpha$ as $\alpha \downarrow 1$. ■

A generalization of Proposition 4.7 to $q \ge 2$ is the following.

**Proposition 4.8:** *Let the assumptions of Proposition 4.7 hold. Suppose additionally that*

$$\sup_{t\in[0,T]} \mathbf{E}||\varphi(t)||^q_{L^2_\rho} < \infty$$

*and that **(QI)** holds for $\eta$ with some $q \in [2, \frac{2}{\zeta})$ with $\zeta$ as in **(A2)** (resp. **(A5)*** with $\nu = 1$) .*
*Then, the mapping $t \mapsto I^{\tilde{N}}_\varphi(t)$ is continuous in $L^q(\Omega; L^2_\rho)$.*

**Proof:** We keep the same notation and repeat the arguments used in proving Proposition 4.7.
Using the Bichteler-Jacod and Hölder's inequalities, similarly to the proof of Proposition 3.4.3 we arrive at the following estimate for
$0 \le r \le t \le T$:

$$
\begin{aligned}
&(4.35)\, \mathbf{E}||\Phi^\alpha(t) - \Phi^\alpha(r)||^q_{L^2_\rho} \\
&\le c(q,T)\left(\int_0^{\frac{r}{\alpha}}\left[\int_{L^2} \mathbf{E}||[U(t,s)-U(r,s)]\mathcal{M}_{\varphi(s)}x||^q_{L^2_\rho}\,\eta(dx)\,ds\right.\right.\\
&\left.\left. +\mathbf{E}\left(\int_{L^2} ||[U(t,s)-U(r,s)]\mathcal{M}_{\varphi(s)}x||^2_{L^2_\rho}\,\eta(dx)\right)^{\frac{q}{2}}\right]\,ds\right)\\
&+ c(q,T)\left(\int_{\frac{r}{\alpha}}^{\frac{t}{\alpha}}\left[\int_{L^2} \mathbf{E}||U(t,s)\mathcal{M}_{\varphi(s)}||^q_{L^2_\rho}\eta(dx)\right.\right.\\
&\left.\left. +\mathbf{E}\left(\int_{L^2} \mathbf{E}||U(t,s)\mathcal{M}_{\varphi(s)}x||^2_{L^2_\rho}\eta(dx)\right)^{\frac{q}{2}}\right]\,ds\right)\\
&\le c(q,T,C_{q,\eta})\left(\int_0^{\frac{r}{\alpha}} \mathbf{E}||[U(t,s)-U(r,s)]\mathcal{M}_{\varphi(s)}||^q_{\mathcal{L}_2(L^2,L^2_\rho)}\,ds\right.\\
&\left. +\int_{\frac{r}{\alpha}}^{\frac{t}{\alpha}} \mathbf{E}||U(t,s)\mathcal{M}_{\varphi(s)}||^q_{\mathcal{L}_2(L^2,L^2_\rho)}\,ds\right).
\end{aligned}
$$

From (4.35), proceeding analogously to the proof of 3.4.7 (the cylindrical case there) we get the convergence to 0 as $r \uparrow t$ resp. $t \downarrow r$, which finishes the proof. ■

Moreover, we have an extension of Proposition 4.7 to the spaces $L^{2\nu}_\rho$ with $\nu > 1$.

**Proposition 4.9:** *Suppose we have **(A0)**–**(A2)** and **(A5)** (or **(A5)***) for an almost strong evolution operator $U$.*
*Furthermore, let $\nu \in [1, \frac{1}{\nu})$ with $\zeta \in [0,1)$ such that $\eta$ fulfills **(QI)** with $q = 2\nu$ and $(\varphi(t))_{t\in[0,T]}$ is an $L^{2\nu}_\rho$-valued predictable process obeying*

*(4.36)*
$$\sup_{t\in[0,T]} \mathbf{E}||\varphi(t)||^{2\nu}_{L^{2\nu}_\rho} < \infty.$$

*Then,*

$$t \mapsto I^{\tilde{N}}_\varphi(t) := \int_0^t \int_{L^2} U(t,s)\mathcal{M}_{\varphi(s)}x\, \tilde{N}(ds,dx)$$

*is continuous in* $L^{2\nu}(\Omega; L^{2\nu}_\rho)$.

**Proof:** As was shown in Proposition 4.3, $I^{\tilde{N}}_\varphi \in L^{2\nu}_\rho$, $P$-a.s., for each $t \in [0,T]$.
Now, the previous scheme of proving Proposition 4.7 and 4.8 runs with $\nu > 1$ if we use the strong continuity of $U(t,s)$ in $L^{2\nu}_\rho$ (see assumption **(A3)**). Indeed, for $\Phi^\alpha(t) \in L^{2\nu}_\rho$ defined by (4.33) and $0 \le r \le t \le T$ we have

$$(4.37)\ \mathbf{E}||\Phi^\alpha(t) - \Phi^\alpha(r)||^{2\nu}_{L^{2\nu}_\rho}$$
$$\le c(\nu)\left[\mathbf{E}\int_0^{\frac{r}{\alpha}}\left[\int_{L^2} ||[U(t,\alpha s) - U(r,\alpha s)]U(\alpha s,s)\mathcal{M}_{\varphi(s)}(x)||^{2\nu}_{L^{2\nu}_\rho}\,\eta(dx)+\right.\right.$$
$$\left.\left(\int_{L^2} ||[U(t,\alpha s) - U(r,\alpha s)]U(\alpha s,s)\mathcal{M}_{\varphi(s)}(x)||^{2}_{L^{2\nu}_\rho}\,\eta(dx)\right)^\nu\right] ds$$
$$\left.+\mathbf{E}\int_{\frac{r}{\alpha}}^{\frac{t}{\alpha}}\left[\int_{L^2} ||U(t,s)\mathcal{M}_{\varphi(s)}(x)||^{2\nu}_{L^{2\nu}_\rho}\,\eta(dx) + \left(\int_{L^2} ||U(t,s)\mathcal{M}_{\varphi(s)}(x)||^{2}_{L^{2\nu}_\rho}\,\eta(dx)\right)^\nu\right] ds\right],$$

where we use the moment estimate (4.27) (cf. Remark 4.6 **(i)**).
Concerning the right hand side of (4.37), let us note that by **(A5)**/ **(A5)*** we have, $P$-a.s., $U(\alpha s,s)\mathcal{M}_{\varphi(s)}(x) \in L^{2\nu}_\rho$ for any $x \in L^2$. Therefore, by the continuity assumption from **(A3)** we have, for any $x \in L^2$ and $s \in [0,T]$,

$\mathbf{1}_{[0,\frac{r}{\alpha}]}(s)[U(t,\alpha s) - U(r,\alpha s)]U(\alpha s,s)\mathcal{M}_{\varphi(s)}(x) \to 0 \in L^{2\nu}_\rho$ as $r \uparrow t$ resp. $t \downarrow r$, $P$-a.s..

Furthermore, by **(A3)** and **(A5)**/ **(A5)***, **(QI)**, Hölder's inequality and the fact that

$$\nu < \tfrac{1}{\zeta} \Rightarrow \zeta\nu < 1,$$

we have the following estimate

$$\mathbf{E}\int_0^{\frac{r}{\alpha}} \int_{L^2} ||[U(t,\alpha s) - U(r,\alpha s)]U(\alpha s,s)\mathcal{M}_{\varphi(s)}(x)||^{2\nu}_{L^{2\nu}_\rho}\,\eta(dx)$$
$$\le 2c(\nu,T)\mathbf{E}\int_0^{\frac{r}{\alpha}} ||U(\alpha s,s)\mathcal{M}_{\varphi(s)}(x)||^{2\nu}_{L^{2\nu}_\rho}\,\eta(dx)\,ds$$

$$
\begin{aligned}
&\leq 2c(\nu,T)\left(\int\limits_{L^2}||x||_{L^2}^{2\nu}\,\eta(dx)\right)\mathbf{E}\int\limits_0^{\frac{r}{\alpha}}||U(\alpha s,s)\mathcal{M}_{\varphi(s)}||_{\mathcal{L}_2(L^2,L_\rho^{2\nu})}^{2\nu}\,ds\\
&\leq 2c(\nu,T)\left(\int\limits_{L^2}||x||_{L^2}^{2\nu}\,\eta(dx)\right)\mathbf{E}\int\limits_0^{\frac{r}{\alpha}}((\alpha-1)s)^{-\zeta}||\varphi(s)||_{L_\rho^{2\nu}}^{2\nu}\,ds\\
&\leq 2c(\nu,T)\left(\int\limits_{L^2}||x||_{L^2}^{2\nu}\,\eta(dx)\right)\left(\sup_{t\in[0,T]}\mathbf{E}||\varphi(s)||_{L_\rho^{2\nu}}^{2\nu}\right)(\alpha-1^{-\zeta\nu}\tfrac{T^{1-\zeta\nu}}{1-\zeta\nu}\\
&<\infty.
\end{aligned}
$$

Thus, we can apply Lebesgue's dominated convergence theorem to get the convergence to 0 of the first integral on the right hand side of (4.37) for $r\uparrow t$ resp. $t\downarrow r$. The proof for the second integral on the right hand side of (4.37) runs completely analogous.
Thus, it remains to consider the third and the fourth integral on the right hand side of (4.37). Concerning the third integral we have

$$
\begin{aligned}
&\mathbf{E}\int\limits_{\frac{r}{\alpha}}^{\frac{t}{\alpha}}\int\limits_{L^2}||U(t,s)\mathcal{M}_{\varphi(s)}(x)||_{L_\rho^{2\nu}}^{2\nu}\,\eta(dx)\,ds\\
&\leq c(T)\left(\int\limits_{L^2}||x||_{L^2}^{2\nu}\,\eta(dx)\right)\mathbf{E}\int\limits_{\frac{r}{\alpha}}^{\frac{t}{\alpha}}(t-s)^{-\zeta\nu}||\varphi(s)||_{L_\rho^{2\nu}}^{2\nu}\,ds\\
&\leq c(c(T),C_{2\nu,\eta})\left(\sup_{t\in[0,T]}\mathbf{E}||\varphi(t)||_{L_\rho^{2\nu}}^{2\nu}\right)\frac{\left(\frac{t-r}{\alpha}\right)^{1-\zeta\nu}}{1-\zeta\nu},
\end{aligned}
$$

which tends to 0 as $r\uparrow t$ resp. $t\downarrow r$. By similar arguments also the fourth integral in (4.37) tends to 0. ■

**Alternative Proof of Proposition 4.9**

Let us first consider regular enough integrands $\varphi(t)$, $t\in[0,T]$. Namely, we suppose that $\varphi\in L^\infty(\Theta)$, P-a.s., for each $t\in[0,T]$, and

$$\sup_{t\in[0,T]}\mathbf{E}||\varphi(t)||_{L_\rho^{2r}}^{2r}<\infty \text{ for some } r>\nu.$$

Let $t\in[0,T]$ be arbitrary and let $(t_n)_{n\in\mathbb{N}}\subset[0,T]$ be a sequence such that $t_n\to t$ as $n\to\infty$.

In the following, we would like to apply the de la Vallée-Poissin theorem, which only holds for finite measures. To this end, we observe that, by integrability condition (4.2) (with $q=2\nu$) for $\eta$,

$$\hat{\eta}_\nu(A):=\int\limits_A||x||_{L^2}^{2\nu}\,\eta(dx),\ A\in\mathcal{B}(L^2)$$

is a finite measure on $L^2$. In particular, we also get a finite measure $\hat{\eta}_1$ by setting

$$\hat{\eta}_1(A) := \int\limits_A ||x||_{L^2}^2 \, \eta(dx).$$

Next, for $s \in [\,0,T\,]$, we define the $L^2_\rho$-valued random variables

$$g_n(s,\omega,x) := \mathbf{1}_{[\,0,t_n\,]}(s)U(t_n,s)\mathcal{M}_{\varphi(s,\omega)}(x)||x||_{L^2}^{-1} \text{ if } ||x||_{L^2} \neq 0$$

and

$$g_n(s,\omega,x) := 0 \text{ if } ||x||_{L^2} = 0$$

for $(s,\omega) \in [\,0,T\,] \times \Omega$.

Thus, applying the Bichteler-Jacod inequality, we get

$$\begin{aligned}(4.38)\ &\mathbf{E}||I_\varphi^{\tilde{N}}(t_n) - I_\varphi^{\tilde{N}}(t)||_{L^{2\nu}_\rho}^{2\nu}\\ &\leq c(\nu,T)\mathbf{E}\int\limits_0^T \Bigg[\int\limits_{L^2} ||g_n(s,x)-g(s,x)||_{L^{2\nu}_\rho}^{2\nu}\,\hat{\eta}_\nu(dx)\\ &+\Bigg(\int\limits_{L^2} ||g_n(s,x)-g(s,x)||_{L^{2\nu}_\rho}^{2}\,\hat{\eta}_1(dx)\Bigg)^{\nu}\Bigg]\,ds.\end{aligned}$$

Therefore, to prove the claim, it suffices to prove that the both integrals on the right hand side converge to 0 as $n \to \infty$.

Since $\varphi(s)x \in L^2$ (recall that at the moment we restrict ourselves to regular processes $\varphi$ such that $\varphi(s) \in L^\infty$ for any $s \in [\,0,T\,]$) and

$$(\,s,T\,] \ni t \mapsto U(t,s) \in \mathcal{L}(L^2, L^{2\nu}_\rho)$$

is strongly continuous, we get that $P$-almost surely

$$||g_n(s,\omega,x) - g(s,\omega,x)||_{L^{2\nu}_\rho}^{2\nu} \to 0 \text{ as } n \to \infty \tag{4.39}$$

for any fixed $s \in [\,0,T\,]$ and $x \in L^2$.

Furthermore, due to the de la Vallée-Poissin theorem we have uniform integrability of $||g_n||_{L^{2\nu}_\rho}^{2\nu}$ w.r.t. $P \otimes ds \otimes \hat{\eta}_\nu$. Indeed, given a small enough $\varepsilon > 0$ such that $\zeta(1+\varepsilon) < 1$ and $\nu(1+\varepsilon) \leq p$, we get by **(A5)**/**(A5)***

$$
\begin{aligned}
&\mathbf{E}\int_0^T \int_{L^2} ||g_n(s,x)||_{L^{2\nu}_\rho}^{2\nu(1+\varepsilon)}\,\hat{\eta}_\nu(dx)\,ds \\
&= \mathbf{E}\int_0^{t_n} \int_{L^2} \frac{||U(t_n,s)\mathcal{M}_{\varphi(s)}(x)||_{L^{2\nu}_\rho}^{2\nu(1+\varepsilon)}}{||x||_{L^2}^{2\nu(1+\varepsilon)}}\,\hat{\eta}_\nu(dx)\,ds \\
&\le c(T)(\hat{\eta}_\nu(L^2))\mathbf{E}\int_0^{t_n}(t_n-s)^{-\zeta(1+\varepsilon)}||\varphi(s)||_{L^{2\nu}_\rho}^{2\nu(1+\varepsilon)}\,ds \\
&\le c(\varepsilon,\nu,T,c(T))(\hat{\eta}_\nu(L^2))\left(\sup_{t\in[0,T]}\mathbf{E}||\varphi(t)||_{L^{2p}_\rho}^{2p}\right)\left(\int_0^T s^{-\zeta(1+\varepsilon)}\,ds\right) \\
&< \infty
\end{aligned}
$$

uniformly for any $n \in \mathbb{N}$. Thus, we have

$$\sup_{n\in\mathbb{N}} \mathbf{E}\int_0^T \int_{L^2} ||g_n(s,x)||_{L^{2\nu}_\rho}^{2\nu(1+\varepsilon)}\,\hat{\eta}_\nu(dx)\,ds < \infty.$$

Together with the $P$-almost sure convergence (4.39), this gives us

$$\lim_{n\to\infty} \mathbf{E}\int_0^T \int_{L^2} ||g_n(s,x) - g(s,x)||_{L^{2\nu}_\rho}^{2\nu}\,\hat{\eta}_\nu(dx) = 0.$$

Analogously, we consider the second term in (4.38).

Let again $\varepsilon > 0$ be such that $\zeta(1+\varepsilon) < 1$ and $\nu(1+\varepsilon) \le p$. Then, by Hölder‘s inequality

$$
\begin{aligned}
&\mathbf{E}\int_0^T \left(\int_{L^2} ||g_n(s,x)||_{L^{2\nu}_\rho}^{2}\,\hat{\eta}_1(dx)\right)^{\nu(1+\varepsilon)} ds \\
&\le c(\nu,\varepsilon)\mathbf{E}\int_0^t \int_{L^2} ||g_n(s,x)||_{L^{2\nu}_\rho}^{2\nu(1+\varepsilon)}\,\hat{\eta}_1(dx),
\end{aligned}
$$

which is uniformly bounded in $n \in \mathbb{N}$ similarly to the previous argument. Thus, we get

$$\sup_{n\in\mathbb{N}} \mathbf{E}\int_0^T \left(\int_{L^2} ||g_n(s,x)||_{L^{2\nu}_\rho}^{2}\hat{\eta}_1(dx)\right)^{\nu(1+\varepsilon)} ds < \infty,$$

which together with (4.39) implies

$$\lim_{n\to\infty} \mathbf{E}\int_0^T \left(\int_{L^2} ||g_n(s,x) - g(s,x)||_{L^{2\nu}_\rho}^{2}\,\hat{\eta}_1(dx)\right)^{\nu} ds = 0.$$

Therefore, the right hand side of the estimate of $\mathbf{E}||I_\varphi^{\tilde{N}}(t_n) - I_\varphi^{\tilde{N}}(t)||_{L^{2\nu}_\rho}^{2\nu}$ tends to 0 as $n \to \infty$, which proves the claim for such regular $\varphi$.

The claim for general $\varphi$ follows from the continuity of $t \mapsto I_\varphi^{\tilde{N}}(t)$ for regular $\varphi$ analogously to the alternative proof of Proposition 3.4.6 from Section 3.4. ■

**Remark 4.10:** *If $(\varphi(t))_{t\in[0,T]}$ is bounded in the sense of (3.81) from Section 3.4., i.e.*

$$\sup_{t\in[0,T]} \mathbf{E}||\varphi||_\infty < \infty,$$

*then the statements of Propositions 4.3, 4.4 and 4.7–4.9 remain valid under the weaker assumption that **(A5)** resp. **(A5)*** holds only for $\varphi \equiv 1 \in L_\rho^{2\nu}$.*

**Remark 4.11:** *From the proof of Claim 3 in the proof of Proposition 4.3 we know that, for each $t \in [0,T]$, there is an $\mathcal{F}_t \otimes \mathcal{B}(\Theta)$-measurable version of*

$$\tilde{I}(t,\theta) = \int_0^t \int_{L^2\setminus\{0\}} \psi(s,x,\theta)\,\tilde{N}(ds,dx),\ \theta \in \Theta \ \textit{(cf (4.20))}$$

*with representation $\psi$ as in (4.17).*
*On the other hand, from Proposition 4.7 it follows that*
*$[0,T] \ni t \mapsto \tilde{I}(t,\omega,\theta) \in \mathbb{R}$ is meansquare continuous (and hence continuous in probability) w.r.t. $P \otimes \mu_\rho$.*
*Thus, by Proposition 2.1.8, there exists a version $(I_\varphi^{\tilde{N}}(t))(\omega,\theta) \in \mathbb{R}$, $(t,\omega,\theta) \in [0,T] \times \Omega \times \Theta$, of the Poisson integral $I_\varphi^{\tilde{N}}(t)$, which is measurable w.r.t. the **predictable** $\sigma$-algebra $\mathcal{P}_T \otimes \mathcal{B}(\Theta)$.*

We finish this section by the following path property.

**Proposition 4.12:** *Under the additional assumption that $\varphi$ is uniformly bounded, i.e.*

$$\sup_{t\in[0,T]} \mathbf{E}||\varphi||_{L^\infty} < \infty, \tag{4.40}$$

*and $U$ is an almost strong evolution operator obeying **(A7)**, the mapping*

$$t \mapsto I_{\varphi(s)}^{\tilde{N}}(t) = \int_0^t \int_{L^2} U(t,s)\mathcal{M}_{\varphi(s)}(x)\,\tilde{N}(ds,dx) \in L_\rho^{2\nu}$$

*has a càdlàg version in $L^2$ resp. $L_\rho^2$.*

**Proof:** Under our assumptions, the multiplication operators $\mathcal{M}_{\varphi(t)}$ are both in $L^2$ and $L_\rho^2$ with

$$\sup_{t\in[0,T]} \mathbf{E}||\mathcal{M}_{\varphi(t)}||_{\mathcal{L}} \leq \sup_{t\in[0,T]} \mathbf{E}||\varphi(t)||_{L^\infty} < \infty.$$

The evolution operator $U$ is pseudo-contractive in $L^2$ resp. $L^2_\rho$.

Recall from Section 2.4 that setting

$$M(t,A) := \int_{A\setminus\{0\}} x\,\tilde{N}(t,dx),\ (t,A) \in [0,T] \times \mathcal{B}(L^2)$$

gives us an orthogonal martingale-valued measure. Then, denoting $M(t) := M(t,L^2)$, $t \in [0,T]$, we get

$$I^{\tilde{N}}_{\varphi(s)}(t) = \int_0^t U(t,s)\mathcal{M}_{\varphi(s)}\,dM(s),\ t \in [0,T].$$

Now, by the contraction property **(A7)** of $U$ in $L^2$ resp. $L^2_\rho$, the uniform moment bound (4.40) and the fact that $M$ defined above is càdlàg in $L^2$, we get the claim in $L^2$ resp. $L^2_\rho$ from Corollary 2.1 from [63] resp. Remark 1.2 1. from [64]. ■

**Remark 4.13:** ***(i)*** *In the case of a contractive semigroup $U(t,s) = e^{-(t-s)A}$, $0 \leq s \leq t$, the existence of a càdlàg version of the stochastic convolution w.r.t. compensated Poisson random measure was proven e.g. by Albeverio, Mandrekar and Rüdiger in [3]( cf. Proposition 2.5., p.840 there).*

***(ii)*** *The proof of the càdlàg property is based on the general maximal inequality for martingales, see Theorem 1.1 in [64]. An essential drawback of this method is that we should assume $\varphi$ to be uniformly bounded (see (4.40)) and $U$ to be pseudo-contractive (see **(A7)**).*
*Furthermore, as was noted in [95], Section 11.4, pp 199/200 there, the factorization method is not applicable to study convolutions of general martingales, in contrast to the special case of a Wiener process (cf. e.g. [26], Chapter 7, Proposition 7.3, p.184).*

**Remark 4.14:** *Let us clarify the relation between càdlàg and predictable versions of $t \mapsto I^{\tilde{N}}_\varphi(t)$.*
*From the considerations above it follows that $I^{\tilde{N}}_\varphi(t)$, $t \in [0,T]$, possesses a predictable version under the general assumptions of Proposition 4.3.*
*Indeeed, from Proposition 4.7, it follows that the mapping $[0,T] \ni t \mapsto I^{\tilde{N}}_\varphi(t) \in L^2_\rho$ is continuous in probability. Since $I^{\tilde{N}}_\varphi(t)$ is $\mathcal{F}_t$-measurable for each $t \in [0,T]$, by Proposition 2.1.8 we get existence of a predictable version of $t \mapsto I^{\tilde{N}}_\varphi(t)$.*

*If we know that* $t \mapsto I_\varphi^{\tilde{N}}(t)$ *obeys a càdlàg version* $\tilde{I}_\varphi^{\tilde{N}}(t)$ *(see Proposition 4.12), then surely* $t \mapsto \tilde{I}_\varphi^{\tilde{N}}(t-)$ *will be predictable.*
*To distinguish between the two versions, some authors use the notation*

$$\int_0^{t+} \int_{L^2} U(t,s)\mathcal{M}_{\varphi(s)} x \, \tilde{N}(ds,dx)$$

*for the càdlàg and*

$$\int_0^{t-} \int_{L^2} U(t,s)\mathcal{M}_{\varphi(s)} x \, \tilde{N}(ds,dx)$$

*for the predictable version. If it does not lead to misunderstandings, we will use the universal notation*

$$\int_0^{t} \int_{L^2} U(t,s)\mathcal{M}_{\varphi(s)} x \, \tilde{N}(ds,dx).$$

# Chapter 5

# Existence and uniqueness of mild solutions in the Lipschitz case

In this chapter, we show several existence and uniqueness results for solutions to equation (1.1) resp. equation (1.2) in the case of Lipschitz coefficients. A precise meaning to such solutions will be given in Section 5.1 below.

Given $\Theta \subset \mathbb{R}^d$ with $d \in \mathbb{N}$, let $\rho \geq 0$ be such that $\mu_\rho(\Theta) < \infty$, i.e. $\rho > d$ for unbounded and $\rho = 0$ for bounded $\Theta$.

For the whole chapter, let $(\Omega, \mathcal{F}, P)$ and $(\mathcal{F}_t)_{t \in [0,T]}$ be as in Section 1.2, where we fix some $T > 0$. Furthermore, we here assume that $e$, $f$, $\sigma$, $\gamma$: $[0,T] \times \Omega \times \mathbb{R} \to \mathbb{R}$ (generating $E$, $F$ and $\Sigma$, $\Gamma$ by **(NEM)**) fulfill the Lipschitz continuity property **(LC)** and the local boundedness property **(LB)**. We further assume that $e$, $f$, $\sigma$ and $\gamma$ are measurable functions mapping $([0,T] \times \Omega \times \mathbb{R}, \mathcal{P}_T \otimes \mathcal{B}(\mathbb{R}))$ to $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. We note that we assume the functions to be predictable just for simplicity. This property is only needed to define the Poisson integral terms. Clearly, for the drift and diffusion coefficients $e$, $f$ and $\sigma$ it suffices to assume only **progressive measurability**.

Given $q \geq 2$ resp. $\nu > 1$, the solutions will be constructed in the Banach spaces $\mathcal{H}^q(T)$ and $\mathcal{G}_\nu(T)$ of predictable $L^2_\rho(\Theta)$ resp. $L^{2\nu}_\rho(\Theta)$-valued processes $(X(t))_{t \in [0,T]}$ with finite $q$-th resp. $2\nu$-th moment (see also Definition 5.1.1 below). We consider solutions both in $L^2_\rho(\Theta)$ and $L^{2\nu}_\rho(\Theta)$ with $\nu > 1$, since later in the case of drifts of polynomial growth, depending on the polynomiality of the drift, the solutions will take their values either in $L^2_\rho(\Theta)$ or in $L^{2\nu}_\rho(\Theta)$ (see Chapter 7).

The explicit setting including the definition of mild predictable solutions is introduced in Section 5.1. Formulation and proof of the existence and uniqueness result are given in Section 5.2.

## 5.1 Definition of mild solutions

In Sections 5.1. and 5.2, we are in the following setting:

1. $(A(t))_{t\in[0,T]}$ generates an almost strong evolution operator $U = (U(t,s))_{0\le s\le t\le T}$ in $L^2_\rho(\Theta)$ in the sense of 2.2.1.
2. $(W(t))_{t\in[0,T]}$ is a $Q$-Wiener process in $L^2(\Theta)$ such that either $Q \in \mathcal{T}^+(L^2(\Theta))$ has the representation (2.4) with an orthonormal basis $(e_n)_n$ obeying (3.1) (called the nuclear case in the following) or $Q = \mathbf{I} \notin \mathcal{T}^+(L^2(\Theta))$ (called the cylindrical case in the following).
3. $(\tilde{N}(t,\cdot))_{t\in[0,T]}$ is a compensated Poisson random measures on $L^2(\Theta)$. We further assume that the corresponding Lévy intensity measure $\eta$ fulfills the integrability condition **(QI)**.
4. $L$ is a Lévy process in $L^2(\Theta)$ with characteristics $(b, W, \eta)$. Concerning $W$ and $\eta$, we assume the properties of the previous items, namely $Q \in \mathcal{T}^+(L^2(\Theta))$ obeys an eigenvector expension with the property (2.4). Note that in general the eigenvector basis in (2.4) does not obey (3.1) (called the general nuclear case later).

To define solutions to equation (1.1) resp. equation (1.2), we introduce the following spaces of predictable processes:

**Definition 5.1.1:** *Let $q \ge 2$ and $\nu \ge 1$ be fixed.*
***(i)*** *By $\mathcal{H}^q(T)$ we denote the Banach space of all predictable (up to a stochastic modification) $L^2_\rho(\Theta)$-valued processes $(Z(t))_{t\in[0,T]}$ such that*

$$\sup_{t\in[0,T]} \mathbf{E}||Z(t)||^q_{L^2_\rho} < \infty. \tag{5.1}$$

*The norm in $\mathcal{H}^q(T)$ is given by*

$$||Z||_{\mathcal{H}^q(T)} := \sup_{t\in[0,T]} \left(\mathbf{E}||Z(t)||^q_{L^2_\rho}\right)^{\frac{1}{q}}. \tag{5.2}$$

***(ii)*** *By $\mathcal{G}_\nu(T)$ we denote the Banach space of all predictable (up to a stochastic modification) $L^{2\nu}_\rho(\Theta)$-valued processes $(Z(t))_{t\in[0,T]}$ such that*

$$\sup_{t\in[0,T]} \mathbf{E}||Z(t)||^{2\nu}_{L^{2\nu}_\rho} < \infty. \tag{5.3}$$

*The norm in $\mathcal{G}_\nu(T)$ is given by*

$$(5.4) \qquad ||Z||_{\mathcal{G}_\nu(T)} := \sup_{t\in[\,0,T\,]} (\mathbf{E}||Z(t)||^{2\nu}_{L^{2\nu}_\rho})^{\frac{1}{2\nu}}.$$

*In particular, we have $\mathcal{G}_1(T) = \mathcal{H}^2(T)$.*
*Note that the same spaces of predictable processes were used in the papers [79] and [81].*

*The above notations are justified by the following*

- *since in **(i)** only the moment changes, the index $q$ is written above;*
- *since in **(ii)** the basic space $L^{2\nu}_\rho(\Theta)$ changes with the index $\nu$, the index $\nu$ is written below.*

To guarantee the completeness w.r.t. norms (5.2) resp. (5.4), actually we consider the *equivalence classes* (up to stochastic modifications) in $\mathcal{H}^q(T)$ resp. $\mathcal{G}_\nu(T)$.

Depending on the choice of initial conditions, which are $L^2_\rho(\Theta)$-valued, $\mathcal{F}_0$-measurable random variables, we split our considerations into two cases:

**Case (A)** $L^2_\rho(\Theta)$-valued initial conditions $\xi$ with $\mathbf{E}||\xi||^q_{L^2_\rho} < \infty$ for some $q \geq 2$;

**Case (B)** $L^{2\nu}_\rho(\Theta)$-valued initial conditions $\xi$ with $\mathbf{E}||\xi||^{2\nu}_{L^{2\nu}_\rho} < \infty$ for some $\nu > 1$.

Now, we rigorously define what we mean by a solution to equation (1.1) resp. equation (1.2) in **Case (A)** resp. in **Case (B)** in the following chapters:

**Definition 5.1.2:** ***(i)*** *In **Case (A)**, an $\mathcal{H}^q(T)$-valued process $X$ is called a **mild** solution to (1.1) resp. (1.2) if the following identity in $L^2_\rho(\Theta)$ holds $P$-almost surely for any $t \in [\,0,T\,]$*

$$\begin{aligned}(5.5)\ X(t) = \quad & U(t,0)\xi + \int_0^t U(t,s)F(s,X(s))ds \\ & + \int_0^t U(t,s)\mathcal{M}_{\Sigma(s,X(s))}dW(s) \\ & + \int_0^t \int_{L^2} U(t,s)\mathcal{M}_{\Gamma(s,X(s))}(x)\,\tilde{N}(ds,dx)\end{aligned}$$

*resp.*

$$(5.6)\ X(t) = \quad U(t,0)\xi + \int_0^t U(t,s)E(s,X(s))ds + \int_0^t U(t,s)\mathcal{M}_{\Sigma(s,X(s))}dL(s),$$

*whereby all the integrals on the right hand side exist.*

***(ii)*** *In* ***Case (B)****, a $\mathcal{G}_\nu(T)$-valued process $X$ is called a* ***mild*** *solution to (1.1) resp. (1.2) if (5.5) resp. (5.6) holds true, $P$-almost surely, for all $t \in [\,0, T\,]$.*

*The well-definedness of the integral terms in (5.5)/(5.6) is discussed in Lemmata 5.1.5–5.1.10 below.*

In later chapters, the spaces $\mathcal{H}^q(T)$ will be used to study equation (1.1) resp. equation (1.2) in the case of the drift coefficients obeying condition **(PG)** with $\nu = 1$, whereas the spaces $\mathcal{G}_\nu(T)$ will be used to study equation (1.1) resp. equation (1.2) in the case of the drift coefficients obeying condition **(PG)** with $\nu > 1$.

**Remark 5.1.3:** ***(i)*** *Note that, in contrast to the solution definition in [76], there is no condition on pathwise continuity in our setting. Indeed, due to the presence of the Poisson resp. Lévy integral we cannot expect pathwise continuity of the solutions anymore.*
*Nevertheless, in view of Proposition 4.12 we get existence of càdlàg versions of the solutions under special assumptions on the jump coefficient $\Gamma$ (case (1.1)) resp. the jump diffusion coefficient $\Sigma$ (case (1.2)) and the evolution operator $U$.*
*The pathwise continuity will be substituted by the meansquare continuity of the solutions resulting from Propositions 3.4.6 and 4.7.*

***(ii)*** *By Definition 5.1.1 it is obvious that $\mathcal{H}^q(T) \subset \mathcal{H}^2(T)$ for $q \geq 2$ and $\mathcal{G}_\nu(T) \subset \mathcal{H}^{2\nu}(T) \subset \mathcal{H}^2(T)$ for $\nu > 1$. Thus, both for $Z \in \mathcal{H}^q(T)$ with $q \geq 2$ and $Z \in \mathcal{G}_\nu(T)$ with $\nu > 1$, we have*

$$\sup_{t\in[\,0,T\,]} \mathbf{E}||Z(t)||^2_{L^2_\rho} < \infty.$$

*Furthermore, for $\nu < \frac{1}{\{\zeta}$ with $\zeta$ as in* ***(A2)*** *resp.* ***(A5)/(A5)****, any process from $\mathcal{G}_\nu(T)$ fulfills the integrability conditions (3.39) and (4.12), which means that 3.4.3 and 4.3 are applicable.*

***(iii)*** *Given any measurable function $\lambda$: $[\,0, T\,] \times \Omega \times \Theta \to \mathbb{R}$ fulfilling the Lipschitz property* ***(LC)*** *and the local boundedness property* ***(LB)****, we see that*

$$\sup_{t\in[\,0,T\,]} \mathbf{E}||\Lambda(t, Z(t))||^q_{L^2_\rho} < \infty$$

*for any $Z \in \mathcal{H}^q(T)$ and*

$$\sup_{t\in[0,T]} \mathbf{E}||\Lambda(t, Z(t))||^{2\nu}_{L^{2\nu}_\rho} < \infty$$

*for any $Z \in \mathcal{G}_\nu(T)$, with $\Lambda$ being defined through $\lambda$ by **(NEM)**. Indeed, we have*

$$
\begin{aligned}
(5.7)\quad \sup_{t\in[0,T]} \mathbf{E}||\Lambda(t, Z(t))||^q_{L^2_\rho} &= \sup_{t\in[0,T]} \mathbf{E}\left(\int_\Theta (|\lambda(t, Z(t,y))|)^2 \mu_\rho(dy)\right)^{\frac{q}{2}} \\
&\le \sup_{t\in[0,T]} \mathbf{E}\left(\int_\Theta (c_\lambda(T)(1+Z(t,y)))^2 \mu_\rho(dy)\right)^{\frac{q}{2}} \\
&\le c(q, c_\lambda(T))\left(1 + \sup_{t\in[0,T]} \mathbf{E}||Z(t)||^q_{L^2_\rho}\right) \\
&= c(q, c_\lambda(T))(1 + ||Z(t)||^q_{\mathcal{H}^q(T)}) \\
&< \infty
\end{aligned}
$$

*for any $Z \in \mathcal{H}^q(T)$ and*

$$
\begin{aligned}
(5.8)\quad \sup_{t\in[0,T]} \mathbf{E}||\Lambda(t, \cdot, Z(t))||^{2\nu}_{L^{2\nu}_\rho} &= \sup_{t\in[0,T]} \mathbf{E}\int_\Theta (|\lambda(t, \cdot, Z(t,y))|)^{2\nu} \mu_\rho(dy) \\
&\le \sup_{t\in[0,T]} \mathbf{E}\int_\Theta (c_\lambda(T)(1+Z(t,y)))^{2\nu} \mu_\rho(dy) \\
&\le c(\nu, c_\lambda(T))\left(1 + \sup_{t\in[0,T]} \mathbf{E}||Z||^{2\nu}_{L^{2\nu}_\rho}\right) \\
&= c(\nu, c_\lambda(T))(1 + ||Z||^{2\nu}_{\mathcal{G}_\nu(T)}) \\
&< \infty
\end{aligned}
$$

*for any $Z \in \mathcal{G}_\nu(T)$, where we used **(LC)**, **(LB)** for $\lambda$ in the second and the finiteness of $\mu_\rho(\Theta)$ in the third step of both cases.*

*So we have shown that under the above conditions the Nemitskii operator preserves finiteness of the $\mathcal{H}^q(T)$- resp. $\mathcal{G}_\nu(T)$-norm.*

The following general proposition concerning existence of measurable realizations for the processes under consideration will be frequently used both in this chapter and the following ones:

**Proposition 5.1.4:** *For any $\mathcal{P}_T/\mathcal{B}(L^2_\rho(\Theta))$-measurable process $\varphi$: $[0,T] \times \Omega \to L^2_\rho(\Theta)$ obeying*

$$(5.9)\qquad \int_0^T \mathbf{E}||\varphi(s)||^2_{L^2_\rho}\, ds < \infty,$$

*there is a $\mathcal{P}_T \otimes \mathcal{B}(\Theta)/\mathcal{B}(\mathbb{R})$-measurable version $\tilde{\varphi}$: $[0,T] \times \Omega \times \Theta \to \mathbb{R}$ such that*

$$\int_0^T \mathbf{E}||\varphi(s) - \tilde{\varphi}(s)||^2_{L^2_\rho}\, ds = 0.$$

**Proof:** Analogously to Steps 1 in the proofs of 3.4.3 and 4.3, we can apply Dirac sequences to construct measurable representatives. Without loss of generality, we may assume here that $\Theta = \mathbb{R}^d$. In the case of $\Theta \subset \mathbb{R}^d$, the function $\varphi(s,\omega) \in L^2_\rho(\Theta)$ should be extended by zero outside $\Theta$.
By analogous arguments as in Step 1 in the proof of 3.4.3, we have the convergence (in $L^2(\Theta)$)

$$conv(\delta_m, \mu_\rho\varphi(s)) \longrightarrow \mu_\rho\varphi(s) \text{ as } m \to \infty, \tag{5.10}$$

for almost all $(s,\omega) \in [\,0,T\,] \times \Omega$ (with *conv* denoting the standard convolution with the Dirac sequence $\delta_m$, $m \in \mathbb{N}$, defined by (3.44) and the weight function $\mu_\rho$ as in the Introduction). Furthermore, we can calculate $conv(\delta_m, \mu_\rho\varphi(s))(\theta)$ for any $\theta \in \Theta$.
Thus, for $m \in \mathbb{N}$, we define $\varphi_m$: $[\,0,T\,] \times \Omega \times L^2(\Theta) \times \Theta \to \mathbb{R}$ by

$$\varphi_m(s,\omega,\theta) := \mu_\rho^{-1}(conv(\delta_m, \mu_\rho\varphi(s)))(\theta). \tag{5.11}$$

By construction, for almost all $(s,\omega) \in [\,0,T\,] \times \Omega$,

$$\varphi_m(s,\omega) \in L^2_\rho(\Theta),\ ||\varphi_m(s,\omega)||_{L^2_\rho} \le ||\varphi(s,\omega)||_{L^2_\rho} \tag{5.12}$$

and

$$\lim_{m\to\infty} ||\varphi_m(s,\omega) - \varphi(s,\omega)||^2_{L^2_\rho} = 0. \tag{5.13}$$

From Theorem 6.1 in [51] we get $\mathcal{P}_T \otimes \mathcal{B}(\Theta)$-measurability of $\varphi_m$, $m \in \mathbb{N}$.

Recall that this needs the following properties to be fulfilled:

- continuity of the mapping

$$\theta \mapsto \varphi_m(s,\omega,\theta)$$

  for almost any fixed $(s,\omega) \in [\,0,T\,] \times \Omega$, and

- predictability of

$$(s,\omega) \mapsto \varphi_m(s,\omega,\theta)$$

for any fixed $\theta \in \Theta$.

The required continuity property holds by (3.45).

For any fixed $\theta \in \Theta$, we have $P \otimes ds$-almost surely

$$\begin{aligned}\varphi_m(s,\omega,\theta) = &\quad \mu_\rho^{-\frac{1}{2}}(\theta) \int_\Theta \delta_m(\xi-\theta)(\mu_\rho^{\frac{1}{2}}\varphi(s,\omega))(\xi)\, d\xi \\ = &\quad \mu_\rho^{-\frac{1}{2}}(\theta) < \delta_m(\cdot-\theta), \mu_\rho^{\frac{1}{2}}\varphi(s) >_{L^2}.\end{aligned}$$

By Fubini's Theorem, we get the required predictability of $(s,\omega) \mapsto \varphi_m(s,\omega,\theta)$ from the predictability of the process $\varphi$.

Thus, for each $m \in \mathbb{N}$, Theorem 6.1 from [51] gives us $\mathcal{P}_T \otimes \mathcal{B}(\Theta)$-measurability of

$$(s,\omega,\theta) \mapsto \varphi_m(s,\omega,x,\theta).$$

By (5.9) and (5.12)/(5.13), $(\varphi_m(s,\omega))_{m\in\mathbb{N}}$ is a Cauchy sequence in $L^2_\rho(\Theta)$ for almost all $(s,\omega) \in [\,0,T\,] \times \Omega$.

By (5.12) we have for all $m, \bar{m} \in \mathbb{N}$ and almost all $(s,\omega) \in [\,0,T\,] \times \Omega$

$$||\varphi_m(s,\cdot) - \varphi_{\bar{m}}(s,\cdot)||^2_{L^2_\rho} \le 4||\varphi(s)||^2_{L^2_\rho}. \tag{5.14}$$

By (5.9) the right hand side of (5.14) is integrable w.r.t. $P \otimes ds$. Thus, Lebesgue's dominated convergence theorem is applicable. Therefore,

$$(s,\omega) \mapsto \varphi_m(s,\omega),\ m \in \mathbb{N},$$

is a Cauchy sequence in $L^2(\Omega \times [\,0,T\,], \mathcal{P}_T, P \otimes dt; L^2_\rho)$.

Furthermore, by Fubini's theorem we have

$$\mathbf{E}\int_0^T\int_\Theta |\varphi_m(s,\theta) - \varphi^{(n)}_{\bar{m}}(s,\theta)|^2\, \mu_\rho(d\theta)\, ds = \mathbf{E}\int_0^t ||\varphi^{(n)}_m(s,\cdot) - \varphi^{(n)}_{\bar{m}}(s,\cdot)||^2_{L^2_\rho}\, ds.$$

Thus,

$$(s,\omega,\theta) \mapsto \varphi_m(s,\omega,\theta),\ m \in \mathbb{N},$$

is a Cauchy sequence in the Hilbert space

$$L^2([\,0,T\,] \times \Omega \times \Theta) := L^2([\,0,T\,] \times \Omega \times \Theta, \mathcal{P}_T \otimes \mathcal{B}(\Theta), P \otimes dt \otimes \mu_\rho).$$

So, there exists a limit function

$$\tilde{\varphi} \in L^2([0,T] \times \Omega \times \Theta)$$

such that

$$\lim_{m\to\infty} \mathbf{E} \int_0^t \int_\Theta |\varphi_m(s,\theta) - \tilde{\varphi}(s,\theta)|^2 \, \mu_\rho(d\theta)\, ds = 0. \tag{5.15}$$

Obviously, this implies that $\tilde{\varphi}(s,\omega,\cdot) \in L^2_\rho(\Theta)$ for $P \otimes dt$-almost all $(s,\omega) \in [0,T] \times \Omega$.

On the other hand, by Lebesgue's theorem and (5.11)/(5.12) we have

$$\lim_{m\to\infty} \mathbf{E} \int_0^t ||\varphi_m(s) - \varphi(s)||^2_{L^2_\rho} \, ds = 0. \tag{5.16}$$

Combined with (5.15) and (5.16), this implies

$$\mathbf{E} \int_0^T \int_{L^2} ||\varphi(s) - \tilde{\varphi(s)}||^2_{L^2_\rho} \, ds = 0.$$

Thus, $\tilde{\varphi}$ is a version of $\varphi$ obeying the required measurability properties. ■

We note that the authors in [76] showed pathwise time-continuity of the solutions to (1.1) in the case $\Gamma = 0$. Because of the jump behaviour of $\tilde{N}$ resp. $L$, we are not able to have this kind of time-continuity in our case. It will be substituted by meansquare continuity or càdlàg properties.

Thus, before we proceed with proving existence and uniqueness of solutions, we need to show that the integrals appearing on the right hand sides of (5.5) and (5.6) map $\mathcal{H}^q(T)$ resp. $\mathcal{G}_\nu(T)$ onto itself and fulfill the required time-continuity properties. This is done by the following six lemmata:

Let $f$ and $\sigma$ be as in the introduction of this chapter.
Everywhere below we assume that $U$ is an almost strong evolution operator in the sense of 2.2.1 obeying **(A0)**–**(A1)**.

We start with the drift terms and define processes $I_F(Z)$, $I_{\Sigma,\eta}(Z)$ by

$$I_F(Z)(t) := \int_0^t U(t,s) F(s, Z(s))\, ds, \tag{5.17}$$

$$I_{\Sigma,m}(Z)(t) := \int_0^t U(t,s)\mathcal{M}_{\Sigma(s,Z(s))}(m)\ ds,\ t \in [\,0, T\,], \tag{5.18}$$

with $m \in L^2$ being defined through $b$ and $\eta$ as described in the refined Lévy-Itô decomposition 2.4.13.
The integration is meant in the Bochner sense in $L^2_\rho(\Theta)$. This is a special case of the Bochner convolutions considered in Section 3.3.

**Lemma 5.1.5: Case (A)**
*Given $q \geq 2$, suppose $(Z(t))_{t\in[\,0,T\,]} \in \mathcal{H}^q(T)$.*

***(i)*** *The process $I_F(Z)$ is adapted, has finite $\mathcal{H}^q(T)$-norm and $t \mapsto I_F(Z)(t)$ is continuous in $L^q(\Omega; L^2_\rho(\Theta))$.*
***(ii)*** *Additionally assuming that $U$ obeys **(A2)** (or even the weaker assumption **(A5)*** with $\nu = 1$), the process $I_{\Sigma,m}(Z)$ is adapted, has finite $\mathcal{H}^q(T)$-norm and $t \mapsto I_{\Sigma,m}(Z)(t)$ is continuous in $L^q(\Omega; L^2_\rho(\Theta))$.*

*Thus, $I_F(Z)$ and $I_{\Sigma,m}(Z)$ obey predictable modifications and hence they are $\mathcal{H}^q(T)$-valued.*
*Furthermore, both $t \mapsto I_F(Z)(t)$ and $t \mapsto I_{\Sigma,m}(Z)(t)$ are pathwise continuous, and there are $\mathcal{P}_T\otimes\mathcal{B}(\Theta)$-measurable versions of the processes by Remark 5.1.4.*

**Proof:** We prove **(i)** and **(ii)** simultaneously following a certain pattern. It involves the following claims:

- predictability, i.e. $\mathcal{P}_T/\mathcal{B}(L^2_\rho())$-measurability of $t \mapsto F(t, Z(t))$ and $t \mapsto \Sigma(t, Z(t))$,
- the well-definedness of $I_F(Z)$ and $I_{\Sigma,m}(Z)$ in $L^2_\rho(\Theta)$,
- finiteness of $\mathcal{H}^q(T)$-norms,
- the required continuity properties of $t \mapsto I_F(Z)(t)$ and $t \mapsto I_{\Sigma,m}(Z)(t)$.
- predictability of $t \mapsto I_F(Z)(t)$ and $t \mapsto I_{\Sigma,m}(Z)(t)$

**Claim 1:** *$t \mapsto F(t, Z(t)) \in L^2_\rho(\Theta)$ is predictable.*

**Proof:** This follows immediately from the assumption $Z \in \mathcal{H}^q(T)$ and the $\mathcal{P}_T \otimes \mathcal{B}(\mathbb{R})/\mathcal{B}(\mathbb{R})$- measurability of $f$. □

**Claim 2:** *$t \mapsto \Sigma(t, Z(t)) \in L^2_\rho(\Theta)$ is predictable.*

**Proof:** This follows immediately from the assumption $Z \in \mathcal{H}^q(T)$ and the $\mathcal{P}_T \otimes \mathcal{B}(\mathbb{R})/\mathcal{B}(\mathbb{R})$- measurability of $\sigma$. □

**Claim 3:** *$I_F(Z)$ is well-defined as a process in $L^2_\rho(\Theta)$.*

**Proof:** Putting

$$\varphi(t) := F(t, Z(t)),\ t \in [\,0, T\,], \tag{5.19}$$

by Claim 1, Remark 5.1.3 **(iii)** and the finiteness of $T > 0$ the process $\varphi = (\varphi(t))_{t\in[\,0,T\,]} \subset L^2_\rho(\Theta)$ fulfills the sufficient condition (3.13) from Proposition 3.3.2. Thus, we get Claim 3 by the well-definedness part in Proposition 3.3.2. □

**Claim 4:** *$I_{\Sigma,m}(Z)$ is well-defined as a process in $L^2_\rho(\Theta)$.*

**Proof:** Putting

$$\varphi(t) := \Sigma(t, Z(t)),\ t \in [\,0, T\,], \tag{5.20}$$

by Claim 1, Remark 5.1.3 **(iii)** and the fact that $0 \le \zeta < 1$ the process $\varphi = (\varphi(t))_{t\in[\,0,T\,]} \subset L^2_\rho(\Theta)$ fulfills condition (3.13). Thus, we get Claim 4 by the well-definedness part in Proposition 3.3.3. □

**Claim 5:** *$I_F(Z)$ has finite $\mathcal{H}^q(T)$-norm.*

**Proof:** Since $\varphi$ defined in (5.19) obeys (3.13) (cf. the proof of Claim 3 above), Claim 5 follows from Proposition 3.3.2. □

**Claim 6:** *$I_{\Sigma,m}(Z)$ has finite $\mathcal{H}^q(T)$-norm.*

**Proof:** Since $\varphi$ defined in (5.20) obeys (3.13) (cf. the proof of Claim 4 above), Claim 6 follows from Proposition 3.3.3. □

**Claim 7:** *$t \mapsto I_F(Z)(t)$ is continuous in $L^q(\Omega; L^2_\rho(\Theta))$.*
*Furthermore, there is a pathwise continuous version of $t \mapsto I_F(Z)(t)$.*

**Proof:** Since $\varphi$ defined in (5.19) obeys (3.13) (cf. the proof of Claim 3 above), Claim 7 follows from the continuity result in Proposition 3.3.2. □

**Claim 8:** *$t \mapsto I_\Sigma(Z)(t)$ is continuous in $L^q(\Omega; L^2_\rho(\Theta))$.*
*Furthermore, there is a pathwise continuous version of $t \mapsto I_{\Sigma,m}(Z)(t)$.*

**Proof:** Since $\varphi$ defined in (5.20) obeys (3.13) (cf. the proof of Claim 3 above), Claim 8 follows from the continuity result in Proposition 3.3.3. $\square$

**Claim 9:** *$I_F(Z)$ obeys a predictable modification.*

**Proof:** Recall that, by Lemma 2.1.8, stochastic continuity and adaptedness imply the existence of a predictable version of the process.
By the continuity property shown in Claim 7, we immediately get stochastic continuity. Thus, it remains to show adaptedness of $t \mapsto I_F(Z)(t)$.
Since $Z$ is predictable, $Z(s)$ is $\mathcal{F}_t$-measurable for all $0 \leq s \leq t \leq T$. The measurability assumption on $f$ then implies $\mathcal{F}_t$-measurability of
$F(s, Z(s)) \in L^2_\rho(\Theta)$. Since $U(t,s) \in \mathcal{L}(L^2_\rho)$, we get the $\mathcal{F}_t$-measurability of $U(t,s)F(s,Z(s))$.
Thus, the Bochner integral

$$I_F(Z)(t) = \int\limits_0^t U(t,s)F(s,Z(s))\,ds$$

is also $\mathcal{F}_t$-measurable. As $t \in [\,0,T\,]$ was chosen arbitrarily, $I_F(Z)$ is adapted as well. $\square$

**Claim 10:** *$t \mapsto I_{\Sigma,m}(Z)(t)$ obeys a predictable modification.*

**Proof:** Analogously to Claim 9, it remains to show adaptedness in order to get predictability by an application of Lemma 2.1.8.
As in the proofs of Section 3.4, we start with $m \in L^\infty(\Theta)$. In this case, we have $\mathcal{M}_{\Sigma(s,Z(s))}m \in L^2_\rho(\Theta)$ for any $0 \leq s \leq t \leq T$. Thus, we get $\mathcal{F}_t$-measurability of

$$I_{\Sigma,m}(Z)(t) = \int\limits_0^t U(t,s)\mathcal{M}_{\Sigma(s,Z(s))}\,ds$$

by Claim 9.
For general $m \in L^2(\Theta)$, we take any sequence $(m_N)_{N\in\mathbb{N}} \subset L^\infty(\Theta)$ such that

$$||m_N - m||_{L^2} \to 0 \text{ as } N \to \infty.$$

Analogously to the proof of 3.3.3, we get

$$\mathbf{E}||I_{\Sigma,m_N}(t) - I_{\Sigma,m}(t)||^2_{L^2_\rho} \to 0 \text{ as } N \to \infty.$$

Thus, for general $m \in L^2(\Theta)$, $I_{\Sigma,m}(t)$ is $\mathcal{F}_t$-measurable as the $L^2(\Omega; L^2_\rho(\Theta))$-limit of $\mathcal{F}_t$-measurable processes. □

By Claims 1–10, we have proven that $I_F$ and $I_{\Sigma,m}$ are well-defined predictable processes (up to stochastic modifications) with finite $\mathcal{H}^q(T)$-norms and that $I_F$ and $I_{\Sigma,m}(Z)$ are time-continuous in $L^q(\Omega; L^2_\rho(\Theta))$. ■

**Lemma 5.1.6: Case (B)**
*Given $\nu > 1$, let $U$ additionally fulfill **(A3)**.*

***(i)*** *For $Z \in \mathcal{G}_\nu(T)$, the process $I_F(Z)$ defined by (5.17) also belongs to $\mathcal{G}_\nu(T)$.*
*Furthermore, $t \mapsto I_F(Z)(t)$ is continuous in $L^{2\nu}(\Omega; L^{2\nu}_\rho(\Theta))$, and there exists a pathwise continuous version of this mapping.*

***(ii)*** *Suppose $U$ also fulfills **(A2)** (or the even weaker assumption **(A5)*** with $\nu = 1$). Then, for any $Z \in \mathcal{G}_\nu(T)$, the process $I_{\Sigma,m}(Z)$ defined by (5.18) has finite $\mathcal{G}_\nu(T)$-norm. Furthermore, $t \mapsto I_{\Sigma,m}(t)$ is continuous in $L^{2\nu}(\Omega; L^{2\nu}_\rho(\Theta))$.*

*In particular, both in **(i)** and **(ii)**, there is a pathwise continuous version of each of the processes.*

**Proof:** As in the proof of 5.1.5, we first prove well-definedness, then finiteness of the $\mathcal{G}_\nu(T)$-norm, then continuity in $L^{2\nu}(\Omega; L^{2\nu}_\rho(\Theta))$ resp. the pathwise continuity property, and finally predictability.

**Claim 1:** *$t \mapsto F(t, Z(t))$ is predictable.*

**Proof:** This follows immediately from the assumption $Z \in \mathcal{G}_\nu(T)$ and the $\mathcal{P}_T \otimes \mathcal{B}(\mathbb{R})/\mathcal{B}(\mathbb{R})$- measurability of $f$. □

**Claim 2:** *$t \mapsto \Sigma(t, Z(t))$ is predictable.*

**Proof:** This follows immediately from the assumption $Z \in \mathcal{G}_\nu(T)$ and the $\mathcal{P}_T \otimes \mathcal{B}(\mathbb{R})/\mathcal{B}(\mathbb{R})$- measurability of $\sigma$. □

**Claim 3:** *$I_F(Z)$ is well-defined as a process in $L^{2\nu}_\rho(\Theta)$.*

**Proof:** The process $\varphi = (\varphi(t))_{t\in[0,T]}$ defined by (5.19) obeys (3.21) from the setting of Proposition 3.3.4 by Claim 1, Remark 5.1.3 **(iii)** and the fact that $T > 0$ is finite. Thus, we get Claim 3 by the well-definedness part from Proposition 3.3.4. □

**Claim 4:** *$I_{\Sigma,m}(Z)$ is well-defined as a process in $L^{2\nu}_\rho(\Theta)$.*

**Proof:** The process $\varphi = (\varphi(t))_{t\in[0,T]}$ defined by (5.20) obeys (3.24) from the setting of Proposition 3.3.5 by Claim 1, Remark 5.1.3 **(iii)** and the fact that $\zeta \in [0,1)$. Thus, we get Claim 4 by the well-definedness part from Proposition 3.3.5. □

**Claim 5:** *$I_F(Z)$ has finite $\mathcal{G}_\nu(T)$-norm.*

**Proof:** Since the process $\varphi = (\varphi(t))_{t\in[0,T]}$ defined by (5.19) obeys (3.21) (cf. the proof of Claim 3), we get Claim 5 from Proposition 3.3.4. □

**Claim 6:** *$I_{\Sigma,m}(Z)$ has finite $\mathcal{G}_\nu(T)$-norm.*

**Proof:** Since the process $\varphi = (\varphi(t))_{t\in[0,T]}$ defined by (5.20) obeys (3.24) (cf. the proof of Claim 4), we get Claim 6 from Proposition 3.3.5. □

**Claim 7:** *$t \mapsto I_F(Z)(t)$ is continuous in $L^{2\nu}(\Omega; L^{2\nu}_\rho(\Theta))$.*
*Furthermore, there is a pathwise continuous version of $t \mapsto I_F(Z)(t)$.*

**Proof:** Since the process $\varphi = (\varphi(t))_{t\in[0,T]}$ defined by (5.19) obeys (3.21) (cf. the proof of Claim 3), we get Claim 7 from the continuity results in Proposition 3.3.4. □

**Claim 8:** *$t \mapsto I_\Sigma(Z)(t)$ is continuous in $L^{2\nu}(\Omega; L^{2\nu}_\rho(\Theta))$.*
*Furthermore, there is a pathwise continuous version of $t \mapsto I_{\Sigma,m}(Z)(t)$.*

**Proof:** Since the process $\varphi = (\varphi(t))_{t\in[0,T]}$ defined by (5.20) obeys (3.24) (cf. the proof of Claim 4), we get Claim 8 from the continuity results in Proposition 3.3.5. □

**Claim 9:** *$I_F(Z)$ obeys a predictable modification.*

**Proof:** This claim holds true by the same arguments as in the proof of Claim 9 in the proof of Lemma 5.1.5. □

**Claim 10:** *$t \mapsto I_{\Sigma,m}(Z)(t)$ obeys a predictable modification.*

**Proof:** This claim holds true by the same arguments as in the proof of Claim 10 in the proof of Lemma 5.1.5. □

By Claims 1–10, we have proven that $I_F$ and $I_{\Sigma,m}$ are well-defined predictable (up to stochastic modifications) processes with finite $\mathcal{G}_\nu(T)$-norms and that $I_F$ and $I_{\Sigma,m}(Z)$ are time-continuous in $L^{2\nu}(\Omega; L^{2\nu}_\rho(\Theta))$. ■

Given a $Q$-Wiener process $(W(t))_{t\in[0,T]}$ in $L^2$, we define the Itô-integral

$$I^W_\Sigma(Z)(t) := \int_0^t U(t,s)\mathcal{M}_{\Sigma(s,Z(s))}\,dW(s),\ t\in[0,T]. \tag{5.21}$$

This is a special case of the Wiener convolution considered in Section 3.4.

**Lemma 5.1.7: Case (A)**
*Suppose $U$ additionally obeys **(A2)** with some $\zeta\in[0,1)$.*
*The claims in this lemma hold*

- *in the nuclear case (which we could prove without assuming **(A2)**),*
- *in the general nuclear case (which we could also prove in case of $U$ obeying the weaker assumption **(A5)*** with $\nu=1$), and*
- *in the cylindrical case.*

*Let us fix some $q\in[2,\frac{2}{\zeta})$ with $\zeta$ as in **(A2)**. For each $(Z(t))_{t\in[0,T]}\in\mathcal{H}^q(T)$, the process $I^W_\Sigma(Z)$ is $\mathcal{H}^q(T)$-valued and $t\mapsto I^W_\Sigma(Z)(t)$ is continuous in $L^q(\Omega; L^2_\rho(\Theta))$.*
*In particular, if $q>\frac{2}{1-\zeta}$ with $\zeta$ as in **(A2)** in the general nuclear and the cylindrical case and $\zeta=0$ in the nuclear case, there is a pathwise continuous version of $[0,T]\ni t\mapsto I^W_\Sigma(Z)(t)\in L^2_\rho(\Theta)$.*

**Proof:** We first show well-definedness, then finiteness of the $\mathcal{H}^q(T)$-norm, then continuity in $L^q(\Omega; L^2_\rho(\Theta))$ and finally predictability.

**Claim 1:** *For any $t \in [0,T]$, $I^W_\Sigma(Z)(t)$ is well-defined in $L^2_\rho(\Theta)$.*

**Proof:** Note that (5.21) is just the stochastic convolution from Section 3.4.
First, by 5.1.3 **(ii)** and $\mathcal{H}^q(T) \subset \mathcal{H}^2(T)$, for any $q \geq 2$ (3.28) from Section 3.4 is fulfilled for $\varphi$ as in (5.20). Thus, we have well-definedness in all three cases by Proposition 3.4.1. □

**Claim 2:** *$I^W_\Sigma(Z)$ has finite $\mathcal{H}^q(T)$-norm.*

**Proof :** Defining $\varphi$ as in (5.20), by the measurability assumption on $\sigma$, Remark 5.1.3 **(iii)** (cf. (5.7) there) and the choice of $\zeta$, the process $\varphi = (\varphi(t))_{t\in[0,T]}$ obeys (3.29). Thus, we can apply Proposition 3.4.1 to get Claim 2 in all three cases. □

**Claim 3:** *Given $q$ from the assumption, $t \mapsto I^W_\Sigma(Z)(t)$ is continuous in $L^q(\Omega; L^2_\rho(\Theta))$.*
*If additionally $q > \frac{2}{1-\zeta}$, there is a pathwise continuous modification.*

**Proof:** Let $\varphi$ be as in (5.20). Then, in view of 5.1.3 **(iii)** (cf. (5.7) there), the process $\varphi$ obeys the assumptions of Proposition 3.4.7, which gives the time-continuity in $L^q(\Omega; L^2_\rho(\Theta))$.
In the case $q > \frac{2}{1-\zeta}$, even the assumptions from Theorem 3.4.5 are fulfilled, which gives us the existence of a pathwise time-continuous modification of $I^W_\Sigma$ in $L^2_\rho(\Theta)$. □

**Claim 4:** *$t \mapsto I^W_\Sigma(Z)(t)$ obeys a predictable version.*

**Proof:** As a stochastic integral, $t \mapsto I^W_\Sigma(Z)(t)$ is adapted. Furthermore, by Claim 3, $t \mapsto I^W_\Sigma(Z)(t)$ is stochastically continuous. Thus, by Lemma 2.1.8, there is a predictable modification of $t \mapsto I^W_\Sigma(Z)(t)$. □

By Claim 1–4 the proof of Lemma 5.1.7 is finished. ■

**Lemma 5.1.8: Case (B)**
*Suppose $U$ additionally obeys **(A2)** with some $\zeta \in [0,1)$, **(A3)**, **(A4)** and **(A5)***. Let us fix some $\nu \in [1, \frac{1}{\zeta})$ with $\zeta$ as in **(A2)**.*
*Again, the claims in this lemma hold*

- *in the nuclear case (which we could prove without assuming **(A2)** and **(A4)**),*

- *in the general nuclear case (which we could also prove in case of $U$ obeying the weaker assumption **(A5)*** with $\nu = 1$ instead of **(A2)** and **(A4)**), and*

- *in the cylindrical case.*

*For $(Z(t))_{t\in[0,T]} \in \mathcal{G}_\nu(T)$ the process $I^W_\Sigma(Z)$ is $\mathcal{G}_\nu(T)$-valued and $t \mapsto I^W_\Sigma(Z)(t)$ is continuous in $L^{2\nu}(\Omega; L^{2\nu}_\rho)$.*
*Additionally assuming that $\nu > \frac{1}{1-\zeta}$ with $\zeta$ as in **(A2)** in the general nuclear and the cylindrical case and $\zeta = 0$ in the nuclear case, there is a pathwise continuous version of $[0,T] \ni t \mapsto I^W_\Sigma(Z)(t) \in L^2_\rho(\Theta)$.*

**Proof:** We proceed analogously to the proof of 5.1.7.

**Claim 1:** *For any $t \in [0,T]$, $I^W_\Sigma(Z)(t)$ is well-defined in $L^{2\nu}_\rho(\Theta)$.*

**Proof:** Note that, by 5.1.3 **(iii)** (cf. (5.8) there), the sufficient condition (3.39) from Section 3.4 is fulfilled for the process $\varphi$ defined by (5.20). Thus, by Proposition 3.4.3, we get Claim 1. $\square$

**Claim 2:** *$I^W_\Sigma(Z)$ has finite $\mathcal{G}_\nu(T)$-norm.*

**Proof :** Defining a process $\varphi$ by (5.20), by the measurability assumption on $\sigma$, Remark 5.1.3 **(iii)** (cf. (5.8) there) and the choice of $\zeta$, we can again apply Proposition 3.4.3 to get Claim 2. $\square$

**Claim 3:** *Given $\nu$ from the assumption, $t \mapsto I^W_\Sigma(Z)(t)$ is continuous in $L^{2\nu}(\Omega; L^{2\nu}_\rho(\Theta))$.*
*If additionally $\nu > \frac{1}{1-\zeta}$, there is a pathwise continuous modification in $L^2_\rho(\Theta)$.*

**Proof:** Let us define a process $\varphi$ by (5.20). Then, in view of 5.1.3(iii) (cf. (5.8) there), the process $\varphi$ obeys the integrability condition from the assumptions of Proposition 3.4.8, which gives the time-continuity in $L^{2\nu}(\Omega; L^{2\nu}_\rho(\Theta))$.
In the case

$$\nu > \tfrac{1}{1-\zeta} \iff 2\nu > \tfrac{2}{1-\zeta} \geq 2,$$

since $||\cdot||_{L^2_\rho} \leq ||\cdot||_{L^{2\nu}_\rho}$, even the assumptions from Proposition 3.4.5 are fulfilled, which gives us the existence of a pathwise time-continuous modification of $I^W_\Sigma$ in $L^2_\rho(\Theta)$. □

**Claim 4:** *$t \mapsto I^W_\Sigma(Z)(t)$ obeys a predictable version.*

**Proof:** As a stochastic integral, $t \mapsto I^W_\Sigma(Z)(t)$ is adapted. Furthermore, by Claim 3 $t \mapsto I^W_\Sigma(Z)(t)$ is stochastically continuous. Thus, by Lemma 2.1.8, there is a predictable modification of $t \mapsto I^W_\Sigma(Z)(t)$. □

By Claim 1–4 we get Lemma 5.1.8. ■

Next, we consider stochastic integrals w.r.t. compensated Poisson random measures $\tilde{N}$.

Let $(\tilde{N}(t,\cdot))_{t\in[0,T]}$ be as described in the introduction of the chapter. We define

$$I^{\tilde{N}}_\Gamma(Z)(t) := \int_0^t \int_{L^2} U(t,s)\mathcal{M}_{\Gamma(s,Z(s))}(x)\,\tilde{N}(ds,dx),\ t\in[0,T]. \tag{5.22}$$

This is a special case of the Poisson stochastic convolutions considered in Chapter 4.

**Lemma 5.1.9: Case (A)**
*Suppose $U$ additionally obeys **(A2)** (or alternatively the weaker assumption **(A5)*** with $\nu = 1$) for some $\zeta \in [0,1)$.*

*For any $Z \in \mathcal{H}^q(T)$ with $q \in [2, \frac{2}{\zeta})$, the process $I^{\tilde{N}}_\Gamma(Z)$ has finite $\mathcal{H}^q(T)$-norm.*

*Furthermore, we have continuity in $L^q(\Omega; L^2_\rho(\Theta))$ for $t \mapsto I^{\tilde{N}}_\Gamma(Z)(t)$.*
*If we additionally assume that $\gamma$ is uniformly bounded on $[0,T]\times\Omega\times\mathbb{R}$ and **(A7)** holds for $U$, there is a càdlàg version of $[0,T] \ni t \mapsto I^{\tilde{N}}_\Gamma(Z)(t) \in L^2_\rho(\Theta)$.*

**Proof:** We first show well-definedness, then finiteness of the $\mathcal{H}^q(T)$-norm, then continuity in $L^q(\Omega; L^2_\rho(\Theta))$ and finally predictability.

**Claim 1:** *$I^{\tilde{N}}_\Gamma(Z)$ is well-defined in $L^2_\rho(\Theta)$.*

**Proof:** By 5.1.3(iii) and the fact that $\zeta \in [0,1)$,

$$\varphi(t) := \Gamma(t, Z(t)),\ t \in [0, T], \tag{5.23}$$

fulfills (4.3) for $Z \in \mathcal{H}^q(T) \subset \mathcal{H}^2(T)$. Thus, by Proposition 4.1, we get the required well-definedness of $I_\Gamma^{\tilde{N}}(Z)$. □

**Claim 2:** *$I_\Gamma^{\tilde{N}}(Z)$ has finite $\mathcal{H}^q(T)$-norm.*

**Proof:** Defining a process $\varphi$ by (5.23), in view of 5.1.3 **(iii)** (cf. (5.7) there) and the choice of $\zeta$ the process $\varphi$ fulfills (4.4) from the assumptions of Proposition 4.1, which gives us Claim 2. □

**Claim 3:** *$t \mapsto I_\Gamma^{\tilde{N}}(t)$ is continuous in $L^q(\Omega; L^2_\rho(\Theta))$.*
*If we additionally assume that $\gamma$ is uniformly bounded on $[0, T] \times \Omega \times \mathbb{R}$ and **(A7)** holds for $U$, we get a càdlàg modification of $t \mapsto I_\varphi^{\tilde{N}}$.*

**Proof:** In view of 5.1.3 **(iii)** (cf. (5.7) there) and the choice of $\zeta$, the process $\varphi$ defined by (5.23) fulfills the assumptions of Proposition 4.8, which gives us time-continuity in $L^q(\Omega; L^2_\rho(\Theta))$.
Concerning the càdlàg property, note that for $\varphi$ as before under the additional assumptions we are just in the situation of Proposition 4.12, which proves Claim 3. □

**Claim 4:** *$t \mapsto I_\Gamma^{\tilde{N}}(Z)(t)$ is predictable.*

**Proof:** As a stochastic integral, $t \mapsto I_\Gamma^{\tilde{N}}(Z)(t)$ is adapted. Furthermore, by Claim 3 $t \mapsto I_\Gamma^{\tilde{N}}(Z)(t)$ is stochastically continuous. Thus, by Lemma 2.1.8, there exists a predictable modification of $t \mapsto I_\Gamma^{\tilde{N}}(Z)(t)$. □

By Claims 1–4 Lemma 5.1.9 is proven. ■

**Lemma 5.1.10: Case (B)**
*Suppose $U$ additionally fulfills **(A3)** and **(A5)** (or **(A5)***) with some $\zeta \in [0, 1)$. Let us fix some $\nu \in [1, \frac{1}{\zeta})$.*
*For $Z \in \mathcal{G}_\nu(T)$, the process $I_\Gamma(Z)$ has finite $\mathcal{G}_\nu(T)$-norm.*
*Furthermore, $t \mapsto I_\Gamma^{\tilde{N}}(Z)(t)$ is continuous in $L^{2\nu}(\Omega; L^{2\nu}_\rho(\Theta))$.*
*Under the additional assumptions that*

*(5.24)*
$$\sup_{[0,T]\times\Omega\times\mathbb{R}} |\gamma| =: K < \infty$$

*and that condition **(A7)** on $U$ is fulfilled, there is a càdlàg version of* $[0,T] \ni t \mapsto I_{\Gamma}^{\tilde{N}}(Z)(t) \in L_{\rho}^{2}$.

**Proof:** We proceed analogously to the proof of 5.1.9.

**Claim 1:** $I_{\Gamma}^{\tilde{N}}(Z)$ *is well-defined.*

**Proof:** Let us define a process $\varphi$ by (5.23). By 5.1.3 **(ii)** $\varphi$ fulfills (4.3), which, analogously to Claim 1 in the proof of 5.1.9, implies the well-definedness of $I_{\Gamma}^{\tilde{N}}(Z)$. □

**Claim 2:** $I_{\Gamma}^{\tilde{N}}(Z)$ *has finite $\mathcal{G}_{\nu}(T)$-norm.*

**Proof:** By 5.1.3 **(iii)** we have

$$\sup_{t\in[0,T]} \mathbf{E}||\Gamma(t,Z(t))||_{L_{\rho}^{2\nu}}^{2\nu} < \infty.$$

By this and the obvious predictability of $t \mapsto \Gamma(t,\cdot,Z(t))$ following from the predictability of $Z$ and the measurability property of $\gamma$, we get that the process $\varphi$ defined by (5.23) is in $\mathcal{G}_{\nu}(T)$. Thus, we can apply 4.4 to get

$$\begin{aligned}
\sup_{t\in[0,T]} \mathbf{E}||I_{\Gamma}^{\tilde{N}}(Z)(t)||_{L_{\rho}^{2\nu}}^{2\nu} &\leq c(\nu, c(T), C_{2\nu,\eta}) \int_0^t (t-s)^{-\zeta\nu} \mathbf{E}||\Gamma(s,Z(s))||_{L_{\rho}^{2\nu}}^{2\nu}\, ds \\
&\leq c(\nu, c(T), C_{2\nu,\eta}) \left(\int_0^t s^{-\zeta\nu}\, ds\right) \left(\sup_{t\in[0,T]} \mathbf{E}||\Gamma(t,Z(t))||_{L_{\rho}^{2\nu}}^{2\nu}\right) \\
&< \infty. \ \square
\end{aligned}$$

**Claim 3:** $t \mapsto I_{\Gamma}^{\tilde{N}}(Z)(t)$ *is continuous in* $L^{2\nu}(\Omega; L_{\rho}^{2\nu}(\Theta))$.
*Under the additional assumption that $\gamma$ obeys (5.24) and **(A7)** holds for $U$, there is a càdlàg modification of $t \mapsto I_{\Gamma}^{\tilde{N}}(Z)(t)$.*

**Proof:** Defining a process $\varphi$ by (5.23), we get inequality (4.36) from Chapter 4 by Remark 5.1.3 **(iii)** (cf. (5.8) there). Thus, by Proposition 4.9, we get the required continuity property.
Finally, by the boundedness assumption on $\gamma$ and **(A7)**, the conditions of Proposition 4.12 are fulfilled. Therefore, we get the required càdlàg versions.
□

**Claim 4:** $t \mapsto I_{\Gamma}^{\tilde{N}}(Z)(t)$ *obeys a predictable modification.*

**Proof:** This holds true by the same arguments as in the proof of Claim 4 from the proof of 5.1.8. □

By Claims 1–4 we get 5.1.10. ■

We finish this section with a remark on Bochner integrals $I_F$ in the case of a non-Lipschitz $f$. This remark will be relevant for Chapters 7 and 8.

**Remark 5.1.11:** *Let us consider $I_F$ defined by (5.17) with $F$ defined by **(NEM)** from a measurable function $f\colon [0,T] \times \Omega \times \mathbb{R} \to \mathbb{R}$, which is continuous in the third variable.*
*Suppose that $f$ obeys **(PG)** and **(LG)** from Section 3.2.*

***(i)** Suppose that $\nu = 1$ and $Z \in \mathcal{H}^q(T)$ for some $q \geq 2$.*
*Then, the process $I_F(Z)$ is adapted, has finite $\mathcal{H}^q(T)$-norm and $[0,T] \ni t \mapsto I_F(Z)(t) \in L^2_\rho$ is continuous in $L^q(\Omega; L^2_\rho)$. Furthermore, there exists a pathwise continuous version of the previous mapping.*

***(ii)** Suppose that $\nu > 1$ and $Z \in \mathcal{G}_\nu(T)$.*
*Then, the process $I_F(Z)$ is adapted, has finite $\mathcal{H}^2(T)$-norm and $[0,T] \ni t \mapsto I_F(Z)(t) \in L^2_\rho$ is continuous in $L^2(\Omega; L^2_\rho)$. Furthermore, there exists a pathwise continuous version of the previous mapping.*

**Proof:** Concerning **(i)**, note that **(PG)** with $\nu = 1$ means that

$$|f(t, \omega, y)| \leq c_f(T)(1 + |y|),\ (t, \omega, y) \in [0,T] \times \Omega \times \mathbb{R}.$$

Thus, we get the same chain of arguments as in (5.7) (of course with $\lambda = f$) from 5.1.3**(iii)** in this case. Then, literally repaeting the arguments from the proof of Lemma 5.1.5 **(i)**, we get **(i)**.

Concerning **(ii)**, note that setting $\varphi(t) := F(t, Z(t))$, $t \in [0,T]$, for $Z = (Z(t))_{t\in[0,T]} \in \mathcal{G}_\nu(T)$ we get, analogously to estimate (5.7) from Remark 5.1.3 **(iii)** (with $q = 2$)

$$
\begin{aligned}
\sup_{t\in[0,T]} \mathbf{E}||\varphi(t)||^2_{L^2_\rho} &= \sup_{t\in[0,T]} \mathbf{E}\int_\Theta (|f(t,\cdot,Z(t,y))|)^2\mu_\rho(dy) \\
&\leq \sup_{t\in[0,T]} \mathbf{E}\int_\Theta (c_f(T)(1+|Z(t,y)|^\nu)^2\mu_\rho(dy) \\
&\leq c(\nu,c_f(T))\left(1+\sup_{t\in[0,T]}\mathbf{E}||Z(t)||^{2\nu}_{L^{2\nu}_\rho}\right) \\
&= c(\nu,c_f(T))(1+||Z||^{2\nu}_{\mathcal{G}_\nu(T)}) \\
&< \infty
\end{aligned}
$$

for any $Z \in \mathcal{G}_\nu(T)$. Thus, by repeating almost literally the arguments from the proof of Lemma 5.1.5 **(i)**, we get **(ii)**. ■

## 5.2 Existence and uniqueness of mild solutions in the case of Lipschitz coefficients

By the lemmata proven in the previous section, we can show existence of unique solutions to equations (1.1) and (1.2) with Lipschitz coefficients $E$, $F$, $\Sigma$ and $\Gamma$.

The proofs will follow the lines of proof of Theorem 3.2.1 from [76]:

**Theorem 5.2.1:** ***Case (A)***
*Suppose the almost strong evolution operator $U$, generated by $(A(t))_{t\in[0,T]}$, has properties **(A0)**–**(A2)** (In the nuclear case **(A2)** can be replaced by **(A5)*** with $\nu = 1$.).*
*Let the initial condition $\xi$ be as in **Case (A)**, whereby $q \in [2, \frac{2}{\zeta})$ is such that the integrability condition **(QI)** for the Lévy measure $\eta$ (corresponding to $\tilde{N}$ (equation (1.1)) resp. $L$ (equation (1.2))) is fulfilled with $q \in [2, \frac{2}{\zeta})$.*

*Then, there exists a unique predicatable mild solution to each of the equations (1.1) and (1.2) in the sense of 5.1.2 **(i)**. These solutions are time-continuous in $L^q(\Omega; L^2_\rho(\Theta))$.*
*Furthermore, we have the moment estimates (concerning (1.1))*

$$(5.25)\quad \sup_{t\in[0,T]} \mathbf{E}||X(t)||^q_{L^2_\rho} \leq c(q,\zeta,T,c(T),c_f(T),c_\sigma(T),c_\gamma(T),C_{q,\eta})(1+\mathbf{E}||\xi||^q_{L^2_\rho})$$

*and (concerning (1.2))*

$$(5.26)\quad \sup_{t\in[0,T]} \mathbf{E}||X(t)||^q_{L^2_\rho} \leq c(q,\zeta,K,T,c(T),c_e(T),c_\sigma(T),C_{q,\eta})(1+\mathbf{E}||\xi||^q_{L^2_\rho})$$

*with a positive constant both on the right hand side of (5.25) and (5.26).*

*Let us assume that* $\zeta \in [0, \frac{1}{2})$ *and* $q \in (\frac{2}{1-\zeta}, \frac{2}{\zeta})$. *Furthermore, let* $\gamma$ *resp.* $\sigma$ *obey (5.24) and let* $U$ *fulfill* ***(A7)***. *Then, there exist càdlàg versions of the mapping* $t \mapsto X(t)$ *both for solutions to (1.1) and (1.2).*

**Theorem 5.2.2:** ***Case (B)***
*Suppose the almost strong evolution operator* $U$, *generated by* $(A(t))_{t\in[0,T]}$, *has properties* ***(A0)–(A5)*** *with some* $\zeta \in [0, 1)$ *(Note that in the nuclear case* ***(A5)*** *can be replaced by the weaker assumption* ***(A5)****.*).*
*Let the initial condition* $\xi$ *be as in* ***Case (B)***, *whereby* $\nu \in [1, \frac{1}{\zeta})$ *is such that the integrability condition* ***(QI)*** *for the Lévy measure* $\eta$ *holds with* $q = 2\nu$.

*Then, there exists a unique solution to each of the equations (1.1) and (1.2) in the sense of 5.1.2* ***(ii)***. *These solutions are time-continuous in* $L^{2\nu}(\Omega; L^{2\nu}_\rho(\Theta))$.
*Furthermore, we have the moment estimates (concerning (1.1))*

$$\text{(5.27)} \quad \sup_{t\in[0,T]} \mathbf{E}||X(t)||^{2\nu}_{L^{2\nu}_\rho} \le c(\nu, \zeta, T, c(T), c_f(T), c_\sigma(T), c_\gamma(T), C_{2\nu,\eta})(1 + \mathbf{E}||\xi||^{2\nu}_{L^{2\nu}_\rho})$$

*and (concerning (1.2))*

$$\text{(5.28)} \quad \sup_{t\in[0,T]} \mathbf{E}||X(t)||^{2\nu}_{L^{2\nu}_\rho} \le c(\nu, \zeta, m, T, c(T), c_e(T), c_\sigma(T), C_{2\nu,\eta})(1 + \mathbf{E}||\xi||^{2\nu}_{L^{2\nu}_\rho})$$

*with positive constants in the right hand sides of (5.27) and (5.28).*

*Let us assume that* $\zeta \in [0, \frac{1}{2})$ *and* $\nu \in (\frac{1}{1-\zeta}, \frac{1}{\zeta})$.
*Under the additional assumption that* $\gamma$ *resp.* $\sigma$ *obeys (5.24) and that* $U$ *obeys* ***(A7)***, *there is a càdlàg version of* $t \mapsto X(t)$ *both for solutions to (1.1) and (1.2).*

**Remark 5.2.3:** ***(i)*** *Note that by the assumptions in Theorem 5.2.1 and 5.2.2 we can treat the case of the Q-Wiener process* $W$ *in (1.1) being*

- ***nuclear***, *i.e.* $Q \in \mathcal{T}^+(L^2)$ *such that (2.4) holds with an orthonormal basis obeying (3.1),*
- ***general nuclear***, *i.e.* $Q \in \mathcal{T}^+(L^2)$ *such that the orthonormal basis in (2.4) does not obey (3.1), and*
- ***cylindrical***, *i.e.* $Q = \mathbf{I}$.

*Recall from Lemma 5.1.7 and Lemma 5.1.8 that the conditions, under which the stochastic integrals w.r.t. Q-Wiener processes are well-defined, are different in the above three cases. Nevertheless under the given assumptions, we have well-definedness of the stochastic integrals in any of the three cases of Q-Wiener processes mentioned above.*
*In the nuclear case **(A2)** and **(A5)** can be substituted by **(A5)***.*

***(ii)*** *By $c_e(T)$, $c_f(T)$ and $c_\sigma(T)$, we denote the common constants in **(LC)** and **(LB)** for the functions e, f and $\sigma$.*
*We need $\zeta \in [\,0, \frac{1}{2}\,)$ in order to have the intervall $(\,\frac{2}{1-\zeta}, \frac{2}{\zeta}\,)$ (Theorem 5.2.1) resp. $(\,\frac{1}{1-\zeta}, \frac{1}{\zeta}\,)$ (Theorem 5.2.2) non-empty. We take q resp. $\nu$ from that intervalls, since, for càdlàg versions, we need both $\nu > \frac{1}{1-\zeta}$ (cf. Lemma 5.1.8) and $\nu < \frac{1}{\zeta}$ (cf. Lemma 5.1.10).*

The two main results will be proven by two different methods.
Theorem 5.2.1 will be proven by a general Banach contraction argument, whereas Theorem 5.2.2 will be proven by a Picard iteration method.
The second method is more general and can also be applied to prove Theorem 5.2.1. Furthermore, it can be used to prove unique solvability even in larger spaces than $\mathcal{H}^q(T)$ and $\mathcal{G}_\nu(T)$.

**Proof of 5.2.1:** Let us start with equation (1.1). On the intervall $[\,0, T\,]$, we look for solutions $X \in \mathcal{H}^q(T)$ to

$$
\begin{aligned}
(5.29)\ X(t) = \quad & U(t,0)\xi + \int_0^t U(t,s)F(s,X(s))\,ds \\
& + \int_0^t U(t,s)\mathcal{M}_{\Sigma(s,X(s))}\,dW(s) \\
& + \int_0^t \int_{L^2} U(t,s)\mathcal{M}_{\Gamma(s,X(s))}(x)\,\tilde{N}(ds,dx) \\
= \quad & U(t,0)\xi + I_F(X)(t) + I_\Sigma^W(X)(t) + I_\Gamma^{\tilde{N}}(X)(t).
\end{aligned}
$$

We note that, by 5.1.5, 5.1.7 and 5.1.9, the $I$-terms preserve $\mathcal{H}^q(T)$.
Furthermore, the $I$-terms are time-continuous in $L^q(\Omega; L^2_\rho(\Theta))$ by the same lemmata.
Thus, any solution to (5.29), if there is one, is $\mathcal{H}^q(T)$-valued and time-continuous in $L^q(\Omega, L^2_\rho(\Theta))$.
In particular, in the case of $\gamma$ being bounded uniformly on $[\,0, T\,] \times \Omega \times \mathbb{R}$ and **(A7)** being fulfilled for $U$, we get the required existence of a càdlàg version of the solution.

Let us denote the right hand side of (5.29) by $I(X)$. Let us check the contraction property of the mapping $I$ acting in the Banach space $\mathcal{H}^q(T)$

(Note that $I$ is well-defined in the sense of stochastic versions). For any pair of processes $X, Y \in \mathcal{H}^q(T)$, we have

$$
\begin{aligned}
&(5.30)\ ||I(X) - I(Y)||_{\mathcal{H}^q(T)} \\
&\leq ||I_F(X) - I_F(Y)||_{\mathcal{H}^q(T)} \\
&+ ||I_\Sigma^W(X) - I_\Sigma^W(Y)||_{\mathcal{H}^q(T)} \\
&+ ||I_\Gamma^{\tilde{N}}(X) - I_\Gamma^{\tilde{N}}(Y)||_{\mathcal{H}^q(T)} \\
&\leq \sup_{t\in[0,T]} (\mathbf{E}||I_F(X)(t) - I_F(Y)(t)||_{L^2_\rho}^q)^{\frac{1}{q}} \\
&+ \sup_{t\in[0,T]} (\mathbf{E}||I_\Sigma^W(X)(t) - I_\Sigma^W(Y)(t)||_{L^2_\rho}^q)^{\frac{1}{q}} \\
&+ \sup_{t\in[0,T]} (\mathbf{E}||I_\Gamma^{\tilde{N}}(X)(t) - I_\Gamma^{\tilde{N}}(Y)(t)||_{L^2_\rho}^q)^{\frac{1}{q}} \\
&= \sup_{t\in[0,T]} \left(\mathbf{E}\left|\left|\int_0^t U(t,s)[F(s,X(s)) - F(s,Y(s))]\,ds\right|\right|_{L^2_\rho}^q\right)^{\frac{1}{q}} \\
&+ \sup_{t\in[0,T]} \left(\mathbf{E}\left|\left|\int_0^t U(t,s)[\mathcal{M}_{\Sigma(s,X(s))} - \mathcal{M}_{\Sigma(s,Y(s))}]\,dW(s)\right|\right|_{L^2_\rho}^q\right)^{\frac{1}{q}} \\
&+ \sup_{t\in[0,T]} \left(\mathbf{E}\left|\left|\int_0^t \int_{L^2} U(t,s)[\mathcal{M}_{\Gamma(s,X(s))} - \mathcal{M}_{\Gamma(s,Y(s))}](x)\,\tilde{N}(ds,dx)\right|\right|_{L^2_\rho}^q\right)^{\frac{1}{q}} \\
&\leq c(T)c_f(T) \sup_{t\in[0,T]} (\mathbf{E}\int_0^t ||X(s) - Y(s)||_{L^2_\rho}^q\,ds)^{\frac{1}{q}} \\
&+ c(q, c(T), c_\sigma(T)) \sup_{t\in[0,T]} \left(\int_0^t (t-s)^{-\zeta}\mathbf{E}||X(s) - Y(s)||_{L^2_\rho}^q\,ds\right)^{\frac{1}{q}} \\
&+ c(q, c(T), c_\gamma(T), C_{q,\eta}) \sup_{t\in[0,T]} \int_0^t \left((t-s)^{-\frac{q\zeta}{2}}\mathbf{E}||X(s) - Y(s)||_{L^2_\rho}^q\,ds\right)^{\frac{1}{q}} \\
&\leq c(q, , T, c(T), c_f(T), c_\sigma(T), c_\gamma(T), C_{q,\eta}) \sup_{t\in[0,T]} (\int_0^t (t-s)^{-\frac{q\zeta}{2}}\mathbf{E}||X(s) - Y(s)||_{L^2_\rho}^q\,ds)^{\frac{1}{q}} \\
&= \left(\frac{T^{1-\frac{q\zeta}{2}}}{1-\frac{q\zeta}{2}}\right)^{\frac{1}{q}} c(q,\zeta,T,c(T),c_f(T),c_\sigma(T),c_\gamma(T),C_{q,\eta})||X-Y||_{\mathcal{H}^q(T)},
\end{aligned}
$$

where we used 2.2.1 **(iii)** for $U$, estimate (3.30) from Proposition 3.4.1, estimate (4.5) from Proposition 4.1, the fact that

$$q < \tfrac{2}{\zeta} \iff \tfrac{q\zeta}{2} < 1,$$

and the Lipschitz property of $f$, $\sigma$ and $\gamma$ in the third step.

Thus, given $\bar{T} > 0$ with

$$c_2(T) := \left(\frac{\bar{T}^{1-\frac{q\zeta}{2}}}{1-\frac{q\zeta}{2}}\right)^{\frac{1}{q}} c(q,\zeta,T,c(T),c_f(T),c_\sigma(T),c_\gamma(T),C_{q,\eta}) < 1,$$

by the Banach fixpoint theorem (5.29) has a unique fixpoint $\bar{X}$, which is a solution to (1.1) on $[\,0,\bar{T}\,]$ by construction. Setting $\xi := \bar{X}(\bar{T})$ leads to a unique solution on $[\,\bar{T},2\bar{T}\,]$. By finite iteration of this procedure, we get a unique solution on the whole intervall $[\,0,T\,]$ in the sense of 5.1.2(i).

The uniqueness in $\mathcal{H}^q(T)$ means uniqueness up to modifications. If $X_1$ and $X_2$ are two predictable solutions from $\mathcal{H}^q(T)$, then $X_1(t) = X_2(t)$ in $L^2_\rho$, $P$-a.s., for any fixed $t \in [\,0,T\,]$. Since Lemma 5.1.7 and 5.1.9 ensure the existence of a continuous resp. càdlàg modification of $I^W_\Sigma$ resp. $I^{\tilde{N}}_\Gamma$, any solution $X$ posesses such a modification, too.
Furthermore, any two càdlàg solutions coincide up to indistinguishability (see p.28), i.e.

$$P\{X_1(t) = X_2(t)\; for\; all\; t \in [\,0,T\,]\} = 1.$$

Thus, it remains to show the estimate (5.25). Similarly to (5.30), for our solution $X \in \mathcal{H}^q(T)$ and arbitrary $t \in [\,0,T\,]$, we have

$$\begin{aligned}
\mathbf{E}||X(t)||^q_{L^2_\rho} \leq\; & c(q)\Big(\mathbf{E}||U(t,0)\xi||^q_{L^2_\rho} + \mathbf{E}||I_F(X)(t)||^q_{L^2_\rho} \\
& +\mathbf{E}||I^W_\Sigma(X)(t)||^q_{L^2_\rho} \\
& +\mathbf{E}||I^{\tilde{N}}_\Gamma(X)(t)||^q_{L^2_\rho}\Big) \\
\leq\; & c(q)\Big(c(q,c(T))\mathbf{E}||\xi||^q_{\rho,2} \\
& +c(q,c(T))\int_0^t \mathbf{E}||F(s,X(s))||^q_{\rho,2}\,ds \\
& +c(q,c(T))\int_0^t (t-s)^{-\zeta}\mathbf{E}||\Sigma(s,X(s))||^q_{\rho,2}\,ds \\
& +c(q,c(T),C_{q,\eta})\int_0^t (t-s)^{-\frac{q\zeta}{2}}\mathbf{E}||\Gamma(s,X(s))||^q_{\rho,2}\,ds\Big) \\
\leq\; & c_1(q,\zeta,T,c(T),c_f(T),c_\sigma(T),c_\gamma(T),C_{q,\eta})(1+\mathbf{E}||\xi||^q_{L^2_\rho}) \\
& +c_2(q,\zeta,T,c(T),c_\sigma(T),c_\gamma(T),C_{q,\eta})\int_0^t (t-s)^{-\frac{q\zeta}{2}}\mathbf{E}||X(s)||^q_{L^2_\rho}\,ds.
\end{aligned}$$

Here, we used 2.2.1 **(iii)** for $U$, estimate (3.30) from Proposition 3.4.1 and estimate (4.5) from Proposition 4.1 in the second and the fact that $f$, $\sigma$ and $\gamma$ obey **(LC)** and **(LB)** in the third step.
Thus, by the Gronwall-Bellman lemma, we get estimate (5.25), which finishes the consideration of the equation (1.1).

Concerning equation (1.2) note that, by the Lévy-Itô decomposition 2.4.13

we are looking for a solution to

$$
\begin{aligned}
(5.31)\ X(t) = \quad & U(t,0)\xi + \int_0^t U(t,s)E(s,\cdot,X(s))\,ds \\
& + \int_0^t U(t,s)\mathcal{M}_{\Sigma(s,X(s))}(m)\,ds \\
& + \int_0^t U(t,s)\mathcal{M}_{\Sigma(s,X(s))}\,dW(s) \\
& + \int_0^t \int_{L^2} U(t,s)\mathcal{M}_{\Sigma(s,X(s))}(x)\,\tilde{N}(ds,dx) \\
= \quad & U(t,0)\xi + I_E(X)(t) + I_{\Sigma,m}(t) + I_\Sigma^W(t) + I_\Sigma^{\tilde{N}}(t).
\end{aligned}
$$

By 5.1.5, 5.1.7 and 5.1.9, all five terms on the right hand side and thus the solution to (5.31), if it exists, are $\mathcal{H}^q(T)$-valued. By 5.1.5, 5.1.7 and 5.1.9, all these terms and hence $X$ will be time-continuous in $L^q(\Omega;L^2_\rho(\Theta))$.

Furthermore, additionally assuming boundedness of $\sigma$ and the pseudo-contraction property **(A7)** for $U$, there is a càdlàg version of $t \mapsto I_\Sigma^{\tilde{N}}(t)$ and thus, by the well-known pathwise continuity properties of the mappings
$t \mapsto I_E(X)(t)$, $t \mapsto I_{\Sigma,m}(t)$ and $t \mapsto I_\Sigma^W(t)$, also of a fixpoint of (5.31).
So, let us denote the right hand side of (5.31) by $I(X)$ and check the contraction property of the mapping $I$ in the Banach space $\mathcal{H}^q(T)$.
Similar to the case of equation (1.1), we get

$$
\begin{aligned}
&(5.32)\ ||I(X) - I(Y)||_{\mathcal{H}^q(T)} \\
&\le ||I_E(X) - I_E(Y)||_{\mathcal{H}^q(T)} \\
&+ ||I_{\Sigma,m}(X) - I_{\Sigma,m}(Y)||_{\mathcal{H}^q(T)} \\
&+ ||I_\Sigma^W(X) - I_\Sigma^W(Y)||_{\mathcal{H}^q(T)} \\
&+ ||I_\Sigma^{\tilde{N}}(X) - I_\Sigma^{\tilde{N}}(Y)||_{\mathcal{H}^q(T)} \\
&= \sup_{t\in[0,T]} (\mathbf{E}||I_E(X)(t) - I_E(Y)(t)||^q_{L^2_\rho})^{\frac{1}{q}} \\
&+ \sup_{t\in[0,T]} (\mathbf{E}||I_{\Sigma,m}(X)(t) - I_{\Sigma,m}(Y)(t)||^q_{L^2_\rho})^{\frac{1}{q}} \\
&+ \sup_{t\in[0,T]} (\mathbf{E}||I_\Sigma^W(X)(t) - I_\Sigma^W(Y)(t)||^q_{L^2_\rho})^{\frac{1}{q}} \\
&+ \sup_{t\in[0,T]} (\mathbf{E}||I_\Sigma^{\tilde{N}}(X)(t) - I_\Sigma^{\tilde{N}}(Y)(t)||^q_{L^2_\rho})^{\frac{1}{q}} \\
&\le c(q,\zeta,m,T,c(T),c_e(T),c_\sigma(T)) \sup_{t\in[0,T]} \left( \mathbf{E} \int_0^t (t-s)^{-\zeta} \mathbf{E}||X(s) - Y(s)||^q_{L^2_\rho}\,ds \right)^{\frac{1}{q}} \\
&+ c(q,c(T),c_\sigma(T),C_{q,\eta}) \sup_{t\in[0,T]} \int_0^t \left( (t-s)^{-\frac{q\zeta}{2}} \mathbf{E}||X(s) - Y(s)||^q_{L^2_\rho}\,ds \right)^{\frac{1}{q}}
\end{aligned}
$$

$$\le c(q,\zeta,m,T,c(T),c_f(T),c_\sigma(T),c_\gamma(T),C_{q,\eta}) \sup_{t\in[0,T]} \left(\int_0^t (t-s)^{-\frac{q\zeta}{2}} \mathbf{E}||X(s)-Y(s)||^q_{L^2_\rho}\,ds\right)^{\frac{1}{q}}$$
$$= \left(\frac{T^{1-\frac{q\zeta}{2}}}{1-\frac{q\zeta}{2}}\right)^{\frac{1}{q}} c(q,\zeta,T,c(T),c_f(T),c_\sigma(T),C_{q,\eta})||X-Y||_{\mathcal{H}^q(T)}.$$

Here, we used 2.2.1 **(iii)** for $U$, estimate (3.18) for $I_{\Sigma,m}$, estimate (3.30) for $I^W_\Sigma$, estimate (4.5) for $I^{\tilde{N}}_\Sigma$ and the Lipschitz property for $e$ and $\sigma$.
Thus, by the same procedure as in **(i)**, we get existence of a unique solution $X\in\mathcal{H}^q(T)$ in the sense of 5.1.2 **(i)**.
Similarly to (5.32), using the fact that $e$ and $\sigma$ obey **(LC)** and **(LB)**, we have for our solution $X\in\mathcal{H}^q(T)$

$$\begin{aligned}\mathbf{E}||X(t)||^q_{L^2_\rho} \le\ & c(q)\Big(\mathbf{E}||U(t,0)\xi||^q_{L^2_\rho}+\mathbf{E}||I_F(X)(t)||^q_{L^2_\rho}+\mathbf{E}||I_{\Sigma,m}(X)(t)||^q_{L^2_\rho}\\ &+\mathbf{E}||I^W_\Sigma(X)(t)||^q_{L^2_\rho}+\mathbf{E}||I^{\tilde{N}}_\Sigma(X)(t)||^q_{L^2_\rho}\Big)\\ \le\ & c(q,\zeta,m,T,K,c(T),c_e(T),C_{q,\eta})(1+\mathbf{E}||\xi||^q_{L^2_\rho})\\ &+c(q,m,\zeta,T,c(T),c_e(T),c_\sigma(T))\int_0^t (t-s)^{-\frac{q\zeta}{2}}\mathbf{E}||X(s)||^q_{L^2_\rho}\,ds.\end{aligned}$$

Now, by the Gronwall-Bellman lemma, we get (5.26), which finishes the proof of Theorem 5.2.1. ■

**Proof of 5.2.2:** Here, we apply a Picard iteration method to prove the unique solvability in the Banach space $\mathcal{G}_\nu(T)$.

**Equation (1.1)**

We define a sequence $(X_n)_{n\in\mathbb{N}}$ of processes by

$$X_0(t) := U(t,0)\xi,\ t\in[0,T],$$

$$\begin{aligned}X_n(t) :=\ & X_0(t)+\int_0^t U(t,s)F(s,,X_{n-1}(s))\,ds\\ &+\int_0^t U(t,s)\mathcal{M}_{\Sigma(s,X_{n-1}(s))}\,dW(s)\\ &+\int_0^t\int_{L^2} U(t,s)\mathcal{M}_{\Gamma(s,X_{n-1}(s))}(x)\,\tilde{N}(ds,dx)\\ =\ & X_0(t)+I_F(X_{n-1})(t)+I^W_\Sigma(X_{n-1})(t)+I^{\tilde{N}}_\Gamma(X)(t),\ t\in[0,T],\ n\in\mathbb{N}.\end{aligned}$$

Let us show that these processes are $\mathcal{G}_\nu(T)$-valued. By **(A3)** and our assumption on $\xi$, $X_0$ is obviously $\mathcal{G}_\nu(T)$-valued.
Now suppose we know that $X_{n-1}\in\mathcal{G}_\nu(T)$ (which we do for $n=1$). We know from 5.1.6, 5.1.8 and 5.1.10 that $I_F(X_{n-1})$, $I^W_\Sigma(X_{n-1})$

and $I_\Gamma^{\tilde{N}}(X_{n-1})$ are in $\mathcal{G}_\nu(T)$. This immediately gives us $\mathcal{G}_\nu(T)$-valuedness of $X_n$.

Next, we show that the sequence $(X_n)_{n\in\mathbb{N}}$ converges in $\mathcal{G}_\nu(T)$.
For any $t \in [\,0,T\,]$ and $n \in \mathbb{N}$, we have

$$
\begin{aligned}
&\mathbf{E}||X_{n+1}(t) - X_n(t)||^{2\nu}_{L^{2\nu}_\rho} \\
&\leq c(\nu)\left(\mathbf{E}\left|\left|\int_0^t U(t,s)[F(s,X_n(s)) - F(s,X_{n-1}(s))]\,ds\right|\right|^{2\nu}_{L^{2\nu}_\rho}\right. \\
&+\mathbf{E}\left|\left|\int_0^t U(t,s)[\mathcal{M}_{\Sigma(s,X_n(s))} - \mathcal{M}_{\Sigma(s,X_{n-1}(s))}]\,dW(s)\right|\right|^{2\nu}_{L^{2\nu}_\rho} \\
&\left.+\mathbf{E}\left|\left|\int_0^t U(t,s)[\mathcal{M}_{\Gamma(s,X_n(s))} - \mathcal{M}_{\Gamma(s,X_{n-1}(s))}]\,\tilde{N}(ds,dx)\right|\right|^{2\nu}_{L^{2\nu}_\rho}\right) \\
&\leq c(\nu,\zeta,T,c(T),c_f(T),c_\sigma(T),c_\gamma(T),C_{2\nu,\eta})\int_0^t (t-s)^{-\zeta\nu}\mathbf{E}||X_n(s)-X_{n-1}(s)||^{2\nu}_{L^{2\nu}_\rho}\,ds,
\end{aligned}
$$

where we used Proposition 3.4.3 and 4.4 and the Lipschitz property **(LC)** for $f$, $\sigma$ and $\gamma$. Herefrom, by the Gronwall-Bellman lemma 2.7.3 we get, for all $n \in \mathbb{N}$,

$$
\begin{aligned}
&\sup_{t\in[\,0,T\,]} \mathbf{E}||X_{n+1}(t) - X_n(t)||^{2\nu}_{L^{2\nu}_\rho} \\
&\leq T^{n(1-\zeta\nu)}c(\nu,\zeta,c(T),c_f(T),c_\sigma(T),c_\gamma(T),C_{2\nu,\eta})\sup_{t\in[\,0,T\,]}\mathbf{E}||X_1(t)-X_0(t)||^{2\nu}_{L^{2\nu}_\rho}
\end{aligned}
$$

with

$$
\begin{aligned}
&\sup_{t\in[\,0,T\,]} \mathbf{E}||X_1(t) - X_0(t)||^{2\nu}_{L^{2\nu}_\rho} \\
&\leq c(\nu)\left(\sup_{t\in[\,0,T\,]}\mathbf{E}\left|\left|\int_0^t U(t,s)F(s,U(s,0)\xi)\,ds\right|\right|^{2\nu}_{L^{2\nu}_\rho}\right. \\
&+\sup_{t\in[\,0,T\,]}\mathbf{E}\left|\left|\int_0^t U(t,s)\mathcal{M}_{\Sigma(s,U(s,0)\xi)}\,dW(s)\right|\right|^{2\nu}_{L^{2\nu}_\rho} \\
&\left.+\sup_{t\in[\,0,T\,]}\mathbf{E}\left|\left|\int_0^t\int_{L^2} U(t,s)\mathcal{M}_{\Gamma(s,U(s,0)\xi)}(x)\,\tilde{N}(ds,dx)\right|\right|^{2\nu}_{L^{2\nu}_\rho}\right) \\
&\leq \bar{c}(\nu,\zeta,T,c(T),c_f(T),c_\sigma(T),c_\gamma(T),C_{2\nu,\eta})(1+\mathbf{E}||\xi||^{2\nu}_{L^{2\nu}_\rho}).
\end{aligned}
$$

Thus, we finally get

$$
||X_{n+1} - X_n||^{2\nu}_{\mathcal{G}_\nu(T)} \leq T^{n(1-\zeta\nu)}c_1 < \infty
$$

with a positive constant
$c_1 := c(\nu, \zeta, c(T), c_f(T), c_\sigma(T), c_\gamma(T), C_{2\nu,\eta})\bar{c}(\nu, \zeta, T, c(T), c_f(T), c_\sigma(T), c_\gamma(T), C_{2\nu,\eta})$.

Since $0 < \nu < \frac{1}{\zeta}$, the right hand side tends to 0 as $n \to \infty$, which gives us the existence of a mild solution to (1.1) on the whole intervall $[0, T]$.

Concerning the uniqueness of the solution, let us note that for any two solutions $X, Y \in \mathcal{G}_\nu(T)$ we have (analogously to the $X_n$-estimate)

$$\begin{aligned}
\mathbf{E}||X(t) - Y(t)||^{2\nu}_{L^{2\nu}_\rho} \leq \quad & c(\kappa) \left( \mathbf{E} \left|\left| \int_0^t U(t,s)[F(s, X(s)) - F(s, Y(s))]\, ds \right|\right|^{2\nu}_{L^{2\nu}_\rho} \right. \\
& + \mathbf{E} \left|\left| \int_0^t U(t,s)[\mathcal{M}_{\Sigma(s,X(s))} - \mathcal{M}_{\Sigma(s,Y(s))}]\, dW(s) \right|\right|^{2\nu}_{L^{2\nu}_\rho} \\
& \left. + \mathbf{E} \left|\left| \int_0^t \int_{L^2} U(t,s)[\mathcal{M}_{\Gamma(s,X(s))} - \mathcal{M}_{\Gamma(s,Y(s))}](x)\, \tilde{N}(ds, dx) \right|\right|^{2\nu}_{L^{2\nu}_\rho} \right) \\
\leq \quad & c(\nu, \zeta, T, c(T), c_f(T), c_\sigma(T), c_\gamma(T), C_{2\nu,\eta}) \\
& \int_0^t (t-s)^{-\zeta\nu} \mathbf{E}||X(s) - Y(s)||^{2\nu}_{L^{2\nu}_\rho}\, ds.
\end{aligned}$$

Then, applying 2.7.3 with $g_n(t) := g(t) := \mathbf{E}||X(t) - Y(t)||^{2\nu}_{L^{2\nu}_\rho}$, gives us

$$\mathbf{E}||X(t) - Y(t)||^{2\nu}_{L^{2\nu}_\rho} \leq 0$$

for any $t \in [0, T]$. Thus, we obviously have

$$||X - Y||_{\mathcal{G}_\nu(T)} = 0,$$

which proves uniqueness in $\mathcal{G}_\nu(T)$. Again, we have continuity in $L^{2\nu}(\Omega; L^{2\nu}_\rho)$ by Lemmata 5.1.6, 5.1.8 and 5.1.10.
Furthermore, for bounded $\gamma$ and **(A7)** for $U$, there is a càdlàg version by Lemmata 5.1.6, 5.1.8 and 5.1.10.
Thus, concerning (1.1), it remains to show the estimate (5.27) for our solution.
To this end, we note that, for any $t \in [0, T]$:

$$\begin{aligned}
& \mathbf{E}||X(t)||^{2\nu}_{L^{2\nu}_\rho} \\
& \leq c(\nu) \left( \mathbf{E}||U(t,0)\xi||^{2\nu}_{L^{2\nu}_\rho} + \mathbf{E} \left|\left| \int_0^t U(t,s)F(s, X(s))\, ds \right|\right|^{2\nu}_{L^{2\nu}_\rho} \right. \\
& + \mathbf{E} \left|\left| \int_0^t U(t,s)\mathcal{M}_{\Sigma(s,X(s))}\, dW(s) \right|\right|^{2\nu}_{L^{2\nu}_\rho}
\end{aligned}$$

$$+\mathbf{E}\left|\left|\int_0^t\int_{L^2} U(t,s)\mathcal{M}_{\Gamma(s,X(s))}(x)\,\tilde{N}(ds,dx)\right|\right|^{2\nu}_{L^{2\nu}_\rho}\Bigg)$$
$$\leq c(\nu)\Big(c(\nu,T)\mathbf{E}||\xi||^{2\nu}_{L^{2\nu}_\rho}$$
$$+c(\nu,\zeta,T,c(T),c_f(T),c_\sigma(T),c_\gamma(T),C_{2\nu,\eta})\left(1+\int_0^t(t-s)^{-\zeta\nu}\mathbf{E}||X(s)||^{2\nu}_{L^{2\nu}_\rho}\,ds\right)\Bigg)$$
$$\leq c(\nu,\zeta,T,c(T),c_f(T),c_\sigma(T),c_\gamma(T),C_{2\nu,\eta})(1+\mathbf{E}||\xi||^{2\nu}_{L^{2\nu}_\rho})$$
$$+\,c(\nu,\zeta,T,c(T),c_f(T),c_\sigma(T),c_\gamma(T),C_{2\nu,\eta})\int_0^t(t-s)^{-\zeta\nu}\mathbf{E}||X(s)||^{2\nu}_{L^{2\nu}_\rho}\,ds.$$

Thus, (5.27) follows by the Gronwall-Bellman lemma 2.7.2/2.7.3 and we are finished with the consideration of equation (1.1).

**Equation (1.2)**

Let us define a sequence of processes $(X_n)_{n\in\mathbb{N}}$ by

$$X_0(t) := U(t,0)\xi$$

and

$$\begin{aligned}
X_n(t) := \;& X_0(t)+\int_0^t U(t,s)E(t,\omega,X_{n-1}(t))\,ds\\
&+\int_0^t U(t,s)\mathcal{M}_{\Sigma(s,X_{n-1}(s))}(m)\,ds\\
&+\int_0^t U(t,s)\mathcal{M}_{\Sigma(s,X_{n-1}(s))}\,dW(s)\\
&+\int_0^t\int_{L^2} U(t,s)\mathcal{M}_{\Sigma(s,X_{n-1}(s))}(x)\,\tilde{N}(ds,dx)\\
=\;& X_0(t)+I_E(X_{n-1})(t)+I_{\Sigma,m}(X_{n-1})(t)+I^W_\Sigma(X_{n-1})(t)+I^{\tilde{N}}_\Sigma(X_{n-1})(t)
\end{aligned}$$

for $t\in[0,T]$ and $n\in\mathbb{N}$, where $W$ is as in the general nuclear case (cf. Sections 2.5 and 3.4).
Again, we have to show that these processes are $\mathcal{G}_\nu(T)$-valued. By **(A3)(i)** and our assumption on $\xi$, $X_0$ is $\mathcal{G}_\nu(T)$-valued.
Suppose we know that $X_{n-1}\in\mathcal{G}_\nu(T)$(which we do for $n=1$). We know from 5.1.6**(ii)**, 5.1.8 and 5.1.10 that $I_E(X_{n-1})$, $I_{\Sigma,m}(X_{n-1})$, $I^W_\Sigma(X_{n-1})$ and $I^{\tilde{N}}_\Sigma(X_{n-1})$ are in $\mathcal{G}_\nu(T)$. Thus $X_n$ is in $\mathcal{G}_\nu(T)$ as a sum of elements from $\mathcal{G}_\nu(T)$.

Next, we show that the sequence $(X_n)_{n\in\mathbb{N}}$ converges in $\mathcal{G}_\nu(T)$.
Indeed, we have

$$\mathbf{E}||X_{n+1}(t)-X_n(t)||^{2\nu}_{L^{2\nu}_\rho} \le c(\kappa)\left(\mathbf{E}\left|\left|\int_0^t U(t,s)[E(s,X_n(s))-E(s,X_{n-1}(s))]\,ds\right|\right|^{2\nu}_{L^{2\nu}_\rho}\right.$$
$$+\mathbf{E}\left|\left|\int_0^t U(t,s)[\mathcal{M}_{\Sigma(s,X_n(s))}-\mathcal{M}_{\Sigma(s,X_{n-1}(s))}]m\,ds\right|\right|^{2\nu}_{L^{2\nu}_\rho}$$
$$+\mathbf{E}\left|\left|\int_0^t U(t,s)[\mathcal{M}_{\Sigma(s,X_n(s))}-\mathcal{M}_{\Sigma(s,X_{n-1}(s))}]\,dW(s)\right|\right|^{2\nu}_{L^{2\nu}_\rho}$$
$$\left.+\mathbf{E}\left|\left|\int_0^t\int_{L^2} U(t,s)[\mathcal{M}_{\Sigma(s,X_n(s))}-\mathcal{M}_{\Sigma(s,X_{n-1}(s))}](x)\,\tilde{N}(ds,dx)\right|\right|^{2\nu}_{L^{2\nu}_\rho}\right)$$
$$\le c(\nu,\zeta,m,T,c(T),c_e(T),c_\sigma(T),C_{2\nu,\eta})$$
$$\int_0^t (t-s)^{-\zeta\nu}\mathbf{E}||X_n(s)-X_{n-1}(s)||^{2\nu}_{L^{2\nu}_\rho}\,ds$$

for all $t \in [\,0,T\,]$ and arbitrary $n \in \mathbb{N}$.

Thus, by the Gronwall-Bellman lemma 2.7.3, we get for all $n \in \mathbb{N}$

$$\sup_{t\in[\,0,T\,]}\mathbf{E}||X_{n+1}(t)-X_n(t)||^{2\nu}_{L^{2\nu}_\rho}$$
$$\le T^{n(1-\zeta\nu)}c(\nu,m,c(T),c_e(T),c_\sigma(T),c_\gamma(T),C_{2\nu,\eta})\sup_{t\in[\,0,T\,]}\mathbf{E}||X_1(t)-X_0(t)||^{2\nu}_{L^{2\nu}_\rho}$$

with

$$\sup_{t\in[\,0,T\,]}\mathbf{E}||X_1(t)-X_0(t)||^{2\nu}_{L^{2\nu}_\rho}$$
$$\le c(\nu)\left(\sup_{t\in[\,0,T\,]}\mathbf{E}\left|\left|\int_0^t U(t,s)\left[E(s,U(s,0)\xi)+\mathcal{M}_{\Sigma(s,\cdot,U(s,0)\xi)}(m)\,\right]\,ds\right|\right|^{2\nu}_{L^{2\nu}_\rho}\right.$$
$$+\sup_{t\in[\,0,T\,]}\mathbf{E}\left|\left|\int_0^t U(t,s)\mathcal{M}_{\Sigma(s,\cdot,U(s,0)\xi)}\,dW(s)\right|\right|^{2\nu}_{L^{2\nu}_\rho}$$
$$\left.+\sup_{t\in[\,0,T\,]}\mathbf{E}\left|\left|\int_0^t\int_{L^2} U(t,s)\mathcal{M}_{\Sigma(s,\cdot,U(s,0)\xi)}(x)\,\tilde{N}(ds,dx)\right|\right|^{2\nu}_{L^{2\nu}_\rho}\right)$$
$$\le c(\nu,\zeta,m,T,K,c(T),c_e(T),c_\sigma(T),c_\gamma(T),C_{2\nu,\eta})(1+\mathbf{E}||\xi||^{2\nu}_{L^{2\nu}_\rho}).$$

Thus, we finally get

$$\sup_{t\in[\,0,T\,]}\mathbf{E}||X_{n+1}(t)-X_n(t)||^{2\nu}_{L^{2\nu}_\rho}$$
$$\le T^{n(1-\zeta\nu)}c(\nu,\zeta,m,c(T),c_e(T),,c_\sigma(T),C_{2\nu,\eta})$$
$$c(\nu,\zeta,T,K,c(T),c_e(T),C_{2\nu,\eta})(1+\mathbf{E}||\xi||^{2\nu}_{L^{2\nu}_\rho})$$
$$<\infty$$

for all $n \in \mathbb{N}$.

Since $0 < \nu < \frac{1}{\zeta}$, we get existence of a mild solution to (1.2). Now, we establish the uniqueness result. Note that, for any two solutions $X, Y \in \mathcal{G}_\nu(T)$, analogously to the $X_n$-estimate we have

$$
\begin{aligned}
&\mathbf{E}||X(t) - Y(t)||^{2\nu}_{L^{2\nu}_\rho} \\
&\leq c(\kappa) \left( \mathbf{E} \left|\left| \int_0^t U(t,s)[E(s,X(s)) - E(s,Y(s))]\, ds \right|\right|^{2\nu}_{L^{2\nu}_\rho} \right. \\
&+ \mathbf{E} \left|\left| \int_0^t U(t,s)[\mathcal{M}_{\Sigma(s,X(s))} - \mathcal{M}_{\Sigma(s,Y(s))}](m)\, ds \right|\right|^{2\nu}_{L^{2\nu}_\rho} \\
&+ \mathbf{E} \left|\left| \int_0^t U(t,s)[\mathcal{M}_{\Sigma(s,X(s))} - \mathcal{M}_{\Sigma(s,Y(s))}]\, dW(s) \right|\right|^{2\nu}_{L^{2\nu}_\rho} \\
&\left. + \mathbf{E} \left|\left| \int_0^t \int_{L^2} U(t,s)[\mathcal{M}_{\Sigma(s,X(s))} - \mathcal{M}_{\Sigma(s,Y(s))}](x)\, \tilde{N}(ds,dx) \right|\right|^{2\nu}_{L^{2\nu}_\rho} \right) \\
&\leq c(\nu, m, T, c(T), c_e(T), c_\sigma(T), C_{2\nu,\eta}) \int_0^t (t-s)^{-\zeta\nu} \mathbf{E}||X(s) - Y(s)||^{2\nu}_{L^{2\nu}_\rho}\, ds.
\end{aligned}
$$

Thus, by the Gronwall-Bellman lemma 2.7.3 we get, for any $t \in [0, T]$,

$$\mathbf{E}||X(t) - Y(t)||^{2\nu}_{L^{2\nu}_\rho} \leq 0.$$

Thus, we obviously have

$$||X - Y||_{\mathcal{G}_\nu(T)} = 0,$$

which proves uniqueness in $\mathcal{G}_\nu(T)$. Again, we have continuity in $L^{2\nu}(\Omega; L^{2\nu}_\rho(\Theta))$ from the fact that $I_E(X)$, $I_{\Sigma,\eta}$, $I^W_\Sigma$ and $I^{\tilde{N}}_\Sigma$ are continuous in $L^{2\nu}(\Omega; L^{2\nu}_\rho(\Theta))$ as was shown in 5.1.6 **(ii)**, 5.1.8 and 5.1.10.
Similar to **(i)**, under the additional assumption **(A7)** we get existence of càdlàg versions of the solutions in **(ii)**.
Thus, it remains to show the a-priori bound (5.28).
Note that we have, for $t \in [0, T]$,

$$
\begin{aligned}
\mathbf{E}||X(t)||^{2\nu}_{L^{2\nu}_\rho} \leq\ & c(\nu) \left( \mathbf{E}||U(t,0)\xi||^{2\nu}_{L^{2\nu}_\rho} + \mathbf{E} \left|\left| \int_0^t U(t,s)E(s,X(s))\, ds \right|\right|^{2\nu}_{L^{2\nu}_\rho} \right. \\
&+ \mathbf{E} \left|\left| \int_0^t U(t,s)\mathcal{M}_{\Sigma(s,X(s))}(m)\, ds \right|\right|^{2\nu}_{L^{2\nu}_\rho}
\end{aligned}
$$

$$+\mathbf{E}\left|\left|\int_0^t U(t,s)\mathcal{M}_{\Sigma(s,X(s))}\,dW(s)\right|\right|^{2\nu}_{L^{2\nu}_\rho}$$

$$+\mathbf{E}\left|\left|\int_0^t\int_{L^2} U(t,s)\mathcal{M}_{\Sigma(s,X(s))}(x)\,\tilde{N}(ds,dx)\right|\right|^{2\nu}_{L^{2\nu}_\rho}\Bigg)$$

$$\leq c(\nu)\Big(c(\nu,T)\mathbf{E}||\xi||^{2\nu}_{L^{2\nu}_\rho}$$

$$+c(\zeta,\nu,m,T,c(T),c_e(T),c_\sigma(T),C_{2\nu,\eta})\left(1+\int_0^t(t-s)^{-\zeta\nu}\mathbf{E}||X(s)||^{2\nu}_{L^{2\nu}_\rho}\,ds\right)\Bigg)$$

$$\leq c_1(\nu,\zeta,m,T,c(T),c_e(T),c_\sigma(T),C_{2\nu,\eta})(1+\mathbf{E}||\xi||^{2\nu}_{L^{2\nu}_\rho})$$

$$+\,c_2(\nu,\zeta,m,T,c(T),c_e(T),c_\sigma(T),C_{2\nu,\eta})\int_0^t(t-s)^{-\zeta\nu}\mathbf{E}||X(s)||^{2\nu}_{L^{2\nu}_\rho}\,ds.$$

Thus, (5.28) follows from the Gronwall-Bellman lemma 2.7.2/2.7.3, which finishes the proof. ■

We complete this section with the following corollaries and remarks, we need later.

**Corollary 5.2.4:** *Under the assumptions of 5.2.1 resp. 5.2.2, there exist mild solutions $V$ to (1.1) resp. (1.2) with $\xi = 0$ and $F = 0$ resp. $E = 0$ such that*

$$\sup_{t\in[0,T]}\mathbf{E}||V(t)||^q_{L^2_\rho} \leq c(q,\zeta,T,c(T),c_\sigma(T),c_\gamma(T),C_{q,\eta}) \tag{5.33}$$

$$\sup_{t\in[0,T]}\mathbf{E}||V(t)||^q_{L^2_\rho} \leq c(q,\zeta,m,T,c(T),c_\sigma(T),C_{q,\eta}) \tag{5.34}$$

*with positive constants on the right hand side resp.*

$$\sup_{t\in[0,T]}\mathbf{E}||V(t)||^{2\nu}_{L^2_\rho} \leq c(\nu,\zeta,T,c(T),c_\sigma(T),c_\gamma(T),C_{2\nu,\eta}) \tag{5.35}$$

$$\sup_{t\in[0,T]}\mathbf{E}||V(t)||^{2\nu}_{L^{2\nu}_\rho} \leq c(\nu,\zeta,m,T,c(T),c_\sigma(T),C_{2\nu,\eta}) \tag{5.36}$$

*with positive constants on the right hand side.*

From Theorem 5.2.1 and 5.2.2, in the special case that $\Gamma$ and $\Sigma$ are solution-independent, we get

**Corollary 5.2.5:** ***(i)*** *In the setting of Theorem 5.2.1, in the special case that $\gamma$: $[0,T]\times\Omega\to\mathbb{R}$ resp. $\sigma$: $[0,T]\times\Omega\to\mathbb{R}$ and $|\gamma|\leq K$ resp. $|\sigma|\leq K$ for some $K>0$ uniformly in $[0,T]\times\Omega$, there exists a unique solution to (1.1) resp. (1.2) in the sense of 5.1.2* ***(i)****. Furthermore, $t\mapsto X(t)$ is continuous in $L^q(\Omega;L^2_\rho(\Theta))$ and obeys estimate (5.25) resp. (5.26).*

***(ii)*** *In the setting of Theorem 5.2.2, in the special case that* $\gamma\colon [0,T] \times \Omega \to \mathbb{R}$ *resp.* $\sigma\colon [0,T] \times \Omega \to \mathbb{R}$ *and* $|\gamma| \leq K$ *resp.* $|\sigma| \leq K$ *for some* $K > 0$ *uniformly in* $[0,T] \times \Omega$*, there exists a unique solution to (1.1) resp. (1.2) in the sense of 5.1.2* ***(ii)****. Furthermore,* $t \mapsto X(t)$ *is continuous in* $L^{2\nu}(\Omega; L^{2\nu}_\rho(\Theta))$ *and obeys estimate (5.27) resp. (5.28).*

**Remark 5.2.6:** *As was proven in the above theorems, under certain assumptions the solutions* $X$ *to the equations (1.1) and (1.2) allow càdlàg modifications in* $L^2_\rho(\Theta)$*. Let us denote them by* $X^{cl}(t)$*,* $t \in [0,T]$*.*

*The processes* $X^{cl}(t)$*,* $t \in [0,T]$*, are in general* ***not*** *predictable. To overcome this, we define*

$$X^{cl}_-(t) := X^{cl}(t-),\ t \in [0,T].$$

*This process is surely left-continuous and hence predictable.*
*It is easy to see that*

$$X(t) = X^{cl}_-(t),\ P\text{-a.s., for any } t \in [0,T]. \tag{5.37}$$

*Indeed, given any* $t \in [0,T]$ *and any sequence* $(t_n)_{n\in\mathbb{N}} \subset [0,T]$ *such that* $t_n \uparrow t$ *as* $n \to \infty$*, by definition we get*

$$X^{cl}_-(t) = \lim_{n\to\infty} X^{cl}(t_n),\ P\text{-a.s.}\ .$$

*On the other hand, in* $L^2(\Omega; L^2_\rho(\Theta))$ *we have*

$$X(t) = \lim_{n\to\infty} X^{cl}(t_n) \text{ in probability.}$$

*Combining both convergences we get (5.37).*

*The Itô isometry shows that the stochastic equivalence of* $X$ *and* $X^{cl}_-$ *implies the stochastic equivalence of all processes on the right hand side of equation (1.1) resp. equation (1.2).*

*Let* $X(t)$*,* $t \in [0,T]$*, be a càdlàg modification of the unique predictable solution to (1.1) resp. (1.2). This modification satisfies the equation*

$$
\begin{aligned}
(5.38)\; X(t) = \quad & U(t,0)\xi + \int_0^t U(t,s)F(s,X(s))ds \\
& + \int_0^t U(t,s)\mathcal{M}_{\Sigma(s,X(s))}dW(s) \\
& + \int_0^t \int_{L^2} U(t,s)\mathcal{M}_{\Gamma(s,X(s-))}(x)\,\tilde{N}(ds,dx)
\end{aligned}
$$

*resp.*

$$
\begin{aligned}
(5.39)\; X(t) = \quad & U(t,0)\xi + \int_0^t U(t,s)E(s,X(s))ds + \int_0^t U(t,s)\mathcal{M}_{\Sigma(s,X(s))}dL(s) \\
= \quad & U(t,0)\xi + \int_0^t U(t,s)E(s,X(s))ds \\
& + \int_0^t U(t,s)\mathcal{M}_{\Sigma(s,X(s))}m\,ds \\
& + \int_0^t U(t,s)\mathcal{M}_{\Sigma(s,X(s))}dW(s) \\
& + \int_0^t \int_{L^2} U(t,s)\mathcal{M}_{\Sigma(s,X(s-))}(x)\,\tilde{N}(ds,dx),
\end{aligned}
$$

*$P$-almost surely, for each fixed $t \in [\,0,T\,]$.*
*Taking the càdlàg modification of all integrals on the right hand sides in (5.38) and (5.39), we have the identity for all $t \in [\,0,T\,]$ on the same universal subset $\Omega_0 \in \mathcal{B}(\Omega)$ with $P(\Omega_0) = 1$.*

# Chapter 6

# Comparison results in the Lipschitz case with additive jump noise

For the whole chapter, let $(\Omega, \mathcal{F}, P)$ and $(\mathcal{F}_t)_{t\in[0,T]}$ for some $T > 0$ be as in Section 1.2.
As in Chapter 5, we consider SDEs in $L^2_\rho(\Theta)$ for $\Theta \subset \mathbb{R}^d$ for some $d \in \mathbb{N}$. Again, we assume $\rho$ to be such that $\mu_\rho(\Theta) < \infty$, i.e. $\rho > d$ for unbounded $\Theta$ and $\rho = 0$ for bounded $\Theta$.

Let $(e_n)_{n\in\mathbb{N}}$ be an orthonormal basis of $L^2(\Theta)$ obeying (3.1).

In this chapter, we show comparison results both for equation (1.1) and for equation (1.2) in the case of additive jumps, i.e. when the coefficients corresponding to the jump parts ($\Gamma$ in (1.1) and $\Sigma$ in (1.2)) are solution-independent (see Section 6.1 below).
Thus, we assume that the drift coefficients $e^{(i)}$, $f^{(i)}$: $[0,T] \times \Omega \times \mathbb{R} \to \mathbb{R}$, $i = 1, 2$, are of the same type as $e$ resp. $f$ in Chapter 5, i.e. they are $\mathcal{P}_T \otimes \mathcal{B}(\mathbb{R})/\mathcal{B}(\mathbb{R})$-measurable and Lipschitz continuous w.r.t. the third variable.
Note that a crucial point to show a comparison theorem for equation (1.1) in the case of an additive jump part is that for our $L^2(\Theta)$-valued $Q$-Wiener process $W$ we suppose that the orthonormal basis $(e_n)_{n\in\mathbb{N}}$ of $L^2(\Theta)$ appearing in the representation

$$W(t) = \sum_{n\in\mathbb{N}} \sqrt{a_n} w_n(t) e_n,\ t \in [0,T],$$

(which exists by Proposition 2.3.7) is uniformly bounded, i.e. $(e_n)_{n\in\mathbb{N}}$ obeys (cf. (3.1) in Section 3.1)

$$\sup_{n\in\mathbb{N}} ||e_n||_\infty < \infty.$$

The proof, whose method is based on Manthey's and Zausinger's proof of their Theorem 3.3.1 in [76], works by showing comparison theorems for finite dimensional approximations of the solutions and establishing their convergence to the solutions of the initial equations. Let us note that the proof of Theorem 3.3.1 in [76] is a simplification of Kotenlenezproof of his comparison result in [65]. For the whole chapter, we assume the existence of a bounded family of operators $((A_N(t))_{t\in[0,T]})_{N\in\mathbb{N}} \subset \mathcal{L}(L^2_\rho(\Theta))$ fulfilling **(A6)** (see Section 3.1 for its definition).

The chapter has the following structure. First, in Section 6.1 we state the main result, a comparison result for some approximating SDE with finite-dimensional Wiener noise and a convergence result for the approximating stochastic differerential equation. Furthermore, we explain, why the method from [76] gives us our main result. Then, we show the comparison result for the approximating SDE in Section 6.2 and the convergence result for the approximating SDE in Section 6.3.

In the subsequent sections, given $X,\ Y \in L^2_\rho(\Theta)$, by writing $X \le Y$ we mean that $X(\theta) \le Y(\theta)$ for $\mu_\rho$-almost all $\theta \in \Theta$.

## 6.1 The main result and the scheme of comparison method for additive jumps

In this section, for $i = 1, 2$, we consider a pair of equations

$$\begin{aligned} dX^{(i)}(t) = \quad & (A(t)X^{(i)}(t) + F^{(i)}(t, X^{(i)}(t)))dt + \mathcal{M}_{\Sigma(t,X^{(i)}(t))}dW(t) \\ & + \int_{L^2} \mathcal{M}_{C(t)}x\, \tilde{N}(dt, dx),\ t \in [0, T], \end{aligned}$$

(6.1)

$$X^{(i)}(0) = \xi^{(i)},$$

resp.

$$dX^{(i)}(t) = (A(t)X^{(i)}(t) + E^{(i)}(t, X^{(i)}(t)))dt + \mathcal{M}_{C(t)}dL(t),\ t \in [0, T],$$

(6.2)

$$X^{(i)}(0) = \xi^{(i)},$$

with $\Sigma$ defined through a $\mathcal{P}_T \otimes \mathcal{B}(\mathbb{R})$-measurable $\sigma$: $[0,T] \times \Omega \times \mathbb{R} \to \mathbb{R}$ as in Chapter 5. Suppose that $c$: $[0,T] \times \Omega \to \mathbb{R}$ is $\mathcal{P}_T$-measurable and uniformly bounded, i.e.

$$\sup_{(t,\omega)\in[0,T]\times\Omega} |c(t,\omega)| < K \tag{6.3}$$

for some $K > 0$. The multiplication operator $\mathcal{M}_C$: $L^2(\Theta) \to L^2(\Theta)$ is given by

$$(\mathcal{M}_{C(t,\omega)}(\psi))(\theta) = c(t,\omega)\psi(\theta),\ (t,\omega) \in [0,T] \times \Omega,\ \theta \in \Theta,\ \psi \in L^2.$$

Thus, (6.1) resp. (6.2) is just a special case of equation (1.1) resp. of equation (1.2) with additive Poisson resp. additive Lévy noise, i.e. the coefficients corresponding to the compensated Poisson random measure $\tilde{N}$ resp. to the Lévy process $L$ are independent of the solution.

As in Chapter 5, for the initial conditions $\xi^{(i)}$ we have the following two cases

**Case (A)** The initial condition $\xi$ is an $L^2_\rho(\Theta)$-valued random variable such that $\mathbf{E}||\xi^{(i)}||^q_{L^2_\rho} < \infty$ for some $q \geq 2$.
The solutions to (6.1) and (6.2) will be constructed in $\mathcal{H}^q(T)$.

**Case (B)** The initial condition $\xi$ is an $L^{2\nu}_\rho(\Theta)$-valued random variable such that $\mathbf{E}||\xi^{(i)}||^{2\nu}_{L^{2\nu}_\rho} < \infty$ for some $\nu \geq 1$.
The solutions to (6.1) and (6.2) will be constructed in $\mathcal{G}_\nu(T)$.

In view of Theorems 5.2.1 and 5.2.2, to have existence of (unique) solutions to (6.1) and (6.2) we also need to assume that $q < \frac{2}{\zeta}$ resp. $\nu < \frac{1}{\zeta}$ with $\zeta \in [0,1)$ from **(A2)**.

It will be enough to get comparison results in **Case (A)**, since the comparison results in **Case (B)** follow immediately. Indeed, since $L^{2\nu}_\rho(\Theta) \subset L^2_\rho(\Theta)$ for $\nu \geq 1$, we have

$$\mathbf{E}||\xi^{(i)}||^{2\nu}_{L^{2\nu}_\rho} < \infty \Rightarrow \mathbf{E}||\xi^{(i)}||^{2\nu}_{L^2_\rho} < \infty.$$

Thus, we are again in **Case (A)** with $q = 2\nu \geq 2$.

For the Wiener process $W$ in (6.1), we have the following basic cases.

- **nuclear case**, i.e. $W$ is a $Q$-Wiener process in the sense of Section

2.3 with $Q \in \mathcal{T}^+(L^2(\Theta))$ such that the operator $Q$ obeys a complete orthonormal system of eigenvectors $(e_n)_{n\in\mathbb{N}}$ fulfilling (3.1), i.e.

$$\sup_{n\in\mathbb{N}} ||e_n||_\infty < \infty.$$

- **general nuclear case**, i.e. $W$ is a $Q$-Wiener process in the sense of Section 2.3 with $Q \in \mathcal{T}^+(L^2(\Theta))$, but $Q$ does not allow an eigenvector expansion fulfilling (3.1).
- **cylindrical case**, i.e. $W$ is a $\mathbf{I}$-Wiener process in $L^2(\Theta)$ in the sense of Section 2.3.

The general nuclear case typically occurs in equations driven by Lévy noise, for which we apply the Lévy-Itô decompostion.

In both cases of bounded and unbounded domain $\Theta$, we write $L^2_\rho$ for the spaces $L^2_\rho(\Theta)$.

A basic assumption on $U$ is that it constitutes an almost strong evolution operator in $L^2_\rho$ obeying **(A0)**–**(A2)**. In the nuclear case, we set $\zeta = 0$. Thus, **(A2)** is surely fulfilled inthis case. In the general nuclear case, we can substitute **(A2)** by the weaker assumption **(A5)*** with $\nu = 1$.

Under the assumptions imposed above, there exist unique mild solutions $X^{(i)} \in \mathcal{H}^q(T)$ (cf. Definition 5.1.1 **(i)** for the notation) to (6.1) and (6.2) (cf. Section 5.2, Corollary 5.2.5).

We prove a comparison result for (6.1) and (6.2) in **Case (A)** by adapting the proof of Theorem 3.3.1 from [76].

**Theorem 6.1.1:** *Let $U$ be an almost strong evolution operator generated by $(A(t))_{t\in[0,T]}$ such that **(A0)**–**(A2)** and **(A6)** hold. Suppose the Lévy measure corresponding to $\tilde{N}$ resp. $L$ obeys the integrability condition **(QI)** with a given $q > \frac{2}{1-\zeta}$, where $\zeta \in [0,1)$ is as in **(A2)**. The coefficients $f$, $\sigma$ and $e$ fulfill the Lipschitz assumption **(LC)** and $c$ obeys (6.3). Furthermore, let $\xi^{(1)}$, $\xi^{(2)} \in L^2_\rho$ be as in **Case (A)**, i.e.*

$$\mathbf{E}||\xi^{(i)}||^q_{L^2_\rho} < \infty,$$

*and let $W$ be as in the nuclear or the cylindrical case. Then,*

***(i)*** $$\xi^{(1)} \leq \xi^{(2)}, \ P\text{-a.s.},$$

*and*

$$f^{(1)}(t,y) \leq f^{(2)}(t,y) \textit{ for all } (t,y) \in [\,0,T\,] \times \mathbb{R},\ P\text{-a.s.},$$

*imply*

$$X^{(1)}(t) \leq X^{(2)}(t),\ P\text{-a.s.},$$

*for all* $t \in [\,0,T\,]$, *where* $X^{(i)} \in \mathcal{H}^q(T)$, $i = 1,2$, *denotes the unique predictable mild solution to (6.1).*

***(ii)*** *Respectively,*

$$\xi^{(1)} \leq \xi^{(2)},\ P\text{-a.s.},$$

*and*

$$e^{(1)}(t,y) \leq e^{(2)}(t,y) \textit{ for all } (t,y) \in [\,0,T\,] \times \mathbb{R},\ P\text{-a.s.},$$

*imply*

$$X^{(1)}(t) \leq X^{(2)}(t),\ P\text{-a.s.},$$

*for all* $t \in [\,0,T\,]$, *where* $X^{(i)} \in \mathcal{H}^q(T)$, $i = 1,2$, *denotes the unique predictable mild solution to (6.2).*

**Remark 6.1.2:** *Assume that the evolution family* $(U(t,s))_{0\leq s\leq t\leq T}$ *fulfills **(A7)**.*
*Then, by Proposition 5.2.1 there are càdlàg solutions* $X^{(i)}$ *with* $X^{(i)}_{-} \in \mathcal{H}^q(T)$ *to each of the equations (6.1) and (6.2). Thus,*

$$P(\{\omega \in \Omega \,|\, X^{(1)}(t,\omega) \leq X^{(2)}(t,\omega)\, for\, all\, t \in [\,0,T\,]\}) = 1.$$

Both in the proof of **(i)** and **(ii)**, we use the following comparison method:

1. We show a comparison result for appropriate finite-dimensional approximations of $X^{(i)}$.

2. We show that these approximations tend to the solution to (6.1) resp. (6.2), which immediately implies the comparison result for the solutions to (6.1) and (6.2).

This will be the content of the Propositions 6.1.3 and 6.1.4 below.

So, let us first of all construct approximations for both equation (6.1) and (6.2).

**Equation (6.1)**

Given the Wiener process $(W(t))_{t\in[0,T]}$ from equation (6.1) introduced above, for $M \in \mathbb{N}$, we define a covariance operator $Q_M \in \mathcal{T}^+(L^2)$ by

$$Q_M\psi := \sum_{n=1}^{M} a_n < \psi, e_n >_{L^2} e_n$$

and the associated $Q_M$-Wiener process $(W_M(t))_{t\in[0,T]} \subset L^2$ by

$$W_M(t) := \sum_{n=1}^{M} \sqrt{a_n} w_n(t) e_n,\ t \in [0,T],$$

where $w_n(t) :=< W(t), e_n >_{L^2}$, $t \in [0,T]$. Obviously $(w_n)_{1\le n\le M}$ is a family of mutually independent real-valued Brownian motions.
Let $X_M^{(i)} \in \mathcal{H}^q(T)$ be a mild solution to (6.1) with $W_M$ substituting $W$, i.e.

$$\begin{aligned} dX_M^{(i)}(t) = {} & (A(t)X_M^{(i)}(t) + F^{(i)}(t, X_M^{(i)}(t)))dt \\ & + \mathcal{M}_{\Sigma(t,X_M^{(i)}(t))}\, dW_M(t) + \int_{L^2} \mathcal{M}_{C(t)}(x)\, \tilde{N}(dt,dx),\ t \in [0,T], \end{aligned}$$

(6.4)

$X_M^{(i)}(0) = \xi^{(i)}$.

By Definition 5.1.1, $X_M^{(i)}$ satisfies the following identity in $L^2_\rho$

$$\begin{aligned} X_M^{(i)}(t) = {} & U(t,0)\xi^{(i)} + \int_0^t U(t,s)F^{(i)}(s, X_M^{(i)}(s))\, ds \\ & + \int_0^t U(t,s)\mathcal{M}_{\Sigma(s,X_M^{(i)}(s))}\, dW_M(s) \\ & + \int_0^t \int_{L^2} U(t,s)\mathcal{M}_{C(s)} x\, \tilde{N}(ds,dx),\ P\text{-a.s., for any } t \in [0,T]. \end{aligned}$$

Next, we fix additionally $N \in \mathbb{N}$ and consider the equation

$$\begin{aligned} dX_{N,M}^{(i)}(t) = {} & (A_N(t)X_{N,M}^{(i)}(t) + F^{(i)}(t, X_{N,M}^{(i)}(t)))dt \\ & + \mathcal{M}_{\Sigma(t,X_{N,M}(t))}\, dW_M(t) + \int_{L^2} \mathcal{M}_{C(t)}(x)\, \tilde{N}(dt,dx),\ t \in [0,T], \end{aligned}$$

(6.5)

$X_{N,M}^{(i)}(0) = \xi^{(i)}$,

where $A_N(t) \in \mathcal{L}(L^2)$ approximates $A(t)$ in the sense of **(A6)**. Due to the boundedness of the operator $A_N(t)$ and the Lipschitz-continuity of all coefficients, equation (6.5) has a unique (mild=strong) solution $X_{N,M}^{(i)} \in \mathcal{H}^q(T)$.

For general results in infinite dimensions about equivalence of globally Lipschitz coefficients see e.g. [97] in the Wiener case, [60] in the Poisson case and Section 9 in [95] in the Lévy case.

Therefore, we have the following identity in $L^2_\rho$

$$\begin{aligned} X_{N,M}^{(i)}(t) = \quad & U_N(t,0)\xi^{(i)} + \int_0^t U_N(t,s)F^{(i)}(s, X_{N,M}^{(i)}(s))\,ds \\ & + \int_0^t U_N(t,s)\mathcal{M}_{\Sigma(s,X_{N,M}^{(i)}(s))}\,dW_M(s) \\ & + \int_0^t \int_{L^2} U_N(t,s)\mathcal{M}_{C(s)}x\,\tilde{N}(ds,dx),\ P\text{-a.s. for any } t \in [\,0,T\,]. \end{aligned}$$

Note that the existence and uniqueness of the $X_M^{(i)}$ and $X_{N,M}^{(i)}$ in $\mathcal{H}^q(T)$ follows from the general solvability results in the Lipschitz case (see Section 5.2, Theorem 5.2.1/Corollary 5.2.5), since $A_N$ and $W_M$ are only special cases of $A$ and $W$ from Sections 5.1/5.2.

The solutions can be constructed e.g. by Picard's iteration method (as in the proof of Theorem 5.2.2). Furthermore, under the above assumptions the classes of strong and mild solutions coincide (see e.g. [37], Sections 8 and 10).

## Equation (6.2)

Let us define approximations similar to those for the equation (6.1). We first note that by the integrability assumption **(QI)** on $\eta$, we have

$$\int_{L^2} ||x||_{L^2}^2\,\eta(dx) < \infty,$$

i.e. **(SI)** holds true.
Thus, we can apply the Lévy-Itô decomposition 2.4.13 in $L^2$ to get

$$L(t) = tm + W(t) + \int_{L^2} x\,\tilde{N}(t,dx),\ t \in [\,0,T\,], \tag{6.6}$$

with $W$ being a $Q$-Wiener process with $Q \in \mathcal{T}^+(L^2)$.

Recall from Section 2.3 that any $Q$-Wiener process $W$ obeys the representation

$$W(t) = \sum_{n\in\mathbb{N}} \sqrt{a_n} w_n(t) e_n,$$

where $(a_n)_{n\in\mathbb{N}} \subset \mathbb{R}_+$ is such that

$$\sum_{n\in\mathbb{N}} a_n < \infty,$$

$(w_n(t))_{t\in[0,T]}$, $n \in \mathbb{N}$, is a family of independent real-valued Brownian motions and $(e_n)_{n\in\mathbb{N}}$ is an orthonormal basis of $L^2$.
Note that, in contrast to equation (6.1), $(e_n)_{n\in\mathbb{N}}$ does not necessarily obey (3.1).

By (6.6), (6.2) becomes

$$\begin{aligned}(6.7)\; dX^{(i)}(t) = {} & (A(t)X^{(i)}(t) + E^{(i)}(t, X^{(i)}(t)) + \mathcal{M}_{C(t)}m)dt \\ & + \mathcal{M}_{C(t)}\, dW(t) + \int_{L^2} \mathcal{M}_{C(t)} x \tilde{N}(dt, dx),\ t \in [0, T] \\ X^{(i)}(0) = {} & \xi^{(i)}.\end{aligned}$$

Given $N \in \mathbb{N}$, let $A_N$ be a bounded operator in $L^2_\rho$ as in condition **(A6)**. For $M \in \mathbb{N}$, let $W_M$ be a finite-dimensional Wiener process as in the approximation of the equation (6.1). We denote by $X^{(i)}_{N,M}, X^{(i)}_M \in \mathcal{H}^q(T)$ the unique mild solutions to

$$\begin{aligned}(6.8)\; dX^{(i)}_{N,M}(t) = {} & (A_N(t)X^{(i)}_{N,M}(t) + E^{(i)}(t, X^{(i)}_{N,M}(t)) + \mathcal{M}_{C(t)}m)dt \\ & + \mathcal{M}_{C(t)}\, dW_M(t) + \int_{L^2} \mathcal{M}_{C(t)} x \tilde{N}(dt, dx),\ t \in [0, T] \\ X^{(i)}_{N,M}(0) = {} & \xi^{(i)},\end{aligned}$$

and

$$\begin{aligned}(6.9)\; dX^{(i)}_M(t) = {} & (A(t)X^{(i)}_M(t) + E^{(i)}(t, X^{(i)}_M(t)) + \mathcal{M}_{C(t)}m)dt \\ & + \mathcal{M}_{C(t)}\, dW_M(t) + \int_{L^2} \mathcal{M}_{C(t)} x \tilde{N}(dt, dx),\ t \in [0, T] \\ X^{(i)}_M(0) = {} & \xi^{(i)},\end{aligned}$$

existing by Corollary 5.2.5 **(i)**.

Having defined the approximations for both equation (6.1) and (6.2), we formulate the following lemmata, which will be proven in Sections 6.2 (Lemma 6.1.3) and 6.3 (Lemma 6.1.4).

**Lemma 6.1.3:** *Let $U$, $W$, the coefficients $f$, $\sigma$ and $e$, and the initial* *conditions $\xi^{(i)}$ be as in Theorem 6.1.1.*

***(i)*** *Let $\xi^{(i)}$, $i = 1, 2$, be as in **Case (A)**, and*

$$\xi^{(1)} \leq \xi^{(2)},\ P\text{-a.s.}.$$

*Furthermore, suppose that*

$$f^{(1)} \leq f^{(2)} \text{ for all } (t, y) \in [\,0, T\,] \times \mathbb{R},\ P\text{-a.s.}.$$

*Then, we have for the corresponding solutions $X^{(i)}_{N,M}$, $i = 1, 2$ of equation (6.5)*

$$X^{(1)}_{N,M}(t) \leq X^{(2)}_{N,M}(t),\ P\text{-a.s.},$$

*for any $t \in [\,0, T\,]$ and all $N, M \in \mathbb{N}$.*

***(ii)*** *Let $\xi^{(i)}$, $i = 1, 2$, as in **Case (A)**, and*

$$\xi^{(1)} \leq \xi^{(2)},\ P\text{-a.s.}.$$

*Furthermore, suppose that*

$$e^{(1)} \leq e^{(2)} \text{ for all } (t, y) \in [\,0, T\,] \times \mathbb{R},\ P\text{-a.s.}.$$

*Then, we have for the corresponding solutions $X^{(i)}_{N,M}$, $i = 1, 2$, of equation (6.8)*

$$X^{(1)}_{N,M}(t) \leq X^{(2)}_{N,M}(t),\ P\text{-a.s.},$$

*for all $t \in [\,0, T\,]$ and all $N, M \in \mathbb{N}$.*

**Lemma 6.1.4:** ***(i)*** *Considering $X^{(i)}$, $X^{(i)}_M$, $X^{(i)}_{N,M}$, $N, M \in \mathbb{N}$, $i = 1, 2$, as defined in (6.1), (6.4) and (6.5), we get the following convergence results:*

$$\lim_{N\to\infty} \mathbf{E}||X^{(i)}_{N,M}(t) - X^{(i)}_M(t)||^2_{L^2_\rho} = 0, \text{ for each } M \in \mathbb{N},$$
$$\lim_{M\to\infty} \mathbf{E}||X^{(i)}_M(t) - X^{(i)}(t)||^2_{L^2_\rho} = 0.$$

***(ii)*** *Considering $X^{(i)}$, $X^{(i)}_M$, $X^{(i)}_{N,M}$, $N, M \in \mathbb{N}$, $i = 1, 2$, as defined in (6.2), (6.9) and (6.8), we get the following convergence results:*

$$\lim_{N\to\infty} \mathbf{E}||X_{N,M}^{(i)}(t) - X_M^{(i)}(t)||_{L^2_\rho}^2 = 0, \textit{ for each } M \in \mathbb{N},$$

$$\lim_{M\to\infty} \mathbf{E}||X_M^{(i)}(t) - X^{(i)}(t)||_{L^2_\rho}^2 = 0.$$

*Thus, we first get*

$$X_{N,M}^{(1)}(t) \le X_{N,M}^{(2)}(t), \ P\text{-a.s.},$$

*for all $t \in [\,0, T\,]$ by 6.1.3* ***(i)*** *(for equation (6.1)) resp. by 6.1.3* ***(ii)*** *(for equation (6.2)). Then, by first letting $N \to \infty$ and then letting $M \to \infty$, we get 6.1.1* ***(i)*** *resp.* ***(ii)*** *by 6.1.4* ***(i)*** *(for equation (6.1)) resp. by 6.1.4* ***(ii)*** *(for equation (6.2)).*

## 6.2 Proof of Lemma 6.1.3

We adapt the proof of 3.3.2 from [76] to our situation.
Both in the proof of **(i)** and **(ii)**, the idea is to construct approximating processes by splitting $[\,0, T\,]$ into smaller intervalls of equal length. We show comparison results for those processes and conclude the required comparison result by letting the length of the subintervalls tend to 0.

The aim of such approximation is to separate stochastic and deterministic terms, which require quite different methods of analysis.

**(i)** For a fixed $j \in \mathbb{N}$, we set $t_k := \frac{kT}{j}$, $k = 0, 1, 2, ..., j$, and thus get a partition of $[\,0, T\,]$ into $j$ intervalls of length $\frac{T}{j}$. We define processes $Z_{k,j}^{(i)}$, $V_{k,j}^{(i)} \in \mathcal{H}^q([\,t_k, tk+1])$ in a recursive way by the following chain of identities holding $P$-almost surely

$$Z_{0,j}^{(i)}(t) := \xi^{(i)} + \int_0^t \mathcal{M}_{\Sigma(s, Z_{0,j}^{(i)}(s))}\, dW_M(s) + \int_0^t \int_{L^2} \mathcal{M}_{C(s)}(x)\, \tilde{N}(ds, dx), \tag{6.10}$$

$$V_{0,j}^{(i)}(t) := Z_{0,j}^{(i)}(t_1) + \int_0^t (A_N(s) V_{0,j}^{(i)}(s) + F^{(i)}(s, V_{0,j}^{(i)}(s)))\, ds,$$

for $t \in [\,0, t_1\,]$ and

$$Z_{k,j}^{(i)}(t) := V_{k-1,j}^{(i)}(t_k) + \int_{t_k}^t \mathcal{M}_{\Sigma(s, Z_{k,j}^{(i)}(s))}\, dW_M(s) + \int_{t_k}^t \int_{L^2} \mathcal{M}_{C(s)}(x)\, \tilde{N}(ds, dx), \tag{6.11}$$

$$V_{k,j}^{(i)}(t) := Z_{k,j}^{(i)}(t_{k+1}) + \int\limits_{t_k}^{t} (A_N(s)V_{k,j}^{(i)}(s) + F^{(i)}(s, V_{k,j}^{(i)}(s)))\, ds,$$

for $t \in [t_k, t_{k+1}]$ and $k = 1, 2, ..., j-1$.

Note that the processes $Z_{k,j}^{(i)}(t)$, $t \in [\,t_k, t_{k+1}\,]$, are described by the SDE driven by the Wiener process and Poisson noise.
For each $0 \le k \le j-1$, the equation for $Z_{k,j}^{(i)}(t)$ has a unique (up to modification) *strong* (and hence also *mild*) predictable solution in $\mathcal{H}^q([\,t_k, t_{k+1}\,])$, which is time-continuous in $L^q(\Omega; L^2_\rho)$.
This is guaranteed by the finite-dimensionality of $W_M$, the Lipschitz property of $\Sigma$, the boundedness of $C$ and the integrability property **(QI)** of the Lévy measure $\eta$ corresponding to $\tilde{N}$.
Then, a standard application of the Banach fixed point theorem (like in the proof of Theorem 5.2.1), as well as the Picard iteration method (like in the proof of Theorem 5.2.2), gives us the unique solvability result in $\mathcal{H}^q([\,t_k, t_{k+1}\,])$.

Let us recall that the Poisson integrals

$$\int\limits_{t_k}^{t} \int\limits_{L^2} \mathcal{M}_{C(s)}(x)\, \tilde{N}(ds, dx)$$

in the right hand sides of (6.10) and (6.11) are càdlàg by their definition. So, to get the versions of $Z_{k,j}^{(i)}$ from $\mathcal{H}^q([\,t_k, t_{k+1}\,])$, one has to take the predictable versions of the above integrals

$$\int\limits_{t_k}^{t-} \int\limits_{L^2} \mathcal{M}_{C(s)}(x)\, \tilde{N}(ds, dx) = \int\limits_{[t_k, t)} \int\limits_{L^2} \mathcal{M}_{C(s)}(x)\, \tilde{N}(ds, dx).$$

The $V$-terms are governed by deterministic equations but with random coefficients. Thus, the predictability of the integrand process is not essential for defining the corresponding Bochner/Lebesgue integrals in the right hand sides of (6.10) and (6.11).
Due to the boundedness of $A_N$ and the Lipschitz property of the $F^{(i)}$, there exists a unique pathwise continuous process $[\,t_k, t_{k+1}\,] \ni t \mapsto V_{k,j}^{(i)}(t) \in L^2_\rho$ solving (6.10), (6.11) $P$-almost surely.
Furthermore, $[\,t_k, t_{k+1}\,] \ni t \mapsto V_{k,j}^{(i)}(t) \in L^2_\rho$ is continuous in $L^q(\Omega; L^2_\rho)$.

Since $V_{k,j}^{(i)}(t)$ is $\mathcal{F}_{t_{k+1}}$-measurable for each $t \in [\,t_k, t_{k+1}\,]$, there exists an $\mathcal{F}_{t_{k+1}} \otimes \mathcal{B}([\,t_k, t_{k+1}\,])$-measurable modification of $V_{k,j}^{(i)}$. However, by its construction $V_{k,j}^{(i)}(t)$ is not $\mathcal{F}_t$-adapted and hence not predictable.

Next, we define $Z_j^{(i)}$, $V_j^{(i)}\colon\, \Omega \times [\,0, T\,] \to L^2_\rho$ by

$$
(6.12)\quad
\begin{aligned}
&Z_j^{(i)}(t) := Z_{k,j}^{(i)}(t),\ t \in [\,t_k, t_{k+1}\,),\ k = 0,1,2,...,j-1,\\
&V_j^{(i)}(0) := \xi^{(i)},\\
&V_j^{(i)}(t) := V_{k,j}^{(i)}(t),\ t \in (t_k, t_{k+1}],\ k = 0,1,...,j-1,\\
&Z_j^{(i)}(T) := V_j^{(i)}(T).
\end{aligned}
$$

One easily checks the following identities (holding $P$-almost surely)

$$
(6.13)\quad
\begin{aligned}
Z_j^{(i)}(t) = \;& \xi^{(i)} + \int_0^{t_k} (A_N(s)V_j^{(i)}(s) + F^{(i)}(s, V_j^{(i)}(s)))\, ds\\
&+ \int_0^t \int_{L^2} \mathcal{M}_{C(s)}(x)\, \tilde{N}(ds, dx)\\
&+ \int_0^t \mathcal{M}_{\Sigma(s, Z_j^{(i)}(s))}\, dW_M(s),
\end{aligned}
$$

for $t \in [\,t_k, t_{k+1}\,)$, $k = 0,1,...,j-1$, and

$$
(6.14)\quad
\begin{aligned}
V_j^{(i)}(t) = \;& \xi^{(i)} + \int_0^t (A_N(s)V_j^{(i)}(s) + F^{(i)}(s, V_j^{(i)}(s)))\, ds\\
&+ \int_0^{t_{k+1}} \mathcal{M}_{\Sigma(s, Z_j^{(i)}(s))}\, dW_M(s) + \int_0^{t_{k+1}} \int_{L^2} \mathcal{M}_{C(s)}(x)\, \tilde{N}(ds, dx),
\end{aligned}
$$

for $t \in (\,t_k, t_{k+1}\,]$, $k = 0,1,...,j-1$.

In particular, $Z_j^{(i)}(t_k) = V_j^{(i)}(t_k)$ for all $0 \le k \le j$.

Note that by this definition, the $V$-terms are of the same structure as in the proof of 3.3.2 in [76], whereas, compared to that proof, our $Z$-terms have an additional jump term (cf. equations (3.1) and (3.2), p. 63 in [76]). By construction, the processes $Z_j^{(i)}$ obey a càdlàg version on the whole intervall $[\,0, T\,]$, whereas the $V_j^{(i)}$'s are càglàd.

Again, we will take a predictable version of $Z_j^{(i)}(t)$, $t \in [\,0, T\,]$, as a limit value at $t-$ on the right hand side in (6.13).
In contrast, the process $V_j^{(i)}(t)$, $t \in [\,0, T\,]$, is not adapted, but obeys an $\mathcal{F}_T \otimes \mathcal{B}([\,0, T\,])$-measurable version.

The proof of Lemma 6.1.3 **(i)** will be splitted into the following two claims:

**Claim 1:** *For our processes defined in (6.10)/(6.11), we have in* $L^2_\rho$

$$V_j^{(1)}(t) \leq V_j^{(2)}(t), \tag{6.15}$$
$$Z_j^{(1)}(t) \leq Z_j^{(2)}(t),$$

*$P$-almost surely, for each $t \in [\,0, T\,]$.*

**Proof:** Let us start with the intervall $[\,0, t_1\,)$.
By (6.13), we have

$$Z_j^{(i)}(t) = Z_{0,j}^{(i)}(t) = \xi^{(i)} + \int_0^t \mathcal{M}_{\Sigma(s, Z_j^{(i)}(s))}\, dW_M(s) + \int_0^t \int_{L^2} \mathcal{M}_{C(s)}(x)\, \tilde{N}(ds, dx), \tag{6.16}$$

for any $t \in [\,0, t_1\,)$.

For a moment, we consider this equation on the whole intervall $[\,0, t_1\,]$. As already mentioned before, it has a unique predictable solution $Z_j^{(i)}(t)$, $t \in [\,0, t_1\,]$.
Thus, by Proposition 5.1.4, similarly to the approximation procedure from Step 1 in the proof of Proposition 3.4.3, we can find $\mathcal{P}_{t_1} \otimes \mathcal{B}(\Theta)$-measurable realizations of the mappings

$$[\,0, t_1\,) \times \Omega \times \Theta \ni (t, \omega, \theta) \mapsto Z_j^{(i)}(t, \omega, \theta) \in \mathbb{R},\ i = 1, 2.$$

We prove the required comparison on $[\,0, t_1\,)$ with the help of Itô's formula applied to a localization of (6.16).

To this end, we take $\mathcal{F}_0 \otimes \mathcal{B}(\Theta)$-measurable realizations of $\xi^{(i)}(\omega, \theta)$ and $\mathcal{P}_{t_1} \otimes \mathcal{B}(\Theta)$-measurable (i.e. predictable) realizations of both the Wiener (cf. the proof of Proposition 3.4.3 in Section 3.4) and the Poisson integral

$$\left( \int_0^t \int_{L^2} \mathcal{M}_{C(s)}(x)\, \tilde{N}(ds, dx) \right)(\omega, \theta) = \int_0^t \int_{L^2} c(s, \omega) x(\theta)\, \tilde{N}(ds, dx).$$

In both cases, the identity holds in $L^2(\Omega; L^2_\rho)$. For the Poisson integral, such realizations exist by Lemma 5.1.8 and Step 3 in the proof of Proposition 4.3.
Thus, (6.16) can be interpreted (see also the proof of Theorem 3.3.1 in [76]) as a family of one-dimensional SDEs

$$\begin{aligned} Z_j^{(i)}(t, \theta) = \quad & \xi^{(i)}(\theta) + \sum_{n=1}^{M} \sqrt{a_n} \int_0^t \sigma(s, Z_j^{(i)}(s, \theta)) e_n(\theta)\, dw_n(s) \\ & + \int_0^t \int_{L^2} c(s, \omega) x(\theta)\, \tilde{N}(ds, dx),\ P\text{-a.s., for } t \in [\,0, t_1\,], \end{aligned} \tag{6.17}$$

where $\theta$ is from some subset $\Theta_0 \in \mathcal{B}(\Theta)$ of full $\mu_\rho$-measure.

For each fixed $\theta \in \Theta_0$, equation (6.17) has a unique predictable solution $Z_j^{(i)}(t,\theta)$, $t \in [\,0,t_1\,]$, such that

$$\sup_{t\in[\,0,t_1\,]} \mathbf{E}|Z_j^{(i)}(t,\theta)|^2 < \infty.$$

Since the coefficient $\sigma(s,\omega,y)$ is Lipschitz continuous in the third variable, this solution can be constructed by the Picard iteration method (as it was done in the proof of Theorem 5.2.2 for the equation (1.1)).
Setting

$$Z_j^{(i,0)}(t,\theta) = \xi^{(i)}(\theta),\; t \in [\,0,t_1\,],$$

and

$$\begin{aligned} Z_j^{(i,n)}(t,\theta) := \quad & Z_j^{(i,0)}(t) + \sum_{n=1}^{M} \sqrt{a_n} e_n(\theta) \int_0^t \sigma(s, Z_j^{(i,n-1)}(t,\theta))\, dw_n(s) \\ & + \int_0^t \int_{L^2} c(s,\omega)x(\theta)\, \tilde{N}(ds,dx),\; t \in [\,0,t_1\,],\; n \in \mathbb{N}, \end{aligned}$$

we get a sequence of processes $(Z_j^{(i,n)}(t,\theta))_{\theta\in\Theta_0, t\in[\,0,t_1\,]}$, $n \in \mathbb{N}$, which obey $\mathcal{P}_{t_1} \otimes \mathcal{B}(\Theta)$-measurable versions (see Proposition A.1 in [13]).
Let us fix $n \in \mathbb{N}$. For any $t \in [\,0,t_1\,]$ and $\theta \in \Theta_0$, by (3.1) and the Lipschitz property **(LC)** for $\sigma$ we get

$$\begin{aligned} &(6.18)\ \mathbf{E}|Z_j^{(i,n+1)}(t,\theta) - Z_j^{(i,n)}(t,\theta)|^2 \\ &= \sum_{n=1}^{M} a_n |e_n(\theta)|^2 \int_0^t \mathbf{E}|\sigma(s, Z_j^{(i,n-1)}(t,\theta)) - \sigma(s, Z_j^{(i,n-1)}(t,\theta))|^2\, dw_n(s) \\ &\le \left(\sum_{n=1}^{M} a_n\right) \left(\sup_{n\in\mathbb{N}} ||e_n||_\infty^2\right) c_\sigma^2(T) \int_0^t \mathbf{E}|Z_j^{(i,n-1)}(t,\theta) - Z_j^{(i,n-1)}(t,\theta)|^2\, ds. \end{aligned}$$

Herefrom, by the Gronwall-Bellman lemma 2.7.2 we conclude that

$$\begin{aligned} &\sup_{t\in[\,0,t_1\,]} \mathbf{E}|Z_j^{(i,n+1)}(t,\theta) - Z_j^{(i,n)}(t,\theta)|^2 \\ &\le t_1^n q_n c(c_\sigma(T)) \sup_{t\in[\,0,t_1\,]} \mathbf{E}|Z_j^{(i,1)}(t,\theta) - Z_j^{(i,0)}(t,\theta)|^2 \end{aligned}$$

with

$$\begin{aligned} &\sup_{t\in[\,0,t_1\,]} \mathbf{E}|Z_j^{(i,1)}(t,\theta) - Z_j^{(i,0)}(t,\theta)|^2 \\ &\le \sup_{t\in[\,0,t_1\,]} \mathbf{E}\left|\sum_{n=1}^{M} \sqrt{a_n} \int_0^t \sigma(s,\xi(\theta)) e_n(\theta)\, dw_n(s)\right|^2 \\ &= \sup_{t\in[\,0,t_1\,]} \sum_{n=1}^{M} a_n \int_0^t \mathbf{E}|\sigma(s,\xi(\theta)) e_n(\theta)|^2\, ds \end{aligned}$$

$$\leq \left(\sum_{n=1}^{M} a_n\right)\left(\sup_{n\in\mathbb{N}} ||e_n||_\infty^2\right) c(T, c_\sigma(T))(1+\mathbf{E}|\xi(\theta)|^2).$$

Thus, we finally get

$$\begin{aligned}\sup_{t\in[\,0,t_1\,]} \mathbf{E}|Z_j^{(i,n+1)}(t,\theta) - Z_j^{(i,n)}(t,\theta)|^2 \leq &\quad t_1^n q_n c(c_\sigma(T))c(T, c_\sigma(T))(1+\mathbf{E}|\xi(\theta)|^2) \\ < &\quad \infty.\end{aligned}$$

Here, $(q_n)_{n\in\mathbb{N}}$ is gained from (6.18) as described in 2.7.2. Note that

$$\int_\Theta c(T, c_\sigma(T))(1+\mathbf{E}|\xi(\theta)|^2)\,\mu_\rho(d\theta) < \infty.$$

Since, in view of the Gronwall-Bellman lemma 2.7.2, we have

$$\sum_{n\in\mathbf{N}} t_1^n q_n < \infty$$

and $t_1^n q_n \to 0$ as $n\to\infty$, and since $\xi^{(i)}$ is a $\mathcal{F}_0 \otimes \mathcal{B}(\Theta)$-measurable version of the initial condition, we get the existence of $Z_j^{(i)}(t,\theta)$, $t\in[\,0,t_1\,]$, for $\mu_\rho$-almost all $\theta\in\Theta$, as the limit of the processes $Z_j^{(i,n)}(t,\theta)$, $n\in\mathbb{N}$, in $L^2(\Omega;\mathbb{R})$.
Furthermore, by similar estimates we have the convergence of $Z_j^{(i)}(t)$ in $L^2(\Omega; L^2_\rho)$ uniformly in $t\in[\,0,t_1\,]$. Applying pairing with cylinder functions (as in Step 3 in the proof of Proposition 3.4.3 resp. 4.3), we can conclude that $Z_j^{(i)}(t)$ solves (6.16) in $L^2_\rho$.

On the other hand, $t\mapsto Z_j^{(i)}(t)\in L^2_\rho$ is a predictable solution to (6.16), which is unique up to modifications in $\mathcal{H}^q(T)$. So, $Z_j^{(i)}(t,\theta)$ is a $\mathcal{P}_{t_1}\otimes\mathcal{B}(\Theta)$-measurable realization of $Z_j^{(i)}(t)\in L^2_\rho$.
Let us consider the difference processes. We have

$$\begin{aligned}(6.19)\ \Delta_j^\theta Z(t) := &\quad Z_j^{(1)}(t,\theta) - Z_j^{(2)}(t,\theta) \\ = &\quad \xi^{(1)}(\theta) - \xi^{(2)}(\theta) \\ &\quad + \sum_{n=1}^{M} \sqrt{a_n} \int_0^t (\sigma(s, Z_j^{(1)}(s,\theta)) - \sigma(s, Z_j^{(2)}(s,\theta)))e_n(\theta)\,dw_n(s),\end{aligned}$$

$P$-almost surely for any $\theta\in\Theta_0$ and $t\in[\,0,t_1\,]$. The right hand side in (6.19) obeys a pathwise time-continuous modification by the standard properties of Wiener integrals.
Thus, we can apply the finite-dimensional Itô's formula (see e.g. Section II.5 in [53]), which gives us for any $C_b^2$-function $\psi$: $\mathbb{R}\to\mathbb{R}$

$$(6.20)\ \psi(\Delta_j^\theta(t))$$

$$= \psi(\Delta_j^\theta(0)) + \int\limits_0^t \psi'(\Delta_j^\theta(s))\, d\Delta_j^\theta(s)$$
$$+ \tfrac{1}{2}\int\limits_0^t \psi''(\Delta_j^\theta(s))\, d{<}\, \Delta_j^\theta(s) \,{>}$$
$$= \sum_{n=1}^M \sqrt{a_n}\int\limits_0^t \psi'(\Delta_j^\theta(s))[\sigma(s, Z^{(1)}(s,\theta)) - \sigma(s, Z^{(2)}(s,\theta))]e_n(\theta)\, dw_n(s)$$
$$+ \tfrac{1}{2}\sum_{n=1}^M \int\limits_0^t \psi''(\Delta_j^\theta(s))[\sigma(s, Z^{(1)}(s,\theta)) - \sigma(s, Z^{(2)}(s,\theta))]^2 e_n^2(\theta)\, ds,$$

for all $t \in [\,0, t_1\,)$, $P$-almost surely.
To make a proper choice of $\psi$, we first need some technical preparations, which can e.g. be found in Sections IV. 3 and VI.1 in [53].
As a Lipschitz function, the diffusion coefficient $\Sigma$ obeys the estimats

$$|\sigma(t,\omega,\xi) - \sigma(t,\omega,\theta)| \le h(|\xi - \theta|),\ \xi,\theta \in \mathbb{R},$$

uniformly on $[\,0,T\,] \times \Omega$ with $h(z) = c_\sigma(T)z$, $z > 0$.
Furthermore, $h$ fulfills $h(0) = 0$ and

$$\int\limits_{(\,0,\varepsilon\,)} h^{-2}(u)\, du = \infty \text{ for any } \varepsilon > 0.$$

Thus, we can find a strictly decreasing sequence $(a_l)_{l\in\mathbb{N}} \subset (\,0,1\,]$ with

$$a_0 = 1,\ \lim_{l\to\infty} a_l = 0 \text{ and } \int\limits_{a_l}^{a_{l-1}} h^{-2}(u)\, du = l \text{ for any } l \in \mathbb{N}.$$

For each $l$, there is a continuous function $\alpha_l$ with support in $(a_l, a_{l-1})$ such that

$$0 \le \alpha_l(z) \le \tfrac{2}{lh^2(z)} \text{ and } \textstyle\int_{a_l}^{a_{l+1}} \alpha_l(z)\, dz = 1.$$

Defining

$$\psi_l(z) := 0,\ z \le 0,$$

and

$$\psi_l(z) := \int\limits_0^z \int\limits_0^y \alpha_l(u) du dy \ge 0,\ z > 0,$$

gives us a nondecreasing sequence $(\psi_l)_{l\in\mathbb{N}}$ of $C_b^2$ functions with $0 \le \psi_l'(z) \le 1$ and $\lim_{l\to\infty} \psi_l(z) = z^+ := z \vee 0$ for each $z \in \mathbb{R}$.
Furthermore, we have

$$|\psi_l''(z)| = |\alpha_l(z)| \le \tfrac{2}{lh^2(z)},\ z > 0. \tag{6.21}$$

Now we substitute into (6.20) the functions $\psi_l$, $l \in \mathbb{N}$, constructed above. Note that, for all $1 \le n \le M$ and $t \in [0, t_1)$, we have

$$(6.22)\ \mathbf{E}\int_0^t |\sigma(s, Z_j^{(i)}(s,\theta))e_n(\theta)|^2\, ds$$
$$\le \left(\sup_{n\in\mathbb{N}} ||e_n||_\infty\right)^2 Tc_\sigma^2(T)\left(1 + \mathbf{E}\int_0^t |Z_j^{(i)}(s,\theta)|^2\, ds\right)$$
$$< \infty$$

for $i = 1, 2$ and $\mu_\rho$-almost all $\theta \in \Theta$.
Without loss of generality, we may assume (6.22) to hold for all $\theta \in \Theta_0 \in \mathcal{B}(\Theta)$ with some $\Theta_0$ having full $\mu_\rho$-measure. Furthermore, we assume that, for all $\theta \in \Theta_0$, $\xi^{(1)}(\theta) \le \xi^{(2)}(\theta)$, $P$-a.s., and $\mathbf{E}|\xi^{(1)}(\theta)| < \infty$, $\mathbf{E}|\xi^{(2)}(\theta)| < \infty$.

Thus, the stochastic integral

$$\int_0^t \psi_j'(\Delta_j^\theta)\sigma(s, Z_j^{(i)})e_n(\theta)dw_n(s)$$

is well-defined for all $\theta \in \Theta_0$. For $t \in [0, t_1)$ and $1 \le n \le M$, we have

$$\mathbf{E}\int_0^t \psi_l'(\Delta_j^\theta(s))(\sigma(s, Z_j^{(2)}(s,\theta)) - \sigma(s, Z_j^{(1)}(s,\theta)))e_n(\theta)\, dw_n(s) = 0$$

and (by (6.21))

$$(6.23)\ \mathbf{E}\sum_{n=1}^M \int_0^t \psi_l''(\Delta_j^\theta(s))(\sigma(s, Z_j^{(1)}(s,\theta)) - \sigma(s, Z_j^{(2)}(s,\theta)))^2 e_n^2(\theta)\, ds$$
$$\le \left(\sup_{n\in\mathbb{N}} ||e_n||_\infty\right)^2 M\mathbf{E}\int_0^t \psi_l''(\Delta_j^\theta(s))h^2(\Delta_j^\theta(s))\, ds$$
$$\le \tfrac{1}{l}2Mt\left(\sup_{n\in\mathbb{N}} ||e_n||_\infty^2\right).$$

Substituting these estimates in the right hand side of (6.20), we obtain

$$\mathbf{E}\psi_l(\Delta_j^\theta(t)) \le \mathbf{E}\psi_l(\Delta_j^\theta(0)) + \tfrac{1}{l}\left(\sup_{n\in\mathbb{N}} ||e_n||_\infty^2\right)Mt.$$

Thus, we get $\lim\limits_{l\to\infty} \mathbf{E}\psi_l(\Delta_j^\theta(t)) \le \lim\limits_{l\to\infty} \mathbf{E}\psi_l(\xi^{(1)}(\theta) - \xi^{(2)}(\theta))$.

By construction we have

$$0 \le \psi_l(z) \uparrow z^+ \text{ as } l \to \infty,$$

which by B.Levi's monotone convergence theorem implies
$\mathbf{E}[(\Delta_j^\theta(t))^+] \le \mathbf{E}[(\xi^{(1)}(\theta) - \xi^{(2)}(\theta))^+]$ for all $\theta \in \Theta_0$ and $t \in [\,0, t_1\,)$.

Since for $\theta \in \Theta_0$, by our assumption, we have $\xi^{(1)}(\theta) \le \xi^{(2)}(\theta)$, $P$-almost surely, we get $\Delta_j^\theta(t) \le 0$, $P$-almost surely, for all $t \in [\,0, t_1\,)$. Herefrom, using the time-continuity of $\Delta_j^\theta(t)$, we conclude that for $\mu_\rho$-almost all $\theta \in \Theta$

$$P(\{\omega \in \Omega \,|\, Z_j^{(1)}(t,\theta) \le Z_j^{(2)}(t,\theta)\, for\, all\, t \in [\,0, t_1\,)\}) = 1.$$

Finally, taking into account the $\mathcal{P}_{t_1} \otimes \mathcal{B}(\Theta)$-measurability of $Z_j^{(i)}(t,\theta)$ and the continuity of the map $[\,0, t_1\,) \ni t \mapsto Z_j^{(1)}(t) - Z_j^{(2)}(t) \in L^2_\rho$, we conclude that

$$P(\{\omega \in \Omega \,|\, Z_j^{(1)}(t) \le Z_j^{(2)}(t)\, in\, L^2_\rho\, for\, all\, t \in [\,0, t_1\,)\}) = 1,$$

which proves that $Z_j^{(1)}(t) \le Z_j^{(2)}(t)$ in $L^2_\rho$ for all $t \in [\,0, t_1\,)$ $P$-almost surely. Similarly, we prove that $Z_{0,j}^{(1)}(t_1) \le Z_{0,j}^{(2)}(t_1)$ $P$-almost surely.

Next, we consider $V_j^{(i)}(t)$ on $[\,0, t_1\,]$.
We will continue to follow the lines of proof of 3.3.2 from [76] and use the analytical tools from operator theory applied there.
Note that the $V$-terms in our proof coincide with the ones from the proof of Lemma 3.3.2 from [76]. The sole difference is that the authors in [76] only consider the case of $\omega$-independent coefficients. Nevertheless, the proof can be done by the same arguments as in [76].
Obviously, we have $V_j^{(1)}(0) = \xi^{(1)} \le \xi^{(2)} = V_j^{(2)}(0)$, $P$-almost surely.
For $V_j^{(i)}(t)$ with $t \in [\,0, t_1\,)$, we have the following deterministic integral equation (with random coefficients) in $L^2_\rho$

$$V_j^{(i)}(t) = Z_{0,j}^{(i)}(t_1) + \int_0^t A_N(s) V_j^{(i)}(s) + F^{(i)}(s, V_j^{(i)}(s))\, ds. \tag{6.24}$$

Since $[\,0, t_1\,) \ni t \mapsto V_j^{(i)}(t) \in L^2_\rho$ is pathwise continuous, one finds a universal subset $\Omega_0 \subset \Omega$ of full $P$-measure such that (6.24) holds for all $\omega \in \Omega_0$, $t \in [\,0, t_1\,)$ and $i = 1, 2$. Without loss of generality, we may assume that $\xi^{(2)} \ge \xi^{(1)}$ and $f^{(2)}(t,\omega,\theta) \ge f^{(1)}(t,\omega,\theta)$ on $[\,0, T\,] \times \mathbb{R}$, for all $\omega \in \Omega_0$.
Next, we fix $\omega \in \Omega_0$ and define a linear operator $B(t) \in \mathcal{L}(L^2_\rho)$ for $t \in [\,0, T\,]$ by

$$B(t)\varphi := \tfrac{F^{(2)}(t, V_j^{(2)}(t)) - F^{(2)}(s, V_j^{(1)}(t))}{V_j^{(2)}(t) - V_j^{(1)}(t)}\,\varphi,\ \varphi \in L^2_\rho,$$

in the case $V_j^{(2)}(t) \ne V_j^{(1)}(t)$ and

$$B(t)\varphi := C_F(T)\varphi,\ \varphi \in L^2_\rho,$$

otherwise. Here, $C_F(T)$ denotes the common Lipschitz-constant of $f^{(1)}$ and $f^{(2)}$, i.e.

$$C_F(T) := \max(c_{f^{(1)}}(T), c_{f^{(2)}}(T)).$$

Obviously, $B(t)$ is a bounded operator in $L^2_\rho$, whose operator norm is less than $C_F(T)$.
From Step 1(ii) in the proof of Lemma 3.3.2 in [76], we know that (with the help of [24])

$$\bar{A}_N(t) := A_N(t) + B(t),\ t \in [\,0, T\,], \tag{6.25}$$

generates a *positivity preserving* evolution operator $\bar{U}_N$ in $L^2_\rho$.

By the definition of the $V$-terms we have, for all $t \in [\,0, t_1\,]$,

$$\begin{aligned}\Delta_j V(t) := V_j^{(2)}(t) - V_j^{(1)}(t) = \quad & Z_{0,j}^{(2)}(t_1) - Z_{0,j}^{(1)}(t_1) + \int_0^t \bar{A}_N(s)(V_j^{(2)}(t) - V_j^{(1)}(t))\,ds \\ & + \int_0^t (F^{(2)}(s, V_j^{(1)}(s)) - F^{(1)}(s, V_j^{(1)}(s)))\,ds.\end{aligned}$$

Next, following the lines of [76], we rewrite the above equation in the mild form, using the evolution family $\bar{U}_N$. Thus, for all $t \in (\,0, t_1]$,

$$\begin{aligned}V_j^{(2)}(t) - V_j^{(1)}(t) = \quad & \bar{U}_N(t,0)(Z_{0,j}^{(2)}(t_1) - Z_{0,j}^{(1)}(t_1)) \\ & + \int_0^t \bar{U}_N(t,s)[F^{(2)}(s, V_j^{(1)}(s)) - F^{(1)}(s, V_j^{(1)}(s))]\,ds.\end{aligned}$$

Recall that $Z_{0,j}^{(i)}(t_1) = Z_j^{(i)}(t_1) = V_j^{(i)}(t_1)$ and, by the previous arguments, $Z_{0,j}^{(1)}(t_1) \leq Z_{0,j}^{(2)}(t_1)$, $P$-almost surely.
Recall that $\bar{U}_N$ is positivity preserving and $f^{(2)}(\omega) \geq f^{(1)}(\omega)$ on $[\,0, T\,] \times \mathbb{R}$, for all $\omega \in \Omega_0$. Thus, we immediately get $V_j^{(2)}(t,\omega) \geq V_j^{(1)}(t,\omega)$ for each $t \in (\,0, t_1]$ and all $\omega \in \Omega_0$.
Since $V_j^{(2)}(0,\omega) - V_j^{(1)}(0,\omega) = \xi^{(2)}(\omega) - \xi^{(1)}(\omega) \geq 0$ for all $\omega \in \Omega_0$, we can thus conclude that $V_j^{(2)}(t,\omega) \geq V_j^{(1)}(t,\omega)$ for each $t \in [\,0, t_1\,]$ and all $\omega \in \Omega_0$.

We also have $Z_j^{(2)}(t_1) = V_j^{(2)}(t_1) \geq V_j^{(1)}(t_1) = Z_j^{(1)}(t_1)$, $P$-almost surely, which yields

$$Z_j^{(1)}(t) \leq Z_j^{(2)}(t)\ P\text{-almost surely on } [\,0, t_1\,].$$

Claim 1 follows by the same arguments on each interval $[\,t_k, t_{k+1}\,]$. $\square$

We finish the proof of **(i)** by the following claim.

**Claim 2:** *For $i = 1, 2$*

$$\lim_{j\to\infty} Z_j^{(i)} = X_{N,M}^{(i)}$$

*in $\mathcal{H}^2(T)$, i.e.*

$$\lim_{j\to\infty} \sup_{t\in[0,T]} \mathbf{E}||Z_j^{(i)}(t) - X_{N,M}^{(i)}(t)||_{L^2_\rho}^2 = 0.$$

*Hence, inequality (6.15) (cf. Claim 1) implies that*
*$X_{N,M}^{(1)}(t) \le X_{N,M}^{(2)}(t)$, $P$-almost surely, for all $t \in [0, T]$.*

**Proof:** Recall that $Z_j^{(i)}$ and $X_{N,M}^{(i)}$ are $\mathcal{H}^q(T)$-valued for all $j, N, M \in \mathbb{N}$ and $i = 1, 2$.
Since $\mathcal{H}^q(T) \subset \mathcal{H}^2(T)$ for any $q \ge 2$, it is enough to consider the processes in this setting.
Furthermore, it is enough to take any modification of the processes $V_j^{(i)}$, $Z_j^{(i)}$ and $X_{N,M}^{(i)}$, because all estimates will be in the meansquare sense.

First, let us note some a-priori estimates for the above processes:

By the moment estimate in the Lipschitz case (cf. Corollary 5.2.5), we get

$$c_{N,M} := \max_{i=1,2} \sup_{t\in[0,T]} \mathbf{E}||X_{N,M}^{(i)}(t)||_{L^2_\rho}^2 < \infty.$$

For arbitrary $j \in \mathbb{N}$ and $i = 1, 2$, we have

$$\begin{aligned} \sup_{t\in[0,T]} \mathbf{E}||V_j^{(i)}(t)||_{L^2_\rho}^2 &\le c\left(\mathbf{E}||\xi^{(i)}||_{L^2_\rho}^2 + \sum_{k=0}^{j-1} \sup_{t\in[t_k,t_{k+1}]} \mathbf{E}||V_{k,j}^{(i)}(t)||_{L^2_\rho}^2\right) \\ &\le c_V(j) < \infty \end{aligned}$$

and similarly

$$\sup_{t\in[0,T]} \mathbf{E}||Z_j^{(i)}(t)||_{L^2_\rho}^2 \le c_Z(j) < \infty.$$

So,

$$(6.26) \quad \sup_{t\in[0,T]} \mathbf{E}[\,||V_j^{(i)}(t) - X_{N,M}^{(i)}(t)||_{L^2_\rho}^2 + ||Z_j^{(i)}(t) - X_{N,M}^{(i)}(t)||_{L^2_\rho}^2\,]$$
$$\le c_{V,Z}(j, N, M) < \infty,\ i = 1, 2.$$

From the definitions (6.13) and (6.14) of $Z_j^{(i)}$ and $V_j^{(i)}$, $i = 1, 2$, we have

$$(6.27)\ Z_j^{(i)}(t) = \quad V_j^{(i)}(t_k) + \int_{t_k}^{t} \mathcal{M}_{\Sigma(s, Z_j^{(i)}(s))}\, dW_M(s) + \int_{t_k}^{t} \int_{L^2} \mathcal{M}_{C(s)} x\, \tilde{N}(ds, dx),\ i = 1, 2,$$

for any $t \in [\, t_k, t_{k+1}\, )$ and $0 \leq k \leq j - 1$.
Furthermore, taking the solutions $X_{N,M}^{(i)}$, $i = 1, 2$, in the strong form, we have

$$(6.28)\ X_{N,M}^{(i)}(t) = \quad X_{N,M}^{(i)}(t_k) + \int_{t_k}^{t} (A_N(s) X_{N,M}^{(i)}(s) + F^{(i)}(s, X_{N,M}^{(i)}(s)))\, ds + \int_{t_k}^{t} \mathcal{M}_{\Sigma(s, \cdot, X_{N,M}^{(i)}(s))}\, dW_M(s) + \int_{t_k}^{t} \int_{L^2} \mathcal{M}_{C(s)}(x)\, \tilde{N}(ds, dx)$$

for any $t \in [\, t_k, t_{k+1}\, )$.

This allows us to express $Z_j^{(i)}(t) - X_{N,M}^{(i)}(t)$ in terms of
$V_j^{(i)}(t) - X_{N,M}^{(i)}(t)$.

By the isometries for stochastic integrals w.r.t. Wiener processes and Poisson random measures, assumption **(QI)** on the Lévy measure corresponding to $\tilde{N}$, Lipschitz property **(LC)** for both $f^{(i)}$ and $\sigma$, the boundedness property **(LB)** for $\sigma$ and the boundedness of the operator $A_N$, we get

$$\begin{aligned}
&\mathbf{E}||V_j^{(i)}(t) - X_{N,M}^{(i)}(t)||_{L^2_\rho}^2 \\
&= \mathbf{E}\Bigg|\Bigg| \xi^{(i)} + \int_0^t (A_N(s) V_j^{(i)}(s) + F^{(i)}(s, V_j^{(i)}(s)))\, ds + \int_0^{t_{k+1}} \mathcal{M}_{\Sigma(s, Z_j^{(i)}(s))}\, dW_M(s) \\
&+ \int_0^{t_{k+1}} \int_{L^2} \mathcal{M}_{C(s)}(x)\, \tilde{N}(ds, dx) \\
&- \Bigg( \xi^{(i)} + \int_0^t (A_N(s) X_{N,M}^{(i)}(s) + F^{(i)}(s, X_{N,M}^{(i)}(s)))\, ds + \int_0^t \mathcal{M}_{\Sigma(s, X_{N,M}^{(i)}(s))}\, dW_M(s) \\
&+ \int_0^t \mathcal{M}_{C(s)}(x)\, \tilde{N}(ds, dx) \Bigg) \Bigg|\Bigg|_{L^2_\rho}^2 \\
&\leq c(C(T), c(N), C_{2,\eta}) \Bigg( \sum_{n=1}^{M} \Bigg[ \int_0^{t_{k+1}} \mathbf{E}||(\mathcal{M}_{\Sigma(s, Z_j^{(i)}(s))} - \mathcal{M}_{\Sigma(s, X_{N,M}^{(i)}(s))})(e_n)||_{L^2_\rho}^2\, ds \\
&+ \int_t^{t_{k+1}} \mathbf{E}||\mathcal{M}_{\Sigma(s, X_{N,M}^{(i)}(s))}(e_n)||_{L^2_\rho}^2\, ds \Bigg] + \int_t^{t_{k+1}} \mathbf{E}||C(s)||_{L^2_\rho}^2\, ds \\
&+ \int_0^t \mathbf{E}||V_j^{(i)}(s) - X_{N,M}^{(i)}(s)||_{L^2_\rho}^2\, ds \Bigg)
\end{aligned}$$

$$
\begin{aligned}
&\le c(T,M,c(N),C_F(T),c_\sigma(T),C_{2,\eta})\left[\int\limits_0^{t_{k+1}}\mathbf{E}||Z_j^{(i)}(s)-X_{N,M}^{(i)}(s)||^2_{L^2_\rho}\,ds\right.\\
&+(t_{k+1}-t_k)\left(1+\sup_{r\in[0,T]}\mathbf{E}||X_{N,M}^{(i)}(r)||^2_{L^2_\rho}\right)+(t_{k+1}-t_k)K^2T\mu_\rho(\Theta)\\
&\left.+\int\limits_0^t\mathbf{E}||V_j^{(i)}(s)-X_{N,M}^{(i)}(s)||^2_{L^2_\rho}\,ds\right]\\
&\le c(T,M,K,c(N),C_F(T),c_\sigma(T),C_{2,\eta})\left[\int\limits_0^{t_{k+1}}\mathbf{E}||Z_j^{(i)}(s)-X_{N,M}^{(i)}(s)||^2_{L^2_\rho}\,ds\right.\\
&\left.+(t_{k+1}-t_k)\left(1+\sup_{r\in[0,T]}\mathbf{E}||X_{N,M}^{(i)}(r)||^2_{L^2_\rho}\right)+\int\limits_0^t\mathbf{E}||V_j^{(i)}(s)-X_{N,M}^{(i)}(s)||^2_{L^2_\rho}\,ds\right]\\
&=:c(T,M,K,c(N),C_F(T),c_\sigma(T),C_{2,\eta})\left[B_j(t_{k+1})+\int\limits_0^t\mathbf{E}||V_j^{(i)}(s)-X_{N,M}^{(i)}(s)||^2_{L^2_\rho}\,ds\right]
\end{aligned}
$$

for $t\in(\,t_k,t_{k+1}\,]$ and $k\in\{0,1,\dots,j-1\}$. With the help of (6.26) the term $B_j$ can be estimated by

$$
\begin{aligned}
&B_j(t_{k+1})\\
&:=\int\limits_0^{t_{k+1}}\mathbf{E}||Z_j^{(i)}(s)-X_{N,M}^{(i)}(s)||^2_{L^2_\rho}\,ds+(t_{k+1}-t_k)\left(1+\sup_{r\in[0,T]}\mathbf{E}||X_{N,M}^{(i)}(r)||^2_{L^2_\rho}\right)\\
&\le T\sup_{t\in[0,T]}\mathbf{E}[\,||V_j^{(i)}(t)-X_{N,M}^{(i)}(t)||^2_{L^2_\rho}+||Z_j^{(i)}(t)-X_{N,M}^{(i)}(t)||^2_{L^2_\rho}\,]\\
&+\tfrac{T}{j}\left(1+\sup_{t\in[0,T]}\mathbf{E}||X_{N,M}^{(i)}(t)||^2_{L^2_\rho}\right)\\
&\le T\left[c_{V,Z}(j,\xi)+\tfrac{1}{j}(1+c(\xi^{(i)}))\right]\\
&<\infty.
\end{aligned}
$$

Now we can apply Gronwall's lemma, which leads to

$$
(6.29)\ \mathbf{E}||V_j^{(i)}(t)-X_{N,M}^{(i)}(t)||^2_{L^2_\rho}\le c(T,M,c(N),C(T),c_\sigma(T),C_{2,\eta})B_j(t_{k+1})
$$

for any $t\in[\,t_k,t_{k+1}\,)$, $k\in\{0,1,\dots,j-1\}$.

Let $t\in[\,t_k,t_{k+1}\,)$ for some $k\in\{0,1,\dots,j-1\}$. By (6.27) and (6.28) we get

$$
\begin{aligned}
&\mathbf{E}||Z_j^{(i)}(t)-X_{N,M}^{(i)}(t)||^2_{L^2_\rho}\\
&\le C\left(\mathbf{E}||V_j^{(i)}(t_k)-X_{N,M}^{(i)}(t_k)||^2_{L^2_\rho}\right.\\
&+\mathbf{E}\left|\left|\int\limits_{t_k}^t(\mathcal{M}_{\Sigma(s,Z_j^{(i)}(s))}-\mathcal{M}_{\Sigma(s,X_{N,M}^{(i)}(s))})\,dW_M(s)\right|\right|^2_{L^2_\rho}
\end{aligned}
$$

$$+\mathbf{E}\left[\int_{t_k}^{t} ||A_N(s)||^2_{\mathcal{L}(L^2_\rho)}\,||X^{(i)}_{N,M}(s)||^2_{L^2_\rho} + ||F^{(i)}(s, X^{(i)}_{N,M}(s))||^2_{L^2_\rho}\,ds\right]\Bigg)$$
$$=: C(I_1 + I_2 + I_3).$$

By the previous considerations, we have (cf. (6.29))

$$I_1 \leq \bar{c}(T, M, c(N), C(T), c_\sigma(T), C_{2,\eta})B_j(t_{k+1}).$$

Concerning $I_2$ and $I_3$, we observe

$$I_2 \leq c(M, c_\sigma(T)) \int_{t_k}^{t} \mathbf{E}||Z^{(i)}_j(s) - X^{(i)}_{N,M}(s)||^2_{L^2_\rho}$$

and

$$\begin{aligned} 0 \leq I_3 \leq \;& \tfrac{T}{j} c(T, N, C(T)) \left(1 + \sup_{r\in[0,T]} \mathbf{E}||X^{(i)}_{N,M}(r)||^2_{L^2_\rho}\right) \\ \leq \;& \tfrac{1}{j} c(T, N, C(T))(1 + c_{N,M}). \end{aligned}$$

Summing all together, by the definition of $B_j(t_{k+1})$ we get

$$\begin{aligned} \mathbf{E}||Z^{(i)}_j(t) - X^{(i)}_{N,M}(t)||^2_{L^2_\rho} \leq \;& \tfrac{T}{j}(1 + \bar{c}(T, N, C(T))(1 + c(\xi^{(i)})) \\ & + c(T, M, c(N), C(T), c_\sigma(T)) \int_0^t \mathbf{E}||Z^{(i)}_j(s) - X^{(i)}_{N,M}(s)||^2_{L^2_\rho}\,ds. \end{aligned}$$

Then, Gronwall's lemma finally implies

$$\mathbf{E}||Z^{(i)}_j(t) - X^{(i)}_{N,M}(t)||^2_{L^2_\rho} \leq \tfrac{T}{j}(1 + \bar{C})(1 + c(\xi^{(i)}))e^{\bar{C}t} < \infty,$$

where $\bar{C}$ denotes the maximum of the two constants $\bar{c}$ and $c$ from the previous inequalities. Thus,

$$\lim_{j\to\infty} \mathbf{E}||Z^{(i)}_j(t) - X^{(i)}_{N,M}(t)||^2_{L^2_\rho} = 0, \tag{6.30}$$

which proves Claim 2. □

From (6.15), we know that for all $t \in [\,0, T\,]$

$$Z^{(1)}_j(t) \leq Z^{(2)}_j(t),\ P\text{-almost surely.}$$

On the other hand, (6.30) implies the existence of a subsequence $(Z^{(i)}_{j(l)}(t))_{l\in\mathbb{N}}$, which $P$-almost surely converges in $L^2_\rho$ to $X^{(i)}_{N,M}(t)$ as $l \to \infty$.
This leads to

$$X^{(1)}_{N,M}(t) \leq X^{(2)}_{N,M}(t) \text{ in } L^2_\rho,\ P\text{-a.s., for all } t \in [\,0,T\,],$$

which proves Lemma 6.1.3 **(i)**. □

**(ii)** As in **(i)**, we fix $j \in \mathbb{N}$ and set $t_k := \frac{kT}{j}$, $k = 0, 1, ..., j$, such that again we have a partition of $[\,0,T\,]$ into $j$ intervalls of length $\frac{T}{j}$. We define processes $Z^{(i)}_{k,j}$, $V^{(i)}_{k,j}$ from $\mathcal{H}^q(T)$ in a recursive way. Compared to (6.10) and (6.11) in the proof of **(i)**, the processes change to

$$Z^{(i)}_{0,j}(t) := \xi^{(i)} + \int_0^t \mathcal{M}_{C(s)}\, dW_M(s) + \int_0^t \int_{L^2} \mathcal{M}_{C(s)}(x)\, \tilde{N}(ds, dx),$$

$$V^{(i)}_{0,j}(t) := Z^{(i)}_{0,j}(t_1) + \int_0^t (A_N(s) V^{(i)}_{0,j}(s) + E^{(i)}(s, V^{(i)}_{0,j}(s)) + \mathcal{M}_{C(s)}(m))\, ds,$$

for $t \in [\,0, t_1\,]$ and

$$Z^{(i)}_{k,j}(t) := V^{(i)}_{k-1,j}(t_k) + \int_{t_k}^t \mathcal{M}_{C(s)}\, dW_M(s) + \int_{t_k}^t \int_{L^2} \mathcal{M}_{C(s)}(x)\, \tilde{N}(ds, dx),$$

$$V^{(i)}_{k,j}(t) := Z^{(i)}_{k,j}(t_{k+1}) + \int_{t_k}^t (A_N(s) V^{(i)}_{k,j}(s) + E^{(i)}(s, V^{(i)}_{k,j}(s)) + \mathcal{M}_{C(s)}(m))\, ds,$$

for $t \in [t_k, t_{k+1}]$ and $k = 1, 2, ..., j-1$.

Thus, we get the required comparison for the processes $Z^{(i)}_j$ (see Claim1 in the proof of **(i)**) as an immediate consequence of the solution indendence of the stochastic integrals, whereas the comparison result for the processes $V^{(i)}_j$ follows analogously to the proof of **(i)**.

The rest of the proof works analogously to the proof of **(i)** and is even simpler, since only the drift coefficients are solution-dependent. □ ■

**Remark 6.2.1:** *So far, in the proof of 6.1.3* ***(i)*** *we needed the special property (3.1) of the eigenvectors $(e_n)_{n\in\mathbb{N}}$ of $Q \in \mathcal{T}^+(L^2)$, i.e.*

$$\sup_{n\in\mathbb{N}} ||e_n||_\infty < \infty,$$

*to control the diffusion terms (see (6.22) and (6.23)) corresponding to $\sigma\colon [\,0,T\,] \times \Omega \times \mathbb{R} \to \mathbb{R}$, which is Lipschitz in the third variable. In particular, this means that we cannot apply this method to show a comparison theorem for equation (6.1) with a $Q$-Wiener process $W$ as in the so-called general nuclear case (cf. Chapter 3) resp. for equation (1.2). In Chapter 8 below, we solve this problem by approximating the $Q_M$-Wiener processes $W_M$ from equation (6.4) by Wiener processes $W_{M,L}$, given by*

$$W_{M,L}(t) := \sum_{n=1}^{M} \sqrt{a_n} e_{n,L} w_n(t),\ t \in [0,T],$$

*where, for any $1 \le n \le M$, $(e_{n,L})_{L\in\mathbb{N}}$ is a sequence of $L^2$-valued functions obeying (3.1) such that*

$$\lim_{L\to\infty} ||e_{n,L} - e_n||_{L^2} = 0.$$

*Thus, we could also prove a comparison theorem for equation (6.1) with $W$ being a $Q$-Wiener process for some $Q \in \mathcal{T}^+(L^2)$ **not** obeying (3.1). This just requires an additional convergence result, namely for $L \to \infty$.*

## 6.3 Proof of Lemma 6.1.4

**(i)** Let us fix $N, M \in \mathbb{N}$:
The difference between the corresponding solutions $X^{(i)}_{N,M}$ and $X^{(i)}_{M}$ can be represented as

$$X^{(i)}_{N,M}(t) - X^{(i)}_{M}(t) = a_N(\xi) + b_N(F) + a_N(F) + b_N(\Sigma) + a_N(\Sigma),\ t \in [0,T],$$

with the terms defined by

$$\begin{aligned}
a_N(\xi) &= [U_N(t,0) - U(t,0)]\xi^{(i)}\\
b_N(F) &= \int_0^t U_N(t,s)[F^{(i)}(s, X^{(i)}_{N,M}(s)) - F^{(i)}(s, X_M(s))]\, ds\\
a_N(F) &= \int_0^t [U_N(t,s) - U(t,s)]F^{(i)}(s, X^{(i)}_{M}(s))\, ds\\
b_N(\Sigma) &= \int_0^t U_N(t,s)[\mathcal{M}_{\Sigma(s,X^{(i)}_{N,M}(s))} - \mathcal{M}_{\Sigma(s,X^{(i)}_{M}(s))}]\, dW_M(s)\\
a_N(\Sigma) &= \int_0^t [U_N(t,s) - U(t,s)]\mathcal{M}_{\Sigma(s,X^{(i)}_{M}(s))}\, dW_M(s).
\end{aligned}$$

Let us first estimate the $a_N$-terms.
By **(A6)**, we have for all $\varphi \in L^2_\rho$

$$\lim_{N\to\infty} \sup_{t\in[0,T]} ||[U_N(t,s) - U(t,s)]\varphi||_{L^2_\rho} = 0.$$

By the Banach-Steinhaus uniform boundedness principle for operators (see e.g. [98], Theorem III.9), this implies

$$\sup_{N\in\mathbb{N}} \sup_{t\in[0,T]} ||U_N(t,s)||_{\mathcal{L}(L^2_\rho)} =: C_U(T) < \infty.$$

Thus, $\lim_{N\to\infty} ||a_N(\xi)||^2_{L^2_\rho} = 0$ and by Lebesgue's dominated convergence theorem

$$\lim_{N\to\infty} \mathbf{E}||a_N(\xi)||^2_{L^2_\rho} = 0.$$

Similarly, $\lim_{N\to\infty} ||a_N(F)||^2_{L^2_\rho} = 0$, where we used the uniform bound

$$\sup_{N\in\mathbb{N}} ||U_N(t,s)F^{(i)}(s, X^{(i)}_{N,M})||^2_{L^2_\rho} \le c(C_U(t), C(T))(1 + ||X^{(i)}_{N,M}||^2_{L^2_\rho}),$$

which is integrable due to the fact that $X^{(i)}_{N,M} \in \mathcal{H}^q(T)$.
Concerning $a_N(\Sigma)$, note that by Itô's isometry

$$\begin{aligned}
\mathbf{E}||a_N(\Sigma)||^2_{L^2_\rho} &= \sum_{n=1}^M a_n \int_0^t \mathbf{E}||(U_N(t,s) - U(t,s))\mathcal{M}_{\Sigma(s,X^{(i)}_M)} e_n||^2_{L^2_\rho}\, ds \\
&\le \left(\sum_{n=1}^M a_n\right)\left(\sup_{n\in\mathbb{N}} ||e_n||^2_\infty\right) \int_0^t \mathbf{E}||[U_N(t,s) - U(t,s)]\Sigma(s, X^{(i)}_M)||^2_{L^2_\rho}\, ds \\
&\to 0, \text{ as } N \to \infty,
\end{aligned}$$

where the integral on the right hand side tends to 0 as $N \to \infty$ by the previous step.
The $b_N$-terms are estimated by the Lipschitz property of $f$ and $\sigma$. Namely,

$$\mathbf{E}||b_N(F)||^2_{L^2_\rho} \le c(c(N), c_f(T)) \int_0^t \mathbf{E}||X^{(i)}_{N,M}(s) - X^{(i)}_M(s)||^2_{L^2_\rho}\, ds$$
$$\mathbf{E}||b_N(\Sigma)||^2_{L^2_\rho} \le c(M, c(N), c_\sigma(T)) \int_0^t \mathbf{E}||X^{(i)}_{N,M}(s) - X^{(i)}_M(s)||^2_{L^2_\rho}\, ds.$$

Alltogether, this gives us

$$\begin{aligned}
&\mathbf{E}||X^{(i)}_{N,M}(t) - X^{(i)}_M(t)||^2_{L^2_\rho} \\
&\le C\Big((\mathbf{E}||a_N(\xi)||^2_{L^2_\rho} + \mathbf{E}||a_N(F)||^2_{L^2_\rho} + \mathbf{E}||a_N(\Sigma)||^2_{L^2_\rho}) \\
&+ c(M, c(N), c_f(T), c_\sigma(T)) \int_0^t \mathbf{E}||X^{(i)}_{N,M}(s) - X^{(i)}_M(s)||^2_{L^2_\rho}\, ds\Big).
\end{aligned}$$

Recall that by Corollary 5.2.5 **(i)**, $X^{(i)}_{N,M}$ and $X^{(i)}_M$ are time-continuous in $L^q(\Omega; L^2_\rho)$.
Applying Gronwall's lemma, the first part of the claim follows by the fact that the $a_N$-terms tend to 0 for $N \to \infty$.

Next, we prove the convergence of $X^{(i)}_M$ to $X^{(i)}$ as $M \to \infty$. We have, for all $t \in [0, T]$,

$$
\begin{aligned}
X_M^{(i)}(t) - X^{(i)}(t) = \;& \int_0^t U(t,s)[F^{(i)}(s,\omega,X_M^{(i)}(s)) - F^{(i)}(s,\omega,X^{(i)}(s))]\,ds \\
&+ \int_0^t U(t,s)[\Sigma(s,X_M^{(i)}(s)) - \Sigma(s,X^{(i)}(s))]\,dW_M(s) \\
&- \sum_{n=M+1}^{\infty} \int_0^t \sqrt{a_n}[U(t,s)\Sigma(s,X^{(i)}(s))](e_n)\,dw_n(s),\ P\text{-a.s..}
\end{aligned}
$$

Analogously to the $b_N$-terms above, by Itô's isometry we have , for all $t \in [\,0,T\,]$,

$$
\begin{aligned}
\mathbf{E}||X_M^{(i)}(t) - X^{(i)}(t)||^2_{L^2_\rho} \le \;& c(M,c(T),c_f(T),c_\sigma(T)) \int_0^t \mathbf{E}||X_M^{(i)}(s) - X^{(i)}(s)||_{L^2_\rho}\,ds \\
&+ \sum_{n=M+1}^{\infty} a_n \int_0^t \mathbf{E}||\;[U(t,s)\mathcal{M}_{\Sigma(s,X^{(i)}(s))}e_n||^2_{L^2_\rho}\,ds.
\end{aligned}
$$

Now, Gronwall's Lemma yields

$$
\mathbf{E}||X_M^{(i)}(t) - X^{(i)}(t)||^2_{L^2_\rho} \le c_M(\Sigma)e^{c(M,c(T),c_f(T),c_\sigma(T))t}
$$

with $c_M(\Sigma)$ given by

$$
c_M(\Sigma) := \sum_{n=M+1}^{\infty} a_n \int_0^t \mathbf{E}||\;[U(t,s)\mathcal{M}_{\Sigma(s,X^{(i)}(s))}e_n||^2_{L^2_\rho}\,ds.
$$

In the nuclear case, we have

$$
trQ := \sum_{n=1}^{\infty} a_n < \infty
$$

and hence

$$
\begin{aligned}
&c_M(\Sigma) = \sum_{n=M+1}^{\infty} a_n \int_0^t \mathbf{E}||\;[U(t,s)\mathcal{M}_{\Sigma(s,X^{(i)}(s))}(e_n)||^2_{L^2_\rho}\,ds \\
&\le T\left(\sum_{n=M+1}^{\infty} a_n\right)\left(\sup_{n\in\mathbb{N}} ||e_n||^2_\infty\right)\left(\sup_{t\in[\,0,T\,]} \mathbf{E}||\Sigma(t,X^{(i)}(t))||^2_{L^2_\rho}\right) \\
&\to 0 \text{ as } M \to \infty.
\end{aligned}
$$

Here, we used the Lipschitz property of $\Sigma$ and (5.1) (with $q = 2$).

In the cylindrical case , we also have

$$
c_M(\Sigma) = \sum_{n=M+1}^{\infty} \int_0^t \mathbf{E}||\;U(t,s)\mathcal{M}_{\Sigma(s,X^{(i)}(s))}(e_n)||^2_{L^2_\rho} \to 0 \text{ as } M \to \infty.
$$

Here, we used the bound

$$(6.31)\quad \sum_{n=M+1}^{\infty} \mathbf{E}||\, U(t,s)\mathcal{M}_{\Sigma(s,X^{(i)}(s))}(e_n)||^2_{L^2_\rho}$$
$$\leq \mathbf{E}||U(t,s)\mathcal{M}_{\Sigma(s,X^{(i)}(s))}||^2_{\mathcal{L}_2(L^2,L^2_\rho)}$$
$$\leq c(T)(t-s)^{-\zeta}\mathbf{E}||\Sigma(s,X^{(i)}(s))||^2_{L^2_\rho}$$

and Lebesgue's dominated convergence theorem. Finally, by the Gronwall-Bellman lemma we get

$$\lim_{M\to\infty} \mathbf{E}||X^{(i)}_M(t) - X^{(i)}(t)||^2_{L^2_\rho} \to 0 \text{ as } M \to \infty,$$

which finishes the proof of **(i)**. □

**(ii)** Let us first fix an arbitrary $M \in \mathbb{N}$.
For any $N \in \mathbb{N}$, setting

$$F^{(i)}(t,\varphi) := E^{(i)}(t,\varphi) + \mathcal{M}_{C(t)}m \text{ for } (t,\varphi) \in [\,0,T\,] \times L^2_\rho,$$

the equations (6.8) and (6.9) become

$$(6.32)\quad \begin{aligned} &dX^{(i)}_{N,M}(t) = (\,A_N(t)X^{(i)}_{N,M}(t) + F^{(i)}(t,X^{(i)}_{N,M}(t))\,)\,dt \\ &+\mathcal{M}_{C(t)}dW_M(t) \\ &+\int_{L^2} \mathcal{M}_{C(t)}x\,\tilde{N}(ds,dx) \\ &X^{(i)}_{N,M}(0) = \xi^{(i)}, \end{aligned}$$

and

$$(6.33)\quad \begin{aligned} &dX^{(i)}_{M}(t) = (\,A(t)X^{(i)}_{M}(t) + F^{(i)}(t,X^{(i)}_{M}(t))\,)\,dt \\ &+\mathcal{M}_{C(t)}dW_M(t) \\ &+\int_{L^2} \mathcal{M}_{C(t)}x\,\tilde{N}(ds,dx) \\ &X^{(i)}_{M}(0) = \xi^{(i)}. \end{aligned}$$

Thus, the convergence of $X^{(i)}_{N,M}$ to $X^{(i)}_M$ as $N \to \infty$ follows from **(i)**.

Concerning the convergence of $X^{(i)}_M$ to $X^{(i)}$ as $M \to \infty$ note that

$$\begin{aligned} X^{(i)}_M(t) - X^{(i)}(t) = \;& \int_0^t U(t,s)[E^{(i)}(s,\omega,X^{(i)}_M(s)) - E^{(i)}(s,\omega,X^{(i)}(s))]\,ds \\ & - \sum_{n=M+1}^{\infty}\int_0^t \sqrt{a_n}U(t,s)c(s)g_n\,dw_n(s) \end{aligned}$$

for all $t \in [\,0,T\,]$ and arbitrary $M \in \mathbb{N}$.

Thus, applying Gronwall's lemma, we get

$$\mathbf{E}||X_M^{(i)}(t) - X^{(i)}(t)||_{L^2_\rho}^2 \leq c_M(C)e^{c_E(T)t}$$

with $c_E(T) := \max\{c_{e^{(1)}}(T), c_{e^{(2)}}(T)\}$ and

$$c_M(C) := \sum_{n=M+1}^{\infty} a_n \int_0^t \mathbf{E}||U(t,s)\mathcal{M}_{C(s)}g_n||_{L^2_\rho}^2 \, ds.$$

Hence,

$$c_M(C) \leq K^2 T \left( \sum_{n=M+1}^{\infty} a_n \right) \left( \underbrace{\sup_{n\in\mathbb{N}} ||g_n||_{L^2}^2}_{=1} \right) \to 0 \text{ as } M \to \infty.$$

Here, we used the boundedness of $c$, the fact that $Q$ is trace class and the fact that $(g_n)_{n\in\mathbb{N}}$ is an orthonormal basis of $L^2 \subset L^2_\rho$.
Thus, we get **(ii)**. □ ■

# Chapter 7

# Main results in the case of non-Lipschitz drift and additive jump noise

This chapter contains the main results of this work in the case of non-Lipschitz drift, Lipschitz diffusion and additive jump resp. jump diffusion coefficients.
We first show existence in the case of an additive Poisson noise added to the equation considered by Manthey and Zausinger in [76]. This is just equation (1.1) with the coefficient $\Gamma$ being independent of the solution. Furthermore, we consider equation (1.2) with the coefficient $\Sigma$ being independent of the solution, which corresponds to the case of additive Lévy noise. More precisely, for $\Theta \subset \mathbb{R}^d$ and $\rho \in \mathbb{N} \cup \{0\}$ such that $\mu_\rho(\Theta) < \infty$, we show existence results for

$$\begin{aligned} dX(t) = \quad & (A(t)X(t) + F(t, X(t)))dt + \mathcal{M}_{\Sigma(t,X(t))}dW(t) \\ & + \int_{L^2} \mathcal{M}_{C(t)} x \, \tilde{N}(dt, dx), \; t \in [\,0, T\,] \end{aligned} \tag{7.1}$$

$$X(0) = \xi^{(i)}$$

resp.

$$\begin{aligned} dX(t) &= (A(t)X(t) + E^{(i)}(t, X(t)))dt + \mathcal{M}_{C(t)} dL(t), \; t \in [\,0, T\,] \\ X(0) &= \xi^{(i)}. \end{aligned} \tag{7.2}$$

In both equations the solution-independent jump resp. jump diffusion coefficient $C$ is defined from a uniformly bounded function $c$: $[\,0, T\,] \times \Omega \to \mathbb{R}$ (for more details on this function, see Section 7.1 below). Since we always

have $\mu_\rho(\Theta) < \infty$, we use the shortened notations $L^2$, $L^2_\rho$ and $L^{2\nu}_\rho$ instead of $L^2(\Theta)$, $L^2_\rho(\Theta)$ and $L^{2\nu}_\rho(\Theta)$.
Our considerations are divided into two cases.

**Case (A)** We suppose that $f$ resp. $e$ generating $F$ resp. $E$ by **(NEM)** fulfills the condition **(PG)** from the introduction with $\nu = 1$, i.e. $f$ resp. $e$ is of at most linear growth.
An $L^2_\rho$-valued initial condition $\xi$ fulfills $\mathbf{E}||\xi||^q_{L^2_\rho} < \infty$ for some $q \geq 2$.

We show existence of a solution $X \in \mathcal{H}^q(T)$ starting from the above $\xi$.

**Case (B)** We suppose that $f$ resp. $e$ generating $F$ resp. $E$ by **(NEM)** fulfills condition **(PG)** from the introduction with $\nu > 1$.
An $L^2_\rho$-valued initial condition $\xi$ obeys $\mathbf{E}||\xi||^{2\nu}_{L^{2\nu}_\rho} < \infty$ for the above $\nu$.

We show existence of a solution $X \in \mathcal{G}_\nu(T)$ starting from the above $\xi$. A crucial point in this proof will be the additional assumption that the Lévy measure associated to the compensated Poisson random measure (equation (1.1)) resp. the Lévy process (equation (1.2)) obeys **(QI)** with $q = 2\nu^2$, which seems to be a natural condition in view of the main existence result in [80].

Finally, given some restrictions on $\Theta$, the evolution operator $U$ and the drift coefficients, with the help of Marinelli's and Röckner's paper [80] (see Proposition 7 there) we even get a uniqueness result (see Theorem 7.1.6 below).

For an exact description of the setting, see Section 7.1 below.

Let us outline the structure of this chapter.
First, in Section 7.1 we present the explicit setting and the main existence and uniqueness results for equation (7.1) and equation (7.2). Sections 7.2 and 7.3 are devoted to the proof of the above results. The scheme of proving the existence result is quite standard and goes along the lines of proving Theorem 3.4.1. in [76]. In particular, we use the same approximation of the non-Lipschitz drifts as Manthey and Zausinger did in [76]. The proof is based on the comparison method derived in Chapter 6. Of course, compared to Manthey's and Zausinger's case, additional technical difficulties are caused by the presence of driving jump terms. Similarly to Chapter 5, this is reflected in the use of the conditions **(A5)**/ **(A5)*** for the almost strong evolution operator $U$. Again, we do not have pathwise time-continuity results but càdlàg versions of the solutions in the case of the evolution operator $U$ obeying **(A7)** and the jump resp. jump diffusion coefficients being uniformly bounded.

For the whole chapter, let $(\Omega, \mathcal{F}, P)$ and $(\mathcal{F}_t)_{t\in[0,T]}$ with some $T > 0$ be as in Section 1.2.

## 7.1 The main results of this chapter

In this section, we give the exact settings and state the main existence and uniqueness results of this chapter.
We assume:

- $(A(t))_{t\in[0,T]}$ generates an almost strong evolution operator in $L^2_\rho$ in the sense of 2.1.1.
- $\sigma$: $[0,T] \times \Omega \times \mathbb{R} \to \mathbb{R}$ generating $\Sigma$ by **(NEM)** is $\mathcal{P}_T \otimes \mathcal{B}(\mathbb{R})/\mathcal{B}(\mathbb{R})$-measurable and fulfills **(LC)** and **(LB)**.
- $e$, $f$: $[0,T] \times \Omega \times \mathbb{R} \to \mathbb{R}$ generating $E$, $F$ by **(NEM)** are $\mathcal{P}_T \otimes \mathcal{B}(\mathbb{R})/\mathcal{B}(\mathbb{R})$-measurable, **continuous** in the third variable and fulfill **(LG)** and**(PG)** with exponent $\nu \geq 1$. (to recall **(PG)** and **(LG)** see Section 3.1).
- $c$: $[0,T] \times \Omega \to \mathbb{R}$ defining $C$ by
$$(C(t,\omega))(\theta) := c(t,\omega),\ \theta \in \Theta$$
is $\mathcal{P}_T/\mathcal{B}(\mathbb{R})$-measurable and bounded, i.e.
$$\sup_{(t,\omega)\in[0,T]\times\Omega} |c(t,\omega)| =: K < \infty. \tag{7.3}$$
- $W$ is a $Q$-Wiener process in $L^2$ such that either $Q \in \mathcal{T}^+(L^2)$ and the system of eigenvectors $(e_n)_{n\in\mathbb{N}}$ of $Q$ obeys (3.1) (called the **nuclear case** below) or $Q = \mathbf{I}$ (called the **cylindrical case** below).
- $L$ is a Lévy process such that the corresponding Lévy measure $\eta$ obeys **(SI)**. This yields the Lévy-Itô decomposition (2.16) (cf. Theorem 2.4.13 above) in $L^2$ with a $Q$-Wiener process such that $Q \in \mathcal{T}^+(L^2)$ (referred to as the **general nuclear case** below).

In this setting, we look for solutions to (7.1) resp. (7.2) in the following sense:

**Definition 7.1.1:** ***(i)*** *In the case $\nu = 1$, given an initial condition*

*$\xi \in L^2_\rho$ as in **Case (A)**, an $\mathcal{H}^q(T)$-valued process $X$ is called a mild solution to (7.1) resp. (7.2) if, $P$-almost surely, we have for all $t \in [0,T]$ in $L^2_\rho$*

$$(7.4)\; X(t) = U(t,0)\xi + \int_0^t U(t,s)F(s,X(s))ds + \int_0^t U(t,s)\mathcal{M}_{\Sigma(s,X(s))}dW(s) + \int_0^t \int_{L^2} U(t,s)\mathcal{M}_{C(s)}(x)\,\tilde{N}(ds,dx)$$

*resp.*

$$(7.5)\; X(t) = U(t,0)\xi + \int_0^t U(t,s)E(s,X(s))ds + \int_0^t U(t,s)\mathcal{M}_{C(s)}dL(s).$$

***(ii)** In the case $\nu > 1$, given an initial condition $\xi \in L^{2\nu}_\rho$ as in **Case (B)**, a $\mathcal{G}_\nu(T)$-valued process $X$ is called a solution to (7.1) resp. (7.2) if (7.4) resp. (7.5) holds true in $L^2_\rho$, $P$-almost surely, for all $t \in [0,T]$.*

This includes the requirement that the right hand sides in (7.4) resp. (7.5) are well-defined.

Our first main result describes the case of drifts having at most linear growth, i.e. when **(PG)** holds with $\nu = 1$.

**Theorem 7.1.2:** *Suppose the almost strong evolution operator $U$ generated by $(A(t))_{t\in[0,T]}$ has properties **(A0)**–**(A2)** and **(A6)**.*
*Let **(PG)** be fulfilled with exponent $\nu = 1$ both for $e$ and $f$. Suppose that $q \in (\frac{2}{1-\zeta}, \frac{2}{\zeta})$ for $\zeta$ from **(A2)** with the additonal assumption that $\zeta \in [0, \frac{1}{2})$ and the initial condition $\xi$ is as in **Case (A)**.*
*Futrhermore, assume that the integrability condition **(QI)** for the Lévy measure $\eta$ is fulfilled with the above $q$.*
*Finally, let $\Gamma$ in (1.1) resp. $\Sigma$ in (1.2) be replaced by $C$, which corresponds to the case of additive driving Lévy noise.*

*Then:*
***(i)** There exists a solution $X \in \mathcal{H}^q(T)$ to (7.1) in the sense of 7.1.1 **(i)**.*
*The process $t \mapsto X(t)$ is continuous in $L^q(\Omega; L^2_\rho)$.*
*Furthermore, we have the estimate*

$$(7.6)\qquad \sup_{t\in[0,T]} \mathbf{E}||X(t)||^q_{L^2_\rho} \le c(q,K,T,c(T),c_f(T),c_\sigma(T),C_{q,\eta})(1+\mathbf{E}||\xi||^q_{L^2_\rho})$$

*with a positive constant on the right hand side.*

***(ii)*** *There exists a solution $X \in \mathcal{H}^q(T)$ to (7.2) in the sense of 7.1.1* ***(i)****. Furthermore, $t \mapsto X(t)$ is continuous in $L^q(\Omega; L^2_\rho)$ and we have the estimate*

$$\sup_{t\in[0,T]} \mathbf{E}||X(t)||^q_{L^2_\rho} \le c(q,K,T,c(T),c_e(T),C_{q,\eta})(1+\mathbf{E}||\xi||^q_{L^2_\rho}) \tag{7.7}$$

*with a positive constant on the right hand side.*

*Both in* ***(i)*** *and* ***(ii)****, under the additional assumption* ***(A7)*** *and the assumption that $\sigma$ obeys (7.3), there is a càdlàg version of the solution process $[0,T] \ni t \mapsto X(t) \in L^2_\rho$.*

**Remark 7.1.3:** *Actually, in the assumptions of the previous theorem, we could also assume* ***(A5)**** *with $\nu = 1$ instead of* ***(A2)*** *in the nuclear case in claims* ***(i)*** *and* ***(ii)*** *(see Remark 3.4.2 (ii) and Theorem 4.1 above).*

The second result covers the case of a drift having at most polynomial growth, i.e. the drift obeys **(PG)** with exponent $\nu > 1$. Let us stress that, in contrast to the existence and uniqueness result in the Lipschitz case (cf. Theorem 5.2.2 above), the solutions take their values in $L^{2\nu}_\rho$ but are time-continuous only in $L^2(\Omega; L^2_\rho)$. This is due to the polynomial growth of order $\nu > 1$ of $f$ resp. $e$ (see also Remark 5.1.11 **(ii)**).

**Theorem 7.1.4:** *Suppose the almost strong evolution operator $U$ generated by $(A(t))_{t\in[0,T]}$ has properties* ***(A0)****–* ***(A4)****,* ***(A5)**** *and* ***(A6)*** *(note that in the nuclear case* ***(A2)*** *and* ***(A4)*** *can be omitted, see Proposition 3.4.3).*
*Furthermore, let $e$, $f$ fullfill* ***(PG)*** *with an exponent $\nu \in (\frac{1}{1-\zeta}, \frac{1}{\zeta})$ with $\zeta$ from* ***(A2)*** *(resp.* ***(A5)****) obeying $\zeta \in [0, \frac{1}{2})$.*
*Suppose the initial condition $\xi$ is as in* ***Case (B)****. Assume that the integrability condition* ***(QI)*** *for the Lévy measure $\eta$ is fulfilled with $q = 2\nu^2$.*

*Then:*
***(i)*** *There exists a solution $X \in \mathcal{G}_\nu(T)$ to (7.1) in the sense of 7.1.1* ***(ii)****.*
*The process $t \mapsto X(t)$ is continuous in $L^2(\Omega; L^2_\rho)$. Furthermore, we have the estimate*

$$\sup_{t\in[0,T]} \mathbf{E}||X(t)||^{2\nu}_{L^{2\nu}_\rho} \le c(\nu,\zeta,K,T,c(T),c_f(T),c_\sigma(T),C_{2\nu,\eta})(1+\mathbf{E}||\xi||^{2\nu}_{L^{2\nu}_\rho}) \tag{7.8}$$

*with a positive constant on the right hand side.*

***(ii)*** *There exists a solution $X \in \mathcal{G}_\nu(T)$ to (7.2) in the sense of 7.1.1* ***(ii)****.*

*Furthermore, $t \mapsto X(t)$ is continuous in $L^2(\Omega; L^2_\rho)$ and we have the estimate*

$$\text{(7.9)} \qquad \sup_{t\in[0,T]} \mathbf{E}||X(t)||^{2\nu}_{L^{2\nu}_\rho} \le c(\nu,\zeta,K,T,c(T),c_e(T),C_{2\nu,\eta})(1+\mathbf{E}||\xi||^{2\nu}_{L^{2\nu}_\rho})$$

*with a positive constant on the right hand side.*

*Again, assuming additionally **(A7)** and that $\sigma$ defining $\Sigma$ by **(NEM)** obeys (7.3), there is a càdlàg version of the process $[0,T] \ni t \mapsto X(t) \in L^2_\rho$ both in **(i)** and **(ii)**.*

**Remark 7.1.5:** ***(i)*** *In the case of equation (1.2), a sufficient condition for well-definedness of the stochastic convolutions in (7.4) is just **(A5)*** with $\varphi = 1$ (see Remarks 3.4.9 and 4.10).*

***(ii)*** *The integrability condition **(QI)** with $q = 2\nu^2$ will be crucial in Step2 in the proof of Theorem 7.1.4 (see the estimate of the term $\bar{I}^{(2)}_M$ on p.245).*

The proofs of Theorem 7.1.2 and 7.1.3 will be done in Sections 7.1.2 and 7.1.3.

Finally, in the special case that $(A(t))_{t\in[0,T]}$ in (7.1) and (7.2) is replaced by the generator $A$ of a $\mathcal{C}_0$-semigroup, we get the following uniqueness result, which is based on the uniqueness results of the two papers [80] and [81] by Marinelli and Röckner.

**Theorem 7.1.6:** *For this theorem, let $\Theta \subset \mathbb{R}^d$ be an open bounded set with smooth boundary $\partial\Theta$. So, as described in the introduction of the chapter, we let $\rho = 0$, i.e. $L^{2\nu}_\rho(\Theta) = L^{2\nu}(\Theta)$ for any $\nu \ge 1$.*

***(i)*** *Let $\nu = 1$. Suppose we have $\mathbf{E}||\xi||^q_{L^2} < \infty$ with some $q \ge 2$ as in Theorem 7.1.2 for the initial condition $\xi$.*
*Suppose that $A \in \mathcal{L}(L^2(\Theta))$ admits a unique extension to a strongly continuous semigroup of positive contractions on $L^2(\Theta)$.*
*Let $f$, $\sigma$, $c$ and $\eta$ obey the assumptions from Theorem 7.1.2 **(i)**. Furthermore, let $f$ be **uniformly maximally monotone on** $[0,T]\times\Omega$, i.e. there is a positive constant $c_f(T)$ such that*

$$\mathbb{R} \ni y \mapsto f(t,\omega,y) + c_f(T)y \in \mathbb{R}$$

*is a monotone function for any $(t,\omega) \in [0,T]\times\Omega$.*
*Now, if the Wiener process $W$ obeys the assumptions of the cylindrical case (cf. Chapter 3), then the solution to (7.1) (in the sense of Definition 7.1.1*

*above) existing by 7.1.2* ***(i)*** *is unique.*

***(ii)*** *Let $\nu > 1$. Suppose we have $\mathbf{E}||\xi||^{2\nu}_{L^{2\nu}} < \infty$ with $\nu$ obeying the assumptions of Theorem 7.1.4 for the initial condition $\xi$.*
*Suppose that $A \in \mathcal{L}(L^2(\Theta))$ admits a unique extension to a strongly continuous semigroup of positive contractions on $L^{2\nu}(\Theta)$ and $L^{2\nu^2}(\Theta)$.*
*Let $f$, $\sigma$, $c$ and $\eta$ obey the assumptions from Theorem 7.1.4* ***(i)****. Furthermore, let $f$ be uniformly maximally monotone on $[0,T] \times \Omega$ in the sense of* ***(i)****.*
*Now, if the Wiener process $W$ obeys the assumptions of the cylindrical case (cf. Chapter 3), then the solution to (7.1) (in the sense of Definition 7.1.1 above) existing by 7.1.4* ***(i)*** *is unique.*

**Proof:** See Section 7.4. below.

In the proofs, we want to apply our knowledge about existence and uniqueness in the case of Lipschitz drift functions $e$, $f$. To this end, we prepare the following definitions and lemmata.

**Definition 7.1.7:** *Consider a real-valued, $\mathcal{P}_T \otimes \mathcal{B}(\mathbb{R})$-measurable function $f\colon [0,T] \times \Omega \times \mathbb{R} \to \mathbb{R}$ and define*

$$f_N(t,\omega,y) := f(t,\omega,y) \vee (-N) \tag{7.10}$$

$$f_{N,M}(t,\omega,y) := \inf_{u \in \mathbf{R}} \{f_N(t,\omega,u) + M|u-y|\} \tag{7.11}$$

*for all $t \in [0,T]$, $\omega \in \Omega$, $y \in \mathbb{R}$ and $N, M \in \mathbb{N}$.*

This construction implies the pointwise monotone convergence

$$f_{N,M}(t,\omega,y) \uparrow f_N(t,\omega,y) \text{ as } M \to \infty \tag{7.12}$$

and

$$f_N(t,\omega,y) \downarrow f(t,\omega,y) \text{ as } N \to \infty. \tag{7.13}$$

Note that (7.12) and (7.13) (and the Comparison theorem 6.1.1) will be crucially used in the proof.
Clearly, $f_N$ and $f_{N,M}$ are $\mathcal{P}_T \otimes \mathcal{B}(\mathbb{R})$-measurable as well.

**Lemma 7.1.8:** *The functions $f_{N,M}$ introduced in 7.1.7 fulfill Lipschitz condition* ***(LC)*** *and the boundedness assumption* ***(LB)****,whereby the corresponding constants can be chosen uniformly for $N \in \mathbb{N}$.*

**Proof:** Let $N, M \in \mathbb{N}$ be arbitrary.

We first prove the Lipschitz property **(LC)**. Indeed, given $y, z, u \in \mathbb{R}$, we have

$$\begin{aligned} M|z-y| = & \; M|y-z| = M|-(u-y)+u-z| \\ \geq & \; M|u-z| - M|u-y| \\ = & \; (f_N(t,\omega,u) + M|u-z|) - (f_N(t,\omega,u) + M|u-y|) \\ \geq & \; f_{N,M}(t,\omega,z) - (f_N(t,\omega,u) + M|u-y|), \end{aligned}$$

which implies $f_{N,M}(t,\omega,z) - f_{N,M}(t,\omega,y) \leq M|z-y|$. By changing the roles of $y$ and $z$, we get

$$f_{N,M}(t,\omega,y) - f_{N,M}(t,\omega,z) \leq M|y-z| = M|z-y|,$$

which shows **(LC)** for $f_{N,M}$ with the Lipschitz constant $M$.
Concerning the boundedness assumption **(LB)**, let us note that by the construction we get

$$-N \leq f_{N,M}(t,\omega,0) \leq f_N(t,\omega,0) \leq c_f(T) \tag{7.14}$$

and thus

$$|f_{N,M}(t,\omega,0)| \leq c(N)$$

with the constant $c(N) := \max\{N, c_f(T)\}$, which is the same for all $M \in \mathbb{N}$ and $\omega \in \Omega$. ■

**Remark 7.1.9:** ***(i)*** *A standard example of drift terms $e(t,\omega,y)$, which fulfill the polynomial growth condition* ***(PG)*** *and the one-sided linear growth condition* ***(LG)****, are polynomials of the form*

$$e(y) = \sum_{k=0}^{n} b_k y^k, \; b_n < 0, \; b_k \in \mathbb{R}, \; 0 \leq k \leq n-1, \; n \text{ odd}.$$

*The coefficients $b_k = b_k(t,\omega)$, $0 \leq k \leq n$ have to be bounded functions of $(t,\omega) \in [\,0,T\,] \times \Omega$.*

***(ii)*** *Let us note that the proof of Theorem 7.1.6 will be based on the uniqueness condition from Marinelli's and Röckner's paper [80]. Recall that in this paper the authors also need the assumption* ***(QI)*** *to be fulfilled with $q = 2\nu^2$ in the case of the drift being of polynomial growth of at most order $\nu$. So this condition from 7.1.4 seems to be quite natural in such framework.*

***(iii)*** *As compared to the existence and uniqueness results in Chapter 5,* *we assume that $\zeta \in (0, \frac{1}{2})$ for $\zeta$ from **(A2)** right from the start. By this assumption the intervall $(\frac{2}{1-\zeta}, \frac{2}{\zeta})$ is non-empty, which is needed to prove the existence of càdlàg versions of the solutions.*

We finish this section by describing the outline of the proofs in the following sections. In general, all proofs run along the lines of the proof of Theorem 3.4.1 from [76], but of course we have to take into account the presence of the jump terms.

**The scheme of proving Theorems 7.1.2 and 7.1.3**

The proofs are devided into five steps:

**Step 1:** We define auxilliary functions $\bar{g}$ and $\bar{h}$, which later will help us to estimate the non-Lipschitz drift term in each of the proofs.

**Step 2:** We show existence and uniquness of solutions $X_{N,M}$ corresponding to the case of $F$ resp. $E$ being replaced by $F_{N,M}$ resp. $E_{N,M}$. Here, $F_{N,M}$ resp. $E_{N,M}$ are defined by **(NEM)** from $f_{N,M}$ resp. $e_{N,M}$ from (7.11).
With the help of the Comparison Theorem 6.1.1, we further establish certain $M$-independent estimates of $X_{N,M}$, which will be crucial for the rest of the proof.

**Step 3:** We construct processes $X_N := \lim_{M\to\infty} X_{N,M}$, which will be our candidates for solutions to the equations with $F$ resp. $E$ being replaced by $F_N$ resp. $E_N$, being defined by **(NEM)** from $f_N$ from (7.10) resp. $e_n$ defined analogously to $f_N$. In this step, we only check the convergence in the appropriate spaces.

**Step 4:** We show that the processes $X_N$ from Step 3 solve the equation, when $F$ is replaced by $F_N$ resp. $E$ is replaced by $E_N$.

**Step 5:** We first show that there are $N$-independent estimates for solutions $X_N$ from Step 4.
Then, we define candidates $X := \lim_{N\to\infty} X_N$ for solutions to the initial equations and prove that they really solve the equations (7.1) and (7.2).
Finally, we prove the required estimates on the moments of the solution.

## 7.2 Proof of Theorem 7.1.2

**Step 1:** Let us define mappings $\bar{g}$, $\bar{h}$: $\mathbb{R} \to \mathbb{R}$ by

$$(7.15)\ \bar{g}(v) := \min\left( \inf_{\substack{0\le u\le v\\(t,\omega)\in[0,T]\times\Omega}} f(t,\omega,u)\mathbf{1}_{[0,\infty)}(v), 0\right) - c_f(T)(1-v)\mathbf{1}_{(-\infty,0)}(v),$$

$$(7.16)\ \bar{h}(v) := \max\left( \sup_{\substack{v\le u\le 0\\(t,\omega)\in[0,T]\times\Omega}} f(t,\omega,u)\mathbf{1}_{(-\infty,0]}(v), 0\right) + c_f(T)(1+v)\mathbf{1}_{(0,\infty)}(v).$$

Note that for all $v \in \mathbb{R}$

$$\inf_{\substack{0\le u\le v\\(t,\omega)\in[0,T]\times\Omega}} f(t,\omega,u) \ge -c_f(T)(1+|v|)$$

and

$$\sup_{\substack{v\le u\le 0\\(t,\omega)\in[0,T]\times\Omega}} f(t,\omega,u) \le c_f(T)(1+|v|),$$

which implies

$$(7.17)\ \bar{g}(v) \ge -c_f(T)[(1+|v|)\mathbf{1}_{[0,\infty)}(v) + (1-v)\mathbf{1}_{(-\infty,0)}(v)]$$

and

$$(7.18)\ \bar{h}(v) \le c_f(T)[(1+|v|)\mathbf{1}_{(-\infty,0]}(v) + (1+v)\mathbf{1}_{[0,\infty)}(v)].$$

Since $f$ fulfills **(LG)**, $\bar{g}$ and $\bar{h}$ obey, for $(t,\omega,v) \in [0,T] \times \Omega \times \mathbb{R}$

$$\bar{g} \le 0\ ,\ \bar{g}(v) \le f(t,\omega,v), \tag{7.19}$$

$$\bar{h} \ge 0\ ,\ \bar{h}(v) \ge f(t,\omega,v). \tag{7.20}$$

Furthermore, $\bar{g}\mathbf{1}_{[0,\infty)}$ and $\bar{h}\mathbf{1}_{(-\infty,0]}$ are decreasing functions on $\mathbb{R}$.
Of course, (7.19) and (7.20) also hold true, when $f$ is replaced by $e$ in (7.15) and (7.16).
These auxiliary functions help us to estimate the integral $I_F(X)$, defined in Section 5.1, in the non-Lipschitz case.

**Step 2:** Given arbitrary $N, M \in \mathbb{N}$, we know that the function $f_{N,M}$ defined by (7.10)/(7.11) from 7.1.5/7.1.6 is $\mathcal{P}_T \otimes \mathcal{B}(\mathbb{R})$-measurable, obeys **(LC)** and **(LB)** and is such that $f_{N,M}$ is $\mathcal{P}_T \otimes \mathcal{B}(\mathbb{R})$-measurable.
Of course, this also holds true for the function $e_{N,M}$ defined analogously to the $f_{N,M}$.

Thus, the existence and uniqueness results from Section 5.2 are applicable. By 5.2.1 (applied to the special cases $\Gamma = C$ resp. $\Sigma = C$), there are processes $X_{N,M} \in \mathcal{H}^q(T)$ solving equations (7.1) resp. (7.2), when $f$ resp. $e$ is replaced by $f_{N,M}$ resp. $e_{N,M}$.
To proceed along the lines of Manthey's and Zausinger's proof, we need to find $M$-independent estimates for the moments of $X_{N,M}$.

**(i) The Poisson noise case - equation (7.1)**

By 5.2.1, the map $t \mapsto X_{N,M}(t)$ is continuous in $L^q(\Omega; L^2_\rho)$. Since $f_{N,M} \leq f_{N,M+1}$, by Theorem 6.1.1 we have

$$X_{N,M}(t) \leq X_{N,M+1}(t),\ P\text{-a.s., for any } t \in [\,0, T\,]. \tag{7.21}$$

We denote solutions to equation (7.1) as follows:

- by $\bar{X}_{0,M}$ in the case of initial condition $\xi^+ := \xi \vee 0$ and drift $F_{0,M}$ resp. $E_{0,M}$,
- by $\underline{X}_{N,M}$ in the case of initial condition $\xi^- := \xi \wedge 0$ and drift $F^-_{N,M}$ resp. $E^-_{N,M}$,
- by $V$ in the case of initial condition $\xi = 0$ and drift $F = 0$ resp. $E = 0$.

We observe that, for all $N, M \in \mathbb{N}$,

$$f^-_{N,M} \leq f_{N,M} \leq f_{0,M}, \tag{7.22}$$

$$f^-_{N,M} \leq 0 \leq f_{0,M}, \tag{7.23}$$

and analogous estimates hold for the $e$-terms. Hence, by 6.1.1 we have in $L^2_\rho$

$$\underline{X}_{N,M}(t) \leq X_{N,M}(t) \leq \bar{X}_{0,M}(t), \tag{7.24}$$
$$\underline{X}_{N,M}(t) \leq V(t) \leq \bar{X}_{0,M}(t), \tag{7.25}$$

$P$-almost surely, for any $t \in [\,0, T\,]$ and $N, M \in \mathbb{N}$.

Note that similarly to Section 5.2, all the solutions above are time-continuous in $L^q(\Omega; L^2_\rho)$ and, by assumption (7.3) on $c$, have a càdlàg version under the additional assumption that $U$ obeys **(A7)**.

In view of (7.24), we show the $M$-independent estimate required for $X_{N,M}$

by showing an $M$-independent estimate both for $\bar{X}_{0,M}$ and $\underline{X}_{N,M}$.

By (5.25) and the boundedness assumption (7.3) on $c$, we first have

$$\sup_{t\in[0,T]} \mathbf{E}||\bar{X}_{0,M}(t)||^q_{L^2_\rho} \le c(q,K,T,c(T),c_{f_{0,M}}(T),c_\sigma(T),C_{q,\eta})(1+\mathbf{E}||\xi^+||^q_{L^2_\rho})$$

with an $M$-dependent constant in the right hand side.
To find an $M$-independent estimate, let us define, for $t \in [\,0,T\,]$,

$$\bar{I}^{(1)}(t) := \mathbf{E}||U(t,0)\xi^+||^q_{L^2_\rho},$$

$$\bar{I}^{(2)}_M(t) := \mathbf{E}\left|\left|\int_0^t U(t,s)F_{0,M}(s,\bar{X}_{0,M}(s))\,ds\right|\right|^q_{L^2_\rho},$$

$$\bar{I}^{(3)}_M(t) := \mathbf{E}\left|\left|\int_0^t U(t,s)\mathcal{M}_{\Sigma(s,\bar{X}_{0,M}(s))}\,dW(s)\right|\right|^q_{L^2_\rho}$$

and

$$I^{(4)}(t) := \mathbf{E}\left|\left|\int_0^t\int_{L^2} U(t,s)\mathcal{M}_{C(s)}(x)\,\tilde{N}(ds,dx)\right|\right|^q_{L^2_\rho}.$$

Thus, we have for each $M \in \mathbb{N}$

$$\mathbf{E}||\bar{X}_{0,M}(t)||^q_{L^2_\rho} \le c(q)[\bar{I}^{(1)}(t)+\bar{I}^{(2)}_M(t)+\bar{I}^{(3)}_M(t)+I^{(4)}(t)]. \tag{7.26}$$

We start with the obvious estimate

$$\bar{I}^{(1)}(t) \le c^q(T)\mathbf{E}||\xi^+||^q_{L^2_\rho} \le c^q(T)\mathbf{E}||\xi||^q_{L^2_\rho}$$

for the first term.
To handle the second term, we note that by (7.16), (7.18) and (7.22) we have

$$f_{0,M}(t,v) \le h(v) \le c_f(T)[(1+|v|)\mathbf{1}_{(-\infty,0]}(v)+(1+v)\mathbf{1}_{[0,\infty)}(v)]. \tag{7.27}$$

This implies

$$\begin{aligned}
\bar{I}_M^{(2)}(t) \leq\ & c(q,T,c(T))\mathbf{E}\int_0^t ||\bar{h}(\bar{X}_{0,M}(s))||_{L^2_\rho}^q\, ds \\
\leq\ & c(q,T,c(T),c_f(T))\mathbf{E}\left(\int_0^t\int_\Theta \Big[(1+\bar{X}^2_{0,M}(s,\theta))\mathbf{1}_{\{\bar{X}_{0,M}(s,\theta)>0\}}(s,\theta)\right. \\
& \left. +(1+|V(s,\theta)|)\mathbf{1}_{\{\bar{X}_{0,M}(s,\theta)<0\}}(s,\theta)\Big]\ \mu_\rho(d\theta)\right)^{\frac{q}{2}}\ ds \\
\leq\ & c(q,T,c(T),c_f(T))\left(1+\int_0^t \mathbf{E}||\bar{X}_{0,M}(s)||_{L^2_\rho}^q\, ds+\int_0^T \mathbf{E}||V(s)||_{L^2_\rho}^q\, ds\right) \\
\leq\ & c(q,K,T,c(T),c_f(T),c_\sigma(T),C_{q,\eta})\left(1+\int_0^t \mathbf{E}||\bar{X}_{0,M}(s)||_{L^2_\rho}^q\, ds\right).
\end{aligned}$$

Here, we used (7.20) in the first, (7.18) in the second and estimate (5.33) for the $V$-term in the fourth step.
By the Burkholder-Davis-Gundy inequality 2.5.4/2.5.6, condition **(A2)** for $U$, Hölder's inequality, the fact that

$$q > \tfrac{2}{1-\zeta} \iff \tfrac{q\zeta}{q-2} > -1$$

and **(LC)**, **(LB)** for $\sigma$, we get (cf. Proposition 3.4.1)

$$\begin{aligned}
\bar{I}_M^{(3)}(t) \leq\ & c(q,c(T))\mathbf{E}\left(\int_0^t (t-s)^{-\zeta}||\Sigma(s,\bar{X}_{0,M}(s))||_{L^2_\rho}^2\, ds\right)^{\frac{q}{2}} \\
\leq\ & c(q,T,\zeta,c(T),c_\sigma(T))\left(1+\int_0^t \mathbf{E}||\bar{X}_{0,M}(s)||_{L^2_\rho}^q\, ds\right).
\end{aligned}$$

By the Bichteler-Jacod inequality 2.6.10, **(QI)** for $\eta$, **(A2)** for $U$, the fact that

$$q < \tfrac{2}{\zeta} \iff \tfrac{\zeta q}{2} < 1$$

and the assumption (7.3) on $c$, we get (cf. Proposition 4.1)

$$\bar{I}^{(4)}(t) \leq C_{q,\eta}^{\frac{1}{q}}\int_0^t (t-s)^{-\frac{\zeta q}{2}}\mathbf{E}||C(s)||_{L^2_\rho}^q\, ds \leq c(q,\zeta,K,T,C_{q,\eta}).$$

Summing up in (7.26), we thus get, for $t\in[\,0,T\,]$,

$$\begin{aligned}
\mathbf{E}||\bar{X}_{0,M}(t)||_{L^2_\rho}^q \leq\ & c(q,\zeta,K,T,c(T),c_f(T),c_\sigma(T),C_{q,\eta})(1+\mathbf{E}||\xi||_{L^2_\rho}^q) \\
& +c(q,\zeta,K,T,c(T),c_f(T),c_\sigma(T),C_{q,\eta})\int_0^t \mathbf{E}||\bar{X}_{0,M}(s)||_{L^2_\rho}^q\, ds.
\end{aligned}$$

Therefore, by Gronwall's Lemma we get

$$\mathbf{E}||\bar{X}_{0,M}(t)||_{L^2_\rho}^q \leq \bar{c}(q,\zeta,K,T,c(T),c_f(T),c_\sigma(T),C_{q,\eta})(1+\mathbf{E}||\xi||_{L^2_\rho}^q)$$

for arbitrary $M \in \mathbb{N}$ and $t \in [0,T]$. Thus, we have proven that

$$\sup_{\substack{t\in[0,T]\\ M\in\mathbb{N}}} \mathbf{E}||\bar{X}_{0,M}(t)||^q_{L^2_\rho} \le \bar{c}(q,\zeta,K,T,c(T),c_f(T),c_\sigma(T),C_{q,\eta})(1+\mathbf{E}||\xi||^q_{L^2_\rho}). \tag{7.28}$$

The fact that (7.20) holds true uniformly in $M \in \mathbb{N}$ was essential for getting the above estimate, which shows that the $\bar{X}_{0,M}$ are uniformly bounded in $M$.

Next, we consider $\underline{X}_{N,M}$ with arbitrary $N, M \in \mathbb{N}$. For any $t \in [0,T]$, we define

$$\underline{I}^{(1)}(t) := \mathbf{E}||U(t,0)\xi^-||^q_{L^2_\rho},$$

$$\underline{I}^{(2)}_{N,M}(t) := \mathbf{E}\left|\left|\int_0^t U(t,s)F^-_{N,M}(s,\underline{X}_{N,M})\,ds\right|\right|^q_{L^2_\rho}$$

and

$$\underline{I}^{(3)}_{N,M}(t) := \mathbf{E}\left|\left|\int_0^t U(t,s)\mathcal{M}_{\Sigma(s,\underline{X}_{N,M}(s))}\,dW(s)\right|\right|^q_{L^2_\rho}.$$

Thus, we have for any $t \in [0,T]$ (with $\bar{I}^{(4)}$ as in (7.26))

$$\mathbf{E}||\underline{X}_{N,M}(t)||^q_{L^2_\rho} \le c(q)[\underline{I}^{(1)}(t) + \underline{I}^{(2)}_{N,M}(t) + \underline{I}^{(3)}_{N,M}(t) + I^{(4)}(t)].$$

Analogously to the consideration of $\bar{I}^{(3)}_M$ above, we get

$$\underline{I}^{(3)}_{N,M}(t) \le c(q,\zeta,T,c(T),c_\sigma(T))\left(1 + \int_0^t \mathbf{E}||\underline{X}_{N,M}(s)||^q_{L^2_\rho}\right).$$

As obviously $\underline{I}^{(1)}(t) \le c(q,c(T))\mathbf{E}||\xi||^q_{L^2_\rho}$ and $I^{(4)}$ has already been calculated before, it remains to estimate $\underline{I}^{(2)}_{N,M}(t)$.
Since by the construction

$$-N \le f^-_{N,M}(t,\omega,y) \le 0 \text{ for any } (t,\omega,y) \in [0,T]\times\Omega\times\mathbb{R},$$

we immediately get

$$\sup_{\substack{t\in[0,T]\\ M\in\mathbb{N}}} \underline{I}^{(2)}_{N,M}(t) \le c(N,q,T,c(T)) < \infty.$$

Thus, putting all the estimates together, we have

$$\begin{aligned}\mathbf{E}||\underline{X}_{N,M}||^q_{L^2_\rho} \le \quad & c(N,q,\zeta,K,T,c(T),c_\sigma(T),C_{q,\eta})(1+\mathbf{E}||\xi||^q_{L^2_\rho})\\ & + c(q,\zeta,T,c(T),c_\sigma(T))\int_0^t \mathbf{E}||\underline{X}_{N,M}(s)||^q_{L^2_\rho}\,ds,\end{aligned}$$

and hence by Gronwall's lemma

$$\mathbf{E}||\underline{X}_{N,M}||^q_{L^2_\rho} \le \underline{c}(N,q,\zeta,K,T,c(T),c_\sigma(T),C_{q,\eta})(1+\mathbf{E}||\xi||^q_{L^2_\rho}).$$

As the previous estimate holds for arbitrary $t \in [0,T]$ and $M \in \mathbb{N}$, we have shown that

$$\sup_{\substack{t\in[0,T]\\ M\in\mathbf{N}}} \mathbf{E}||\underline{X}_{N,M}||^q_{L^2_\rho} \le \underline{c}(N,q,\zeta,K,T,c(T),c_\sigma(T),C_{q,\eta})(1+\mathbf{E}||\xi||^q_{L^2_\rho}). \tag{7.29}$$

Finally, by (7.24), (7.28) and (7.29) we get

$$\sup_{\substack{t\in[0,T]\\ M\in\mathbb{N}}} \mathbf{E}||X_{N,M}||^q_{L^2_\rho} \le c(N,q,\zeta,K,T,c(T),c_\sigma(T),C_{q,\eta})(1+\mathbf{E}||\xi||^q_{L^2_\rho}) \tag{7.30}$$

with $c = \bar{c} + \underline{c}$.

### (ii) The Lévy noise case - equation (7.2)

Now, we replace $\Sigma$ by the solution independent coefficient $C$.
For the $Q$-Wiener process $W$ appearing in the Lévy-Itô decomposition of $L$ we cannot guarantee the representation (2.5) with an orthonormal basis $(e_n)_n \subset L^2$ obeying (3.1). So, $W$ is as in the general nuclear case (cf. Section 3.4) but not necessarily as in the nuclear case.
We denote the solutions to (7.2) as follows:

- by $\bar{X}_{0,M}$ if $\xi^+$ and $E_{0,M}$ replace $\xi$ and $E$,
- by $\underline{X}_{N,M}$ if $\xi^-$ and $E^-_{N,M}$ replace $\xi$ and $E$, and
- by $V$ if 0 and 0 replace $\xi$ and $E$.

Obviously, we get the relations (7.24) and (7.25) again. By (5.25), we have

$$\sup_{t\in[0,T]} \mathbf{E}||\bar{X}_{0,M}(t)||^q_{L^2_\rho} \le c(q,M)(1+\mathbf{E}||\xi^+||^q_{L^2_\rho}) < \infty.$$

For any $t \in [0,T]$, we define

$$\bar{I}^{(1)}(t) := \mathbf{E}||U(t,0)\xi^+||^q_{L^2_\rho},$$

$$\bar{I}^{(2)}_M(t) := \mathbf{E}\left|\left|\int_0^t U(t,s)E_{0,M}(s,\bar{X}_{0,M}(s))\,ds\right|\right|^q_{L^2_\rho}$$

and

$$I^{(3)}(t) := \mathbf{E}\left|\left|\int_0^t U(t,s)\mathcal{M}_{C(s)}\,dL(s)\right|\right|^q_{L^2_\rho}.$$

Thus, for $t \in [\,0,T\,]$, we have

$$\text{(7.32)} \quad \mathbf{E}||\bar{X}_{0,M}(t)||^q_{L^2_\rho} \le c(q)[\bar{I}^{(1)}(t) + \bar{I}^{(2)}_M(t) + \bar{I}^{(3)}(t)].$$

Obviously $\bar{I}^{(1)}(t) \le c(q, c(T))\mathbf{E}||\xi||^q_{L^2_\rho}$.
By the same arguments as in **(i)**, we get

$$\bar{I}^{(2)}_M(t) \le c(q, K, T, c(T), c_e(T), C_{q,\eta}) \left(1 + \int_0^t \mathbf{E}||\bar{X}_{0,M}(s)||^q_{L^2_\rho}\, ds\right).$$

Applying first the Lévy-Itô decomposition 2.4.13 and then the Burkholder-Davis-Gundy and Bichteler-Jacod inequalities, we obtain for $q < \frac{2}{\zeta}$ that

$$I^{(3)}(t) \le c(q, \zeta, T, K, c(T), C_{q,\eta}).$$

Thus, by (7.32), we have for $t \in [\,0,T\,]$

$$\begin{aligned}\mathbf{E}||\bar{X}_{0,M}(t)||^q_{L^2_\rho} \le \quad & c(q, \zeta, K, T, c(T), c_e(T), C_{q,\eta})(1 + \mathbf{E}||\xi||^q_{L^2_\rho}) \\ & + c(q, K, T, c(T), c_e(T), C_{q,\eta}) \int_0^t, \mathbf{E}||\bar{X}_{0,M}(s)||^q_{L^2_\rho}\, ds.\end{aligned}$$

Hence, by Gronwall's Lemma we conclude that

$$\text{(7.33)} \quad \sup_{\substack{t\in[\,0,T\,]\\ M\in\mathbb{N}}} \mathbf{E}||\bar{X}_{0,M}(t)||^q_{L^2_\rho} \le c(q, \zeta, K, T, c(T), c_e(T), C_{q,\eta})(1 + \mathbf{E}||\xi||^q_{L^2_\rho}).$$

Next, we consider $\underline{X}_{N,M}$ for arbitrary $N, M \in \mathbb{N}$. Setting

$$\underline{I}^{(1)}(t) := \mathbf{E}||U(t,0)\xi^-||^q_{L^2_\rho}$$

and

$$\underline{I}^{(2)}_{N,M}(t) := \mathbf{E}\left|\left|\int_0^t U(t,s)E^-_{N,M}(s, \underline{X}_{N,M})\, ds\right|\right|^q_{L^2_\rho},\ t \in [\,0,T\,],$$

we get (with $I^{(3)}$ as before)

$$\mathbf{E}||\underline{X}_{N,M}(t)||^q_{L^2_\rho} \le c(q)[\underline{I}^{(1)}(t) + \underline{I}^{(2)}_{N,M}(t) + \bar{I}^{(3)}(t)].$$

Since obviously $\underline{I}^{(1)}(t) \le c(q, c(T))\mathbf{E}||\xi||^q_{L^2_\rho}$, it remains to consider $\underline{I}^{(2)}_{N,M}(t)$.
Recall that by construction

$$-N \le e^-_{N,M}(t, \omega, y) \le 0$$

for any $(t,\omega,y) \in [0,T] \times \Omega \times \mathbb{R}$. Thus, there is a constant depending on $N$, $q$, $T$ and $U$ such that

$$\underline{I}^{(2)}_{N,M}(t) \leq c(N,q,T,c(T)) < \infty.$$

Putting all the estimates together, we get for any $t \in [0,T]$

$$\mathbf{E}||\underline{X}_{N,M}(t)||^q_{L^2_\rho} \leq \underline{c}(N,q,\zeta,K,T,c(T),c_\sigma(T),C_{q,\eta})(1+\mathbf{E}||\xi||^q_{L^2_\rho}).$$

Since the previous estimate holds for arbitrary $t \in [0,T]$ and $M \in \mathbb{N}$, we have proven that

$$\sup_{\substack{t\in[0,T]\\ M\in\mathbb{N}}} \mathbf{E}||\underline{X}_{N,M}||^q_{L^2_\rho} \leq \underline{c}(N,q,\zeta,K,T,c(T),c_\sigma(T),C_{q,\eta})(1+\mathbf{E}||\xi||^q_{L^2_\rho}). \tag{7.34}$$

Finally, by (7.24), (7.33) and (7.34) we conclude that

$$\sup_{\substack{t\in[0,T]\\ M\in\mathbb{N}}} \mathbf{E}||X_{N,M}||^q_{L^2_\rho} \leq c(N,q,\zeta,K,T,c(T),c_\sigma(T),C_{q,\eta})(1+\mathbf{E}||\xi||^q_{L^2_\rho}) \tag{7.35}$$

with $c = \bar{c} + \underline{c}$, which finishes Step 2.

**Step 3:** Our aim is to show that $X_{N,M}$ converges, as $M \to \infty$, to a process $X_N$, which shall solve equation (7.1) resp. (7.2) with $F$ resp. $E$ being replaced by $F_N$ resp. $E_N$, $N \in \mathbb{N}$. Recall that $F_N$ and $E_N$ are defined by **(NEM)** with $f_N$ from (7.10) resp. $e_N$ defined analogously to $f_N$. In this step we only check that the limit process $X_N$ exists and belongs to $\mathcal{H}^q(T)$.

Let us define

$$Z_{N,M}(t) := X_{N,M}(t) - X_{N,1}(t),\ N,M \in \mathbb{N},\ t \in [0,T].$$

Thus, we have (by (7.24) and (7.32), (7.35))

$$0 \leq Z_{N,M}(t) \leq Z_{N,M+1}(t),\ t \in [0,T], \tag{7.36}$$

and

$$\begin{aligned}\sup_{\substack{t\in[0,T]\\ M\in\mathbb{N}}} \mathbf{E}||Z_{N,M}(t)||^q_{L^2_\rho} \leq{}& c(q)\left[\sup_{\substack{t\in[0,T]\\ M\in\mathbb{N}}} \mathbf{E}||X_{N,M}(t)||^q_{L^2_\rho} + \sup_{t\in[0,T]} \mathbf{E}||X_{N,1}(t)||^q_{L^2_\rho}\right]\\ <{}& \infty.\end{aligned}$$

Next, we define

$$0 \le Z_N(t) := \sup_{M\in\mathbb{N}} Z_{N,M}(t),\ N \in \mathbb{N},\ t \in [\,0,T\,]. \tag{7.37}$$

Actually, for each $N \in \mathbb{N}$ and $t \in [\,0,T\,]$, the random variable $Z_N(t)$ is uniquely defined up to a $P \otimes \mu_\rho$-zero set in $\Omega \times \Theta$ (which depends on the $\mathcal{B}(\Omega) \otimes \mathcal{B}(\Theta)$-measurable representations chosen for $Z_{N,M}$).
By (7.24) and B.Levi's monotone convergence theorem, we get

$$\sup_{t\in[\,0,T\,]} \mathbf{E}||Z_N(t)||^q_{L^2_\rho} = \sup_{t\in[\,0,T\,]} \sup_{M\in\mathbb{N}} \mathbf{E}||Z_{N,M}(t)||^q_{L^2_\rho} < \infty. \tag{7.38}$$

By construction, $t \mapsto Z_N(t) \in L^2_\rho$ obeys a predictable modification. Thus, $(Z_N(t))_{t\in[\,0,T\,]}$ is a process in $\mathcal{H}^q(T)$ for any $N \in \mathbb{N}$.
Finally, we define

$$X_N(t) := Z_N(t) + X_{N,1}(t),\ t \in [\,0,T\,],\ N \in \mathbb{N}. \tag{7.39}$$

Obviously, $[\,0,T\,] \ni t \mapsto X_N(t) \in L^2_\rho$ is again predictable as a sum of predictable processes. From (7.35) and (7.38), we get

$$\sup_{t\in[\,0,T\,]} \mathbf{E}||X_N(t)||^q_{L^2_\rho} < \infty \tag{7.40}$$

such that $X_N \in \mathcal{H}^q(T)$ for any $N \in \mathbb{N}$.

Now, we check that for each fixed $N \in \mathbb{N}$ $X_{N,M}$ converges to $X_N$ in $\mathcal{H}^q(T)$ as $M \to \infty$.
Indeed, by (7.37) and B.Levi's monotone convergence theorem we have, for each $t \in [\,0,T\,]$,

$$\lim_{M\to\infty} \mathbf{E}||X_{N,M}(t) - X_N(t)||^q_{L^2_\rho} = \lim_{M\to\infty} \mathbf{E}||Z_{N,M}(t) - Z_N(t)||^q_{L^2_\rho} = 0. \tag{7.41}$$

Herefrom, by (7.38) and Lebesgue's dominated convergence theorem we immediately get

$$\lim_{M\to\infty} \int_0^T \mathbf{E}||X_{N,M}(t) - X_N(t)||^q_{L^2_\rho}\, dt = \lim_{M\to\infty} \int_0^T \mathbf{E}||Z_{N,M}(t) - Z_N(t)||^q_{L^2_\rho}\, dt = 0. \tag{7.42}$$

In the same manner, we construct processes $\underline{X}_N$, $\bar{X} \in \mathcal{H}^q(T)$ such that

$$\begin{aligned} &\lim_{M\to\infty} \int_0^T \mathbf{E}||\underline{X}_{N,M}(t) - \underline{X}_N(t)||^q_{L^2_\rho}\, ds = 0, \\ &\lim_{M\to\infty} \int_0^T \mathbf{E}||\bar{X}_{0,M}(t) - \bar{X}(t)||^q_{L^2_\rho}\, ds = 0, \end{aligned} \tag{7.43}$$

and (by (7.24), (7.25))

$$\begin{aligned} &\underline{X}_N(t) \le X_N(t) \le \bar{X}(t), \\ (7.44) \qquad & \\ &\underline{X}_N(t) \le V(t) \le \bar{X}(t), \end{aligned}$$

$P$-almost surely, for all $t \in [0,T]$.

**Step 4:** We show that for each $N \in \mathbb{N}$, the process $X_N$ defined in Step 3 solves (7.1) resp. (7.2) in the case of $F$ resp. $E$ being replaced by $F_N$ resp. $E_N$ described in the beginning of Step 3.
Furthermore, we show that $t \mapsto X_N(t)$ is continuous in $L^q(\Omega; L^2_\rho)$ and that, under the additional assumption **(A7)**, there is a càdlàg version of $[0,T] \ni t \mapsto X_N(t) \in L^2_\rho$.

By (7.42), there is a subsequence of $(X_{N,M})_{M\in\mathbb{N}}$ that converges $P\otimes ds\otimes d\mu_\rho$-almost everywhere to $X_N$. Without loss of generality, we assume $(X_{N,M})_{M\in\mathbb{N}}$ itself to be this sequence.

**(i) The Poisson case - equation (7.1)**

Putting

$$\begin{aligned} I^{(1)}_{N,M}(t) &:= \mathbf{E}||X_N(t) - X_{N,M}(t)||^2_{L^2_\rho}, \\ I^{(2)}_{N,M}(t) &:= \mathbf{E}\left|\left|\int_0^t U(t,s)[F_N(s,X_N(s)) - F_{N,M}(s,X_{N,M}(s))]\,ds\right|\right|^2_{L^2_\rho}, \\ I^{(3)}_{N,M}(t) &:= \mathbf{E}\left|\left|\int_0^t U(t,s)[\mathcal{M}_{\Sigma(s,X_N(s))} - \mathcal{M}_{\Sigma(s,X_{N,M}(s))}]\,dW(s)\right|\right|^2_{L^2_\rho}, \ t \in [0,T], \end{aligned}$$

we have, for a fixed $t \in [0,T]$,

$$\begin{aligned} &\mathbf{E}\Bigg|\Bigg|X_N(t) - U(t,0)\xi - \int_0^t U(t,s)F_N(s,X_N(s))\,ds - \int_0^t U(t,s)\mathcal{M}_{\Sigma(s,X_N(s))}\,dW(s) \\ &- \int_0^t \int_{L^2} U(t,s)\mathcal{M}_{C(s)}(x)\,\tilde{N}(ds,dx)\Bigg|\Bigg|^2_{L^2_\rho} \\ &\le C[I^{(1)}_{N,M}(t) + I^{(2)}_{N,M}(t) + I^{(3)}_{N,M}(t)]. \end{aligned}$$

Thus, by (7.41) at least the first term tends to 0 as $M \to \infty$. Let us consider the second term.

$$\begin{aligned} I^{(2)}_{N,M}(t) \leq\ & 2c(T)\left(\int_0^T \mathbf{E}||F_N(s,X_N(s)) - F_N(s,X_{N,M}(s))||^2_{L^2_\rho}\,ds\right. \\ & \left.+\int_0^T \mathbf{E}||F_N(s,X_{N,M}(s)) - F_{N,M}(s,X_{N,M}(s))||^2_{L^2_\rho}\,ds\right) \\ =:\ & 2c(T)(I^{(21)}_{N,M}(T) + I^{(22)}_{N,M}(T)). \end{aligned}$$

Note that, by the continuity of $f_N$ and the convergence property of $X_N$, we have for almost all $s \in [\,0,T\,]$

$$|f_N(s,\omega,X_N(s,\omega,\theta)) - f_N(s,\omega,X_{N,M}(s,\omega,\theta))| \to 0 \text{ as } M \to \infty$$

for $P \otimes d\mu_\rho$ almost all $(\omega,\theta) \in \Omega \times \Theta$.
Condition **(PG)** with exponent $\nu = 1$ and the relation

$$X_{N,1}(s) \leq X_{N,M}(s) \leq X_N(s),\ N,M \in \mathbb{N}$$

imply, for almost all $s \in [\,0,T\,]$,

$$\begin{aligned} &|f_N(s,\omega,X_N(s,\omega,\theta)) - f_N(s,\omega,X_{N,M}(s,\omega,\theta))| \\ &\leq 2c(N,c_f(T))(1+|X_N(s,\omega,\theta)|+|X_{N,1}(s,\omega,\theta)|),\ P\otimes d\mu_\rho\text{-almost surely.} \end{aligned}$$

To apply Lebesgue's theorem, we need integrability of the majorizing mapping

$$\begin{aligned} (7.45)\quad & \sup_{s\in[\,0,T\,]} \int_\Omega\int_\Theta |X_N(s,\omega,\theta)|^2 + |X_{N,1}(s,\omega,\theta)|^2\,\mu_\rho(d\theta)P(d\omega) \\ = & \sup_{s\in[\,0,T\,]} (\mathbf{E}||X_N(s)||^2_{L^2_\rho} + \mathbf{E}||X_{N,1}(s)||^2_{L^2_\rho}) < \infty. \end{aligned}$$

The right hand side in (7.45) is finite, since $X_N, X_{N,1} \in \mathcal{H}^q(T) \subset \mathcal{H}^2(T)$ by Step 3.
Thus, Lebesgue's theorem is applicable and gives us first

$$\lim_{M\to\infty} \mathbf{E}||F_N(s,X_N(s)) - F_N(s,X_{N,M}(s))||^2_{L^2_\rho} = 0 \text{ for almost all } s \in [\,0,T\,]$$

and hence

$$\lim_{M\to\infty} \int_0^T \mathbf{E}||F_N(s,X_N(s)) - F_N(s,X_{N,M}(s))||^2_{L^2_\rho}\,ds = 0.$$

To estimate $I^{(22)}_{N,M}$, we use the following trick, which was already implemented in [76].

Let us fix some $L \leq M$, $L,M \in \mathbb{N}$. In full analogy to the consideration of $I^{(21)}_{N,M}$, the fact that $f_{N,M} \uparrow f_N$ (and thus $f_N - f_{N,M} \downarrow 0$) as $M \to \infty$

implies

$$\begin{aligned}
\lim_{M\to\infty} I^{(22)}_{N,M} \le & \quad \lim_{M\to\infty} \int_0^T \mathbf{E}||F_N(s, X_{N,M}(s)) - F_{N,L}(s, X_{N,M}(s))||^2_{L^2_\rho}\, ds \\
= & \quad \int_0^T \mathbf{E}||F_N(s, X_N(s)) - F_{N,L}(s, X_N(s))||^2_{L^2_\rho}\, ds,
\end{aligned}$$

which holds for any $L \in \mathbb{N}$. Letting $L \to \infty$ and noting that $f_{N,L} \uparrow f_N$ as $L \to \infty$ gives us, by Lebesgue's convergence theorem, that

$$\lim_{M\to\infty} I^{(22)}_{N,M}(T) = 0.$$

Thus, we have

$$\lim_{M\to\infty} I^{(2)}_{N,M}(t) = 0,\ t \in [\,0, T\,].$$

Finally, we examine $I^{(3)}_{N,M}$.
By Itô's isometry, **(A2)**, **(LC)**, Hölder's inequality and the fact that

$$q > \tfrac{2}{1-\zeta} \iff \tfrac{\zeta q}{q-2} < 1,$$

we have

$$\begin{aligned}
I^{(3)}_{N,M}(t) \le & \quad c^2(T)\mathbf{E}\int_0^t (t-s)^{-\zeta}||\Sigma(s, X_N(s)) - \Sigma(s, X_{N,M}(s))||^2_{L^2_\rho}\, ds \\
\le & \quad c(q, \zeta, T, c(T), c_\sigma(T)) \left( \mathbf{E}\int_0^T ||X_N(s) - X_{N,M}(s)||^q_{L^2_\rho}\, ds \right)^{\frac{2}{q}},
\end{aligned}$$

which by (7.42) tends to 0 as $M \to \infty$.

So, $X_N$ solves the equation in the sense of 7.1.1 **(i)**, when $F$ is replaced by $F_N$ for arbitrary $N \in \mathbb{N}$.

Substituting $X_N$ in the equation (7.1), we now obtain from the above estimates

$$\begin{aligned}
||X_N - X_{N,M}||^2_{\mathcal{H}^2(T)} = & \quad \sup_{t\in[\,0,T\,]} \mathbf{E}||X_N(t) - X_{N,M}(t)||^2_{L^2_\rho} \\
\le & \quad 3[I^{(1)}_{N,M}(t) + I^{(2)}_{N,M}(t) + I^{(3)}_{N,M}(t)] \to 0 \text{ as } M \to \infty.
\end{aligned}$$

Similar reasoning shows that $\underline{X}_N$ solves equation (7.1) with $\xi^-$ and $f_N^-$ and $\bar{X}$ solves equation (7.1) with $\xi^+$ and $f^+$.
The required continuity properties of the solutions $X_N$, $\underline{X}_N$ and $\bar{X}$ follow immediately from the corresponding properties of the integrals in the right hand side of (7.4), which were established in Section 5.1.

**(ii) The Lévy case - equation (7.2)**
Setting

$$I^{(1)}_{N,M}(t) := \mathbf{E}||X_N(t) - X_{N,M}(t)||^2_{L^2_\rho},$$
$$I^{(2)}_{N,M}(t) := c^2(T)\mathbf{E}\left|\left|\int_0^t E_N(s, X_N(s)) - E_{N,M}(s, X_{N,M}(s))\, ds\right|\right|^2_{L^2_\rho},\ t \in [\,0, T\,],$$

we have, for each $t \in [\,0, T\,]$,

$$\mathbf{E}\left|\left|X_N(t) - U(t,0)\xi - \int_0^t U(t,s)E_N(s, X_N(s))\, ds - \int_0^t U(t,s)\mathcal{M}_{C(s)}\, dL(s)\right|\right|^2_{L^2_\rho}$$
$$\leq 2(I^{(1)}_{N,M}(t) + I^{(2)}_{N,M}(t)).$$

But $\lim\limits_{M\to\infty} I^{(1)}_{N,M}(t) = 0$ by (7.41), whereas $\lim\limits_{M\to\infty} I^{(2)}_{N,M}(t) = 0$ just by replacing $F$-terms by $E$-terms in the above reasoning for (7.1).
Thus, $X_N$ solves (7.2) in the sense of 7.1.1 **(i)** with $E$ being replaced by $E_N$.
The continuity properties of $X_N$ follow analogously to the case **(i)**.

**Step 5:** In this final step, we shall check that

$$X(t) := \inf_{N\in\mathbb{N}} X_N(t),\ t \in [\,0, T\,] \tag{7.46}$$

solves equation (7.1) resp. (7.2).
To this end, we first show that

$$\sup_{N\in\mathbb{N}} \sup_{t\in[\,0,T\,]} \mathbf{E}||X_N(t)||^q_{L^2_\rho} < \infty. \tag{7.47}$$

Recall that by the above construction

$$\underline{X}_N(t) \leq X_N(t) \leq \bar{X}(t) \text{ in } L^2_\rho.$$

The process $\bar{X} \in \mathcal{H}^q(T)$ was defined in Step 3. Thus, in particular, we have

$$\sup_{t\in[\,0,T\,]} \mathbf{E}||\bar{X}(t)||^q_{L^2_\rho} < \infty.$$

Therefore, it would suffice to establish the $N$-independent estimate for $\underline{X}_N$.

From Step 4 we already know that

$$\sup_{t\in[\,0,T\,]} \mathbf{E}||\underline{X}_N(t)||^q_{L^2_\rho} \leq c(N,q)(1 + \mathbf{E}||\xi||^q_{L^2_\rho}),$$

where the constant on the right hand side depends on $N$. Now, we will improve the estimate by using Gronwall's lemma.

**(i) The case of Poisson noise - equation (7.1)**

Setting

$$\begin{aligned}
\underline{I}^{(1)}(t) &:= \mathbf{E}||U(t,0)\xi^-||^q_{L^2_\rho} \le c^q(T)\mathbf{E}||\xi||^q_{L^2_\rho},\\
\underline{I}^{(2)}_N(t) &:= \mathbf{E}\left|\left|\int_0^t U(t,s)F^-_N(s,\underline{X}_N(s))\,ds\right|\right|^q_{L^2_\rho},\\
\underline{I}^{(3)}_N(t) &:= \mathbf{E}\left|\left|\int_0^t U(t,s)\mathcal{M}_{\Sigma(s,\underline{X}_N(s))}\,dW(s)\right|\right|^q_{L^2_\rho}
\end{aligned}$$

and

$$I^{(4)}(t) := \mathbf{E}\left|\left|\int_0^t\int_{L^2} U(t,s)\mathcal{M}_{C(s)}(x)\,\tilde{N}(ds,dx)\right|\right|^q_{L^2_\rho},\ t\in[0,T],$$

we have, for each $t \in [0,T]$,

$$\mathbf{E}||\underline{X}_N(t)||^q_{L^2_\rho} \le c(q)[\underline{I}^{(1)}(t) + \underline{I}^{(2)}_N(t) + \underline{I}^{(3)}_N(t) + I^{(4)}(t)].$$

We proceed analogously to the case of $\bar{I}^{(2)}_M$ and $\bar{I}^{(3)}_M$ considered in Step 2. By means of (7.17) and (7.19), we get

$$\begin{aligned}
\underline{I}^{(2)}_N(t) \le\ & c(q,c(T))\int_0^t ||F^-_N(s,\underline{X}_N(s))||^q_{L^2_\rho}\,ds\\
\le\ & c(q,c(T))\int_0^t \mathbf{E}||\bar{g}(\underline{X}_N(s))||^q_{L^2_\rho}\,ds\\
\le\ & c(q,T,c(T),c_f(T))\left(1+\int_0^t \mathbf{E}||\underline{X}_N(s)||^q_{L^2_\rho}\,ds\right),
\end{aligned}$$

where we used that $\underline{X}_N(t) \le V(t)$ and $\bar{g} \le f^-$.
Then, by the Burkholder-Davis-Gundy inequality 2.5.4/2.5.6, **(A2)**, Hölder‘s inequality, the fact that

$$q > \tfrac{q}{1-\zeta} \iff \tfrac{q}{q-2} < 1$$

and the Lipschitz and monotonicity assumption **(LC)**, **(LB)**, we get

$$\begin{aligned}\underline{I}_N^{(3)}(t) \leq \quad & c(q,c(T))\mathbf{E}\left(\int\limits_0^t (t-s)^{-\zeta}||\Sigma(s,\underline{X}_N(s))||_{L^2_\rho}^2\,ds\right)^{\frac{q}{2}}\\ \leq \quad & c(q,\zeta,T,c(T),c_\sigma(T))\left(1+\int\limits_0^t \mathbf{E}||\underline{X}_N(s)||_{L^2_\rho}^q\,ds\right).\end{aligned}$$

Recall from Step 2 that

$$I^{(4)}(t) \leq c(q,\zeta,K,T,C_{q,\eta}),\ t\in[\,0,T\,].$$

Putting the four estimates together, we get for all $t\in[\,0,T\,]$

$$\begin{aligned}\mathbf{E}||\underline{X}_N(t)||_{L^2_\rho}^q \leq \quad & c(q,\zeta,K,T,c(T),c_f(T),c_\sigma(T),C_{q,\eta})(1+\mathbf{E}||\xi||_{L^2_\rho}^q)\\ & +c(q,\zeta,T,c(T),c_f(T),c_\sigma(T))\int\limits_0^t \mathbf{E}||\underline{X}_N(s)||_{L^2_\rho}^q\,ds,\end{aligned}$$

and herefrom by Gronwall's lemma

$$\mathbf{E}||\underline{X}_N(t)||_{L^2_\rho}^q \leq c(q,\zeta,K,T,c(T),c_f(T),c_\sigma(T),C_{q,\eta})(1+\mathbf{E}||\xi||_{L^2_\rho}^q).$$

Hence, we have proven that

$$\sup_{\substack{t\in[\,0,T\,]\\ N\in\mathbb{N}}} \mathbf{E}||\underline{X}_N(t)||_{L^2_\rho}^q \leq c_1(q,\zeta,K,T,c(T),c_f(T),c_\sigma(T),C_{q,\eta})(1+\mathbf{E}||\xi||_{L^2_\rho}^q),$$

which implies (7.47).

**(ii) The case of Lévy noise - equation (7.2)**

We set

$$\begin{aligned}\underline{I}^{(1)}(t) &:= \mathbf{E}||U(t,0)\xi^-||_{L^2_\rho}^q \leq c^q(T)\mathbf{E}||\xi||_{L^2_\rho}^q,\\ \underline{I}_N^{(2)}(t) &:= \mathbf{E}\left|\left|\int\limits_0^t U(t,s)E_N^-(s,\underline{X}_N(s))\,ds\right|\right|_{L^2_\rho}^q,\\ I^{(3)}(t) &:= \mathbf{E}\left|\left|\int\limits_0^t U(t,s)\mathcal{M}_{C(s)}\,dL(s)\right|\right|_{L^2_\rho}^q,\ t\in[\,0,T\,],\end{aligned}$$

and obtain, for any $t\in[\,0,T\,]$,

$$\mathbf{E}||\underline{X}_N(t)||_{L^2_\rho}^q \leq c(q)\left(\underline{I}^{(1)}(t)+\underline{I}_N^{(2)}(t)+I^{(3)}(t)\right).$$

Replacing $F_N$ by $E_N$ in the previous arguments, we get

$$\underline{I}_N^{(2)}(t) \leq c(q,T,c(T),c_e(T))\left(1+\int\limits_0^t \mathbf{E}||\underline{X}_N(s)||_{L^2_\rho}^q\,ds\right).$$

Analogously to Step 2, we have

$$\underline{I}^{(3)} \le c(q, \zeta, m, K, T, c(T), C_{q,\eta}).$$

Putting the three estimates together and applying Gronwall's lemma, we conclude that

$$\mathbf{E}||\underline{X}_N(t)||^q_{L^2_\rho} \le c_1(q, \zeta, K, T, c(T), c_e(T), C_{q,\eta})(1 + \mathbf{E}||\xi||^q_{L^2_\rho})$$

for all $N \in \mathbb{N}$ and $t \in [\,0, T\,]$. Thus,

$$\sup_{\substack{t\in[\,0,T\,]\\ N\in\mathbb{N}}} \mathbf{E}||\underline{X}_N(t)||^q_{L^2_\rho} \le c_1(q, \zeta, K, T, c(T), c_e(T), C_{q,\eta})(1 + \mathbf{E}||\xi||^q_{L^2_\rho}),$$

which is the $N$-independent estimate on the moments of $\underline{X}_N$, we needed to prove.

Next, we consider the process $\bar{X} \in \mathcal{H}^q(T)$. Recall that this process was defined in Step 3 as a limit of $\bar{X}_{0,M}$ for $M \to \infty$. More precisely, by (7.41) we have

$$\lim_{M\to\infty} \mathbf{E}||\bar{X}_{0,M}(t) - \bar{X}(t)||^q_{L^2_\rho} = 0 \text{ for each } t \in [\,0, T\,].$$

By Step 2, we know that

$$\sup_{\substack{t\in[\,0,T\,]\\ M\in\mathbb{N}}} \mathbf{E}||\bar{X}_{0,M}(t)||^q_{L^2_\rho} \le c(q, \zeta, K, T, c(T), c_f(T), c_\sigma(T), C_{q,\eta})(1 + \mathbf{E}||\xi||^q_{L^2_\rho})$$

in the Poisson noise case resp.

$$\sup_{\substack{t\in[\,0,T\,]\\ M\in\mathbb{N}}} \mathbf{E}||\bar{X}_{0,M}(t)||^q_{L^2_\rho} \le c(q, \zeta, m, K, T, c(T), c_e(T), C_{q,\eta})(1 + \mathbf{E}||\xi||^q_{L^2_\rho})$$

in the Lévy noise case.

Thus, we get

$$\sup_{t\in[\,0,T\,]} \mathbf{E}||\bar{X}(t)||^q_{L^2_\rho} \le c_3(q, \zeta, K, T, c(T), c_f(T), c_\sigma(T), C_{q,\eta})(1 + \mathbf{E}||\xi||^q_{L^2_\rho})$$

in the Poisson noise case resp.

$$\sup_{t\in[\,0,T\,]} \mathbf{E}||\bar{X}(t)||^q_{L^2_\rho} \le c_4(q, \zeta, K, T, c(T), c_e(T), C_{q,\eta})(1 + \mathbf{E}||\xi||^q_{L^2_\rho})$$

in the Lévy noise case.
By construction (cf. (7.44)) we have for all $t \in [\,0, T\,]$

$$\underline{X}_N(t) \leq X_N(t) \leq \bar{X}(t) \text{ in } L^2_\rho,$$

which leads to

$$\begin{aligned}
&\sup_{\substack{t\in[0,T]\\ N\in\mathbb{N}}} \mathbf{E}||X_N(t)||^q_{L^2_\rho}\\
&\leq (c_1(q,\zeta,K,T,c(T),c_f(T),c_\sigma(T),C_{q,\eta})\\
&+c_3(q,\zeta,K,T,c(T),c_f(T),c_\sigma(T),C_{q,\eta}))\,(1+\mathbf{E}||\xi||^q_{L^2_\rho})
\end{aligned}$$

(in the Poisson noise case) and, respectively, (in the Lévy noise case)

$$\begin{aligned}
&\sup_{\substack{t\in[0,T]\\ N\in\mathbb{N}}} \mathbf{E}||X_N(t)||^q_{L^2_\rho}\\
&\leq (c_2(q,\zeta,K,T,c(T),c_e(T),c_\sigma(T),C_{q,\eta})\\
&+c_4(q,\zeta,K,T,c(T),c_e(T),C_{q,\eta}))\,(1+\mathbf{E}||\xi||^q_{L^2_\rho}).
\end{aligned}$$

Thus, we have shown that there are $N$-independent estimates for the $X_N$ both in the Poisson and the Lévy noise case.
Next, we define our candidates for the solution to (7.1) resp. (7.2).
Since $f_N \downarrow f$ resp. $e_N \downarrow e$, the comparison from Theorem 6.1.1 implies

$$\begin{aligned}
&X_{N+1}(t) \leq X_N(t),\\
&\underline{X}_{N+1}(t) \leq \underline{X}_N(t),
\end{aligned} \tag{7.48}$$

$P$-a.s., for all $t \in [\,0,T\,]$, $N \in \mathbb{N}$.

We claim that

$$X(t) := \inf_{N\in\mathbb{N}} X_N(t)\,,\; t \in [\,0,T\,],$$

is a solution in the sense of 7.1.2 **(i)** both in the Poisson and the Lévy noise case.
We would like to proceed similarly to Step 3 and 4. But, in contrast to the $(X_{N,M})_{M\in\mathbb{N}}$, the sequence $(X_N)_{N\in\mathbb{N}}$ is decreasing.
Thus, we define

$$Y_N(t) := X_1(t) - X_N(t),\; t \in [\,0,T\,],\; N \in \mathbb{N}.$$

By (7.48), this is a sequence of random variables in $L^2_\rho$ with the properties

$$0 \leq Y_N(t) \leq Y_{N+1}(t),\; P\text{-a.s.},\; t \in [\,0,T\,],\; N \in \mathbb{N},$$

and

$$\begin{aligned}\sup_{t\in[0,T]} \mathbf{E}||Y_N(t)||^q_{L^2_\rho} &= \sup_{t\in[0,T]} \mathbf{E}||X_1(t) - X_N(t)||^q_{L^2_\rho} \\ &\leq c(q)\left(\sup_{t\in[0,T]} \mathbf{E}||X_1(t)||^q_{L^2_\rho} + \sup_{t\in[0,T]} \mathbf{E}||X_N(t)||^q_{L^2_\rho}\right) \\ &< \infty.\end{aligned}$$

Hence, $Y_N \in \mathcal{H}^q(T)$ for any $N \in \mathbb{N}$.
Now, analogously to the definition of $X_N$ in Step 3, we set

$$Y(t) := \sup_{N\in\mathbb{N}} Y_N(t) = X_1(t) - X(t),\ t \in [0, T]. \tag{7.49}$$

Note that, for each $t \in [0, T]$, the random variable $Y(t)$ is uniquely defined up to a zero set in $\Omega \times \Theta$ .
By its construction, $t \mapsto Y(t)$ obeys a predictable modification. By B.Levi's monotone convergence theorem and the previous estimate on $Y_N$, we get

$$\begin{aligned}\sup_{t\in[0,T]} \mathbf{E}||Y(t)||^q_{L^2_\rho} &= \sup_{t\in[0,T]}\left[\sup_{N\in\mathbb{N}} \mathbf{E}||Y_N(t)||^q_{L^2_\rho}\right] \\ &\leq 2c(q) \sup_{\substack{t\in[0,T]\\ N\in\mathbb{N}}} \mathbf{E}||X_N(t)||^q_{L^2_\rho} < \infty,\end{aligned} \tag{7.50}$$

where the last term is finite by (7.47).
Since by (7.49)

$$X(t) = Y(t) - X_1(t),\ t \in [0, T],$$

we have proven that $X \in \mathcal{H}^q(T)$.

Next, we show that

$$\begin{aligned}&\lim_{N\to\infty} \int_0^T \mathbf{E}||X_N(t) - X(t)||^q_{L^2_\rho}\, dt \\ &= \lim_{N\to\infty} \int_0^T \mathbf{E}||Y_N(t) - Y(t)||^q_{L^2_\rho}\, dt = 0.\end{aligned}$$

As $|Y_N(t) - Y(t)| \leq 2|Y(t)|$ $P$-almost surely, by (7.50) Lebesgue's theorem is applicable and gives us

$$\lim_{N\to\infty} \mathbf{E}||Y_N(t) - Y(t)||^q_{L^2_\rho} = 0,\ t \in [0, T], \tag{7.51}$$

which in turn implies

$$\lim_{N\to\infty} \mathbf{E}||X_N(t) - X(t)||^q_{L^2_\rho} = 0,\ t \in [0, T], \tag{7.52}$$

and

$$\lim_{N\to\infty} \mathbf{E} \int_0^T ||X_N(t) - X(t)||^q_{L^2_\rho}\, dt = 0. \tag{7.53}$$

It remains to show that $X$ solves equation (7.1) resp. (7.2). We apply the method used in Step 4 for $X_N$.
We denote the process on the right hand side of (7.3) by $K(X)$ and the process on the right hand side of (7.4) by $\bar{K}(X)$. Then, by setting

$$I_N^{(1)}(t) := \mathbf{E}||X(t) - X_N(t)||^2_{L^2_\rho},$$

$$I_N^{(2)}(t) := \mathbf{E}\left|\left|\int_0^t U(t,s)[F(s,X(s)) - F_N(s,X_N(s))]\, ds\right|\right|^2_{L^2_\rho},$$

$$\bar{I}_N^{(2)}(t) := \mathbf{E}\left|\left|\int_0^t U(t,s)[E(s,X(s)) - E_N(s,X_N(s))]\, ds\right|\right|^2_{L^2_\rho}$$

and

$$I_N^{(3)}(t) := \mathbf{E}\left|\left|\int_0^t U(t,s)[\mathcal{M}_{\Sigma(s,X(s))} - \mathcal{M}_{\Sigma(s,X_N(s))}]\, dW(s)\right|\right|^2_{L^2_\rho},\ t \in [\,0,T\,],$$

we get, for each $t \in [\,0,T\,]$,

$$\mathbf{E}||X(t) - K(X)(t)||^2_{L^2_\rho} \le 3(I_N^{(1)}(t) + I_N^{(2)}(t) + I_N^{(3)}(t))$$

for (7.1) resp.

$$\mathbf{E}||X(t) - \bar{K}(X)(t)||^2_{L^2_\rho} \\ \le 2(I_N^{(1)}(t) + \bar{I}_N^{(2)}(t))$$

for (7.2). Analogously to Step 4, we find that

$$I_N^{(1)}(t) \le c(q,\rho)\left(\mathbf{E}||X(t) - X_N(t)||^q_{L^2_\rho}\right)^{\frac{2}{q}}$$

$$\begin{aligned} I_N^{(2)}(t) \le \quad & 2c(t)\left(\int_0^T \mathbf{E}||F(s,X(s)) - F(s,X_N(s))||^2_{L^2_\rho}\, ds \right. \\ & \left. + \int_0^T \mathbf{E}||F(s,X_N(s)) - F_N(s,X_N(s))||^2_{L^2_\rho}\, ds\right) \\ =: \quad & I_N^{(21)}(T) + I_N^{(22)}(T). \end{aligned}$$

Note that, by (7.53), there is a subsequence of $(X_N)_{N\in\mathbb{N}}$ converging to $X$ $P\otimes ds\otimes\mu_\rho$-everywhere on $[\,0,T\,]\times\Omega\times\Theta$. Let us suppose that $(X_N)_N$ itself is this sequence.
By the continuity of $f$ and the convergence of $X_N$, we have for almost all $s\in[\,0,T\,]$

$$f(s,\omega,X_N(s,\omega,\theta))\to f(s,\omega,X(s,\omega,\theta)) \text{ as } N\to\infty$$

for $P\otimes\mu_\rho$-almost all $(\omega,\theta)\in\Omega\times\Theta$.
Condition **(PG)** with exponent $\nu=1$ and the bound

$$X(t)\le X_N(t)\le X_1(t),\ t\in[\,0,T\,],\ N\in\mathbb{N},$$

imply, for any $s\in[\,0,T\,]$,

$$\begin{aligned}&|f(s,\omega,X(s,\omega,\theta))-f(s,\omega,X_N(s,\omega,\theta))|\\ &\le c(c_f(T))(1+|X(s,\omega,\theta)|+|X_N(s,\omega,\theta)|)\\ &\le 2c(c_f(T))(1+|X(s,\omega,\theta)|+|X_1(s,\omega,\theta)|)\end{aligned}$$

for $P\otimes\mu_\rho$-almost all $(\omega,\theta)\in\Omega\times\Theta$.
It is easy to check that

$$\begin{aligned}&\sup_{s\in[\,0,T\,]}\int_\Omega\int_\Theta(|X(s,\omega,\theta)|^2+|X_1(s,\omega,\theta)|^2)\mu_\rho(d\Theta)\,P(d\omega)\\ &\le\sup_{s\in[\,0,T\,]}(\mathbf{E}||X(s)||^2_{L^2_\rho}+\mathbf{E}||X_1(s)||^2_{L^2_\rho})<\infty,\end{aligned}$$

since $X,X_1\in\mathcal{H}^q(T)\subset\mathcal{H}^2(T)$. Thus, Lebesgue's theorem is applicable and gives us

$$\begin{aligned}&\lim_{N\to\infty}I_N^{(21)}(T)\\ &=\int_0^T\int_\Omega\int_\Theta(f(s,\omega,X(s,\omega,\theta))-f(s,\omega,X_N(s,\omega,\theta)))^2\,\mu_\rho(d\theta)\,P(d\omega)\,ds\\ &=0.\end{aligned}$$

To estimate $I_N^{(22)}(T)$, let us fix some $K\le N$, $K,N\in\mathbb{N}$. By the definition $f_N:=f\vee(-N)\downarrow f$ as $N\to\infty$, we always have

$$f^2\ge f_N^2 \text{ and } (f-f_N)^2\le(f-f_K)^2 \tag{7.54}$$

whenever $K\le N$.
By the preceeding arguments, used to prove that $I_N^{(21)}(t)\to 0$ as $N\to\infty$, we get

$$\begin{aligned}\lim_{N\to\infty} I_N^{(22)}(T) \leq & \lim_{N\to\infty} \int_0^T \mathbf{E}||F(s,X_N(s)) - F_K(s,X_N(s))||^2_{L^2_\rho}\, ds \\ = & \int_0^T \mathbf{E}||F(s,X(s)) - F_K(s,X(s))||^2_{L^2_\rho}\, ds,\end{aligned}$$

which holds for any $K \in \mathbb{N}$.
Letting here $K \to \infty$ and noting that $f_K \downarrow f$ gives us that $\lim\limits_{N\to\infty} I_N^{(22)}(T) \to 0$.
Replacing the $F$-terms by $E$-terms, we know that $\bar{I}_N^{(2)}(t) \to 0$ as $N \to \infty$. Therefore, $X = K(X)$ solves equation (7.2) in the sense of 7.1.1 **(i)**.
By (7.53) we also get $\lim\limits_{N\to\infty} I_N^{(3)}(T) = 0$ by the same arguments as in Step 4. Thus, $X = \bar{K}(X)$ solves (7.1) in the sense of 7.1.1 **(i)**.

Concerning the required continuity properties of $t \mapsto X(t) \in L^2_\rho$, we note that, analogously to the $X_N(t)$ from Step 4, the Lipschitz property (LC) for $\sigma$ and $c$ gives us similar properties for the stochastic integrals on the right hand side of (7.4) resp. (7.5).

It remains to show the a-priori bounds (7.6), (7.7). We note that, both in the Poisson and the Lévy noise case, we have for any $t \in [\,0,T\,]$

$$\mathbf{E}||X(t)||^q_{L^2_\rho} \leq c(q) \left[\mathbf{E}||Y(t)||^q_{L^2_\rho} + \mathbf{E}||X_1(t)||^q_{L^2_\rho}\right],$$

and thus

$$\begin{aligned}\sup_{t\in[\,0,T\,]} \mathbf{E}||X(t)||^q_{L^2_\rho} \leq & \; c(q) \left[\sup_{t\in[\,0,T\,]} \mathbf{E}||Y(t)||^q_{L^2_\rho} + \sup_{t\in[\,0,T\,]} \mathbf{E}||X_1(t)||^q_{L^2_\rho}\right] \\ \leq & \; c(q) \left[\sup_{t\in[\,0,T\,]} \mathbf{E}||Y(t)||^q_{L^2_\rho} + \sup_{\substack{t\in[\,0,T\,]\\ N\in\mathbf{N}}} \mathbf{E}||X_N(t)||^q_{L^2_\rho}\right] \\ \leq & \; c(q,\zeta,K,T,c(T),c_f(T),c_\sigma(T),C_{q,\eta})(1+\mathbf{E}||\xi||^q_{L^2_\rho})\end{aligned}$$

in the Poisson noise case resp.

$$\begin{aligned}\sup_{t\in[\,0,T\,]} \mathbf{E}||X(t)||^q_{L^2_\rho} \leq & \; c(q) \left[\sup_{t\in[\,0,T\,]} \mathbf{E}||Y(t)||^q_{L^2_\rho} + \sup_{t\in[\,0,T\,]} \mathbf{E}||X_1(t)||^q_{L^2_\rho}\right] \\ \leq & \; c(q) \left[\sup_{t\in[\,0,T\,]} \mathbf{E}||Y(t)||^q_{L^2_\rho} + \sup_{\substack{t\in[\,0,T\,]\\ N\in\mathbf{N}}} \mathbf{E}||X_N(t)||^q_{L^2_\rho}\right] \\ \leq & \; c(q,\zeta,m,K,T,c(T),c_e(T),C_{q,\eta})(1+\mathbf{E}||\xi||^q_{L^2_\rho})\end{aligned}$$

in in the Lévy noise case, which finishes the proof. ■

## 7.3 Proof of Theorem 7.1.4

As in the proof of 7.1.2, we adapt the procedure from the proof of Theorem 3.4.1 in [76]. To shorten the proof, we proceed simultaneously for the Poisson and the Lévy noise case.

**Step 1:** Compared to Step 1 in the proof of Theorem 7.1.2 (cf. Section 7.2), (7.15)/(7.16) and (7.19)/(7.20) remain valid, whereas (7.17)/(7.18) changes to

(7.55) $\bar{g}(v) \geq -c_f(T)[(1+|v|^\nu)\mathbf{1}_{[0,\infty)}(v) + (1-v)\mathbf{1}_{(-\infty,0)}(v),$

(7.56) $\bar{h}(v) \geq c_f(T)[(1+|v|^\nu)\mathbf{1}_{(-\infty,0]}(v) + (1+v)\mathbf{1}_{(0,\infty)}(v),\ v \in \mathbb{R}.$

Again, all inequalities also hold true, when $f$ is replaced by $e$.

**Step 2:** Here, we establish the $M$-independent estimates for $X_{N,M} \in \mathcal{G}_\nu(T)$ solving (7.1) resp. (7.2) with $F$ resp. $E$ being replaced by $F_{N,M}$ resp. $E_{N,M}$. Analogously to Step 2 in the proof of 7.1.2, we refer to the results of Section 5.2 for the unique solvability of equation (7.1) resp. equation (7.2) with Lipschitz coefficients.
By 5.2.2 applied to the special case of $\Gamma = C$ resp. $\Sigma = C$, there are unique (up to modifications) mild solutions $X_{N,M} \in \mathcal{G}_\nu(T)$ to the equations (7.1) resp. (7.2), when $f$ resp. $e$ is replaced by $f_{N,M}$ resp. $e_{N,M}$.
The solution processes are time-continuous in $L^{2\nu}(\Omega; L^{2\nu}_\rho)$. Moreover, they obey càdlàg versions if $U$ additionally fulfills **(A7)**.
Similarly to (7.24), we have

$$X_{N,M}(t) \leq X_{N,M+1}(t)\ P\text{-a.s.},\ t \in [0,T],$$

by the comparison result Theorem 6.1.1 both in the Poisson and the Lévy noise case.

We denote by

- $\bar{X}_{0,M}$ the unique solution to (7.1) resp. (7.2) with initial condition $\xi^+$ and drift $F_{0,M}$,
- $\underline{X}_{N,M}$ the unique solution to (7.1) resp. (7.2) with initial condition $\xi^-$ and drift $F^-_{N,M}$,

- $V$ the unique solution to (7.1) resp. (7.2) with initial condition $\xi = 0$ and drift $F = 0$.

Then, we have (cf. (7.24),(7.25))

$$\begin{gathered}\underline{X}_{N,M}(t) \le X_{N,M}(t) \le \bar{X}_{0,M}(t),\\ \underline{X}_{N,M}(t) \le V(t) \le \bar{X}_{0,M}(t)\end{gathered}$$

$P$-almost surely, for each $t \in [\,0,T\,]$ and $N, M \in \mathbb{N}$.

**(i) The Poisson noise case - equation (7.1)**

We first prove that

$$\sup_{\substack{t\in[\,0,T\,]\\ M\in\mathbb{N}}} \mathbf{E}||\bar{X}_{0,M}(t)||^{2\nu}_{L^{2\nu}_\rho} < \infty. \tag{7.57}$$

Setting

$\bar{I}^{(1)}(t) := \mathbf{E}||U(t,0)\xi^+||^{2\nu}_{L^{2\nu}_\rho}$,

$\bar{I}^{(2)}_M(t) := \mathbf{E}\left|\left|\int\limits_0^t U(t,s)F_{0,M}(s,\bar{X}_{0,M}(s))\,ds\right|\right|^{2\nu}_{L^{2\nu}_\rho}$,

$\bar{I}^{(3)}_M(t) := \mathbf{E}\left|\left|\int\limits_0^t U(t,s)\mathcal{M}_{\Sigma(s,\bar{X}_{0,M}(s))}\,dW(s)\right|\right|^{2\nu}_{L^{2\nu}_\rho}$

and

$I^{(4)}(t) := \mathbf{E}\left|\left|\int\limits_0^t \int\limits_{L^2} U(t,s)\mathcal{M}_{C(s)}(x)\,\tilde{N}(ds,dx)\right|\right|^{2\nu}_{L^{2\nu}_\rho}$, $t \in [\,0,T\,]$,

we get, for any $t \in [\,0,T\,]$,

$\mathbf{E}||\bar{X}_{0,M}(t)||^{2\nu}_{L^{2\nu}_\rho} \le c(\nu)(\bar{I}^{(1)}(t) + \bar{I}^{(2)}_M(t) + \bar{I}^{(3)}_M(t) + I^{(4)}(t))$.

First, by **(A3)** we have

$$\begin{aligned}\bar{I}^{(1)}(t) \le\ & c(\nu,T)\mathbf{E}||\xi^+||^{2\nu}_{L^{2\nu}_\rho}\\ \le\ & c(\nu,T)\mathbf{E}||\xi||^{2\nu}_{L^{2\nu}_\rho}.\end{aligned}$$

**(A1)** and **(A3)** imply

$\bar{I}^{(2)}_M(t) \le c(\nu,T)\mathbf{E}\left|\left|\int\limits_0^t U(t,s)|F_{0,M}(s,\bar{X}_{0,M}(s))|^\nu\,ds\right|\right|^2_{L^2_\rho}$

and thus

$$\begin{aligned}
\bar{I}_M^{(2)}(t) \leq\ & c(\nu, T, c(T))\mathbf{E}\int_0^t ||\bar{h}(\bar{X}_{0,M}(s))||_{L_\rho^{2\nu}}^{2\nu}\, ds \\
\leq\ & c(\nu, T, c(T), c_f(T))\mathbf{E}\int_0^t \int_{\bar{\Theta}} \Big[(1 + |\bar{X}_{0,M}|^{2\nu}(s,\theta))\mathbf{1}_{\{\bar{X}_{0,M}(s,\theta)>0\}}(s,\theta) \\
& +(1 + |V|^{2\nu^2}(s,\theta))\mathbf{1}_{\{\bar{X}_{0,M}(s,\theta)<0\}}(s,\theta)\Big]\ \mu_\rho(d\theta)\, ds \\
\leq\ & c(\nu, T, c(T), c_f(T))\left(1 + \int_0^t \mathbf{E}||\bar{X}_{0,M}(s)||_{L_\rho^{2\nu}}^{2\nu}\, ds + \int_0^T \mathbf{E}||V(s)||_{L_\rho^{2\nu^2}}^{2\nu^2}\, ds\right) \\
\leq\ & c(\nu, T, c(T), c_f(T), c_\sigma(T))\left(1 + \int_0^t \mathbf{E}||\bar{X}_{0,M}(s)||_{L_\rho^{2\nu}}^{2\nu}\, ds\right),
\end{aligned}$$

where (7.20) was used in the first, (7.18) and (7.25) were used in the second and estimate (5.35) for the $V$-term was used in the fourth step.

Let us note that to estimate $\mathbf{E}||V(t)||_{L_\rho^{2\nu^2}}^{2\nu^2}$ in a way similar to Corollary 5.2.4, we need to impose assumption **(QI)** with $q = 2\nu^2$.
This is in full consistency with the assumption imposed on the jump coefficient in [80].

To estimate the third term, we apply Proposition 3.4.3 to the process $\bar{\varphi}(t) := \Sigma(t, \bar{X}_{0,M}(t)) \in L_\rho^{2\nu}$, $t \in [0, T]$.
Then, (3.40) and **(LC)**, **(LB)** for $\sigma$ give us

$$\begin{aligned}
\bar{I}_M^{(3)}(t) \leq\ & c(\nu, T)\int_0^t (t-s)^{-\zeta}\mathbf{E}||\Sigma(s, \bar{X}_{0,M}(s))||_{L_\rho^{2\nu}}^{2\nu}\, ds \\
\leq\ & c(\nu, \zeta, T, c_\sigma(T))\left(1 + \int_0^t (t-s)^{-\zeta}\mathbf{E}||\bar{X}_{0,M}(s)||_{L_\rho^{2\nu}}^{2\nu}\, ds\right).
\end{aligned}$$

Finally, by Proposition 4.4 we have

$$\begin{aligned}
\bar{I}^{(4)}(t) \leq\ & c(\nu, T, c(T), C_{2\nu,\eta})\int_0^t (t-s)^{-\zeta\nu}\mathbf{E}||C(s)||_{L_\rho^{2\nu}}^{2\nu}\, ds \\
\leq\ & c(\nu, \zeta, T, c(T), K, C_{2\nu,\eta}),
\end{aligned}$$

where we take into account the boundedness of $c$ and the fact that

$$\nu < \tfrac{1}{\zeta} \iff \nu\zeta < 1.$$

Putting these four estimates together, we get

$$\begin{aligned}\mathbf{E}||\bar{X}_{0,M}(t)||^{2\nu}_{L^{2\nu}_\rho} \leq \quad & c(\nu,T)\mathbf{E}||\xi||^{2\nu}_{L^{2\nu}_\rho} \\ & +c(\nu,\zeta,K,T,c(T),c_f(T),C_{2\nu,\eta}) \\ & +c(\nu,\zeta,T,c(T),c_\sigma(T),C_{2\nu,\eta})\left(1+\int\limits_0^t (t-s)^{-\zeta}\mathbf{E}||\bar{X}_{0,M}(s)||^{2\nu}_{L^{2\nu}_\rho}\,ds\right)\end{aligned}$$

for arbitrary $t \in [\,0,T\,]$. Thus, by the Gronwall-Bellman lemma 2.7.3,

$$\mathbf{E}||\bar{X}_{0,M}(t)||^{2\nu}_{L^{2\nu}_\rho} \leq c(\nu,\zeta,T,c(T),c_f(T),c_\sigma(T),c_\gamma(T),C_{2\nu,\eta})(1+\mathbf{E}||\xi||^{2\nu}_{L^{2\nu}_\rho}),$$

for arbitrary $M \in \mathbb{N}$ and $t \in [\,0,T\,]$.
Thus, we have proven that

$$\begin{aligned}\sup_{\substack{t\in[\,0,T\,]\\ M\in\mathbf{N}}} \mathbf{E}||\bar{X}_{0,M}(t)||^{2\nu}_{L^{2\nu}_\rho} \leq \quad & c(\nu,\zeta,T,c(T),c_f(T),c_\sigma(T),c_\gamma(T),C_{2\nu,\eta})(1+\mathbf{E}||\xi||^{2\nu}_{L^{2\nu}_\rho}) \\ < \quad & \infty\end{aligned}$$

as it is required.

Next, we consider $\underline{X}_{N,M}$ for arbitrary $N,M \in \mathbb{N}$. Setting

$$\underline{I}^{(1)}(t) := \mathbf{E}||U(t,0)\xi^-||^{2\nu}_{L^{2\nu}_\rho},$$

$$\underline{I}^{(2)}_{N,M}(t) := \mathbf{E}\left|\left|\int\limits_0^t U(t,s)F^-_{N,M}(s,\underline{X}_{N,M})\,ds\right|\right|^{2\nu}_{L^{2\nu}_\rho}$$

and

$$\underline{I}^{(3)}_{N,M}(t) := \mathbf{E}\left|\left|\int\limits_0^t U(t,s)\mathcal{M}_{\Sigma(s,\underline{X}_{N,M}(s))}\,dW(s)\right|\right|^{2\nu}_{L^{2\nu}_\rho},\ t\in[\,0,T\,],$$

we get, for $t \in [\,0,T\,]$,

$$\mathbf{E}||\underline{X}_{N,M}||^{2\nu}_{L^{2\nu}_\rho} \leq c(\nu)[\underline{I}^{(1)}(t)+\underline{I}^{(2)}_{N,M}(t)+\underline{I}^{(3)}_{N,M}(t)+I^{(4)}(t)].$$

Analogously to the consideration of $\bar{I}^{(3)}_M$ above, we get

$$\underline{I}^{(3)}_{N,M}(t) \leq c(\nu,c(T),c_\sigma(T))\left(1+\int\limits_0^t (t-s)^{-\zeta}\mathbf{E}||\underline{X}_{N,M}(s)||^{2\nu}_{L^{2\nu}_\rho}\right).$$

Since obviously $\underline{I}^{(1)}(t) \leq c(\nu,c(T))\mathbf{E}||\xi||^{2\nu}_{L^{2\nu}_\rho}$ and $I^{(4)}(t)$ is the same as in the estimate for $\bar{X}_{0,M}$, it remains to estimate $\underline{I}^{(2)}_{N,M}$.
Since by construction we have

$$-N \leq f^-_{N,M}(t,\omega,y) \leq 0 \text{ for any } (t,\omega,y)\in[\,0,T\,]\times\Omega\times\mathbb{R},$$

we get by Definition 2.2.1 that

$$\sup_{\substack{t\in[0,T]\\ M\in\mathbb{N}}} \underline{I}^{(2)}_{N,M}(t) \le c(N,\nu,T,c(T)) < \infty.$$

Thus, putting all the estimates together, we have

$$\begin{aligned}\mathbf{E}||\underline{X}_{N,M}||^{2\nu}_{L^{2\nu}_\rho} \le \quad & c(N,\nu,T,c(T),c_\sigma(T),c_\gamma(T),C_{2\nu,\eta})(1+\mathbf{E}||\xi||^{2\nu}_{L^{2\nu}_\rho}) \\ & + c(\nu,\zeta,T,c(T),c_\sigma(T),c_\gamma(T),C_{2\nu,\eta}) \int_0^t (t-s)^{-\zeta}\mathbf{E}||\underline{X}_{N,M}(s)||^{2\nu}_{L^{2\nu}_\rho}\,ds.\end{aligned}$$

Again by the Gronwall-Bellman lemma 2.7.3, we get

$$\mathbf{E}||\underline{X}_{N,M}||^{2\nu}_{L^{2\nu}_\rho} \le \underline{c}(N,\nu,\zeta,K,T,c(T),c_\sigma(T),c_\gamma(T),C_{2\nu,\eta})(1+\mathbf{E}||\xi||^{2\nu}_{L^{2\nu}_\rho})$$

and hence

$$\sup_{\substack{t\in[0,T]\\ M\in\mathbb{N}}} \mathbf{E}||\underline{X}_{N,M}||^{q}_{L^{2}_\rho} \le \underline{c}(N,\nu,K,T,c(T),c_\sigma(T),C_{2\nu,\eta})(1+\mathbf{E}||\xi||^{2\nu}_{L^{2\nu}_\rho}).$$

Finally, by (7.23) we conclude

$$\sup_{\substack{t\in[0,T]\\ M\in\mathbb{N}}} \mathbf{E}||X_{N,M}||^{2\nu}_{L^{2\nu}_\rho} \le c(N,\nu,K,T,c(T),c_\sigma(T),C_{2\nu,\eta})(1+\mathbf{E}||\xi||^{2\nu}_{L^{2\nu}_\rho})$$

with $c = \bar{c} + \underline{c}$.

### (ii) The case of Lévy noise - equation (7.2)

Let us first consider $\bar{X}_{0,M}$. Setting

$$\tilde{I}^{(2)}_M(t) := \mathbf{E}\left|\left|\int_0^t U(t,s)E_{0,M}(s,\bar{X}_{0,M}(s))\,ds\right|\right|^{2\nu}_{L^{2\nu}_\rho}$$

and

$$\tilde{I}^{(3)}_C(t) := \mathbf{E}\left|\left|\int_0^t U(t,s)\mathcal{M}_{C(s)}\,dL(s)\right|\right|^{2\nu}_{L^{2\nu}_\rho},\ t\in[0,T],$$

we get, for $t\in[0,T]$,

$$\mathbf{E}||\bar{X}_{0,M}(t)||^{2\nu}_{L^{2\nu}_\rho} \le c(\nu)(\bar{I}^{(1)}(t) + \tilde{I}^{(2)}_M(t) + \tilde{I}^{(3)}_C(t)),$$

with $\bar{I}^{(1)}(t) \le c(\nu,c(T))\mathbf{E}||\xi||^{2\nu}_{L^{2\nu}_\rho}$ as in the Poisson noise case.
To estimate the second term, we use the same chain of arguments as in **(i)** to get

$$
\begin{aligned}
\tilde{I}_M^{(2)}(t) \leq\ & c(\nu, T, c(T))\mathbf{E}\int_0^t ||\bar{h}(\bar{X}_{0,M}(s))||_{L_\rho^{2\nu}}^{2\nu}\, ds \\
\leq\ & c(\nu, T, c(T), c_e(T))\mathbf{E}\int_0^t \int_{\bar{\Theta}} \Big[(1+\bar{X}_{0,M}^{2\nu}(s,\theta))\mathbf{1}_{\{\bar{X}_{0,M}(s,\theta)>0\}}(s,\theta) \\
& +(1+V^{2\nu^2}(s,\theta))\mathbf{1}_{\{\bar{X}_{0,M}(s,\theta)<0\}}(s,\theta)\Big]\ \mu_\rho(d\theta)\, ds \\
\leq\ & c(\nu, T, c(T), c_e(T))\left(1+\int_0^t \mathbf{E}||\bar{X}_{0,M}(s)||_{L_\rho^{2\nu}}^{2\nu}\, ds + \int_0^T \mathbf{E}||V(s)||_{L_\rho^{2\nu}}^{2\nu^2}\, ds\right) \\
\leq\ & c(\nu, T, c(T), c_e(T), C_{2\nu,\eta})\left(1+\int_0^t \mathbf{E}||\bar{X}_{0,M}(s)||_{L_\rho^{2\nu}}^{2\nu}\, ds\right).
\end{aligned}
$$

Note that again the estimate on $\mathbf{E}||V(t)||_{L_\rho^{2\nu}}^{2\nu^2}$ requires that the Lévy measure $\eta$ obeys **(QI)** with $q = 2\nu^2$.
Now, by **(QI)** and **(A5)**/**(A5)***, the Lévy-Itô-decomposition 2.4.12 and the stochastic convolution results 3.4.3 and 4.4 we get

$$
\begin{aligned}
\tilde{I}_C^{(3)}(t) \leq\ & c(\nu, m, c(T), C_{2\nu,\eta})\int_0^t (t-s)^{-\nu\zeta}\mathbf{E}||C(s)||_{L_\rho^{2\nu}}^{2\nu}\, ds \\
& +c(\nu, c(T))\int_0^t (t-s)^{-\nu\zeta}\mathbf{E}||C(s)||_{L_\rho^{2\nu}}^{2\nu}\, ds \\
\leq\ & c(\nu, \zeta, m, K, c(T), C_{2\nu,\eta}).
\end{aligned}
$$

Putting all the estimates together, we get

$$
\begin{aligned}
&\mathbf{E}||\bar{X}_{0,M}(t)||_{L_\rho^{2\nu}}^{2\nu} \\
&\leq c(\nu, T)\mathbf{E}||\xi||_{L_\rho^{2\nu}}^{2\nu} \\
&+ c(\nu, \zeta, m, K, T, c(T), c_e(T), c_\gamma(T), C_{2\nu,\eta})\left(1+\int_0^t \mathbf{E}||\bar{X}_{0,M}(s)||_{L_\rho^{2\nu}}^{2\nu}\, ds\right)
\end{aligned}
$$

for arbitrary $t \in [\,0, T\,]$.
Thus, by Gronwall's Lemma we have

$$
\sup_{\substack{t\in[\,0,T\,]\\ M\in\mathbb{N}}} \mathbf{E}||\bar{X}_{0,M}||_{L_\rho^{2\nu}}^{2\nu} \leq \bar{c}(\nu, \zeta, m, K, T, c(T), c_e(T), C_{2\nu,\eta})(1+\mathbf{E}||\xi||_{L_\rho^{2\nu}}^{2\nu}).
$$

Next, we consider $\underline{X}_{N,M}$ for arbitrary $N, M \in \mathbb{N}$. Setting

$$
\underline{I}^{(1)}(t) := \mathbf{E}||U(t,0)\xi^-||_{L_\rho^{2\nu}}^{2\nu},
$$

$$
\underline{I}_{N,M}^{(2)}(t) := \mathbf{E}\left|\left|\int_0^t U(t,s)E_{N,M}^-(s,\underline{X}_{N,M})\, ds\right|\right|_{L_\rho^{2\nu}}^{2\nu},
$$

and $I^{(3)}$ as before, we get for $t \in [\,0, T\,]$

$$
\mathbf{E}||\underline{X}_{N,M}(t)||_{L_\rho^{2\nu}}^{2\nu} \leq c(\nu)[\underline{I}^{(1)}(t) + \underline{I}_{N,M}^{(2)}(t) + I^{(3)}(t)].
$$

We already know that

$$\underline{I}^{(1)}(t) \leq c(\nu, c(T))\mathbf{E}||\xi||^{2\nu}_{L^{2\nu}_\rho}$$

and

$$I^{(3)}(t) \leq c(\nu, \zeta, m, K, T, c(T), C_{2\nu,\eta}),$$

so it remains to estimate $\underline{I}^{(2)}_{N,M}(t)$.
Since by the construction

$$-N \leq e^-_{N,M}(t, \omega, y) \leq 0 \text{ for any } (t, \omega, y) \in [\,0, T\,] \times \Omega \times \mathbb{R},$$

by Definition 2.2.1 we immediately see that

$$\underline{I}^{(2)}_{N,M}(t) \leq c(N, \nu, T, c(T)) < \infty.$$

Thus, putting all the estimates together, we have

$$\begin{aligned}
\mathbf{E}||\underline{X}_{N,M}(t)||^{2\nu}_{L^{2\nu}_\rho} \leq \quad & c(\nu, c(T))\mathbf{E}||\xi||^{2\nu}_{L^{2\nu}_\rho} + c(N, \nu, T, c(T)) \\
& + c(\nu, \zeta, m, K, T, c(T), C_{2\nu,\eta}) \\
& + c(N, \nu, T, c(T)) \\
\leq \quad & \underline{c}(N, \nu, \zeta, m, K, T, c(T), C_{2\nu,\eta})(1 + \mathbf{E}||\xi||^{2\nu}_{L^{2\nu}_\rho})
\end{aligned}$$

and thus

$$\sup_{\substack{t\in[\,0,T\,]\\ M\in\mathbb{N}}} \mathbf{E}||\underline{X}_{N,M}(t)||^{q}_{L^2_\rho} \leq \underline{c}(N, \nu, \zeta, K, m, T, c(T), C_{2\nu,\eta})(1 + \mathbf{E}||\xi||^{2\nu}_{L^{2\nu}_\rho}).$$

By (7.24), we then conclude that

$$\sup_{\substack{t\in[\,0,T\,]\\ M\in\mathbb{N}}} \mathbf{E}||X_{N,M}(t)||^{2\nu}_{L^{2\nu}_\rho} \leq c(N, \nu, \zeta, m, K, T, c(T), C_{2\nu,\eta})(1 + \mathbf{E}||\xi||^{2\nu}_{L^{2\nu}_\rho})$$

with $c = \bar{c} + \underline{c}$.

**Step 3:** Analogously to Step 3 in the proof of 7.1.2, we construct the processes $X_N := \lim_{M\to\infty} X_{N,M}$, which later shall solve our equations in the case of $F$ resp. $E$ being replaced by $F_N$ resp. $E_N$.

To this end, we define the random variables

$$Z_{N,M}(t) := X_{N,M}(t) - X_{N,1}(t) \in L^{2\nu}_\rho,\ t \in [\,0, T\,],\ N, M \in \mathbb{N},$$

such that (cf. (7.36))

$$0 \leq Z_{N,M}(t) \leq Z_{N,M+1}(t).$$

With the help of the $M$-independent estimate from Step 2, we get

$$\begin{aligned} \sup_{\substack{t\in[0,T]\\ M\in\mathbb{N}}} \mathbf{E}||Z_{N,M}(t)||^{2\nu}_{L^{2\nu}_\rho} \leq & \quad c(\nu)\left[\sup_{\substack{t\in[0,T]\\ M\in\mathbb{N}}} \mathbf{E}||X_{N,M}(t)||^{2\nu}_{L^{2\nu}_\rho} + \sup_{t\in[0,T]} \mathbf{E}||X_{N,1}(t)||^{2\nu}_{L^{2\nu}_\rho}\right] \\ < & \quad \infty. \end{aligned}$$

Next, we define

$$Z_N(t) := \sup_{M\in\mathbb{N}} Z_{N,M}(t),\ t\in[0,T],\ N\in\mathbb{N},$$

and check that there exists a modification $Z_N \in \mathcal{G}_\nu(T)$, which fulfills

$$\sup_{t\in[0,T]} \mathbf{E}||Z_N(t)||^{2\nu}_{L^{2\nu}_\rho} = \sup_{t\in[0,T]} \sup_{M\in\mathbb{N}} \mathbf{E}||Z_{N,M}(t)||^{2\nu}_{L^{2\nu}_\rho} < \infty. \tag{7.58}$$

Thereafter, we can define $X_N \in \mathcal{G}_\nu(T)$ by

$$X_N(t) := Z_N(t) + X_{N,1}(t) \in L^{2\nu}_\rho,\ N\in\mathbb{N},\ t\in[0,T]$$

and check that

$$\begin{aligned} &\lim_{M\to\infty} \int_0^T \mathbf{E}||X_{N,M}(t) - X_N(t)||^{2\nu}_{L^{2\nu}_\rho}\,dt \\ &= \lim_{M\to\infty} \int_0^T \mathbf{E}||Z_{N,M}(t) - Z_N(t)||^{2\nu}_{L^{2\nu}_\rho}\,dt = 0. \end{aligned} \tag{7.59}$$

Indeed, by B.Levi's monotone convergence theorem we have for each $t\in[0,T]$

$$\lim_{M\to\infty} \mathbf{E}||X_{N,M}(t) - X_N(t)||^{2\nu}_{L^{2\nu}_\rho} = \lim_{M\to\infty} \mathbf{E}||Z_{N,M}(t) - Z_N(t)||^{2\nu}_{L^{2\nu}_\rho} = 0.$$

Herefrom, by (7.58) and Lebesgue's dominated convergence theorem we immediately get (7.59).

In the same manner, we construct processes $\underline{X}_N$, $\bar{X} \in \mathcal{G}_\nu(T)$ such that

$$\lim_{M\to\infty} \int_0^T \mathbf{E}||\underline{X}_{N,M}(t) - \underline{X}_N(t)||^{2\nu}_{L^{2\nu}_\rho}\,ds = 0,$$

$$\lim_{M\to\infty} \int_0^T \mathbf{E}||\bar{X}_{0,M}(t) - \bar{X}(t)||^{2\nu}_{L^{2\nu}_\rho}\,ds = 0$$

and (by (7.24), (7.25))

$$\underline{X}_N(t) \le X_N(t) \le \bar{X}(t), \qquad (7.60)$$
$$\underline{X}_N(t) \le V(t) \le \bar{X}(t),$$

$P$-almost surely, for any $t \in [\,0, T\,]$.

**Step 4:** The aim of this step is to show that, for any $N \in \mathbb{N}$, the process $X_N \in \mathcal{G}_\nu(T)$ defined in Step 3 solves (7.1) resp. (7.2) in the case of $F$ resp. $E$ being replaced by $F_N$ resp. $E_N$.
This implies that $t \mapsto X_N(t)$ is continuous in $L^{2\nu}(\Omega, \mathcal{F}, P; L^{2\nu}_\rho)$ and, under the additional assumption **(A7)** on $U$, has a càdlàg modification.

By (7.59), there is a subsequence of $(X_{N,M})_{M\in\mathbb{N}}$ that converges $P\otimes ds\otimes d\mu_\rho$-almost everywhere to $X_N$. For simplicity, we assume $(X_{N,M})_{M\in\mathbb{N}}$ itself to be this sequence.

**(i) The case of Poisson noise - equation (7.1)**

We define for $t \in [\,0, T\,]$

$$I^{(1)}_{N,M}(t) := \mathbf{E}||X_N(t) - X_{N,M}(t)||^2_{L^2_\rho}$$

$$I^{(2)}_{N,M}(t) := \mathbf{E}\left|\left|\int_0^t U(t,s)[F_N(s, X_N(s)) - F_{N,M}(s, X_{N,M}(s))]\, ds\right|\right|^2_{L^2_\rho},$$

and

$$I^{(3)}_{N,M}(t) := \mathbf{E}\left|\left|\int_0^t U(t,s)[\mathcal{M}_{\Sigma(s,X_N(s))} - \mathcal{M}_{\Sigma(s,X_{N,M}(s))}]\, dW(s)\right|\right|^2_{L^2_\rho}.$$

Thus, for each $t \in [\,0, T\,]$, we have

$$\mathbf{E}\left|\left|X_N(t) - U(t,0)\xi - \int_0^t U(t,s)F_N(s, X_N(s))\, ds - \int_0^t U(t,s)\mathcal{M}_{\Sigma(s,X_N(s))}\, dW(s)\right.\right.$$
$$\left.\left. - \int_0^t \int_{L^2} U(t,s)\mathcal{M}_{C(s)}(x)\, \tilde{N}(ds, dx)\right|\right|^2_{L^2_\rho}$$
$$\le 3[I^{(1)}_{N,M}(t) + I^{(2)}_{N,M}(t) + I^{(3)}_{N,M}(t)].$$

First of all, by Hölder's inequality we get

$$I^{(1)}_{N,M}(t) := \mathbf{E}||X_N(t) - X_{N,M}(t)||^2_{L^2_\rho} \leq c(\nu,\rho)[\mathbf{E}||X_N(t) - X_{N,M}(t)||^{2\nu}_{L^{2\nu}_\rho}]^{\frac{1}{\nu}}$$

such that by construction the first term tends to 0 for $M \to \infty$. By 2.1.1 **(iii)** for $U$, we have for the second term

$$\begin{aligned} I^{(2)}_{N,M}(t) \leq \quad & 2c(T)\left(\mathbf{E}\int_0^T ||F_N(s, X_N(s)) - F_N(s, X_{N,M}(s))||^2_{L^2_\rho}\, ds \right. \\ & \left. +\mathbf{E}\int_0^T ||F_N(s, X_{N,M}(s)) - F_{N,M}(s, X_{N,M}(s))||^2_{L^2_\rho}\, ds\right) \\ =: \quad & 2c(T)[I^{(21)}_{N,M}(T) + I^{(22)}_{N,M}(T)]. \end{aligned}$$

By the continuity of $f_N$ and the convergence property of $X_{N,M}$, we have for almost all $s \in [\,0,T\,]$

$$|f_N(s,\omega,X_N(s,\omega,\theta)) - f_N(s,\omega,X_{N,M}(s,\omega,\theta))| \to 0 \text{ as } M \to \infty$$

$P \otimes d\mu_\rho$ almost surely on $\Omega \times \Theta$.
Condition **(PG)** and the relation

$$X_{N,1}(t) \leq X_{N,M}(t) \leq X_N(t),\ t \in [\,0,T\,],\ N, M \in \mathbb{N}$$

imply for almost all $s \in [\,0,T\,]$

$$\begin{aligned} & |f_N(s,\omega,X_N(s,\omega,\theta)) - f_N(s,\omega,X_{N,M}(s,\omega,\theta))| \\ & \leq 2c(N,c_f(T))(1 + |X_N(s,\omega,\theta)|^\nu + |X_{N,1}(s,\omega,\theta)|^\nu), \end{aligned}$$

$P \otimes d\mu_\rho$ almost surely.
To apply Lebesgue's theorem, we check that

$$\begin{aligned} & \sup_{s\in[\,0,T\,]} \int_\Omega \int_\Theta (|X_N(s,\omega,\theta)|^{2\nu} + |X_{N,1}(s,\omega,\theta)|^{2\nu})\, \mu_\rho(d\theta))\, P(d\omega)\, ds \\ & = \sup_{s\in[\,0,T\,]} \left(\mathbf{E}||X_N(s)||^{2\nu}_{L^{2\nu}_\rho} + \mathbf{E}||X_{N,1}(s)||^{2\nu}_{L^{2\nu}_\rho}\right) < \infty, \end{aligned}$$

since $X_N,\ X_{N,1} \in \mathcal{G}_\nu(T)$. Thus, Lebesgue's theorem is applicable and gives us

$$\begin{aligned} & \lim_{M\to\infty} I^{(21)}_{N,M}(T) \\ & = \lim_{M\to\infty} \int_0^T \left[\int_\Omega \int_\Theta [f_N(s,\omega,X_N(s,\omega,\theta)) - f_N(s,\omega,X_{N,M}(s,\omega,\theta))]^2\, d\mu_\rho\, P(d\omega)\right] ds \\ & = 0. \end{aligned}$$

To estimate $I^{(22)}_{N,M}(T)$, we fix $L \leq M,\ L, M \in \mathbb{N}$. In full analogy to the consideration of $I^{(21)}_{N,M}$ with $f_{N,M} \uparrow f_N$ (and thus $f_N - f_{N,M} \downarrow 0$), we get

$$\begin{aligned}\lim_{M\to\infty} I^{(22)}_{N,M} \leq & \quad \lim_{M\to\infty} \int_0^T \mathbf{E}||F_N(s, X_{N,M}(s)) - F_{N,L}(s, X_{N,M}(s))||^2_{L^2_\rho}\, ds \\ = & \quad \int_0^T \mathbf{E}||F_N(s, X_N(s)) - F_{N,L}(s, X_N(s))||^2_{L^2_\rho}\, ds.\end{aligned}$$

Letting here $L \to \infty$ gives us

$$\lim_{M\to\infty} I^{(22)}_{N,M}(T) = 0.$$

Thus, we have

$$\lim_{M\to\infty} I^{(2)}_{N,M}(T) = 0.$$

By **(A4)**, **(LC)**, Itô's isometry and Hölder's inequality, we get the following estimate for the third term

$$\begin{aligned}I^{(3)}_{N,M}(T) \leq & \quad c(c(T), c_\sigma(T)) \int_0^t (t-s)^{-\zeta}\mathbf{E}||X_N(s) - X_{N,M}(s)||^2_{L^2_\rho}\, ds \\ \leq & \quad c(\nu, T, c(T), c_\sigma(T)) \left(\int_0^T \mathbf{E}||X_N(s) - X_{N,M}(s)||^{2\nu}_{L^{2\nu}_\rho}\, ds\right)^{\frac{1}{\nu}}.\end{aligned}$$

Thus, by (7.59) $I^{(3)}_{N,M}(T)$ tends to 0 as $M \to \infty$.

So we have shown that $X_N \in \mathcal{G}_\nu(T)$ is the mild solution to the equation (7.1), when $F$ is replaced by $F_N$.

The required continuity properties of $X_N$ follow immediately from the properties of the integrals on the right hand side of (7.1).
Indeed, the Bochner convolution integral in (7.1) is time-continuous in $L^2(\Omega; L^2_\rho)$ by Remark 5.1.11 **(ii)**, whereas the stochastic convolutions integrals in (7.1) are continuous by the continuity results 5.1.8 and 5.1.10 and the simple (Hölder's) estimate

$$\mathbf{E}||\xi||^2_{L^2_\rho} \leq (\mathbf{E}||\xi||^{2\nu}_{L^{2\nu}_\rho})^{\frac{1}{\nu}}$$

holding for any $L^{2\nu}_\rho$-valued random variable.

Similar arguments show that $\underline{X}_N \in \mathcal{G}_\nu(T)$ solves (7.1) with $\xi^-$ and $f^-_N$, whereas $\bar{X}$ solves (7.1) with $\xi^+$ and $f^+$.

## (ii) The case of Lévy noise - equation (7.2)

Defining

$$I^{(1)}_{N,M}(t) := \mathbf{E}||X_N(t) - X_{N,M}(t)||^2_{L^2_\rho}$$

and

$$I^{(2)}_{N,M}(t) := \mathbf{E}\left|\left|\int_0^t U(t,s)[E_N(s,X_N(s)) - E_{N,M}(s,X_{N,M}(s))]\,ds\right|\right|^2_{L^2_\rho}, \; t \in [\,0,T\,],$$

we have for $t \in [\,0,T\,]$

$$\mathbf{E}\left|\left|X_N(t) - U(t,0)\xi - \int_0^t U(t,s)E_N(s,X_N(s))\,ds - \int_0^t U(t,s)\mathcal{M}_{C(s)}\,dL(s)\right|\right|^2_{L^2_\rho}$$
$$\le 2[I^{(1)}_{N,M}(t) + I^{(2)}_{N,M}(t)].$$

But $I^{(1)}_{N,M}(t)$ tends to 0 for $M \to \infty$ by construction, whereas $I^{(2)}_{N,M}(t)$ tends to 0 for $M \to \infty$ by replacing the $F$-terms by the $E$-terms in the proof of the same claim in the Poisson noise case.

Thus, for all $N \in \mathbb{N}$, $X_N \in \mathcal{G}_\nu(T)$ is a solution to (7.2) in the sense of 7.1.1.

Again, the stochastic integrals obey the required continuity properties by the results from Section 5.1.
Indeed, the Bochner convolution integral in (7.2) is time-continuous in $L^2(\Omega; L^2_\rho)$ by Remark 5.1.11 **(ii)**, whereas the stochastic convolutions integral in (7.2) is continuous by the continuity results 5.1.6 **(ii)**, 5.1.8 and 5.1.10 and the simple estimate

$$\mathbf{E}||\xi||^2_{L^2_\rho} \le (\mathbf{E}||\xi||^{2\nu}_{L^{2\nu}_\rho})^{\frac{1}{\nu}}$$

holding for any $L^{2\nu}_\rho$-valued random variable (by taking a $\mathcal{F}\otimes\mathcal{B}(\Theta)$-measurable version and applying Hölder's inequality twice).

**Step 5:** As in Step 5 of the proof of 7.1.2, we first establish $N$-independent estimates for the moments of $\underline{X}_N$. Then, the required $N$-independent estimate for $X_N$ will follow from the inequality in $L^{2\nu}_\rho$ (cf. (7.21))

$$\underline{X}_N(t) \le X_N(t) \le \bar{X}(t),$$

which holds for any $t \in [\,0,T\,]$ and any $N \in \mathbb{N}$.

From Steps 2 and 4, we already know that

$$\sup_{t\in[0,T]} \mathbf{E}||\underline{X}_N(t)||^{2\nu}_{L^{2\nu}_\rho} \le c(N,\nu,\zeta,K,T,c(T),c_f(T),c_\sigma(T),C_{2\nu,\eta})(1+\mathbf{E}||\xi||^{2\nu}_{L^{2\nu}_\rho})$$

in the Poisson noise case and (in the Lévy noise case)

$$\sup_{t\in[0,T]} \mathbf{E}||\underline{X}_N(t)||^{2\nu}_{L^{2\nu}_\rho} \le c(N,\nu,\zeta,m,K,T,c(T),c_e(T),C_{2\nu,\eta})(1+\mathbf{E}||\xi||^{2\nu}_{L^{2\nu}_\rho}).$$

In the Poisson noise case, setting

$$\underline{I}^{(1)}(t) := \mathbf{E}||U(t,0)\xi^-||^{2\nu}_{L^{2\nu}_\rho},$$

$$\underline{I}^{(2)}_N(t) := \mathbf{E}\left|\left|\int_0^t U(t,s)F^-_N(s,\underline{X}_n(s))\,ds\right|\right|^{2\nu}_{L^{2\nu}_\rho},$$

$$\underline{I}^{(3)}_N(t) := \mathbf{E}\left|\left|\int_0^t U(t,s)\mathcal{M}_{\Sigma(s,\underline{X}_N(s))}\,dW(s)\right|\right|^{2\nu}_{L^{2\nu}_\rho},$$

$$I^{(4)}(t) := \mathbf{E}\left|\left|\int_0^t\int_{L^2} U(t,s)\mathcal{M}_{C(s)}(x)\,\tilde{N}(ds,dx)\right|\right|^{2\nu}_{L^{2\nu}_\rho},\ t\in[0,T],$$

we have for, $t\in[0,T]$,

$$\mathbf{E}||\underline{X}_N(t)||^{2\nu}_{L^{2\nu}_\rho} \le c(\nu)\left[\underline{I}^{(1)}(t)+\underline{I}^{(2)}_N(t)+\underline{I}^{(3)}_N(t)+I^{(4)}(t)\right].$$

In the Lévy noise case, setting

$$\underline{I}^{(1)}(t) := \mathbf{E}||U(t,0)\xi^-||^{2\nu}_{L^{2\nu}_\rho},$$

$$\underline{I}^{(2)}_N(t) := \mathbf{E}\left|\left|\int_0^t U(t,s)E^-_N(s,\underline{X}_n(s))\,ds\right|\right|^{2\nu}_{L^{2\nu}_\rho},$$

$$I^{(3)}(t) := \mathbf{E}\left|\left|\int_0^t U(t,s)\mathcal{M}_{C(s)}\,dL(s)\right|\right|^{2\nu}_{L^{2\nu}_\rho},\ t\in[0,T]$$

we have for $t\in[0,T]$

$$\mathbf{E}||\underline{X}_N(t)||^{2\nu}_{L^{2\nu}_\rho} \le c(\nu)[\underline{I}^{(1)}(t)+\underline{I}^{(2)}_N(t)+I^{(3)}(t)].$$

Now, by **(A3)** (both in the Poisson and the Lévy noise case)

$$\begin{aligned}\underline{I}^{(1)}(t) \le&\quad c(\nu,T)\mathbf{E}||U(t,0)|\xi^-|^\nu||^2_{L^2_\rho}\\ \le&\quad c(\nu,T,c(T))\mathbf{E}||\xi||^{2\nu}_{L^{2\nu}_\rho}.\end{aligned}$$

In the Poisson noise case, we get by (7.20), (7.25) and (7.60)

$$\begin{aligned}
\underline{I}_N^{(2)}(t) \leq\ & c(\nu, c(T)) \int_0^t ||F_N^-(s, \underline{X}_N(s))||_{L_\rho^{2\nu}}^{2\nu}\, ds \\
\leq\ & c(\nu, c(T)) \int_0^t \mathbf{E}||\bar{g}(\underline{X}_N(s))||_{L_\rho^{2\nu}}^{2\nu}\, ds \\
\leq\ & c(\nu, T, c(T), c_f(T)) \left(1 + \int_0^t \mathbf{E}||\underline{X}_N(s)||_{L_\rho^{2\nu}}^{2\nu}\, ds + \int_0^t \mathbf{E}||V(s)||_{L_\rho^{2\nu}}^{2\nu^2}\, ds\right) \\
\leq\ & c(\nu, T, c(T), c_f(T), C_{2\nu^2,\eta} \left(1 + \int_0^t \mathbf{E}||\underline{X}_N(s)||_{L_\rho^{2\nu}}^{2\nu}\, ds\right),
\end{aligned}$$

where we used $\underline{X}_N(t) \leq V(t)$ and $\bar{g} \leq f$.
Respectively, in the Lévy noise case, just replacing $F_N$ by $E_N$ in the above estimate, we get

$$\underline{I}_N^{(2)}(t) \leq c(\nu, T, c(T), c_e(T), C_{2\nu^2,\eta}) \left(1 + \int_0^t \mathbf{E}||\underline{X}_N(s)||_{L_\rho^{2\nu}}^{2\nu}\, ds\right).$$

Next, we consider the third term in the Poisson noise case. By 3.4.3 and **(LC)**, **(LB)** for $\sigma$, we have

$$\underline{I}_N^{(3)}(t) \leq c(\nu, T, c(T), c_\sigma(T)) \left(1 + \int_0^t (t-s)^{-\zeta} \mathbf{E}||\underline{X}_N(s)||_{L_\rho^{2\nu}}^{2\nu}\, ds\right).$$

Combining this with the estimate

$$\begin{aligned}
&\mathbf{E}\left|\left|\int_0^t \int_{L^2} U(t,s)\mathcal{M}_{C(s)}(x)\, N(ds, dx)\right|\right|_{L_\rho^{2\nu}}^{2\nu} \\
&\leq c(\nu, T, c(T), C_{2\nu,\eta}) \int_0^t (t-s)^{-\zeta\nu} ||C(s)||_{L_\rho^{2\nu}}^{2\nu}\, ds \\
&\leq c(\nu, \zeta, K, T, c(T), C_{2\nu,\eta})
\end{aligned}$$

following from 4.4 and the boundedness assumption on $c$, we get for all $t \in [0, T]$

$$\begin{aligned}
\mathbf{E}||\underline{X}_N(t)||_{L_\rho^{2\nu}}^{2\nu} \leq\ & c(\nu, T)\mathbf{E}||\xi||_{L_\rho^{2\nu}}^{2\nu} \\
& + c(\nu, T, c(T), c_f(T), c_\sigma(T)) \left(1 + \int_0^t (t-s)^{-\zeta} \mathbf{E}||\underline{X}_N(s)||_{L_\rho^{2\nu}}^{2\nu}\, ds\right) \\
& + c(\nu, \zeta, K, T, c(T), C_{2\nu,\eta}) \\
\leq\ & c(\nu, \zeta, K, T, c(T), c_f(T), c_\sigma(T), C_{2\nu,\eta})(1 + \mathbf{E}||\xi||_{L_\rho^{2\nu}}^{2\nu}) \\
& + c(\nu, T, c(T), c_f(T), c_\sigma(T)) \int_0^t (t-s)^{-\zeta} \mathbf{E}||\underline{X}_N(s)||_{L_\rho^{2\nu}}^{2\nu}.
\end{aligned}$$

Thus, by the Gronwall-Bellman lemma 2.7.3, we conclude that

$$\sup_{\substack{t \in [0,T] \\ N \in \mathbb{N}}} \mathbf{E}||\underline{X}_N(t)||_{L_\rho^{2\nu}}^{2\nu} \leq c_1(\nu, \zeta, K, T, c(T), c_f(T), c_\sigma(T), C_{2\nu,\eta})(1 + \mathbf{E}||\xi||_{L_\rho^{2\nu}}^{2\nu}).$$

The above arguments are modified to the Lévy noise case in the following way. By the Lévy-Itô decomposition 2.4.13 together with 3.4.3 and 4.4, we get

$$I^{(3)}(t) \leq c(\nu, \zeta, m, K, T, c(T), C_{2\nu,\eta}).$$

Combining this with the above estimates on $\underline{I}^{(1)}(t)$ and $\underline{I}_N^{(2)}(t)$, we get for all $t \in [\,0, T\,]$

$$\begin{aligned}\mathbf{E}||\underline{X}_N(t)||_{L_\rho^{2\nu}}^{2\nu} \leq \quad & c(\nu, T)\mathbf{E}||\xi||_{L_\rho^{2\nu}}^{2\nu} \\ & + c(\nu, \zeta, m, K, T, c(T), c_e(T), C_{2\nu,\eta}) \left(1 + \int_0^t (t-s)^{-\zeta} \mathbf{E}||\underline{X}_N(s)||_{L_\rho^{2\nu}}^{2\nu}\, ds\right).\end{aligned}$$

Then, the Gronwall-Bellman lemma is applicable, which yields

$$\mathbf{E}||\underline{X}_N(t)||_{L_\rho^{2\nu}}^{2\nu} \leq c_2(\nu, \zeta, m, K, T, c(T), c_e(T), C_{2\nu,\eta})(1 + \mathbf{E}||\xi||_{L_\rho^{2\nu}}^{2\nu})$$

and hence

$$\sup_{\substack{t \in [\,0,T\,] \\ N \in \mathbb{N}}} \mathbf{E}||\underline{X}_N(t)||_{L_\rho^{2\nu}}^{2\nu} \leq c(\nu, \zeta, m, K, T, c(T), c_e(T), C_{2\nu,\eta})(1 + \mathbf{E}||\xi||_{L_\rho^{2\nu}}^{2\nu})$$

also in the Lévy noise case.

As we have shown in Step 2, for any $t \in [\,0, T\,]$ and $M \in \mathbb{N}$,

$$\mathbf{E}||\bar{X}_{0,M}(t)||_{L_\rho^{2\nu}}^{2\nu} \leq c_3(\nu, K, T, c(T), c_f(T), c_\sigma(T), C_{2\nu,\eta})(1 + \mathbf{E}||\xi||_{L_\rho^{2\nu}}^{2\nu})$$

in the Poisson noise case respectively

$$\mathbf{E}||\bar{X}_{0,M}(t)||_{L_\rho^{2\nu}}^{2\nu} \leq c_4(\nu, K, T, m, c(T), c_e(T), C_{2\nu,\eta})(1 + \mathbf{E}||\xi||_{L_\rho^{2\nu}}^{2\nu})$$

in the Lévy noise case.

Next, let us consider the upper solution $\bar{X}$.
Recall that, by its definition in Step 3, $\bar{X}$ is a process in $\mathcal{G}_\nu(T)$ such that

$$\lim_{M \to \infty} \mathbf{E}||\bar{X}_{0,M}(t) - \bar{X}(t)||_{L_\rho^{2\nu}}^{2\nu} = 0 \text{ for each } t \in [\,0, T\,].$$

Thus, we get

$$\sup_{t \in [\,0,T\,]} \mathbf{E}||\bar{X}(t)||_{L_\rho^{2\nu}}^{2\nu} \leq c(\nu, \zeta, K, T, c(T), c_f(T), c_\sigma(T), C_{2\nu,\eta})(1 + \mathbf{E}||\xi||_{L_\rho^{2\nu}}^{2\nu})$$

in the Poisson noise case respectively

$$\sup_{t\in[0,T]} \mathbf{E}||\bar{X}(t)||^{2\nu}_{L^{2\nu}_\rho} \le c(\nu,\zeta,m,K,T,c(T),c_e(T),C_{2\nu,\eta})(1+\mathbf{E}||\xi||^{2\nu}_{L^{2\nu}_\rho})$$

in the Lévy noise case. Since by (7.44)

$$\underline{X}_N(t) \le X_N(t) \le \bar{X}(t) \text{ in } L^{2\nu}_\rho,$$

we immediately obtain that

$$\sup_{\substack{t\in[0,T]\\N\in\mathbb{N}}} \mathbf{E}||X_N(t)||^{2\nu}_{L^{2\nu}_\rho}$$
$$\le (c_1(\nu,K,T,c(T),c_f(T),c_\sigma(T),C_{2\nu,\eta})$$
$$+c_3(\nu,\zeta,K,T,c(T),c_f(T),c_\sigma(T),C_{2\nu,\eta}))\,(1+\mathbf{E}||\xi||^{2\nu}_{L^{2\nu}_\rho})$$

resp.

$$\sup_{\substack{t\in[0,T]\\N\in\mathbf{N}}} \mathbf{E}||X_N(t)||^{2\nu}_{L^{2\nu}_\rho}$$
$$\le (c_2(\nu,K,T,c(T),c_e(T),C_{2\nu,\eta})+c_4(\nu,\zeta,K,T,c(T),c_e(T),C_{2\nu,\eta}))(1+\mathbf{E}||\xi||^{2\nu}_{L^{2\nu}_\rho}),$$

which proves the $N$-independent estimates for moments of $X_N$.

By $f_N \downarrow f$ resp. $e_N \downarrow e$, Theorem 6.1.1 implies (cf. (7.44))

$$X_{N+1}(t) \le X_N(t),\ P\text{-almost surely, for all } t\in[0,T],\ N\in\mathbb{N}.$$

Now, we can define our solution candidates.

We claim that

$$X(t) := \inf_{N\in\mathbb{N}} X_N(t),\ t\in[0,T],$$

defines a solution in the sense of 7.1.1**(ii)** both for (7.1) and (7.2).
As in Step 5 in the proof of 7.1.2, we have to overcome the problem that $(X_N)_{N\in\mathbb{N}}$ is a decreasing but not necessarily positive sequence in $L^{2\nu}_\rho$.
So, we first fix $t\in[0,T]$ and define

$$Y_N(t) := X_1(t) - X_N(t),\ N\in\mathbb{N}. \tag{7.61}$$

Since

$$\begin{aligned}\sup_{t\in[0,T]} \mathbf{E}||Y_N(t)||^{2\nu}_{L^{2\nu}_\rho} &= \sup_{t\in[0,T]} \mathbf{E}||X_1(t)-X_N(t)||^{2\nu}_{L^{2\nu}_\rho}\\ &\le c(\nu)\left(\sup_{t\in[0,T]} \mathbf{E}||X_1(t)||^{2\nu}_{L^{2\nu}_\rho} + \sup_{t\in[0,T]} \mathbf{E}||X_N(t)||^{2\nu}_{L^{2\nu}_\rho}\right)\\ &< \infty,\end{aligned}$$

we have $Y_N \in \mathcal{G}_\nu(T)$ for any $N \in \mathbb{N}$.
Obviously, $0 \le Y_N(t) \le Y_{N+1}(t)$, $P$-almost surely, for each $t \in [0,T]$. Thus, defining a process $Y$ by

$$[0,T] \ni t \mapsto Y(t) := \sup_{N\in\mathbb{N}} Y_N(t),\ N \in \mathbb{N},$$

by B.Levi's monotone convergence theorem we get that

$$\sup_{t\in[0,T]} \mathbf{E}||Y(t)||^{2\nu}_{L^{2\nu}_\rho} = \sup_{t\in[0,T]}\sup_{N\in\mathbb{N}} \mathbf{E}||Y_N(t)||^{2\nu}_{L^{2\nu}_\rho} < \infty. \tag{7.62}$$

Since $t \mapsto Y(t)$ is predictable by its construction, it is also an element of $\mathcal{G}_\nu(T)$.
Since by (7.61)

$$X(t) = Y(t) - X_1(t),$$

we have proven that $X \in \mathcal{G}_\nu(T)$.
As $|Y_N(t) - Y(t)| \le 2|Y(t)|$ and $Y \in \mathcal{G}_\nu(T)$, by Lebesgue's dominated convergence theorem we conclude that, for each $t \in [0,T]$,

$$\lim_{N\to\infty} \mathbf{E}||X_N(t)-X(t)||^{2\nu}_{L^{2\nu}_\rho} = \lim_{N\to\infty} \mathbf{E}||Y_N(t)-Y(t)||^{2\nu}_{L^{2\nu}_\rho} = 0. \tag{7.63}$$

Taking into account (7.62) and applying Lebesgue's theorem ones more, we get

$$\lim_{N\to\infty} \mathbf{E}\int_0^T ||X_N(t)-X(t)||^{2\nu}_{L^{2\nu}_\rho}\,dt = \lim_{N\to\infty} \mathbf{E}\int_0^T ||Y_N(t)-Y(t)||^{2\nu}_{L^{2\nu}_\rho}\,dt = 0. \tag{7.64}$$

It remains to show that $X$ solves equation (7.1) resp. (7.2) in the sense of 7.1.1 **(ii)**.
Defining

$$I_N^{(1)}(t) := \mathbf{E}||X(t)-X_N(t)||^2_{L^2_\rho},$$

$$I_N^{(2)}(t) := \mathbf{E}\left|\left|\int_0^t U(t,s)[F(s,X(s)) - F_N(s,X_N(s))]\,ds\right|\right|^2_{L^2_\rho},$$

$$\tilde{I}_N^{(2)}(t) := \mathbf{E}\left|\left|\int_0^t U(t,s)[E(s,X(s)) - E_N(s,X_N(s))]\,ds\right|\right|^2_{L^2_\rho}$$

and

$$I_N^{(3)}(t) := \mathbf{E}\left|\left|\int_0^t U(t,s)[\mathcal{M}_{\Sigma(s,X(s))} - \mathcal{M}_{\Sigma(s,X_N(s))}]\, dW(s)\right|\right|^2_{L^2_\rho},\ t \in [\,0, T\,],$$

we get, for each $t \in [\,0, T\,]$,

$$\mathbf{E}\left|\left|X(t) - U(t,0)\xi - \int_0^t U(t,s)F(s,X(s))\, ds - \int_0^t U(t,s)\mathcal{M}_{\Sigma(s,X(s))}\, dW(s)\right.\right.$$
$$\left.\left. - \int_0^t \int_{L^2} U(t,s)\mathcal{M}_{C(s)}(x)\, \tilde{N}(ds,dx)\right|\right|^2_{L^2_\rho}$$
$$\leq C[I_N^{(1)}(t) + I_N^{(2)}(t) + I_N^{(3)}(t)]$$

in the Poisson noise case and, respectively in the Lévy noise case,

$$\mathbf{E}\left|\left|X(t) - U(t,0)\xi - \int_0^t U(t,s)E(s,X(s))\, ds - \int_0^t U(t,s)\mathcal{M}_{C(s)}\, dL(s)\right|\right|^2_{L^2_\rho}$$
$$\leq C[I_N^{(1)}(t) + \tilde{I}_N^{(2)}(t)].$$

Analogously to Step 4, we have (by (7.63))

$$I_N^{(1)}(t) \leq c(\nu,\rho)\left(\mathbf{E}||X(t) - X_N(t)||^{2\nu}_{L^{2\nu}_\rho}\right)^{\frac{1}{\nu}} \to 0, \text{ as } N \to \infty,$$

both in the Poisson and the Lévy noise case.
By (7.64) there is a subsequence of $X_N$, which converges to $X$, $P \otimes dt \otimes d\mu_\rho$-almost everywhere on $[\,0, T\,] \times \Omega \times \Theta$. For simplicity, we assume $X_N$ itself to be this sequence.
Herefrom, by the continuity of $f$ we have for almost all $s \in [\,0, T\,]$

$$f(s,\omega,X(s,\omega,\theta)) - f(s,\omega,X_N(s,\omega,\theta)) \to 0, \text{ as } N \to \infty,$$

$P \otimes \mu_\rho$-almost surely on $\Omega \times \Theta$.
Condition **(PG)** with exponent $\nu > 1$ and

$$X(t) \leq X_N(t) \leq X_1(t),\ t \in [\,0, T\,],\ N \in \mathbb{N},$$

imply that for all $s \in [\,0, T\,]$

$$|f(s,\omega,X(s,\omega,y)) - f(s,\omega,X_N(s,\omega,y))|$$
$$\leq c(c_f(T))(1 + |X(s,\omega,y)|^\nu + |X_N(s,\omega,y)|^\nu)$$
$$\leq 2c(c_f(T))(1 + |X(s,\omega,y)|^\nu + |X_1(s,\omega,y)|^\nu),$$

$P \otimes \mu_\rho$-almost surely on $\Omega \times \Theta$.
Obviously

$$\begin{aligned}
&\sup_{s\in[0,T]} \int_\Omega \int_\Theta (|X(s,\omega,\theta)|^{2\nu} + |X_1(s,\omega,\theta)|^{2\nu})\, \mu_\rho(dy)\, P(d\omega) \\
&\leq \sup_{s\in[0,T]} \left( \mathbf{E}||X(s)||^{2\nu}_{L^{2\nu}_\rho}\, ds + \mathbf{E}||X_1(s)||^{2\nu}_{L^{2\nu}_\rho} \right) < \infty,
\end{aligned}$$

since $X, X_1 \in \mathcal{G}_\nu(T)$. Thus, Lebesgue's theorem is applicable and gives us

$$\begin{aligned}
\lim_{N\to\infty} I_N^{(21)}(T) = & \int_0^T \left[ \int_\Omega \int_\Theta (f(s,\omega,X(s,\omega,\theta)) - f(s,\omega,X_N(s,\omega,\theta)))^2\, \mu_\rho(d\theta)\, P(d\omega) \right] ds \\
= & \ 0.
\end{aligned}$$

To estimate $I^{(22)}_{N,M}(T)$, let us fix some $K \leq N$, $K, N \in \mathbb{N}$.
Using the relation (7.54), we find that

$$\begin{aligned}
(7.65)\quad \lim_{N\to\infty} I_N^{(22)}(T) \leq & \lim_{N\to\infty} \int_0^T \mathbf{E}||F(s,\cdot,X_N(s)) - F_K(s,\cdot,X_N(s))||^2_{L^2_\rho}\, ds \\
= & \int_0^T \mathbf{E}||F(s,\cdot,X(s)) - F_K(s,\cdot,X(s))||^2_{L^2_\rho}\, ds.
\end{aligned}$$

Since (7.65) holds for any $K \in \mathbb{N}$, letting $K \to \infty$ we get $I_N^{(22)}(T) \to 0$.
Replacing the $F$-terms by the $E$-terms, we know that $\tilde{I}_N^{(2)}(t) \to 0$ as $N \to \infty$.

In view of (7.64), we can apply the same arguments as in Step 4 to get

$$\lim_{N\to\infty} I_N^{(3)}(t) = 0 \text{ for any } t \in [0,T].$$

Thus, $X$ solves (7.1) resp. (7.2) in the sense of 7.1.1 **(ii)**.

The required continuity properties of the solution follow from the similar properties of the integral terms in the right hand side of the equations (7.1) and (7.2), which were considered in Section 5.1.

Thus, it remains to show estimates (7.8), (7.9).
We note that in both cases, we have for any $t \in [0,T]$

$$\mathbf{E}||X(t)||^{2\nu}_{L^{2\nu}_\rho} \leq c(\nu) \left[ \mathbf{E}||Y(t)||^{2\nu}_{L^{2\nu}_\rho} + \mathbf{E}||X_1(t)||^{2\nu}_{L^{2\nu}_\rho} \right]$$

and thus

$$\begin{aligned}
\sup_{t\in[0,T]} \mathbf{E}||X(t)||^{2\nu}_{L^{2\nu}_\rho} &\leq c(\nu)\left[\sup_{t\in[0,T]} \mathbf{E}||Y(t)||^{2\nu}_{L^{2\nu}_\rho} + \sup_{t\in[0,T]} \mathbf{E}||X_1(t)||^{2\nu}_{L^{2\nu}_\rho}\right] \\
&\leq c(\nu)\left[\sup_{t\in[0,T]} \mathbf{E}||Y(t)||^{2\nu}_{L^{2\nu}_\rho} + \sup_{\substack{t\in[0,T]\\ N\in\mathbb{N}}} \mathbf{E}||X_N(t)||^{2\nu}_{L^{2\nu}_\rho}\right] \\
&\leq c(\nu,\zeta,K,T,c(T),c_f(T),c_\sigma(T),C_{2\nu,\eta})(1+\mathbf{E}||\xi||^{2\nu}_{L^{2\nu}_\rho})
\end{aligned}$$

in the Poisson noise case resp.

$$\begin{aligned}
\sup_{t\in[0,T]} \mathbf{E}||X(t)||^{2\nu}_{L^{2\nu}_\rho} &\leq c(\nu)\left[\sup_{t\in[0,T]} \mathbf{E}||Y(t)||^{2\nu}_{L^{2\nu}_\rho} + \sup_{t\in[0,T]} \mathbf{E}||X_1(t)||^{2\nu}_{L^{2\nu}_\rho}\right] \\
&\leq c(\nu)\left[\sup_{t\in[0,T]} \mathbf{E}||Y(t)||^{2\nu}_{L^{2\nu}_\rho} + \sup_{\substack{t\in[0,T]\\ N\in\mathbf{N}}} \mathbf{E}||X_N(t)||^{2\nu}_{L^{2\nu}_\rho}\right] \\
&\leq c(\nu,\zeta,m,K,T,c(T),c_e(T),C_{2\nu,\eta})(1+\mathbf{E}||\xi||^{2\nu}_{L^{2\nu}_\rho})
\end{aligned}$$

in the Lévy noise case, which finishes the proof. ■

## 7.4 Proof of Theorem 7.1.6

So far, we have only established existence of a solution, but not uniqueness. Before we start with the proof of Theorem 7.1.6, let us recall two related uniqueness results on SDEs in infinite dimensions with non-Lipschitz drift. First, in the framework of Section 7.1, Manthey and Zausinger proved pathwise uniqueness of mild solutions to the Wiener noise driven SDE

$$\begin{aligned}
&dX(t) = (A(t)X(t) + F(t,X(t)))dt + \mathcal{M}_{\Sigma(t,X(t))}\,dW(t),\ t\in[0,T],\\
&X(0) = \xi,
\end{aligned}$$

which is just equation (7.1) with $C = 0$.
Their result requires the following additional assumptions:

- $\Theta$ is bounded;
- $W$ is a cylindrical Wiener process in $L^2$;
- $f$ is bounded, i.e. $\sup_{(t,y)\in[0,T]\times\mathbb{R}} |f(t,y)| < \infty$;
- $\sigma$ is strictly positive, i.e.

$$\inf_{(t,y)\in[0,T]\times\mathbb{R}} |\sigma(t,y)| \geq \varepsilon > 0,\ (t,y)\in[0,T]\times\Theta.$$

To prove their uniqueness result, Manthey and Zausinger used Girsanov's theorem, which is not applicable in our case. Therefore, our proof of Theorem 7.1.6 is based on the following uniqueness result in infinite dimensions with both Wiener and Poisson noise.

In Section 2.1 of their paper [80], Marinelli and Röckner consider the SDE

$$(7.66)\quad \begin{aligned} dX(t) = {} & (A(t)X(t) + F(X(t)))dt + B(t)dW(t) \\ & + \int_Z G(t,z)\,\tilde{N}(dt,dz),\ t \in [\,0,T\,], \\ X(0) = {} & \xi. \end{aligned}$$

In their setting, solutions to equation (7.66) take values in $L^2(\Theta)$ with $\Theta \subset \mathbb{R}^d$ being open and bounded with a smooth boundary. Here, $W$ is a cylindrical Wiener process in $L^2$ and $\tilde{N}$ is a compensated Poisson random measure on $[\,0,T\,] \times Z$ with compensator $dt \otimes m$, where $(Z, \mathcal{Z}, m)$ is a measurable space.
Since the results in [80] are restricted to open bounded $\Theta$ with smooth boundary $\partial\Theta$, we will **not** use the shortened notations from the previous sections.

**Definition 7.4.1:** *Assume that $f$ defining $F$ by **(NEM)** obeys **(PG)** with exponent $\nu \geq 1$. Given an initial condition $\xi \in L^{2\nu}(\Theta)$, an adapted process $(X(t))_{t\in[\,0,T\,]}$ such that*

$$\mathbf{E} \sup_{t\in[\,0,T\,]} ||X(t)||_{L^2}^2 < \infty$$

*is called a **mild solution** (7.66) if (cf. Definition 2, p.1531 in [80])*

- *$X(t) \in L^{2\nu}(\Theta)$, $P$-a.s., for all $t \in [\,0,T\,]$, and*
- $$\begin{aligned} X(t) = {} & e^{tA}\xi + \int_0^t e^{(t-s)A} F(X(s))\,ds \\ & + \int_0^t e^{(t-s)A} B(s)\,dW(s) + \int_0^t \int_Z e^{(t-s)A} G(s,z)\,\tilde{N}(ds,dz), \end{aligned}$$
  *$P$-a.s., for all $t \in [\,0,T\,]$ and the integrals on the right hand side are well-defined.*

Besides the assumption that $f$ is of at most polynomial growth, it is assumed that (cf. p.1531 in [80]):

- $A$ admits a unique extension to a strongly continuous semigroup of positive contractions on $L^{2\nu}(\Theta)$ and $L^{2\nu^2}(\Theta)$.
- $f$ is a continuous maximal monotone function, i.e. there is some $\mu \in \mathbb{R}$ such that $\mathbb{R} \ni \theta \mapsto f(\theta) + \mu\theta \in \mathbb{R}$ is monotone.

- $G$: $[0,T] \times \Omega \times Z \to L^{2\nu}(\Theta)$ obeys

$$\mathbf{E}\int_0^T \left[\int_Z ||G(t,z)||_{L^{2\nu^2}}^{2\nu^2}\, m(dz) + \left(\int_Z ||G(t,z)||_{L^{2\nu^2}}^{2}\, m(dz)\right)^{\nu^2}\right] dt < \infty. \tag{7.67}$$

Under these assumptions, the uniqueness result proven in [80] (cf. Proposition 7, p.1539 there) can be stated as follows

**Proposition 7.4.2:** *There is a unique càdlàg mild solution to (7.66) in the sense of Definition 7.4.1. It satisfies the estimate*

$$\mathbf{E}\sup_{t\in[0,T]} ||X(t)||_{L^{2\nu}}^{2\nu} \leq c(\nu,T)(1+\mathbf{E}||\xi||_{L^{2\nu}}^{2\nu}),$$

*where $c(\nu,T)$ is a positive constant.*

Now, let us prove Theorem 7.1.6 with the help of Proposition 7.4.2.

**Proof of 7.1.6(i):** First, we recall that by Theorem 7.1.2 **(i)** we have the existence of a mild predictable solution $[0,T] \ni t \mapsto X(t) \in L^2(\Theta)$, which obeys a càdlàg version.
As was mentioned in [80] (cf. Remark 13, p.1546 there), Proposition 7.4.2 extends to the case of an additional solution-dependent Wiener noise with a Lipschitz coefficient. This gives us the possibility to apply Proposition 7.4.2 to the solutions of equation (7.1) if the other conditions from 7.4.2 are fulfilled.
Note that the original result in [80] was formulated for nonrandom and time-independent $f$. Since by assumption $f$ is maximal monotone uniformly in $(t,\omega)$, the dependence of $f$ on $(t,\omega)$ does not cause any problem in extending Proposition 7.4.2 to our setting.
By assumption **(QI)** for $\eta$ and the fact that $c$ is bounded uniformly in $[0,T]\times\Omega$, we immediately get (7.67) with $G(t,x) = \mathcal{M}_{C(t)}(x)$, $(t,x) \in [0,T] \times L^2(\Theta)$.
Now, since the assumptions on $A$ are such that the assumptions from above are fulfilled, Proposition 7.4.2 gives us the claim. □

**Proof of 7.1.6(ii):** In this case, by Theorem 7.1.3 **(i)** we have the existence of a mild predictable solution $[0,T] \ni t \mapsto X(t) \in L^{2\nu}(\Theta)$, which obeys a càdlàg version. Similarly to the proof of 7.1.6 **(i)**, we get that Proposition 7.4.2 is applicable. Again, by the assumption that $f$ is maximal monotone uniformly in $(t,\omega)$ and the fact that $G$ from the proof of 7.1.6 **(i)** obeys (7.67), since **(QI)** holds with $q = 2\nu^2$ for $\eta$ and $c$ is bounded uniformly in $[0,T]\times\Omega$, we can apply Proposition 7.4.2. Thus, we get the claim. □ ■

**Remark 7.4.3:** *Theorem 7.4.2 can also be proven by the abstract unique-*

*ness result for dissipative stochastic evolution equations in Hilbert spaces proved in [81] (see Theorem 1, p.365/366 there). Also, this approach allows us to consider infinite-dimensional SDEs with nuclear Wiener processes.*

**Remark 7.4.4:** *Unfortunately, in the solution-dependent case, under the assumptions of the existence theorem for equations (1.1) and (1.2) (cf. Section 8.1) the uniqueness result given by Theorem 12 in [80] and Theorem 3 in [81] is not directly applicable as we will see in Remark 8.1.6* ***(v)*** *below.*

# Chapter 8

# Existence in the case of non-Lipschitz drift with multiplicative jump noise

In this chapter, we will present the main result of the thesis. We will prove the existence of mild solutions to equations (1.1) and (1.2) in the case of non-Lipschitz drift, i.e. we allow the jump coefficients in the equations to be solution-dependent.

The proofs will be done analogously to the proofs of Theorems 7.1.2 and 7.1.4. To shorten the presentation, we will merely discuss the parts that differ from the proofs in Chapter 7. Of course, this procedure requires comparison results for equations (1.1) and (1.2) in the case of Lipschitz drifts. These results will be proven similarly to the results in the additive case in Chapter 6, namely by using finite-dimensional approximations of the initial equations. To compare the corresponding solutions to the finite-dimensional equations driven by multiplicative Poisson resp. Lévy noise, we consider the approximations as equations in Sobolev spaces, since then we can apply the Sobolev embedding theorem to get solutions, which are continuous and bounded in space. With the help of this technique we can apply the comparison results from [92], [68] and [67] collected in Appendix C. To make use of the Sobolev embedding theorem, we need the restriction that the domain $\Theta \subset \mathbb{R}^d$ obeys the weak cone property (cf. Appendix A, Theorem A.6). The application of the finite-dimensional comparison results mentioned before forces us to assume that the function defining the jump resp. jump diffusion coefficient is monotonically increasing (resp. decreasing), the Lévy measure is supported by the set $L^2_{\geq 0}(\Theta)$ of nonnegative functions (resp. by the set $L^2_{\leq 0}(\Theta)$ of nonpositive functions). Furthermore, the family $(U_N)_{N\in\mathbb{N}}$ approximating the almost strong evolution operator $U$ in the sense of **(A6)** should obey the regularity property **(A8)** in Sobolev spaces $W^{m,2}(\Theta)$. This is the main new issue compared to Chapters 6 and 7.

Let us recall the basic framework.
For the whole chapter, let $(\Omega, \mathcal{F}, P)$ and $(\mathcal{F}_t)_{t\in[0,T]}$ for some $T > 0$ be as in Section 1.2.
We suppose that $\rho \in \mathbb{N} \cup \{0\}$ is such that $\mu_\rho(\Theta) < \infty$, i.e. we have the two basic cases

- $\rho > d$ for unbounded $\Theta \subset \mathbb{R}^d$ and
- $\rho = 0$ for bounded $\Theta$.

Again, given $X$, $Y \in L^2_\rho(\Theta)$, by writing $X \leq Y$ we mean that $X(\theta) \leq Y(\theta)$ for $\mu_\rho$-almost all $\theta \in \Theta$.
Analogously to the formulation of Theorem 7.1.6, in order to emphasize that we have a restriction on the domain $\Theta$, we will not use the shortened notation in this chapter, i.e. we will always write $L^2(\Theta)$ or $L^2_\rho(\Theta)$.

## 8.1 The main results of this chapter

First of all, let us give the exact setting.
Recall that according to the general framework from Chapter 7:

- $(A(t))_{t\in[0,T]}$ generates an almost strong evolution operator in $L^2_\rho(\Theta)$ in the sense of 2.1.1.
- $\sigma$, $\gamma$: $[0,T] \times \Omega \times \mathbb{R} \to \mathbb{R}$ generating $\Sigma$, $\Gamma$ by **(NEM)** are $\mathcal{P}_T \otimes \mathcal{B}(\mathbb{R})$-measurable and fulfill the Lipschitz property **(LC)** and the local boundedness property **(LB)**.
- $e$, $f$: $[0,T] \times \Omega \times \mathbb{R} \to \mathbb{R}$ generating $E$, $F$ by **(NEM)** are $\mathcal{P}_T \otimes \mathcal{B}(\mathbb{R})$-measurable, continuous in the third variable and fulfill the polynomial growth condition **(PG)** with exponent $\nu \geq 1$ and the one-sided linear growth condition **(LG)**.
- $W$ is a $Q$-Wiener process in $L^2(\Theta)$ such that either $Q \in \mathcal{T}^+(L^2(\Theta))$ and the system of eigenvectors $(e_n)_{n\in\mathbb{N}}$ of $Q$ obeys (3.1) (referred to as the **nuclear case**) or $Q = \mathbf{I}$ (referred to as the **cylindrical case**).
- $\tilde{N}$ is a compensated Poisson random measure and $L$ is a Lévy process such that the corresponding intensity measure $\eta$ obeys the square integrablility condition **(SI)**.

Additionally to the previous conditions, we need to assume that

**(M)** Let $\gamma$, $\sigma$: $[0,T]\times\Omega\times\mathbb{R}\to\mathbb{R}$ be $\mathcal{P}_T\otimes\mathcal{B}(\mathbb{R})$-measurable functions such that

$$\gamma(t,\omega,y_1)\leq\gamma(t,\omega,y_2)\text{ for any }(t,\omega)\in[0,T]\times\Omega\text{ and }y_1\leq y_2$$

resp.

$$\sigma(t,\omega,y_1)\leq\sigma(t,\omega,y_2)\text{ for any }(t,\omega)\in[0,T]\times\Omega\text{ and }y_1\leq y_2.$$

**(P)** The intensity measure $\eta$ is supported on

$$L^2_{\geq 0}:=\{\psi\in L^2\,|\,\psi\geq 0\,,\ d\theta-a.e.\}.$$

Assuming **(M)** and **(P)** is crucial to apply the comparison method in the case of multiplicative Poisson resp. Lévy noise.
Assumption **(M)** means that $\gamma$ resp. $\sigma$ is **nondecreasing** in the last variable.
**(P)** is surely satisfied if $\eta$ corresponds to a Lévy process $L$ of **positive jumps**, i.e., $\Delta L(t)\in L^2_{\geq 0}$.
Alternatively, we could assume that $\gamma$ resp. $\sigma$ is nonincreasing in the last variable and that $\eta$ is supported on $L^2_{\leq 0}:=\{\psi\in L^2\,|\,\psi\leq\ 0\ d\theta-a.e.\}$. E.g. this is the case if $\eta$ corresponds to a Lévy process $L$ of nonpositive jumps, i.e. $\Delta L(t)\in L^2_{\leq 0}$. Lévy processes with nonpositive jumps are e.g. used to model queing, insurance risk and dam theory.
For further examples of application and a closer look at Lévy processes with nonpositive jumps, we refer to [12], Chapter 7.
In the proof of the comparison theorem 8.1.5 below, we have to work in a Hilbert space setting in order to develop proper stochastic analysis and define Wiener and Poisson stochastic integral via Itô's isometry, which is not evident in the case of general Banach spaces. We further need to evaluate the equations pointwise. Thus, we will work in Sobolev spaces $W^{m,2}(\Theta)$ (for the definition see Appendix A) with large enough $m>\frac{d}{2}$, since

1. $W^{m,2}(\Theta)$ is a Hilbert space (in contrast to the general Sobolev spaces $W^{m,p}(\Theta)$) and
2. $W^{m,2}(\Theta)\underset{\rightarrow}{\subseteq}C_b(\Theta)$, i.e. $W^{m,2}(\Theta)$ is continuously embedded in $C_b(\Theta)$.

Recall from Appendix A (cf. Theorem A.6 Case 1 (with $j=0$) there) that in the case of $\Theta$ being a domain obeying the **weak cone property**(cf. Definition A.1.1 in Appendix A), the second item holds for any $m\in\mathbb{N}$ such that

$$mp = 2m > d \iff m > \tfrac{d}{2}.$$

The weak cone property is a standard assumption in the theory of Sobolev spaces.

Below, we always suppose that $\Theta \in \mathcal{B}(\mathbb{R}^d)$ obeys the weak cone property (for examples of spaces fulfilling this property, see Appendix A).

For a comparison theorem it is enough to consider the solutions in $\mathcal{W}_m^2(T)$, the space of predictable, $W^{m,2}(\Theta)$-valued processes $X = (X(t))_{t\in[0,T]}$ such that

$$\sup_{t\in[0,T]} \mathbf{E}||X(t)||^2_{W^{m,2}} < \infty.$$

Furthermore, we need an additional assumption on the approximations of the generating family of operators $(A(t))_{t\in[0,T]}$:

**(A8)** The family $(A_N(t))_{t\in[0,T]}$ from **(A6)** is such that, for any $N \in \mathbb{N}$, we have $(A_N(t))_{t\in[0,T]} \subset \mathcal{L}(W^{m,2}(\Theta))$ and, for the corresponding evolution operators $U_N$,

$$\sup_{0\le s\le t\le T} ||U_N(t,s)||_{\mathcal{L}(W^{m,2})} := \bar{c}_N(T) < \infty.$$

Now, we can formulate the main results of this chapter, which will be proven in Sections 8.2–8.4 below.

As in the case of Lipschitz coefficients considered in Chapter 5, we split our considerations into the following two cases:

**Case (A)** We suppose that $f$ resp. $e$ generating $F$ resp. $E$ by **(NEM)** fulfills the condition **(PG)** from the introduction with $\nu = 1$, i.e. $f$ resp. $e$ is of at most linear growth.
An $L^2_\rho$-valued initial condition $\xi$ fulfills $\mathbf{E}||\xi||^q_{L^2_\rho} < \infty$ for some $q \ge 2$.
We show existence of a solution $X \in \mathcal{H}^q(T)$ starting from the above $\xi$.

**Case (B)** We suppose that $f$ resp. $e$ generating $F$ resp. $E$ by **(NEM)** fulfills condition **(PG)** from the introduction with $\nu > 1$.

An $L_\rho^{2\nu}$-valued initial condition $\xi$ obeys $\mathbf{E}||\xi||_{L_\rho^{2\nu}}^{2\nu} < \infty$ for the above $\nu$. We show existence of a solution $X \in \mathcal{G}_\nu(T)$ starting from the above $\xi$.

Analogously to the situation in Chapter 7, the first theorem describes the case of at most linear growth for the drift terms.

**Theorem 8.1.1:** *Suppose that, additionally to* ***(A8)****, the almost strong evolution operator* $U$ *generated by* $(A(t))_{t\in[0,T]}$ *has properties* ***(A0)****–****(A2)****, and* ***(A6)****.*
*Suppose that in* ***(A2)*** *we have* $\zeta \in [0, \frac{1}{2})$.
*Furthermore, let* ***(PG)*** *be fulfilled with exponent* $\nu = 1$ *both for* $e$ *and* $f$.
*Suppose that* $q \in (\frac{2}{1-\zeta}, \frac{2}{\zeta})$ *(Note that, by the choice of* $\zeta$*, this intervall is non-empty!) and the initial condition is as in* ***Case (A)****.*
*Finally, assume that the integrability condition* ***(QI)*** *for the Lévy measure* $\eta$ *is fulfilled with the above* $q$*.*

***(i)*** *There exists a predictable mild solution to (1.1) in the sense of 5.1.2* ***(i)****. The process* $t \mapsto X(t)$ *is continuous in* $L^q(\Omega; L_\rho^2(\Theta))$*.*
*Furthermore, we have the moment bound*

$$\sup_{t\in[0,T]} \mathbf{E}||X(t)||_{L_\rho^2}^q \le c(q, K, T, c(T), c_f(T), c_\sigma(T), C_{q,\eta})(1 + \mathbf{E}||\xi||_{L_\rho^2}^q) \tag{8.1}$$

*with a positive constant on the right hand side. Additionally assuming that* $\gamma$ *resp.* $U$ *obeys*

$$\sup_{(t,\omega,y)\in[0,T]\times\Omega\times\mathbb{R}} |\gamma(t, \omega, y)| := K < \infty \tag{8.2}$$

*resp.* ***(A7)****, there exists a cádlág version of the solution.*

***(ii)*** *There exists a predictable mild solution to (1.2) in the sense of 5.1.2* ***(i)****.*
*The process* $t \mapsto X(t)$ *is continuous in* $L^q(\Omega; L_\rho^2(\Theta))$*.*
*Furthermore, we have the moment bound*

$$\sup_{t\in[0,T]} \mathbf{E}||X(t)||_{L_\rho^2}^q \le c(q, K, T, c(T), c_e(T), C_{q,\eta})(1 + \mathbf{E}||\xi||_{L_\rho^2}^q) \tag{8.3}$$

*with a positive constant on the right hand side. Additionally assuming that* $\sigma$ *resp.* $U$ *obeys*

$$\sup_{(t,\omega,y)\in[0,T]\times\Omega\times\mathbb{R}} |\sigma(t, \omega, y)| := K < \infty \tag{8.4}$$

*resp.* ***(A7)****, there exists a cádlág version of the solution.*

**Remark 8.1.2:** *Actually, in the nuclear case we could also assume* ***(A5)**** *with* $\nu = 1$ *instead of* ***(A2)*** *(see Remark 3.4.2* ***(ii)*** *and Theorem 4.1 above).*

The second theorem describes the general case of drift terms having polynomial growth of order $\nu > 1$. Analogously to Theorem 7.1.4, the solutions take their values in $L^{2\nu}_\rho$ but are only time-continuous in $L^2(\Omega; L^2_\rho)$.

**Theorem 8.1.3:** *Suppose that, additionally to* ***(A8)****, the almost strong evolution operator* $U$*, generated by* $(A(t))_{t\in[0,T]}$*, has properties* ***(A0)****–* ***(A4)****,* ***(A5)**** *and* ***(A6)*** *(Note that, analogously to Theorem 7.1.4, we could also assume* ***(A5)*** *instead of* ***(A2)*** *and* ***(A4)****. Furthermore, in the nuclear case we can drop* ***(A2)*** *and* ***(A4)*** *at all).*
*Suppose that in* ***(A2)*** *(resp.* ***(A5)****) we have* $\zeta \in [0, \frac{1}{2})$*.*
*Let* $e$ *and* $f$ *fullfill* ***(PG)*** *with an exponent* $\nu \in (\frac{1}{1-\zeta}, \frac{1}{\zeta})$ *with* $\zeta$ *from* ***(A2)*** *(Note that, by the choice of* $\zeta$*, this intervall is non-empty!).*
*Suppose the initial condition* $\xi$ *is as in* ***Case (B)****. Assume that the integrability condition* ***(QI)*** *for the Lévy measure* $\eta$ *is fulfilled with* $q = 2\nu^2$*.*

***(i)*** *There exists a predictable mild solution* $X$ *to (1.1) in the sense of 5.1.2* ***(ii)****.*
*The process* $t \mapsto X(t)$ *is continuous in* $L^2(\Omega; L^2_\rho(\Theta))$*. Furthermore, we have the estimate*

$$\text{(8.5)} \qquad \sup_{t\in[0,T]} \mathbf{E}||X(t)||^{2\nu}_{L^{2\nu}_\rho} \le c(\nu, \zeta, K, T, c(T), c_f(T), c_\sigma(T), C_{2\nu,\eta})(1 + \mathbf{E}||\xi||^{2\nu}_{L^{2\nu}_\rho})$$

*with a positive constant on the right hand side. Additionally assuming that* $\gamma$ *resp.* $U$ *obeys (8.2) resp.* ***(A7)****, there exists a càdlàg version of the solution.*

***(ii)*** *There exists a predictable mild solution* $X$ *to (1.2) in the sense of 5.1.2* ***(ii)****.*
*The process* $t \mapsto X(t)$ *is continuous in* $L^2(\Omega; L^2_\rho(\Theta))$*. Furthermore, we have the estimate*

$$\text{(8.6)} \qquad \sup_{t\in[0,T]} \mathbf{E}||X(t)||^{2\nu}_{L^{2\nu}_\rho} \le c(\nu, \zeta, K, T, c(T), c_e(T), C_{2\nu,\eta})(1 + \mathbf{E}||\xi||^{2\nu}_{L^{2\nu}_\rho})$$

*with a positive constant on the right hand side. Additionallly assuming that* $\sigma$ *resp.* $U$ *obeys (8.4) resp.* ***(A7)****, there exists a càdlàg version of the solution.*

The proofs of Theorems 8.1.1 and 8.1.3 will be done in Sections 8.3 resp. 8.4.

**Remark 8.1.4:** *Analogously to Theorem 7.1.4 in the previous chapter, the assumption that* ***(QI)*** *holds with* $q = 2\nu^2$ *will be crucial in Step 2 in the* *proof of Theorem 8.1.3 (see estimate of the term $\bar{I}_M^{(2)}$ from the proof of 7.1.4 on p.245 (which accurs again on p.321 below)).*

To prove Theorem 8.1.1 and 8.1.2, similarly to the proofs of Theorems 7.1.2 and 7.1.4, we need the following comparison results in the case of Lipschitz drifts.

**Theorem 8.1.5:** *Let $U$ be an almost strong evolution operator generated by $(A(t))_{t\in[0,T]}$ such that, additionally to* ***(A8)****, also* ***(A0)*–*(A2)****,* ***(A6)*** *and* ***(A7)*** *hold. Let* ***(QI)*** *hold with the given $q \in (\frac{2}{1-\zeta}, \frac{2}{\zeta})$ for $\zeta$ from* ***(A2)*** *in $[0, \frac{1}{2})$. Furthermore, let $\xi^{(1)}$, $\xi^{(2)}$ be as in* ***Case (A)****:*

***(i)*** *Let $f^{(i)}$, $i = 1, 2$, $\sigma$ and $\gamma$ fulfill the Lipschitz property* ***(LC)*** *and the local boundedness property* ***(LB)****. Suppose that $\gamma$ additionally fulfills* ***(M)****. Furthermore, assume that*

$$\xi^{(1)} \leq \xi^{(2)},\ P\text{-a.s.},$$

*and*

$$f^{(1)}(t, y) \leq f^{(2)}(t, y) \text{ for all } (t, y) \in [0, T] \times \mathbb{R},\ P\text{-a.s.}.$$

*Then, we have*

$$X^{(1)}(t) \leq X^{(2)}(t),\ P\text{-a.s.},$$

*for any $t \in [0, T]$, where $X^{(i)} \in \mathcal{H}^q(T)$ denotes the unique predictable mild solution to (1.1).*

***(ii)*** *Let $e^{(i)}$, $i = 1, 2$, and $\sigma$ fulfill the Lipschitz property* ***(LC)*** *and the local boundedness property* ***(LB)****. Suppose that $\sigma$ additionally fulfills* ***(M)****. Furthermore, assume that*

$$\xi^{(1)} \leq \xi^{(2)},\ P\text{-a.s.},$$

*and*

$$e^{(1)}(t, y) \leq e^{(2)}(t, y) \text{ for all } (t, y) \in [0, T] \times \mathbb{R},\ P\text{-a.s.}.$$

*Then, we have*

$$X^{(1)}(t) \leq X^{(2)}(t),\ P\text{-a.s.},$$

*for all $t \in [0,T]$, where $X^{(i)} \in \mathcal{H}^q(T)$ denotes the unique predictable mild solution to (1.2).*

**Proof:** See Section 8.2 below.

**Remark 8.1.6:** ***(i)*** *By Proposition 5.2.1, **(A7)** implies the existence of càdlàg solutions $X^{(i)}$ with $X_-^{(i)} \in \mathcal{H}^q(T)$ to each of the equations (1.1) resp. (1.2). Thus,*

$$P(\{\omega \in \Omega \,|\, X^{(1)}(t,\omega) \leq X^{(2)}(t,\omega) for \quad all\ t \in [0,T]\}) = 1.$$

***(ii)*** *Repeating literally the arguments from Section 6.1 a comparison result with initial conditions $\xi^{(i)}$ as in **Case (B)** follows immediately from the comparison theorem 8.1.5 in **Case (A)**.*

***(iii)*** *Recall that already in Chapter 4 we had integrability conditions on the Lévy measure $\eta$, which of course also have to be fulfilled in this special case.*
*For a class of examples fulfilling both **(P)** and **(QI)** from Chapter 4, see Appendix E at the end of this thesis.*

***(iv)*** *The assumption $q \in (\frac{2}{1-\zeta}, \frac{2}{\zeta})$ and the boundedness assumption on $\gamma$ resp. $\sigma$ are needed to guarantee the existence of a càdlàg mild solution (cf. the results on the pathwise properties of the Bochner convolutions and the stochastic convolutions w.r.t. Wiener processes and compensated Poisson random measures from Sections 3.3, 3.4 and 4.2).*

***(v)*** *Note that in this chapter, we do not have a uniqueness result by the means used so far.*
*Indeed, in their paper [80] Marinelli and Röckner get the additional assumption*

$$\mathbf{E} \int\limits_{L^2(\Theta)} ||\mathcal{M}_{\Gamma(s,\varphi_1)} - \mathcal{M}_{\Gamma(s,\varphi_2)} x||^2_{L^2_\rho}\, \eta(dx) \leq h(s) ||\varphi_1 - \varphi_2||^2_{L^2_\rho}$$

*for the jump coefficient in the case, where the jump coefficient is solution-dependent (cf. Theorem 12, p.1544 in [80]). Here, $h$ is supposed to be in $L^1([0,T])$.*
*To have such an estimate, we need that $\eta$ is supported on the Sobolev space $W^{m,2}(\Theta)$ with $m > \frac{d}{2}$ as above and with $\Theta \subset \mathbb{R}^d$ obeying the weak cone property.*
*Then, by Theorem A.6, we have $W^{m,2}(\Theta) \underset{\rightarrow}{\subseteq} C_b(\Theta)$ and thus*

$$\sup_{\theta \in \Theta} |x(\theta)| \leq C ||x||_{W^{m,2}} \text{ for any } x \in W^{m,2}(\Theta).$$

*Therefore, for any* $s \in [\,0, T\,]$ *and* $\varphi_1,\ \varphi_2 \in L^2_\rho(\Theta)$*, given a Lipschitz function* $\Gamma$ *we have*

$$\begin{aligned}
&\int_{L^2} ||(\mathcal{M}_{\Gamma(s,\varphi_1)} - \mathcal{M}_{\Gamma(s,\varphi_2)})x||^2_{L^2_\rho}\,\eta(dx)\\
&= \int_{L^2} ||(\mathcal{M}_{\Gamma(s,\varphi_1)-\Gamma(s,\varphi_2)}x||^2_{L^2_\rho}\,\eta(dx)\\
&= \int_{W^{m,2}} \int_{\Theta} [\gamma(s,\varphi_1(\theta)) - \gamma(s,\varphi_2(\theta))]^2 x^2(\theta)\,\mu_\rho(d\theta)\,\eta(dx)\\
&\le C^2 \left( \int_{W^{m,2}} ||x||^2_{W^{m,2}}\,\eta(dx) \right) ||\Gamma(s,\varphi_1) - \Gamma(s,\varphi_2)||^2_{L^2_\rho}\\
&\le C^2 c^2_\gamma(T) \left( \int_{W^{m,2}} ||x||^2_{W^{m,2}}\,\eta(dx) \right) ||\varphi_1 - \varphi_2||^2_{L^2_\rho},
\end{aligned}$$

*and obviously*

$$h(s) := C^2 c^2_\gamma(T) \left( C \int_{W^{m,2}} ||x||^2_{W^{m,2}}\,\eta(dx) \right),\ s \in [\,0, T\,]$$

*defines an element in* $L^1([\,0, T\,])$ *such that the assumptions of Theorem 12 in [80] are fulfilled.*
*Thus, to be able to have an estimate of the required form we may strengthen the assumption **(QI)** in the sense that it holds in* $W^{m,2}(\Theta)$ *instead of* $L^2(\Theta)$.

## 8.2 Proof of Theorem 8.1.5

Recall that in Chapter 6 the jump coefficients were just additive, i.e. solution-independent, whereas in Theorem 8.1.5 we allow for multiplicative, i.e. solution-dependent, jump coefficients.

To this end, we need to apply the corresponding finite-dimensional comparison results for jump diffusions shown by Peng and Zhu (cf. [92]) resp. Krasin and Melnikov (cf. [67]). To apply those results, we need to evaluate our solution processes pointwise, which gives reason to consider the Sobolev spaces $W^{m,2}(\Theta)$ described in the introduction of this chapter.

We try to keep the structure of Chapter 6, but, compared to that chapter, we need further approximations to evaluate the equations pointwise.

The approximations will be described in Subsections 8.2.1 (for equation (1.1)) and 8.2.2 (for equation (1.2)). The comparison results for the approximating equations from Subsections 8.2.1 and 8.2.2 will be shown in Subsection 8.2.3, whereas the convergence of the approximation is shown in Subsection 8.2.4.

To shorten the presentation, in Subsections 8.2.3 and 8.2.4 we only present the issues that principally differ from that of Chapter 6.

### 8.2.1 Approximations of equation (1.1)

It has already been emphasized before that, in order to prove Theorem 8.1.5, we need to evaluate the solutions pointwise. Therefore, we will work in Sobolev spaces $W^{m,2}(\Theta)$ with $m > \frac{d}{2}$ and $\Theta \subset \mathbb{R}^d$ obeying the weak cone property.
The main difference to the proof of the comparison result in Chapter 6 is that we first approximate equation (1.1) by equations that are uniquely solvable in $W^{m,2}(\Theta) \underset{\rightarrow}{\subseteq} L^2(\Theta) \underset{\rightarrow}{\subseteq} L^2_\rho(\Theta)$. After getting the comparison result for the regularized equations, by taking limits in $L^2_\rho(\Theta)$ we can conclude the similar result for the initial equations. The latter will be done by reasoning close to that from Chapter 6.

To be able to approximate in $W^{m,2}(\Theta)$, we need the some technical preparations:

For a fixed $\varphi \in C_0^\infty(\mathbb{R}^d)$, let $(\delta_J)_{J\in\mathbb{N}}$ be given by

$$\delta_J(\theta) := \tfrac{1}{J}\varphi(J\theta),\ \theta \in \mathbb{R}^d,\ J \in \mathbb{N}. \tag{8.7}$$

This is a Dirac sequence (cf. 2.13 2 from [6]) and for any $\psi \in L^2(\mathbb{R}^d)$ we have $\psi_J := conv(\psi, \delta_J) \in C_0^\infty(\mathbb{R}^d)$ (cf. 2.12 4 from [6]), where *conv* denotes the standard convolution mapping in $\mathbb{R}^d$ (More about the properties of convolutions can be found in the proof of Proposition 3.4.3 in Chapter 3.).

Recall that we also need boundedness of the approximating functions. In the following, we describe the procedure of getting such approximations by considering the initial conditions $\xi^{(i)} \in L^2_\rho(\Theta)$, $i = 1, 2$. We have

$$\xi^{(i)} \in L^2_\rho(\Theta) \iff \psi^{(i)} := \mu_\rho^{-\frac{1}{2}}\xi^{(i)} \in L^2(\Theta).$$

In the case $\Theta \neq \mathbb{R}^d$, denoting the trivial extension of $\xi^{(i)}$ on $\mathbb{R}^d$ again by $\xi^{(i)}$, we get that $\psi^{(i)} := \mu_\rho^{-\frac{1}{2}}\xi^{(i)} \in L^2(\mathbb{R}^d)$.
Next, we define a sequence of cut-off functions $\chi_J$: $\mathbb{R}_+ \to [0, 1]$,

$\chi_J \in C_0^\infty(\mathbb{R}_+)$, $J \in \mathbb{N}$, with the properties $\chi_J(r) = 1$ for $r \in [0, J]$, $\chi_J(r) = 0$ for $r \geq J+1$ and $\chi_{J+1}(r) = \chi_J(r-1)$ for $r \geq 1$.
Thus, by setting

$$\psi_J^{(i)} := \mu_\rho^{\frac{1}{2}} conv(\chi_J \psi^{(i)}, \delta_J),\ J \in \mathbb{N}, \tag{8.8}$$

(and, if necessary, taking a subsequence $\delta_{J'(J)}$, $J \to \infty$) we get families $(\psi_J^{(i)})_{J\in\mathbb{N}} \subset C_0^\infty(\mathbb{R}^d)$, $i = 1, 2$, with the properties

$$\begin{gathered} ||\xi_J^{(i)}||_{L^2_\rho} \leq ||\xi^{(i)}||_{L^2_\rho}, \\ ||\psi_J^{(i)} - \psi^{(i)}||_{L^2_\rho} = ||\mu_\rho^{-\frac{1}{2}} \xi_J^{(i)} - \mu_\rho^{-\frac{1}{2}} \xi^{(i)}||_{L^2} \to 0 \text{ as } J \to \infty. \end{gathered} \tag{8.9}$$

By Lebesgue's dominated convergence theorem, this immediately gives us

$$\mathbf{E}||\xi_J^{(i)} - \xi^{(i)}||^2_{L^2_\rho} \to 0 \text{ as } J \to \infty.$$

Obviously for each $J \in \mathbb{N}$ $\psi_J^{(i)}$ restricted to $\Theta$ is an element of $W^{m,2}(\Theta)$ and $\psi_J^{(1)} \leq \psi_J^{(2)}$ provided $\psi^{(1)} \leq \psi^{(2)}$.

Next, we approximate the identity function

$$L^2(\Theta) \ni \psi \mapsto I(\psi) := \psi \in L^2(\Theta)$$

by the family of mappings

$$L^2(\Theta) \ni \psi \mapsto I_J(\psi) := conv(\psi, \delta_J) \in W^{m,2}(\Theta).$$

Therefore, for any $\psi \in L^2(\Theta)$, $I_J(\psi) \to I(\psi) = \psi$ in $L^2(\Theta)$ as $J \to \infty$ and

$$||I_J(\psi)||_{L^2} \leq ||\psi||_{L^2}. \tag{8.10}$$

Obviously, $\psi \in L^2(\Theta)$ implies $I_J(\psi) \in C^\infty(\mathbb{R}^d)$ and , for arbitrary $\alpha \in \mathbb{N}_0^d$, for the corresponding partial derivative we have
$\partial^\alpha I_J = conv(\psi, \partial^\alpha \delta_J) \in L^2(\Theta)$.
Furthermore, we need to approximate the coefficients $f^{(i)}$, $\sigma$ and $\gamma$, which define the operators $F^{(i)}$, $\Sigma$ and $\Gamma$ by **(NEM)**, by smooth functions with the following properties (cf. e.g. [65], items (2.15)–(2.17) in Section 2 there):

1. The $k$-th derivatives of $f_J^{(i)}$, $\sigma_J$ and $\gamma_J$ w.r.t. $y \in \mathbb{R}$ are bounded and continuous for $k = 0, 1, ..., m+1$ and $i = 1, 2$ with all partial deriva-

tives taking values in $L^2(\Theta)$;

2. $f_J^{(1)}(t,y) \le f_J^{(2)}(t,y)$ for all $(t,y) \in [0,T] \times \mathbb{R}$, $P$-a.s.;

3. $f_J^{(i)}(t,\omega,y) \to f^{(i)}(t,\omega,y)$ as $J \to \infty$ for any $(t,\omega,y) \in [0,T] \times \Omega \times \mathbb{R}$ and $i = 1,2$.
   The same holds true for $\sigma_J$ and $\gamma_J$;

4. The functions $f_J^{(i)}$, $i = 1,2$, $\sigma_J$ and $\gamma_J$ fulfill **(LC)** and **(LB)** uniformly in $J \in \mathbb{N}$.

Such functions can e.g. be gained by using cut-off and mollifiers in $\mathbb{R}^d$ (like in the previous constructions). A crucial moment here is that the convolution operator is a contraction not only in $L^2(\Theta)$ but also in the Lipschitz norm (see e.g. [108]), which guarantees that **(LC)** holds for all $f_J^{(i)}$, $J \in \mathbb{N}$, with the same constant, which does not depend on $t \in [0,T]$ and $\omega \in \Omega$. Furthermore, the convolution operator preserves the monotonicity property, i.e. given any $t \in [0,T]$ and any $y \in \mathbb{R}$, $f^{(1)}(t,y) \le f^{(2)}(t,y)$ implies $f_J^{(1)}(t,y) \le f_J^{(2)}(t,y)$ for all $J \in \mathbb{N}$.

For a fixed $J \in \mathbb{N}$, we consider the SDE

$$\begin{aligned} dX_J^{(i)}(t) = \;& (A(t)X_J^{(i)}(t) + F_J^{(i)}(t, X_J^{(i)}(t)))dt \\ & + \mathcal{M}_{\Sigma_J(t,X_J(t))}\, dW(t) + \int_{L^2} \mathcal{M}_{\Gamma_J(t,X_J^{(i)}(t))} I_J(x)\, \tilde{N}(dt,dx),\ t \in [0,T], \end{aligned}$$

(8.11)

$$X_J^{(i)}(0) = \xi_J^{(i)}.$$

We look for solutions in the mild sense, i.e. for each $t \in [0,T]$ we have in $L^2_\rho(\Theta)$, $P$-a.s.,

$$\begin{aligned} X_J^{(i)}(t) = \;& U(t,0)\xi_J^{(i)} + \int_0^t U(t,s)F_J^{(i)}(s, X_J^{(i)}(s))\, ds \\ & + \int_0^t U(t,s)\mathcal{M}_{\Sigma_J(s,X_J^{(i)}(s))}\, dW(s) \\ & + \int_0^t \int_{L^2} U(t,s)\mathcal{M}_{\Gamma_J(s,X_J^{(i)}(t))} I_J(x)\, \tilde{N}(ds,dx). \end{aligned} \tag{8.12}$$

Since by construction the $f_J^{(i)}$, $\sigma_J$ and $\gamma_J$ fulfill **(LC)** and **(LB)** and $\gamma_J$ is bounded, we get the existence of càdlàg modifications of mild solutions to (8.11) in $L^2_\rho(\Theta)$ from the existence and uniqueness results in Chapter 5.

Starting from (8.11), we define the further approximations similarly to the procedure in Chapter 6.

Analogously to equation (6.4), for a fixed $L \in \mathbb{N}$, we solve in $L^2_\rho(\Theta)$, for $t \in [0,T]$

$$\begin{aligned} dX^{(i)}_{L,J}(t) = \quad & (A(t)X^{(i)}_{L,J}(t) + F^{(i)}_J(t, X^{(i)}_{L,J}(t)))dt \\ & + \mathcal{M}_{\Sigma_J(t,X^{(i)}_{L,J}(t))}\, dW_L(t) + \int_{L^2} \mathcal{M}_{\Gamma_J(t,X^{(i)}_{L,J}(t))} I_J(x)\, \tilde{N}(dt,dx), \end{aligned}$$

(8.13)

$$X^{(i)}_{L,J}(0) = \xi^{(i)}_J$$

in the mild sense. Recall that, given the representation (2.5) from Section 2.3, $W_L$ was defined as

$$W_L(t) := \sum_{n=1}^{L} \sqrt{a_n} e_n w_n(t),\ t \in [0,T]. \tag{8.14}$$

Note that in general we do not assume that the orthonormal basis $(e_n)_{n\in\mathbb{N}}$ obeys (3.1).
The solution $X^{(i)}_{L,J} \in \mathcal{H}^q(T)$ satisfies $P$-a.s. the identity in $L^2_\rho(\Theta)$

$$\begin{aligned} X^{(i)}_{L,J}(t) = \quad & U(t,0)\xi^{(i)}_J + \int_0^t U(t,s)F^{(i)}_J(s, X^{(i)}_{L,J}(s))\, ds \\ & + \int_0^t U(t,s)\mathcal{M}_{\Sigma_J(s,X^{(i)}_{L,J}(s))}\, dW_L(s) \\ & + \int_0^t \int_{L^2} U(t,s)\mathcal{M}_{\Gamma_J(s,X^{(i)}_{L,J}(t))} I_J(x)\, \tilde{N}(ds,dx),\ t \in [0,T]. \end{aligned} \tag{8.15}$$

Next, we approximate any element $e_n \in L^2(\Theta)$ of the orthonormal basis in the representation (8.14) by a sequence $(e_{n,M})_{M\in\mathbb{N}} \subset C^\infty_0(\Theta)$ in the sense of (8.9).
Let us fix some $M \in \mathbb{N}$. We define

$$W_{M,L}(t) := \sum_{n=1}^{L} \sqrt{a_n} e_{n,M} w_n(t),\ t \in [0,T]. \tag{8.16}$$

In $L^2_\rho(\Theta)$, we solve, for $t \in [0,T]$

$$\begin{aligned} dX^{(i)}_{M,L,J}(t) = \quad & (A(t)X^{(i)}_{M,L,J}(t) + F^{(i)}_J(t, X^{(i)}_{M,L,J}(t)))dt \\ & + \mathcal{M}_{\Sigma_J(t,X^{(i)}_{M,L,J}(t))}\, dW_{M,L}(t) + \int_{L^2} \mathcal{M}_{\Gamma_J(t,X^{(i)}_{M,L,J}(t))} I_J(x)\, \tilde{N}(dt,dx), \end{aligned}$$

(8.17)

$X^{(i)}_{M,L,J}(0) = \xi^{(i)}_J$

in the mild sense. The solution process $X^{(i)}_{M,L,J} \in \mathcal{H}^q(T)$ satisfies $P$-a.s. the identity in $L^2_\rho(\Theta)$

$$(8.18)\ X^{(i)}_{M,L,J}(t) = \begin{aligned}[t] & U(t,0)\xi^{(i)}_J + \int_0^t U(t,s)F^{(i)}_J(s, X^{(i)}_{M,L,J}(s))\,ds \\ & + \int_0^t U(t,s)\mathcal{M}_{\Sigma_J(s,X^{(i)}_{M,L,J}(s))}\,dW_{M,L}(s) \\ & + \int_0^t \int_{L^2} U(t,s)\mathcal{M}_{\Gamma_L(s,X^{(i)}_{M,L,J}(t))} I_J(x)\,\tilde{N}(ds,dx),\ t \in [\,0,T\,]. \end{aligned}$$

Finally, we additionally fix some $N \in \mathbb{N}$ and consider the equation

$$\begin{aligned} dX^{(i)}_{N,M,L,J}(t) = \ & (A_N(t)X^{(i)}_{N,M,L,J}(t) + F^{(i)}_J(t, X^{(i)}_{N,M,L,J}(t)))dt \\ & + \mathcal{M}_{\Sigma_J(t,X_{N,M,L,J}(t))}\,dW_{M,L}(t) \\ & + \int_{L^2} \mathcal{M}_{\Gamma_J(t,X^{(i)}_{N,M,L,J}(t))} I_J(x)\,\tilde{N}(dt,dx),\ t \in [\,0,T\,], \end{aligned}$$

(8.19)

$X^{(i)}_{N,M,L,J}(0) = \xi^{(i)}_J,$

where $A_N(t) \in \mathcal{L}(L^2)$ approximates $A(t)$ in the sense of **(A6)** and obeys **(A8)**.

Since all coefficients are Lipschitz continuous with a uniform Lipschitz constant for all $(t,\omega) \in [\,0,T\,] \times \Omega$, analogously to equation (6.5) we get the existence of a unique (strong=mild) solution $X^{(i)}_{N,M,L,J} \in \mathcal{H}^q(T)$ to (8.19). Being rewritten in the mild form, $P$-a.s., for each $t \in [\,0,T\,]$ we have the following identity in $L^2_\rho(\Theta)$

$$(8.20)\ X^{(i)}_{N,M,L,J}(t) = \begin{aligned}[t] & U_N(t,0)\xi^{(i)}_J + \int_0^t U_N(t,s)F^{(i)}_J(s, X^{(i)}_{N,M,L,J}(s))\,ds \\ & + \int_0^t U_N(t,s)\mathcal{M}_{\Sigma_J(s,X^{(i)}_{N,M,L,J}(s))}\,dW_{M,L}(s) \\ & + \int_0^t \int_{L^2} U_N(t,s)\mathcal{M}_{\Gamma_J(s,X^{(i)}_{N,M,L,J}(s))} I_J(x)\,\tilde{N}(ds,dx). \end{aligned}$$

To summarize, the aim of the construction is to consider approximations of $W$ by finite-dimensional Wiener processes $W_{M,L}$ and the approximation of $A(t)$ by the bounded operators $A_N(t)$. Furthermore, all functions defining Nemitskii operators are chosen to have bounded smooth derivatives.

Analogously to the consideration of (6.4) and (6.5) in Section 6.1, we get the existence and uniqueness of the $X^{(i)}_{M,L,J}$ and $X^{(i)}_{N,M,L,J}$ in $\mathcal{H}^q(T)$ from the general solvability results in the Lipschitz case (see Section 5.2, Theorem 5.2.1/Corollary 5.2.5), since $A_N$ and $W_{M,L}$ can be seen as special cases of $A$ and $W$ from Sections 5.1/5.2.
Furthermore, we have a unique strong solution to equation (8.19) considered in the Sobolev spaces $W^{m,2}(\Theta)$. This follows by general results about SDEs with Poisson noise and Lipschitz coefficients in Hilbert spaces (see e.g. [95] or [80]). We point out that according to the above approximation procedure all operator coefficients in (8.19) are Lipschitzian in the space $W^{m,2}(\Theta)$, whereby the Lipschitz constant for them can be chosen to be the same for all $(t,\omega) \in [\,0,T\,] \times \Omega$.
Thus, we also have $X^{(i)}_{N,M,L,J} \in \mathcal{W}^2_m(T)$ and it obeys a càdlàg modification in $W^{m,2}(\Theta)$.
Furthermore, we have the existence of càdlàg versions in $L^2_\rho(\Theta)$ of the solutions to (8.11), (8.17) and (8.19).

In Section 8.2.3, for fixed $N, M, L, J \in \mathbb{N}$ we will show a pointwise comparison result for the processes $Y^{(i)} := X^{(i)}_{N,M,L,J} \in \mathcal{W}^2_m(T)$.

### 8.2.2 Approximations of equation (1.2)

Taking into account the Lévy-Itô decomposition, we note that (1.2) becomes

$$(8.21)\quad \begin{aligned} dX^{(i)}(t) = \;& (A(t)X^{(i)}(t) + E^{(i)}(t, X^{(i)}(t)) + \mathcal{M}_{\Sigma(t,X^{(i)}(t))}m)dt \\ & + \mathcal{M}_{\Sigma(t,X^{(i)}(t))}\, dW(t) + \int_{L^2} \mathcal{M}_{\Sigma(t,X^{(i)}(t))} x \tilde{N}(dt,dx), \\ X^{(i)}(0) = \;& \xi^{(i)}. \end{aligned}$$

Analogously to (8.11), (8.13), (8.17) and (8.19), we get the following SDEs:

Given $J \in \mathbb{N}$, let us consider

$$\begin{aligned} dX^{(i)}_J(t) = \;& (A(t)X^{(i)}_J(t) + E^{(i)}_J(t, X^{(i)}_J(t)) + \mathcal{M}_{\Sigma_J(t,X^{(i)}_J(t))} I_J(m))dt \\ & + \mathcal{M}_{\Sigma_J(t,X_J(t))}\, dW(t) + \int_{L^2} \mathcal{M}_{\Sigma_J(t,X^{(i)}_J(t))} I_J(x)\, \tilde{N}(dt,dx),\ t \in [\,0,T\,], \end{aligned}$$

(8.22)

$$X^{(i)}_J(0) = \xi^{(i)}_J.$$

Here, the $e^{(i)}_J$ are constructed analogously to the $f^{(i)}_J$.
Since by the construction the $e^{(i)}_J$ and $\sigma_J$ fulfill **(LC)** and **(LB)**, by the

existence and uniquenes results from Chapter 5 there are mild predictable solutions $X_J^{(i)} \in \mathcal{H}^q(t)$ to (8.22). In particular, due to the boundedness of $\sigma_J$ we get the existence of a càdlàg version of this solution.
Given additionally $L \in \mathbb{N}$, let $X_{L,J}^{(i)} \in \mathcal{H}^q(T)$ be the mild solution to the equation

$$\begin{aligned} dX_{L,J}^{(i)}(t) = \quad & (A(t)X_{L,J}^{(i)}(t) + E_J^{(i)}(t, X_{L,J}^{(i)}(t)) + \mathcal{M}_{\Sigma_J(t,X_{L,J}^{(i)}(t))} I_J(m))dt \\ & + \mathcal{M}_{\Sigma_J(t,X_{L,J}^{(i)}(t))}\, dW_L(t) + \int\limits_{L^2} \mathcal{M}_{\Gamma_J(t,X_{L,J}^{(i)}(t))} I_J(x)\, \tilde{N}(dt, dx), \end{aligned}$$

(8.23)

$$X_{L,J}^{(i)}(0) = \xi_J^{(i)}$$

with $W_L$ as in (8.14).
Next, for $M \in \mathbb{N}$ and $W_{M,L}$ as in (8.16), we uniquely solve in $\mathcal{H}^q(T)$

$$\begin{aligned} dX_{M,L,J}^{(i)}(t) = \quad & (A(t)X_{M,L,J}^{(i)}(t) + E_J^{(i)}(t, X_{M,L,J}^{(i)}(t)) + \mathcal{M}_{\Sigma_J(t,X_{M,L,J}^{(i)}(t))} I_J(m))dt \\ & + \mathcal{M}_{\Sigma_J(t,X_{M,L,J}^{(i)}(t))}\, dW_{M,L}(t) + \int\limits_{L^2} \mathcal{M}_{\Sigma_J(t,X_{M,L,J}^{(i)}(t))} I_J(x)\, \tilde{N}(dt, dx), \end{aligned}$$

(8.24)

$$X_{M,L,J}^{(i)}(0) = \xi_J^{(i)}.$$

Finally, given $N \in \mathbb{N}$, for $t \in [\,0, T\,]$ we consider

$$\begin{aligned} dX_{N,M,L,J}^{(i)}(t) = \quad & (A_N(t)X_{N,M,L,J}^{(i)}(t) + E_J^{(i)}(t, X_{N,M,L,J}^{(i)}(t)) + \mathcal{M}_{\Sigma_J(t,X_{N,M,L,J}^{(i)}(t))} I_J(m))dt \\ & + \mathcal{M}_{\Sigma_J(t,X_{N,M,L}(t))}\, dW_{M,L}(t) + \int\limits_{L^2} \mathcal{M}_{\Sigma_J(t,X_{N,M,L,J}^{(i)}(t))} I_J(x)\, \tilde{N}(dt, dx), \end{aligned}$$

(8.25)

$$X_{N,M,L,J}^{(i)}(0) = \xi_J^{(i)},$$

where $A_N(t) \in \mathcal{L}(L^2)$ approximates $A(t)$ in the sense of **(A6)** and obeys **(A8)**.

We get the existence and uniqueness of the $X_{M,L,J}^{(i)}$ and $X_{N,M,L,J}^{(i)}$ in $\mathcal{H}^q(T)$ from the general solvability results in the Lipschitz case (see Section 5.2, Theorem 5.2.1/Corollary 5.2.5), since $A_N$ and $W_M$ are only special cases of $A$ and $W$ from Sections 5.1/5.2.
Analogously to the case of equation (8.19), we have a unique strong solution $X_{N,M,L,J}^{(i)} \in \mathcal{W}_m^2(T)$ to (8.25). In particular, we get the existence of càdlàg versions in $L_\rho^2(\Theta)$ of the solutions to (8.24) and (8.25). Obviously, these

solutions also obey càdlàg modifications in $W^{m,2}(\Theta)$.

In Section 8.2.3, for fixed $N, M, L, J \in \mathbb{N}$ we will show a pointwise comparison result for the processes $Y^{(i)} := X^{(i)}_{N,M,L,J}$.

### 8.2.3 Comparison results for the approximations of (1.1) and (1.2)

To prove Theorem 8.1.3, we proceed analogously to the proof of 6.1.1. Similar to Section 6.2, we first prove a comparison result for the approximations in (8.20) and (8.25).

Given arbitrary $N, M, L, J \in \mathbb{N}$ and $i = 1, 2$, we shorten notation by setting

$$Y^{(i)}(t) := X^{(i)}_{N,M,L,J}(t),\ t \in [\,0, T\,]. \tag{8.26}$$

The main result of this subsection is the following comparison result for the processes $Y^{(i)}$, $i = 1, 2$, as in (8.26).

**Lemma 8.2.3.1:** ***(i)*** *Let $\xi^{(i)} \in L^2_\rho(\Theta)$, $i = 1, 2$, as in **Case (A)** and*

$$\xi^{(1)} \leq \xi^{(2)},\ P\text{-a.s.}.$$

*Furthermore, suppose that*

$$f^{(1)} \leq f^{(2)} \text{ for all } (t, y) \in [\,0, T\,] \times \mathbb{R},\ P\text{-a.s.}.$$

*Then, for càdlàg processes (8.20) we get*

$$Y^{(1)}(t) \leq Y^{(2)}(t), \text{ for all } t \in [\,0, T\,],\ P\text{-a.s.}.$$

***(ii)*** *Let $\xi^{(i)}$, $i = 1, 2$, as in **Case (A)**, and*

$$\xi^{(1)} \leq \xi^{(2)},\ P\text{-a.s.}.$$

*Furthermore suppose that*

$$e^{(1)} \leq e^{(2)}, \text{ for all } (t, y) \in [\,0, T\,] \times \mathbb{R},\ P\text{-a.s.}.$$

*Then, for càdlàg processes (8.25) we have*

$$Y^{(1)}(t) \leq Y^{(2)}(t), \textit{ for all } t \in [\,0,T\,],\ P\textit{-a.s..}$$

**Remark 8.2.3.2:** *Note that in contrast to Section 6.2, in the proof of 8.2.3.1 **(i)** we will not use the special property (3.1) for the orthonormal basis $(e_n)_{n\in\mathbb{N}} \subset L^2(\Theta)$ from (8.14) (since by construction, the approximating vectors $(e_{n,M})_{n\in\mathbb{N}}$ obey this property). Thus, we do not need to restrict the covariance operator $Q$ corresponding to $W$ to the nuclear case.*
*Therefore, we can conclude the comparison result **(ii)** for (1.2) from that for (1.1) (in the case $F = E + \mathcal{M}_{\Sigma_J}(m)$ and $\Sigma$=$\Gamma$).*

**Proof of 8.2.3.1:**

By Remark 8.2.3.2, it suffices to show **(i)**.

We use a discritization scheme similar to that of Section 6.2.
For a fixed $j \in \mathbb{N}$, we set $t_k := \frac{kT}{j}$, $k = 0, 1, 2, ..., j$, and thus get a partition of $[\,0,T\,]$ into $j$ intervalls of length $\frac{T}{j}$. We define processes $Z^{(i)}_{k,j}$ and $V^{(i)}_{k,j}$ by

$$Z^{(i)}_{0,j}(t) := \xi^{(i)}_J + \int_0^t \mathcal{M}_{\Sigma_J(s,Z^{(i)}_{0,j}(s))}\, dW_{M,L}(s) + \int_0^t \int_{L^2} \mathcal{M}_{\Gamma_J(s)} I_J(x)\, \tilde{N}(ds,dx),$$
$$V^{(i)}_{0,j}(t) := Z^{(i)}_{0,j}(t_1) + \int_0^t (A_N(s)V^{(i)}_{0,j}(s) + F^{(i)}_J(s, V^{(i)}_{0,j}(s)))\, ds$$

for $t \in [\,0,t_1\,]$ and

$$Z^{(i)}_{k,j}(t) := V^{(i)}_{k-1,j}(t_k) + \int_{t_k}^t \mathcal{M}_{\Sigma_J(s,Z^{(i)}_{k,j}(s))}\, dW_{M,L}(s) + \int_{t_k}^t \int_{L^2} \mathcal{M}_{\Gamma_J(s)} I_J(x)\, \tilde{N}(ds,dx),$$
$$V^{(i)}_{k,j}(t) := Z^{(i)}_{k,j}(t_{k+1}) + \int_{t_k}^t (A_N(s)V^{(i)}_{k,j}(s) + F^{(i)}_J(s, V^{(i)}_{k,j}(s)))\, ds$$

for $t \in [t_k, t_{k+1}]$ and $k = 1, 2, ..., j-1$.

Due to the Lipschitz continuity of the coefficients, the above equations obey unique strong solutions $V^{(i)}_{k,j}$, $Z^{(i)}_{k,j} \in \mathcal{W}^2_m([\,t_k, t_{k+1}\,])$.
Next, analogously to the proof of Theorem 6.1.1 in Section 6.2, we define processes $V^{(i)}_j$ and $Z^{(i)}_j$, $i = 1, 2$, taking values in $\mathcal{W}^2_m(T)$, from the above processes by setting (cf. (6.12) in Chapter 6)

$$Z^{(i)}_j(t) := Z^{(i)}_{k,j}(t),\ t \in [\,t_k, t_{k+1}\,),\ k = 0, 1, 2, ..., j-1,$$

$$V^{(i)}_j(0) := \xi^{(i)},$$

$V_j^{(i)}(t) := V_{k,j}^{(i)}(t)$, $t \in (t_k, t_{k+1}]$, $k = 0, 1, ..., j-1$,

$Z_j^{(i)}(T) := V_j^{(i)}(T)$.

Furthermore, $Z_j$ obeys a càdlàg version in $W^{m,2}(\Theta)$. Because of the continuous embedding $W^{m,2}(\Theta) \underset{\to}{\subseteq} C_b(\Theta)$, we can evaluate $V_j^{(i)}(t,\theta)$ resp. $Z_j^{(i)}(t,\theta)$ for any $t \in [\,0, T\,]$ and any $\theta \in \Theta$.

First, we prove

**Claim 1:** *For the processes $V_j$, $Z_j$ defined above we have, $P$-almost surely,*

$$\begin{aligned} &V_j^{(1)}(t,\theta) \le V_j^{(2)}(t,\theta), \\ &Z_j^{(1)}(t,\theta) \le Z_j^{(2)}(t,\theta), \end{aligned} \tag{8.27}$$

*for any $\theta \in \Theta$ and any $t \in [\,0, T\,]$.*

**Proof:** Let us start with the intervall $[\,0, t_1\,]$.
By (6.13), we have for $t \in [\,0, t_1\,)$

$$(8.28)\; Z_j^{(i)}(t) = \quad Z_{0,j}^{(i)}(t) = \xi_J^{(i)} + \int_0^t \mathcal{M}_{\Sigma_J(s, Z_j^{(i)}(s))}\, dW_{M,L}(s) + \int_0^t \int_{L^2} \mathcal{M}_{\Gamma_J(s, Z_j^{(i)}(s))} I_J(x)\, \tilde{N}(ds, dx).$$

For a moment, we consider (8.28) on the whole intervall $[\,0, t_1\,]$.
Now, for the first time we need a comparison result in the jump noise case. Note that, by construction, there exists a unique strong solution to (8.28) in $W^{m,2}(\Theta) \underset{\to}{\subseteq} C_b(\Theta)$. Thus, we can really evaluate $Z_j^{(i)}(t)$ pointwise in any $\theta \in \Theta$.

By the boundedness of $\gamma$, there is a càdlàg version $\tilde{Z}_j^{(i)}(t)$ of $Z_j^{(i)}(t)$.

To estimate the value of $Z_j^{(i)}(t,\theta)$, we consider the pairing of $Z_j^{(i)}(t)$ with $\delta_\theta$, where $\delta_\theta$ denotes the $\delta$-function at a fixed $\theta \in \mathbb{R}^d$. Due to the embedding theorem $W^{m,2}(\Theta) \underset{\to}{\subseteq} C_b(\Theta)$, this is a linear bounded functional in $W^{m,2}(\Theta)$.

Thus, by Propositions B.8, 2.5.3 resp. 2.6.8 for the Bochner- Wiener- resp. Poisson stochastic integral, we get

$$
\begin{aligned}
(8.29)\ Z_j^{(i)}(t,\theta) = \quad & < Z_j^{(i)}(t), \delta_\theta >_{L^2(\Theta)} \\
= \quad & \xi_J^{(i)}(\theta) + \sum_{n=1}^{L} \sqrt{a_n} \int_0^t < \mathcal{M}_{\Sigma_J(s,Z_j^{(i)}(s))} e_{n,M}, \delta_\theta >_{L^2} \ dw_n(s) \\
& + \int_0^t \int_{L^2} < \mathcal{M}_{\Gamma_J(s,Z_j^{(i)}(s))} x_J, \delta_\theta >_{L^2} \ \tilde{N}(ds,dx).
\end{aligned}
$$

Obviously, for any $1 \le n \le L$ and $\theta \in \Theta$ we have

$$
\int_0^t < \mathcal{M}_{\Sigma_J(s,Z_j^{(i)}(s))} e_{nM}, \delta_\theta >_{L^2} \ dw_n(s) = \int_0^t \sigma_J(s, Z_j^{(i)}(s,\theta)) e_{n,M}(\theta)\, dw_n(s).
$$

The integral w.r.t. the compensated Poisson random measure can be rewritten as follows:

$$
\begin{aligned}
(8.30)\ & \int_0^t \int_{L^2(\Theta)} < \mathcal{M}_{\Gamma_J(s,Z_j^{(i)}(s))} I_J(x), \delta_\theta >_{L^2(\Theta)} \ \tilde{N}(ds,dx) \\
= & \int_0^t \int_{L^2(\Theta)} (\gamma_J(s, Z_j^{(i)}(s)) I_J(x))(\theta)\, \tilde{N}(ds,dx).
\end{aligned}
$$

Note that $I_J(x)(\theta) := \int_{\mathbb{R}^d} x(y)\delta_J(\theta - y)\, dy =< x, \delta_{J,\theta} >_{L^2}$, where $0 \le \delta_{J,\theta} \in C_0^\infty(\mathbb{R}^d)$ is defined by $\delta_{J,\theta}(y) := \delta_J(\theta - y)$, $y \in \mathbb{R}^d$. Thus, the integral in (8.30) can be rewritten as

$$
\int_0^t \int_{\mathbb{R}} \gamma_J(s, Z_j^{(i)}(s,\theta)) u\, \tilde{N}_\theta(ds,du),
$$

where $\tilde{N}_\theta(ds,du)$ is the projection of $\tilde{N}(ds,dx)$ (cf. Section 2.4 for general projections) on the one-dimensional subspace of functions

$$
L_\theta^2 := \{< x, \delta_{J,\theta} >_{L^2} \delta_{J,\theta} \,|\, x \in L^2(\Theta)\} \subset L^2(\Theta).
$$

Since $\eta$ corresponding to $\tilde{N}$ is supported on $L^2_{\ge 0}$, $\eta_\theta$ corresponding to $\tilde{N}_\theta$ is supported on $\mathbb{R}_+$. Here, we crucially use that $< x, \delta_{J,\theta} >_{L^2} \ge 0$ for any $x \in L^2_{\ge 0}$.
Now, by the Lipschitz properties of $\sigma$ and $\gamma$ and the monotonicity property **(M)** for $\gamma$, we can apply the finite dimensional comparison theory for càdlàg solutions of SDEs from [92] resp.[67] (for more details, cf. Appendix C).

From (8.29) and (8.30), $Z_j^{(i)}(t,\theta) =< \tilde{Z}_j^{(i)}(t), \delta_\theta >_{L^2(\Theta)} \in \mathbb{R}$ is a càdlàg solution to the equation

$$
\begin{aligned}
Z_j^{(i)}(t,\theta) = \quad & \xi_J^{(i)}(\theta) + \sum_{n=1}^{L} \sqrt{a_n} \int_0^t \sigma_J(s, Z_j^{(i)}(s,\theta)) e_{n,M}(\theta)\, dw_n(s) \\
& + \int_0^t \int_{\mathbb{R}} \gamma_J(s, Z_j^{(i)}(s-,\theta)) u\, \tilde{N}_\theta(ds,du).
\end{aligned}
$$

Then, by Theorem C.2.1 (with $f_j = 0$) we have $Z_j^{(1)}(t,\theta) \leq Z_j^{(2)}(t,\theta)$, $P$-almost surely, for each fixed $\theta \in \Theta$ and $t \in [\,0, t_1\,]$.
Since $\theta \mapsto Z_j^{(i)}(t,\theta)$ is a continuous function, there exists a subset $\Omega_t$ of full $P$-measure such that

$$Z_j^{(1)}(t,\omega,\theta) \leq Z_j^{(2)}(t,\omega,\theta)$$

for all $\omega \in \Omega_t$ and $\theta \in \Theta$. By considering càdlàg versions of $Z_j^{(i)}(t) \in L^2_\rho(\Theta)$, we can get this inequality for each $t \in [\,0, T\,]$ on a universal subset $\Omega_0$ of full $P$-measure.

Analogously to Section 6.2, we define operators $B_J$: $W^{m,2}(\Theta) \to W^{m,2}(\Theta)$ by

$$B_J(t)\varphi := \tfrac{F_J^{(2)}(t, Z^{(2)}(t)) - F_J^{(2)}(s, Z^{(1)}(t))}{Z^{(2)}(t) - Z^{(1)}(t)}\,\varphi,\ \varphi \in W^{m,2}(\Theta),$$

in the case $Z^{(2)}(t) \neq Z^{(1)}(t)$ and

$$B_J(t)\varphi := C(T)\varphi,\ \varphi \in W^{m,2}(\Theta),$$

otherwise. Here, $C(T)$ is as in the proof of 6.1.1 in Section 6.2. Due to the existence of a common Lipschitz constant for all $F_J^{(i)}$. By the additional assumption that $A_N$ maps $W^{m,2}(\Theta)$ onto itself, analogously to the proof of Claim 1 in Section 6.2, we get

$$V_j^{(1)}(t) \leq V_j^{(2)}(t) \text{ in } W^{m,2}(\Theta) \text{ for all } t \in [\,0, t_1\,]\ P\text{-a.s..}$$

Now, by the continuous embedding property $W^{m,2}(\Theta) \underset{\to}{\subseteq} C_b(\Theta)$, this gives us (8.27) on $[\,0, t_1\,]$ and analogously to the proof of Claim 1 in Section 6.2, we get (8.27) on $[\,0, T\,]$ by iterating the previous procedure on all intervalls $[t_k, t_{k+1}]$, $k = 1, 2, ..., j-1$. $\square$

Let us note that, by the boundedness assumption on $\gamma$, the processes $Z_j^{(i)}$ and $V_j^{(i)}$ are also elements of $\mathcal{H}^q(T)$. This is shown analogously to the proof of Lemma 6.1.3 in Section 6.2.
Thus, to finish the proof of Lemma 8.2.3.1, we need the following claim, which is the analogon to Claim 2 from Section 6.2:

**Claim 2:** *For* $i = 1, 2$

$$\lim_{j \to \infty} Z_j^{(i)} = Y^{(i)}$$

*in* $\mathcal{H}^2(T)$, *i.e.*

$$\lim_{j\to\infty} \sup_{t\in[0,T]} \mathbf{E}||Z_j^{(i)}(t) - Y^{(i)}(t)||^2_{L^2_\rho} = 0.$$

*Hence, inequality (8.27) (cf. Claim 1) implies that* $Y^{(1)}(t) \le Y^{(2)}(t)$, *P-almost surely, for all* $t \in [0,T]$.

**Proof:** Actually, we have $Y^{(1)}(t,\theta) \le Y^{(2)}(t,\theta)$, $P$-almost surely, for all $t \in [0,T]$ and $\theta \in \Theta$. This follows for the càdlàg solutions $Y^{(1)}$, $Y^{(2)}$ due to the continuous embedding $W^{m,2}(\Theta) \underset{\rightarrow}{\subseteq} C_b(\Theta)$.

We only describe the necesssary modifications to the proof of Claim 2 in Section 6.2.

We express the difference $Z_j^{(i)}(t) - Y^{(i)}(t)$ in terms of the difference $V_j^{(i)}(t) - Y^{(i)}(t)$.

By the Lipschitz property of $\gamma$, the latter can be estimated as follows.

$$\begin{aligned}
&\mathbf{E}||V_j^{(i)}(t) - Y^{(i)}(t)||^2_{L^2_\rho}\\
&= \mathbf{E}\Bigg|\Bigg|\xi^{(i)} + \int_0^t (A_N(s)V_j^{(i)}(s) + F_J^{(i)}(s, V_j^{(i)}(s)))\,ds + \int_0^{t_{k+1}} \mathcal{M}_{\Sigma_J(s,Z_j^{(i)}(s))}\,dW_{M,L}(s)\\
&+ \int_0^{t_{k+1}}\int_{L^2} \mathcal{M}_{\Gamma_J(s,Z_j^{(i)}(s))}(I_J(x))\,\tilde{N}(ds,dx)\\
&- \Bigg(\xi_J^{(i)} + \int_0^t (A_N(s)Y^{(i)}(s) + F_J^{(i)}(s, Y^{(i)}(s)))\,ds + \int_0^t \mathcal{M}_{\Sigma_J(s,Y^{(i)}(s))}\,dW_{M,L}(s)\\
&+ \int_0^t \mathcal{M}_{\Gamma_J(s,Y^{(i)}(s))}(I_J(x))\,\tilde{N}(ds,dx)\Bigg)\Bigg|\Bigg|^2_{L^2_\rho}\\
&\le c(C(T), c(N), C_{2,\eta}) \Bigg(\sum_{n=1}^M a_n \Bigg[\int_0^{t_{k+1}} \mathbf{E}||(\mathcal{M}_{\Sigma_J(s,Z_j^{(i)}(s))} - \mathcal{M}_{\Sigma_J(s,Y^{(i)}(s))})(e_{n,M})||^2_{L^2_\rho}\,ds\\
&+ \int_t^{t_{k+1}} \mathbf{E}||\mathcal{M}_{\Sigma_J(s,Y^{(i)}(s))}(e_{n,M})||^2_{L^2_\rho}\,ds\Bigg] + \int_0^{t_{k+1}} \mathbf{E}||\Gamma_J(s, Z_j^{(i)}(s)) - \Gamma_J(s, Y^{(i)}(s))||^2_{L^2_\rho}\,ds\\
&+ \int_t^{t_{k+1}} \mathbf{E}||\Gamma_J(s, Y^{(i)}(s))||^2_{L^2_\rho}\,ds + \int_0^t \mathbf{E}||V_j^{(i)}(s) - Y^{(i)}(s)||^2_{L^2_\rho}\,ds\Bigg)\\
&\le c(T, M, c(N), C(T), c_\sigma(T), c_\gamma(T), C_{2,\eta}) \Bigg[\int_0^{t_{k+1}} \mathbf{E}||Z_j^{(i)}(s) - Y^{(i)}(s)||^2_{L^2_\rho}\,ds\\
&+ (t_{k+1} - t_k)\Bigg(1 + \sup_{r\in[0,T]} \mathbf{E}||Y^{(i)}(r)||^2_{L^2_\rho}\Bigg)
\end{aligned}$$

$$+\int_0^t \mathbf{E}||V_j^{(i)}(s) - Y^{(i)}(s)||^2_{L^2_\rho}\, ds\Big]$$

$$\leq c(T, M, K, c(N), C(T), c_\sigma(T), c_\gamma(T), C_{2,\eta}) \left[\int_0^{t_{k+1}} \mathbf{E}||Z_j^{(i)}(s) - Y^{(i)}(s)||^2_{L^2_\rho}\, ds\right.$$

$$\left. +(t_{k+1} - t_k)\left(1 + \sup_{r\in[0,T]} \mathbf{E}||Y^{(i)}(r)||^2_{L^2_\rho}\right) + \int_0^t \mathbf{E}||V_j^{(i)}(s) - Y^{(i)}(s)||^2_{L^2_\rho}\, ds\right]$$

$$=: c(T, M, K, c(N), C(T), c_\sigma(T), c_\gamma(T), C_{2,\eta}) \left[B_j(t_{k+1}) + \int_0^t \mathbf{E}||V_j^{(i)}(s) - Y^{(i)}(s)||^2_{L^2_\rho}\, ds\right]$$

with $B_j$ as in Section 6.2 (replacing $X_{N,M}$-terms by $Y = X_{N,M,L,J}$-terms).

As in that Section, we apply Gronwall's lemma to get

$$\mathbf{E}||V_j^{(i)}(t) - Y^{(i)}(t)||^2_{L^2_\rho} \leq c(T, M, c(N), C(T), c_\sigma(T), c_\gamma(T), C_{2,\eta}) B_j(t_{k+1})$$

for any $t \in [\,t_k, t_{k+1}\,)$, $k \in \{0, 1, \ldots, j-1\}$.

The estimate of $Z_j^{(i)}(t) - Y^{(i)}(t)$ from that proof changes as follows.

Given $t \in [\,t_k, t_{k+1}\,)$, $k \in \{0, 1, \ldots, j-1\}$, we get

$$\begin{aligned}
&\mathbf{E}||Z_j^{(i)}(t) - Y^{(i)}(t)||^2_{L^2_\rho} \\
&\leq C\Bigg(\mathbf{E}||V_j^{(i)}(t_k) - Y^{(i)}(t_k)||^2_{L^2_\rho} \\
&+\mathbf{E}\left|\left|\int_{t_k}^t (\mathcal{M}_{\Sigma_J(s, Z_j^{(i)}(s))} - \mathcal{M}_{\Sigma_J(s, Y^{(i)}(s))}\, dW_{M,L}(s)\right|\right|^2_{L^2_\rho} \\
&+\mathbf{E}\left|\left|\int_{t_k}^t \int_{L^2} (\mathcal{M}_{\Gamma_J(s, Z_j^{(i)}(s))} - \mathcal{M}_{\Gamma_J(s, Y^{(i)}(s))})(I_J(x))\, \tilde{N}(ds, dx)\right|\right|^2_{L^2_\rho} \\
&+\mathbf{E}\left[\int_{t_k}^t ||A_N(s)||^2\, ||Y^{(i)}(s)||^2_{L^2_\rho} + ||F^{(i)}(s, \cdot, Y^{(i)}(s))||^2_{L^2_\rho}\, ds\right]\Bigg) \\
&=: C(I_1 + I_2 + I_2^{ad} + I_3).
\end{aligned}$$

Since the $\sigma_J$ are uniformly Lipschitz, except for the term $I_2^{ad}$, all terms in the estimate are as the ones from the proof of Claim 2 in Section 6.2 (again with $X_{N,M}$-terms being replaced by $Y = X_{N,M,L,J}$-terms).

But, by Itô's isometry w.r.t. compensated Poisson random measures, the fact that the $\gamma_j$ are uniformly Lipschitz and the fact that

$$\int_{L^2(\theta)} ||I_J(x)||^2_{L^2}\, \eta(dx) \leq \int_{L^2(\Theta)} ||x||^2_{L^2}\, \eta(dx) < \infty,$$

we immediately get

$$\begin{aligned} I_2^{ad} \leq \quad & c(c_\gamma(T), C_{2,\eta}) \left[ \int_{t_k}^{t} \mathbf{E}||Z_j^{(i)}(s) - Y^{(i)}(s)||_{L_\rho^2}^2 \, ds \right] \\ \leq \quad & \bar{c}(T, c(N), C(T), c_\gamma(T), C_{2,\eta}) \left[ \int_0^t \mathbf{E}||Z_j^{(i)}(s) - Y^{(i)}(s)||_{L_\rho^2}^2 \, ds \right]. \end{aligned}$$

Thus, summing all the estimates together and recalling the definition of $B_j(t_{k+1})$, we get

$$\begin{aligned} & \mathbf{E}||Z_j^{(i)}(t) - Y^{(i)}(t)||_{L_\rho^2}^2 \\ & \leq \tfrac{T}{j}(1 + \bar{c}(T, N, C(T))(1 + c_{N,M,L})) \\ & + c(T, M, c(N), C(T), c_\sigma(T), c_\gamma(T), C_{2,\eta}) \int_0^t \mathbf{E}||Z_j^{(i)}(s) - Y^{(i)}(s)||_{L_\rho^2}^2 \, ds. \end{aligned}$$

Then, Gronwall's lemma finally implies

$$\mathbf{E}||Z_j^{(i)}(t) - Y^{(i)}(t)||_{L_\rho^2}^2 \leq \tfrac{T}{j}(1 + \bar{C})(1 + c_{N,M,L})e^{\bar{C}t} < \infty,$$

where $\bar{C}$ denotes the maximum of the two constants from the equations before. Thus,

$$\lim_{j \to \infty} \mathbf{E}||Z_j^{(i)}(t) - Y^{(i)}(t)||_{L_\rho^2}^2 = 0,$$

which was needed to prove Claim 2. □ ■

### 8.2.4 Convergence of the approximations

As already mentioned in the introduction of this section, in the proofs of this subsection we restrict ourselves to the issues that differ from the proofs in Section 6.3.
But let us first formulate the main result of this subsection:

**Lemma 8.2.4.1:** *(i) Considering predictable solutions $X^{(i)}$, $X_J^{(i)}$, $X_{L,J}^{(i)}$, $X_{M,L,J}^{(i)}$ and $X_{N,M,L,J}^{(i)}$, $N, M, L, J \in \mathbb{N}$, $i = 1, 2$, to the equations (1.1), (8.11), (8.13), (8.17) and (8.19), we get the following convergence results:*

$$\lim_{N \to \infty} \mathbf{E}||X_{N,M,L,J}^{(i)}(t) - X_{M,L,J}^{(i)}(t)||_{L_\rho^2}^2 = 0, \tag{8.31}$$

$$\lim_{M \to \infty} \mathbf{E}||X_{M,L,J}^{(i)}(t) - X_{L,J}^{(i)}(t)||_{L_\rho^2}^2 = 0, \tag{8.32}$$

$$\lim_{L \to \infty} \mathbf{E}||X_{L,J}^{(i)}(t) - X_J^{(i)}(t)||_{L_\rho^2}^2 = 0, \tag{8.33}$$

*(8.34)*
$$\lim_{J\to\infty} \mathbf{E}||X_J^{(i)}(t) - X^{(i)}(t)||_{L^2_\rho}^2 = 0.$$

***(ii)*** *Considering* $X^{(i)}$, $X_J^{(i)}$, $X_{L,J}^{(i)}$, $X_{M,L,J}^{(i)}$ *and* $X_{N,M,L,J}^{(i)}$, $N, M, L, J \in \mathbb{N}$, $i = 1, 2$, *as defined in (1.2) and (8.22)–(8.25), we get the following convergence results:*

*(8.35)*
$$\lim_{N\to\infty} \mathbf{E}||X_{N,M,L,J}^{(i)}(t) - X_{M,L,J}^{(i)}(t)||_{L^2_\rho}^2 = 0,$$

*(8.36)*
$$\lim_{M\to\infty} \mathbf{E}||X_{M,L,J}^{(i)}(t) - X_{L,J}^{(i)}(t)||_{L^2_\rho}^2 = 0,$$

*(8.37)*
$$\lim_{L\to\infty} \mathbf{E}||X_{L,J}^{(i)}(t) - X_J^{(i)}(t)||_{L^2_\rho}^2 = 0,$$

*(8.38)*
$$\lim_{J\to\infty} \mathbf{E}||X_J^{(i)}(t) - X^{(i)}(t)||_{L^2_\rho}^2 = 0.$$

From Lemma 8.2.4.1 and Lemma 8.2.3.1 we immediately get Theorem 8.1.3. Indeed, we first get

$$X_{N,M,L,J}^{(1)}(t) \le X_{N,M,L,J}^{(2)}(t),\ P\text{-a.s.},$$

for all $t \in [\,0, T\,]$ by 8.2.3.1 **(i)**(equation (1.1)) resp. **(ii)** (equation (1.2)). Then, by taking $N \to \infty$, $M \to \infty$, $L \to \infty$ and finally $J \to \infty$, we get Theorem 8.1.3.

**Proof:** To shorten the proof, we only consider the issues that appear additionally to the proof of Lemma 6.1.4 in Section 6.3.

**(i) The limit properties of the approximations of (1.1)**

For fixed $M, L, J \in \mathbb{N}$, the difference between solutions $X_{N,M,L,J}^{(i)}$ and $X_{M,L,J}^{(i)}$ can be represented as

$$\begin{aligned} X_{N,M,L,J}^{(i)}(t) - X_{M,L,J}^{(i)}(t) = \quad & a_N(\xi) + a_N(F) + b_N(F) + a_N(\Sigma) + b_N(\Sigma) \\ & + a_N(\Gamma) + b_N(\Gamma) \end{aligned}$$

for fixed $t \in [\,0, T\,]$ and $N \in \mathbb{N}$, with the terms defined by

$$a_N(\xi) := [U_N(t,0) - U(t,0)]\xi_J^{(i)},$$

$$a_N(F) := \int_0^t [U_N(t,s) - U(t,s)] F_J^{(i)}(s, X_{M,L,J}^{(i)}(s))\, ds,$$

$$b_N(F) := \int_0^t U_N(t,s)[F_J^{(i)}(s, X_{N,M,L,J}^{(i)}(s)) - F_J^{(i)}(s, X_{M,L,J}(s))]\, ds,$$

$$a_N(\Sigma) := \sum_{n=1}^{L} \sqrt{a_n} \int_0^t [U_N(t,s) - U(t,s)] \mathcal{M}_{\Sigma_J(s, X_{M,L,J}^{(i)}(s))} e_{n,M}\, dw_n(s),$$

$$b_N(\Sigma) := \sum_{n=1}^{L} \sqrt{a_n} \int_0^t U_N(t,s)[\mathcal{M}_{\Sigma_J(s, X_{N,M,L,J}^{(i)}(s))} - \mathcal{M}_{\Sigma_J(s, X_{M,L,J}^{(i)}(s))}] e_{n,M}\, dw_n(s),$$

$$a_N(\Gamma) := \int_0^t \int_{L^2} [U_N(t,s) - U(t,s)] \mathcal{M}_{\Gamma_J(s, X_{N,M,L,J}^{(i)}(s))} I_J(x) \tilde{N}(ds, dx)$$

and

$$b_N(\Gamma) := \int_0^t \int_{L^2} U(t,s)[\mathcal{M}_{\Gamma_J(s, X_{N,M,L,J}^{(i)}(s))} - \mathcal{M}_{\Gamma_J(s, X_{M,L,J}^{(i)}(s))}] I_J(x)\, \tilde{N}(ds, dx).$$

Analogously to the proof of Lemma 6.1.4 (cf. Section 6.3), we get

$$\lim_{N\to\infty} \mathbf{E}||a_N(\xi)||_{L_\rho^2}^2 = 0,\ \lim_{N\to\infty} \mathbf{E}||a_N(F)||_{L_\rho^2}^2 = 0$$

and the estimate

$$\mathbf{E}||b_N(F)||_{L_\rho^2}^2 \leq c(c(N), c_f(T)) \int_0^t \mathbf{E}||X_{N,M,L,J}^{(i)}(s) - X_{M,L,J}^{(i)}(s)||_{L_\rho^2}^2\, ds.$$

Concerning $a_N(\Sigma)$ and $b_N(\Sigma)$, note that the system $(e_{n,M})_{1\leq n\leq L} \subset C_0^\infty(\Theta)$ is not orthonormal in $L^2(\Theta)$. Thus, compared to the part **(i)** in the proof of Lemma 6.1.4 in Section 6.3, the proof changes as follows.

First of all, by Itô's isometry and the boundedness of $\sigma_J$ we get

$$\begin{aligned}
\mathbf{E}||a_N(\Sigma)||_{L_\rho^2}^2 &= \sum_{n=1}^{L} a_n \int_0^t \mathbf{E}||[U_N(t,s) - U(t,s)] \mathcal{M}_{\Sigma_J(s, X_{N,M,L,J}^{(i)}(s))} e_{n,M}||_{L_\rho^2}^2\, ds \\
&\leq c(L,K) \int_0^t \underbrace{\mathbf{E}||[U_N(t,s) - U(t,s)] e_{n,M}||_{L_\rho^2}^2}_{\to 0\, as\, N\to\infty}\, ds \\
&\leq 2Tc(L,K,c(N),c(T)) \left( \max_{1\leq n\leq L} ||e_{n,M}||_{L^2} \right) \\
&< \infty.
\end{aligned}$$

By Lebesgue's dominated convergence theorem we get
$\lim_{N\to\infty} \mathbf{E}||a_N(\Sigma)||_{L_\rho^2}^2 = 0.$
On the other hand, for $b_N(\Sigma)$ we get

$$\begin{aligned}\mathbf{E}||b_N(\Sigma)||^2_{L^2_\rho} &= \sum_{n=1}^{L} a_n \int_0^t \mathbf{E}||U_N(t,s)[\mathcal{M}_{\Sigma_J(s,X^{(i)}_{N,M,L,J}(s))} - \mathcal{M}_{\Sigma_J(s,X^{(i)}_{M,L,J}(s))}]e_{n,M}||^2_{L^2_\rho} \\ &\le c(L, c(N), c_\sigma(T)) \int_0^t \mathbf{E}||X^{(i)}_{N,M,L,J}(s) - X^{(i)}_{M,L,J}(s)||^2_{L^2_\rho}\, ds,\end{aligned}$$

where we applied the fact that $\max_{1\le n\le L} ||e_{n,M}||_{L^\infty} < \infty$, **(A6)** and the Lipschitz property **(LC)** for $\sigma_J$ (recall that by construction this property holds uniformly in $J$).

By Itô's isometry for the stochastic integration w.r.t. compensated Poisson random measures and the boundedness of $\gamma_L$ we get

$$\begin{aligned}\mathbf{E}||a_N(\Gamma)||^2_{L^2_\rho} &= \int_0^t \int_{L^2} \mathbf{E}||[U_N(t,s) - U(t,s)]\mathcal{M}_{\Gamma_J(s,X^{(i)}_{N,M,L,J}(s))} I_J(x)||^2_{L^2_\rho}\, \eta(dx)\, ds \\ &\le c(L,K) \int_0^t \int_{L^2} \underbrace{\mathbf{E}||[U_N(t,s) - U(t,s)]I_J(x)||^2_{L^2_\rho}}_{\to 0\, as\, N\to\infty}\, \eta(dx)\, ds.\end{aligned}$$

By the fact that

$$\mathbf{E}||[U_N(t,s) - U(t,s)]I_J(x)||^2_{L^2_\rho} \le c(c(N), c(T))||I_J(x)||^2_{L^2}$$

and (cf. **(QI)** and (8.10))

$$\int_0^t \int_{L^2} c(c(N), c(T))||I_J(x)||^2_{L^2}\, \eta(dx)\, ds \le T c(c(N), c(T)) \int_{L^2} ||I_J(x)||^2_{L^2}\, \eta(dx) \le c(c(N), c(T), C_{q,\eta}) < \infty,$$

we can apply Lebesgue's theorem to get $\mathbf{E}||a_N(\Gamma)||^2_{L^2_\rho} \to 0$ as $N \to \infty$.

Finally, applying Itô's isometry for the stochastic integration w.r.t. compensated Poisson random measures, **(A2)** (or **(A5)*** with $\nu = 1$) and **(QI)** we get

$$\begin{aligned}\mathbf{E}||b_N(\Gamma)||^2_{L^2_\rho} &= \int_0^t \int_{L^2} \mathbf{E}||U(t,s)[\mathcal{M}_{\Gamma_J(s,X^{(i)}_{N,M,L,J}(s))} - \mathcal{M}_{\Gamma_J(s,X^{(i)}_{M,L,J}(s))}]I_J(x)||^2_{L^2_\rho}\, \eta(dx)\, ds \\ &\le c(c(T)) \left( \int_{L^2} ||x_J||^2_{L^2}\, \eta(dx) \right) \\ &\quad \int_0^t (t-s)^{-\zeta} \mathbf{E}||\Gamma_J(s, X^{(i)}_{N,M,L,J}(s)) - \Gamma_J(s, X^{(i)}_{M,L,J}(s))||^2_{L^2_\rho}\, ds \\ &\le c(c(T), c_\gamma, C_{q,\eta}) \int_0^t (t-s)^{-\zeta} \mathbf{E}||X^{(i)}_{N,M,L,J}(s) - X^{(i)}_{M,L,J}(s)||^2_{L^2_\rho}\, ds.\end{aligned}$$

Thus, given any $t \in [0, T]$, we can estimate

$\mathbf{E}||X^{(i)}_{N,M,L,J}(s) - X^{(i)}_{M,L,J}(s)||^2_{L^2_\rho}$

$$\leq (\mathbf{E}||a_N(\xi)||^2_{L^2_\rho} + \mathbf{E}||a_N(F)||^2_{L^2_\rho} + \mathbf{E}||a_N(\Sigma)||^2_{L^2_\rho} + \mathbf{E}||a_N(\Gamma)||^2_{L^2_\rho})$$
$$+c(\zeta, L, T, c(N), c_f(T), c_\sigma(T), c_\gamma(T), C_{q,\eta}) \int_0^t (t-s)^{-\zeta} \mathbf{E}||X^{(i)}_{N,M,L,J}(s) - X^{(i)}_{M,L,J}(s)||^2_{L^2_\rho}\, ds.$$

Thus, by the Gronwall-Bellman lemma 2.7.2 and the corresponding Remark 2.7.3, we get

$$\mathbf{E}||X^{(i)}_{N,M,L,J}(s) - X^{(i)}_{M,L,J}(s)||^2_{L^2_\rho}$$
$$\leq (\mathbf{E}||a_N(\xi)||^2_{L^2_\rho} + \mathbf{E}||a_N(F)||^2_{L^2_\rho} + \mathbf{E}||a_N(\Sigma)||^2_{L^2_\rho} + \mathbf{E}||a_N(\Gamma)||^2_{L^2_\rho})$$
$$c(\zeta, L, T, c(N), c_f(T), c_\sigma(T), c_\gamma(T), C_{q,\eta}).$$

Now, since the $a_N$-terms tend to 0 as $N \to \infty$, we get the convergence result (8.31) for $N \to \infty$.

Next, we consider the $M$-approximation. For any $t \in [0, T]$, we have

$$X^{(i)}_{M,L,J}(t) - X^{(i)}_{L,J}(t) = b_M(F) + a_M(\Sigma) + b_M(\Sigma) + b_M(\Gamma)$$

with the terms defined by

$$b_M(F) := \int_0^t U(t,s)[F^{(i)}_J(s, X^{(i)}_{M,L,J}(s)) - F^{(i)}_J(s, X_{L,J}(s))]\, ds,$$
$$a_M(\Sigma) := \sum_{n=1}^{L} \sqrt{a_n} \int_0^t U(t,s)[\mathcal{M}_{\Sigma_J(s, X^{(i)}_{L,J}(s))}(e_{n,M} - e_n)\, dw_n(s),$$
$$b_M(\Sigma) := \sum_{n=1}^{L} \sqrt{a_n} \int_0^t U(t,s)[\mathcal{M}_{\Sigma_J(s, X^{(i)}_{M,L,J}(s))} - \mathcal{M}_{\Sigma_J(s, X^{(i)}_{L,J}(s))}] e_{n,M}\, dw_n(s),$$

and

$$b_M(\Gamma) := \int_0^t \int_{L^2} U(t,s)[\mathcal{M}_{\Gamma_J(s, X^{(i)}_{M,L,J}(s))} - \mathcal{M}_{\Gamma_J(s, X^{(i)}_{L,J}(s))}] I_J(x)\, \tilde{N}(ds, dx).$$

Analogously to the $N$-convergence case, we show that the $a_M$-term tends to 0 as $M \to \infty$. Indeed, by **(A2)** and the uniform boundedness of the $\sigma_J$ we get

$$\begin{aligned}
\mathbf{E}||a_M(\Sigma)||^2_{L^2_\rho} &= \sum_{n=1}^{L} a_n \int_0^t \mathbf{E}||U(t,s)\mathcal{M}_{\Sigma_J(s, X^{(i)}_{L,J}(s))}(e_{n,M} - e_n)||^2_{L^2_\rho}\, ds \\
&\leq \sum_{n=1}^{L} a_n \left( \int_0^t (t-s)^{-\zeta} \mathbf{E}||\Sigma_J(s, X^{(i)}_{L,J}(s))||^2_{L^2_\rho}\, ds \right) ||e_{n,M} - e_n||^2_{L^2} \\
&\leq c(\zeta, K, T) \sum_{n=1}^{M} \underbrace{||e_{n,M} - e_n||^2}_{\to 0\, as\, M \to \infty\, for\, 1 \leq n \leq L} \\
&\to 0 \text{ as } M \to \infty.
\end{aligned}$$

Furthermore, similar to the $N$-convergence we have

$$\mathbf{E}||b_M(F)||^2_{L^2_\rho} \leq c(c(T), c_f(T)) \int_0^t \mathbf{E}||X^{(i)}_{M,L,J}(s) - X^{(i)}_{L,J}(s)||^2_{L^2_\rho}\, ds,$$

$$\mathbf{E}||b_M(\Sigma)||^2_{L^2_\rho} \leq c(L, c(T), c_\sigma(T)) \int_0^t \mathbf{E}||X^{(i)}_{M,L,J}(s) - X^{(i)}_{L,J}(s)||^2_{L^2_\rho}\, ds$$

and

$$\mathbf{E}||b_M(\Gamma)||^2_{L^2_\rho} \leq c(c(T), c_\gamma, C_{q,\eta}) \int_0^t (t-s)^{-\zeta}\mathbf{E}||X^{(i)}_{M,L,J}(s) - X^{(i)}_{L,J}(s)||^2_{L^2_\rho}.$$

Thus, given any $t \in [\,0, T\,]$, we can estimate

$$\begin{aligned}
&\mathbf{E}||X^{(i)}_{M,L,J}(s) - X^{(i)}_{L,J}(s)||^2_{L^2_\rho}\\
&\leq \mathbf{E}||a_M(\Sigma)||^2_{L^2_\rho}\\
&+c(\zeta, L, T, c(T), c_f(T), c_\sigma(T), c_\gamma(T), C_{q,\eta}) \int_0^t (t-s)^{-\zeta}\mathbf{E}||X^{(i)}_{M,L,J}(s) - X^{(i)}_{L,J}(s)||^2_{L^2_\rho}\, ds.
\end{aligned}$$

Thus, by the Gronwall-Bellman lemma 2.7.2 and the corresponding Remark 2.7.3, we get

$$\begin{aligned}
&\mathbf{E}||X^{(i)}_{M,L,J}(s) - X^{(i)}_{L,J}(s)||^2_{L^2_\rho}\\
&\leq (\mathbf{E}||a_M(\Sigma)||^2_{L^2_\rho}) c(\zeta, M, T, c(N), c_f(T), c_\sigma(T), c_\gamma(T), C_{q,\eta}).
\end{aligned}$$

Now, since the $a_M$-term tends to 0 as $M \to \infty$, we get the convergence result (8.32) for $M \to \infty$.

Let us now proceed with the convergence for $L \to \infty$.
For any $t \in [\,0, T\,]$, we have

$$\begin{aligned}
X^{(i)}_{L,J}(t) - X^{(i)}_J(t) = \;& \int_0^t U(t,s)[F^{(i)}_J(s, X^{(i)}_{L,J}(s)) - F^{(i)}_J(s, X^{(i)}_J(s))]\, ds\\
&+ \int_0^t \int_{L^2} U(t,s)[\mathcal{M}_{\Gamma_J(s, X^{(i)}_{L,J}(s))} - \mathcal{M}_{\Gamma_J(s, X^{(i)}_J(s))}] I_J(x)\, \tilde{N}(ds, dx)\\
&+ \sum_{n=1}^{L} \sqrt{a_n} \int_0^t U(t,s)[\mathcal{M}_{\Sigma_J(s, X^{(i)}_{L,J}(s))} - \mathcal{M}_{\Sigma_J(s, X^{(i)}_J(s))}] e_n\, dw_n(s)\\
&+ \sum_{n=L+1}^{\infty} \sqrt{a_n} \int_0^t U(t,s)\mathcal{M}_{\Sigma_J(s, X^{(i)}_J(s))} e_n\, dw_n(s).
\end{aligned}$$

Analogously to the $b_N$-terms above, we get

$$\begin{aligned}\mathbf{E}||X_{L,J}^{(i)}(t)-X_J^{(i)}(t)||_{L^2_\rho}^2 \leq\ & c(\zeta,L,T,c(T),c_f(T),c_\sigma(T),c_\gamma(T),C_{q,\eta}) \\ & \int_0^t (t-s)^{-\zeta}\mathbf{E}||X_{L,J}^{(i)}(s)-X_J^{(i)}(s)||_{L^2_\rho}^2\,ds \\ & +\sum_{n=L+1}^{\infty} a_n \int_0^t \mathbf{E}||U(t,s)\mathcal{M}_{\Sigma_J(s,X_J^{(i)}(s))}e_n||_{L^2_\rho}^2\,ds.\end{aligned}$$

Thus, the Gronwall-Bellman lemma yields

$$\mathbf{E}||X_{L,J}^{(i)}(t)-X_J^{(i)}(t)||_{L^2_\rho}^2 \leq c_L(\Sigma)e^{c(\zeta,L,T,c(T),c_f(T),c_\sigma(T),c_\gamma(T),C_{q,\eta})t}$$

with $c_L(\Sigma)$ as in the proof of Lemma 6.1.4 **(i)** (cf. Section 6.3), i.e.

$$c_L(\Sigma) := \sum_{n=L+1}^{\infty} a_n \int_0^t \mathbf{E}||U(t,s)\mathcal{M}_{\Sigma_J(s,X_J^{(i)}(s))}e_n||_{L^2_\rho}^2\,ds.$$

Let us first check the nuclear case, when

$$\sum_{n=1}^{\infty} a_n, \infty.$$

Since each $\sigma_J$ is bounded (cf. Section 8.2.1), we get

$$\begin{aligned}c_L(\Sigma) :=\ & \sum_{n=L+1}^{\infty} a_n \int_0^t \mathbf{E}||U(t,s)\mathcal{M}_{\Sigma_J(s,X_J^{(i)}(s))}e_n||_{L^2_\rho}^2\,ds \\ \leq\ & \sum_{n=L+1}^{\infty} a_n \left(\sup_{n\in\mathbb{N}} \int_0^t \mathbf{E}||U(t,s)\mathcal{M}_{\Sigma_J(s,X_J^{(i)}(s))}e_n||_{L^2_\rho}^2\,ds\right) \\ \leq\ & c(T,c(T),c_\sigma(T)) \sum_{n=L+1}^{\infty} a_n \\ \longrightarrow\ & 0 \text{ as } L\to\infty,\end{aligned}$$

which proves $\mathbf{E}||X_{L,J}^{(i)}(t)-X_J^{(i)}(t)||_{L^2_\rho}^2 \to 0$ as $L\to\infty$ in this case. Note that here we did not apply property (3.1) such that the claim indeed holds in the general nuclear case.

In the cylindrical case, we get $c_L(\Sigma)\to 0$ as $L\to\infty$ analogously to the consideration of $c_L(\Sigma)$ in the proof of Lemma 6.1.4 **(i)**. Thus, also in the cylindrical case we have $\mathbf{E}||X_{L,J}^{(i)}(t)-X_J^{(i)}(t)||_{L^2_\rho}^2 \to 0$ as $L\to\infty$. Therefore, the proof of (8.33) is finished.

Finally, for fixed $J\in\mathbb{N}$ we have

$$
\begin{aligned}
X_J^{(i)}(t)-X^{(i)}(t) = \; & U(t,0)[\xi_J^{(i)}-\xi^{(i)}]+\int_0^t U(t,s)[F_J^{(i)}(s,X_J^{(i)}(s))-F^{(i)}(s,X^{(i)}(s))]\,ds \\
& +\sum_{n\in\mathbb{N}}\sqrt{a_n}\int_0^t U(t,s)[\mathcal{M}_{\Sigma_J(s,X_J^{(i)}(s))}-\mathcal{M}_{\Sigma(s,X^{(i)}(s))}]e_n\,dw_n(s) \\
& +\int_0^t\int_{L^2} U(t,s)[\mathcal{M}_{\Gamma_J(s,X_J^{(i)}(s))}I_J(x)-\mathcal{M}_{\Gamma(s,X^{(i)}(s))}x]\,\tilde{N}(ds,dx) \\
= \; & a_J(\xi)+a_J(F)+b_J(F)+a_J(\Sigma)+b_J(\Sigma) \\
& +a_J(\Gamma)+b_JL(\Gamma)
\end{aligned}
$$

with the terms defined by

$$a_J(\xi):=U(t,0)(\xi_J^{(i)}-\xi^{(i)}),$$

$$
\begin{aligned}
a_J(F) &:= \int_0^t U(t,s)[F_J^{(i)}(s,X^{(i)}(s))-F^{(i)}(s,X^{(i)}(s))]\,ds, \\
b_J(F) &:= \int_0^t U(t,s)[F_J^{(i)}(s,X_J^{(i)}(s))-F_J^{(i)}(s,X^{(i)}(s))]\,ds, \\
a_J(\Sigma) &:= \int_0^t U(t,s)[\mathcal{M}_{\Sigma_J(s,X^{(i)}(s))}-\mathcal{M}_{\Sigma(s,X^{(i)}(s))}]\,dW(s), \\
b_J(\Sigma) &:= \int_0^t U(t,s)[\mathcal{M}_{\Sigma_J(s,X_J^{(i)}(s))}-\mathcal{M}_{\Sigma_J(s,X^{(i)}(s))}]\,dW(s), \\
a_J(\Gamma) &:= \int_0^t\int_{L^2} U(t,s)[\mathcal{M}_{\Gamma_J(s,X^{(i)}(s))}I_J(x)-\mathcal{M}_{\Gamma(s,X^{(i)}(s))}x]\,\tilde{N}(ds,dx)
\end{aligned}
$$

and

$$b_J(\Gamma):=\int_0^t\int_{L^2} U(t,s)[\mathcal{M}_{\Gamma_J(s,X_J^{(i)}(s))}-\mathcal{M}_{\Gamma_J(s,X^{(i)}(s))}]I_J(x)\,\tilde{N}(ds,dx).$$

Analogously to the proof of Lemma 6.1.4 in Section 6.3, we show convergence of the $a_J$ and estimate the $b_J$-terms with the help of the Lipschitz property of the coefficients.

First, concerning $a_J(\xi)$ we note that for any fixed $t\in[\,0,T\,]$

$$
\begin{aligned}
\mathbf{E}||U(t,0)(\xi_J^{(i)}-\xi^{(i)})||_{L^2_\rho}^2 \leq \; & c(T)\mathbf{E}||\xi_J^{(i)}-\xi^{(i)}||_{L^2_\rho}^2 \\
\to 0 \; & \text{ as } J\to\infty.
\end{aligned}
$$

For $a_J(F)$, we know that the integrand converges to 0 as $J\to\infty$, since $f_J\to f$ by the choice of the approximating functions.
Furthermore, by the uniform Lipschitz property of the $f_J$, $J\in\mathbb{N}$, we can estimate

$$\mathbf{E}||U(t,s)[F_J^{(i)}(s,X^{(i)}(s))-F^{(i)}(s,X^{(i)}(s))]||_{L^2_\rho}^2\leq c(c(T),c_f(T))(1+\mathbf{E}||X^{(i)}(s)||_{L^2_\rho}^2),$$

where the right hand side is intergrable by the fact that
$X^{(i)} \in \mathcal{H}^q(T) \subset \mathcal{H}^2(T)$.
Thus, by Lebesgue's dominated convergence theorem, we get $a_J(F) \to 0$ as $J \to \infty$.

Concerning $a_J(\Sigma)$, note that by the fact that $\sigma_J \to \sigma$ as $J \to \infty$ we get, for any $s \in [\,0, T\,]$,

$$\lim_{j\to\infty} \mathbf{E}||\Sigma_J(s, X^{(i)}(s)) - \Sigma(s, X^{(i)}(s))||_{L^2_\rho} = 0.$$

Furthermore, since the $\sigma_J$ fulfill **(LC)** and **(LB)** with constants uniformly in $J$, we have, for any $s \in [\,0, T\,]$,

$$\lim_{j\to\infty} \mathbf{E}||\Sigma_J(s, X^{(i)}(s)) - \Sigma(s, X^{(i)}(s))||_{L^2_\rho} \le c(c_\sigma(T))(1 + \mathbf{E}||X^{(i)}(s)||_{L^2_\rho})$$

and, for any $t \in [\,0, T\,]$,

$$(t-s)^{-\zeta} c(c_\sigma(T))(1 + \mathbf{E}||X^{(i)}(s)||_{L^2_\rho})$$

is integrable on $[\,0, t\,]$, since $X^{(i)}(s) \in \mathcal{H}^q(T) \subset \mathcal{H}^2(T)$.
Thus, for any $t \in [\,0, T\,]$, by the estimate

$$\begin{aligned}\mathbf{E}||a_J(\Sigma)||^2_{L^2_\rho} = &\int_0^t \mathbf{E}||U(t,s)[\mathcal{M}_{\Sigma_J(s,X^{(i)}(s))} - \mathcal{M}_{\Sigma(s,X^{(i)}(s))}]||^2_{\mathcal{L}_2}\, ds \\ \le &\int_0^t (t-s)^{-\zeta} \mathbf{E}||\Sigma_J(s, X^{(i)}(s)) - \Sigma(s, X^{(i)}(s))||^2_{L^2_\rho}\, ds\end{aligned}$$

we get that $\mathbf{E}||a_J(\Sigma)||^2_{L^2_\rho} \to 0$ as $J \to \infty$.

Concerning $a_J(\Gamma)$, note that

$$\begin{aligned}&(8.39)\ \mathbf{E}||a_J(\Gamma)||^2_{L^2_\rho} \\ &= \int_0^t \int_{L^2} \mathbf{E}||U(t,s)[\mathcal{M}_{\Gamma_J(s,X^{(i)}(s))} I_J(x) - \mathcal{M}_{\Gamma(s,X^{(i)}(s))} x]||^2_{L^2_\rho}\, \eta(dx)\, ds \\ &\le \int_0^t \int_{L^2} \mathbf{E}||U(t,s)\mathcal{M}_{\Gamma_J(s,X^{(i)}(s))}[I_J(x) - x]||^2_{L^2_\rho}\, \eta(dx)\, ds \\ &+ \int_0^t \int_{L^2} \mathbf{E}||U(t,s)[\mathcal{M}_{\Gamma_J(s,X^{(i)}(s))} - \mathcal{M}_{\Gamma(s,X^{(i)}(s))}]x||^2_{L^2_\rho}\, \eta(dx)\, ds \\ &\le \left( \int_{L^2} ||I_J(x) - x||^2_{L^2}\, \eta(dx) \right) \int_0^t (t-s)^{-\zeta} \mathbf{E}||\Gamma_J(s, X^{(i)}(s))||^2_{L^2_\rho}\, ds\end{aligned}$$

$$+\left(\int\limits_{L^2} ||x||^2_{L^2}\,\eta(dx)\right)\int\limits_0^t (t-s)^{-\zeta}\mathbf{E}||\Sigma_J(s,X^{(i)}(s))-\Sigma(s,X^{(i)}(s))||^2_{L^2_\rho}\,ds.$$

Note that the first term on right hand side of the above estimate tends to 0 as $J\to\infty$ by Lebesgue's dominated convergence theorem, since $\eta$ is $\sigma$-finite, $||x_J-x||^2_{L^2}\le 2||x||^2_{L^2}$ for any $x\in L^2(\Theta)$,

$$\int\limits_{L^2} 2||x||_{L^2}\,\eta(dx)<\infty,$$

and since, by the the fact that $|\gamma_J|\le|\gamma|$ for all $J\in\mathbb{N}$ and $\gamma$ obeys the Lipschitz property **(LC)** and the local boundedness property **(LB)**, we have

$$\int\limits_0^t (t-s)^{-\zeta}\mathbf{E}||\Gamma_J(s,X^{(i)}(s))||^2_{L^2_\rho}\,ds\le c_\gamma(T)(1+||X^{(i)}||^2_{\mathcal{H}^2(T)})\frac{T^{1-\zeta}}{1-\zeta}<\infty.$$

The second term on the right hand side tends to 0 for $J\to\infty$ by **(QI)** and since

$$\lim_{J\to\infty}\int\limits_0^t (t-s)^{-\zeta}\mathbf{E}||\Gamma_J(s,X^{(i)}(s))-\Gamma(s,X^{(i)}(s))||^2_{L^2_\rho}\,ds=0$$

analogously to the considerations of $a_J(\Sigma)$ before.
Thus, we get $\lim\limits_{J\to\infty}\mathbf{E}||a_J(\Gamma)||^2_{L^2_\rho}=0$.

So all the $a_J$-terms tend to 0 as $J\to\infty$.

Concerning the $b_J$-terms note that with the help of the Itô isometries w.r.t. Wiener processes and compensated Poisson random measures and the uniform Lipschitz properties, we get

$$\mathbf{E}\left|\left|\int\limits_0^t U(t,s)[F_J^{(i)}(s,\cdot,X_J^{(i)}(s))-F_J^{(i)}(s,\cdot,X^{(i)}(s))]\,ds\right|\right|^2_{L^2_\rho}$$

$$\le c(c(T),c_f(T))\int\limits_0^t \mathbf{E}||X_J^{(i)}(s)-X^{(i)}(s)||^2_{L^2_\rho}\,ds,$$

$$\mathbf{E}\left|\left|\int\limits_0^t U(t,s)[\mathcal{M}_{\Sigma_J(s,X_J^{(i)}(s))}-\mathcal{M}_{\Sigma_J(s,X^{(i)}(s))}]\,dW(s)\right|\right|^2_{L^2_\rho}$$

$$\le c(c(T),c_\sigma(T))\int\limits_0^t (t-s)^{-\zeta}\mathbf{E}||X_J^{(i)}(s)-X^{(i)}(s)||^2_{L^2_\rho}\,ds,$$

and by **(A2)** (or **(A5)*** with $\nu=1$)

$$\mathbf{E}\left|\left|\int\limits_0^t\int\limits_{L^2} U(t,s)[\mathcal{M}_{\Gamma_J(s,X_J^{(i)}(s))}-\mathcal{M}_{\Gamma_J(s,X^{(i)}(s))}]I_J(x)\,\tilde{N}(ds,dx)\right|\right|^2_{L^2_\rho}$$

$$\leq c(c(T), c_\gamma(T), C_{q,\eta}) \int_0^t (t-s)^{-\zeta} \mathbf{E}||X_J^{(i)}(s) - X^{(i)}(s)||^2_{L^2_\rho}\, ds.$$

Thus, given any $t \in [0, T]$, we can estimate

$$\begin{aligned}
&\mathbf{E}||X_J^{(i)}(s) - X^{(i)}(s)||^2_{L^2_\rho} \\
&\leq (\mathbf{E}||a_J(\xi)||^2_{L^2_\rho} + \mathbf{E}||a_J(F)||^2_{L^2_\rho} + \mathbf{E}||a_J(\Sigma)||^2_{L^2_\rho} + \mathbf{E}||a_J(\Gamma)||^2_{L^2_\rho}) \\
&+ c(c(T), c_f(T), c_\sigma(T), c_\gamma(T), C_{q,\eta}) \int_0^t (t-s)^{-\zeta} \mathbf{E}||X_J^{(i)}(s) - X^{(i)}(s)||^2_{L^2_\rho}\, ds.
\end{aligned}$$

Thus, by the Gronwall-Bellman lemma 2.7.2 and the corresponding Remark 2.7.3, we get

$$\begin{aligned}
&\mathbf{E}||X_J^{(i)}(s) - X^{(i)}(s)||^2_{L^2_\rho} \\
&\leq (\mathbf{E}||a_J(\xi)||^2_{L^2_\rho} + \mathbf{E}||a_J(F)||^2_{L^2_\rho} + \mathbf{E}||a_J(\Sigma)||^2_{L^2_\rho} + \mathbf{E}||a_J(\Gamma)||^2_{L^2_\rho}) \\
&c(c(T), c_f(T), c_\sigma(T), c_\gamma(T), C_{q,\eta}).
\end{aligned}$$

Now, since the $a_J$-terms tend to 0 as $J \to \infty$ we get the convergence result (8.34) for $J \to \infty$, which finishes part **(i)**.

### (ii) The limit properties of the approximations of (1.2)

Fixing $M, L, J \in \mathbb{N}$, the difference between solutions $X^{(i)}_{N,M,L,J}$ and $X^{(i)}_{M,L,J}$ can be represented as

$$\begin{aligned}
&X^{(i)}_{N,M,L,J}(t) - X^{(i)}_{M,L,J}(t) \\
&= a_N(\xi) + a_N(E) + b_N(E) + a_N(\Sigma, m) + b_N(\Sigma, m) + a_N(\Sigma) + b_N(\Sigma) \\
&+ a_N(\Sigma_2) + b_N(\Sigma_2)
\end{aligned}$$

for $t \in [0, T]$ and fixed $N \in \mathbb{N}$ with the terms defined by

$$a_N(\xi) := [U_N(t, 0) - U(t, 0)]\xi_J^{(i)},$$

$$a_N(E) := \int_0^t [U_N(t, s) - U(t, s)] E_J^{(i)}(s, X^{(i)}_{M,L,J}(s))\, ds,$$

$$b_N(E) := \int_0^t U_N(t, s)[E_J^{(i)}(s, X^{(i)}_{N,M,L,J}(s)) - E_J^{(i)}(s, X_{M,L,J}(s))]\, ds,$$

$$a_N(\Sigma, m) := \int_0^t [U_N(t, s) - U(t, s)] \mathcal{M}_{\Sigma_J(s,\cdot,X^{(i)}_{M,L,J}(s))} I_J(m)\, ds,$$

$$b_N(\Sigma, m) := \int_0^t U_N(t, s)[\mathcal{M}_{\Sigma_J(s,\cdot,X^{(i)}_{N,M,L,J}(s))} - \mathcal{M}_{\Sigma_J(s,\cdot,X^{(i)}_{M,L}(s))}] I_J(m)\, ds,$$

$$a_N(\Sigma) := \sum_{n=1}^{L} \sqrt{a_n} \int_0^t [U_N(t,s) - U(t,s)]\mathcal{M}_{\Sigma_J(s,X^{(i)}_{M,L,J}(s))} e_{n,M}\, dw_n(s),$$

$$b_N(\Sigma) := \sum_{n=1}^{L} \sqrt{a_n} \int_0^t U_N(t,s)[\mathcal{M}_{\Sigma_J(s,X^{(i)}_{N,M,L,J}(s))} - \mathcal{M}_{\Sigma_J(s,X^{(i)}_{M,L,J}(s))}]e_{n,M}\, dw_n(s),$$

$$a_N(\Sigma_2) := \int_0^t \int_{L^2} [U_N(t,s) - U(t,s)]\mathcal{M}_{\Sigma_J(s,X^{(i)}_{N,M,L,J}(s))} I_J(x)\tilde{N}(ds,dx)$$

and

$$b_N(\Sigma_2) := \int_0^t \int_{L^2} U_N(t,s)[\mathcal{M}_{\Sigma_J(s,X^{(i)}_{N,M,L,J}(s))} - \mathcal{M}_{\Sigma_J(s,X^{(i)}_{M,L}(s))}]I_J(x)\,\tilde{N}(ds,dx).$$

Again, the aim is to show that the $a_N$-terms tend to 0 as $N \to \infty$.
Clearly, we have

$$\lim_{N\to\infty} \mathbf{E}||a_N(\xi)||^2_{L^2_\rho} 0, \lim_{N\to\infty} \mathbf{E}||a_N(E)||^2_{L^2_\rho} = 0$$

analogously to the proof of Lemma 6.1.4 (cf. Section 6.3). Since the $\sigma_J$ are uniformly Lipschitz in $J$ and $||I_J(m)||_{L^2} \le ||m||_{L^2}$, we get

$$\lim_{N\to\infty} \mathbf{E}||a_N(\Sigma,m)||^2_{L^2_\rho} = 0$$

analogously to the consideration of $a_N(m)$ in the proof of 6.1.4 **(ii)**.
Analogously to the proof of **(i)**, we also have

$$\lim_{N\to\infty} \mathbf{E}||a_N(\Sigma)||^2_{L^2_\rho} = 0.$$

Finally, analogously to the consideration of $a_N(\Gamma)$ in the proof of **(i)**, we get

$$\lim_{N\to\infty} \mathbf{E}||a_N(\Sigma_2)||^2_{L^2_\rho} = 0.$$

Concerning the $b_N$-terms note that, applying the uniform Lipschitz properties in $J$ of $e_J$ and $\sigma_J$, we get

$$\mathbf{E}||b_N(E)||^2_{L^2_\rho} \le c(c(N), c_e(T)) \int_0^t \mathbf{E}||X^{(i)}_{N,M,L,J}(s) - X^{(i)}_{M,L,J}(s)||^2_{L^2_\rho}\, ds,$$

$$\mathbf{E}||b_N(\Sigma,m)||^2_{L^2_\rho} \le c(m, c(N), c_\sigma(T))(T)) \int_0^t (t-s)^{-\zeta}\mathbf{E}||X^{(i)}_{N,M,L,J}(s) - X^{(i)}_{M,L,J}(s)||^2_{L^2_\rho}\, ds,$$

$$\mathbf{E}||b_N(\Sigma)||^2_{L^2_\rho} \le c(m, c(N), c_\sigma(T)) \int_0^t \mathbf{E}||X^{(i)}_{N,M,L,J}(s) - X^{(i)}_{M,L,J}(s)||^2_{L^2_\rho}\, ds$$

and

$$\mathbf{E}||b_N(\Sigma_2)||^2_{L^2_\rho} \le c(c(N), c_\sigma(T), C_{q,\eta}) \int_0^t (t-s)^{-\zeta} \mathbf{E}||X^{(i)}_{N,M,L,J}(s) - X^{(i)}_{M,L,J}(s)||^2_{L^2_\rho}\, ds.$$

Thus, for any $t \in [0, T]$, we have the estimate

$$\begin{aligned}
&\mathbf{E}||X^{(i)}_{N,M,L,J}(s) - X^{(i)}_{M,L,J}(s)||^2_{L^2_\rho} \\
&\le (\mathbf{E}||a_N(\xi)||^2_{L^2_\rho} + \mathbf{E}||a_N(E)||^2_{L^2_\rho} + \mathbf{E}||a_N(\Sigma, m)||^2_{L^2_\rho} + \mathbf{E}||a_N(\Sigma)||^2_{L^2_\rho} + \mathbf{E}||a_N(\Sigma_2)||^2_{L^2_\rho}) \\
&+ c(\zeta, M, m, T, c(N), c_e(T), c_\sigma(T), c_\gamma(T), C_{q,\eta}) \int_0^t (t-s)^{-\zeta} \mathbf{E}||X^{(i)}_{N,M,L,J}(s) - X^{(i)}_{M,L,J}(s)||^2_{L^2_\rho}\, ds.
\end{aligned}$$

Thus, by the Gronwall-Bellman lemma 2.7.2 and the corresponding Remark 2.7.3, we get

$$\begin{aligned}
&\mathbf{E}||X^{(i)}_{N,M,L,J}(s) - X^{(i)}_{M,L,J}(s)||^2_{L^2_\rho} \\
&\le (\mathbf{E}||a_N(\xi)||^2_{L^2_\rho} + \mathbf{E}||a_N(E)||^2_{L^2_\rho} + \mathbf{E}||a_N(\Sigma, m)||^2_{L^2_\rho} + \mathbf{E}||a_N(\Sigma) + \mathbf{E}||a_N(\Sigma_2)||^2_{L^2_\rho}) \\
&c(\zeta, M, m, T, c(N), c_f(T), c_\sigma(T), c_\gamma(T), C_{q,\eta}).
\end{aligned}$$

Now, since the $a_N$-terms tend to 0 as $N \to \infty$, we get the convergence result (8.35) for $N \to \infty$.

Next, we consider the $M$-approximation. For any $t \in [0, T]$, we have

$$X^{(i)}_{M,L,J}(t) - X^{(i)}_{L,J}(t) = b_M(F) + b_M(\Sigma, m) + a_M(\Sigma) + b_M(\Sigma) + b_M(\Sigma_2)$$

with the terms defined by

$$\begin{aligned}
b_M(E) &:= \int_0^t U(t,s)[E^{(i)}_J(s, X^{(i)}_{M,L,J}(s)) - E^{(i)}_J(s, X_{L,J}(s))]\, ds, \\
b_M(\Sigma, m) &:= \int_0^t U(t,s)[\mathcal{M}_{\Sigma_J(s,\cdot,X^{(i)}_{M,L,J}(s))} - \mathcal{M}_{\Sigma_J(s,\cdot,X^{(i)}_{L,J}(s))}] I_J(m)\, ds, \\
a_M(\Sigma) &:= \sum_{n=1}^L \sqrt{a_n} \int_0^t U(t,s) \mathcal{M}_{\Sigma_J(s,X^{(i)}_{L,J}(s))}(e_{n,M} - e_n)\, dw_n(s), \\
b_M(\Sigma) &:= \sum_{n=1}^L \sqrt{a_n} \int_0^t U(t,s)[\mathcal{M}_{\Sigma_J(s,X^{(i)}_{M,L,J}(s))} - \mathcal{M}_{\Sigma_J(s,X^{(i)}_{L,J}(s))}] e_{n,M}\, dw_n(s)
\end{aligned}$$

and

$$b_M(\Sigma_2) := \int_0^t \int_{L^2} U(t,s)[\mathcal{M}_{\Sigma_J(s,X^{(i)}_{M,L,J}(s))} - \mathcal{M}_{\Sigma_J(s,X^{(i)}_{L,J}(s))}] I_J(x)\, \tilde{N}(ds, dx).$$

Analogously to the $N$-convergence case, we show that the $a_M$-term tends to 0 as $M \to \infty$. Indeed, this holds true by the same arguments as in the proof of **(i)**.
Furthermore, similar to the $N$-convergence we have

$$\mathbf{E}||b_M(E)||^2_{L^2_\rho} \le c(c(T), c_e(T)) \int\limits_0^t \mathbf{E}||X^{(i)}_{M,L,J}(s) - X^{(i)}_{L,J}(s)||^2_{L^2_\rho}\, ds,$$

$$\mathbf{E}||b_M(\Sigma, m)||^2_{L^2_\rho} \le c(m, c(N), c_\sigma(T))(T)) \int\limits_0^t (t-s)^{-\zeta}\mathbf{E}||X^{(i)}_{M,L,J}(s) - X^{(i)}_{L,J}(s)||^2_{L^2_\rho}\, ds,$$

$$\mathbf{E}||b_M(\Sigma)||^2_{L^2_\rho} \le c(L, c(T), c_\sigma(T)) \int\limits_0^t \mathbf{E}||X^{(i)}_{M,L,J}(s) - X^{(i)}_{L,J}(s)||^2_{L^2_\rho}\, ds$$

and

$$\mathbf{E}||b_M(\Sigma_2)||^2_{L^2_\rho} \le c(c(T), c_\sigma, C_{q,\eta}) \int\limits_0^t (t-s)^{-\zeta}\mathbf{E}||X^{(i)}_{M,L,J}(s) - X^{(i)}_{L,J}(s)||^2_{L^2_\rho}.$$

Thus, for any $t \in [\,0, T\,]$, we can estimate

$$\begin{aligned}
&\mathbf{E}||X^{(i)}_{M,L,J}(s) - X^{(i)}_{L,J}(s)||^2_{L^2_\rho} \\
&\le \mathbf{E}||a_M(\Sigma)||^2_{L^2_\rho} \\
&+c(\zeta, L, T, c(T), c_f(T), c_\sigma(T), c_\gamma(T), C_{q,\eta}) \int\limits_0^t (t-s)^{-\zeta}\mathbf{E}||X^{(i)}_{M,L,J}(s) - X^{(i)}_{L,J}(s)||^2_{L^2_\rho}\, ds.
\end{aligned}$$

Therefore, by the Gronwall-Bellman lemma 2.7.2 and the corresponding Remark 2.7.3 we get

$$\begin{aligned}
&\mathbf{E}||X^{(i)}_{M,L,J}(s) - X^{(i)}_{L,J}(s)||^2_{L^2_\rho} \\
&\le (\mathbf{E}||a_M(\Sigma)||^2_{L^2_\rho}) c(\zeta, M, T, c(N), c_f(T), c_\sigma(T), c_\gamma(T), C_{q,\eta}).
\end{aligned}$$

Now, since the $a_M$-term tends to 0 as $M \to \infty$, we get the convergence result (8.36) for $M \to \infty$.

Let us now proceed with the convergence for $L \to \infty$.
For any $t \in [\,0, T\,]$, we have

$$\begin{aligned}
X^{(i)}_{L,J}(t) - X^{(i)}_J(t) = \;& \int\limits_0^t U(t,s)[E^{(i)}_J(s, X^{(i)}_{L,J}(s)) - E^{(i)}_J(s, X^{(i)}_J(s))]\, ds \\
&+ \int\limits_0^t \int\limits_{L^2} U(t,s)[\mathcal{M}_{\Gamma_J(s, X^{(i)}_{L,J}(s))} - \mathcal{M}_{\Gamma_J(s, X^{(i)}_{L,J}(s))}] I_J(x)\, \tilde{N}(ds, dx) \\
&+ \int\limits_0^t U(t,s)[\mathcal{M}_{\Sigma_J(s, X^{(i)}_{L,J}(s))} - \mathcal{M}_{\Sigma_J(s, X^{(i)}_{L,J}(s))}] I_J(m)\, ds \\
&+ \sum_{n=1}^{L} \sqrt{a_n} \int\limits_0^t U(t,s)[\mathcal{M}_{\Sigma_J(s, X^{(i)}_{L,J}(s))} - \mathcal{M}_{\Sigma_J(s, X^{(i)}_J(s))}] e_n\, dw_n(s) \\
&+ \sum_{n=L+1}^{\infty} \sqrt{a_n} \int\limits_0^t U(t,s)\mathcal{M}_{\Sigma_J(s, X^{(i)}_J(s))} e_n\, dw_n(s).
\end{aligned}$$

Analogously to the $b_N$-terms from the previous steps we get

$$\begin{aligned}\mathbf{E}||X^{(i)}_{L,J}(t) - X^{(i)}_J(t)||^2_{L^2_\rho} \leq \quad & c(\zeta, M, m, T, c(T), c_f(T), c_\sigma(T), c_\gamma(T), C_{q,\eta}) \\ & \int_0^t (t-s)^{-\zeta} \mathbf{E}||X^{(i)}_{L,J}(s) - X^{(i)}_J(s)||^2_{L^2_\rho}\, ds \\ & + \sum_{n=L+1}^{\infty} a_n \int_0^t \mathbf{E}||U(t,s)\mathcal{M}_{\Sigma_J(s, X^{(i)}_J(s))} e_n||^2_{L^2_\rho}\, ds.\end{aligned}$$

Thus, the Gronwall-Bellman lemma yields

$$\mathbf{E}||X^{(i)}_{M,L}(t) - X^{(i)}_L(t)||^2_{L^2_\rho} \leq c_L(\Sigma) e^{c(\zeta, M, T, c(T), c_f(T), c_\sigma(T), c_\gamma(T), C_{q,\eta})t}$$

with $c_L(\Sigma)$ as in the proof of Lemma 6.1.4 **(i)** (cf. Section 6.3). By the same arguments as in the proof of part **(i)** we get $c_L(\Sigma) \to 0$ as $L \to \infty$. This proves $\mathbf{E}||X^{(i)}_{L,J}(t) - X^{(i)}_J(t)||^2_{L^2_\rho} \to 0$ as $L \to \infty$, i.e. (8.37) holds.

Finally, for fixed $J \in \mathbb{N}$ we have

$$\begin{aligned}X^{(i)}_J(t) - X^{(i)}(t) = \quad & U(t,0)[\xi^{(i)}_J - \xi^{(i)}] + \int_0^t U(t,s)[E^{(i)}_J(s, X^{(i)}_J(s)) - E^{(i)}(s, X^{(i)}(s))]\, ds \\ & + \int_0^t U(t,s)[\mathcal{M}_{\Sigma_J(s, X^{(i)}_J(s))} I_J(m) - \mathcal{M}_{\Sigma(s, X^{(i)}(s))} m]\, ds \\ & + \int_0^t U(t,s)[\mathcal{M}_{\Sigma_J(s, X^{(i)}_J(s))} - \mathcal{M}_{\Sigma(s, X^{(i)}(s))}]\, dW(s) \\ & + \int_0^t \int_{L^2} U(t,s)[\mathcal{M}_{\Sigma_J(s, X^{(i)}_J(s))} I_J(x) - \mathcal{M}_{\Sigma(s, X^{(i)}(s))} x]\, \tilde{N}(ds, dx) \\ = \quad & a_J(\xi) + a_J(E) + b_J(E) + a_J(\Sigma, m) + b_J(\Sigma, m) \\ & + a_J(\Sigma) + b_J(\Sigma) + a_J(\Sigma_2) + b_J(\Sigma_2).\end{aligned}$$

Here, $a_J(\xi)$, $a_J(\Sigma)$ and $b_J(\Sigma)$ are as in the proof of **(i)**, $a_J(E)$ resp. $b_J(E)$ is just $a_J(F)$ resp. $b_J(F)$ from the proof of **(i)** with $F$ being replaced by $E$ and $a_J(\Sigma_2)$ resp. $b_J(\Sigma_2)$ is just $a_J(\Gamma)$ resp. $b_J(\Gamma)$ from the proof of **(i)** with $\Gamma$ being replaced by $\Sigma$. Finally, $a_J(\Sigma, m)$ resp. $b_J(\Sigma, m)$ is defined by

$$a_J(\Sigma, m) := \int_0^t U(t,s)[\mathcal{M}_{\Sigma^{(i)}_J(s, X^{(i)}(s))} I_J(m) - \mathcal{M}_{\Sigma(s, X^{(i)}(s))} m]\, ds$$

resp.

$$b_J(\Sigma, m) := \int_0^t U(t,s)[\mathcal{M}_{\Sigma^{(i)}_J(s, X^{(i)}_J(s))} - \mathcal{M}_{\Sigma^{(i)}_J(s, X^{(i)}(s))}] I_J(m)\, ds.$$

Analogously to the proof of **(i)** we get

$$\lim_{J\to\infty} \mathbf{E}||a_J(\xi)||^2_{L^2_\rho} = 0,$$

$$\lim_{J\to\infty} \mathbf{E}||a_J(E)||^2_{L^2_\rho} = 0,$$

and

$$\lim_{J\to\infty} \mathbf{E}||a_J(\Sigma)||^2_{L^2_\rho} = 0.$$

Concerning $a_J(\Sigma, m)$, note that

$$\begin{aligned}
\mathbf{E}||a_J(\Sigma,m)||^2_{L^2_\rho} \leq & \int_0^t \mathbf{E}||U(t,s)[\mathcal{M}_{\Sigma_J(s,X^{(i)}(s))}I_J(m) - \mathcal{M}_{\Sigma(s,X^{(i)}(s))}m]||^2_{L^2_\rho}\,ds \\
\leq & \int_0^t \mathbf{E}||U(t,s)\mathcal{M}_{\Sigma_J(s,X^{(i)}(s))}[I_J(m) - m]||^2_{L^2_\rho}\,ds \\
& + \int_0^t \mathbf{E}||U(t,s)[\mathcal{M}_{\Sigma_J(s,X^{(i)}(s))} - \mathcal{M}_{\Sigma(s,X^{(i)}(s))}]m||^2_{L^2_\rho}\,ds \\
\leq & \left(||I_J(m) - m||^2_{L^2}\right)\int_0^t (t-s)^{-\zeta}\mathbf{E}||\Sigma_J(s,X^{(i)}(s))||^2_{L^2_\rho}\,ds \\
& + ||m||^2_{L^2}\int_0^t (t-s)^{-\zeta}\mathbf{E}||\Sigma_J(s,X^{(i)}(s)) - \Sigma(s,X^{(i)}(s))||^2_{L^2_\rho}\,ds.
\end{aligned}$$

The first term on the right hand side tends to 0 as $J \to \infty$, since $\lim_{J\to\infty} ||I_J(m) - m||^2_{L^2} = 0$ and

$$\int_0^t (t-s)^{-\zeta}\mathbf{E}||\Sigma_J(s,X^{(i)}(s))||^2_{L^2_\rho}\,ds \leq c(c_\sigma(T))\frac{T^{1-\zeta}}{1-\zeta} < \infty.$$

The second term obvously tends to 0 as $J \to \infty$ by the uniform Lipschitz convergence of $\sigma_J$, $J \in \mathbb{N}$.
This yields $\mathbf{E}||a_J(\Sigma,m)||^2_{L^2_\rho} \to 0$ as $J \to \infty$.

Analogously to the consideration of $a_J(\Gamma)$ in the proof of **(i)**, we get $\mathbf{E}||a_J(\Sigma_2)||^2_{L^2_\rho} \to 0$ as $J \to \infty$.

So all $\mathbf{E}||a_J||_{L^2_\rho}$-terms tend to 0 as $J \to \infty$.

Concerning the $b_J$-terms note that with the help of the Itô isometries w.r.t. Wiener processes and compensated Poisson random measures and the uniformness of the Lipschitz properties, we get

$$\begin{aligned}
&\mathbf{E}\left|\left|\int_0^t U(t,s)[E_J^{(i)}(s,X_J^{(i)}(s)) - E_J^{(i)}(s,X^{(i)}(s))]\,ds\right|\right|^2_{L^2_\rho} \\
&\leq c(c(T),c_e(T))\int_0^t \mathbf{E}||X_J^{(i)}(s) - X^{(i)}(s)||^2_{L^2_\rho}\,ds,
\end{aligned}$$

$$\mathbf{E}\left|\left|\int_0^t U(t,s)[\mathcal{M}_{\Sigma_J(s,X_J^{(i)}(s))} - \mathcal{M}_{\Sigma_J(s,X^{(i)}(s))}]I_J(m)\,ds\right|\right|^2_{L^2_\rho}$$
$$\leq c(m,c(T),c_\sigma(T))\int_0^t (t-s)^{-\zeta}\mathbf{E}||X_J^{(i)}(s) - X^{(i)}(s)||^2_{L^2_\rho}\,ds,$$
$$\mathbf{E}\left|\left|\int_0^t U(t,s)[\mathcal{M}_{\Sigma_J(s,X_J^{(i)}(s))} - \mathcal{M}_{\Sigma_J(s,X^{(i)}(s))}]\,dW(s)\right|\right|^2_{L^2_\rho}$$
$$\leq c(c(T),c_\sigma(T))\int_0^t (t-s)^{-\zeta}\mathbf{E}||X_J^{(i)}(s) - X^{(i)}(s)||^2_{L^2_\rho}\,ds,$$

and

$$\mathbf{E}\left|\left|\int_0^t\int_{L^2} U(t,s)[\mathcal{M}_{\Sigma_J(s,X_J^{(i)}(s))} - \mathcal{M}_{\Sigma_J(s,X^{(i)}(s))}]I_J(x)\,\tilde{N}(ds,dx)\right|\right|^2_{L^2_\rho}$$
$$\leq c(c(T),c_\sigma(T),C_{q,\eta})\int_0^t (t-s)^{-\zeta}\mathbf{E}||X_J^{(i)}(s) - X^{(i)}(s)||^2_{L^2_\rho}\,ds.$$

Thus, for any $t \in [0,T]$, we can estimate

$$\mathbf{E}||X_J^{(i)}(s) - X^{(i)}(s)||^2_{L^2_\rho}$$
$$\leq (\mathbf{E}||a_J(\xi)||^2_{L^2_\rho} + \mathbf{E}||a_J(E)||^2_{L^2_\rho} + \mathbf{E}||a_J(\Sigma,m)||^2_{L^2_\rho} + \mathbf{E}||a_J(\Sigma)||^2_{L^2_\rho} + \mathbf{E}||a_J(\Sigma,2)||^2_{L^2_\rho})$$
$$+\, c(m,c(T),c_e(T),c_\sigma(T),C_{q,\eta})\int_0^t (t-s)^{-\zeta}\mathbf{E}||X_J^{(i)}(s) - X^{(i)}(s)||^2_{L^2_\rho}\,ds.$$

Thererfore, by the Gronwall-Bellman lemma 2.7.2 and the corresponding Remark 2.7.3 we get

$$\mathbf{E}||X_J^{(i)}(s) - X^{(i)}(s)||^2_{L^2_\rho}$$
$$\leq (\mathbf{E}||a_J(\xi)||^2_{L^2_\rho} + \mathbf{E}||a_J(F)||^2_{L^2_\rho} + \mathbf{E}||a_J(\Sigma,m)||^2_{L^2_\rho} + \mathbf{E}||a_J(\Sigma)||^2_{L^2_\rho} + \mathbf{E}||a_J(\Sigma_2)||^2_{L^2_\rho})$$
$$c(m,c(T),c_f(T),c_\sigma(T),C_{q,\eta}).$$

Now, since the $a_J$-terms tend to 0 as $J \to \infty$, we get the convergence result (8.38) for $J \to \infty$. ■

## 8.3 Proof of Theorem 8.1.1

**Step 1:** This is just Step 1 from the proof of Theorem 7.1.2, i.e. by (7.15) and (7.16) we get functions $\bar{g}$, $\bar{h}$: $\mathbb{R} \to \mathbb{R}$ obeying

$$\bar{g} \leq 0\ ,\ \bar{g}(v) \leq f(t,\omega,v),\ (t,\omega,v) \in [0,T] \times \Omega \times \mathbb{R}, \tag{8.40}$$

$$\bar{h} \geq 0 \;,\; \bar{h}(v) \geq f(t,\omega,v),\; (t,\omega,v) \in [\,0,T\,] \times \Omega \times \mathbb{R}. \tag{8.41}$$

Of course, (8.40), (8.41) also hold true, when $f$ is replaced by $e$.
These auxiliary functions help us to estimate the integral $I_F(X)$ (defined in Section 5.1) in the non-Lipschitz case.

**Step 2:** Given arbitrary $N, M \in \mathbb{N}$, let $f_{N,M}$ be defined by (7.10), (7.11) from 7.1.7. Then, $f_{N,M}$ obeys **(LC)** and **(LB)** and is such that $f_{N,M}$ is $\mathcal{P}_T \otimes \mathcal{B}(\mathbb{R})$-measurable by Lemma 7.1.8. Of course, this also holds true for functions $e_{N,M}$ defined analogously to the $f_{N,M}$.

Thus, the theory from Sections 5.1/5.2 is applicable. By 5.2.1 there are processes $X_{N,M} \in \mathcal{H}^q(T)$ solving equations (1.1) resp. (1.2), when $f$ resp. $e$ is replaced by $f_{N,M}$ resp. $e_{N,M}$.
To proceed along the lines of Manthey's and Zausinger's proof, we need to find $M$-independent estimates for the moments of $X_{N,M}$.

**(i) Equation (1.1) - the Poison case**

By Theorem 5.2.1 $t \mapsto X_{N,M}(t)$ is continuous in $L^q(\Omega; L^2_\rho)$. Furthermore, we have

$$X_{N,M}(t) \leq X_{N,M+1}(t) \text{ for all } t \in [\,0,T\,], \tag{8.42}$$

$P$-almost surely, by Theorem 8.1.5. We denote solutions to the equations with initial conditions $\xi^+$, $\xi^-$ resp. 0 and drifts $F_{0,M}$, $F^-_{N,M}$ resp. 0 by $\bar{X}_{0,M}$, $\underline{X}_{N,M}$ resp. $V$. Of course, by 8.1.5 we have

$$\underline{X}_{N,M}(t) \leq X_{N,M}(t) \leq \bar{X}_{0,M}(t), \tag{8.43}$$
$$\underline{X}_{N,M}(t) \leq V(t) \leq \bar{X}_{0,M}(t), \tag{8.44}$$

$P$-almost surely, for each $t \in [\,0,T\,]$ and arbitrary $N, M \in \mathbb{N}$.

Note that similar to Section 5.2, all the solutions above are time-continuous in $L^q(\Omega; L^2_\rho)$ and, by the uniform boundedness of $\gamma$, have a càdlàg version under the additional assumption that $U$ obeys **(A7)**.

In view of (8.43), we show the required $M$-independent estimate for $X_{N,M}$ by showing $M$-independent estimates for $\bar{X}_{0,M}$ and $\underline{X}_{N,M}$.

Let us fix $t \in [\,0,T\,]$. To find an $M$-independent estimate, we note that for each $M \in \mathbb{N}$

$$\text{(8.45)} \quad \mathbf{E}||\bar{X}_{0,M}(t)||^q_{L^2_\rho} \leq c(q)[\bar{I}^{(1)}(t) + \bar{I}^{(2)}_M(t) + \bar{I}^{(3)}_M(t) + \bar{I}^{(4)}_M(t)].$$

Here, $\bar{I}^{(1)}$, $\bar{I}^{(2)}_M$ and $\bar{I}^{(3)}_M$ are as in Step 2 of the proof of 7.1.2 **(i)**, whereas $\bar{I}^{(4)}_M$ is defined by

$$\bar{I}^{(4)}(t) := \mathbf{E}\left|\left|\int_0^t \int_{L^2} U(t,s)\mathcal{M}_{\Gamma(s,\bar{X}_{0,M}(s))}(x)\,\tilde{N}(ds,dx)\right|\right|^q_{L^2_\rho}.$$

Analogously to Step 2 in the proof of Theorem 7.1.2 **(i)**, we have

$$\bar{I}^{(1)}(t) \leq c^q(T)\mathbf{E}||\xi||^q_{L^2_\rho},$$

$$\bar{I}^{(2)}_M(t) \leq c(q,T,c(T),c_f(T),c_\sigma(T),c_\gamma(T),C_{q,\eta})\left(1 + \int_0^t \mathbf{E}||\bar{X}_{0,M}(s)||^q_{L^2_\rho}\,ds\right)$$

and

$$\bar{I}^{(3)}_M(t) \leq c(q,T,\zeta,c(T),c_\sigma(T))\left(1 + \int_0^t \mathbf{E}||\bar{X}_{0,M}(s)||^q_{L^2_\rho}\,ds\right).$$

Finally, by the Bichteler-Jacod inequality 2.6.10, **(QI)** for $\eta$, **(LC)** and **(LB)** for $\gamma$, **(A2)** for $U$ and the fact

$$q < \tfrac{2}{\zeta} \iff \tfrac{\zeta q}{2} < 1,$$

we get

$$\begin{aligned}
\bar{I}^{(4)}_M(t) \leq \quad & C^{\frac{1}{q}}_{q,\eta}\int_0^t (t-s)^{-\frac{\zeta q}{2}}\mathbf{E}||\Gamma(s,\bar{X}_{0,M}(s))||^q_{L^2_\rho}\,ds \\
\leq \quad & c(q,\zeta,T,c_\gamma(T),C_{q,\eta})\left(1 + \int_0^t (t-s)^{-\frac{\zeta q}{2}}\mathbf{E}||\bar{X}_{0,M}(s)||^q_{L^2_\rho}\,ds\right).
\end{aligned}$$

Thus, by (8.45) we have

$$\begin{aligned}
&\mathbf{E}||\bar{X}_{0,M}(t)||^q_{L^2_\rho} \\
&\leq c(q,\zeta,T,c(T),c_f(T),c_\sigma(T),c_\gamma(T),C_{q,\eta})(1 + \mathbf{E}||\xi||^q_{L^2_\rho}) \\
&+ c(q,\zeta,T,c(T),c_f(T),c_\sigma(T),c_\gamma(T),C_{q,\eta})\int_0^t (t-s)^{-\frac{\zeta q}{2}}\mathbf{E}||\bar{X}_{0,M}(s)||^q_{L^2_\rho}\,ds
\end{aligned}$$

for arbitrary $t \in [0,T]$. Therefore, by the Gronwall-Bellman lemma we get

$$\mathbf{E}||\bar{X}_{0,M}(t)||^q_{L^2_\rho} \leq \bar{c}(q,\zeta,T,c(T),c_f(T),c_\sigma(T),c_\gamma(T),C_{q,\eta})(1 + \mathbf{E}||\xi||^q_{L^2_\rho})$$

for arbitrary $M \in \mathbb{N}$ and $t \in [0,T]$. Thus, we have

$$\sup_{\substack{t\in[0,T]\\ M\in\mathbb{N}}} \mathbf{E}||\bar{X}_{0,M}(t)||^q_{L^2_\rho} \leq \bar{c}(q,\zeta,T,c(T),c_f(T),c_\sigma(T),c_\gamma(T),C_{q,\eta})(1+\mathbf{E}||\xi||^q_{L^2_\rho}).$$

Next, we consider $\underline{X}_{N,M}$ for arbitrary $N, M \in \mathbb{N}$. For any $t \in [0,T]$, we have

$$\mathbf{E}||\underline{X}_{N,M}||^q_{L^2_\rho} \leq c(q)(\underline{I}^{(1)}(t) + \underline{I}^{(2)}_{N,M}(t) + \underline{I}^{(3)}_{N,M}(t) + \underline{I}^{(4)}_{N,M}(t))$$

with $\underline{I}^{(1)}$, $\underline{I}^{(2)}_{N,M}$ and $\underline{I}^{(3)}_{N,M}$ as in the proof of Theorem 7.1.2 **(i)** and

$$\underline{I}^{(4)}_{N,M}(t) := \mathbf{E}\left|\left|\int_0^t \int_{L^2} U(t,s)\mathcal{M}_{\Gamma(s,\underline{X}_{N,M}(s))}(x)\,\tilde{N}(ds,dx)\right|\right|^q_{L^2_\rho}.$$

Similarly to Step 2 in the proof of Theorem 7.1.2 **(i)**, we get

$$\underline{I}^{(1)}(t) \leq c(q,c(T))\mathbf{E}||\xi||^q_{L^2_\rho},$$

$$\underline{I}^{(2)}_{N,M}(t) \leq c(N,q,T,c(T))$$

and

$$\underline{I}^{(3)}_{N,M}(t) \leq c(q,\zeta,c(T),c_\sigma(T))\left(1+\int_0^t \mathbf{E}||\underline{X}_{N,M}(s)||^q_{L^2_\rho}\right).$$

Furthermore, by the Bichteler-Jacod inequality 2.6.10, **(QI)** for $\eta$, **(LC)** and **(LB)** for $\gamma$, **(A2)** for $U$ and the fact

$$q < \tfrac{2}{\zeta} \iff \tfrac{\zeta q}{2} < 1$$

we get

$$\begin{aligned} \bar{I}^{(4)}_M(t) \leq\ & C^{\frac{1}{q}}_{q,\eta}\int_0^t (t-s)^{-\frac{\zeta q}{2}}\mathbf{E}||\Gamma(s,\underline{X}_{N,M}(s))||^q_{L^2_\rho}\,ds \\ \leq\ & c(q,\zeta,T,c_\gamma(T),C_{q,\eta})\left(1+\int_0^t (t-s)^{-\frac{\zeta q}{2}}\mathbf{E}||\underline{X}_{N,M}(s)||^q_{L^2_\rho}\,ds\right). \end{aligned}$$

Putting the estimates together we conclude

$$\begin{aligned}\mathbf{E}||\underline{X}_{N,M}||^q_{L^2_\rho} \leq\ & c(q,c(T))\mathbf{E}||\xi||^q_{L^2_\rho} + c(N,q,T,c(T)) \\ & c(q,c(T),c_\sigma(T),c_\gamma(T))\left(1+\int_0^t (t-s)^{-\frac{q\zeta}{2}}\mathbf{E}||\underline{X}_{N,M}(s)||^q_{L^2_\rho}\right) \\ \leq\ & c(N,q,\zeta,T,c(T),c_\sigma(T),c_\gamma(T),C_{q,\eta})(1+\mathbf{E}||\xi||^q_{L^2_\rho}) \\ & + c(q,\zeta,c(T),c_\sigma(T),c_\gamma(T),C_{q,\eta})\int_0^t (t-s)^{-\frac{q\zeta}{2}}\mathbf{E}||\underline{X}_{N,M}(s)||^q_{L^2_\rho}\,ds.\end{aligned}$$

Herefrom, by the Gronwall-Bellman lemma we get

$$\mathbf{E}||\underline{X}_{N,M}||^q_{L^2_\rho} \leq \underline{c}(N,q,\zeta,T,c(T),c_\sigma(T),c_\gamma(T),C_{q,\eta})(1+\mathbf{E}||\xi||^q_{L^2_\rho}).$$

Since the previous estimate holds for arbitrary $t\in[0,T]$ and $M\in\mathbb{N}$, we get

$$\sup_{\substack{t\in[0,T]\\ M\in\mathbb{N}}}\mathbf{E}||\underline{X}_{N,M}||^q_{L^2_\rho} \leq \underline{c}(N,q,\zeta,T,c(T),c_\sigma(T),c_\gamma(T),C_{q,\eta})(1+\mathbf{E}||\xi||^q_{L^2_\rho}).$$

Finally, by (8.43) we get

$$\sup_{\substack{t\in[0,T]\\ M\in\mathbb{N}}}\mathbf{E}||X_{N,M}||^q_{L^2_\rho} \leq c(N,q,\zeta,T,c(T),c_\sigma(T),c_\gamma(T),C_{q,\eta})(1+\mathbf{E}||\xi||^q_{L^2_\rho})$$

with $c=\bar{c}+\underline{c}$.

**(ii) Equation (1.2) - the Lévy noise**

Denoting solutions to (1.2) with $\xi^+$ and $E_{0,M}$ resp. $\xi^-$ and $E^-_{N,M}$ resp. 0 and 0 replacing $\xi$ and $E$ by $\bar{X}_{0,M}$ resp. $\underline{X}_{N,M}$ resp. $V$, we get relations (8.43) and (8.44) again.

Concerning $\bar{X}_{0,M}$ note that we have, for any $t\in[0,T]$,

$$\text{(8.46)}\quad \mathbf{E}||\bar{X}_{0,M}(t)||^q_{L^2_\rho} \leq c(q)(\bar{I}^{(1)}(t)+\bar{I}^{(2)}_M(t)+\bar{I}^{(3)}_M(t))$$

with $\bar{I}^{(1)}$ and $\bar{I}^{(2)}_M$ as in Step 2 in the proof of Theorem 7.1.2 **(ii)** and

$$\bar{I}^{(3)}_M(t) := \mathbf{E}\left|\left|\int_0^t U(t,s)\mathcal{M}_{\Sigma(s,\bar{X}_{0,M})}\,dL(s)\right|\right|^q_{L^2_\rho}.$$

Analogously to Step 2 in the proof of Theorem 7.1.2 **(ii)**,

$$\bar{I}^{(1)}(t) \leq c(q,c(T))\mathbf{E}||\xi||^q_{L^2_\rho}.$$

and

$$\bar{I}^{(2)}_M(t) \leq c(q,T,c(T),c_e(T),c_\sigma(T),C_{q,\eta})\left(1+\int_0^t \mathbf{E}||\bar{X}_{0,M}(s)||^q_{L^2_\rho}\,ds\right)$$

hold true for any $t \in [0, T]$.
Applying first the Lévy-Itô decomposition 2.4.13 and then estimate (3.30) from Proposition 3.4.1, estimate (4.5) from Proposition 4.1, the integrability condition **(QI)**, **(LC)** and **(LB)** for $\sigma$ and the fact that

$$q < \tfrac{2}{\zeta} \iff \tfrac{\zeta q}{2} < 1,$$

we easily get

$$\bar{I}_M^{(3)}(t) \leq c(q, \zeta, T, c(T), c_\sigma(T), C_{q,\eta}) \left(1 + \int_0^t (t-s)^{-\frac{q\zeta}{2}} \mathbf{E}||\bar{X}_{0,M}(s)||_{L_\rho^2}^q \, ds\right).$$

Thus, by (8.46) we have, for arbitrary $t \in [0, T]$,

$$\begin{aligned} \mathbf{E}||\bar{X}_{0,M}(t)||_{L_\rho^2}^q \leq \quad & c(q, \zeta, T, c(T), c_e(T), c_\sigma(T), C_{q,\eta})(1 + \mathbf{E}||\xi||_{L_\rho^2}^q) \\ & + c(q, T, c(T), c_e(T), c(T), C_{q,\eta}) \int_0^t (t-s)^{-\frac{q\zeta}{2}} \mathbf{E}||\bar{X}_{0,M}(s)||_{L_\rho^2}^q \, ds. \end{aligned}$$

From this estimate, analogously to the consideration of (1.1) in **(i)**, we get

$$\sup_{\substack{t \in [0,T] \\ M \in \mathbb{N}}} \mathbf{E}||\bar{X}_{0,M}(t)||_{L_\rho^2}^q \leq c(q, \zeta, T, c(T), c_e(T), c_\sigma(T), C_{q,\eta})(1 + \mathbf{E}||\xi||_{L_\rho^2}^q).$$

Next, we consider $\underline{X}_{N,M}$ for arbitrary $N, M \in \mathbb{N}$. For any $t \in [0, T]$, we get

$$\mathbf{E}||\underline{X}_{N,M}||_{L_\rho^2}^q \leq c(q)(\underline{I}^{(1)}(t) + \underline{I}_{N,M}^{(2)}(t) + \underline{I}_{N,M}^{(3)}(t))$$

with $\underline{I}^{(1)}$ and $\underline{I}_{N,M}^{(2)}$ as in Step 2 in the proof of 7.1.2(ii) and

$$\underline{I}_{N,M}^{(3)}(t) := \mathbf{E}\left|\left|\int_0^t U(t,s) \mathcal{M}_{\Sigma(s, \underline{X}_{N,M}(s))} \, dL(s)\right|\right|_{L_\rho^2}^q.$$

From Step 2 in the proof of 7.1.2 **(ii)**, we immediately get

$$\underline{I}^{(1)}(t) \leq c(q, c(T)) \mathbf{E}||\xi||_{L_\rho^2}^q$$

and

$$\underline{I}_{N,M}^{(2)}(t) \leq c(N, q, T, c(T)).$$

Finally, applying first the Lévy-Itô decomposition 2.4.13 and then estimate (3.30) from Proposition 3.4.1, estimate (4.5) from Proposition 4.1, the integrability condition **(QI)**, **(LC)** and **(LB)** for $\sigma$ and the fact that

$$q < \tfrac{2}{\zeta} \iff \tfrac{\zeta q}{2} < 1,$$

we obey

$$\underline{I}^{(3)}_{N,M}(t) \le c(q,\zeta,T,c(T),c_\sigma(T),C_{q,\eta})\left(1+\int_0^t (t-s)^{-\frac{q\zeta}{2}}\mathbf{E}||\underline{X}_{N,M}(s)||^q_{L^2_\rho}\,ds\right).$$

Thus, putting the estimates together, we have

$$\begin{aligned}\mathbf{E}||\underline{X}_{N,M}(t)||^q_{L^2_\rho} \le\quad & c(q,c(T))\mathbf{E}||\xi||^q_{L^2_\rho} + c(N,q,T,c(T))\\ & + c(q,\zeta,T,c_\sigma(T),C_{q,\eta})\left(1+\int_0^t (t-s)^{-\frac{q\zeta}{2}}\mathbf{E}||\underline{X}_{N,M}(s)||^q_{L^2_\rho}\,ds\right),\end{aligned}$$

which by the Gronwall-Bellman lemma gives us

$$\mathbf{E}||\underline{X}_{N,M}(t)||^q_{L^2_\rho} \le \underline{c}(N,q,\zeta,T,c(T),c_\sigma(T),C_{q,\eta})(1+\mathbf{E}||\xi||^q_{L^2_\rho}).$$

Since the previous estimate holds for arbitrary $t \in [0,T]$ and $M \in \mathbb{N}$, we get

$$\sup_{\substack{t\in[0,T]\\ M\in\mathbb{N}}} \mathbf{E}||\underline{X}_{N,M}||^q_{L^2_\rho} \le \underline{c}(N,q,\zeta,T,c(T),c_\sigma(T),C_{q,\eta})(1+\mathbf{E}||\xi||^q_{L^2_\rho}).$$

Now, by (8.43) we get

$$\sup_{\substack{t\in[0,T]\\ M\in\mathbb{N}}} \mathbf{E}||X_{N,M}||^q_{L^2_\rho} \le c(N,q,\zeta,T,c(T),c_\sigma(T),C_{q,\eta})(1+\mathbf{E}||\xi||^q_{L^2_\rho})$$

with $c = \bar{c} + \underline{c}$, which finishes Step 2.

**Step 3:** Recall that in Step 3 in the proof of 7.1.2 we had the following procedure.

For $N, M \in \mathbb{N}$, define

$$Z_{N,M}(t) := X_{N,M}(t) - X_{N,1}(t),\ t \in [0,T].$$

Furthermore, we set

$$Z_N(t) := \sup_{M\in\mathbb{N}} Z_{N,M}(t),\ t \in [0,T]$$

and

$$X_N(t) := Z_N(t) + X_{N,1}(t),\ t \in [0,T].$$

By the well-definedness of the $X_{N,M}$, (8.42) and the $M$-independent estimates on the $X_{N,M}$ shown in Step 2, by almost literally repeating Step 3 from the proof of 7.1.2 we get $X_N \in \mathcal{H}^q(T)$ for any $N \in \mathbb{N}$ both in **(i)** and **(ii)**. Furthermore, denoting $\bar{X}$ and $V$ as in Step 3 in the proof of 7.1.2, the processes obey

$$\begin{aligned} \underline{X}_N(t) \le X_N(t) \le \bar{X}(t), \\ \underline{X}_N(t) \le V(t) \le \bar{X}(t), \end{aligned}$$

$P$-almost surely, for any $t \in [0, T]$.
Finally, we have

$$\lim_{M\to\infty} \mathbf{E}||X_{N,M}(t) - X_N(t)||^q_{L^2_\rho} = 0,\ t \in [0, T], \tag{8.47}$$

and

$$\lim_{M\to\infty} \int_0^T \mathbf{E}||X_{N,M}(t) - X_N(t)||^q_{L^2_\rho}\, dt = 0, \tag{8.48}$$

both in **(i)** and **(ii)**, and there are processes $\underline{X}_N$, $\bar{X} \in \mathcal{H}^q(T)$ such that

$$\begin{aligned} \lim_{M\to\infty} \int_0^T \mathbf{E}||\underline{X}_{N,M}(t) - \underline{X}_N(t)||^q_{L^2_\rho}\, ds = 0 \\ \lim_{M\to\infty} \int_0^T \mathbf{E}||\bar{X}_{0,M}(t) - \bar{X}(t)||^q_{L^2_\rho}\, ds = 0. \end{aligned}$$

**Step 4:** We show that, given arbitrary $N \in \mathbb{N}$, the processes $X_N$ defined in Step 3 solve (1.1) resp. (1.2) in case of $F$ resp. $E$ being replaced by $F_N$ resp. $E_N$.
Furthermore, we show that $t \mapsto X_N(t)$ is continuous in $L^q(\Omega; L^2_\rho)$ and that, additionally assuming (8.2) for $\gamma$ (for equation (1.1)) resp. (8.4) for $\sigma$ (for equation (1.2)) and **(A7)** for $U$, there is a càdlàg version of $t \mapsto X_N(t)$.

By (8.48), there is a subsequence of $(X_{N,M})_{M\in\mathbb{N}}$ that converges $P\otimes ds\otimes d\mu_\rho$-almost everywhere to $X_N$. We assume $(X_{N,M})_{M\in\mathbb{N}}$ itself to be this sequence.

**(i) Equation (1.1) - the Poisson case**
We have, for each $t \in [0, T]$,

$$\begin{aligned} &\mathbf{E}\Bigg|\Bigg|X_N(t) - U(t,0)\xi - \int_0^t U(t,s)F_N(s, X_N(s))\, ds - \int_0^t U(t,s)\mathcal{M}_{\Sigma(s,X_N(s))}\, dW(s) \\ &- \int_0^t \int_{L^2} U(t,s)\mathcal{M}_{\Gamma(s,X_N(s))}(x)\, \tilde{N}(ds,dx)\Bigg|\Bigg|^2_{L^2_\rho} \\ &\le C(I^{(1)}_{N,M}(t) + I^{(2)}_{N,M}(t) + I^{(3)}_{N,M}(t) + I^{(4)}_{N,M}(t)). \end{aligned}$$

Here, $I^{(1)}_{N,M}$, $I^{(2)}_{N,M}$ and $I^{(3)}_{N,M}$ are as in Step 4 in the proof of Theorem 7.1.2, whereas

$$I^{(4)}_{N,M}(t) := \mathbf{E}\left|\left|\int_0^t \int_{L^2} U(t,s)[\mathcal{M}_{\Gamma(s,X_N(s))} - \mathcal{M}_{\Gamma(s,X_{N,M}(s))}](x)\,\tilde{N}(ds,dx)\right|\right|^2_{L^2_\rho}.$$

In view of (8.47) and (8.48), which is just (7.22) and (7.23) from Step 4 in the proof of Theorem 7.1.2**(i)**, we immediately get (for $1 \le i \le 3$)

$$\lim_{M\to\infty} I^{(i)}_{N,M}(t) = 0.$$

Applying Itô's isometry for the stochastic integration w.r.t. compensated Poisson random measures, **(A2)**, **(QI)** and the fact that

$$q > \tfrac{2}{1-\zeta} \iff \tfrac{q\zeta}{q-2} < 1,$$

we get

$$\begin{aligned} I^{(4)}_{N,M}(t) \le \quad & c(c(T), C_{q,\eta})\mathbf{E}\int_0^t (t-s)^{-\zeta}||\Gamma(s,X_N(s)) - \Gamma(s,X_{N,M}(s))||^2_{L^2_\rho}\,ds \\ \le \quad & c(q,\zeta,T,c(T),c_\sigma(T),C_{q,\eta})\left(\mathbf{E}\int_0^t ||X_N(s) - X_{N,M}(s)||^q_{L^2_\rho}\,ds\right)^{\frac{2}{q}}, \end{aligned}$$

which tends to 0 for $M \to \infty$ by (8.48).
Thus, $X_N$ solves the equation in the sense of 5.1.2 **(i)**, when $F$ is replaced by $F_N$ for arbitrary $N \in \mathbb{N}$.

Similar to Step 4 in the proof of Theorem 7.1.2 **(i)**, to prove the required continuity property we only need to consider the drift term, but this follows by literally repeating the arguments from Step 4 in the proof of Theorem 7.1.2 **(i)**.

**(ii) Equation (1.2) - the Lévy case**

For any fixed $t \in [\,0,T\,]$ we have

$$\mathbf{E}\left|\left|X_N(t) - U(t,0)\xi - \int_0^t U(t,s)E_N(s,X_N(s))\,ds - \int_0^t U(t,s)\mathcal{M}_{\Sigma(s,X_N(s))}\,dL(s)\right|\right|^2_{L^2_\rho}$$
$$\le 3(I^{(1)}_{N,M}(t) + I^{(2)}_{N,M}(t) + I^{(3)}_{N,M}(t))$$

with $I^{(i)}_{N,M}$, $i = 1,2$, as in Step 4 in the proof of 7.1.2 **(ii)** and

$$I^{(3)}_{N,M}(t) := \mathbf{E}\left|\left|\int_0^t U(t,s)[\mathcal{M}_{\Sigma(s,X_N(s))} - \mathcal{M}_{\Sigma(s,X_{N,M}(s))}]\, dL(s)\right|\right|^2_{L^2_\rho},\ t \in [\,0,T\,].$$

Thus, $I^{(i)}_{N,M}(t)$, $i = 1,2$, tends to 0 for $M \to \infty$ by the same arguments as in Step 4 in the proof of 7.1.2 **(ii)**, whereas, by first applying the Lévy-Itô decomposition 2.4.13 and then estimate (3.30) from Proposition 3.4.1., estimate (4.5) from Proposition 4.1, **(QI)** for $\eta$, **(LC)** for $\sigma$ and the fact

$$q > \tfrac{2}{1-\zeta} \iff \tfrac{\zeta q}{q-2} < 1,$$

we obtain

$$I^{(3)}_{N,M}(t) \le c(q,\zeta,T,c(T),c_\sigma(T),C_{q,\eta}) \left(\int_0^t \mathbf{E}||X_N(s) - X_{N,M}(s)||^q_{L^2_\rho}\, ds\right)^{\frac{2}{q}},$$

which tends to 0 as $M \to \infty$ by (8.48).
Thus, $X_N$ solves (1.2) in the sense of 5.1.2 **(i)**, when $E$ is replaced by $E_N$. By simply replacing $F_N$ by $E_N$, the continuity property follows analogously to **(i)**.

**Step 5:** In this final step, we first obtain $N$-independent estimates for the $X_N$.
By construction, we have

$$\underline{X}_N(t) \le X_N(t) \le \bar{X}(t) \text{ in } L^2_\rho(\Theta).$$

Therefore, we have the required $N$-independent estimate for the moments of $X_N$ if we show $N$-independent estimates for $\underline{X}_N(t)$ and $\bar{X}(t)$.
We consider separately equations (1.1) and (1.2).

**(i) Equation (1.1) - The Poisson case**

Fix an arbitrary $t \in [\,0,T\,]$. We estimate

$$\mathbf{E}||\underline{X}_N(t)||^q_{L^2_\rho} \le c(q)(\underline{I}^{(1)}(t) + \underline{I}^{(2)}_N(t) + \underline{I}^{(3)}_N(t) + \underline{I}^{(4)}_N(t))$$

with $\underline{I}^{(1)}$, $\underline{I}^{(2)}_N$ and $\underline{I}^{(3)}_N$ as in Step 5 in the proof of Theorem 7.1.2 **(i)** and with

$$\underline{I}^{(4)}_N(t) := \mathbf{E}\left|\left|\int_0^t \int_{L^2} U(t,s)\mathcal{M}_{\Gamma(s)}(x)\, \tilde{N}(ds,dx)\right|\right|^q_{L^2_\rho}.$$

Thus, from Step 5 in the proof of Theorem 7.1.2 **(i)** we get

$$\underline{I}^{(1)}(t) \le c^q(T)\mathbf{E}||\xi||^q_{L^2_\rho},$$

$$\underline{I}_N^{(2)}(t) \le c(q,T,c(T),c_f(T)) \left(1+\int_0^t \mathbf{E}||\underline{X}_N(s)||^q_{L^2_\rho}\, ds\right)$$

and

$$\underline{I}_N^{(3)}(t) \le c(q,\zeta,T,c(T),c_\sigma(T)) \left(1+\int_0^t \mathbf{E}||\underline{X}_N(s)||^q_{L^2_\rho}\, ds\right).$$

Finally, by the Bichteler-Jacod inequality 2.6.10, **(A2)** for $U$, the integrability property **(QI)** for $\eta$, **(LC)** and **(LB)** for $\gamma$ and the fact that

$$q < \tfrac{2}{\zeta} \iff \tfrac{q\zeta}{2} < 1,$$

we get

$$\underline{I}_N^{(4)}(t) \le c(q,\zeta,T,c(T),c_\gamma(T),C_{q,\eta}) \left(1+\int_0^t (t-s)^{-\frac{q\zeta}{2}} \mathbf{E}||\underline{X}_N(s)||^q_{L^2_\rho}\, ds\right).$$

Putting the four estimates together we get, for all $t \in [\,0,T\,]$,

$$\begin{aligned}&\mathbf{E}||\underline{X}_N(t)||^q_{L^2_\rho}\\ &\le c(q,\zeta,T,c(T),c_f(T),c_\sigma(T),c_\gamma(T),C_{q,\eta})(1+\mathbf{E}||\xi||^q_{L^2_\rho})\\ &+ c(q,\zeta,T,c(T),c_f(T),c_\sigma(T),c_\Gamma(T),C_{q,\eta}) \int_0^t (t-s)^{-\frac{q\zeta}{2}} \mathbf{E}||\underline{X}_N(s)||^q_{L^2_\rho}\, ds.\end{aligned}$$

By the Gronwall-Bellman lemma 2.7.2/2.7.3 this yields

$$\mathbf{E}||\underline{X}_N(t)||^q_{L^2_\rho} \le c(q,\zeta,T,c(T),c_f(T),c_\sigma(T),c_\gamma(T),C_{q,\eta})(1+\mathbf{E}||\xi||^q_{L^2_\rho})$$

for all $N \in \mathbb{N}$ and $t \in [\,0,T\,]$ and hence

$$\sup_{\substack{t\in[\,0,T\,]\\ N\in\mathbb{N}}} \mathbf{E}||\underline{X}_N(t)||^q_{L^2_\rho} \le c_1(q,\zeta,T,c(T),c_f(T),c_\sigma(T),c_\gamma(T),C_{q,\eta})(1+\mathbf{E}||\xi||^q_{L^2_\rho}).$$

**(ii) Equation (1.2) - The Lévy case**

For arbitrary $t \in [\,0,T\,]$ we have

$$\mathbf{E}||\underline{X}_N(t)||^q_{L^2_\rho} \le c(q) \left(\underline{I}^{(1)}(t) + \underline{I}_N^{(2)}(t) + \underline{I}_N^{(3)}(t)\right)$$

with $\underline{I}^{(1)}$ and $\underline{I}_N^{(2)}$ as in Step 5 in the proof of 7.1.2 **(ii)** and

$$\underline{I}_N^{(3)}(t) := \mathbf{E}\left|\left|\int_0^t U(t,s)\mathcal{M}_{\Sigma(s)}\, dL(s)\right|\right|^q_{L^2_\rho}.$$

Thus, from Step 5 in the proof of 7.1.2(ii), we immediately get

$$\underline{I}^{(1)}(t) \leq c^q(T)\mathbf{E}||\xi||^q_{L^2_\rho}$$

and

$$\underline{I}^{(2)}_N(t) \leq c(q,T,c(T),c_e(T)) \left(1 + \int_0^t \mathbf{E}||\underline{X}_N(s)||^q_{L^2_\rho}\, ds\right).$$

Applying first the Lévy-Itô decomposition 2.4.12 and then estimate (3.30) from Proposition 3.4.1, estimate (4.5) from Proposition 4.1, integrability condition **(QI)** for $\eta$, **(LC)** and **(LB)** for $\sigma$ and the fact that

$$q < \tfrac{2}{\zeta} \iff \tfrac{q\zeta}{2} < 1,$$

we obey

$$\underline{I}^{(3)}_N(t) \leq c(q,\zeta,T,c(T),c_\sigma(T),C_{q,\eta}) \left(1 + \int_0^t (t-s)^{-\frac{q\zeta}{2}}\mathbf{E}||\underline{X}_{N,M}(s)||^q_{L^2_\rho}\, ds\right).$$

Putting the three estimates together we get, for all $t \in [\,0,T\,]$,

$$\begin{aligned}\mathbf{E}||\underline{X}_N(t)||^q_{L^2_\rho} \leq \quad & c(q,\zeta,T,c(T),c_e(T),c_\sigma(T),C_{q,\eta})(1+\mathbf{E}||\xi||^q_{L^2_\rho}) \\ & + c(q,T,c(T),c_e(T),c_\sigma(T),C_{q,\eta}) \int_0^t (t-s)^{-\frac{q\zeta}{2}}\mathbf{E}||\underline{X}_N(s)||^q_{L^2_\rho}\, ds.\end{aligned}$$

Herefrom, by the Gronwall-Bellman lemma 2.7.2/2.7.3 we get

$$\sup_{t\in[\,0,T\,]} \mathbf{E}||\underline{X}_N(t)||^q_{L^2_\rho} \leq c_2(q,\zeta,T,c(T),c_e(T),c_\sigma(T),C_{q,\eta})(1+\mathbf{E}||\xi||^q_{L^2_\rho}).$$

Since this estimate holds for any $N \in \mathbb{N}$, we conclude

$$\sup_{\substack{t\in[\,0,T\,]\\ N\in\mathbb{N}}} \mathbf{E}||\underline{X}_N(t)||^q_{L^2_\rho} \leq c_2(q,\zeta,T,c(T),c_e(T),c_\sigma(T),C_{q,\eta})(1+\mathbf{E}||\xi||^q_{L^2_\rho}),$$

which is the required $N$-independent estimate on the moments of $\underline{X}_N$.

Let $\bar{X}$ be as in Step 3 in the proof of 7.1.2. Recall from Step 2 that

$$\sup_{\substack{t\in[\,0,T\,]\\ M\in\mathbb{N}}} \mathbf{E}||\bar{X}_{0,M}(t)||^q_{L^2_\rho} \leq c(q,\zeta,T,c(T),c_f(T),c_\sigma(T),c_\gamma(T),C_{q,\eta})(1+\mathbf{E}||\xi||^q_{L^2_\rho})$$

in **(i)** resp.

$$\sup_{\substack{t\in[\,0,T\,]\\ M\in\mathbb{N}}} \mathbf{E}||\bar{X}_{0,M}(t)||^q_{L^2_\rho} \leq c(q,\zeta,T,c(T),c_e(T),c_\sigma(T),C_{q,\eta})(1+\mathbf{E}||\xi||^q_{L^2_\rho})$$

in **(ii)**.

Thus, we get

$$\sup_{t\in[\,0,T\,]} \mathbf{E}||\bar{X}(t)||^q_{L^2_\rho} \leq c_3(q,\zeta,T,c(T),c_f(T),c_\sigma(T),c_\gamma(T),C_{q,\eta})(1+\mathbf{E}||\xi||^q_{L^2_\rho})$$

in **(i)** resp.

$$\sup_{t\in[\,0,T\,]} \mathbf{E}||\bar{X}(t)||^q_{L^2_\rho} \leq c_4(q,\zeta,K,T,c(T),c_e(T),c_\sigma(T),C_{q,\eta})(1+\mathbf{E}||\xi||^q_{L^2_\rho})$$

in **(ii)**.
By construction, for all $t \in [\,0,T\,]$ we have

$$\underline{X}_N(t) \leq X_N(t) \leq \bar{X}(t) \text{ in } L^2_\rho$$

and this leads to

$$\begin{aligned}
&\sup_{\substack{t\in[\,0,T\,]\\ N\in\mathbb{N}}} \mathbf{E}||X_N(t)||^q_{L^2_\rho}\\
&\leq [c_1(q,\zeta,T,c(T),c_f(T),c_\sigma(T),c_\gamma(T),C_{q,\eta})\\
&+c_3(q,\zeta,K,T,c(T),c_f(T),c_\sigma(T),c_\gamma(T),C_{q,\eta})]\,(1+\mathbf{E}||\xi||^q_{L^2_\rho})
\end{aligned}$$

(in **(i)**) and (in **(ii)**)

$$\begin{aligned}
&\sup_{\substack{t\in[\,0,T\,]\\ N\in\mathbb{N}}} \mathbf{E}||X_N(t)||^q_{L^2_\rho}\\
&\leq [c_2(q,\zeta,T,c(T),c_e(T),c_\sigma(T),C_{q,\eta})\\
&+c_4(q,\zeta,T,c(T),c_e(T),c_\sigma(T),C_{q,\eta})]\,(1+\mathbf{E}||\xi||^q_{L^2_\rho}).
\end{aligned}$$

Thus, we have shown that there are $N$ independent estimates for the $X_N$ both in **(i)** and **(ii)**.
Next, we define our candidates for the solution to (1.1) resp. (1.2).
Note that, by $f_N \downarrow f$ resp. $e_N \downarrow e$, 8.1.5 implies

$$X_{N+1}(t) \leq X_N(t) \ P\text{-a.s.},\ t\in[\,0,T\,],\ N\in\mathbb{N} \tag{8.49}$$

both in **(i)** and **(ii)**.

Analogously to Step 5 in the proof of Theorem 7.1.2, we claim that

$$X(t) := \inf_{N\in\mathbb{N}} X_N(t)\,,\ t\in[\,0,T\,],$$

is a solution in the sense of 5.1.2 **(i)** in both cases.

With the help of the $N$-independent estimates shown above, we get

$$\lim_{N\to\infty} \mathbf{E}||X_N(t) - X(t)||^q_{L^2_\rho} = 0 \tag{8.50}$$

for all $t \in [0,T]$ and

$$\lim_{N\to\infty} \mathbf{E} \int_0^T ||X_N(t) - X(t)||^q_{L^2_\rho}\, dt = 0 \tag{8.51}$$

both in **(i)** and **(ii)** by literally repeating the arguments from Step 5 in the proof of Theorem 7.1.2.

Let us show that $X$ solves equation (1.1) resp. (1.2).
Let us fix $t \in [0,T]$. We denote the process in the right hand side of (5.5) by $K(X)$ and the process in the right hand side of (5.6) by $\bar{K}(X)$. Then, by setting

$$I_N^{(1)} := \mathbf{E}||X(t) - X_N(t)||^2_{L^2_\rho},$$

$$I_N^{(2)} := \mathbf{E}\left|\left|\int_0^t U(t,s)[F(s,X(s)) - F_N(s,X_N(s))]\, ds\right|\right|^2_{L^2_\rho},$$

$$\bar{I}_N^{(2)} := \mathbf{E}\left|\left|\int_0^t U(t,s)[E(s,X(s)) - E_N(s,X_N(s))]\, ds\right|\right|^2_{L^2_\rho},$$

$$I_N^{(3)} := \mathbf{E}\left|\left|\int_0^t U(t,s)[\mathcal{M}_{\Sigma(s,X(s))} - \mathcal{M}_{\Sigma(s,X_N(s))}]\, dW(s)\right|\right|^2_{L^2_\rho},$$

$$\bar{I}_N^{(3)} := \mathbf{E}\left|\left|\int_0^t U(t,s)[\mathcal{M}_{\Sigma(s,X(s))} - \mathcal{M}_{\Sigma(s,X_N(s))}]\, dL(s)\right|\right|^2_{L^2_\rho}$$

and

$$I_N^{(4)} := \mathbf{E}\left|\left|\int_0^t \int_{L^2} U(t,s)[\mathcal{M}_{\Gamma(s,X(s))} - \mathcal{M}_{\Gamma(s,X_N(s))}]x\, \tilde{N}(ds,dx)\right|\right|^2_{L^2_\rho},$$

we get

$$\mathbf{E}||X(t) - K(X)(t)||^2_{L^2_\rho} \le C(I_N^{(1)} + I_N^{(2)} + I_N^{(3)} + I_N^{(4)})$$

for (1.1) resp. for (1.2)

$$\mathbf{E}||X(t) - \bar{K}(X)(t)||^2_{L^2_\rho}$$
$$\le 2(I_N^{(1)} + \bar{I}_N^{(2)} + \bar{I}_N^{(3)}).$$

Analogously to the procedure in Step 5 in the proof of Theorem 7.1.2, we get $I_N^{(i)} \to 0$ as $N \to \infty$ for $i = 1, 2, 3$ and $\bar{I}_N^{(i)} \to 0$ as $N \to \infty$ for $i = 1, 2$.

Thus, it remains to consider $I_N^{(4)}$ and $\bar{I}_N^{(3)}$.
Concerning $I_N^{(4)}$, note that, by the Bichteler-Jacod inequality 2.6.10, **(A2)**, **(QI)** and the Lipschitz property of $\gamma$, we get

$$I_N^{(4)} \le c(q, T, \zeta, c_\gamma(T), C_{q,\eta}) \left( \int_0^t \mathbf{E}||X(s) - X_N(s)||_{L^2_\rho}^q \, ds \right)^{\frac{2}{q}},$$

which converges to 0 as $N \to \infty$ by (8.51).
Concerning $\bar{I}_N^{(3)}$, note that applying the Lévy-Itô- decomposition 2.4.13, analogously to Step 4, we get

$$\bar{I}_N^{(3)} \le c(q, T, \zeta, c_\sigma(T), C_{q,\eta}) \left( \int_0^t \mathbf{E}||X(s) - X_N(s)||_{L^2_\rho}^q \, ds \right)^{\frac{2}{q}},$$

which again converges to 0 as $N \to \infty$ by (8.51).

Thus, $X$ solves (1.1) resp. (1.2) in the sense of 5.1.2 **(i)**.

The required continuity property in $L_\rho^{2\nu}(\Theta)$ follows from the continuity results for stochastic convolutions presented in Section 5.1. In particular, by Remark 5.1.11 **(i)** for the Bochner convolution integral we have continuity of the mapping

$$[\,0, T\,] \ni t \mapsto \int_0^t U(t, s) F(s, X(s)) \, ds \in L_\rho^{2\nu}(\Theta)$$

even in the case of a non-Lipschitz $F$.

Analogously to the proof of the estimates (7.6) and (7.7) in the proof of 7.1.2, we get the estimates (8.1) and (8.3) with the help of the $N$-independent estimates for $X_N$. ■

## 8.4 Proof of Theorem 8.1.3

**Step 1:** This step is completely identical with Step 1 from the proof of 7.1.2.

**Step 2:** We show that we have $M$-independent estimates for $X_{N,M}$, where $X_{N,M}$ is the solution to (1.1) resp. (1.2) with $F$ resp. $E$ being replaced by

$F_{N,M}$ resp. $E_{N,M}$. Analogously to Step 2 in the proof of 7.1.4, we get that the theory from Section 5.2 is applicable.
By 5.2.2 there are processes $X_{N,M} \in \mathcal{G}_\nu(T)$ solving equations (1.1) resp. (1.2), when $f$ resp. $e$ is replaced by $f_{N,M}$ resp. $e_{N,M}$. The solutions are time-continuous in $L^{2\nu}(\Omega; L^{2\nu}_\rho)$. In the case of $\gamma$ resp. $\sigma$ obeying (8.2) resp. (8.4) and $U$ fulfilling **(A7)** they have a càdlàg version.

### (i) Equation (1.1) - The Poisson case

By Theorem 8.1.5 we have

$$X_{N,M}(t) \leq X_{N,M+1}(t) \ P\text{-a.s., for each } t \in [\,0,T\,],$$

where the processes $X_{N,M} \in \mathcal{G}_\nu(T)$ solve (1.1), when $f$ is replaced by $f_{N,M}$. We denote solutions to the equations with initial conditions $\xi^+$, $\xi^-$ resp. 0 and drift $F_{0,M}$, $F^-_{N,M}$ resp. 0 by $\bar{X}_{0,M}$, $\underline{X}_{N,M}$ resp. $V$. Again, by Theorem 8.1.5 we get (8.43) and (8.44) $P$-almost surely for each $t \in [\,0,T\,]$ and arbitrary $N, M \in \mathbb{N}$. This allows us to show the wanted $M$-independent estimate of $X_{N,M}$ by showing $M$-independent estimates for $\bar{X}_{0,M}$ and $\underline{X}_{N,M}$.

Let us first consider $\bar{X}_{0,M}$. We get, for any $t \in [\,0,T\,]$,

$$\mathbf{E}||\bar{X}_{0,M}(t)||^{2\nu}_{L^{2\nu}_\rho} \leq c(\nu)(\bar{I}^{(1)}(t) + \bar{I}^{(2)}_M(t) + \bar{I}^{(3)}_M(t) + \bar{I}^{(4)}_M(t)).$$

Here, $\bar{I}^{(1)}$, $\bar{I}^{(2)}_M$ and $\bar{I}^{(3)}_M$ are as in the proof of 7.1.4. Recall that it is crucial to have the integrability condition **(QI)** with $q = 2\nu^2$ to estimate $\bar{I}^{(2)}_M$.
Finally, for any $t \in [\,0,T\,]$, we have

$$\bar{I}^{(4)}_M(t) := \mathbf{E}\left|\left|\int_0^t \int_{L^2} U(t,s)\mathcal{M}_{\Gamma(s,\cdot,\bar{X}_{0,M}(s))}(x)\,\tilde{N}(ds,dx)\right|\right|^{2\nu}_{L^{2\nu}_\rho}.$$

By Proposition 4.4, **(LC)** and **(LB)** for $\gamma$ and the fact that $\nu < \frac{1}{\zeta}$, we get, for any $t \in [\,0,T\,]$,

$$\begin{aligned}\bar{I}^{(4)}_M(t) \leq &\quad c(\nu,T,c(T),C_{2\nu,\eta})\int_0^t (t-s)^{-\zeta\nu}\mathbf{E}||\Gamma(s,\bar{X}_{0,M}(s))||^{2\nu}_{L^{2\nu}_\rho}\,ds \\ \leq &\quad c(\nu,T,\zeta,T,c(T),c_\gamma(T),C_{2\nu,\eta})\left(1+\int_0^t (t-s)^{-\zeta\nu}\mathbf{E}||\bar{X}_{0,M}(s)||^{2\nu}_{L^{2\nu}_\rho}\,ds\right).\end{aligned}$$

Putting the four estimates together we get, for arbitrary $t \in [\,0,T\,]$.

$$\begin{aligned}&\mathbf{E}||\bar{X}_{0,M}(t)||^{2\nu}_{L^{2\nu}_\rho} \\ &\leq c(\nu,T)\mathbf{E}||\xi||^{2\nu}_{L^{2\nu}_\rho}\end{aligned}$$

$$+c(\nu,\zeta,T,c(T),c_f(T),c_\sigma(T),c_\gamma(T),C_{2\nu,\eta})\left(1+\int_0^t(t-s)^{-\zeta\nu}\mathbf{E}||\bar{X}_{0,M}(s)||^{2\nu}_{L^{2\nu}_\rho}\,ds\right).$$

Thus, by the Gronwall-Bellman lemma 2.7.3, we get

$$\mathbf{E}||\bar{X}_{0,M}(t)||^{2\nu}_{L^{2\nu}_\rho}\leq c(\nu,\zeta,T,c(T),c_f(T),c_\sigma(T),c_\gamma(T),C_{2\nu,\eta})(1+\mathbf{E}||\xi||^{2\nu}_{L^{2\nu}_\rho})$$

for arbitrary $M\in\mathbb{N}$ and $t\in[\,0,T\,]$. So, the $M$-independence of the constant implies

$$\sup_{\substack{t\in[\,0,T\,]\\ M\in\mathbb{N}}}\mathbf{E}||\bar{X}_{0,M}(t)||^{2\nu}_{L^{2\nu}_\rho}\leq c(\nu,\zeta,T,c(T),c_f(T),c_\sigma(T),c_\gamma(T),C_{2\nu,\eta})(1+\mathbf{E}||\xi||^{2\nu}_{L^{2\nu}_\rho}).$$

Next, we consider $\underline{X}_{N,M}$ for arbitrary $N,M\in\mathbb{N}$. We have, for any $t\in[\,0,T\,]$,

$$\mathbf{E}||\underline{X}_{N,M}||^{2\nu}_{L^{2\nu}_\rho}\leq c(\nu)(\underline{I}^{(1)}(t)+\underline{I}^{(2)}_{N,M}(t)+\underline{I}^{(3)}_{N,M}(t)+\underline{I}^{(4)}_{N,M}(t))$$

with $\underline{I}^{(1)}$, $\underline{I}^{(2)}_{N,M}$ and $\underline{I}^{(3)}_{N,M}$ as in Step 2 in the proof of 7.1.3 and

$$\underline{I}^{(4)}_{N,M}(t):=\mathbf{E}\left|\left|\int_0^t\int_{L^2}U(t,s)\mathcal{M}_{\Gamma(s,\underline{X}_{N,M}(s))}\,\tilde{N}(ds,dx)\right|\right|^{2\nu}_{L^{2\nu}_\rho}.$$

By the same arguments as in the case $\bar{I}^{(4)}_M$, for any $t\in[\,0,T\,]$ we get

$$\begin{aligned}\underline{I}^{(4)}_{N,M}(t)\leq\quad & c(\nu,T,c(T),C_{2\nu,\eta})\int_0^t(t-s)^{-\zeta\nu}\mathbf{E}||\Gamma(s,\underline{X}_{N,M}(s))||^{2\nu}_{L^{2\nu}_\rho}\,ds\\ \leq\quad & c(\nu,T,\zeta,T,c(T),c_\gamma(T),C_{2\nu,\eta})\left(1+\int_0^t(t-s)^{-\zeta\nu}\mathbf{E}||\underline{X}_{N,M}(s)||^{2\nu}_{L^{2\nu}_\rho}\,dss\right).\end{aligned}$$

Putting the estimates together, we have

$$\begin{aligned}\mathbf{E}||\underline{X}_{N,M}||^{2\nu}_{L^{2\nu}_\rho}\leq\quad & c(\nu,c(T))\mathbf{E}||\xi||^{2\nu}_{L^{2\nu}_\rho}+c(N,\nu,T,c(T))\\ & c(\nu,\zeta,T,c(T),c_\sigma(T),c_\gamma(T),C_{2\nu,\eta})\left(1+\int_0^t(t-s)^{-\zeta\nu}\mathbf{E}||\underline{X}_{N,M}(s)||^{2\nu}_{L^{2\nu}_\rho}\right)\\ \leq\quad & c(N,\nu,K,T,c(T),c_\sigma(T),c_\gamma(T),C_{2\nu,\eta})(1+\mathbf{E}||\xi||^{2\nu}_{L^{2\nu}_\rho})\\ & +c(\nu,\zeta,T,c(T),c_\sigma(T),c_\gamma(T))\int_0^t(t-s)^{-\zeta}\mathbf{E}||\underline{X}_{N,M}(s)||^{2\nu}_{L^{2\nu}_\rho}\,ds.\end{aligned}$$

Again, by the Gronwall-Bellman lemma 2.7.3 we get

$$\mathbf{E}||\underline{X}_{N,M}||^{2\nu}_{L^{2\nu}_\rho}\leq\underline{c}(N,\nu,\zeta,K,T,c(T),c_\sigma(T),C_{2\nu,\eta})(1+\mathbf{E}||\xi||^{2\nu}_{L^{2\nu}_\rho}).$$

Since the previous estimate holds for arbitrary $t\in[\,0,T\,]$ and $M\in\mathbb{N}$, we get

$$\sup_{\substack{t\in[0,T]\\ M\in\mathbb{N}}} \mathbf{E}||\underline{X}_{N,M}||^q_{L^2_\rho} \le \underline{c}(N,\nu,K,T,c(T),c_\sigma(T),C_{2\nu,\eta})(1+\mathbf{E}||\xi||^{2\nu}_{L^{2\nu}_\rho}).$$

Now, by (8.43) we get

$$\sup_{\substack{t\in[0,T]\\ M\in\mathbb{N}}} \mathbf{E}||X_{N,M}||^{2\nu}_{L^{2\nu}_\rho} \le c(N,\nu,K,T,c(T),c_\sigma(T),C_{2\nu,\eta})(1+\mathbf{E}||\xi||^{2\nu}_{L^{2\nu}_\rho})$$

with $c = \bar{c} + \underline{c}$.

**(ii) Equation (1.2) - The Lévy case**

Let us first consider $\bar{X}_{0,M}$. For any $t \in [\,0,T\,]$ we get

$$\mathbf{E}||\bar{X}_{0,M}(t)||^{2\nu}_{L^{2\nu}_\rho} \le c(\nu)(\bar{I}^{(1)}(t) + \tilde{I}^{(2)}_M(t) + \tilde{I}^{(3)}_M(t))$$

with $\bar{I}^{(1)}$ and $\tilde{I}^{(2)}_M$ as in Step 2 in the proof of 7.1.4 **(ii)** and

$$\tilde{I}^{(3)}_M(t) := \mathbf{E}\left|\left|\int_0^t U(t,s)\mathcal{M}_{\Sigma(s,\cdot,\bar{X}_{0,M}(s))}\,dL(s)\right|\right|^{2\nu}_{L^{2\nu}_\rho}.$$

By the Lévy-Itô decomposition 2.4.13 and with the help of 3.3.5 resp. 3.4.3 (with $\phi = \Gamma$ and $\zeta = 0$ since $W$ is nuclear) resp. 4.4 and the Lipschitz property of $\sigma$, we get

$$\begin{aligned}\tilde{I}^{(3)}_M(t) \le\;& c(\nu,T,c(T))\int_0^t (t-s)^{-\zeta}\mathbf{E}||\Sigma(s,\bar{X}_{0,M}(s))||^{2\nu}_{L^{2\nu}_\rho}\,ds\\ &+c(\nu,\zeta,m,T,c(T),C_{2\nu,\eta})\int_0^t (t-s)^{-\zeta\nu}\mathbf{E}||\Sigma(s,\bar{X}_{0,M}(s))||^{2\nu}_{L^{2\nu}_\rho}\,ds\\ \le\;& c(\nu,\zeta,m,K,T,c(T),c_\sigma(T),C_{2\nu,\eta})\left(1+\int_0^t (t-s)^{-\zeta\nu}\mathbf{E}||\bar{X}_{0,M}(s)||^{2\nu}_{L^{2\nu}_\rho}\,ds\right).\end{aligned}$$

Thus, we get

$$\begin{aligned}&\mathbf{E}||\bar{X}_{0,M}(t)||^{2\nu}_{L^{2\nu}_\rho}\\ &\le c(\nu,T)\mathbf{E}||\xi||^{2\nu}_{L^{2\nu}_\rho}\\ &+c(\nu,\zeta,m,T,c(T),c_e(T),c_\sigma(T),C_{2\nu,\eta})\left(1+\int_0^t (t-s)^{-\zeta\nu}\mathbf{E}||\bar{X}_{0,M}(s)||^{2\nu}_{L^{2\nu}_\rho}\,ds\right)\end{aligned}$$

for arbitrary $t \in [\,0,T\,]$.
Therefore, by the Gronwall-Bellman lemma we have

$$\mathbf{E}||\bar{X}_{0,M}||^{2\nu}_{L^{2\nu}_\rho} \le \bar{c}(\nu,\zeta,m,K,T,c(T),c_e(T),c_\sigma(T),C_{2\nu,\eta})(1+\mathbf{E}||\xi||^{2\nu}_{L^{2\nu}_\rho})$$

for any $t \in [\,0,T\,]$ and any $M \in \mathbb{N}$. This implies

$$\sup_{\substack{t\in[0,T]\\M\in\mathbb{N}}} \mathbf{E}||\bar{X}_{0,M}||^{2\nu}_{L^{2\nu}_\rho} \leq \bar{c}(\nu,\zeta,m,K,T,c(T),c_e(T),c_\sigma(T),C_{2\nu,\eta})(1+\mathbf{E}||\xi||^{2\nu}_{L^{2\nu}_\rho}).$$

Next, we consider $\underline{X}_{N,M}$ for arbitrary $N,M\in\mathbb{N}$. For any $t\in[0,T]$ we get

$$\mathbf{E}||\underline{X}_{N,M}||^{2\nu}_{L^{2\nu}_\rho} \leq c(\nu)(\underline{I}^{(1)}(t)+\underline{I}^{(2)}_{N,M}(t)+\underline{I}^{(3)}_{N,M}(t))$$

with $\underline{I}^{(1)}$ and $\underline{I}^{(2)}_{N,M}$ as in the proof of Theorem 7.1.4 and

$$\underline{I}^{(3)}_{N,M}(t) := \mathbf{E}\left|\left|\int_0^t U(t,s)\mathcal{M}_{\Sigma(s,\underline{X}_{N,M})}\,dL(s)\right|\right|^{2\nu}_{L^{2\nu}_\rho}.$$

Analogously to the consideration of $\tilde{I}^{(3)}_M$ above, we get

$$\underline{I}^{(3)}_{N,M}(t) \leq c(\nu,\zeta,m,K,T,c(T),c_\sigma(T),C_{2\nu,\eta})\left(1+\int_0^t (t-s)^{-\zeta\nu}\mathbf{E}||\underline{X}_{N,M}(s)||^{2\nu}_{L^{2\nu}_\rho}\,ds\right).$$

Putting the estimates together, we have

$$\begin{aligned}&\mathbf{E}||\underline{X}_{N,M}||^{2\nu}_{L^{2\nu}_\rho}\\ &\leq c(\nu,c(T))\mathbf{E}||\xi||^{2\nu}_{L^{2\nu}_\rho}+c(N,\nu,T,c(T))\\ &+c(\nu,\zeta,K,m,T,c(T),c_\sigma(T),C_{2\nu,\eta})\left(1+\int_0^t (t-s)^{-\zeta\nu}\mathbf{E}||\underline{X}_{N,M}(s)||^{2\nu}_{L^{2\nu}_\rho}\,ds\right),\end{aligned}$$

which by the Gronwall-Bellman lemma 2.7.3 gives us

$$\mathbf{E}||\underline{X}_{N,M}||^{2\nu}_{L^{2\nu}_\rho} \leq \underline{c}(N,\nu,\zeta,m,K,T,c(T),c_\sigma(T),C_{2\nu,\eta})(1+\mathbf{E}||\xi||^{2\nu}_{L^{2\nu}_\rho}).$$

Since the previous estimate holds for arbitrary $t\in[0,T]$ and $M\in\mathbb{N}$, we get

$$\sup_{\substack{t\in[0,T]\\M\in\mathbb{N}}} \mathbf{E}||\underline{X}_{N,M}||^{q}_{L^{2}_\rho} \leq \underline{c}(N,\nu,\zeta,m,K,T,c(T),c_\sigma(T),C_{2\nu,\eta})(1+\mathbf{E}||\xi||^{2\nu}_{L^{2\nu}_\rho}).$$

By (8.43) we get

$$\sup_{\substack{t\in[0,T]\\M\in\mathbb{N}}} \mathbf{E}||X_{N,M}||^{2\nu}_{L^{2\nu}_\rho} \leq c(N,\nu,\zeta,m,K,T,c(T),c_\sigma(T),C_{2\nu,\eta})(1+\mathbf{E}||\xi||^{2\nu}_{L^{2\nu}_\rho})$$

with $c=\bar{c}+\underline{c}$.

**Step 3:** For $N,M\in\mathbb{N}$ we define

$$Z_{N,M}(t) := X_{N,M}(t)-X_{N,1}(t)\in L^{2\nu}_\rho(\Theta),\ t\in[0,T].$$

Furthermore, we set

$$Z_N(t) := \sup_{M\in\mathbb{N}} Z_{N,M}(t),\ t \in [\,0,T\,]$$

and

$$X_N(t) := Z_N(t) + X_{N,1}(t),\ t \in [\,0,T\,].$$

By the well-definedness of the $X_{N,M}$, the monotonicity property (8.42) and the $M$-independent estimates on the $X_{N,M}$ shown in Step 2, by almost literally repeating Step 3 from the proof of 7.1.4, we get $X_N \in \mathcal{G}_\nu(T)$ for any $N \in \mathbb{N}$ both in **(i)** and **(ii)**. Furthermore, denoting $\bar{X}$ and $V$ as in Step 3 in the proof of 7.1.2, the processes obey

$$\begin{aligned}\underline{X}_N(t) \le X_N(t) \le \bar{X}(t),\\ \underline{X}_N(t) \le V(t) \le \bar{X}(t),\end{aligned}$$

$P$-almost surely, for any $t \in [\,0,T\,]$.
Finally,

$$\lim_{M\to\infty} \mathbf{E}||X_{N,M}(t) - X_N(t)||^{2\nu}_{L^{2\nu}_\rho} = 0,\ t \in [\,0,T\,], \tag{8.52}$$

and

$$\lim_{M\to\infty} \int_0^T \mathbf{E}||X_{N,M}(t) - X_N(t)||^{2\nu}_{L^{2\nu}_\rho}\, dt = 0 \tag{8.53}$$

both in **(i)** and **(ii)**.
Furthermore, there exist processes $\underline{X}_N$, $\bar{X} \in \mathcal{G}_\nu(T)$ such that

$$\begin{aligned}\lim_{M\to\infty} \int_0^T \mathbf{E}||\underline{X}_{N,M}(t) - \underline{X}_N(t)||^{2\nu}_{L^{2\nu}_\rho}\, ds = 0\\ \lim_{M\to\infty} \int_0^T \mathbf{E}||\bar{X}_{0,M}(t) - \bar{X}(t)||^{2\nu}_{L^{2\nu}_\rho}\, ds = 0.\end{aligned}$$

**Step 4:** The aim of this step is to show that, for any $N \in \mathbb{N}$, the process $X_N \in \mathcal{G}_\nu(T)$ defined in Step 3 solves (1.1) resp. (1.2) in case of $F$ resp. $E$ being replaced by $F_N$ resp. $E_N$.
Furthermore, by the results from Section 5.1 $t \mapsto X_N(t)$ is continuous in $L^{2\nu}(\Omega, \mathcal{F}, P; L^{2\nu}_\rho)$. In particular, by Remark 5.1.11 we have the required continuity property of the Bochner convolutions even in the case of non-Lipschitz $f$ resp. $e$.
Under the additional assumptions that $\gamma$ resp. $\sigma$ obeys (8.2) resp. (8.4) and

$U$ obeys **(A7)**, there even exist càdlàg modifications of the solutions.

**(i) Equation (1.1) - The Poisson case**

For any given $t \in [0,T]$ we have

$$\mathbf{E}\left|\left|X_N(t) - U(t,0)\xi - \int_0^t U(t,s)F_N(s,\cdot,X_N(s))\,ds - \int_0^t U(t,s)\mathcal{M}_{\Sigma(s,\cdot,X_N(s))}\,dW(s)\right.\right.$$
$$\left.\left. - \int_0^t \int_{L^2} U(t,s)\mathcal{M}_{\Gamma(s,\cdot,X_N(s))}(x)\,\tilde{N}(ds,dx)\right|\right|^2_{L^2_\rho}$$
$$\le C[I^{(1)}_{N,M}(t) + I^{(2)}_{N,M}(t) + I^{(3)}_{N,M}(t) + I^{(4)}_{N,M}(t)]$$

with $I^{(1)}_{N,M}$, $I^{(2)}_{N,M}$ and $I^{(3)}_{N,M}$ as in the proof of 7.1.4 and

$$I^{(4)}_{N,M}(t) := \mathbf{E}\left|\left|\int_0^t \int_{L^2} U(t,s)[\mathcal{M}_{\Gamma(s,\cdot,X_N(s))} - \mathcal{M}_{\Gamma(s,\cdot,X_{N,M}(s))}]\,\tilde{N}(ds,dx)\right|\right|^2_{L^2_\rho}.$$

Thus, we have $\lim\limits_{M\to\infty} I^{(i)}_{N,M}(t) \to 0$ for each $i = 1,2,3$, by the same arguments as in the proof of 7.1.4.
Finally, by Itô's isometry w.r.t. compensated Poisson random measures, the integrability condition **(QI)** (which particularly implies the square integrability of $\eta$), **(LC)** for $\gamma$, Hölder's inequality and the fact that

$$\nu > \tfrac{1}{1-\zeta} \iff \tfrac{\nu\zeta}{\nu-1} < 1,$$

we get

$$\begin{aligned} I^{(4)}_{N,M}(T) \le\ & c(c(T), c_\gamma(T), C_{2\nu,\eta}) \int_0^t (t-s)^{-\zeta}\mathbf{E}||X_N(s) - X_{N,M}(s)||^2_{L^2_\rho}\,ds \\ \le\ & c(\nu,\zeta,T,c(T),c_\gamma(T),C_{2\nu,\eta}) \left(\int_0^T \mathbf{E}||X_N(s) - X_{N,M}(s)||^{2\nu}_{L^{2\nu}_\rho}\,ds\right)^{\frac{1}{\nu}}, \end{aligned}$$

which tends to 0 for $M \to \infty$ by (8.53).

Thus, $X_N$ solves the equation, when $F$ is replaced by $F_N$.

**(ii) Equation (1.2) - The Lévy case**

For any given $t \in [0,T]$ we have

$$\mathbf{E}\left|\left|X_N(t) - U(t,0)\xi - \int_0^t U(t,s)E_N(s,\cdot,X_N(s))\,ds - \int_0^t U(t,s)\mathcal{M}_{\Sigma(s,\cdot,X_N(s))}\,dL(s)\right|\right|^2_{L^2_\rho}$$

$\leq C[I^{(1)}_{N,M}(t) + I^{(2)}_{N,M}(t) + I^{(3)}_{N,M}(t)]$

with $I^{(1)}_{N,M}$ and $I^{(2)}_{N,M}$ as in the proof of 7.1.4 and

$$I^{(3)}_{N,M}(t) := \mathbf{E}\left|\left|\int_0^t \int_{L^2} U(t,s)[\mathcal{M}_{\Sigma(s,\cdot,X_N(s))} - \mathcal{M}_{\Sigma(s,\cdot,X_{N,M}(s))}]\, dL(s)\right|\right|^2_{L^2_\rho}.$$

Thus, we have $\lim\limits_{M\to\infty} I^{(i)}_{N,M}(t) \to 0$ for $i = 1,2$ by the same arguments as in the proof of 7.1.4.
By the Lévy-Itô decomposition 2.4.13, the Itô isometries w.r.t. Wiener processes and compensated Poisson random measures, the Lipschitz property of $\sigma$, Hölder's inequality and the fact that

$$\nu > \tfrac{1}{1-\zeta} \iff \tfrac{\nu\zeta}{\nu-1} < 1,$$

we get

$$I^{(3)}_{N,M}(t) \leq c(\nu,\zeta,T,c(T),c_\sigma(T),C_{2\nu,\eta}) \left(\int_0^t \mathbf{E}||X_{N,M}(s) - X_N(s)||^{2\nu}_{L^{2\nu}_\rho}\, ds\right)^{\frac{1}{\nu}},$$

which tends to 0 for $M \to \infty$ by (8.53).

Thus, $X_N$ is a solution to (1.2) in the sense of 5.1.2 for all $N \in \mathbb{N}$.

**Step 5:** As in Step 5 in the proof of 7.1.2, we first show an $N$-independent estimate for the moments of $\underline{X}_N$. Then, we get the required $N$-independent estimate by the fact that $P$-almost surely

$$\underline{X}_N(t) \leq X_N(t) \leq \bar{X}(t)$$

for any $t \in [\,0,T\,]$ and $N \in \mathbb{N}$.
From Steps 2 and 4 we already know that in **(i)**

$$\sup_{t\in[\,0,T\,]} \mathbf{E}||\underline{X}_N(t)||^{2\nu}_{L^{2\nu}_\rho} \leq c(N,\nu,\zeta,T,c(T),c_f(T),c_\sigma(T),c_\gamma(T),C_{2\nu,\eta})(1+\mathbf{E}||\xi||^{2\nu}_{L^{2\nu}_\rho})$$

and, resp. in **(ii)**

$$\sup_{t\in[\,0,T\,]} \mathbf{E}||\underline{X}_N(t)||^{2\nu}_{L^{2\nu}_\rho} \leq c(N,\nu,\zeta,m,T,c(T),c_e(T),c_\sigma(T),C_{2\nu,\eta})(1+\mathbf{E}||\xi||^{2\nu}_{L^{2\nu}_\rho}).$$

**(i) Equation (1.1) - The Poisson case**

We have

$$\mathbf{E}||\underline{X}_N(t)||^{2\nu}_{L^{2\nu}_\rho} \leq c(\nu)\left(\underline{I}^{(1)}(t)+\underline{I}^{(2)}_N(t)+\underline{I}^{(3)}_N(t)+\underline{I}^{(4)}_N(t)\right)$$

for all $t \in [\,0,T\,]$ with $\underline{I}^{(1)}$, $\underline{I}^{(2)}_N$ and $\underline{I}^{(3)}_N$ as in the proof of Theorem 7.1.4 **(i)** and

$$\underline{I}^{(4)}_N(t) := \mathbf{E}\left|\left|\int\limits_0^t\int\limits_{L^2} U(t,s)\mathcal{M}_{\Gamma(s,\cdot,\underline{X}_N(s))}(x)\,\tilde{N}(ds,dx)\right|\right|^{2\nu}_{L^{2\nu}_\rho}.$$

From the proof of Theorem 7.1.4 we get

$$\underline{I}^{(1)}(t) \leq c(\nu,T,c(T))\mathbf{E}||\xi||^{2\nu}_{L^{2\nu}_\rho},$$

$$\underline{I}^{(2)}_N(t) \leq c(\nu,T,c(T),c_f(T))\left(1+\int\limits_0^t \mathbf{E}||\underline{X}_N(s)||^{2\nu}_{L^{2\nu}_\rho}\,ds\right)$$

and

$$\underline{I}^{(3)}_N(t) \leq c(\nu,T,c(T),c_\sigma(T))\left(1+\int\limits_0^t (t-s)^{-\zeta}\mathbf{E}||\underline{X}_N(s)||^{2\nu}_{L^{2\nu}_\rho}\,ds\right).$$

Finally, by **(LC)** and **(LB)** for $\gamma$ and the integrability condition **(QI)** for $\eta$ we get

$$\mathbf{E}\left|\left|\int\limits_0^t\int\limits_{L^2} U(t,s)\mathcal{M}_{\Gamma(s,\cdot,\underline{X}_N(s))}(x)\,N(ds,dx)\right|\right|^{2\nu}_{L^{2\nu}_\rho}$$
$$\leq c(\nu,\zeta,T,c(T),c_\gamma(T),C_{2\nu,\eta})\left(1+\int\limits_0^t (t-s)^{-\zeta\nu}\mathbf{E}||\underline{X}_N(s)||^{2\nu}_{L^{2\nu}_\rho}\,ds\right),$$

which yields

$$\begin{aligned}\mathbf{E}||\underline{X}_N(t)||^{2\nu}_{L^{2\nu}_\rho} \leq\;& c(\nu,\zeta,T,c(T),c_f(T),c_\sigma(T),c_\gamma(T),C_{2\nu,\eta})(1+\mathbf{E}||\xi||^{2\nu}_{L^{2\nu}_\rho})\\ &+c(\nu,T,c(T),c_f(T),c_\sigma(T),c_\gamma(T),C_{2\nu,\eta})\int\limits_0^t (t-s)^{-\zeta\nu}\mathbf{E}||\underline{X}_N(s)||^{2\nu}_{L^{2\nu}_\rho}\end{aligned}$$

for all $t \in [\,0,T\,]$. Thus, by the Gronwall-Bellman lemma 2.7.3, we get

$$\sup_{\substack{t\in[\,0,T\,]\\ N\in\mathbb{N}}} \mathbf{E}||\underline{X}_N(t)||^{2\nu}_{L^{2\nu}_\rho} \leq c_1(\nu,\zeta,T,c(T),c_f(T),c_\sigma(T),c_\gamma(T),C_{2\nu,\eta})(1+\mathbf{E}||\xi||^{2\nu}_{L^{2\nu}_\rho}).$$

**(ii) Equation (1.2) - The Lévy case**

For all $t \in [\,0,T\,]$ we have

$$\mathbf{E}||\underline{X}_N(t)||^{2\nu}_{L^{2\nu}_\rho} \leq c(\nu)\left(\underline{I}^{(1)}(t)+\underline{I}^{(2)}_N(t)+\underline{I}^{(3)}_N(t)\right)$$

with $\underline{I}^{(1)}$ and $\underline{I}^{(2)}_N$ as in **(i)** and

$$\underline{I}_N^{(3)}(t) := \mathbf{E}\left|\left|\int_0^t U(t,s)\mathcal{M}_{\Sigma(s,\cdot,\underline{X}_N(s))}\,dL(s)\right|\right|_{L_\rho^{2\nu}}^{2\nu}.$$

By the Lévy-Itô decomposition 2.4.13, 3.3.5, 3.4.3 (with $\phi = \Sigma$ and $\zeta = 0$ since $W$ is nuclear) and 4.4 we get

$$\underline{I}_N^{(3)}(t) \leq c(\nu,\zeta,m,T,c(T),c_\sigma(T),C_{2\nu,\eta}) \leq \left(1+\int_0^t (t-s)^{-\zeta\nu}\mathbf{E}||\underline{X}_N(s)||_{L_\rho^{2\nu}}^{2\nu}\,ds\right).$$

Together with the estimates on $\underline{I}^{(1)}(t)$ and $\underline{I}_N^{(2)}(t)$ from above, this implies

$$\begin{aligned}\mathbf{E}||\underline{X}_N(t)||_{L_\rho^{2\nu}}^{2\nu} \leq \quad & c(\nu,T)\mathbf{E}||\xi||_{L_\rho^{2\nu}}^{2\nu} \\ & + c(\nu,\zeta,m,T,c(T),c_e(T),c_\sigma(T),C_{2\nu,\eta})\left(1+\int_0^t (t-s)^{-\zeta\nu}\mathbf{E}||\underline{X}_N(s)||_{L_\rho^{2\nu}}^{2\nu}\,ds\right)\end{aligned}$$

for all $t \in [\,0,T\,]$. Then, by Gronwall's lemma, we get

$$\mathbf{E}||\underline{X}_N(t)||_{L_\rho^{2\nu}}^{2\nu} \leq c_2(\nu,\zeta,m,T,c(T),c_e(T),c_\sigma(T),C_{2\nu,\eta})(1+\mathbf{E}||\xi||_{L_\rho^{2\nu}}^{2\nu})$$

for all $N \in \mathbb{N}$. Therefore,

$$\sup_{\substack{t\in[\,0,T\,]\\ N\in\mathbb{N}}} \mathbf{E}||\underline{X}_N(t)||_{L_\rho^{2\nu}}^{2\nu} \leq c(\nu,\zeta,m,T,c(T),c_e(T),c_\sigma(T),C_{2\nu,\eta})(1+\mathbf{E}||\xi||_{L_\rho^{2\nu}}^{2\nu})$$

in **(ii)**.

Next, let us consider the moments of $\bar{X}$.
Recall from Step 2 that

$$\mathbf{E}||\bar{X}_{0,M}(t)||_{L_\rho^{2\nu}}^{2\nu} \leq c_3(\nu,T,c(T),c_f(T),c_\sigma(T),c_\gamma(T),C_{2\nu,\eta})(1+\mathbf{E}||\xi||_{L_\rho^{2\nu}}^{2\nu})$$

in **(i)** and

$$\mathbf{E}||\bar{X}_{0,M}(t)||_{L_\rho^{2\nu}}^{2\nu} \leq c_4(\nu,T,c(T),c_e(T),c_\sigma(T),C_{2\nu,\eta})(1+\mathbf{E}||\xi||_{L_\rho^{2\nu}}^{2\nu})$$

in **(ii)** for any $t \in [\,0,T\,]$ and any $M \in \mathbb{N}$.
Thus, we get

$$\sup_{t\in[\,0,T\,]} \mathbf{E}||\bar{X}(t)||_{L_\rho^{2\nu}}^{2\nu} \leq c(\nu,\zeta,T,c(T),c_f(T),c_\sigma(T),c_\gamma(T),C_{2\nu,\eta})(1+\mathbf{E}||\xi||_{L_\rho^{2\nu}}^{2\nu})$$

in **(i)** and

$$\sup_{t\in[\,0,T\,]} \mathbf{E}||\bar{X}(t)||_{L_\rho^{2\nu}}^{2\nu} \leq c(\nu,\zeta,m,K,T,c(T),c_e(T),c_\sigma(T),C_{2\nu,\eta})(1+\mathbf{E}||\xi||_{L_\rho^{2\nu}}^{2\nu})$$

in **(ii)**. Both in **(i)** and **(ii)** we have

$$\underline{X}_N(t) \leq X_N(t) \leq \bar{X}(t) \text{ in } L^{2\nu}_\rho(\Theta).$$

Thus,

$$\sup_{\substack{t\in[0,T] \\ N\in\mathbb{N}}} \mathbf{E}||X_N(t)||^{2\nu}_{L^{2\nu}_\rho}$$
$$\leq [c_1(\nu, T, c(T), c_f(T), c_\sigma(T), c_\gamma(T), C_{2\nu,\eta})$$
$$+c_3(\nu, \zeta, T, c(T), c_f(T), c_\sigma(T), c_\gamma(T), C_{2\nu,\eta})]\,(1 + \mathbf{E}||\xi||^{2\nu}_{L^{2\nu}_\rho})$$

resp.

$$\sup_{\substack{t\in[0,T] \\ N\in\mathbf{N}}} \mathbf{E}||X_N(t)||^{2\nu}_{L^{2\nu}_\rho}$$
$$\leq [c_2(\nu, T, c(T), c_e(T), c_\sigma(T), C_{2\nu,\eta}) + c_4(\nu, \zeta, T, c(T), c_e(T), c_\sigma(T), C_{2\nu,\eta})]\,(1+$$
$$\mathbf{E}||\xi||^{2\nu}_{L^{2\nu}_\rho}),$$

which proves the required $N$-independent estimates for $X_N$.

We claim that

$$X(t) := \inf_{N\in\mathbb{N}} X_N(t),\ t \in [\,0, T\,],$$

defines a solution in the sense of 5.1.2 **(ii)** both for (1.1) and (1.2).

First of all, the $N$-independent estimates of $X_N$ give us the possibility to get

$$\lim_{N\to\infty} \mathbf{E}||X_N(t) - X(t)||^{2\nu}_{L^{2\nu}_\rho} = 0 \tag{8.54}$$

for all $t \in [\,0, T\,]$ and

$$\lim_{N\to\infty} \mathbf{E}\int_0^T ||X_N(t) - X(t)||^{2\nu}_{L^{2\nu}_\rho}\, dt = 0 \tag{8.55}$$

both in **(i)** and **(ii)**.
Defining $K$, $\bar{K}$, $I^{(i)}_N$, $1 \leq i \leq 3$, $\bar{I}^{(i)}_N$, $1 \leq i \leq 2$, as in the proof of 7.1.4 and $I^{(4)}_N$ and $\bar{I}^{(3)}_N$ as in the proof of 8.1.1, we get

$$\mathbf{E}||X(t) - K(X)(t)||^2_{L^2_\rho} \leq C(I^{(1)}_N + I^{(2)}_N + I^{(3)}_N + I^{(4)}_N)$$

for (1.1) resp. (for (1.2))

$$\mathbf{E}||X(t) - \bar{K}(X)(t)||^2_{L^2_\rho} \qquad \leq 2(I_N^{(1)} + \bar{I}_N^{(2)} + \bar{I}_N^{(3)}).$$

Analogously to the procedure in Step 5 in the proof of Theorem 7.1.4, we get $I_N^{(i)} \to 0$ as $N \to \infty$ for $i = 1, 2, 3$ and $\bar{I}_N^{(i)} \to 0$ as $N \to \infty$ for $i = 1, 2$.

Thus, it remains to consider $I_N^{(4)}$ and $\bar{I}_N^{(3)}$.

By Itô's isometry w.r.t. compensated Poisson random measures, **(A2)** for $U$, **(QI)** for $\eta$, **(LC)** for $\gamma$, Hölder's inequality and the fact that

$$\nu > \tfrac{1}{1-\zeta} \iff \tfrac{\nu\zeta}{\nu-1} < 1,$$

we get

$$\begin{aligned} I_N^{(4)}(t) \leq \quad & c(c(T), c_\gamma(T), C_{2\nu,\eta}) \int_0^t (t-s)^{-\zeta} \mathbf{E}||X(s) - X_N(s)||^2_{L^2_\rho}\, ds \\ \leq \quad & c(\nu, \zeta, T, c(T), c_\gamma(T), C_{2\nu,\eta}) \left( \int_0^T \mathbf{E}||X(s) - X_N(s)||^{2\nu}_{L^{2\nu}_\rho}\, ds \right)^{\frac{1}{\nu}}, \end{aligned}$$

which tends to 0 for $N \to \infty$ by (8.55).

Analogously, by the Lévy-Itô decomposition 2.4.13, the Itô isometries w.r.t. Wiener processes and compensated poisson random measures, **(A2)** for $U$, **(QI)** for $\eta$, **(LC)** for $\gamma$, Hölder's inequality and the fact that

$$\nu > \tfrac{1}{1-\zeta} \iff \tfrac{\nu\zeta}{\nu-1} < 1,$$

we get

$$\bar{I}_N^{(3)}(t) \leq c(\nu, \zeta, T, c(T), c_\sigma(T), C_{2\nu,\eta}) \left( \int_0^t \mathbf{E}||X(s) - X_N(s)||^{2\nu}_{L^{2\nu}_\rho}\, ds \right)^{\frac{1}{\nu}},$$

which tends to 0 for $N \to \infty$ by (8.55).

Thus, $X$ solves (1.1) resp. (1.2) in the sense of 5.1.2 **(ii)**.
Again, the requested continuity properties follow from Section 5.1. In particular, the continuity property for the Bochner stochastic convolution follows from Remark 5.1.11 **(ii)**.

Analogously to the proof of the estimates (7.8) and (7.9) in the proof of 7.1.2, we get the estimates (8.5) and (8.6) with the help of the $N$-independent estimates for $X_N$. ■

# Appendix A

# Sobolev spaces on general domains $\Theta \subset \mathbb{R}^d$ and a version of Sobolev's embedding theorem

In this section, we recall the definition of Sobolev spaces $W^{m,p}(\Theta)$ for integer $m \geq 1$ and real $p \geq 1$, on domains $\Theta \subset \mathbb{R}^d$ for arbitrary $d \in \mathbb{N}$. As usual under the term **domain** we understand non-empty, open subsets of $\mathbb{R}^d$.
Thereafter, we discuss a version of Sobolev's embedding theorem presented in the paper [2] and the monograph [1] by Adams and Fournier. This version of Sobolev's embedding theorem holds for a large class of domains obeying the so-called *weak cone property* (see Definition A.1 below for the explanation of this property).
The general theory was first introduced by Sobolev in [104] and later refined e.g. by Gagliardo in [43] and Morrey in [86]. For a more recent overview on Sobolev's embedding theorem, see Chapter 4 in the book [1] by Adams and Fournier.

Let us first recall the following regularity condition for the domain $\Theta \subset \mathbb{R}^d$ (cf. Section 1 of [2]), which is supposed to hold in the main result in [2].

**Definition A.1:** *Given $\theta \in \Theta \subset \mathbb{R}^d$, denote by $R(\theta)$ the set of all points $\xi \in \Theta$ such that the line segment joining $\theta$ to $\xi$ lies entirely in $\Theta$. Setting*

$$\Gamma(\theta) := \{\xi \in R(\theta) \,:\, |\xi - \zeta| < 1\},$$

*we say that $\Theta$ fulfills the* ***weak cone property*** *if there exists a constant $\delta > 0$ such that the Lebesgue measure in $\mathbb{R}^d$ of $\Gamma(\theta)$ is at least $\delta$ for all $\theta \in \Theta$.*

**Remark A.2:** ***(i)*** *Obviously, the weak cone property is fulfilled for any open ball of finite radius in $\mathbb{R}^d$. In particular, $\mathbb{R}^d$ itself fulfills the weak cone property.*

***(ii)*** *Let us compare the above property with similar ones that are most often considered in the literature(see e.g. 4.3–4.7 in Chapter IV in [1] or [43] and [86]). Recall the following three classes of domains:*

1. *$\Theta \subset \mathbb{R}^d$ has the **uniform cone property** if there exists a locally finite open cover $(U_n)_{n\in\mathbb{N}}$ of the boundary $\partial\Theta$ and a corresponding sequence $(C_n)_{n\in\mathbb{N}}$ of finite cones, each congruent to some fixed finite cone $C$, such that*

   *(a) For some finite $M$, every $U_n$ has diameter less than $M$.*

   *(b) For some $\delta > 0$, $\Theta_\delta := \{x \in \Theta \,|\, d(x, \partial\Theta) < \delta\} \subset \cup_{n+1}^{\infty} U_n$.*

   *(c) For every $n \in \mathbb{N}$, $\cup_{\theta\in\Theta\cap U_n}(\theta + U_n) =: Q_n \subset \Theta$*

   *(d) For some finite $R$, every collection of $R + 1$ of the sets $Q_n$ from (c) has empty intersection.*

   *The subset $C \subset \mathbb{R}^d$ is called a **finite cone** if there exist $x \in \mathbb{R}^d$ and open balls $B_1$, $B_2$ such that $x$ is the center of $B_1$, $x$ is not contained in $B_2$ and $C = B_1 \cap \{x + \lambda(y - x) \,|\, y \in B_2, \lambda > 0\}$.*

2. *$\Theta \subset \mathbb{R}^d$ has a **strong local Lipschitz boundary** if there exist positive numbers $\delta$ and $M$, a locally finite open cover $(U_n)_{n\in\mathbb{N}}$ of $\partial\Theta$, and for each $U_n$ a real-valued function $f_n$ of $d-1$ real variables, such that the following conditions hold:*

   *(a) For some finite $R$, the collection of $R+1$ of the sets $U_n$ has empty intersection.*

   *(b) For every pair of points $\theta$, $\xi \in \Theta_\delta$ (with $\Theta_\delta$ as in (b) in 1.) such that $|\theta - \xi| < \delta$, there exists an $n$ such that*

   $$\theta, \xi \in V_n := \{x \in U_n \,|\, d(\theta, \partial\Theta) > \delta\}.$$

   *(c) Each function $f_n$ satisfies the Lipschitz condition with the same constant $M$.*

(d) *For some cartesian system* $(\xi_{n,1},...,\xi_{n,N})$ *in* $U_n$ *the set* $\Theta \cap U_n$ *is represented by the inequality* $\xi_{n,N} < f_n(\xi_{n,1},...,\xi_{n,N-1})$.

3. *A bounded domain* $\Theta$ *has the* ***class*** $\mathcal{C}^k$***-regularity property*** *if there exists a locally finite open cover* $(U_n)$ *of* $\partial\Theta$ *and a corresponding sequence* $(\Phi_n)$ *of k-smooth one-to-one transformations (see Section 3.34 in [1] for the definition of this term) taking* $U_n$ *onto* $B_1$*, the open ball of radius* 1 *with center* $0 \in \mathbb{R}^d$*, such that*

   (a) *For some* $\delta > 0$ *and* $\Theta_\delta$ *as defined in 1. we have*

   $$\Theta_\delta \subset \cup_{n=1}^{\infty} \Psi_n(\{\xi \in \mathbb{R}^d \,|\, |\xi| < \tfrac{1}{2}\}),$$

   *where* $\Psi_n := \Phi_n^{-1}$.

   (b) *For some finite* $R$*, every collection of* $R+1$ *of the sets* $U_n$ *has empty intersection.*

   (c) *For each* $n \in \mathbb{N}$, $\Phi_n(U_n \cap \Theta) = \{\xi \in B_1 \,|\, \xi_d > 0\}$.

   (d) *If* $(\Phi_{n,1},...,\Phi_{n,N})$ *and* $(\Psi_{n,1},...,\Psi_{n,N})$ *denote the components of* $\Phi_n$ *and* $\Psi_n$ *respectively, then there exists a finite* $M$ *such that for all* $\alpha$, $|\alpha| \le m$*, for every* $1 \le i \le N$ *and for evry* $n$*, we have*

   $$|D^\alpha \Phi_{n,i}(\theta)| \le M,\ \theta \in U_n,$$

   $$|D^\alpha \Psi_{n,i}(\xi)| \le M,\ \xi \in B_1.$$

*Between these three classes we have the relation* 3. $\Rightarrow$ 2. $\Rightarrow$ 1.*, and the two-dimensional domain*

$$\Theta := \{(x,y) \in \mathbb{R}^2 \,|\, 0 < |x| < 1\,,\ 0 < y < 1\}$$

*is an example of a domain obeying 1. but not 2. and 3..*

***(iii)*** *Obviously, the cone condition from item 1. in* ***(ii)*** *implies the weak cone property. Furthermore, there are many domains satisfying the latter property but not the former (see e.g. Section 1 in [2]).*

Next, we repeat the general definition of Sobolev spaces (cf. Section 2 of [2]).

**Definition A.3:** *Let $\Theta \subset \mathbb{R}^d$ be a domain in $\mathbb{R}^d$. For integer $m \geq 1$ and real $p \geq 1$, the Sobolev space $W^{m,p}(\Theta)$ consists of (equivalence classes of) functions $u \in L^p(\Theta)$, whose distributional derivatives $D^\alpha u$ of orders $|\alpha| \leq m$ also belong to $L^p(\Theta)$, where the $D^\alpha u$ are distributional in the following sense (cf. the section on distribution and weak derivatives in Chapter 1 in [1], p.19–22 there):*
*Given $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d) \in \mathbb{N}^d$ such that $\sum_{i=1}^d \alpha_i = |\alpha|$, a function $v_\alpha \in L^1_{loc}(\Theta)$ is called* ***weak*** *or* ***distributional derivative*** *of u if*

$$\int_\Theta u(\theta) D^\alpha \phi(\theta)\, d\theta = (-1)^{|\alpha|} \int_\Theta v_\alpha(\theta)\phi(\theta)\, d\theta$$

*for every $\phi \in \mathcal{D}(\Theta)$, where $\mathcal{D}(\Theta)$ denotes the space of distributions on $\Theta$. $W^{m,p}(\Theta)$ is a Banach space with norm*

$$||u||_{W^{m,p}(\Theta)} := \left( \sum_{|\alpha| \leq m} ||D^\alpha u||^p_{L^p(\Theta)} \right)^{\frac{1}{p}}. \tag{A.1}$$

In the Sobolev embedding theorem, we also need the following definition of the spaces of smooth bounded functions (cf. Section 2 in [2]):

**Definition A.4:** *For integer $j \geq 0$ we denote by $C_b^j(\Theta)$ the Banach space of functions u posessing bounded, continuous partial derivatives $D^\alpha u$ on $\Theta$ for $0 \leq |\alpha| \leq j$. The norm on $C_b^j(\Theta)$ is*

$$||u||_{C_b^j(\Theta)} = \max_{|\alpha| \leq j} \sup_{y \in \Theta} |D^\alpha u(y)|.$$

*In particular, for $j = 0$ we have $C_b(\Theta) := C_b^0(\Theta)$ with*

$$||u||_{C_b(\Theta)} = \sup_{y \in \Theta} |u(y)|.$$

**Remark A.5:** ***(i)*** *It is well-known (see e.g. [84] and p.52 in [1]) that the intersection $C^{m,p}(\Theta) := W^{m,p}(\Theta) \cap C^\infty$ is dense in $W^{m,p}(\Theta)$.*
*This means that we can define $W^{m,p}(\Theta)$ as the completion of $C^{m,p}(\Theta)$ w.r.t. the norm (A.1).*

***(ii)*** *Given any Banach space X of functions on $\Theta$, we write $W^{m,p}(\Theta) \underset{\rightarrow}{\subseteq} X$ for the embedding of $W^{m,p}(\Theta)$ into X. This embedding is equivalent to the existence of a finite constant C such that, for every $u \in C^{m,p}(\Theta)$, we have*

$$||u||_X \leq C ||u||_{W^{m,p}(\Theta)}.$$

*C is called the embedding constant.*

Adams and Fournier prove the following version of Sobolev's embedding theorem (cf. Theorem 1 in Section 2 of [2]):

**Theorem A.6:** *Let $\Theta$ be a domain in $\mathbb{R}^d$ satisfying the weak cone property.*
*Let $H$ be a $k$-dimensional plane in $\mathbb{R}^d$ with $1 \leq k \leq d$. (If $k = d$, then $H = \mathbb{R}^d$.)*

***Case 1:*** *If either $mp > d$ or $m = d$ and $p = 1$, then*

$$W^{m+j,p}(\Theta) \underset{\rightarrow}{\subseteq} C_b^j(\Theta) \text{ for } j \geq 0.$$

*Moreover,*

$$W^{m,p}(\Theta) \underset{\rightarrow}{\subseteq} L^q(\Theta \cap H) \text{ for } p \leq q \leq \infty.$$

***Case 2:*** *If $mp = d$, then*

$$W^{m,p}(\Theta) \underset{\rightarrow}{\subseteq} L^q(\Theta \cap H) \text{ for } p \leq q \leq \infty,$$

*and in particular*

$$W^{m,p}(\Theta) \underset{\rightarrow}{\subseteq} L^q(\Theta) \text{ for } p \leq q \leq \infty.$$

***Case 3:*** *If $mp < d$ and either $d - mp \leq d$, or $p = 1$ and $d - m \leq k \leq d$, then*

$$W^{m,p}(\Theta) \underset{\rightarrow}{\subseteq} L^q(\Theta \cap H) \text{ for } p \leq q \leq p* := \tfrac{kp}{d-mp}.$$

*In particular,*

$$W^{m,p}(\Theta) \underset{\rightarrow}{\subseteq} L^q(\Theta) \text{ for } p \leq q \leq p* := \tfrac{dp}{d-mp}.$$

*The embedding constants for all the above embeddings depend only on $m$, $n$, $p$, $q$, $j$, $k$, and the constant $\delta$ of the weak cone property.*

**Remark A.7:** (cf. Remark 1 in [2])
***(i)*** *We stress that, in the formulation of Theorem A.6, it is not relevant whether the domain $\Theta$ is bounded or not.*
*The boundedness of $\Theta$ is necessary for the compactness of the embedding operator $W^{m,p}(\Theta) \underset{\rightarrow}{\subseteq} C_b^j(\Theta)$.*

***(ii)*** *It is a natural question, whether the further basic results of Sobolev's embedding theory are still valid for domains obeying the weak cone property.*

*Such results are commonly known for the domains obeying the cone property. Nevertheless the following claims extend to the domains obeying the weak cone property:*

- *the Rellich-Kondrachov theorem, asserting the compactness of certain embeddings of* $W^{m,p}(\Theta)$ *if* $\Theta$ *is bounded;*
- *the closure of* $W^{m,p}(\Theta)$ *under pointwise multiplication of its elements, provided* $mp > d$*;*
- *the analog of Sobolev's embedding theorem for Orlicz-Sobolev spaces.*

# Appendix B

# The Bochner integral in Banach spaces

We collect a general definition and some properties of Bochner integral in Banach spaces; in our context this will be the weighted Lebesgue spaces $L^{2\nu}_{\rho}$, $\nu \geq 1$. The presentation is based on Appendix A from [97] and Chapter II from [30], which treat the more general case of Bochner integration in (not necessarily separable) Banach spaces.

Let $(B, ||\cdot||_B)$ be a Banach space, $\mathcal{B} := \mathcal{B}(B)$ the Borel $\sigma$-algebra of $B$ and $(\Omega, \mathcal{F}, \mu)$ a measure space with finite measure $\mu$.

**Definition B.1:** *By $S_\mu$ we denote the set of functions $f\colon \Omega \to B$ of the form*

$$f = \sum_{k=1}^{n} x_k \mathbf{1}_{A_k},\ x_k \in B,\ A_k \in \mathcal{F},\ 1 \leq k \leq n,\ n \in \mathbb{N}. \tag{B.1}$$

*For any $f \in S_\mu$ there is a representation of form (B.1) such that the $A_k$ are pairwise disjoint. Thus, a seminorm on $S_\mu$ is defined by*

$$\begin{aligned} ||f||_{S_\mu} &:= \int_\Omega ||f||_B \, d\mu \\ &:= \sum_{k=1}^{n} ||x_k||_B \mu(A_k),\ f \in S_\mu. \end{aligned}$$

*In the following, we call elements of $S_\mu$* ***simple functions****. Furthermore, we define the Bochner integral w.r.t. a simple function $f \in S_\mu$ by*

$$\int_\Omega f \, d\mu := \sum_{k=1}^{n} x_k \mu(A_k).$$

*Obviously, we have the bound*

$$\left|\left|\int_{\Omega} f\, d\mu\right|\right|_B \leq \int_{\Omega} ||f||_B\, d\mu, \tag{B.2}$$

*which implies that*

$$(S_\mu, ||\cdot||_{S_\mu}) \ni f \mapsto \int_{\Omega} f\, d\mu \in (B, ||\cdot||_B)$$

*is a linear bounded mapping.*

**Definition B.2:** ***(i)*** *A function $f\colon \Omega \to B$ is called* ***(strongly) measurable*** *if it is Borel-measurable, i.e. $f^{-1}(B) \in \mathcal{F}$ for any $B \in \mathcal{B}$.*

***(ii)*** *Given $1 \leq p < \infty$, by $L^p(\Omega, \mathcal{F}, \mu; B)$ we denote the set of equivalence classes of measurable mappings $f\colon \Omega \to B$ obeying*

$$\int_{\Omega} ||f||_B^p d\mu < \infty. \tag{B.3}$$

*Being equipped with the norm*

$$||f||_{L^p} := \left(\int_{\Omega} ||f||_B^p\, d\mu\right)^{\frac{1}{p}},$$

*$L^p(\Omega, \mathcal{F}, \mu; B)$ is a Banach space.*

Note that $\Omega \ni \omega \mapsto ||f(\omega)||_B \in \mathbb{R}$ is measurable due to the continuity of the norm-function $B \ni x \mapsto ||x||_H \in \mathbb{R}$.

**Proposition B.3:** *The set of $\mathcal{F}/\mathcal{B}$-measurable functions from $\Omega$ to $B$ is closed under the formation of pointwise limits.*

**Proof:** [22], Proposition E.1, p.350.

Furthermore, to construct the Bochner integral for (strongly) measurable functions the following lemma is applied in [97].

**Lemma B.4:** *Let $f\colon \Omega \to B$ be measurable (in the sense of Definiton B.2).*
*Then, there exists a sequence $(f_n)_{n\in\mathbb{N}} \subset S_\mu$ such that for each $\omega \in \Omega$, $||f_n(\omega) - f(\omega)||_H$ is monotonically decreasing to 0.*

**Proof:** [26], Lemma 1.1, p.16.

Now, the Bochner integral can be constructed with the help of the following result; cf. Step 2b in the construction of the Bochner integral in [97] (Appendix A, p. 106 there).

**Proposition B.5:** *$S_\mu$ is a dense subset of $L^1(\Omega, \mathcal{F}, \mu; B)$ w.r.t. $||\cdot||_{L^1}$.*

Lemma B.4 and Proposition B.5 lead to the following definition.

**Definition B.6:** *Let $f\colon \Omega \to B$ be a measurable function (in the sense of Definition B.2 **(i)**) such that $f \in L^1(\Omega, \mathcal{F}, \mu; B)$, i.e. $f$ obeys (B.3) with $p = 1$.*
*By Proposition B.5 there exists a sequence $(f_n)_{n\in\mathbb{N}} \subset S_\mu$ such that*

$$||f_n(\omega) - f(\omega)||_B \downarrow 0 \text{ for all } \omega \in \Omega \text{ as } n \to \infty$$

*and thus, by Lebesgue's dominated convergence theorem,*

$$||f_n - f||_{L^1} \to 0 \text{ as } n \to \infty.$$

*We define the **Bochner integral of** $f$ w.r.t. $\mu$ by*

$$\int_\Omega f\,d\mu := \lim_{n\to\infty} \int_\Omega f_n\,d\mu \text{ (in } L^1(\Omega, \mathcal{F}, \mu; H)). \tag{B.4}$$

*By (B.2) we see that the limit in (B.4) is the same for any sequence $(f_n)_{n\in\mathbb{N}} \subset S_\mu$ approximating $f$ in the $||\cdot||_{L^1}$-norm.*

In this thesis we crucially use the following properties of the Bochner integral, which are taken from Section A.2 in [97].

**Proposition B.7: (Bochner inequality)**
*Let $f \in L^1(\Omega, \mathcal{F}, \mu; B)$. Then,*

$$\left|\left|\int_\Omega f\,d\mu\right|\right|_B \leq \int_\Omega ||f||_B d\mu.$$

**Proposition B.8: (Continuity)**
*Let $f \in L^1(\Omega, \mathcal{F}, \mu; B)$. Then,*

$$\int_\Omega L \circ f\,d\mu = L\left(\int_\Omega f\,d\mu\right),$$

*where $L \in \mathcal{L}(B, \tilde{B})$ with $\tilde{B}$ being another Banach space.*

We have an analogue of Lebesgue's dominated convergence theorem for Bochner integrals.

**Proposition B.9:** *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of Bochner integrable $B$-valued functions on $\Omega$.*
*If $f = \lim\limits_{n\to\infty} f_n$ in $\mu$-measure, i.e.*

$$\lim_{n\to\infty} \mu(\{||f_n - f||_B > \varepsilon\}) = 0 \text{ for all } \varepsilon > 0,$$

*and if there exists a $\mu$-integrable function $g\colon \Omega \to \mathbb{R}_+$ such that $||f_n||_B \leq g$, $\mu$-a.s., then $f$ is Bochner integrable and*

$$\lim_{n\to\infty} \left|\left| \int\limits_{\Omega} f_n \, d\mu - \int\limits_{\Omega} f \, d\mu \right|\right|_B \leq \lim_{n\to\infty} \int\limits_{\Omega} ||f_n - f||_B \, d\mu = 0.$$

# Appendix C

# Comparison theorems in the case of finite-dimensional SDE with Poisson and Lévy noise

In this chapter, we collect some comparison results for finite-dimensional stochastic differential equations (SDEs) with jump noise.

In the theory of SDEs, comparison theorems describe the relation between solutions for a pair of equations with the same noises but different coefficients. At first, comparison theorems have been done in the case of finite-dimensional SDEs with Wiener noise. A standard reference here is Chapter VI in the book [53] of Ikeda and Watanabe. These classical results can also be extended to the case of discontinuous noise. Below we briefly review the main contributions to the comparison theorems for jump diffusions obtained independently by Galchuk (cf. [44], [45]), Krasin and Melnikov (cf. [67]) resp. by Peng and Zhu (cf. [92] and [113]).

We divide our considerations into two subsections treating the above mentioned groups of results.
In Section C.1, we follow the ideas of Peng and Zhu from [92] and [113]. Peng and Zhu([92]) first in the one-dimensional and later Zhu ([113]) in the multi-dimensional case show a comparison result for certain SDEs driven by Wiener and Poisson noise (see equation (C.1) below) by using the concept of viscosity solutions (which we will briefly explain below). Assuming the unique solvability of the equations under consideration, which is achieved by assumptions (C.2)-(C.3) below, they prove **necessary(!)** and **sufficient** conditions for a one- resp. multi-dimensional comparison theorem in the case of deterministic initial conditions and coefficients. Since the one-

dimensional result in [92] follows from the multi-dimensional result in [113] (cf. Corollary C.1.3 below), we just present the setting and results in the multi-dimensional case.

In Section C.2, we consider the comparison theorem of Krasin and Melnikov from [67]. On the one hand, in contrast to Section C.1, the results from Section C.2 offer just sufficient but not necessary conditions on the coefficients in the case of a finite-dimensional SDE with Wiener and Poisson noises. On the other hand, in contrast to Section C.1, the results from Section C.2 allow for random coefficients. We stress that [67] extends the corresponding one-dimensional comparison results obtained in the earlier papers of Rong (cf. [101]) and Galchuk (cf. [44]). The way of proving follows the standard scheme presented in the book [53] by Ikeda and Watanabe (cf. Chapter VI there). This scheme is based on Itô's formula.
As we will see, the conditions on the coefficients in Sections C.1 and C.2 will be in full consistency with each other (see Remark C.2.2 **(ii)** below).

## C.1 The scheme and main result from [92] and [113]

Given some finite time horizon $T > 0$ and a starting time $t \in [\,0, T\,]$, in [113] Zhu considers a pair of $d$-dimensional SDEs (with $d \in \mathbb{N}$)

$$\begin{aligned} X_s^1 &= x_1 + \int_t^s b^1(r, X_r^1)\,dr + \int_t^s \sigma^1(r, X_r^1)\,dW_r + \int_t^s \int_Z \gamma^1(r, X_{r-}^1, z)\,\tilde{N}(dz, dr) \\ X_s^2 &= x_2 + \int_t^s b^2(r, X_r^2)\,dr + \int_t^s \sigma^2(r, X_r^2)\,dW_r + \int_t^s \int_Z \gamma^2(r, X_{r-}^2, z)\,\tilde{N}(dz, dr) \end{aligned} \tag{C.1}$$

on the intervall $[\,t, T\,]$. Here, $W$ is a $d$-dimensional standard Brownian motion and $\tilde{N}$ is a compensated Poisson random measure on $\mathbb{R}_+ \times Z$, where $Z \subset \mathbb{R}^l$ is equipped with its Borel $\sigma$-algebra $\mathcal{B}(Z) := \mathcal{B}(\mathbb{R}^l) \cap Z$, $l \in \mathbb{N}$. For the corresponding Lévy intensity measure $n$ we have

$$\int_Z n(dz) < \infty.$$

Solutions $(X_s^i)_{s\in[\,t,T\,]} \subset \mathbb{R}^d$, $i = 1, 2$, to (C.1) take values in the Banach space $\mathbb{H}_d^2(T)$ of càdlàg adapted $d$-dimensional processes such that

$$\mathbf{E}\left[\sup_{t\in[\,0,T\,]} |X_t|^2\right] < \infty.$$

Furthermore, they obey $P$-almost surely for all $s \in [t, T]$

$$X_s^i = x_i + \int_t^s b^1(r, X_r^i)\, dr + \int_t^s \sigma^1(r, X_r^i)\, dW_r + \int_t^s \int_Z \gamma^i(r, X_{r-}^i, z)\, \tilde{N}(dz, dr).$$

The (nonrandom) drift and diffusion coefficients are assumed to have the following properties:

(C.2) $b^i$: $\mathbb{R}_+ \times \mathbb{R}^m \to \mathbb{R}^m$, $\sigma^i$: $\mathbb{R}_+ \times \mathbb{R}^m \to \mathbb{R}^{m\times d}$ and $\gamma^i$: $\mathbb{R}_+ \times \mathbb{R}^m \times \mathbb{R}^l \to \mathbb{R}^m$ are continuous functions in $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^m$;

(C.3) for all $x, x' \in \mathbb{R}^m$, $z \in Z$ and $t \geq 0$,

$$\begin{aligned}
&|b^i(t,x) - b^i(t,x')| + |\sigma^i(t,x) - \sigma^i(t,x')| \leq \mu|x - x'|,\\
&|\gamma^i(t,x,z) - \gamma^i(t,x',z)| \leq \rho(z)|x - x'|,\\
&|b^i(t,x)| + |\sigma^i(t,x)| \leq \mu(1 + |x|),\\
&|\gamma^i(t,x,z)| \leq \rho(z)(1 + |x|),
\end{aligned}$$

for sufficiently large $\mu > 0$ and a function $\rho$: $\mathbb{R}^l \to \mathbb{R}_+$ with

$$\int_Z \rho^2(z) n(dz) < \infty.$$

It is well-known that under these assumptions there are unique solutions to both equations in (C.1).

As already mentioned in the introduction to this appendix, the proof of the comparison result in [95] is based on the concept of viscosity solution, which we will now describe briefly.

**Definition C.1.1:** *Consider the linear parabolic PDE*

$$\text{(C.4)}\quad \begin{aligned}&\mathcal{L}u(t,x) + \mathcal{N}u(t,x) - Cu(t,x) + d_K^2(x) = 0,\ (t,x) \in (0,T) \times \mathbb{R}^d,\\ &u(T,x) = d_K^2(x),\end{aligned}$$

*where $C$ is a positive constant and $d_K$ denotes the distance function of a closed set $K \subset \mathbb{R}^m$. The operators $\mathcal{L}$ and $\mathcal{N}$ are defined for $\varphi \in C^{1,2}([0,T] \times \mathbb{R}^m)$ (i.e. for functions, which are continuously differentable in $t$ and twice continuously differentiable in $x$) by*

$$\mathcal{L}\varphi(t,x) = \frac{\partial\varphi(t,x)}{\partial t} + < D\varphi(t,x), b(t,x) > + \frac{1}{2} tr[D^2\varphi(t,x)\sigma\sigma^T(t,x)]$$

*and*

$$\mathcal{N}\varphi(t,x) = \int_Z [\varphi(t, x + \gamma(t,x,z)) - \varphi(t,x) - < D\varphi(t,x), \gamma(t,x,z) >]\, n(dz).$$

*A function $u$: $[0,T] \times \mathbb{R}^d \to \mathbb{R}$, which is continuous in $(t,x)$, uniformly continuous in $t$ and uniformly continuous in $x$, is called a* ***viscosity supersolution (resp. subsolution) to (C.4)*** *if $u(T,x) \geq d_K^2(x)$ (resp. $u(T,x) \leq d_K^2(x)$) and for any $\varphi \in C^{1,2}([0,T] \times \mathbb{R}^d)$ such that $\varphi$ is of at most quadratic growth in $x$ and at any point $(t,x) \in [0,T] \times \mathbb{R}^d$ at which $u - \varphi$ attains its maximum (resp. minimum)*

$$\mathcal{L}u(t,x) + \mathcal{N}u(t,x) - Cu(t,x) + d_K^2(x) \leq 0$$

*resp.*

$$\mathcal{L}u(t,x) + \mathcal{N}u(t,x) - Cu(t,x) + d_K^2(x) \geq 0.$$

*Furthermore, $u$ is called a* ***viscosity solution to (C.4)*** *if $u$ is both a viscosity supersolution and a viscosity subsolution.*

The following proposition is crucial for the theory of Peng and Zhu in [92].

**Proposition C.1.2:** (cf. Proposition 2.2 in [92])
*Consider the SDE (C.1), where $b$, $\sigma$ and $\gamma$ are as in (C.2), (C.3). Setting, in (C.4),*

$$C := 1 + 2\mu + \mu^2 + \int_Z \rho^2(z)\, n(dz)$$

*and*

$$u(t,x) := \mathbf{E}\left[\int_t^T e^{-C(s-t)} d_K^2(X_s^{t,x})\, ds + e^{-C(T-t)} d_K^2(X_T^{t,x})\right],\ (t,x) \in [0,T] \times \mathbb{R}^d,$$

*for some closed set $K \subset \mathbb{R}^d$, $u$ is a viscosity solution to (C.4) with $C$ as above* ***if and only if*** *for each $(t,x) \in [0,T] \times K$ we have $X_s^{t,x} \in K$ for all $s \in [t,T]$ $P$-almost surely.*

Based on Proposition C.1.2, in [113] Zhu shows the following comparison theorem:

**Theorem C.1.3:** (cf. Theorem 3.1 from [113], p. 4)
*Let $b^i$, $\sigma^i$ and $\gamma^i$ be as above and let us set $K := \mathbb{R}_+^m \times \mathbb{R}^m$ and $\bar{X} := (X^1 - X^2, X^2)$, $\bar{x} := (x_1 - x_2, x_2)$, $\bar{b} := (b^1 - b^2, b^2)$, $\bar{\sigma} := (\sigma^1 - \sigma^2, \sigma^2)$ and $\bar{\gamma} := (\gamma^1 - \gamma^2, \gamma^2)$.*
*Then, the following two properties are equivalent:*

*(1) $\bar{X}$ solves equation (C.1) with $b$, $\sigma$ and $\gamma$ being replaced by $\bar{b}$, $\bar{\sigma}$ and $\bar{\gamma}$.*

*Furthermore,* $\bar{X}_s^{t,\bar{x}} \in K$ *for all* $s \in [\,t_0, T\,]$ *P-a.s.;*

*(2)* $\sigma^1 = \sigma^2$ *and for any* $t \in [\,0, T\,]$, $k = 1, 2, \ldots, d$,

*(a) the k-th row of* $\sigma^i(x)$ *only depends on the k-th component* $x_k$ *of* $x \in \mathbb{R}^d$*;*

*(b) for all* $x' \in \mathbb{R}^d$ *and* $x \in \mathbb{R}^d_+$
$x_k + \gamma_k^1(t, x + x', z) - \gamma_k^2(t, x + x', z) \geq 0$, $n(dz)$*-a.s.;*

*(c) for all* $x', \delta^k x \in \mathbb{R}^d$ *such that* $\delta^k x \geq 0$ *and the k-th component of* $\delta^k x$ *is zero, we have*
$b_k^1(t, \delta^k x + x') - \int_Z \gamma_k^1(t, \delta^k x + x', z)\, n(dz) \geq b_k^2(t, x') - \int_Z \gamma_k^2(t, x', z)\, n(dz).$

We proceed with two corollaries in the one-dimensional case, in order to establish a relation to the foregoing paper [92].

**Corollary C.1.4:** (cf. Corollary 3.3 from [113], p. 7)
*Let* $m = d = 1$ *and let the coefficients fulfill (C.2) and (C.3). Then, given initial conditions* $x_1 \leq x_2 \in \mathbb{R}$, $K := \mathbb{R}_+ \times \mathbb{R}$ *and* $\bar{X}$, $\bar{x}$, $\bar{b}$, $\bar{\sigma}$ *and* $\bar{\gamma}$ *as in the formulation of Theorem C.1.3, the following are equivalent:*

- $\bar{X}$ *solves equation (C.1) with* $b$, $\sigma$ *and* $\gamma$ *being replaced by* $\bar{b}$, $\bar{\sigma}$ *and* $\bar{\gamma}$*. Furthermore,* $\bar{X}_s^{t,\bar{x}} \in K$ *for all* $s \in [\,t_0, T\,]$ *P-a.s.;*
- *For any* $t \in [\,0, T\,]$, $x \in \mathbb{R}$,

  *(1)* $\sigma^1(t, x) = \sigma^2(t, x)$*;*

  *(2)* $x_1 + \gamma^1(t, x_1, z) \geq x_2 + \gamma^2(t, x_2, z)$, *for all* $x_1 \geq x_2$, $n(dz)$*-a.s.;*

  *(3)* $b^1(t, x) - \int_Z \gamma^1(t, x, z)\, n(dz) \geq b^2(t, x) - \int_Z \gamma^2(t, x, z)\, n(dz)$.

Note that in Corollary C.1.4 we do not require that the jump coefficients of the equations in (C.1) coincide as it was the case in the main result of [92] (see Theorem 3.1, p. 375 there). Nevertheless, we also get Theorem 3.1 from [92] resp. the main comparison result in the Wiener case (cf. Theorem 6.1.1, p.352/353 in [53]) in the following way.

**Corollary C.1.5:** (cf. Corollary 3.4 and 3.5 in [113], p.7/8 there)
***(i)*** *Additionally assuming* $\gamma^1 = \gamma^2$ *in Corollary C.1.4, the necessary and sufficient conditions change as follows:*
*For any* $t \in [\,0, T\,]$, $x \in \mathbb{R}$, *we have*

- $\sigma^1(t, x) = \sigma^2(t, x)$*;*

- $x_1 + \gamma^1(t, x_1, z) \geq x_2 + \gamma^1(t, x_2, z)$, *for all* $x_1 \geq x_2$, $n(dz)$*-a.s.;*
- $b^1(t, x) \geq b^2(t, x)$.

*These are just the conditions from Theorem 3.1 in [92].*

***(ii)*** *Additionally assuming* $\gamma^1 = \gamma^2 = 0$ *in Corollary C.1.4, the necessary and sufficient conditions become even simpler, since the second assumption from* ***(i)*** *becomes superfluous and we are left with the following:*
*For any* $t \in [0, T]$, $x \in \mathbb{R}$, *we have*

- $\sigma^1(t, x) = \sigma^2(t, x)$,
- $b^1(t, x) \geq b^2(t, x)$.

*These are just the classical assumptions from Theorem 6.1.1 in [53].*

**Remark C.1.6:** ***(i)*** *Note that the equation (C.1) is particularly useful to get comparison theorems for equation (1.2) in finite dimensions. Indeed, assuming that* $(L(t))_{t\in[0,T]}$ *is a Lévy process in* $\mathbb{R}^l$ *for some* $l \in \mathbb{N}$ *obeying*

$$\int_{\mathbb{R}^l} |x|^2 \, \eta(dx) < \infty, \tag{C.5}$$

*by the special Lévy-Itô decomposition from Lemma 2.4.13 equation (1.2) becomes*

$$\begin{aligned} dX(t) = &\; b(X(t))\, ds + \sigma(X(t))\, dL(s) \\ = &\; (b(X(t)) + \sigma(X(t))m)\, dt + \sigma(X(t))dW(t) \\ &+ \int_{\mathbb{R}^l} \sigma(t, X(t))x\tilde{N}(dt, dx). \end{aligned}$$

***(ii)*** *Obviously, the above results hold true if we consider* $\mathcal{F}_0$*-measurable random initial conditions* $x_1$, $x_2$ *such that* $x_1 \leq x_2$ $P$*-a.s.. This follows from the scheme of proof in [92]. The same can also be derived from the statement of Theorem C.1.3 by using the Markov property of the of the solution* $X$ *resp.* $\bar{X}$ *to (C.4) resp. (C.1).*
*Indeed, keeping the notation from Theorem C.1.3, let us denote by* $p_{t,s}(x)$ *the transition probability of the random initial value at time* $t$*, i.e. the distribution at time* $s \in [t, T]$ *of the solution starting at point* $\bar{x} \in \mathbb{R}^d_+ \times \mathbb{R}^d$ *at time* $t \in [0, T]$*. Furthermore, let* $\bar{\nu}_t$ *be the law at time* $t$ *of the initial value* $\bar{x}$ *and let* $\bar{\nu}_s$ *be the law of the corresponding solution* $\bar{X}^{t,\bar{x}}_s$ *at times* $s \geq t$*.*
*Then, by the Markov property we have*

$$\bar{\nu}_s(d\bar{\xi}) = \int_{\mathbb{R}^d \times \mathbb{R}^d} p_{t,s}(\bar{x}, d\bar{\xi})\, \bar{\nu}_t(d\bar{x}).$$

*Since by Theorem C.1.3 all* $p_{t,s}(\bar{x}, K) = 1$ *for* $\bar{x} \in K$,

$$\bar{\nu}_s(K) = \int_{\mathbb{R}^d\times\mathbb{R}^d} p_{t,s}(\bar{x}, K)\,\bar{\nu}_t(d\bar{x}) = \int_{\mathbb{R}^d\times\mathbb{R}^d} \mathbf{I}_K\,\bar{\nu}_t(d\bar{x}) = \bar{\nu}_t(K) = 1.$$

*Thus, $\bar{\nu}_s$ is concentrated on $K$ if $\bar{\nu}_t$ is, which means that for all $s \in [\,t, T\,]$*

$$P\left(\{\, X_1(s) \geq X_2(s)\,\}\right) = 1.$$

## C.2 The scheme and main result from [67]

In this section, we discuss the comparison results obtained by Galchuk and Rong in the one-dimensional case and by Melnikov in the multi-dimensional case. These results concern a rather general setting of discontinuous semimartingales. Namely suppose that (cf. [67]) for $1 \leq d \in \mathbb{N}$

- $A = (A_1, A_2, ..., A_d)$ is a $d$-dimensional non-decreasing continuous process,
- $M = (M_1, M_2, ..., M_d)$ is a $d$-dimensional continuous local martingale,
- $\mu = (\mu_1, \mu_2, ..., \mu_d)$ is a $d$-dimensional random jump measure with compensator $(\nu_1, \nu_2, ..., \nu_d)$.

Furthermore, all processes are supposed to be càdlàg and the $\nu_i$ are supposed to be continuous.

Consider a pair of SDEs (written in a coordinate form)

$$(C.6)\ dX_l^i(t) = \begin{array}{l} \sum\limits_{j=1}^{d} f_{lj}^{(i)}(X^{(i)}(t-))\,dA_j(t) + \sum\limits_{j=1}^{d} g_{lj}(X_l^{(i)}(t-))\,dM_j(t) \\ +\mathbf{1}_{|u|\leq 1} h_l(u, X_l^{(i)}(t-))\,d(\mu_l(t) - \nu_l(t)) + \mathbf{1}_{|u|>1} k_l(u, X_l^{(i)}(t-))\,d\mu_l(t), \\ l = 1, 2, ..., d, \end{array}$$

with different $f_{lj}^{(i)}$ and initial conditions $\xi_l^{(i)}$, $i = 1, 2$.
Here, the $f_{lj}$, $g_{lj}$,$h_l$ and $k_l$ are predictable functions, which depend on $t$, $y$ and $\omega$ and are continuous in $(t, y)$. So, as compared to Peng and Zhu in [92], Krasin and Melnikov allow for random coefficients.
They extend the method of proof of Ikeda and Watanabe (see [53], Chapter VI there), which is based on Itô's formula, to the jump case.
Of course, one needs some additional technical assumptions to guarantee that solutions exist and are unique.
Without loss of generality, we can assume that all coefficients are globally Lipschitz continuous in $y$ uniformly in $(t, \omega)$, which covers the assumptions

from the comparison results in [53] (see Theorem 6.1.3 there) or [67] (see Theorem 1 there).
In particular, the Lipschitz condition guarantees the existence and uniqueness of solutions to (C.6) in the Banach space $\mathbb{H}_d^2(T)$ of càdlàg adapted, square summable processes introduced above.

Under these assumptions, the following holds.

**Theorem C.2.1:** *Assume that for all $l, j = 1, 2, ..., d$, the functions $f_{l,j}^{(1)}$, $f_{l,j}^{(2)}$, $g_{l,j}$, $h_l$ and $k_l^{(1)}$, $k_l^{(2)}$ are Lipschitz continuous in $x$ uniformly in $(t, \omega)$. Furthermore, suppose that we have*

- *for the initial conditions*

*(C.7)* $$\xi^{(1)} \geq \xi^{(2)}, \ P\text{-a.s.},$$

- *for the drift coefficients for all $\tilde{x}_k \geq x_k$, $k = 1, 2, ..., d$*

*(C.8)* $$f_{lj}^{(2)}(\tilde{x}_1, ..., \tilde{x}_{l-1}, x_l, \tilde{x}_{l+1}, ...) \geq f_{lj}^{(1)}(x_1, ..., x_{l-1}, x_l, \tilde{x}_{l+1}, ...), \ P\text{-a.s., and}$$

- *for the jump coefficients, for all $y \geq x$ and $u \in \mathbb{R}^d \setminus \{0\}$,*

*(C.9)* $$h_l(u, y) \geq h_l(u, x), \ P\text{-a.s.},$$

*(C.10)* $$k_l^{(2)}(u, y) \geq k_l^{(1)}(u, x), \ P\text{-a.s.}.$$

*Then, we have $X^1(t) \geq X^2(t)$ for all $t \in [\,0, T\,]$ $P$-almost surely.*

**Remark C.2.2:** ***(i)*** *In the original result [67] of Krasin and Melnikov there is a strict inequality in (C.8) (cf. Theorem 1, p.173 in [67]). This is a standard formulation if we have existence but not yet uniqueness of the corresponding solutions $X^1$, $X^2$. In particular, this formulation also appears in the one-dimensional comparison result for SDEs with Wiener noise in the book of Ikeda and Watanabe (cf. Theorem 6.1.1 in [53]). As soon as we have uniqueness of the solutions $X^1$, $X^2$, e.g. under the Lipschitz condition (C.3) above, by approximating $f^{(i)}$ by $f^{(i)} + \varepsilon$, $\varepsilon > 0$, we can immediately derive the comparison theorem under the non-strict inequality (C.8) (see also the* *arguments in proving Theorem 6.1.1 in [53]).*

***(ii)*** *The assumptions of Theorem C.2.1 are in full consistency with the ones discussed in Section C.1. Namely, in both theorems we assume the inequality* $b^1 \geq b^2$ *for the drift coefficients and the monotonicity property of the jump coefficient.*

**Remark C.2.3:** ***(i)*** *Equation (C.6) is particularly useful to get comparison theorems for equation (1.2) in finite dimensions. Indeed, suppose that in the setting of Remark C.1.6* ***(i)*** *the Lévy measure* $\eta$ *corresponding to* $L$ ***does not*** *obey the square-integrability property*

$$\int_{\mathbb{R}^l} |z|^2 \, \eta(dz) < \infty.$$

*Then, we do not have the special Lévy-Ito decomposition from Lemma 2.4.13 but just the one from Theorem 2.4.10.*
*Thus, in* $\mathbb{R}^d$*, equation (1.2) becomes*

$$\begin{aligned} dX(t) = \;& b(X(t))\, ds + \sigma(X(t))\, dL(s) \\ = \;& (b(X(t)) + \sigma(X(t))m)\, dt + \sigma(X(t)) dW(t) \\ & + \int_{\mathbb{R}^l} \sigma(t, X(t)) x N(dt, dx) \\ & + \int_{\mathbb{R}^l} \sigma(t, X(t)) x \tilde{N}(dt, dx). \end{aligned}$$

***(ii)*** *In this thesis, we apply Theorem C.2.1 in Section 8.1 to have a comparison result for the processes* $Z_j^{(i)}$ *given for any* $\theta \in \Theta$ *by (cf. equation (8.29))*

$$\begin{aligned} Z_j^{(i)}(t, \theta) = \;& \xi_J^{(i)}(\theta) + \sum_{n=1}^{L} \sqrt{a_n} \int_0^t \sigma_J(s, Z_j^{(i)}(s, \theta)) e_{n,M}(\theta)\, dw_n(s) \\ & + \int_0^t \int_{\mathbb{R}} \gamma_J(s, Z_j^{(i)}(s-, \theta)) u \, \tilde{N}_\theta(ds, du). \end{aligned}$$

*For a fixed* $\theta \in \Theta$*, we can apply Theorem C.2.1 with* $f_j = 0$*,* $j = 1, 2, ..., d$*,* $h = k = \sigma$*,* $\mu = N$ *and* $\nu = \eta$*.*

# Appendix D

# Some remarks on evolution operators fulfilling conditions (A0)–(A8) from Section 3.1

In this chapter, we present examples of evolution operators obeying the assumptions necessary to apply our main results from Chapters 7 and 8.

First, in Section D.1 we present a large class of evolution operators $U$ obeying the conditions **(A0)**–**(A4)** and **(A5)***. These operators are constructed via integral kernels obeying certain regularity properties. For such operators we can apply e.g. the results of Chapter 5 to get existence and uniqueness of solutions to (1.1) resp. (1.2) with Lipschitz coefficients.
Afterwards, in Section D.2 we first present a special case of a one-parameter semigroup $U$ obeying conditions **(A0)**–**(A4)**, **(A5)*** and **(A6)**–**(A8)**. Given such operators we can apply the existence and uniqueness results both from Chapter 7 and 8. Then, we generalize the previous case to the case of a two-parameter semigroup $U$ by allowing for additional time-dependence of the generator. Again, the assumptions **(A0)**–**(A4)**, **(A5)*** and **(A6)**–**(A8)** are fulfilled. Therefore, the theory from Chapters 7 and 8 is applicable for such operators.
The construction in Section D.2 assures that the generator $A(t)$ is an elliptic differential operator of order $m \geq 2$ in $\mathbb{R}^d$.

## D.1 An evolution operator obeying the conditions of Chapter 5

Recall from Section 3.1 (cf. Remark 3.1.1) that conditions **(A0)**–**(A4)** (as well as **(A6)**) have already been used in [76]. In that paper Manthey and Zausinger present two examples (see Examples 2.4 and 2.5 there), which we treat in the following.

Recall the notation $\alpha(\theta) := (1 + |\theta|^2)^{\frac{1}{2}}$, $\theta \in \mathbb{R}^d$, from the Introduction.
For this weight function we have (cf. Lemma 2.4.1, p.51 in [76]) the following lemma.

**Lemma D.1.1:** *There exists a positive constant* $c(\rho)$ *such that*

$$\alpha^{\rho}(\xi)\alpha^{-\rho}(\theta) \leq c(\rho)\alpha^{\rho}(\theta - \xi) \text{ for all } \theta,\, \xi \in \Theta.$$

A large class of evolution operators $U$ satisfying the conditions required in Section 5.1 can be constructed in the following way (see also Example 2.4 in [76]):

Let $U = (U(t,s))_{0 \leq s \leq t \leq T}$ be an almost strong evolution operator fulfilling **(A0)** and **(A1)**. Given $\Theta \subset \mathbb{R}^d$, we assume that $U$ obeys a representation

$$(U(t,s)\varphi)(\theta) = \int_{\Theta} G(t,s,\theta,\xi)\varphi(\xi)\, d\xi,\ \theta \in \Theta,\ \varphi \in L^2_{\rho}, \tag{D.1}$$

with some integral kernel $G$: $\{(t,s) \in \mathbb{R}^2_+ \,|\, 0 \leq s < t \leq T\} \times \Theta \times \Theta \to \mathbb{R}_+$.

Let us note that such construction of an almost strong evolution operator had already been applied by Kotelenez in [65], see Example 1.2 there.

Furthermore, we suppose that

$$\int_{\Theta} G(t,s,\theta,\xi)\alpha^{\rho}(\theta - \xi)\, d\xi + \int_{\Theta} G(t,s,\theta,\xi)\alpha^{\rho}(\xi - \theta) d\theta \leq c(T) \tag{D.2}$$

and

$$\int_{\Theta} G^2(t,s,\theta,\xi)\alpha^{\rho}(\theta - \xi)\, d\xi + \int_{\Theta} G^2(t,s,\theta,\xi)\alpha^{\rho}(\xi - \theta)\, d\theta \leq c(T)(t-s)^{-\zeta} \tag{D.3}$$

uniformly for any $0 \leq s < t \leq T$, $\theta, \xi \in \Theta$, with some $\zeta \in [\,0, 1\,)$ and $c(T) \in \mathbb{R}$. Finally, let $U(t,s) = \mathbf{I}$ if $s = t$.

Applying first Hölder's inequality and (D.2) and then the definition of $\mu_{\rho}$ from the Introduction, Lemma D.1.1 and again (D.2), we have for any $\varphi \in L^2_{\rho}$

and $0 \le s \le t \le T$ (cf. inequality (0.3), p. 51 in [76])

$$
\begin{aligned}
(D.4)\ ||U(t,s)\varphi||^2_{L^2_\rho} &= \int_\Theta \left( \int_\Theta G(t,s,\theta,\xi)\varphi(\xi)\, d\xi \right)^2 \mu_\rho(d\theta) \\
&= \int_\Theta \left( \int_\Theta G^{\frac{1}{2}}(t,s,\theta,\xi) G^{\frac{1}{2}}(t,s,\theta,\xi)\varphi(\xi)\, d\xi \right)^2 \mu_\rho(d\theta) \\
&\le \int_\Theta \left( \int_\Theta G(t,s,\theta,\xi)\, d\xi \right) \left( \int_\Theta G(t,s,\theta,\xi)\varphi^2(\xi)\, d\xi \right) \mu_\rho(d\theta) \\
&\le c(T) \int_\Theta \int_\Theta G(t,s,\theta,\xi)\varphi^2(\xi)\, d\xi \mu_\rho(d\theta) \\
&= c(T) \int_\Theta \int_\Theta G(t,s,\theta,\xi)\varphi^2(\xi)\, d\xi \alpha^{-\rho}(\theta)\, d\theta \\
&\le c(T) \int_\Theta \left[ \int_\Theta G(t,s,\theta,\xi)\alpha^{\rho}(\theta-\xi)\, d\theta \right] \varphi^2(\xi)\alpha^{-\rho}(\xi)\, d\xi \\
&\le c(T) \int_\Theta \varphi^2(\xi)\, \mu_\rho(d\xi) \\
&= c(T)||\varphi||^2_{L^2_\rho} < \infty.
\end{aligned}
$$

Thus, (D.2) guarantees that we have $U(t,s) \in \mathcal{L}(L^2_\rho)$ with an operator norm uniformly bounded in $0 \le s \le t \le T$ as required in Definition 2.1.1 **(iii)**.

Concerning **(A2)** note the following (see also p. 51 in [76]):
Given an arbitrary $\varphi \in L^2_\rho$ with the corresponding multiplication operator $\mathcal{M}_\varphi$: $L^2 \to L^1_\rho$, for any orthonormal basis $(g_n)_{n\in\mathbb{N}} \subset L^2$ and any $0 \le s < t \le T$ we have

$$
\begin{aligned}
\sum_{n\in\mathbb{N}} ||U(t,s)\mathcal{M}_\varphi g_n||^2_{L^2_\rho} &= \sum_{n\in\mathbb{N}} \int_\Theta \left( \int_\Theta G(t,s,\theta,\xi)\varphi(\xi) g_n(\xi)\, d\xi \right)^2 \mu_\rho(d\theta) \\
&= \int_\Theta \sum_{n\in\mathbb{N}} < G(t,s,\theta,\cdot)\varphi, g_n >^2_{L^2}\ \mu_\rho(d\theta) \\
&= \int_\Theta ||G(t,s,\theta,\cdot)\varphi||^2_{L^2}\, \mu_\rho(d\theta) \\
&= \int_\Theta \left( \int_\Theta G^2(t,s,\theta,\xi)\varphi^2(\xi)\, d\xi \right) \alpha^{-\rho}(\theta)\, d\theta \\
&\le c(\rho) \int_\Theta \left[ \int_\Theta G^2(t,s,\theta,\xi)\alpha^{\rho}(\theta-\xi)\, d\theta \right] \varphi^2(\xi)\alpha^{-\rho}(\xi)\, d\xi \\
&\le c(\rho, c(T))(t-s)^{-\zeta}||\varphi||^2_{L^2_\rho} < \infty.
\end{aligned}
$$

Here, we used Lemma D.1.1 in the second last and (D.3) in the last step. Thus, we have $U(t,s)\mathcal{M}_\varphi \in \mathcal{L}_2(L^2, L^2_\rho)$ and the estimate (3.2) on its Hilbert-Schmidt norm, so that $U$ defined by (D.1) really obeys **(A2)**.

Let us now check **(A3)** (see again p. 51 in [76]):
Given $\varphi \in L^{2\kappa}_\rho$, Hölder's inequality and (D.2) imply

$$\begin{aligned}
|U(t,s)\varphi|^\nu &= \left|\int_\Theta G(t,s,\cdot,\xi)\varphi(\xi)\,d\xi\right|^\nu \\
&= \left|\int_\Theta G^{\frac{\nu-1}{\nu}}(t,s,\cdot,\xi)G^{\frac{1}{\nu}}(t,s,\cdot,\xi)\varphi(\xi)\,d\xi\right|^\nu \\
&\le \left(\int_\Theta G(t,s,\cdot,\xi)\,d\xi\right)^{\nu-1}\int_\Theta G(t,s,\cdot,\xi)|\varphi(\xi)|^\nu\,d\xi \\
&= c(\nu,T)U(t,s)|\varphi|^\nu,
\end{aligned}$$

which is just **(A3)**.

Concerning **(A4)** we note that, by Remark 3.1.2.1 **(iii)** (see also Remark 2.3 **(ii)** in [76]), it suffices to consider the cylindrical case, i.e. $Q = \mathbf{I}$ (see also Lemma 2.4.2, p. 52 in [76]):

Let $(\varphi(t))_{t\in[0,T]}$ be an $L^{2\nu}_\rho$-valued predictable process. Similarly to the consideration of **(A2)** we get **(A4)** by the following chain of inequalities:

$$\begin{aligned}
&\mathbf{E}\int_\Theta \left[\sum_{n\in\mathbb{N}}\int_0^t (U(t,s)\mathcal{M}_{\varphi(s)}e_n)^2(\theta)\,ds\right]^\nu \mu_\rho(d\theta) \\
&= \mathbf{E}\int_\Theta \left[\sum_{n\in\mathbb{N}}\int_0^t \left(\int_\Theta G(t,s,\theta,\xi)\varphi(s,\xi)e_n(\xi)\,d\xi\right)^2 ds\right]^\nu \mu_\rho(d\theta) \\
&= \mathbf{E}\int_\Theta \left[\int_0^t ||G(t,s,\theta,\cdot)\varphi(s)||^2_{L^2}\,ds\right]^\nu \mu_\rho(d\theta) \\
&= \mathbf{E}\int_\Theta \left[\int_0^t\int_\Theta G^2(t,s,\theta,\xi)\varphi^2(s,\xi)\,d\xi\,ds\right]^\nu \mu_\rho(d\theta) \\
&= \mathbf{E}\int_\Theta \left[\int_0^t\int_\Theta G^{2-\frac{2}{\nu}+\frac{2}{\nu}}(t,s,\theta,\xi)\varphi^2(s,\xi)\,d\xi\,ds\right]^\nu \mu_\rho(d\theta) \\
&\le \mathbf{E}\int_\Theta \left(\int_0^t \left(\int_\Theta G^2(t,s,\theta,\xi)\,d\xi\right)^{\frac{\nu-1}{\nu}} \left(\int_\Theta G^2(t,s,\theta,\xi)\varphi^{2\nu}(s,\xi)\,d\xi\right)^{\frac{1}{\nu}} ds\right)^\nu \mu_\rho(d\theta) \\
&\le \mathbf{E}\int_\Theta \left(\int_0^t (t-s)^{-\frac{\zeta(\nu-1)}{\nu}} \left(\int_\Theta G^2(t,s,\theta,\xi)\varphi^{2\nu}(s,\xi)\,d\xi\right)^{\frac{1}{\nu}} ds\right)^\nu \mu_\rho(d\theta) \\
&\le \left(\int_0^t s^{-\zeta}\,ds\right)^{\nu-1} \mathbf{E}\int_\Theta \left(\int_0^t\int_\Theta G^2(t,s,\theta,\xi)\varphi^{2\nu}(s,\xi)\,d\xi\,ds\right) \mu_\rho(d\theta) \\
&\le c(\zeta,\nu,T)\mathbf{E}\int_\Theta \left(\int_0^t\int_\Theta G^2(t,s,\theta,\xi)\varphi^{2\nu}(s,\xi)\,d\xi\,ds\right) \mu_\rho(d\theta) \\
&= c(\zeta,\nu,T)\mathbf{E}\int_0^t\int_\Theta\int_\Theta G^2(t,s,\theta,\xi)\varphi^{2\nu}(s,\xi)\,d\xi\,\alpha^{-\rho}(\theta)\,d\theta\,ds \\
&\le c(\rho,\zeta,\nu,T)\mathbf{E}\int_0^t\int_\Theta\int_\Theta G^2(t,s,\theta,\xi)\alpha^\rho(\theta-\xi)\,d\theta\,\varphi^{2\nu}(s,\xi)\alpha^{-\rho}(\xi)\,d\xi\,ds \\
&\le c(\rho,\zeta,\nu,c(T),T)\mathbf{E}\int_0^t (t-s)^{-\zeta}||\varphi(s)||^{2\nu}_{L^{2\nu}_\rho}\,ds.
\end{aligned}$$

Here, we used Hölder's inequality w.r.t. $d\xi$ in the fifth, (D.3) in the sixth, Hölder's inequality w.r.t. $ds$ in the seventh and Lemma D.1.1 in the second last step.
Thus, **(A4)** holds.

So far, we have properties **(A2)**– **(A4)** for $U$ given by (D.1).

Recall from Section 3.1 that to deal with the jump-terms in equations (1.1) and (1.2), we need **(A5)**/ **(A5)***.
For our evolution operator $U$ defined by (D.1), we first show the weaker condition **(A5)*** in the following.

**Lemma D.1.2:** *Let $U$ be defined by (D.1) from $G$ obeying (D.2) and (D.3). Then, $U$ obeys **(A5)***.*

**Proof:** Let $0 \le s < t \le T$ and let $\varphi \in L^{2\nu}_\rho$, $\psi \in L^2$ be fixed.
We claim that $U(t,s)\mathcal{M}_\varphi(\psi) \in L^{2\nu}_\rho$.
Indeed, the following chain of inequalities holds

$$
\begin{aligned}
&\int_\Theta (U(t,s)\mathcal{M}_\varphi(\psi))^{2\nu}(\theta)\,\mu_\rho(d\theta)\\
&= \int_\Theta \left(\int_\Theta G(t,s,\theta,\xi)\varphi(\xi)\psi(\xi)\,d\xi\right)^{2\nu} \mu_\rho(d\theta)\\
&\le \int_\Theta \left[\left(\int_\Theta G^2(t,s,\theta,\xi)\varphi^2(\xi)\,d\xi\right)\left(\underbrace{\int_\Theta \psi^2(\xi)\,d\xi}_{=||\psi||^2_{L^2}}\right)\right]^\nu \mu_\rho(d\theta)\\
&\le ||\psi||^{2\nu}_{L^2}\int_\Theta \left[\int_\Theta G(t,s,\theta,\xi)^{\frac{2(\nu-1)}{\nu}} G(t,s,\theta,\xi)^{\frac{2}{\nu}}\varphi^2(\xi)\,d\xi\right]^\nu \mu_\rho(d\theta)\\
&\le ||\psi||^{2\nu}_{L^2}\int_\Theta \left[\left(\int_\Theta G^2(t,s,\theta,\xi)\,d\xi\right)^{\frac{\nu-1}{\nu}}\left(\int_\Theta G^2(t,s,\theta,\xi)\varphi^{2\nu}(\xi)\,d\xi\right)^{\frac{1}{\nu}}\right]^\nu \mu_\rho(d\Theta)\\
&\le c(\rho,\nu,c(T))||\psi||^{2\nu}_{L^2}(t-s)^{-\zeta(\nu-1)}\int_\Theta\left[\int_\Theta G^2(t,s,\theta,\xi)\alpha^\rho(\theta-\xi)d\theta\right]\varphi^{2\nu}(\xi)\,\mu_\rho(d\xi)\\
&\le c(\nu,T)||\psi||^{2\nu}_{L^2}(t-s)^{-\zeta\nu}||\varphi||^{2\nu}_{L^{2\nu}_\rho},
\end{aligned}
$$

which means that we have

$$||U(t,s)\mathcal{M}_\varphi||^{2\nu}_{\mathcal{L}(L^2,L^2_\rho)} \le c(\nu,T)(t-s)^{-\zeta\nu}||\varphi||^{2\nu}_{L^{2\nu}_\rho}$$

and thus (3.6).

Here, we used Hölder's inequality in the second and third, Lemma D.1.1 in the fourth and (D.3) in the fifth and sixth step. ■

**Remark D.1.3:** ***(i)*** *Based on* ***(A2)*** *and* ***(A5)**** *we can also show* ***(A5)****.* *Note that for any $0 \le s < t \le T$*

$$||U(t,s)\mathcal{M}_\varphi||_{\mathcal{L}_2(L^2,L^{2\nu}_\rho)} \le \left|\left|U\left(t,\tfrac{t+s}{2}\right)\right|\right|_{\mathcal{L}_2(L^2)} \left|\left|U\left(\tfrac{t+s}{2},s\right)\right|\right|_{\mathcal{L}(L^2,L^{2\nu}_\rho)}, \tag{D.5}$$

*where the second term in the right hand side of (D.5) was estimated in* ***(A5)****.
*For the operator $U(t,s)$, given by the integral representation (D.1), we have*

$$||U(t,s)||^2_{\mathcal{L}_2(L^2)} = \int_\Theta\int_\Theta |G(t,s,\theta,\xi)|^2\, d\theta\, d\xi. \tag{D.6}$$

*So, additionally to (D.2) and (D.3), we have to assume that*

$$\int_\Theta\int_\Theta |G(t,s,\theta,\xi)|^2\, d\theta\, d\xi \le c(T)(t-s)^{-\zeta},\ 0 \le s < t \le T, \tag{D.7}$$

*for some $\zeta \in [\,0,1\,)$. Such condition is also typical for evolution families generated by elliptic operators. Then, both (D.5) and (D.6)/(D.7) applies* ***(A5)*** *with $\zeta' = 2\zeta$.*

***(ii)*** *Another approach to* ***(A5)*** *is to prove the estimate*

$$||U(t,s)\mathcal{M}_\varphi||_{\mathcal{L}_2(L^2,L^{2\nu}_\rho)} \le \left|\left|U\left(t,\tfrac{t+s}{2}\right)\right|\right|_{\mathcal{L}_2(L^2,L^2_{\rho_0})} \left|\left|U\left(\tfrac{t+s}{2},s\right)\right|\right|_{\mathcal{L}(L^2_{\rho_0},L^{2\nu}_\rho)}, \tag{D.8}$$

*where $\rho_0 > d$ and $\rho - \rho_0 > d$.*
*Then, the first norm in the right hand side in (D.8) is estimated by* ***(A2)*** *and the second norm is estimated in full analogy with* ***(A5)****. Again we will get* ***(A5)*** *with the new exponent $\zeta' = 2\zeta$.*

***(iii)*** *In Section D.2 below we consider the example of an operator generating a semigroup, which can be represented similar to (D.1) by setting $G(t,s,\theta,\xi) := G(t-s,\theta,\xi)$.*

## D.2 Two examples of evolution operators obeying the conditions of Chapters 7 and 8

In this section we give two examples of evolution operators obeying the conditions from Chapters 7 and 8. The first example is a (one-parameter) $\mathcal{C}_0$-semigroup. This example is built on Example 2.5 from [76], where Manthey and Zausinger construct a $\mathcal{C}_0$-semigroup obeying the properties needed in their theory on SDEs with Wiener noise. The second example is generated from the first example by assuming additional time-depenence of the generator.

**A $\mathcal{C}_0$-semigroup obeying the properties from Chapters 7 and 8**

Let $\Theta = \mathbb{R}^d$ and let $A$ be a second order elliptic differential operator given by

(D.9) $A\varphi(\theta) := \sum\limits_{i,j,=1}^{d} a_{ij}(\theta)\frac{\partial^2}{\partial\theta_i\partial\theta_j}\varphi(\theta) + \sum\limits_{i=1}^{d} b_i(\theta)\frac{\partial}{\partial\theta_i}\varphi(\theta) + c(\theta)\varphi(\theta)$, $\theta \in \mathbb{R}^d$, $\varphi \in \mathcal{D}(A) := W^{2,2}(\mathbb{R}^d)$,

with coefficient functions $a_{ij} = a_{ji}$, $b_i$ and $c$ from the set $C_b^\infty(\mathbb{R}^d)$ of bounded, infinitely differentiable functions on $\mathbb{R}^d$.

Note that this example has already been presented by Manthey and Zausinger (cf. Example 2.5 in [76] or in Appendix B in [92]).

Let us assume (taking $m = 1$ on p.17 in [95]) that the standard ellipticity conditon holds, i.e. there is some $\delta > 0$ such that for all $\theta$, $\xi \in \mathbb{R}^d$

$$\sum_{i,j=1}^{d} a_{ij}(\theta)\xi_i\xi_j \geq \delta|\xi|^2. \tag{D.10}$$

Then, as described in Section B.2 in [95], there exists a continuous Green function $G$: $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ corresponding to the parabolic operator $\frac{\partial}{\partial t} + A$ such that defining $U$ by $U(0) = \mathbf{I}$ and

$$(U(t)\varphi)(\theta) := \int_\Theta G(t,\theta,\xi)\varphi(\xi)\,d\xi,\ \theta \in \Theta,\ \varphi \in L^2_\rho(\Theta),\ t > 0, \tag{D.11}$$

gives us a $\mathcal{C}_0$-semigroup in $L^2_\rho$ (cf. Theorem B.9 in [95]) such that (cf. Theorem B.7 in [95])

$$\sup_{t\in[0,T]} ||U(t)||_{\mathcal{L}(L^2_\rho)} < \infty.$$

It is a well-known fact (see e.g. Lemma 2.1 in [42] (in the case $j = |\alpha| = |\beta| = 0$)) that $G(t,\theta,\xi)$ obeys a sub-Gaussian growth, i.e. there are positive constants $c_1$, $c_2$ such that

$$0 < G(t,\theta,\xi) \le c_1 t^{-\frac{d}{2}} exp\left(-c_2 \frac{|\theta-\xi|^2}{t}\right),\ (t,\theta,\xi) \in (\,0,T\,] \times \Theta \times \Theta. \tag{D.12}$$

To prove the properties **(A2)**–**(A4)** and **(A5)***, we want to apply the theory developed in Section D.1 with $G(t,s,\theta,\xi) := G(t-s,\theta,\xi)$. To this end, cf. the conditions of Lemma D.1.2, we need (D.2) and (D.3) to hold.

Let us first show that $G$ obeys property (D.2).
With the help of (D.12) one gets

$$\begin{aligned}
\int\limits_\Theta G(t,\theta,\xi)\alpha^\rho(\theta-\xi)\,d\xi \le\ & \int\limits_\Theta c_1 t^{-\frac{d}{2}} exp\left(-c_2\frac{|\theta-\xi|^2}{t}\right)(1+|\theta-\xi|^2)^{\frac{\rho}{2}}\,d\xi \\
\le\ & c(\rho)\left[\int\limits_{\mathbb{R}^d} t^{-\frac{d}{2}} exp\left(-\frac{|\xi|^2}{t}\right)\,d\xi\right. \\
& \left. + \int\limits_\Theta |\xi|^\rho t^{-\frac{d}{2}} exp\left(-\frac{|\xi|^2}{t}\right)\,d\xi\right] \\
\le\ & c(\rho,d,T) < \infty.
\end{aligned}$$

Since $\alpha^\rho(\theta-\xi)$ is a symmetric function, we also get

$$\int\limits_\Theta G(t,\theta,\xi)\alpha^\rho(\theta-\xi)\,d\theta \le c(\rho,d,T) < \infty,\ t>0,$$

which implies (D.2).
Thus, it remains to show the estimate (D.3). As we will see from (D.12), in the given example this causes a restriction to the case $\frac{d}{2} < 1$, i.e. $d = 1$ (cf. also Example 2.5, p.54 in [76]).

So let us assume that $d = 1$. Then, (D.12) becomes

$$0 < G(t,\theta,\xi) \le c_1 t^{-\frac{1}{2}} exp\left(-\frac{|\theta-\xi|^2}{t}\right),\ (t,\theta,\xi) \in (\,0,T\,] \times \Theta \times \Theta. \tag{D.13}$$

Applying (D.13) we get

$$\begin{aligned}
\int\limits_\Theta G^2(t,\theta,\xi)\alpha^\rho(\theta-\xi)\,d\xi \le\ & \int\limits_\Theta c_1 t^{-1} exp\left(-\frac{|\theta-\xi|^2}{t}\right)(1+|\theta-\xi|^2)^{\frac{\rho}{2}}\,d\xi \\
\le\ & c(\rho)\left[\int\limits_{\mathbb{R}^d} t^{-1} exp\left(-\frac{|\xi|^2}{t}\right)\,d\xi\right. \\
& \left. + \int\limits_\Theta |\xi|^\rho t^{-1} exp\left(-\frac{|\xi|^2}{t}\right)\,d\xi\right] \\
\le\ & c(\rho,d,T)t^{-\frac{1}{2}} < \infty,\ t>0.
\end{aligned}$$

Similarly to the above consideration of (D.2), by the symmetry of $\alpha^\rho(\theta-\xi)$, we also get

$$\int_\Theta G^2(t,\theta,\xi)\alpha^\rho(\theta-\xi)\,d\theta \le c(\rho,d,T)t^{-\frac{1}{2}} < \infty,\ t>0,$$

which implies (D.3) with $\zeta = \frac{1}{2}$ in the case $d = 1$.

Therefore, we can apply Lemma D.1.2 (cf. also Example 2.4 from [76]) to get the following:

**Lemma D.2.2:** *Suppose that $\Theta = \mathbb{R}$. For functions $a$, $b$ and $c$ from $C_b^\infty(\mathbb{R})$, let $A$ be given by*

$$A := a\frac{\partial^2}{\partial\theta^2} + b\frac{\partial}{\partial\theta} + c with\ \mathcal{D}(A) := W^{2,2}(\mathbb{R}).$$

*Furthermore, suppose that $a$ obeys (D.10), i.e. $A$ is an elliptic operator. Let $G$ be the Green function corresponding to the operator $\frac{\partial}{\partial t} + A$ on $\mathbb{R}^d$. Then, $U$ defined by (D.11) constitutes an almost strong evolution operator fulfilling* ***(A0)–(A4)*** *and* ***(A5)****.

So far, we have constructed a $\mathcal{C}_0$-semigroup in $L^2_\rho(\Theta)$ obeying **(A0)–(A4)** and **(A5)***. These properties are just enough to apply the theory from Chapter 5 (without having càdlàg properties of the solutions (this would additionally require **(A7)**)). As was shown in Remark D.1.3, **(A5)*** implies **(A5)** with a modified parameter $\zeta$.

To be able to apply the comparison theory from Chapter 6, we need the approximation property **(A6)**. We refer to Example 2.6 in [76], where Manthey and Zausinger describe how to gain a family $(A_N)_{N\in\mathbb{N}}$ of operators approximating $A$ in the sense of **(A6)**.

A standard way is to put

$$A_N := N\left(U\left(\tfrac{1}{N}\right) - \mathbf{I}\right) \in \mathcal{L}(L^2_\rho),\ N\in\mathbb{N}.$$

Then, the corresponding evolution family in $L^2_\rho$ is given by

$$U_N(t) := exp(tA_N) = exp\left(tNU\left(\tfrac{1}{N}\right)\right) exp(-tN),\ t\in(\,0,T\,],\ N\in\mathbb{N}. \tag{D.14}$$

The operator $U_N$ is obviuosly positivity preserving if $U$ is positivity preserving. For the rest of the properties needed to have **(A6)**, we refer to Example 2.6, p.55 in [76] resp. Theorem 1.8.1 in [89]. Concerning the uniform norm bound for the $U_N$ (not being part of the condition in [76]), we note that (as described in the proof of Theorem 6.1.4 in Section 6.3,) this immediately follows by the Banach-Steinhaus uniform boundedness principle for linear operators (cf. e.g. Theorem III.9 in [98]).

Thus, Lemma D.2.2 can be developed further to the following:

**Lemma D.2.3:** *Suppose that* $\Theta = \mathbb{R}$. *Let* $A$ *and* $G$ *be as in Lemma D.2.2.*
*Then,* $U$ *defined by (D.11) is an almost strong evolution operator fulfilling* ***(A0)–(A4)**, **(A5)*** and **(A6)**.*

Thus, operator $A$ defined in Lemma D.2.2 is an example of an operator obeying all properties required on the operator family $(A(t))_{t\in[0,T]}$ in the comparison result of Chapter 6 (cf. Theorem 6.1.1 there).
To prove the existence of càdlàg solutions and to treat the case of multiplicative jump noise in equations (1.1) and (1.2), we need **(A7)** and **(A8)**. Let us analyse how to achieve these conditions.
Recall that **(A7)** means the pseudo contractivity of $U(t,s)$ either in $L^2_\rho$ or in $L^2$. In the case $d = 1$ we refer to the classical results of [9].
The condition (D.10) imposed above means that there is some $\delta > 0$ such that

$$a(\theta) \geq \delta \text{ for any } \theta \in \mathbb{R}^d.$$

Thus, $A$ obeys the assumptions of Proposition 2.7 in [9]. By this proposition we get that the $\mathcal{C}_0$-semigroup in $L^2_\rho$ generated by $A$ and having the representation (D.11) is positive.
Furthermore, $U$ is contractive in $L^2(\mathbb{R}^d)$ if and only if $c \geq 0$ for the function $c$ in (D.9). In view of Remark 3.1.2.1 **(vii)**, this also gives us **(A7)**.

Concerning **(A8)**, we note the following:
By the definition of the semigroup $U$ (see (D.11)) and the properties of the derivatives of a convolution, we immediately get $U(t)\varphi \in W^{2,2}(\mathbb{R})$ for any $\varphi \in L^2(\mathbb{R})$. Thus, setting $\mathcal{D}(A_N) = W^{2,2}(\mathbb{R})$ for any $N \in \mathbb{N}$ and $A_N := N\left(U\left(\frac{1}{N}\right) - \mathbf{I}\right)$, $N \in \mathbb{N}$, as before, we get a family $(A_N)_{N\in\mathbb{N}}$ of linear bounded operators on $W^{2,2}(\mathbb{R})$. Furthermore, for the evolution family (D.14) we have

$$\sup_{t\in[0,T]} ||U_N(t,s)\varphi||_{W^{2,2}} \leq c_N(T)||\varphi||_{W^{2,2}}.$$

Thus, **(A8)** is fulfilled.

So, we get the following result:

**Proposition D.2.5:** *Let* $\Theta = \mathbb{R}$ *and let*

$$A := a\tfrac{\partial^2}{\partial\theta^2} + b\tfrac{\partial}{\partial\theta} + c,\ \mathcal{D}(A) := W^{2,2}(\mathbb{R})$$

*with coefficients a, b, c from $C_b^\infty(\mathbb{R})$ such that a obeys (D.10), i.e. A is a strictly elliptic operator. Furthermore, suppose that $c \geq 0$. Then, given the Green function G corresponding to the operator $\frac{\partial}{\partial\theta} + A$ on $\Theta$, U defined by (D.11) is an almost strong evolution operator fulfilling **(A0)–(A8)** (with $m = 2$ in **(A8)**).*

Unfortunately, in the above case we have $\zeta = \frac{1}{2}$.
For such an operator we can apply the theory of e.g. Manthey and Zausinger in the case of a stochastic evolution equation with Wiener noise (see [76]). But recall from the main existence, uniqueness and comparison results in this thesis that, both in the case of drifts of at most linear growth and in the case of drifts of at most polynomial growth, we need $\frac{1}{1-\zeta} < \frac{1}{\zeta}$, i.e. $\zeta < \frac{1}{2}$. To have this property fulfilled, we can e.g. consider higher order differential operators.

Let us assume that $d = 1$ and $1 < m \in \mathbb{N}$. Instead of $A$ from (D.9), we will consider the differential operator

$$A(\theta) = \sum_{\alpha \leq 2m} a_\alpha(\theta) \frac{\partial^\alpha}{\partial\theta^\alpha},\ \theta \in \mathbb{R},\ \mathcal{D}(A) := W^{2m,2}(\mathbb{R}), \tag{D.15}$$

of order $2m$. Here, the coefficient functions $a_\alpha$ are from $C_b^\infty(\mathbb{R})$.
Let us assume that there is some $\delta > 0$ such that (cf. e.g. Section 2.5 on p.17 in [95])

$$(-1)^m a_{2m}(\theta)\xi^{2m} \leq -\delta|\xi|^{2m},\ \theta,\ \xi \in \mathbb{R}. \tag{D.16}$$

(A particular case of this is (D.10) with $m = 1$.)
Let $G : \mathbb{R}_+ \backslash \{0\} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be the Greeen function corresponding to $\frac{\partial}{\partial t} + A$.

To proceed, we need the following general estimate for Green functions, which is due to Aronson (cf. Theorem 2.6 in Section 2.5 in [95]):

**Lemma D.2.6:** *For each $T > 0$ and $0 \leq |\alpha| \leq 2m$, there exist constants $c_1$, $c_2 > 0$ such that*

$$\left|\frac{\partial^{|\alpha|}}{\partial\xi^\alpha} G(t,\theta,\xi)\right| \leq c_1 t^{-\frac{|\alpha|}{2m}} t^{-\frac{d}{2m}} exp\left(-c_2|\theta-\xi|^{\frac{2m}{2m-1}} t^{-\frac{1}{2m-1}}\right),\ t \in [0,T],\ \theta,\xi \in \mathbb{R}^d.$$

In the particular case $d = 1$ and $\alpha = 0$, we have

$$|G(t,\theta,\xi)| \leq c_1 t^{-\frac{1}{2m}} exp\left(-c_2|\theta-\xi|^{\frac{2m}{2m-1}} t^{-\frac{1}{2m-1}}\right),\ t \in [0,T],\ \theta,\xi \in \mathbb{R}, \tag{D.17}$$

for some $c_1$, $c_2 > 0$.

With the help (D.17) one gets

$$\begin{aligned}\int_\Theta G(t,\theta,\xi)\alpha^\rho(\theta-\xi)\,d\xi \leq\ & \int_{\mathbb{R}} c_1 t^{-\frac{1}{2m}} exp\left(-c_2|\theta-\xi|^{\frac{2m}{2m-1}} t^{-\frac{1}{2m-1}}\right)(1+|\theta-\xi|^2)^{\frac{\rho}{2}}\,d\xi \\ \leq\ & c(\rho)\left[\int_{\mathbb{R}} t^{-\frac{1}{2m}} exp\left(-c_2|\xi|^{\frac{2m}{2m-1}} t^{-\frac{1}{2m-1}}\right)d\xi\right. \\ & \left.+\int_{\mathbb{R}} |\xi|^\rho t^{-\frac{1}{2m}} exp\left(-c_2|\xi|^{\frac{2m}{2m-1}} t^{-\frac{1}{2m-1}}\right)d\xi\right] \\ \leq\ & c(\rho,T)<\infty.\end{aligned}$$

Again by the symmetry of $\alpha^\rho(\theta-\xi)$, we get

$$\int_{\mathbb{R}} G(t,\theta,\xi)\alpha^\rho(\theta-\xi)\,d\theta \leq c(\rho,d,T)<\infty,\ t>0,$$

which implies (D.2).
Furthermore, we get

$$\begin{aligned}\int_{\mathbb{R}} G^2(t,\theta,\xi)\alpha^\rho(\theta-\xi)\,d\xi \leq\ & \int_\Theta c_1 t^{-\frac{1}{m}} exp\left(-c_2|\xi|^{\frac{2m}{2m-1}} t^{-\frac{1}{2m-1}}\right)(1+|\theta-\xi|^2)^{\frac{\rho}{2}}\,d\xi \\ \leq\ & c(\rho)\left[\int_{\mathbb{R}} t^{-\frac{1}{m}} expexp\left(-c_2|\xi|^{\frac{2m}{2m-1}} t^{-\frac{1}{2m-1}}\right)d\xi\right. \\ & \left.+\int_{\mathbb{R}} |\xi|^\rho t^{-\frac{1}{m}} exp\left(-c_2|\xi|^{\frac{2m}{2m-1}} t^{-\frac{1}{2m-1}}\right)d\xi\right] \\ \leq\ & c(\rho,T)t^{-\frac{1}{2m}}<\infty,\ t>0.\end{aligned}$$

With the help of this estimate, we get (D.3), but with $\zeta=\frac{1}{2m}<\frac{1}{2}$ (recall that $m>1$) as requested in the main results of Chapters 7 and 8.
Again, we get a $\mathcal{C}_0$-semigroup by (D.11).

So we have proven the following extension of Lemma D.2.2:

**Lemma D.2.7:** *Let the operator $A$ be given by (D.15) with continuous, bounded coefficient functions $a_\alpha$ such that (D.16) is fulfilled.*

*Let $G$ be the Green function corresponding to the operator $\frac{\partial}{\partial t}+A$ on $\mathbb{R}$. Then, $U$ defined by (D.11) is an almost strong evolution operator fulfilling **(A0)–(A4)** and **(A5)*.***

Similarly to the previous considerations, we also can check that **(A6)** and **(A8)** are fulfilled. However, to check **(A7)**, we cannot apply Proposition 2.7 from [9], since the paper of Arendt and his coauthors was directly concerned with second order differential operators. But we can assume that $A$ is a self-adjoint positive operator in $L^2(\mathbb{R})$, which immediately implies the contractivity property of the semigroup $U$ in $L^2(\mathbb{R})$. Thus, we have the following lemma:

**Lemma D.2.8:** *Let the operator $A$ be as in Lemma D.2.7 and let $G$ be the Green function corresponding to the operator $\frac{\partial}{\partial t} + A$ on $\mathbb{R}$.*
*Then, $U$ defined by (D.11) is an almost strong evolution operator fulfilling **(A0)–(A4)**, **(A5)***, **(A6)** and **(A8)**. If $A$ is self-adjoint and positive in $L^2(\mathbb{R})$, then it also fulfills **(A7)**.*

**Remark D.2.9:** *By Lemma D.2.8 we can apply the theory of Chapters 7 and 8 in the case $\Theta = \mathbb{R}^d$ for differential operators of any even order greater than 2 if the coefficient functions are from $C_b^\infty(\mathbb{R})$.*

### A strong evolution operator obeying the assumptions from Chapters 7 and 8

We start with an example of a general second-order elliptic differential operator taken from the paper [33] by Eidelman and Porper.
Again, let $\Theta = \mathbb{R}^d$. Compared to (D.9), we allow the second order elliptic operator to be time-dependent in the sense that

$$\text{(D.18)}\quad A(t)\varphi(\theta) := \sum_{i,j,=1}^{d} a_{ij}(t,\theta)\frac{\partial^2}{\partial\theta_i\partial\theta_j}\varphi(\theta) + \sum_{i=1}^{d} b_i(t,\theta)\frac{\partial}{\partial\theta_i}\varphi(\theta) + c(t,\theta)\varphi(\theta),$$
$t \in [0,T]$, $\theta \in \Theta$, $\varphi \in \mathcal{D}(A) := W^{2,2}(\mathbb{R}^d)$.

Here, the coefficients obey the following properties (cf. $A_1$–$A_3$ on p.122 in [33])

- There is a constant $\mu \geq 1$ such that for any $(t,\theta) \in [0,T] \times \mathbb{R}^d$

$$\text{(D.19)}\qquad \tfrac{1}{\mu}|\xi|^2 \leq a_{ij}(t,\theta)\xi_i\xi_j \leq \mu|\xi|^2 \text{ for all } \xi \in \mathbb{R}^d.$$

- The coefficient functions $a_{ij}$, $b_i$ and $c$ are continuous and uniformly bounded by a constant $M_0 > 0$.

- The coefficient functions satisfy the so-called *Dini condition uniformly in* $(t,\theta) \in [0,T] \times \mathbb{R}^d$, i.e. there is a function $w$: $[0,1] \to \mathbb{R}_+$ obeying

$$\int_0^1 \frac{w(r)}{r}\,dr < \infty$$

  such that for the moduli $w_{t,\theta}$ of continuity we have $w_{t,\theta}(r) \leq w(r)$.

By Theorem 1.1 from [33] resp. pp.23-28 in the book [32] by Eidelman, these assumptions guarantee the existence of a Green function

$$G\colon \{(t,s)\,|\,0 \le s < t \le T\} \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$$

corresponding to the operators $\frac{\partial}{\partial t} + A(t)$ obeying the inequality

(D.20) $G(t,s,\theta,\xi) \le c_1(t-s)^{\frac{d}{2}} exp\left(-c_2|\theta-\xi|^2(t-s)^{-1}\right)$, $0 \le s < t \le T$, $\theta, \xi \in \mathbb{R}$,

with positive constants $c_1$, $c_2$ depending on $\mu$, $d$, $M_0$ and $T$.
Next, we define a family $(U(t,s))_{0\le s<t\le T}$ of operators by (D.1), i.e.

$$(U(t,s)\varphi)(\theta) = \int\limits_{\Theta} G(t,s,\theta,\xi)\varphi(\xi)\,d\xi,\ 0 \le s < t \le T,\ \theta \in \Theta,\ \varphi \in L^2_\rho,\ \theta \in \Theta.$$

By the construction we have, for $0 \le s < r < t \le T$, (cf. Property 2 on p.125 in [33])

$$G(t,s,\theta,\xi) = \int\limits_{\mathbb{R}^d} G(t,r,\theta,\bar\theta)G(r,s,\bar\theta,\xi)\,d\bar\theta,$$

which implies $U(t,s) = U(t,r)U(r,s)$ as required in Definition 2.2.1 **(ii)**.
Fixing an arbitrary $t \in [\,0,T\,]$, analogously to the case of the $\mathcal{C}_0$-semigroup discussed before (cf. (D.11)), defining $U(t,t) = \mathbf{I}$ gives us an evolution operator $(U(t,s))_{0\le s\le t}$ in $L^2_\rho$ (cf. Theorem B.9 in [95]) such that (cf. Theorem B.7 in [95])

$$\sup_{0\le s\le t} ||U(t,s)||_{\mathcal{L}(L^2_\rho)} < \infty.$$

Completely analogously we also get a $\mathcal{C}_0$-semigroup $(U(t,s))_{s\le t\le T}$ in $L^2_\rho$ (cf. Theorem B.9 in [95]) such that (cf. Theorem B.7 in [95])

$$\sup_{s\le t\le T} ||U(t,s)||_{\mathcal{L}(L^2_\rho)} < \infty.$$

Thus, $U$ is an almost strong evolution operator in the sense of Definition 2.2.1.
By Property 3 from p.125 in [33], the kernel $G$ is positive, which immediately gives us the positivity preserving property for $U$ defined by (D.1).
Again, to prove the properties **(A2)**–**(A4)** and **(A5)***, we want to apply the theory developed in Section D.1. To this end, cf. the conditions of Lemma D.1.2, we need (D.2) and (D.3) to hold.

First, we fix an arbitrary $t \in [\,0,T\,]$ and show that property (D.2) is fulfilled. We note that, by the above consideration, all constants below are uniformly in $t \in [\,0,T\,]$.
By (D.20) we have

$$
\begin{aligned}
\int_\Theta G(t,s,\theta,\xi)\alpha^\rho(\theta-\xi)\,d\xi \le\ & \int_\Theta c_1(t-s)^{-\frac{d}{2}} exp\left(-c_2\tfrac{|\theta-\xi|^2}{t-s}\right)(1+|\theta-\xi|^2)^{\frac{\rho}{2}}\,d\xi \\
\le\ & c(\rho)\left[\int_{\mathbb{R}^d}(t-s)^{-\frac{d}{2}} exp\left(-\tfrac{|\xi|^2}{t-s}\right)\,d\xi\right. \\
& \left.+\int_\Theta |\xi|^\rho (t-s)^{-\frac{d}{2}} exp\left(-\tfrac{|\xi|^2}{t-s}\right)\,d\xi\right] \\
\le\ & c(\rho,d,T)<\infty.
\end{aligned}
$$

Since $\alpha^\rho(\theta-\xi)$ is a symmetric function, we also get

$$\int_\Theta G(t,s,\theta,\xi)\alpha^\rho(\theta-\xi)\,d\theta \le c(\rho,d,T)<\infty,\ t>0,$$

which implies (D.2).
Thus, it remains to show the estimate (D.3). By the same reasoning as in the previous example of a $\mathcal{C}_0$-semigroup, we have to restrict our considerations to the case $d=1$ in the following.

So, let $d=1$. Then, (D.20) becomes (recall the positivity of $G$)

$$\text{(D.21)}\quad 0<G(t-s,\theta,\xi)\le c_1(t-s)^{-\frac{1}{2}} exp\left(-\tfrac{|\theta-\xi|^2}{t-s}\right),\ (s,\theta,\xi)\in[\,0,t\,)\times\Theta\times\Theta.$$

Applying (D.21), we get for $0\le s<t\le T$

$$
\begin{aligned}
\int_\Theta G^2(t,s,\theta,\xi)\alpha^\rho(\theta-\xi)\,d\xi \le\ & \int_\Theta c_1(t-s)^{-1} exp\left(-\tfrac{|\theta-\xi|^2}{t-s}\right)(1+|\theta-\xi|^2)^{\frac{\rho}{2}}\,d\xi \\
\le\ & c(\rho)\left[\int_{\mathbb{R}^d}(t-s)^{-1} exp\left(-\tfrac{|\xi|^2}{t-s}\right)\,d\xi\right. \\
& \left.+\int_\Theta |\xi|^\rho (t-s)^{-1} exp\left(-\tfrac{|\xi|^2}{t-s}\right)\,d\xi\right] \\
\le\ & c(\rho,d,T)(t-s)^{-\frac{1}{2}}<\infty.
\end{aligned}
$$

Similarly to the above consideration of (D.2), by the symmetry of $\alpha^\rho(\theta-\xi)$, we also get

$$\int_\Theta G^2(t,s,\theta,\xi)\alpha^\rho(\theta-\xi)\,d\theta \le c(\rho,d,T)(t-s)^{-\frac{1}{2}}<\infty,\ 0\le s<t,$$

which implies (D.3) with $\zeta=\frac{1}{2}$ in the case $d=1$.

Thus, we can apply Lemma D.1.2 to get:

**Lemma D.2.10:** *Suppose that $\Theta=\mathbb{R}$. Let $a(t)$, $b(t)$ and $c(t)$, $t\in[\,0,T\,]$ be functions from $C_b^\infty(\mathbb{R})$ satisfying the Dini conditions. Furthermore, let $a(t)$ obey (D.19). Let us define a family of uniformly elliptic operators $(A(t))_{t\in[\,0,T\,]}$*

$$A(t) := a(t)\frac{\partial^2}{\partial\theta^2} + b(t)\frac{\partial}{\partial\theta} + c(t),\ \mathcal{D}(A(t)) = W^{2,2}(\mathbb{R}),\ t \in [\,0,T\,].$$

*Given $t \in [\,0,T\,]$, let $G$ be a Green function corresponding to the operator $\frac{\partial}{\partial t} + A(t)$.*
*Then, $U$ defined by (D.1) constitutes an almost strong evolution operator fulfilling **(A0)–(A4)** and **(A5)***.*

So far, we have constructed an evolution operator in $L^2_\rho(\Theta)$ obeying **(A0)–(A4)** and **(A5)***. Again this is just enough to apply the theory from Chapter 5 (without càdlàg properties, which would require **(A7)**).

To be able to apply the comparison theory from Chapter 6, we need the approximation property **(A6)**. We modify Example 2.6 from [76] to gain a family $(A_N(t))_{N\in\mathbb{N}}$ of operators approximating $A(t)$ in the sense of **(A6)**.

Given $N \in \mathbb{N}$, we put

$$A_N(t) := N\left(U\left(t, t-\tfrac{1}{N}\right) - \mathbf{I}\right) \in L^2_\rho,\ t \in [\,0,T\,].$$

For $0 \le s < t \le T$ the corresponding evolution family is given by

$$U_N(t,s) := exp(sA_N(t)) = exp\left(sNU\left(t,t-\tfrac{1}{N}\right)\right) exp(-sN),\ s \in [\,0,t\,).$$

The operator $U_N$ is obviously positivity preserving if $U$ is positivity preserving.

The strong convergence property follows from the uniform boundedness of the coefficient functions $a$, $b$ and $c$. Concerning the uniform norm bound of the $U_N$ (not being part of the condition in [76]), we note that (as described in the proof of Theorem 6.1.4 in Section 6.3,) this immediately follows by the Banach-Steinhaus uniform boundedness principle for linear operators (cf. e.g. Theorem III.9 in [98]).

Thus, Lemma D.2.10 can be developed further to the following:

**Lemma D.2.11:** *Suppose that $\Theta = \mathbb{R}$. Let $(A(t))_{t\in[\,0,T\,]}$ and $G$ be as in Lemma D.2.10.*
*Then, $U$ defined by (D.1) is an almost strong evolution operator fulfilling **(A0)–(A4)**, **(A5)*** and **(A6)**.*

Thus, the operator family $(A(t))_{t\in[\,0,T\,]}$ fulfills all the conditions needed in the comparison result of Chapter 6 (cf. Theorem 6.1.1 there).
To prove the existence of càdlàg solutions and to treat the case of multi-

plicative jump noise in equations (1.1) and (1.2), we need **(A7)** and **(A8)**. Let us analyse how to achieve these conditions.
Recall that **(A7)** means the pseudo contractivity of $U(t,s)$ in $L^2(\mathbb{R})$ or $L^2_\rho(\mathbb{R})$.
As in (D.15), we just assume that the operators $(A(t))_{t\in[0,T]}$ are self-adjoint and positive in $L^2(\mathbb{R})$. This implies the contractivity property of the correaponding semigroups $(U(t,s))_{0\le s\le t}$ in $L^2(\mathbb{R})$.
According to Remark 3.1.1 **(vi)**, this gives us **(A7)**.

Concerning **(A8)**, we note that, by the definition of the evolution operator $(U(t,s))_{0\le s\le t}$ (see (D.1)) and the properties of the derivatives of a convolution, we immediately get $U(t,s)\varphi \in W^{2,2}(\mathbb{R})$ for any $\varphi \in L^2(\mathbb{R})$. Thus, setting $\mathcal{D}(A_N(t)) := W^{2,2}(\mathbb{R})$ and $A_N(t) := N\left(U\left(t, t-\frac{1}{N}\right) - \mathbf{I}\right)$ for any $N \in \mathbb{N}$ and $t \in [0,T]$, we get a family $(A_N(t))_{N\in\mathbb{N}}$ of linear bounded operators on $W^{2,2}(\mathbb{R})$. Thus, **(A8)** is fulfilled.

Therefore, we get the following result:

**Proposition D.2.12:** *Let $\Theta = \mathbb{R}$. We define a family of operators*

$$A(t) := a(t)\tfrac{\partial^2}{\partial\theta^2} + b(t)\tfrac{\partial}{\partial\theta} + c(t),\ t \in [0,T],\ \mathcal{D}(A(t)) = W^{2,2}(\mathbb{R})$$

*with coefficient functions $a(t)$, $b(t)$, $c(t)$ from $C_b^\infty(\mathbb{R})$ with a uniform bound obeying the Dini property and $(a(t))_{\{t\in[0,T]}$ fulfilling (D.19), i.e. $(A(t))_{t\in[0,T]}$ is a family of strictly elliptic operators. Furthermore, suppose that $c(t) \ge 0$ for any $t \in [0,T]$. Then, given the the function $G$ corresponding to the family of operators $(\frac{\partial}{\partial\theta} + A(t))_{t\in[0,T]}$, $U$ defined by (D.1) is an almost strong evolution operator fulfilling **(A0)**–**(A4)**, **(A5)*** and **(A6)**–**(A8)** (with $m = 2$ in **(A8)**).*

If we want to avoid the case $\zeta = \frac{1}{2}$, e.g. to apply the main existence, uniqueness and comparison results from Chapters 5–8, which need that $\frac{1}{1-\zeta} < \frac{1}{\zeta}$, we have to consider higher order differential operators.

To this end, we refer to an example from the classical book [40] of Friedman (cf. Chapter 9, Section 2 there).

For the rest of this section, let $d = 1$ and $1 < m \in \mathbb{N}$.
In the following let the operator family $(A(t))_{t\in[0,T]}$ be given by

(D.22) $$A(t)\varphi(\theta) := \sum_{|\alpha|\le 2m} a_\alpha(t)(D_x^\alpha\varphi)(\theta),\ t \in [0,T],\ \theta \in \mathbb{R},\ \mathcal{D}(A(t)) = W^{2m,2}(\mathbb{R}),$$

with the coefficient functions $a_\alpha(t)$ only depending on $t$. More precisely, the mapping $t \mapsto a_\alpha(t)$ is supposed to be continuous in $[0,T]$. Given the function $G$: $\{(t,s)\,|\,0 \le s < t \le T\} \times \mathbb{R}^d \times \mathbb{R}^d$ corresponding to the family of operators $(\frac{\partial}{\partial t} + A(t))_{t\in[0,T]}$, we again get an evolution operator $U = (U(t,s))_{0\le s\le t\le T}$ with the help of (D.1). Furthermore, we can estimate $G$ in the following way (cf. Theorem 2, (2.5) on p.241 in [40] or [28])

$$|G(t,s,\theta,\xi)| \le c_1(t-s)^{-\frac{1}{2m}} exp\left(-c_2\left(\frac{|\theta-\xi|^{2m}}{(t-s)}\right)^{\frac{1}{2m-1}}\right),\ 0 \le s < t \le T,\ \theta,\xi \in \mathbb{R}, \tag{D.23}$$

with positive constants $c_1$, $c_2$.

With the help (D.23), we have

$$\begin{aligned}
&\int_\Theta G(t,s,\theta,\xi)\alpha^\rho(\theta-\xi)\,d\xi \\
&\le \int_\Theta c_1(t-s)^{-\frac{1}{2m}} exp\left(-c_2|\theta-\xi|^{\frac{2m}{2m-1}}(t-s)^{-\frac{1}{2m-1}}\right)(1+|\theta-\xi|^2)^{\frac{\rho}{2}}\,d\xi \\
&\le c(\rho)\left[\int_{\mathbb{R}}(t-s)^{-\frac{1}{2m}} exp\left(-c_2|\xi|^{\frac{2m}{2m-1}}(t-s)^{-\frac{1}{2m-1}}\right)\,d\xi\right. \\
&\left.+\int_\Theta |\xi|^\rho(t-s)^{-\frac{1}{2m}} exp\left(-c_2|\xi|^{\frac{2m}{2m-1}}(t-s)^{-\frac{1}{2m-1}}\right)\,d\xi\right] \\
&\le c(\rho,T) < \infty.
\end{aligned}$$

Again, by the symmetry of $\alpha^\rho(\theta-\xi)$, we get

$$\int_\Theta G(t,s,\theta,\xi)\alpha^\rho(\theta-\xi)\,d\theta \le c(\rho,d,T) < \infty,\ t > 0,$$

which implies (D.2).
Furthermore, we get

$$\begin{aligned}
&\int_\Theta G^2(t,\theta,\xi)\alpha^\rho(\theta-\xi)\,d\xi \\
&= \int_\Theta G_t^2(t-s,\theta,\xi)\alpha^\rho(\theta-\xi)\,d\xi \\
&\le \int_\Theta c_1(t-s)^{-\frac{1}{m}} exp\left(-c_2|\xi|^{\frac{2m}{2m-1}}(t-s)^{-\frac{1}{2m-1}}\right)(1+|\theta-\xi|^2)^{\frac{\rho}{2}}\,d\xi \\
&\le c(\rho)\left[\int_{\mathbb{R}}(t-s)^{-\frac{1}{m}} exp\left(-c_2|\xi|^{\frac{2m}{2m-1}}(t-s)^{-\frac{1}{2m-1}}\right)\,d\xi\right. \\
&\left.+\int_\Theta |\xi|^\rho(t-s)^{-\frac{1}{m}} exp\left(-c_2|\xi|^{\frac{2m}{2m-1}}(t-s)^{-\frac{1}{2m-1}}\right)\,d\xi\right] \\
&\le c(\rho,T)(t-s)^{-\frac{1}{2m}} < \infty,\ t > 0.
\end{aligned}$$

With the help of this estimate, we get (D.3) with $\zeta = \frac{1}{2m} < \frac{1}{2}$ (recall that $m > 1$) as requested in the main results of Chapters 7 and 8.
Thus, $U$ defined by (D.1) is an evolution operator with the requested prop-

erties.

So, we have proven the following extension of Lemma D.2.2:

**Lemma D.2.13:** *Let the operator $A(t)$ be given by (D.22) with coefficient functions $a_\alpha(t)$ only depending on $t$ such that $t \mapsto a_\alpha(t)$ is continuous in $[0,T]$.*
*Let $G$ be the Green function corresponding to the operator family $(\frac{\partial}{\partial\theta} + A(t))_{t\in[0,T]}$ on $\mathbb{R}$.*
*Then, $U$ defined by (D.1) is an almost strong evolution operator fulfilling* ***(A0)–(A4)*** *and* ***(A5)****.

Similarly to the previous considerations, we can also check that **(A6)** and **(A8)** are fulfilled. However, to check **(A7)** we cannot apply Proposition 2.7 from [9], since the paper of Arendt and his colaborators was directly concerned with second order differential operators. But we can assume that $A(t)$ is a self-adjoint positive operator in $L^2(\mathbb{R})$, which immediately implies the contractivity property of the semigroup $(U(t,s))_{0\leq s\leq t}$ in $L^2(\mathbb{R})$. Thus, we have:

**Lemma D.2.14:** *Let the operator $A(t)$ be as in Lemma D.2.13 and, given arbitrary $t \in [0,T]$, let $G$ be the Green function corresponding to the operator family $(\frac{\partial}{\partial\theta} + A(t))_{t\in[0,T]}$ on $\mathbb{R}$.*
*Then, $U$ defined by (D.1) is an almost strong evolution operator fulfilling* ***(A0)–(A4)****,* ***(A5)****,* ***(A6)*** *and* ***(A8)****. If $A(t)$ is self-adjoint and positive in $L^2(\mathbb{R})$ for any $t \in [0,T]$, then it also fulfills* ***(A7)****.*

**Remark D.2.15:** *By Lemma D.2.14 we can apply the theory of Chapters 7 and 8 in the case $\Theta = \mathbb{R}$ for differential operators of any even order greater than 2 if the coefficient functions depend only on time in a continuous manner.*

# Appendix E

# A general construction of positive measures in $L^2(\Theta)$ with summable weak moments

The aim of this appendix is to show a general method to construct examples of (Lévy) measures in $L^2(\Theta)$ fulfilling the integrability condition (4.2) from Chapter 4 and the positivity condition **(P)** from Chapter 8 such that we can claim there is a class of examples, for which we can apply Theorems 8.1.1 and 8.1.2 .

Now, let us first assume that given $\mathbb{N} \ni q \geq 2$ $\mu$ is an arbitrary $\sigma$-finite, square-integrable measure on $L^2(\Theta)$ additionally obeying

$$\int_{L^2(\Theta)} ||x||_{L^2}^q \, \mu(dx) < \infty.$$

Let $Q : L^2(\Theta) \to L^2(\Theta)$ be a symmetric nonnegative operator, which additionally fulfills $Q \geq 0$, and is positivity preserving in the sense that it maps $L^2_+(\Theta)$ (for the definition of $L^2_+(\Theta)$ see the introduction of Chapter 8) onto itself. Furthermore, let $Q$ be of form (2.4) with $(e_n)_{n\in\mathbb{N}}$ fulfilling (3.1) and $(a_n)_{n\in\mathbb{N}}$ being summable, i.e. $Q \in \mathcal{T}(L^2(\Theta))$.
Finally, with the help of the absolute value in $\mathbb{R}$ we define the following mapping av on $L^2(\Theta)$

$$(\mathrm{av}(\psi))(\theta) := |\psi(\theta)| \; ; \theta \in \Theta \; , \psi \in L^2(\Theta). \tag{E.1}$$

Now, we have

**Theorem E.1 :** *Under the previous assumptions*

$$\eta := \mu \circ Q^{-1} \circ av^{-1} \tag{E.2}$$

*fulfills the integrability condition (4.2) from Chapter 4.*
*By the definition, we have for any functional $F\colon L^2(\Theta) \to \mathbb{R}$*

$$\int_{L^2(\Theta)} F(x)\,\eta(dx) := \int_{L^2(\Theta)} F(Q(av(x)))\,\mu(dx).$$

**Proof:** To prove the claim it suffices to consider

$$\sum_{n\in\mathbb{N}} \left( \int_{L^2} |(x,e_n)_{L^2}|^2\,\eta(dx) \right)^{\frac{1}{2}}$$

and

$$\sum_{n\in\mathbb{N}} \left( \int_{L^2} |(x,e_n)_{L^2}|^q\,\eta(dx) \right)^{\frac{1}{q}},$$

where $(e_n)_{n\in\mathbb{N}}$ is an orthonormal basis of $L^2(\Theta)$ obeying (cf. (3.1) in Chapter 3)

$$\sup_{n\in\mathbb{N}} ||e_n||_\infty < \infty.$$

Indeed, we have

$$\begin{aligned}
\int_{L^2} ||x||_{L^2}^q\,\eta(dx) &= \int_{L^2} \left( \int_{\Theta} \left( \sum_{n\in\mathbf{N}} (x,e_n)_{L^2} e_n \right)^2 (\theta)\,d\theta \right)^{\frac{q}{2}} \eta(dx) \\
&\leq \left( \sup_{n\in\mathbf{N}} ||e_n||_{L^\infty} \right)^q \left( \int_{L^2} \left( \sum_{n\in\mathbf{N}} |(x,e_n)_{L^2}| \right)^q \eta(dx) \right) \\
&= C \left( \left|\left| \sum_{n\in\mathbf{N}} |(x,e_n)_{L^2}| \right|\right|_{L^q(L^2,\eta)} \right)^q \\
&\leq C \left( \sum_{n\in\mathbf{N}} ||\,|(x,e_n)_{L^2}|\,||_{L^q(L^2,\eta)} \right)^q \\
&= C \left( \sum_{n\in\mathbf{N}} \left( \int_{L^2} |(x,e_n)_{L^2}|^q\,\eta(dx) \right)^{\frac{1}{q}} \right)^q
\end{aligned}$$

with the decomposition of x in $L^2$ used in the second and Minkowski's inequality used in the fifth step , and (by analogous arguments)

$$\left(\int\limits_{L^2} ||x||^2_{L^2_\rho}\,\eta(dx)\right)^{\frac{q}{2}} \le \left[\left(\sum_{n\in\mathbf{N}}\left(\int\limits_{L^2}|(x,e_n)_{L^2}|^2\,\eta(dx)\right)^{\frac{1}{2}}\right)^2\right]^{\frac{q}{2}}$$
$$= \left(\sum_{n\in\mathbf{N}}\left(\int\limits_{L^2}|(x,e_n)_{L^2}|^2\,\eta(dx)\right)^{\frac{1}{2}}\right)^q .$$

Let us first consider the general case. For any $q \ge 2$ we have

$$\begin{aligned}
\sum_{n\in\mathbb{N}}\left(\int\limits_{L^2}|(x,e_n)_{L^2}|^q\,\eta(dx)\right)^{\frac{1}{q}} &= \sum_{n\in\mathbb{N}}\left(\int\limits_{L^2}|(Q\mathrm{av}(x),e_n)_{L^2}|^q\,\mu(dx)\right)^{\frac{1}{q}} \\
&= \sum_{n\in\mathbb{N}}\left(\int\limits_{L^2}|(\mathrm{av}(x),Qe_n)_{L^2}|^q\,\mu(dx)\right)^{\frac{1}{q}} \\
&= \sum_{n\in\mathbb{N}} a_n\left(\int\limits_{L^2}|(\mathrm{av}(x),e_n)_{L^2}|^q\,\mu(dx)\right)^{\frac{1}{q}} \\
&\le \sum_{n\in\mathbb{N}} a_n\left(\underbrace{||e_n||^q_{L^2}}_{=1\,for\,any\,n}\int\limits_{L^2}||x||^q_{L^2}\,\mu(dx)\right)^{\frac{1}{q}} \\
&= tr\,Q\left(\int\limits_{L^2}||x||^q_{L^2}\,\mu(dx)\right)^{\frac{1}{q}} \\
&< \infty,
\end{aligned}$$

where we used (E.2) in the first, symmetry of the inner product in $L^2(\Theta)$ in the second, the Cauchy-Schwartz inequality in the fourth and the assumption of finite $q$-th moment in the last step.
Now, by the special case $q = 2$, we also have finiteness of the first sum of integrals. ■

We finish the appendix by the following theorem, which we need to have condition **(P)** from Chapter 8 fulfilled.

**Theorem E.2 :** *$\eta$ defined by (E.2) is a positive measure, i.e. it is concentrated on the cone $L^2_+(\Theta)$ introduced in Chapter 8.*

**Proof:** To prove this let us show that for

$$B := L^2(\Theta)\setminus L^2_{\ge 0}(\Theta) = \{\psi \in L^2(\Theta)\,|\,\psi(\theta) < 0\,d\theta - a.s.\}$$

we have $\eta(B) = 0$ .
By (E.2) we have

(E.3) $\eta(B) = \int\limits_{L^2(\Theta)} \mathbf{1}_B(x)\,\eta(dx) = \int\limits_{L^2(\Theta)} \mathbf{1}_B(Q(\mathrm{av}(x)))\,\mu(dx).$

By (E.1) it is obvious that $\mathrm{av}(x) \geq 0$ for all $x \in L^2(\Theta)$, i.e. $\mathrm{av}(x) \in L^2_{\geq 0}(\Theta)$ for any $x \in L^2(\Theta)$. Therefore, by the fact that $Q$ maps $L^2_{\geq 0}(\Theta)$ onto itself, we get $Q(\mathrm{av}(x)) \geq 0$ for any $x \in L^2(\Theta)$, and hence $\mathbf{1}_B(Q(\mathrm{av}(x))) = 0$ for any $x \in L^2(\Theta)$. By (E.3) this implies the claim. ■

# Bibliography


[1] R.Adams, J.Fournier, *Sobolev Spaces*, Academic Press, 1991.

[2] R.Adams, J.Fournier, *Cone Conditions and Properties of Sobolev Spaces*, Journal of Mathematical Analysis and Applications **61** (1977), pp. 713–734.

[3] S.Albeverio, V.Mandrekar, B.Rüdiger, *Existence of Mild Solutions for Stochastic Differential Equations and Semilinear Equations with Non-Gaussian Lévy Noise*, Stochastic Processes and Their Applications **119** (2009), pp.835–863.

[4] S.Albeverio, B.Rüdiger, *Stochastic Integrals in the Levy-Ito Decomposition Theorem on Separable Banach Spaces*, Stochastic Analysis and Applications, **23(2)**(2005), pp.217–253.

[5] S.Alberverio, J.Wu, T.Zhang, *Parabolic SPDEs driven by Poisson white noise*, Stochastic Processes and Their Applications **74** (1998), pp.21-36.

[6] H.W.Alt, *Lineare Funktionalanalysis*, Springer, 1999.

[7] D.Applebaum, *Lévy processes and stochastic calculus*, Cambridge Studies in Advanced Mathematics, 2004.

[8] D.Applebaum, *Martingale-Valued Measures,Ornstein-Uhlenbeck Processes with Jumps and Operator Self-Decomposability in Hilbert Space*, Lecture Notes in Mathematics **1874**, Springer, Berlin, 2006.

[9] W.Arendt, C.Batty, D.Robinson, *Positive Semigroups Generated by Elliptic Operators on Lie Groups*, Journal of Operator Theory **23** (1990), pp.369–407.

[10] H.Bauer, *Wahrscheinlichkeitstheorie (4.Auflage)*, Walter de Gruyter-Verlag, Berlin, 1991.

[11] V.Bally, I.Gyöngy, E.Pardoux, *White Noise Driven Parabolic SPDEs with Measureable Drift*, Journal of Functional Analysis **120** (1994), pp.484–510.

[12] J.Bertoin, *Lévy Processes*, Cambridge University Press, 1996.

[13] T.Björk, G.DiMasi, Y.Kabanov, W.Ruggaldier, *Towards a General Theory of Bond Markets*, Finance and Stochastics **1** (1997), pp.141–174.

[14] L.J.Bo, K.Shi, Y.J.Wang, *Approximating Solutions of Neutral Stochastic Evolution Equations With Jumps*, Science in China Series A: Mathematics, **52(5)** (2009), pp. 895–907.

[15] V.Bogachev, M.Röckner, B.Schmuland, *Generalized Mehler semigroups and applications*, Probability Theory and Related Fields, **105** (1996), pp. 193-225.

[16] S.Bonnacorsi, C.Marinelli, G.Ziglio, *Stochastic FitzHugh-Nagumo Equations on Networks With Impulsive Noise*, Electric Journal of Probability **13(49)** (2008), pp.1362–1379.

[17] Z.Brezniak, E.Hausenblas, *Martingale Solutions for Stochastic Equation of Reaction Diffusion Type Driven by Lévy Noise or Poisson Random Measure*, arXiv-Server **1010.5933v1**, 2010.

[18] G.Cao, K.He, X.Zhang, *Successive Aproximations of Infinite Dimensional SDEs with Jump*, Stochastics and Dynamics **5(4)** (2005), pp.609–619.

[19] R.Carmona, M.Tehranchi, *Interest Rate Models: An Infinite-Dimensional Stochastic Analysis Perspective*, Springer, Berlin, 2006.

[20] S.Cerrai, *Second order PDEs in finite and infinite dimensions. A probabilistic approach*, Lecture Notes in Mathematics **1762**, Springer, New York - Berlin - Heidelberg, 2001.

[21] A.Chojnowska-Michalik, *On processes of Ornstein-Uhlenbeck-type in Hilbert spaces*, Stochastics **21** (1987), pp. 251–286.

[22] D.L.Cohn, *Measure Theory*, Birkhäuser-Verlag, 1980.

[23] P.Courrège, *Sur la forme integro-diffèrentielle des opèrateurs de $C_K^\infty$ dans $C_0$ satisfaisant au principe du maximum*, Sèminaire Thèorie du Potentiel, **2**, 1965.

[24] R.F.Curtain, A.J.Pritchard, *Infinite Dimensional Linear System Theory*, Lecture Notes in Control and Information Sciences ($8^{\underline{th}}$ edition), Springer, 1978.

[25] G.Da Prato, S.Kwapien, J.Zabczyk, *Regularity of solutions of linear stochstic equations in Hilbert Spaces*, Stochastics **23** (1987), pp.1–23.

[26] G.Da Prato, J.Zabczyk, *Stochastic Equations in Infinite Dimensions*, Cambridge University Press, Cambridge, 1992.

[27] G.DaPrato, J.Zabczyk, *Convergence to Equilibrium for Classical and Quantum Spin Systems*, Probability Theory Related Fields **103** (1995), pp.529–552.

[28] E.B. Davies, *Heat kernel bounds for higher order elliptic operators*, Journées Équations aux dérivées partielles, pp.1–11, 1995.

[29] C.Dellacherie, P.A. Meyer, *Probability and Potential. Theory of martingales*, North Holland Publishing Company, 1982.

[30] J.Diestel, J.J.Uhl, *Vector Measures*, Mathematical Surveys Volume **15**, American Mathematical Society, Providence, 1977.

[31] S.Dirksen, *Noncommutative and Vector-valued Rosenthal Inequalities*, Technische Universiteit Delft, PhD thesis, 2011.

[32] S.Eidelman, *Parabolic Systems*, Noordhoff, Groningen, North-Holland, Amsterdam, 1969

[33] S.Eidelman, F.Porper, *Two-sided Estimates of Fundamental Solutions of Secon-Order Parabolic Equations, and Some Applications*, Russian Mathematical Surveys **39(3)** (1984), pp.119–178.

[34] S.N. Ethier, T.G. Kurtz, *Charactarization and Convergence*, John Wiley Sons, 1986.

[35] W.Faris, G.Jona-Lasinio, *Large Fluctuations for a Non-Linear Heat Equation with Noise*, Journal Physics A, **15** (1982), pp.3025–3055.

[36] D.Filipovic, *Consistency Problems for Heath-Jarrow-Morton Interest Rate Models*, Lecture Notes in Mathematics, **1760**, Springer, 2001.

[37] D.Fillipovic, S.Tappe, J.Teichmann, *Jump Diffusions in Hilbert Spaces: Existence, Uniqueness and Numerics*, arXiv-Server **0810.5023v1**, 2008.

[38] D.Filipovic, S.Tappe, J.Teichmann, *Term Structure Models Driven by Wiener Processes and Poisson Measures: Existence and Positivity*, SIAM Journal of Financial Mathematics **1** (2010), pp. 523-554.

[39] O.Forster, *Analysis 3*, Vieweg-Verlag, 1991.

[40] A.Friedman, *Partial Differential Equations: The parabolic case*, Holt, Rinehart and Winston, 1969.

[41] M.Fuhrmann, M.Röckner, *Generalized Mehler semigroups: the non-Gaussian case*, Potential Analysis **12** (2000), pp. 1–47.

[42] T.Funaki, *Regularity Properties for Stochastic Partial Differential Equations of Parabolic Type*, Osaka Journal of Mathematics **28** (1991), p.495–516.

[43] E.Gagliardo, *Proprieta di Alcune Classi di Unzioni*, Piu Variabili Ricerche di matematica **7** (1958), pp.102-137.

[44] L.I.Galchuk, *A comparison theorem for stochastic equations with integrals with respect to martingales and random measures*, Theory of Probability and its Applications **27(3)** (1982), 450–460.

[45] L.I.Galchuk, *On the change of variables formula*, Mateamtica Zametki **26(4)** (1979), pp.633-641.

[46] I.Gyöngy, *On stochastic equations with respect to semimartingales III*, Stochastics, **7(4)** (1982), pp.231–254.

[47] I.Gyöngy, *On Nondegenerated Quasilinear Stochastic Partial Differential Equations*, Journal of Potential Analysis, **4** (1995), pp.157–171.

[48] I.Gyöngy, N.Krylov, *On Stochastic Equation with Respect to Semimartingales*, Stochastics **4** (1980) , pp.1–21.

[49] I.Göngy, E.Pardoux, *On Quasi-Linear Stochastic Partial Differential Equations*, Probability Theory and Related Fields **94** (1993), pp.413–425.

[50] I.Gyöngy, E.Pardoux, *On the Regularization Effect of Space-Time White Noise on quasi-linear Parabolic Partial Differential Equations*, Probability Theory and Related Fields **97** (1993), pp.211–229.

[51] C.J. Himmelberg, *Measurable Relations*, Fundamenta Mathematica **87** (1976), pp.53–72.

[52] W.Hoh, *A symbolic calculus for pseudo-differential operators generating Feller semigroups*, Osaka Journal of Mathematics, **35(4)** (1998), pp.789–820.

[53] N.Ikeda, S.Watanabe, *Stochastic Differential Equations and Diffusion Processes*, North-Holland Publishing Company/Kodansha Ltd., 1981.

[54] K.Iwata, *An Infinite Dimensional Stochastic Differential Equation with State Space* $\mathbb{C}(\mathbb{R})$, Probability Theory Related Fields **74** (1987), pp. 141–159.

[55] J.Jakubowski, J.Zabczyk, *Exponential Moments for HJM Models With Jumps*, Finance and Stochastics **11** (2007), pp. 429–449.

[56] I.Karatzas, S.Shreve, *Brownian Motion and Stochastic Calculus ($2^{\underline{nd}}$ edition)*, *Springer*, 1991.

[57] H.Kawabi, M.Röckner, *Essential self-adjointness of Dirichlet operators on a path space with Gibbs measures via an SPDE approach*, Journal of Functional Analysis **242** (2007), pp.486–518.

[58] F.Knäble, *Ornstein-Uhlenbeck Equations with Time-Dependent Coefficients with Levy Noise in Finite and Infinite Dimensions*, Diploma thesis, BiBos preprint server **E09-01-310**, 2008.

[59] K.Knäble, *Stochastic Evolution Equations with Levy Noise and Applications to the Heath-Jarrow-Morton Model*, Diploma thesis, BiBos preprint server **E06-11-235**, 2006.

[60] C.Knoche, *Stochastic Integration and Stochastic Differential Equation with Respect to Compensated Poisson Random Measure in Infinite Dimensional Hilbert Space*, BiBos preprint server **03-09-126**, 2003.

[61] C.Knoche, *Mild Solutions of SPDEs Driven by Poisson Noise in Infinite Dimensions and Their Dependence on Initial Conditions*, PhD thesis, BiBos preprint server **E05-10-194**, 2005.

[62] C.Knoche, K.Frieler, *Solutions of Stochastic Differential Equations in Infinite Dimensional Hilbert Spaces and their Dependence on Initial Data*, Diploma thesis, BiBos preprint server **E02-04-083**, 2001.

[63] P.Kotelenez, *A Submartingale Type Inequality With Applications to Stochastic Partial Differential Equations*, Stochastics **8** (1982), pp.139–151.

[64] P.Kotelenez, *A Stopped Doob Inequality for Stochastic Convolution Integrals and Stochastic Evolution Equations*, Stochastic Analysis and Applications **2(3)** (1984), pp. 245–265.

[65] P.Kotelenez, *Comparison Methods for a Class of Function Valued Stochastic Partial Differential Equations*, Probability Theory Related Fields **93(1)** (1992), pp.1–19.

[66] P.Kotelenez, *Existence, Uniqueness and Smoothness for a Class of Function Valued Stochastic Partial Differential Equations*, Stochastics and Stochastic Reports **41(3)** (1992), pp.177-199.

[67] V.Krasin, A.Melnikov, *On Comparison Theorem and its Applications to Finance in Optimality and Risk*, Modern Trends in Mathematical Finance, pp. 171–181, 2009.

[68] V.Krasin, *Comparison Theorem and its Applications to Finance*, PhD-thesis, University of Alberta, 2010.

[69] E.Lieb, M.Loss, *Analysis Second Edition*, Graduate Studies in Mathematics **14**, 2001.

[70] W.Liu, M.Röckner, *SPDE in Hilbert space with locally monotone coefficients*, Journal of Functional Analysis, **259(11)** (2010), pp.2902–2922.

[71] V.Mandrekar, B.Rüdiger, *Existence and Uniqueness of Pathwise Solutions for Stochastic Integral Equations Driven by non Gaussian Noise on Separable Banach Spaces*, Stochastics **78(4)** (2006), pp.189–212.

[72] R.Manthey, *On the Cauchy Problem for Reaction Diffusion Equations with White Noise*, Mathematische Nachrichten **136** (1988), pp. 209–228.

[73] R.Manthey, *On Semilinear Stochastic Partial Differential Equations on the Real Line*, Stochastics and Stochastic Reports **57** (1996), pp.213-234.

[74] R.Manthey, K.Mittmann, *Stochastic Reaction Diffusion Equations with Continuous Reaction*, Stochastic Analysis and Applications **15(4)** (1994), pp. 555-583.

[75] R.Manthey, C.Stiewe, *Existence and Uniqueness of Solutions to Volterras Population Equation with Diffusion and Noise*, Stochastics and Stochastics Reports **41** (1992), pp.135–161.

[76] R.Manthey , T.Zausinger, *Stochastic Evolution Equations in $L^{2\nu}_{\rho}$*, Stochastics and Stochastic Reports **66** (1999), pp.37–85.

[77] R.Marcus, *Stochastic Diffusion on an Unbounded Domain*, Pacific Journal of Mathematics **84** (1979), pp.143–151.

[78] C.Marinelli, *Well-Posedness and Invariant Measures for HJM Models with Deterministic Volatility and Levy Noise*, SFB 611-Bonn, Preprint server **323**, 2007 or Quantitative Finance **10(1)**, pp.39–47.

[79] C.Marinelli, C.Prévôt, M.Röckner, *Regular Dependence on Initial Data for Stochastic Evolution Equations with Multiplicative Poisson Noise*, arXiv-Server **0808.1509v1**, 2008.

[80] C.Marinelli, M.Röckner, *Well-Posedness and Asymptotic Behaviour for Stochastic Recation Diffusion Equations With Multiplicative Poisson Noise*, Electronic Journal of Probability **15**, pp. 1528–1555, 2010.

[81] C.Marinelli, M.Röckner, *On Uniqueness of Mild Solutions for Dissipative Stochastic Evolution Equations*, Infinite Dimenmsional Analysis and Applcations **13(3)**, 2010.

[82] R.C.Merton, *Option Pricing, When Underlying Stock Returns Are Discontinuous*, Journal of Financial Economics **3(1)**, pp. 125–144.

[83] M.Metvier, J.Pellaumail, *Stochastic Integration*, Accademic Press, New York, 1980.

[84] N.Meyers, J.Serrin, $H = W$, Proceedings in Natural Academic Science **51** (1964), pp. 1055-1056.

[85] S.Michel, *Stochastic Evolution Equations in Weighted $L^p$ Spaces and Applications to the Heath-Jarrow-Morton Model*, Diploma thesis, BiBos preprint server **E06-12-236**, 2006.

[86] C.Morrey, *Functions on Several Variables and Absolute Continuity II*, Duke Journal of Mathematics **6** (1940), pp. 187–215.

[87] R.Oszepian, A.Pelczyinski, *On the Existence of a Fundamental Total Biorthogonal Sequence in Every Separable Banach Space and Related Constructions of Uniformly Bounded Orthonormal Systems in $L_2$*, Studia Mathematica LIV (1975), pp.149–159.

[88] F.Özkan,T.Schmidt, *Credit Risk With Infinite Dimensional Lévy Processes*, Statistics and Dynamics **23** (2005), pp.281–299.

[89] A.Pazy, *Semigroups of linear operators and applications to partial differential equations*, *Springer*, 1983.

[90] G.Parisi, Y.Wu, *Perturbation Theory Without Gauge Fixing*, Scientia Sinica **24** (1981), pp. 469–483.

[91] K.R. Pathasarathy, *Probability Measures on Metric Spaces*, Academic Press, New York-London, 1967.

[92] S.Peng, X.Zhu, *Necessary and Sufficient Condition for Comparison Theorem of 1-dimensional Stochastic Differential Equations*, Stochastic Processes and Applications **116(3)** (2006), pp.370–380.

[93] S.Peszat, *Existence and Uniqueness of the Solution for Stochastic Equations on Banach space*, Stochastics and Stochastic Reports **55** (1995), pp.167–193.

[94] S.Peszat, J.Zabczyk, *Stochastic Evolution Equation with a Spatially Homogeneous Wiener Process*, Stochastic Processes and Applications **72(2)** (1997), pp.187–204.

[95] S.Peszat, J.Zabczyk, *Stochastic Differential Equations with Levy Noise*, Cambridge University Press, 2007.

[96] P.Protter, *Stochastic Integration and Differential Equations*, Springer, 1990.

[97] C.Prévôt, M.Röckner, *A Concise Course on Stochastic Partial Differential Equations*, Lecture Notes in Mathematics **1905**, Springer, 2006.

[98] M.Reed,B.Simon, *Methods of Modern Mathematical Physics Volume I (Functional Analysis)*, Academic Press, 1972.

[99] M.Riedle, D.Applebaum, *Cylindrical Lévy processes in Banach spaces*, Proceedings of the London Mathematical Society **101(3)** (2010), pp. 697–726.

[100] M.Röckner, Y.Wang, *A Note on Variational Solutions to SPDE Perturbed by Gaussian Noise in a General Class*, arXiv-server **0902.4324**, 2009.

[101] S.Rong, *On Strong Solution, Uniqueness, Stability and Comparison Theorems for a Stochastic System with Poisson Jumps*, Lecture Notes in Control and Information Theory **75** (1985), pp.352–381.

[102] B.Rüdiger, G.Ziglio, *Ito Formula for Stochastic Integrals w.r.t. Compensated Poisson Random Measures on Separable Banach Spaces*, Stochastics and Stochastic Reports **78(6)** (2006), pp.377–410.

[103] J.Seidler, *Da Prato-Zabczyk's Maximal Inequality Revisited I*, Mathematica Bohemica **118** (1993), pp.67–106.

[104] S.Sobolev, *On a Theorem of Functional Analysis*, American Mathematical Society Translations **34** (1938), pp. 39–68.

[105] M.Stephan, *Yosida approximations for multivalued, stochastic differential equations on Banach spaces via Gelfand triple*, University of Bielefeld, PhD thesis, 2012.

[106] S.Stolze, *Stochastic Equations in Hilbert Spaces and Their Applications in Finance*, Diploma thesis, BiBos preprint server **E05-07-188** , 2005.

[107] H.Tanabe, *Equations of Evolution*, Pitman, London, 1979.

[108] G.M.Troianello, *Elliptic Differential Equations and Obstacle Problems*, Plenum Press, 1987.

[109] J.van Neerven, M.Veraar, L.Weis, *Stochastic Integration in UMD Banach Spaces*, Annals of Probability **35(4)**, pp. 1438–1478, 2007.

[110] J.B.Walsh, *An Introduction to Stochastic Partial Differential Equations*, Ecole d'ete probabilites de Saint-Flour XIV, pp. 266–439, 1986.

[111] S.L.Yadava, *Stochastic Equations in Locally Convex Space*, Proceedings Mathematical Sciences, **95(2)** (1986), pp.79–94.

[112] W.Zhang, S.Deng, *Projected Gronwall-Bellman's Inequality for Integrable Functions*, Mathematical and Computer Modelling **34** (2001), pp. 393–402.

[113] X.Zhu, *On the Comparison Theorem for Multidimensional SDEs with Jumps*, arXiv-server **1006.1454v1**, 2010.