Average Harmonic Spectrum of the Whole-Plane SLE

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Abstract

We consider unbounded version of the whole-plane Shramm-Loewner evolution. Using exact solutions of differential equations for moments of derivative of conformal mappings we determine its average integral means beta-spectrum.

1 Introduction

Recently, the area of two-dimensional critical phenomena has enjoyed a breakthrough due to a radical development by O. Schramm in stochastic Loewner evolution, as an approach to description of boundaries of critical clusters (see for a review e.g. [6] and [10]). Among the most important characteristics of such boundaries are their multi-fractal spectra. In this article we give exact description of the average $\beta-$spectrum for unbounded conformal mappings of the whole-plane Schramm-Loevner Evolution (SLE).

An initial motivation for the present work was study of the coefficient problem for injective conformal mappings from the interior of the unit circle to the plane determined by the whole-plane Stochastic Loewner Evolution [5], [11]. Investigation of the asymptotic behavior of the moments of the Taylor coefficients of such mappings led us to finding the exact multi-fractal spectrum of unbounded conformal mappings related to the whole-plane SLE.

In this section we introduce notations and describe main results.

Radial stochastic Schramm-Loewner Evolution with parameter κ (see e.g. [3], [9], [10]) describes dynamics of a slit domain in the z-plane that can be represented by growth of a random planar curve $\Gamma = \Gamma(t)$ starting from a point on a unit circle $|z| = 1$ at $t = 0$. The principal idea of the theory of SLE is that the growth of the curve can be presented by the time dependent conformal mapping $z = F(w, t)$ which satisfies the Loewner equation with stochastic driving:

$$\frac{\partial F(w,t)}{\partial t} = w \frac{\partial F(w,t)}{\partial w} \frac{w + e^{i\tilde{B}(t)}}{w - e^{i\tilde{B}(t)}}, \quad t \geq 0, \quad F(w,0) = w,$$

where $\tilde{B}(t)$ denotes the Brownian motion with "temperature" κ:

$$\langle (\tilde{B}(t) - \tilde{B}(t'))^2 \rangle = \kappa |t - t'|.$$

Everywhere through the article $\langle \; \rangle$ denotes expectation (ensemble averaging).

One can consider either the "exterior" problem, where the complement of the unit disc $\mathbb{D}_+ = \{w : |w| > 1\}$ in the w-plane is mapped by $F = F_+$ to the "exterior" slit domain $\mathbb{D}_+ \backslash \Gamma(t)$ in the z-plane (i.e. the curve starting from the unit circle is growing in the exterior of the unit disc)

$$F_+(w,t) = e^t \left(w + \sum_{i=0}^{\infty} \frac{F_i^+(t)}{w^i} \right), \quad |w| > 1,$$

or the "interior" problem, where the curve starting from the unit circle is growing in the interior of the unit disc. In this problem the unit disc in the w-plane $\mathbb{D}_- = \{w : |w| < 1\}$ is mapped by $F = F_-$ to the "interior" slit domain $\mathbb{D}_- \backslash \Gamma(t)$ in the z-plane

$$F_-(w,t) = e^{-t} \left(w + \sum_{i=2}^{\infty} F_i^-(t) w^i \right), \quad |w| < 1.$$

Since SLE is a conformally invariant stochastic process [3], [6], [9], the exterior and interior problems are related by the inversion $F_-(w,t) = 1/F_+(1/w,t)$.

In the present article we study the whole-plane SLE, which is an infinite-time limit of the radial SLE: There are two versions of the whole-plane SLE:

$$\frac{\partial \mathcal{F}_\pm(w,t)}{\partial t} = \pm w \frac{\partial \mathcal{F}_\pm(w,t)}{\partial w} \frac{w + e^{iB(t)}}{w - e^{iB(t)}}, \quad \mathcal{F}_\pm(w,t) = e^t \left(w + \sum_{i=1 \mp 1}^{\infty} \mathcal{F}_i^\pm(t) w^{\mp i} \right). \tag{1}$$

The first version is the "interior" whole plane SLE which can be viewed as the limit of the interior problem

$$\mathcal{F}_-(w,t) = \lim_{T \to \infty} e^T F_-(w, T - t), \quad |w| < 1, \quad B(t) = \tilde{B}(T - t) \tag{2}$$

describing the growth process in an infinite slit domain by "erasing" in time a curve/slit that starts at some point on the plane and goes to infinity.

The second version of the whole-plane SLE can be viewed as the limit of the exterior problem

$$\mathcal{F}_+(w,t) = \lim_{T \to \infty} e^{-T} F_+(w, T + t), \quad |w| > 1, \quad B(t) = \tilde{B}(T + t) \tag{3}$$

In the present article we find explicit expression for average $\beta(q)$-spectrum for the unbounded-map (interior) whole-plane SLE. In more details, the β-spectrum quantitatively describes subsets of the domain boundary, where certain scaling laws apply to harmonic measure. This describes singularities of derivatives of conformal mappings related to distribution of the "wedge angles" along the curve (see e.g. [1], [2], [4] and for general introduction to multi-fractal analysis see e.g. [7]). Below we study the integral means $\beta(q)$ spectrum which is defined through the moments of the derivatives at $|w| \to 1$:

$$\beta(q) = \lim_{\epsilon \to 0+} \frac{\log \int_0^{2\pi} \langle |\mathcal{F}'_\pm \left(e^{\pm \epsilon + i\varphi}, t \right)|^q \rangle d\varphi}{-\log \epsilon}, \tag{4}$$

where the prime denotes the w-derivative and $\pm$ stand for the exterior or interior cases respectively. Below we present arguments related to the following result: The average integral means β-spectrum of the interior whole-plane SLE equals

$$\beta = \begin{cases} \kappa \frac{\gamma(q,\kappa)^2}{2} - 2\gamma(q,\kappa) - 1, & q \leq -1 - \frac{3\kappa}{8} \\ \kappa \frac{\gamma(q,\kappa)^2}{2}, & -1 - \frac{3\kappa}{8} \leq q \leq Q(\kappa) \\ 3q - \frac{1}{2} - \frac{1}{2}\sqrt{1 + 2q\kappa}, & q \geq Q(\kappa) \end{cases} \tag{5}$$

where

$$\gamma(q,\kappa) = \frac{\kappa + 4 - \sqrt{(\kappa+4)^2 - 8q\kappa}}{2\kappa} \tag{6}$$

and

$$Q(\kappa) = \frac{\kappa^2 + 8\kappa + 12 - 2\sqrt{2\kappa^2 + 16\kappa + 36}}{16\kappa} \tag{7}$$

The arguments are based on the infinite number of one-parametric families of exact solutions for moments of derivatives of conformal mappings and general facts from analysis of the harmonic measures, such as continuity and non-negativity of the β-spectrum. This is the first known explicit description of multi-fractal spectrum for non-trivial unbounded conformal mappings.

The spectrum (5) shows two transitions, one of which happens at negative $q = -1 - \frac{3\kappa}{8}$ due to singularity of the conformal mappings at the tip of the curve. Another transition happens at positive $q = Q(\kappa)$.

Note that these transitions can already be traced in the simplest example of the $\kappa = 0$ evolution: In such a case we have a deterministic growth of the ray on the real line and $\mathcal{F}_- = e^t \frac{w}{(1+w)^2}$. Derivative of this map $\mathcal{F}_-' = e^t \frac{(1-w)}{(1+w)^3}$ vanishes on the tip of the ray at $w = 1$, which, according to (4), results in transition at $q = -1$. On the other hand the derivative of this map goes to infinity as $w \to -1$ and we have transition at $q = Q(0) = 1/3$.

It worth to mention that several points of this spectrum (interior problem for $q = 2$ and $\kappa = 2$, $\kappa = 6$) have been recently found using computer algebra by B.Duplantier et al in [5].

Finally, we found that consideration of the interior version of the whole-plane SLE turns out to be useful for complete derivation of the version-independent multi-fractal spectrum of the bulk of SLE (see Appendix).

2 Derivative expectations

To find β-spectrum (4), one needs to evaluate moments of derivatives of conformal mappings. It turns out that moments of derivatives satisfy linear partial differential equations of the second order. To derive such equations it is convenient to change variable $w \to we^{iB(t)}$ and define the conformal transformations $z = f_\pm(w,t)$ in the "rotating frame", in which the immovable point $w = 1$ on the unit circle is mapped to the moving tip of the curve:

$$f_\pm(w,t) = \mathcal{F}_\pm(we^{iB(t)}, t) = e^{t+iB(t)} \left(w + \sum_{j=1\mp1}^{\infty} f_j^\pm(t) w^{\mp j} \right). \tag{8}$$

We remind that the unbounded mapping f_-, corresponding to the interior whole-plane SLE (1), (2), sends the unit disc in the w-plane to the complement of a curve that starts at some point and goes to infinity in the z-plane. On the other hand, the bounded mapping f_+, corresponding to the exterior whole-plane SLE (1), (3), sends the complement of the unit disc in the w-plane to the complement of a bounded curve that starts and ends at some finite points in the z-plane.

We now define the following functions

$$\rho(w,\bar{w}|q;\kappa) = e^{-qt}\langle(f_\pm'(w,t)\bar{f}_\pm'(\bar{w},t))^{q/2}\rangle = \lim_{T\to\infty} e^{\mp qT}\langle(F_\pm'(we^{i\tilde{B}(T)}, T)\bar{F}_\pm'(\bar{w}e^{-i\tilde{B}(T)}, T))^{q/2}\rangle, \tag{9}$$

where the prime denotes derivative wrt w or $\bar{w}$ (in general, we do not suppose that the second argument $\bar{w}$ of the above function is a complex conjugate of the first one w).

To estimate ρ one can use the following differential equation

$$L[\rho](w, \bar{w}|q; \kappa) = \sigma q \rho(w, \bar{w}|q; \kappa), \tag{10}$$

$$L = -\frac{\kappa}{2} \left(w \frac{\partial}{\partial w} - \bar{w} \frac{\partial}{\partial \bar{w}} \right)^2 + \frac{w+1}{w-1} w \frac{\partial}{\partial w} + \frac{\bar{w}+1}{\bar{w}-1} \bar{w} \frac{\partial}{\partial \bar{w}} - \frac{q}{(w-1)^2} - \frac{q}{(\bar{w}-1)^2} + q$$

where $\sigma = -1$ for the interior problem and $\sigma = 1$ for the exterior problem respectively.

Below, we present a simple derivation of equation (10) which relies on procedure first introduced by Hastings [8] (for a derivation using formal Ito calculus one can apply e.g. approach presented in [1]).

We start with the exterior whole-plane SLE. From (8), (9) it follows that

$$\frac{\partial}{\partial t} \langle g \rangle = q \langle g \rangle, \quad g = (f'_+(w,t) \bar{f}'_+(\bar{w},t))^{q/2} \tag{11}$$

Using the fact that for the Loewner chain (1) we have the composition rule $\mathcal{F}_+(w, t + \delta t) = \mathcal{F}_+(\delta \mathcal{F}(w, \delta t), t)$, where $\delta \mathcal{F}$ itself satisfies the Loewner chain equation

$$\frac{\partial \delta \mathcal{F}(w, \delta t)}{\partial \delta t} = w \frac{\partial \delta \mathcal{F}(w, \delta t)}{\partial w} \frac{w + e^{iB(t+\delta t)}}{w - e^{iB(t+\delta t)}}, \quad \delta \mathcal{F}(w, \delta t = 0) = w, \tag{12}$$

in the "rotating frame" (8) we get

$$f_+(w, t + \delta t) = f_+ \left(e^{-iB(t)} \delta \mathcal{F} \left(w e^{iB(t+\delta t)}, \delta t \right), t \right). \tag{13}$$

Solving the differential equation (12) up to the first order in δt (up to the second order in $\delta B = B(t + \delta t) - B(t)$) we obtain

$$e^{-iB(t)} \delta \mathcal{F} \left(w e^{iB(t+\delta t)}, \delta t \right) = w + \delta \phi(w) + \ldots, \quad \delta \phi(w) = w \frac{w+1}{w-1} \delta t + iw \delta B - \frac{w}{2} (\delta B)^2.$$

Then, with the help of (11), (13) we conclude that in the first order in δt:

$$g(w, \bar{w}; t + \delta t) = g \left(w + \delta \phi(w), \bar{w} + \delta \bar{\phi}(\bar{w}); t \right) (1 + \delta \phi'(w))^{q/2} \left(1 + \delta \bar{\phi}'(\bar{w}) \right)^{q/2} + \ldots$$

Equating expectations of RHS and LHS of the above and taking into account that $< \delta B >= 0$, $< (\delta B)^2 >= \kappa \delta t$ we find the time derivative of $\langle g \rangle$

$$\frac{\partial}{\partial t} \langle g \rangle = L [\langle g \rangle],$$

where differential operator L is given in (10). Finally, with the help of (9) and (11) we arrive at the differential equation (10) with $\sigma = 1$ for the exterior whole-plane SLE.

Since, according to (2), we defined the interior whole-plane SLE as a reversed Markovian process, equation for the interior whole-plane SLE has to be derived in the reversed time. This derivation coincides with that of the exterior case except $-t$ replaces t in (8), (9) and $-q$ replaces q in (11). So we arrive at equation (10) with $\sigma = -1$.

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3 Exact Solutions and the β-spectrum

In the main body of the article we study the interior problem ($\sigma = -1$) only, i.e. the unbounded mapping version of the whole plane SLE. So, we drop all $\pm$ sub/superscripts from our notations.

It is convenient to make the following change of dependent variable ρ that eliminates most singular terms in equation (10)

$$\rho(w, \bar{w}|q; \kappa) = ((1 - w)(1 - \bar{w}))^{\gamma} \, \Theta(w, \bar{w}|\gamma; \kappa), \quad q = 2\gamma + \frac{1}{2}\kappa\gamma - \frac{1}{2}\kappa\gamma^2, \tag{14}$$

where γ is given by (6). Then (10) becomes

$$\left(-\frac{\kappa}{2}(w - 1)(\bar{w} - 1)\left(w\frac{\partial}{\partial w} - \bar{w}\frac{\partial}{\partial \bar{w}} \right)^2 + (\kappa\gamma - 1)(w - \bar{w})\left(w\frac{\partial}{\partial w} - \bar{w}\frac{\partial}{\partial \bar{w}} \right) \right.$$

$$\left. + (w\bar{w} - 1)\left(w\frac{\partial}{\partial w} + \bar{w}\frac{\partial}{\partial \bar{w}} \right) + \left((\kappa - \kappa\gamma + 6)w\bar{w} + \frac{2\kappa\gamma - \kappa - 6}{2}(w + \bar{w}) \right)\gamma \right)\Theta = 0, \tag{15}$$

According to (8), (9) and (14), function $\Theta(w, \bar{w})$ has the following Taylor expansion around $w = 0$

$$\Theta(w, \bar{w}|\gamma; \kappa) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \theta_{i,j}(\gamma, \kappa)w^{i-1}\bar{w}^{j-1}, \quad \theta_{1,1} = 1. \tag{16}$$

Substituting expansions (16) into equation (15), we get the "two-dimensional", four-term recurrence relation for $\theta_{i,j}$:

$$\sum_{k=0}^{1}\sum_{l=0}^{1} C_{i,j}^{l,k}\theta_{i-l,j-k} = 0, \quad \theta_{1,1} = 1, \quad \theta_{i,j<1} = \theta_{i<1,j} = 0 \tag{17}$$

$$C_{i,j}^{0,0} = -\frac{\kappa}{2}(i - j)^2 - i - j + 2, \quad C_{i,j}^{1,1} = -\frac{\kappa}{2}(i - j)^2 + i + j - \kappa\gamma^2 + \kappa\gamma + 6\gamma - 4,$$

$$C_{i,j}^{0,1} = -\frac{\kappa}{2}(i - j + 1)^2 + (1 - \kappa\gamma)(i - j + 1) + \kappa\gamma^2 - \frac{1}{2}\kappa\gamma - 3\gamma, \quad C_{i,j}^{1,0} = C_{j,i}^{0,1},$$

Note, that any element $\theta_{i,j}(\gamma, \kappa)$ can be found in a consecutive manner by expressing $\theta_{i,j}$ as a linear combination of 3 elements: $\theta_{i,j-1}$, $\theta_{i-1,j}$ and $\theta_{i-1,j-1}$: At the first step one finds $\theta_{1,2}$, then $\theta_{1,3}$ etc up to $\theta_{1,l}$. Repeating similar procedure for the next row $\theta_{2,j}$ etc, up to the nth row, one gets all $\theta_{1..l,1..l}$.

Since $w = 0$ is a regular point of the conformal mappings, the moments of derivative of conformal mappings are analytic in $w, \bar{w}$ at $w = 0$. It then follows from the above described recurrent procedure that once a solution of equation (10) analytic and non vanishing at $w = 0$ is found for a given q, it will correspond to expectation of the q-th moment of the absolute value of derivative, since such a solution is unique (up to a constant factor).

To determine β-spectrum we will use an infinite number of families of exact solutions of equation (15) related to finite-band reductions of matrices of expansion coefficients $\theta_{i,j}$.

In more details: As easy to see from the above described recurrent procedure and relation (17), the expansion coefficients $\theta_{i,j}$ vanish for $|i - j| > M$, where $M = 0, 1, 2 \ldots$, if the recurrence coefficient $C_{i,j}^{0,1}$ vanishes at $j = i + M + 1$. As follows from (17) and (14), such a $2M + 1$-band truncation happens when

$$\kappa = 2\frac{M + 3\gamma}{M^2 + 2M\gamma + 2\gamma^2 - \gamma}, \quad q = \frac{\gamma(M + \gamma)(2M + 1 + \gamma)}{M^2 + 2M\gamma + 2\gamma^2 - \gamma} \tag{18}$$
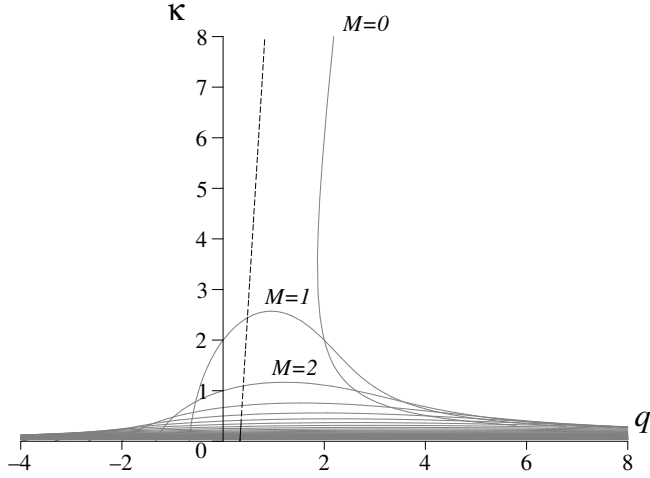
Figure 1: Gray curves show families (18) in the parametric (q, κ)-plane along which the $2M + 1$-diagonal truncations of solutions take place. Dashed dark curve marks transition in the β-spectrum at $q = Q(\kappa)$.

The above equations determine infinite number of curves on the parametric (q, κ)-plane (see Figure 1). In the rest of the paper, with the help of exact solutions, we find means β-spectrum along these curves and extrapolate it on the whole (q, κ)-plane.

Note that due to non-negativity of κ and according to (18)

$$\gamma \geq -\frac{M}{3}.$$

In the case of the above $2M + 1$-diagonal truncations, solutions to (15) can be represented as the Fourier polynomials

$$\Theta(w, \bar{w}) = \sum_{n=-M}^{M} w^n f_n (w\bar{w}), \tag{19}$$

where coefficients $f_n(\xi), \xi := w\bar{w}$ are regular at $\xi = 0$. Note, that since $\theta(w, \bar{w})$ is real and symmetric, the expansion coefficients are also real and symmetric, i.e. $\bar{\theta}_{i,j} = \theta_{i,j}$, $\theta_{i,j} = \theta_{i,j}$ and

$$f_{-n}(\xi) = \xi^n f_n(\xi), \quad \bar{f}_n = f_n. \tag{20}$$

Substituting Fourier expansion (19) into equation (10) we get the three-term recurrence relation for $f_n(\xi)$

$$\xi A_{n+1} f_{n+1} + A_{-n+1} f_{n-1} + (B_n + (1-\xi)C_n) f_n + 2\xi(\xi-1)\frac{df_n}{d\xi} = 0, \tag{21}$$

where

$$A_n = \frac{\kappa}{2}(n-\gamma)^2 + n - 3\gamma - \frac{\kappa}{2}\gamma(1-\gamma),$$

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$$B_n = -\kappa(n^2 + \gamma^2 - \gamma) + 6\gamma, \quad C_n = \kappa \frac{n^2 - 2\gamma + 2\gamma^2}{2} - n - 6\gamma.$$

Taking into account the symmetry relation (20), we see that in the case of the $2M + 1$-band truncation, $2M + 1$ equations (21) are reduced to a system of $M + 1$ first order linear ODEs for $M + 1$ functions $f_0(\xi), f_1(\xi), \ldots, f_M(\xi)$. By consecutive differentiations and linear transformations, this system can be always "diagonalized", i.e. decoupled into separate $M + 1$th order ODEs for each function $f_i(\xi)$. These equations of $M + 1$th order have polynomial in ξ coefficients and their singular points are $\xi = 0$, $\xi = 1$ and $\xi = \infty$.

As examples, we write down solutions for $M = 0$ and $M = 1$ cases:

The simplest case is the $M = 0$ family of solutions, which corresponds to the one-diagonal truncation (i.e. matrix $\theta_{i,j}$ is diagonal) and (15) is reduced to a single first order linear ODE with the following solution

$$\Theta(w, \bar{w}) = (1 - w\bar{w})^{\frac{-3\gamma^2}{2\gamma - 1}}, \quad \kappa = \frac{6}{2\gamma - 1}, \quad q = \frac{\gamma(\gamma + 1)}{2\gamma - 1},$$

where values of q and κ are taken from equations (18) with $M = 0$. According to (14), this corresponds to the one-parametric family of solutions $\rho = \rho_0(w, \bar{w})$ of equation (10) along the $q = \frac{(2+\kappa)(6+\kappa)}{8\kappa}$ curve in the parametric (q, κ)-plane

$$\rho_0 = \frac{((1 - w)(1 - \bar{w}))^{\frac{6+\kappa}{2\kappa}}}{(1 - w\bar{w})^{\frac{(6+\kappa)^2}{8\kappa}}}, \quad q = \frac{(2 + \kappa)(6 + \kappa)}{8\kappa}. \tag{22}$$

The second example is the $M = 1$ family of solutions (i.e. the case of the three-diagonal truncation) $\rho = \rho_1(w, \bar{w})$. Here $f_0(\xi)$ and $f_1(\xi)$ each satisfies a second order equation. These equations are amenable to the Gauss hypergeometric type equations and we get

$$\rho_1 = ((1 - w)(1 - \bar{w}))^{\gamma} (1 - w\bar{w})^{-\frac{(\gamma+1)(3\gamma^2+6\gamma-1)}{2\gamma^2+\gamma+1}} \left(\left(1 - \frac{w + \bar{w}}{2}\right) \Phi_1 + \frac{1 - 3\gamma}{1 + \gamma}(1 - w\bar{w})\frac{w + \bar{w}}{2}\Phi_2 \right)$$

$$\Phi_1 = {}_2F_1\left(\begin{array}{c} \frac{(\gamma+1)(1-3\gamma)}{2\gamma^2+\gamma+1}, \frac{1-\gamma-4\gamma^2}{2\gamma^2+\gamma+1}; \frac{(\gamma+1)^2}{2\gamma^2+\gamma+1} \end{array} \middle| w\bar{w} \right), \quad \Phi_2 = {}_2F_1\left(\begin{array}{c} \frac{(1-\gamma)(2+\gamma)}{2\gamma^2+\gamma+1}, \frac{2(1-\gamma^2)}{2\gamma^2+\gamma+1}; \frac{3\gamma^2+3\gamma+2}{2\gamma^2+\gamma+1} \end{array} \middle| w\bar{w} \right),$$

$$\kappa = \frac{2(3\gamma + 1)}{2\gamma^2 + \gamma + 1}, \quad q = \frac{\gamma(\gamma + 1)(\gamma + 3)}{2\gamma^2 + \gamma + 1}. \tag{23}$$

The goal of this work is determination of the β-spectrum of the whole plane SLE. For this purpose we need to look for asymptotic behavior of $2M + 1$-band solutions (19) in vicinity of the unit circle $w\bar{w} = 1$ (i.e. at $\xi \to 1$).

The system of $2M + 1$ linear ODEs (21) for $f := \{f_n(\xi), n = -M..M\}$ has $2M + 1$ linearly independent solutions that behave at $\xi \to 1$ as

$$f(\xi) \to (1 - \xi)^{-\beta_l} \mathcal{V}_l,$$

where $\beta_l, l = 0..2M$, are $2M + 1$ eigenvalues of a $2M + 1 \times 2M + 1$ tri-diagonal matrix and $\mathcal{V}_l$ is a ξ-independent $2M + 1$ component eigenvector corresponding to the eigenvalue β_l. Indeed, expanding solutions of (21) at $\xi \to 1$ as follows

$$f_n(\xi) = (1 - \xi)^{-\tilde{\beta}} \psi_n + \ldots, \quad n = -M..M \tag{24}$$

and substituting the above expansion into (21) we get the following three-term recurrence relation for ψ_n

$$R[\psi] = \tilde{\beta}\psi, \quad R[\psi]_n := \frac{1}{2}\left(A_{n+1}\psi_{n+1} + A_{-n+1}\psi_{n-1} + B_n\psi_n\right) \tag{25}$$

with $\tilde{\beta}$ being an eigenvalue of the $2M + 1 \times 2M + 1$ three-diagonal matrix R.

It is convenient to introduce the following function

$$\Psi(\varphi) := \sum_{n=-M}^{M} \psi_n e^{\imath n\varphi}, \quad \bar{\Psi} = \Psi \tag{26}$$

As follows from (19), (24) this is the "angular part" of $\Theta(w, \bar{w})$ at the unit circle $w = e^{\imath\varphi}$, i.e.

$$\Psi(\varphi) = \lim_{r \to 1-} \frac{\Theta\left(re^{\imath\varphi}, re^{-\imath\varphi}\right)}{(1 - r^2)^{\tilde{\beta}}} \tag{27}$$

To relate the above eigenvalue to the β-spectrum, we note that according to definitions (4) and (9)

$$\beta = \lim_{\epsilon \to 0+} \frac{\log \int_0^{2\pi} \rho\left(e^{-\epsilon + \imath\varphi}, e^{-\epsilon - \imath\varphi}|q;\kappa\right)d\varphi}{-\log\epsilon},$$

Then, from the fact that factor $((1-w)(1-\bar{w}))^\gamma$ in (14) ceases to be integrable on the unit circle $w = e^{\imath\varphi}$ when $\gamma < -1/2$, from the previous equation and eq. (27) we have

$$\beta = \begin{cases} \tilde{\beta} - 2\gamma - 1, & \gamma \leq -\frac{1}{2} \\ \tilde{\beta}, & \gamma \geq -\frac{1}{2} \end{cases}, \quad \tilde{\beta} \in \{\beta_l, l = 0..2M\}, \tag{28}$$

provided $\Psi(\varphi)$ does not have zeros at $\varphi = 0$ when $\gamma < -1/2$ (we will see later that this is the case). Therefore, the β-spectrum of the whole-plane SLE is determined by an eigenvalue of the three diagonal matrix (25).

The eigenvalues $\{\beta_l, l = 0..2M\}$ can be found exactly: the spectrum $\beta_l = \beta_l(M, \gamma)$ is a quadratic function of l with the biggest eigenvalue being either β_0 or β_{2M} depending on values M and γ. Below we will show that $\tilde{\beta} = \max(\beta_l, l = 0..2M)$, i.e., in the case of $2M + 1$ truncation, the β-spectrum is determined by the maximal eigenvalue of the three-diagonal matrix (25).

As a consequence one can observe transition in the means β-spectrum at points of the parametric (q, κ)-plane where $\beta_0 = \beta_{2M}$. Such a transition happens when $q = Q(\kappa)$ (see Eqs (5-7)).

It is interesting to note that, in an addition to $\kappa = 0$ example given in the introduction, another simple explicit example of such a transition can be observed on exact solution (23) for the $M = 1$ family, where at $|w| \to 1$, $f_0 \to \text{Const}\,(1 - w\bar{w})^{-\frac{(\gamma+1)(3\gamma^2+6\gamma-1)}{2\gamma^2+\gamma+1}}$ when $\gamma > \frac{\sqrt{57}-5}{16}$ and $f_0 \to \text{Const}\,(1 - w\bar{w})^{-\frac{\gamma^2(3\gamma+1)}{2\gamma^2+\gamma+1}}$ when $-1/3 < \gamma < \frac{\sqrt{57}-5}{16}$.

4 Computation of the spectrum

To get the β-spectrum we need to determine eigenvalues of the three-diagonal matrix (25). Due to mirror symmetry (i.e. symmetry w.r.t. reflections $n \to -n$) its eigenvectors are either symmetric or antisymmetric in n. By reality and symmetry of ρ and Θ (i.e. by eq (20)) we need to consider only symmetric eigenvectors

$$\psi_n = \psi_{-n}, \quad \Psi(\varphi) = \Psi(-\varphi)$$

It is now convenient to rewrite the three-term relation (25) in the form of the second order linear ODE for $\Psi(\varphi)$:

$$\frac{\kappa}{2}\left(1-\cos(\varphi)\right)\Psi''(\varphi)-(1-\kappa\gamma)\sin(\varphi)\Psi'(\varphi)+\left(\left(\kappa\frac{2\gamma-1}{2}-3\right)\gamma\cos(\varphi)-\left(\kappa\frac{\gamma-1}{2}-3\right)\gamma-\tilde{\beta}\right)\Psi(\varphi)=0 \tag{29}$$

It is also convenient to make the change of the independent variable $x = \frac{1-\cos(\varphi)}{2}$ in the above equation and also reparametrize $\tilde{\beta} = \tilde{\beta}(\lambda)$ as

$$\tilde{\beta} = 3q - 1 - \frac{q}{M+\gamma} + \left(\kappa\frac{1-\lambda-2M-4\gamma}{4}+1\right)(2M-\lambda) \tag{30}$$

After such changes, equation (29) is amenable to the hypergeometric form and has the following general solution
$$\Psi(\varphi) = \mathcal{C}_1 x^{\lambda/2} g_1(x) + \mathcal{C}_2 x^{(\lambda-1)/2} g_2(x)$$

with

$$g_1(x) = {}_2F_1\left(a,b;\frac{1}{2}|1-x\right), \quad g_2(x) = \sqrt{x(1-x)}\,{}_2F_1\left(\frac{1}{2}+a,\frac{1}{2}+b;\frac{3}{2}|1-x\right) \tag{31}$$

where

$$a = \frac{\lambda}{2}+\gamma-\frac{1}{\kappa}-\frac{\sqrt{1+2q\kappa}}{\kappa}, \quad b = \frac{\lambda}{2}+\gamma-\frac{1}{\kappa}+\frac{\sqrt{1+2q\kappa}}{\kappa},$$

or in terms of γ and M (see Eq.(18))

$$a = \frac{\lambda}{2}-M, \quad b = \frac{\lambda}{2}+\gamma\frac{3M+1+4\gamma}{M+3\gamma} \tag{32}$$

As seen from (31) and (32) g_1 has a form of the Fourier polynomial (26) which is an even function of φ if λ is an even non-negative integer. On the other hand, when λ is an odd positive integer, g_2 is a Fourier polynomial which is odd in φ. Therefore, for the $2M+1$-diagonal reduction we have to choose an eigenvalue from the $\lambda = l \in \{0,2,4,\dots 2M\}$ subset of (30) with even eigenfunctions which (up to a constant factor) equal

$$\Psi = (1-\cos(\varphi))^{l/2}{}_2F_1\left(\frac{l}{2}-M,\frac{l}{2}+\gamma\frac{3M+1+4\gamma}{M+3\gamma};\frac{1}{2}\left|\frac{1+\cos(\varphi)}{2}\right.\right), \quad l=0,2,4,\dots 2M \tag{33}$$

Corresponding eigenvalues equal

$$\beta_l = \frac{2(M+3\gamma)\gamma^2-(2M^2+M-8\gamma^2+\gamma)l+(M+3\gamma)l^2}{2(M^2+2M\gamma+2\gamma^2-\gamma)} \tag{34}$$

The biggest eigenvalue, depending on M and γ, is either

$$\beta_0 = \frac{M+3\gamma}{M^2+2M\gamma+2\gamma^2-\gamma}\gamma^2 = \frac{\kappa\gamma^2}{2} \tag{35}$$

or

$$\beta_{2M} = \frac{(M+\gamma)(6M\gamma-M+3\gamma^2)}{M^2+2M\gamma+2\gamma^2-\gamma} = 3q-\frac{1}{2}-\frac{1}{2}\sqrt{1+2q\kappa} \tag{36}$$

These two eigenvalues are equal at the transition point

$$\gamma = \gamma_M := \frac{\sqrt{36M^2+20M+1}-6M+1}{16} \tag{37}$$

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or, equivalently, at $q = Q(\kappa)$.

Now, we will show that in the case of the $2M + 1$-diagonal reduction $\tilde{\beta}$ equals the biggest eigenvalue.

Indeed, by non-negativity of derivative moments, $\Psi(\varphi)$ should be non-negative i.e. cannot oscillate on the unit circle. In the case when $\gamma > \gamma_M$, (i.e. in the case when $\beta_{2M} > \beta_0$) the only eigenfunction that does not oscillate on the unit circle is that corresponding to the biggest eigenvalue β_{2M}. According to (33), up to a constant factor, it equals

$$\Psi = (1 - \cos(\varphi))^M \qquad (38)$$

Rather than proving directly that this is the only non-oscillating eigenfunction when $\gamma > \gamma_M$, we note that, as follows from (34), the only positive even eigenvalues in the vicinity of point $\gamma = \gamma_M$ are β_0 and β_{2M}. The eigenfunction that corresponds to $l = 0$ in (33) equals to 1 at $x = 0$ and is negative at $x = 1$, when $\gamma > \gamma_M$. Therefore the only choice for $\tilde{\beta}$ in some vicinity of $\gamma = \gamma_M$ for $\gamma > \gamma_M$ is $\tilde{\beta} = \beta_{2M}$. Also, for all $\gamma > \gamma_M$ the rest of eigenvalues $\beta_{2i}(\gamma), i = 0..M - 1$ is strictly less than β_{2M}. Therefore, by continuity of the β-spectrum, to the right from the transition point $\gamma = \gamma_M$ we have

$$\tilde{\beta} = \beta_{2M} \quad \text{if} \quad \gamma \geq \gamma_M.$$

To the left from the transition point $\gamma = \gamma_M$, the bigger eigenvalue is β_0. Here, not only the eigenfunction (38) is non-negative. To proceed, we note that, as follows from (34), all the β_{2i} except β_0 are negative when $\gamma < \sqrt{M^2 + \frac{M}{3}} - M$. Therefore, in the range $-\frac{M}{3} \leq \gamma \leq \sqrt{M^2 + \frac{M}{3}} - M$, $\tilde{\beta} = \beta_0$. On the rest of the interval on the left of $\gamma = \gamma_M$, i.e. at $\sqrt{M^2 + \frac{M}{3}} - M < \gamma < \gamma_M$, the only positive eigenvalues are β_0 and β_{2M}, but since $\beta_0 > \beta_{2M}$ for $\gamma < \gamma_M$, by continuity of the beta spectrum, we have

$$\tilde{\beta} = \beta_0 \quad \text{if} \quad -\frac{M}{3} \leq \gamma \leq \gamma_M.$$

Therefore, in the case of the $2M + 1$-diagonal families of solutions β-spectrum corresponds to the biggest eigenvalue of the three-diagonal matrix (25) and according to (35), (36), (37) along the infinite number of curves (18) in the parametric (q, κ)-plane we have

$$\tilde{\beta} = \begin{cases} \frac{1}{2}\kappa\gamma^2, & q \leq Q(\kappa) \\ 3q - \frac{1}{2} - \frac{1}{2}\sqrt{1 + 2q\kappa}, & q \geq Q(\kappa) \end{cases}$$

which, together with (28), leads to the conclusion that along these curves, the integral means β-spectrum of the whole-plane SLE is given by equation (5). Therefore, we have derived the main result of the paper (i.e. equation (5)) for (q, κ) belonging to these curves (see Figure 1).

Finally, we put forward the hypothesis that this result can be continued from the infinite family of curves to the upper $\kappa \geq 0$ parametric (q, κ)-plane.

5 Concluding Remarks

In summary: From elementary consideration it follows that any analytic and non-vanishing at $w = 0$ solution of equation for moments (10) is unique (up to constant factor) and therefore provides a point in the β-spectrum. Continuing $\beta(q, \kappa)$ spectrum derived along the infinite number of curves in the parametric (q, κ)-plane, we have presented exact description (5-7) of the multifractal spectrum for the unbounded whole-plane SLE.

Note that continuation of the spectrum from the curves to the plane can be shown for $q \leq 0$ using method presented in [1] (see Appendix).

The question of continuation of the spectrum from the curves to the plane for $q > 0$ is open.

Another interesting question is to study the Loewner evolution driven by non-Brownian Levy processes. In difference from the case of the Schramm-Loewner Evolution the moments of derivatives for such processes satisfy equations which are not differential. However, one can rewrite such equations in the form of a nine-term "two-dimensional" recurrence relation for the expansion coefficients of function $\rho(w, \bar{w})$ which also admits some finite-diagonal reductions [11], leading to exact solutions for some Levy processes. It will be interesting to study such processes.

Appendix

In the case of the exterior problem ($\sigma = 1$) one needs to consider solutions of equation (10) that are analytic at $w = \infty$. Similarly to the interior case, by the change of dependent variable

$$\rho = \left((1 - w^{-1})(1 - \bar{w}^{-1})\right)^\gamma \Theta \tag{39}$$

one eliminates most singular terms in (10) and comes to a four-term recurrence relation for the coefficients of the expansion of $\Theta(w, \bar{w})$ in $1/w, 1/\bar{w}$ at $w \to \infty$. This relation shows $2M + 1$-band truncation when

$$\kappa = 2\frac{M - \gamma}{M^2 + 2M\gamma + \gamma}, \quad q = \frac{\gamma(M + \gamma)(2M + 1 + \gamma)}{M^2 + 2M\gamma + \gamma}, \tag{40}$$

or, in terms of q and κ, when

$$q = \frac{M(2 - \kappa M)(\kappa M + \kappa + 2)(\kappa M + \kappa + 4)}{2(2\kappa M + \kappa + 2)^2}$$

Note that due to non-negativity of κ, the first non-trivial reduction in the exterior case corresponds to $M = 1$, and in (40)

$$\frac{-M^2}{2M + 1} < \gamma \leq M, \quad M = 1, 2, 3, \dots$$

In the case of the above $2M + 1$-diagonal truncations, solutions Θ can be represented as the Fourier polynomials

$$\Theta(w, \bar{w}) = \sum_{n=-M}^{M} w^{-n} f_n \left(1/(w\bar{w})\right),$$

where coefficients $f_n(\xi), \xi = 1/(w\bar{w})$, also obey symmetry relation (20) and are solutions of a system of $M + 1$ linear first order ODE's.

The β-spectrum is determined by an eigenvalue of a three-diagonal matrix. In the case of $2M + 1$-diagonal truncation of the exterior problem these eigenvalues equal

$$\beta_l = \frac{\kappa}{2}\gamma^2 + \frac{4\kappa\gamma - \kappa - 4}{4}l + \frac{\kappa}{4}l^2, \quad l = 0, 2, 4, \dots, 2M$$

In contrast to the interior problem, the β-spectrum is determined by the only one eigenvalue corresponding to $l = 0$,

$$\beta_0 = \frac{\kappa}{2}\gamma^2,$$

which is not always maximal (this is in accordance with the β-spectrum proposed in [1]). Indeed, the above eigenvalue is maximal only when

$$q > \mathcal{Q}^+(\kappa), \quad \mathcal{Q}^+(\kappa) = -(\kappa + 4)^2(\kappa + 8)/128 \tag{41}$$

Although for smaller values of q the bigger eigenvalue is

$$\beta_{2M} = -q - \frac{1}{2} - \frac{1}{2}\sqrt{1 - 2q\kappa},$$

the β-spectrum is nevertheless defined by β_0: Due to the fact that factor $\left((1 - w^{-1})(1 - \bar{w}^{-1})\right)^\gamma$ in equation (39) ceases to be integrable on the unit circle when $\gamma < -1/2$, β corresponding to β_0 gets additional contribution due to "angular" singularity of solution ρ at $\xi \to 1$, $\varphi \to 0$ leading to

$$\beta = \beta_0 - 2\gamma - 1, \quad \gamma < -1/2$$

and this value of β is always bigger than β_{2M}. On the other hand, there is no such additional contribution in the case of eigenvalue β_{2M} since, similarly to the interior problem, the "angular part" of the eigenfunction corresponding to eigenvalue β_{2M} has form (38). It vanishes at $\varphi \to 0$ cancelling effect of singularity of factor $\left((1 - w^{-1})(1 - \bar{w}^{-1})\right)^\gamma$ at $\varphi \to 0$.

Note that, although the β-spectrum of the exterior problem does not show an additional transition at $q = \mathcal{Q}^+(\kappa)$, there is the transition for the blow-up rates of solution of (10) at $|w| \to 1$ and φ away from zero at $q = \mathcal{Q}^+(\kappa)$.

In the case (41), when q exceeds critical value, the analysis of solution of equation (10) turns out to be simple, and, as shown in [1] the β-spectrum can be found in this region of the parametric (κ, q)-plane by using facts from general theory of parabolic PDE and the harmonic measure theory. Approach of [1] mainly relies on asymptotic analysis of equation (10) in vicinity of the singular point $w = 1$, corresponding to the tip of the curve: Taking the $\epsilon \to 0$, $w = 1 + \imath\epsilon\zeta$, $\bar{w} = 1 - \imath\epsilon\bar{\zeta}$ limit of (10), one gets the following equation

$$\left(-\frac{\kappa}{2}\left(\frac{\partial}{\partial\zeta} + \frac{\partial}{\partial\bar{\zeta}}\right)^2 + \frac{2}{\zeta}\frac{\partial}{\partial\zeta} + \frac{2}{\bar{\zeta}}\frac{\partial}{\partial\bar{\zeta}} - q\left(\frac{1}{\zeta^2} + \frac{1}{\bar{\zeta}^2}\right)\right)\rho(\zeta, \bar{\zeta}) = 0,$$

which has a solution of the form

$$\rho = (\zeta - \bar{\zeta})^{-\kappa\gamma^2/2}(\zeta\bar{\zeta})^\gamma.$$

A part of the β-spectrum is then determined by blow-up rates of such a solution, i.e. by exponents γ and $-\kappa\gamma^2/2$.

In more details: Rewriting equation (10) in the polar coordinates $w = re^{\imath\varphi}$, one gets the equation for $\mu(r, \varphi) = \rho\left(re^{\imath\varphi}, re^{-\imath\varphi}|q; \kappa\right)$, which for $\sigma = 1$ writes as

$$\frac{\kappa}{2}\mu_{\varphi\varphi} + r\frac{(r^2 - 1)\mu_r - 2\sin(\varphi)\mu_\varphi}{r^2 + 1 - 2r\cos(\varphi)} + q\left(\frac{r^4 + 4r^2(1 - r\cos(\varphi)) - 1}{(r^2 + 1 - 2r\cos(\varphi))^2} - 1\right)\mu = 0 \tag{42}$$

This is a parabolic PDE with the radius $r \to 1$ playing the role of a "time" variable and angle φ being a "spatial" variable.

Function μ is bounded on any circle of radius $r > 1$ and is singular at the unit circle $r = 1$. The authors of [1] introduced the functions

$$u = \left(-\log(r - 1)\right)^\delta \Phi, \quad v = \frac{\Phi}{\left(-\log(r - 1)\right)^\delta},$$

$$\Phi = (r - 1)^{-\frac{\kappa\gamma^2}{2}}(r^2 - 2r\cos(\varphi) + 1)^\gamma G(r^2 - 2r\cos(\varphi) + 1) \tag{43}$$

which are bounded on the circle of radius $r_0 > 1$ that is sufficiently close to 1. In the above equations δ is a positive which can be arbitrarily small and $G(x)$ is bounded and positive on the unit circle (i.e. $G(x) > 0$ when $x = 2 - 2\cos(\varphi)$).

Functions u and v are positive in the annulus $1 < r \leq r_0$. They are constructed in such a way that $\Lambda v > 0$, while $\Lambda u < 0$ in the annulus $1 < r \leq r_0$, where Λ is the parabolic operator corresponding to equation (42). By the maximum principle for parabolic equations and positivity of μ, there exist constants U and V, such that $Vv < \mu < Uu$ for all $1 < r < r_0$. Then u and v have the same blow-up rates and the β-spectrum is defined by the singular part $(r-1)^{-\frac{\kappa\gamma^2}{2}}(r^2 - 2r\cos(\varphi) + 1)^\gamma$ of Φ in (43).

Function G in (43) is obtained from the condition that $\Lambda\Phi$ vanishes on the unit circle by substitution of (43) into (10), factoring $(r-1)^{-\frac{\kappa\gamma^2}{2}}(r^2 - 2r\cos(\varphi)+1)^\gamma$ out, and fixing $r = 1$. This results in the following ODE for $\Psi(\varphi) = G(2 - 2\cos(\varphi))$

$$\kappa(1 - \cos(\varphi))\Psi''(\varphi) + 2(\kappa\gamma - 1)\sin(\varphi)\Psi'(\varphi) - \gamma(\kappa+2)(1 - \cos(\varphi))\Psi(\varphi) = 0.$$

Changing variable from φ to $x = 2 - 2\cos(\varphi)$, $0 \leq x \leq 4$ leads to the hypergeometric equation for $G(x)$

$$x(x-4)G''(x) + ((a+b+1)x - 2 - 4a - 4b)\,G'(x) + abG(x) = 0,$$

$$a = \gamma - \frac{1 + \sqrt{1 - 2q\kappa}}{\kappa}, \quad b = \gamma - \frac{1 - \sqrt{1 - 2q\kappa}}{\kappa}$$

The solution of this equation that is even in φ up to a constant factor equals

$$G(x) = {}_2F_1\left(\frac{1}{2} + a, \frac{1}{2} + b; \frac{1}{2}|1 - \frac{x}{4}\right), \tag{44}$$

It turns out that (44) attains its minimum at endpoints $x = 0$ or $x = 4$ (i.e. at $\varphi = 0$ or $\varphi = \pi$) [1]. Since, according to (44), $G(4) = 1$, this minimum is positive if $G(0) > 0$, i.e. if $G(0) = {}_2F_1\left(\frac{1}{2} + a, \frac{1}{2} + b; \frac{1}{2}|1\right) = \frac{\Gamma(1/2)\Gamma(1/2 - a - b)}{\Gamma(1/2 - a)\Gamma(1/2 - b)} > 0$. Therefore, $G(x)$ ceases to be positive on the unit circle when $1/2 - a - b = 0$, which happens at $q = \frac{3(\kappa+4)^2}{32\kappa}$, or when $1/2 - b = 0$, which happens at $q = \mathcal{Q}^+(\kappa)$. Summing up: function $G(x)$ is positive on the unit circle when

$$\mathcal{Q}^+(\kappa) < q < \frac{3(\kappa+4)^2}{32\kappa},$$

and in the above interval of q, the β-spectrum is defined by exponents (blow-up rates) of Φ in (43).

For $q > \frac{3(\kappa+4)^2}{32\kappa}$ the $\beta(q)$-spectrum continues as a linear function of unit slope, which follows from the theory of harmonic measure: The $\beta(q)$-spectrum must be a continuous convex function of q with $|\beta'(q)| \leq 1$. Since $\beta' = 1$ at $q = \frac{3(\kappa+4)^2}{32\kappa}$, at bigger q it can only be continued as a line with the unit slope.

As have been discussed above, the β-spectrum does not show transition at $q = \mathcal{Q}^+(\kappa)$ and, therefore, in the exterior problem

$$\beta = \begin{cases} \kappa\frac{\gamma(q,\kappa)^2}{2} - 2\gamma(q,\kappa) - 1, & q \leq -1 - \frac{3\kappa}{8} \\ \kappa\frac{\gamma(q,\kappa)^2}{2}, & -1 - \frac{3\kappa}{8} \leq q \leq \frac{3(\kappa+4)^2}{32\kappa} \\ q - \frac{(\kappa+4)^2}{16\kappa}, & q \geq \frac{3(\kappa+4)^2}{32\kappa} \end{cases}$$

Note that for $q < \mathcal{Q}^+(\kappa)$ the analysis of the β-spectrum in [1] turns out not to be complete due to non-positiveness of $G(x)$ in this region. So, the only foundations so far for the values of the spectrum here follow from exact solutions on families of curves (40).

Moreover, the solution of the PDE for moments in the exterior case itself does not provide complete derivation for the bulk spectrum for the whole range of q. Recall, that the bulk spectrum

is defined for generic points of the curve away from the tip. However, in this version of the whole-plane SLE the curve has two tips: Apart of the growing end at $w = 0$ (i.e. $r = 1, \varphi = 0$) the conformal mapping has additional singularity at the origin of the curve, that gives own contribution to the integral (4). An effect of this contribution results in the above mentioned transition in the blow-up rates away from $\varphi = 0$ at $q = \mathcal{Q}^+(\kappa)$.

Therefore, to complete the derivation of the bulk spectrum one should consider blow up-rates away from $\varphi = 0$ for both versions of the whole plane SLE: the exterior version for $q \geq 0$ and the interior version for $q \leq 0$. The Legendre transform of so derived bulk spectrum corresponds to the multi-fractal harmonic spectrum predicted from quantum gravity by B.Duplantier [4].

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References

[1] D. Beliaev, S. Smirnov, *Harmonic measure and SLE*, Commun. Math. Phys. 290, 577595 (2009).

[2] D. Beliaev and S. Smirnov, *Harmonic measure on fractal sets*, Eur. Congress of Math., pages 41-59. Eur. Math. Soc., Zurich, (2005).

[3] John Cardy, *SLE for theoretical physicists*, Ann.Phys. 318 (2005) 81-118

[4] B.Duplantier, *Conformally invariant fractals and potential theory*, Phys.Rev.Lett., 84(7): 1363-1367 ,(2000).

[5] Bertrand Duplantier, Thi Phuong Chi Nguyen, Thi Thuy Nga Nguyen, Michel Zinsmeister, *Coefficient estimates for whole-plane SLE processes*, `http://hal.inria.fr/hal-00609774`, (2011).

[6] Ilya A. Gruzberg, *Stochastic geometry of critical curves, Schramm–Loewner evolutions and conformal field theory*, J. Phys. A: Math. Gen. **39**, no. 41 (2006) 12601–12655.

[7] T.C. Halsey, M.H. Jensen, L.P. Kadanoff, I. Procaccia, and B.I. Shraiman, *Fractal measures and their singularities: the characterization of strange sets*, Phys. Rev. A (3), 33(2): p. 1141, 1986.

[8] Matthew B. Hastings, *Exact Multifractal Spectra for Arbitrary Laplacian Random Walks*, Phys. Rev. Lett. **88** (2002) 055506.

[9] Gregory F. Lawler, *Conformally invariant processes in the plane*, Mathematical Surveys and Monographs, **114**, Amer. Math. Soc., Providence, RI, (2005).

[10] Gregory F. Lawler, *Conformal invariance and 2D statistical physics*, *Bull. Amer. Math. Soc.* **46** (2009) 35–54.

[11] Igor Loutsenko, SLE_κ: *correlation functions in the coefficient problem*, J. Phys. A: Math. Theor. 45 275001, 2012,(2012)