On estimates of solutions of Fokker–Planck–Kolmogorov equations with potential terms and non uniformly elliptic diffusion matrices

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Abstract

We consider Fokker–Planck–Kolmogorov equations with unbounded coefficients and obtain upper estimates of solutions. We also obtain new estimates involving Lyapunov functions.

1. Introduction

The goal of this work is to obtain upper estimates of solutions of the Fokker–Planck–Kolmogorov equation

$$\partial_t\mu = \partial_{x_i}\partial_{x_j}\bigl(a^{ij}\mu\bigr) - \partial_{x_i}\bigl(b^i\mu\bigr) + c\mu. \tag{1.1}$$

Throughout the summation over repeated indices is meant. Let $T>0$. We shall say that a locally finite Borel measure μ on $\mathbb{R}^d\times(0,T)$ is given by a flow of Borel measures $(\mu_t)_{t\in(0,T)}$ if for every Borel set $B\subset\mathbb{R}^d$ the mapping $t\to\mu_t(B)$ is measurable and for every function $u\in C_0^\infty(\mathbb{R}^d\times(0,T))$ one has

$$\int_{\mathbb{R}^d\times(0,T)} u(x,t)\,\mu(dx\,dt) = \int_0^T\int_{\mathbb{R}^d} u(x,t)\,\mu_t(dx)\,dt.$$

A typical example is $\mu(B) = P(x_t\in B)\,dt$, where x_t is a random process. Set

$$Lu = a^{ij}\partial_{x_i}\partial_{x_j}u + b^i\partial_{x_i}u + cu.$$

We shall say that a measure $\mu=(\mu_t)_{t\in(0,T)}$ satisfies equation (1.1) if a^{ij}, b^i and c are locally integrable with respect to the measure $|\mu|$ (the total variation of μ) and

$$\int_0^T\int_{\mathbb{R}^d}\bigl[\partial_t u(x,t) + Lu(x,t)\bigr]\,\mu_t(dx)\,dt = 0$$

for every $u\in C_0^\infty(\mathbb{R}^d\times(0,T))$. The measure μ satisfies the initial condition $\mu|_{t=0}=\nu$, where ν is a Borel locally finite measure on $\mathbb{R}^d$, if for every function $\zeta\in C_0^\infty(\mathbb{R}^d)$ there holds the equality

$$\lim_{t\to0}\int_{\mathbb{R}^d}\zeta(x)\,\mu_t(dx) = \int_{\mathbb{R}^d}\zeta(x)\,\nu(dx).$$

The following assertion is trivial and can be found in [3], [9].

Lemma 1.1. *Let $\mu=(\mu_t)_{t\in(0,T)}$ be a solution of equation* (1.1)*, let $u\in C^{1,2}(\mathbb{R}^d\times(0,T))$ be such that $u(t,x)=0$ if $x\notin U$ for some ball $U\subset\mathbb{R}^d$. Then there exists a set $J_u\subset(0,T)$ of full Lebesgue measure in $(0,T)$ such that for all $s,t\in J_u$*

$$\int_{\mathbb{R}^d} u(x,t)\,\mu_t(dx) = \int_{\mathbb{R}^d} u(x,s)\,\mu_s(dx) + \int_s^t\int_{\mathbb{R}^d}\bigl[\partial_\tau u(x,\tau) + Lu(x,\tau)\bigr]\,\mu_\tau(dx)\,d\tau.$$

Moreover, if, in addition, $u\in C(\mathbb{R}^d\times[0,T))$, the measure $\mu=(\mu_t)_{0<t<T}$ satisfies the initial condition $\mu|_{t=0}=\nu$ and a^{ij}, $b^i, c\in L^1(U\times[0,T],\mu)$, then we may assume that for every $t\in J_u$

$$\int_{\mathbb{R}^d} u(x,t)\,\mu_t(dx) = \int_{\mathbb{R}^d} u(x,0)\,\nu(dx) + \int_0^t\int_{\mathbb{R}^d}\bigl[\partial_\tau u(x,\tau) + Lu(x,\tau)\bigr]\,\mu_\tau(dx)\,d\tau.$$

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We shall say that a Borel measure σ is a subprobability on $\mathbb{R}^d$ if $\sigma \geq 0$ and $\sigma(\mathbb{R}^d) \leq 1$. A subprobability measure σ on $\mathbb{R}^d$ is probability if $\sigma(\mathbb{R}^d) = 1$.

A function $V \in C^{1,2}(\mathbb{R}^d \times (0,T)) \bigcap C(\mathbb{R}^d \times [0,T))$ is termed a Lyapunov function if for every closed interval $[a,b] \subset (0,J)$ one has

$$\lim_{|x|\to+\infty} \min_{t\in[a,b]} V(x,t) = +\infty.$$

We shall obtain L^p and L^∞ local and global estimates of the densities of solutions of equation (1.1). Our main interest is in the case of unbounded coefficients of the operator L. If the coefficients are globally bounded or have a linear growth, then there are well-known Gaussian estimates (see, e.g., [1] and [13]).

Global boundedness of the densities (with upper estimates) for solutions of the Cauchy problem for equation (1.1) without any restrictions on the growth of coefficients is established in [6] for sufficiently regular initial conditions. More precisely, the existence of a density of the initial condition with finite entropy is required. In [2], [10], [11] and [16] the transition kernels of the semigroup $\{T_t\}$ are investigated such that for every nonnegative bounded continuous function f the function $T_t f$ is the minimal nonnegative solution of the Cauchy problem $\partial_t u = Lu$, $u|_{t=0} = f$. It is assumed there that the coefficients are locally Hölder continuous and the diffusion matrix A is uniformly nondegenerate and continuously differentiable. Moreover, the coefficients do not depend on t. The principal results of the cited papers give certain upper estimates of the kernel densities and the continuity of semigroup T_t in various functional classes. The conditions on the coefficients in these papers are formulated in terms of certain Lyapunov functions. The kernel of $\{T_t\}$ satisfies equation (1.1), but the initial condition is Dirac's measure, so the results from [6] do not apply.

In [14] and [15], some estimates of densities are obtained for arbitrary initial conditions. The main idea of these works is to deduce global bounds from local estimates in [4] by using appropriate scalings. Note that in [14] and [15] the coefficients b and c are assumed to be only integrable, but the diffusion matrix is assumed to uniformly bounded, uniformly nondegenerate and uniformly Lipschitzian.

In the present work we generalize the results from [14] and [15] to the case where the diffusion matrix can be unbounded and need not be uniformly elliptic. Moreover, we generalize the estimates from [11] and [16] involving Lyapunov functions. The main difference between the estimates from [11], [16] and the usual estimates with Lypunov functions is that the former do not depend on the initial condition.

It is worth mentioning that various lower estimates are considered in [7]. The existence and uniqueness problems are investigated in [3] and [9]. A recent survey on elliptic and parabolic equations for measures is given in [5].

The next section is concerned with estimates involving Lyapunov functions. In the last section we obtain local and global L^p and L^∞ estimates and investigate the behavior of densities at infinity.

2. Estimates with Lyapunov functions

In this section we assume that $c \leq 0$ and investigate a solution μ that is given by a family of nonnegative measures μ_t such that $|c| \in L^1(\mu)$ and

$$\mu_t(\mathbb{R}^d) \leq \nu(\mathbb{R}^d) + \int_0^t \int_{\mathbb{R}^d} c(x,s)\, \mu_s(dx)\, ds. \tag{2.1}$$

In particular, μ_t are subprobability measures on $\mathbb{R}^d$. There are no other restrictions on the coefficients a^{ij}, b^i and c.

Note that the kernels considered in [10] satisfy condition (2.1). Moreover, in the case of globally bounded coefficients any solution μ which is given by a family of subprobability measures $(\mu_t)_{t\in(0,T)}$ satisfies condition (2.1). Note also that if c is continuous and μ_t is a weak limit of a sequence of measures μ_t^n satisfying (2.1), then condition (2.1) is fulfilled for each μ_t. Hence this condition is fulfilled for every solution μ obtained as a limit of solutions of equations with bounded coefficients. Thus, this is a natural condition that is a generalization of the hypothesis that μ_t is a subprobability measure for almost all t in the case $c = 0$.

Theorem 2.1. *Let $\mu = (\mu_t)_{0<t<T}$ be a solution of the Cauchy problem $\partial_t\mu = L^*\mu$, $\mu|_{t=0} = \nu$ such that $c \le 0$, μ_t and ν are subprobability measures on $\mathbb{R}^d$ and condition* (2.1) *holds. Assume that there exists a Lyapunov function V such that for some positive functions $K, H \in L^1((0,T))$ one has*

$$\partial_t V(x,t) + LV(x,t) \le K(t) + H(t)V(x,t).$$

Assume also that $V(\,\cdot\,,0) \in L^1(\nu)$. Then for almost all $t \in (0,T)$

$$\mu_t(\mathbb{R}^d) = \nu(\mathbb{R}^d) + \int_0^t \int_{\mathbb{R}^d} c(x,s)\,\mu_s(dx)\,ds$$

and

$$\int_{\mathbb{R}^d} V(x,t)\,\mu_t(dx) \le Q(t) + R(t)\int_{\mathbb{R}^d} V(x,0)\,\nu(dx),$$

where

$$R(t) = \exp\Bigl(\int_0^t H(s)\,ds\Bigr), \quad Q(t) = R(t)\int_0^t \frac{K(s)}{R(s)}\,ds.$$

Proof. Let $\zeta_N \in C^2([0,+\infty))$ be such that $0 \le \zeta' \le 1$, $\zeta'' \le 0$, and $\zeta_N(s) = s$ if $s \le N-1$ and $\zeta(s) = N$ if $s > N+1$. Substitute the function $u = \zeta_N(V) - N$ in the equality in Lemma 1.1. We obtain

$$\begin{aligned}\int_{\mathbb{R}^d} \zeta_N(V(x,t))\,\mu_t(dx) = &\int_{\mathbb{R}^d} \zeta_N(V(x,s))\,\mu_s(dx) + \\ &+ \Bigl(\mu_t(\mathbb{R}^d) - \nu(\mathbb{R}^d) - \int_0^t\int_{\mathbb{R}^d} c(x,\tau)\,\mu_\tau(dx)\,d\tau\Bigr) N + \\ &+ \int_s^t\int_{\mathbb{R}^d} \Bigl(\zeta_N'(V)(\partial_t V + LV + \zeta_N''(V)|\sqrt{A}\nabla V|^2\Bigr)\,\mu_\tau(dx)\,d\tau + \\ &+ \int_s^t\int_{\mathbb{R}^d} c\,(\zeta_N(V) - \zeta_N'(V)V)\ \mu_\tau(dx)\,d\tau.\end{aligned}$$

Noting that $z\zeta_N'(z) \le \zeta_N(z)$, we have

$$\begin{aligned}\int_{\mathbb{R}^d} \zeta_N(V(x,t))\,\mu_t(dx) \le &\int_{\mathbb{R}^d} \zeta_N(V(x,s))\,\mu_s(dx) + \\ &\Bigl(\mu_t(\mathbb{R}^d) - \nu(\mathbb{R}^d) - \int_0^t\int_{\mathbb{R}^d} c(x,\tau)\,\mu_\tau(dx)\,d\tau\Bigr) N + \\ &+ \int_s^t K(\tau) + H(\tau)\int_{\mathbb{R}^d} \zeta_N(V(x,\tau))\,\mu_\tau(dx)\,d\tau,\end{aligned}$$

Letting $s \to 0$, we arrive at the inequality

$$\int_{\mathbb{R}^d} \zeta_N(V(x,t))\, d\mu_t \leq$$
$$\leq \int_{\mathbb{R}^d} \zeta_N(V(x,0))\, d\nu + \left(\mu_t(\mathbb{R}^d) - \nu(\mathbb{R}^d) - \int_0^t \int_{\mathbb{R}^d} c(x,\tau)\, \mu_\tau(dx)\, d\tau\right) N +$$
$$+ \int_0^t K(\tau) + H(\tau) \int_{\mathbb{R}^d} \zeta_N(V(x,\tau))\, \mu_\tau(dx)\, d\tau. \quad (2.2)$$

Since

$$\mu_t(\mathbb{R}^d) \leq \nu(\mathbb{R}^d) + \int_0^t \int_{\mathbb{R}^d} c(x,\tau)\, \mu_\tau(dx)\, d\tau,$$

the last inequality can be rewritten as

$$\int_{\mathbb{R}^d} \zeta_N(V(x,t))\, \mu_t(dx) \leq \int_{\mathbb{R}^d} \zeta_N(V(x,0))\, \nu(dx) +$$
$$+ \int_0^t K(\tau) + H(\tau) \int_{\mathbb{R}^d} \zeta_N(V(x,\tau))\, \mu_\tau(dx)\, d\tau.$$

Applying Gronwall's inequality we obtain

$$\int_{\mathbb{R}^d} \zeta_N(V(x,t))\, \mu_t(dx) \leq Q(t) + R(t) \int_{\mathbb{R}^d} \zeta_N(V(x,0))\, \nu(dx).$$

Letting $N \to \infty$, we obtain the required estimate. Note that if

$$\mu_t(\mathbb{R}^d) < \nu(\mathbb{R}^d) + \int_0^t \int_{\mathbb{R}^d} c(x,\tau)\, \mu_\tau(dx)\, d\tau,$$

then

$$\int_{\mathbb{R}^d} V(x,t)\, \mu_t(dx) - \int_{\mathbb{R}^d} V(x,0)\, \nu(dx) - \int_0^t K(\tau) + H(\tau) \int_{\mathbb{R}^d} V(x,\tau)\, \mu_\tau(dx)\, d\tau = -\infty,$$

which is impossible. Hence

$$\mu_t(\mathbb{R}^d) = \nu(\mathbb{R}^d) + \int_0^t \int_{\mathbb{R}^d} c(x,\tau)\, \mu_\tau(dx)\, d\tau,$$

which completes the proof. □

Corollary 2.2. *Let $\mu = (\mu_t)_{0<t<T}$ be a solution of the Cauchy problem $\partial_t \mu = L^*\mu$, $\mu|_{t=0} = \nu$, where $c \leq 0$, μ_t and ν are subprobability measures on $\mathbb{R}^d$ and condition* (2.1) *holds. Let a positive function $W \in C^2(\mathbb{R}^d)$ be such that $\lim\limits_{|x|\to+\infty} W(x) = +\infty$.*

(i) *If for some number $C > 0$ and almost every $(x,t) \in \mathbb{R}^d \times (0,T)$ there holds the inequality*

$$LW(x,t) \leq C + CW(x),$$

then for almost every $t \in (0,T)$ we have

$$\int_{\mathbb{R}^d} W(x)\, \mu_t(dx) \leq \exp(Ct) + \exp(Ct) \int_{\mathbb{R}^d} W(x)\, \nu(dx).$$

(ii) *Let G be a positive continuous increasing function on $[0,+\infty)$ such that*

$$\int_1^{+\infty} \frac{ds}{sG(s)} < +\infty.$$

Let η be a continuous function on $[0,T)$ defined by the equality

$$t=\int_0^{\eta(t)}\frac{ds}{sG(s^{-\delta})},\qquad \delta\in(0,1).$$

If for some number $C>0$ and almost every $(x,t)\in\mathbb{R}^d\times(0,T)$ there holds the inequality

$$LW(x,t)\le C-W(x)G(W(x)),$$

then for almost every $t\in(0,T)$ we have

$$\int_{\mathbb{R}^d}W(x)\,\mu_t(dx)\le\frac{1}{(1-\delta)\eta^\delta(t)}+\frac{C}{\eta(t)}\int_0^t\eta(s)\,ds.$$

(iii) *Let G and η be the functions mentioned in* (ii). *Assume that for some number $C>0$ and almost every $(x,t)\in\mathbb{R}^d\times(0,T)$ there holds the inequality*

$$LW(x,t)+\eta(t)|\sqrt{A(x,t)}\nabla W(x)|^2\le C-W(x)G(W(x)).$$

Then for almost every $t\in(0,T)$

$$\int_{\mathbb{R}^d}\exp\bigl(\eta(t)W(x)\bigr)\,\mu_t(dx)\le\exp\Bigl((1-\delta)^{-1}\eta^{1-\delta}(t)+C\int_0^t\eta(s)\,ds\Bigr).$$

Proof. In order to prove (i) it is enough to apply Theorem 2.1 with $H(t)=K(t)=C$ and $V(x,t)=W(x)$.

Let us prove (ii). Let $V(x,t)=\eta(t)W(x)$. Set

$$\partial_tV(x,t)+LV(x,t)\le\eta'(t)W(x)-\eta(t)W(x)G(W(x))+C\eta(t).$$

Note that for all nonnegative numbers α and β

$$\alpha\beta\le\alpha G^{-1}(\alpha)+\beta G(\beta),$$

where G^{-1} is the inverse function to G. Applying this inequality with $\alpha=\eta'/\eta$ and $\beta=W$, we obtain

$$\partial_tV(x,t)+LV(x,t)\le\eta'(t)G^{-1}\bigl(\eta'(t)/\eta(t)\bigr)+C\eta(t)=\frac{\eta'(t)}{\eta^\delta(t)}+C\eta(t),$$

because our assumptions imply that $\eta'(t)=\eta(t)G(\eta^{-\delta}(t))$.

Applying Theorem 2.1 with $H(t)=0$ and $K(t)=\frac{\eta'(t)}{\eta^\delta(t)}+C\eta(t)$, we arrive at to the required inequality.

Let us prove (iii). Let $V(x,t)=\exp\bigl(\eta(t)W(x)\bigr)$. Then

$$\partial_tV(x,t)+LV(x,t)\le\Bigl[\eta'(t)W(x)-\eta(t)W(x)G(W(x))+C\eta(t)\Bigr]\exp\bigl(\eta(t)W(x)\bigr).$$

Hence

$$\partial_tV(x,t)+LV(x,t)\le\Bigl[\frac{\eta'(t)}{\eta^\delta(t)}+C\eta(t)\Bigr]\exp\bigl(\eta(t)W(x)\bigr).$$

Applying Theorem 2.1 with $K(t)=0$ and

$$H(t)=\frac{\eta'(t)}{\eta^\delta(t)}+C\eta(t),$$

we obtain the required assertion. $\square$

Let us consider several examples.

Example 2.3. Set $V(x,t)=|x|^r$, where $r\ge 2$. Then

$LV(x,t)=r|x|^{r-2}\mathrm{trace}\,A(x,t)+r(r-2)|x|^{r-4}(A(x,t)x,x)+r|x|^{r-2}(b(x,t),x)+|x|^r c(x,t)$.

Assume that for some numbers $C_1>0$, $C_2>0$ and all $(x,t)\in\mathbb{R}^d\times[0,T]$ we have

$$r\mathrm{trace}\,A(x,t)+r(r-2)|x|^{-2}(A(x,t)x,x)+r(b(x,t),x)+|x|^2c(x,t)\le C_1+C_2|x|^2.$$

Let $|x|^r\in L^1(\nu)$. Then

$$\int_{\mathbb{R}^d}|x|^r\,\mu_t(dx)\le e^{C_3t}+e^{C_3t}\int_{\mathbb{R}^d}|x|^r\,\nu(dx)$$

for almost every $t\in(0,T)$ and some $C_3>0$.

Example 2.4. Set $V(x,t)=\exp(\alpha|x|^r)$, where $r\ge 2$. Then

$$\begin{aligned}LV(x,t)=\exp(\alpha|x|^r)\Bigl[&\alpha r|x|^{r-2}\mathrm{trace}\,A(x,t)+\\&+\alpha r(r-2)|x|^{r-4}(A(x,t)x,x)+\alpha^2r^2|x|^{2r-4}(A(x,t)x,x)+\\&+\alpha r|x|^{r-2}(b(x,t),x)+c(x,t)\Bigr].\end{aligned}$$

Suppose that there exists a number C_1 such that for every $(x,t)\in\mathbb{R}^d\times[0,T]$ we have

$$\begin{aligned}&\alpha r|x|^{r-2}\mathrm{trace}\,A(x,t)+\\&+\alpha r(r-2)|x|^{r-4}(A(x,t)x,x)+\alpha^2r^2|x|^{2r-4}(A(x,t)x,x)+\\&+\alpha r|x|^{r-2}(b(x,t),x)+c(x,t)\le C_1.\end{aligned}$$

If $\exp(|x|^r)\in L^1(\nu)$, then

$$\int_{\mathbb{R}^d}\exp(\alpha|x|^r)\,\mu_t(dx)\le e^{C_2t}+e^{C_2t}\int_{\mathbb{R}^d}\exp(\alpha|x|^r)\,\nu(dx).$$

for almost all $t\in(0,T)$.

Example 2.5. Let $k>2$ and $r\ge 2$. Assume that

$$r\mathrm{trace}\,A(x,t)+r(r-2)|x|^{-2}(A(x,t)x,x)+r(b(x,t),x)+|x|^2c(x,t)\le C_1-C_2|x|^k,$$

where $C_1>0$ and $C_2>0$. Then

$$L|x|^r\le C_3-C_3|x|^{r+k-2}$$

for some $C_3>0$. Set $W(x)=|x|^r$ and $G(z)=C_3z^\sigma$, where $\sigma=(k-2)/r>0$. Hence

$$LW(x,t)\le C_3-WG(W(x)).$$

Then $\eta(t)=C_4t^{\frac{1}{\delta\sigma}}$, where C_4 depends on C_3, δ and σ. By Corollary 2.2 we obtain the estimate

$$\int_{\mathbb{R}^d}|x|^r\,\mu_t(dx)\le\frac{\gamma}{t^{\frac{r}{k-2}}},$$

where γ depends on C_1, C_2, δ, σ.

Example 2.6. Let $r>2$ and $k>r$. Assume that

$$\begin{aligned}&\alpha r|x|^{r-2}\mathrm{trace}\,A(x,t)+\\&+\alpha r(r-2)|x|^{r-4}(A(x,t)x,x)+\alpha^2r^2|x|^{2r-4}(A(x,t)x,x)+\\&+\alpha r|x|^{r-2}(b(x,t),x)+c(x,t)\le C_1-C_2|x|^k,\end{aligned}$$

where $C_1>0$ and $C_2>0$. Then

$$L\exp(\alpha|x|^r)\le C_3-C_3|x|^k\exp(\alpha|x|^r)$$

for some $C_3>0$. Set $W(x)=\exp(\alpha|x|^r)$ and $G(z)=C_3|\ln z|^\sigma$ if $z\geq 2$, where $\sigma=\frac{k}{r}>1$. We obtain

$$LW(x,t)\leq C_3-WG(W(x)).$$

Then $\eta(t)=C_4\exp(-C_5t^{\frac{-1}{\sigma-1}})$, where $C_4>0$ and $C_5>0$ depend on C_3, δ and σ. By Corollary 2.2 we have

$$\int_{\mathbb{R}^d}\exp(\alpha|x|^r)\,\mu_t(dx)\leq\gamma_1\exp\bigl(\frac{\gamma_2}{t^{\frac{r}{k-r}}}\bigr),$$

where γ_1 and γ_2 depend on C_1, C_2, δ and σ.

Example 2.7. Let $r>2$, $k>2$ and $\alpha>0$. Assume that

$$\begin{aligned}&\alpha r\operatorname{trace}A(x,t)+\alpha r(r-2)|x|^{-2}(A(x,t)x,x)+\\&\qquad+\alpha r(b(x,t),x)+\alpha|x|^2c(x,t)+\alpha^2r^2|x|^{r-2}(A(x,t)x,x)\leq C_1-C_2|x|^k,\end{aligned}$$

where $C_1>0$ and $C_2>0$. Then

$$\alpha L|x|^r+\alpha^2r^2|x|^{2r-4}(A(x,t)x,x)\leq C_3-C_3|x|^{k+r-2}.$$

Set $W(x)=\alpha|x|^r$ and $G(z)=C_3\alpha^{-(1+\sigma)/\sigma}z^\sigma$, where $\sigma=\frac{k-2}{r}>0$. We obtain

$$LW(x,t)+|\sqrt{A(x,t)}\nabla W(x)|^2\leq C_3-WG(W(x)).$$

Hence we can apply Corollary 2.2 with $\delta\in(0,1)$, $\eta(t)=C_4t^{\frac{1}{\delta\sigma}}$, where C_4 depends on C_3, δ and σ. Thus, for every $\beta>\frac{r}{k-2}$ we obtain the estimate

$$\int_{\mathbb{R}^d}\exp(\alpha t^\beta|x|^r)\,\mu_t(dx)\leq\gamma_1\exp\Bigl(\gamma_2(t^{\beta-\frac{r}{k-2}}+t^{\beta+1})\Bigr),$$

where the numbers γ_1 and γ_2 depend on C_1, C_2, r and β.

Note that the estimates in Examples 2.6 and 2.7 do not depend on the initial condition. If we apply these estimates to the transition probabilities $P(y,0,t,\,dx)$ of the corresponding processes, then the resulting estimates will be uniform in y. Such estimates for kernels of diffusion semigroups (with possibly rapidly growing drifts) were first obtained in [11] and [16].

3. Local and global bounds of solutions

In this section we obtain local and global L^p and L^∞ estimates of densities of solutions. The main idea is to use a modification of Moser's iteration method (see [12]). We start with local estimates and then we obtain global estimates by using local one and a suitable scaling.

Let $\mu=(\mu_t)_{t\in(0,T)}$ be a nonnegative solution of equation (1.1).

We assume that $A=(a^{ij})$ is a symmetric matrix satisfying the following condition:

(H1) for some number $p>d+2$, every ball $U\subset\mathbb{R}^d$ and every segment $J\subset(0,T)$ one has

$$\sup_{t\in J}\|a^{i,j}(\,\cdot\,,t)\|_{W^{1,p}(U)}<\infty$$

and

$$0<\lambda(U,J):=\inf\bigl\{(A(x,t)\xi,\xi):\,|\xi|=1,\,(x,t)\in U\times J\bigr\}.$$

We also suppose that

(H2) for some number $p>d+2$, every ball $U\subset\mathbb{R}^d$ and every closed interval $J\subset(0,T)$ one has

$$b,c\in L^p(U\times J)\quad\text{or}\quad b,c\in L^p(U\times J,\mu).$$

According to [4, Corollary 3.9] and [8, Corollary 2.2], conditions (H1) and (H2) yield existence of a Hölder continuous density ϱ of the solution μ with respect to Lebesgue measure. Moreover, for every ball $U \subset \mathbb{R}^d$ and every closed interval $J \subset (0,T)$ we have $\varrho(\,\cdot\,,t) \in W^{1,p}(U)$ and

$$\int_J \|\varrho(\,\cdot\,,t)\|^p_{W^{1,p}(U)}\,dt < \infty.$$

Set $B^i = b^i - \partial_{x_j} a^{ij}$. Then we can rewrite equation (1.1) in the divergence form

$$\partial_t \varrho = \operatorname{div}\bigl(A\nabla\varrho - B\varrho\bigr) + c\varrho, \tag{3.1}$$

which is understood in the sense of the integral identity

$$\int_0^T \int_{\mathbb{R}^d} \bigl[-\varrho\partial_t\varphi + (A\nabla\varrho, \nabla\varphi)\bigr]\,dx\,dt = \int_0^T \int_{\mathbb{R}^d} \bigl[(B,\nabla\varphi)\varrho + c\varrho\varphi\bigr]\,dx\,dt \tag{3.2}$$

for every function $\varphi \in C_0^\infty(\mathbb{R}^d \times (0,T))$.

Recall the following embedding theorem (see [6, Lemma 3.1] or [1]).

Lemma 3.1. *Let J be a closed interval in $(0,T)$ and let $u(\,\cdot\,,t) \in W^{1,2}(\mathbb{R}^d)$ be such that $x \mapsto u(x,t)$ has compact support for almost all $t \in J$. Then there exists a constant $C>0$ depending only on d such that*

$$\|u\|_{L^{2(d+2)/2}(\mathbb{R}^d\times J)} \le C\bigl(\sup_{t\in J}\|u(\,\cdot\,,t)\|_{L^2(\mathbb{R}^d)} + \|\nabla u\|_{L^2(\mathbb{R}^d\times J)}\bigr).$$

Note that now we do not assume that c is a nonpositive function.

Let $c^+(x,t) = \max\{c(x,t),0\}$.

The following lemma is the key step of our proof.

Lemma 3.2. *Let $m \ge 1$. Let $U \subset \mathbb{R}^d$ be a ball and let $[s_1,s_2] \subset (0,T)$. Assume that $\psi \in C_0^\infty(\mathbb{R}^d \times (0,T))$ is such that the support of ψ is contained in $U \times (0,T)$ and $\psi(x,s_1) = 0$ for every x. Then there exists a constant $C(d)$ depending only on d such that*

$$\Bigl(\int_{s_1}^{s_2}\int_U |\varrho^m\psi|^{2(d+2)/d}\,dx\,dt\Bigr)^{d/(d+2)} \le$$
$$\le 32C(d)m^2(1+\lambda^{-1})\int_{s_1}^{s_2}\int_U \bigl[|\psi||\psi_t| + \|A\||\nabla\psi|^2 + |\sqrt{A^{-1}}B|^2\psi^2 + c^+\psi^2\bigr]\varrho^{2m}\,dx\,dt, \tag{3.3}$$

where $\|A(x,t)\| = \min_{|\xi|=1}(A(x,t)\xi,\xi)$ and $\lambda = \lambda(U,[s_1,s_2])$ is defined as above.

Proof. Let f be a smooth function on $[0,+\infty)$ such that $f \ge 0$, $f' \ge 0$, $f'' \ge 0$. Substituting the function $\varphi = f'(\varrho)\psi^2$ in equality (3.2), for any $t \in [s_1,s_2]$ we obtain

$$\int_{\mathbb{R}^d} f(\varrho(x,t))\psi^2(x)\,dx - \int_{\mathbb{R}^d} f(\varrho(x,s_1))\psi^2(x)\,dx +$$
$$+\frac{1}{3}\int_{s_1}^t\int_{\mathbb{R}^d} |\sqrt{A}\nabla\varrho|^2 f''(\varrho)\psi^2\,dx\,d\tau \le$$
$$\int_{s_1}^t\int_{\mathbb{R}^d} 2|\psi||\psi_t|f(\varrho) + 3|\sqrt{A}\nabla\psi|^2\frac{f'(\varrho)^2}{f''(\varrho)} + 3|\sqrt{A^{-1}}B|^2\varrho^2 f''(\varrho)\psi^2 +$$
$$+ 2|(B,\nabla\psi)|\psi\varrho f'(\varrho) + c^+\varrho f'(\varrho)\psi^2\,dx\,d\tau.$$

To this end, it is enough to note that

$$2(A\nabla\varrho,\nabla\psi)\psi f'(\varrho) \le 3^{-1}|\sqrt{A}\nabla\varrho|^2 f''(\varrho)\psi^2 + 3|\sqrt{A}\nabla\psi|^2\frac{f'(\varrho)^2}{f''(\varrho)},$$

$$(B,\nabla\varrho)\varrho f''(\varrho)\psi^2 \le 3^{-1}|\sqrt{A}\nabla\varrho|^2 f''(\varrho)\psi^2 + 3|\sqrt{A^{-1}}B|^2\varrho^2 f''(\varrho)\psi^2.$$

Set $f(\varrho)=\varrho^{2m}$. Recall that $\psi(x,s_1)=0$. We have

$$\sup_{t\in[s_1,s_2]}\int_{\mathbb{R}^d}\varrho^{2m}(x,t)\psi^2(x)\,dx+\frac{4m-2}{3m}\int_{s_1}^{s_2}\int_{\mathbb{R}^d}|\sqrt{A}\nabla(\varrho^m\psi)|^2\,dx\,d\tau\le$$

$$\le 32m^2\int_{s_1}^{s_2}\int_{\mathbb{R}^d}\bigl[|\psi||\psi_t|+|\sqrt{A}\nabla\psi|^2+|\sqrt{A^{-1}}B|^2\psi^2+c^+\psi^2\bigr]\varrho\,dx\,d\tau.$$

Now our assertion follows from Lemma 3.1. □

Theorem 3.3. (L^p-estimates) *Let $p\ge 2(d+2)/d$. Let U and U' be balls in $\mathbb{R}^d$ with $\overline{U'}\subset U$. Let also $[s_1,s_2]\subset(0,T)$. Then, for every $s\in(s_1,s_2)$, there exists a number $C>0$ depending on U, U', s, s_1, d and p such that*

$$\|\varrho\|_{L^p(U'\times[s,s_2])}\le C(1+\lambda^{-1})^{\gamma}\int_{s_1}^{s_2}\int_U\bigl[1+\|A\|^{\gamma}+|c^+|^{\gamma}+|\sqrt{A^{-1}}B|^{2\gamma}\bigr]\varrho\,dx\,dt,$$

where $\gamma=(d+2)/2p'$, $p'=p/(p-1)$ and $\lambda=\lambda(U,[s_1,s_2])$, $\|A\|$ are defined as above.

Proof. Set $m=dp/2(d+2)$ and

$$\alpha=1+\frac{4m}{(2m-1)d},\quad \alpha'=1+\frac{(2m-1)d}{4m},\quad \delta=\frac{4}{d(2m-1)+4m}.$$

Note that $m\ge 1$. Let us fix a function $\psi=\zeta(x)\eta(t)$, where $\zeta\in C_0^\infty(U)$, $\zeta(x)=1$ if $x\in U'$, $0\le\psi\le 1$, $\eta\in C_0^\infty((s_1,T))$, $\eta(t)=1$ if $t\in[s,s_2]$, $0\le\eta\le 1$ and

$$|\partial_t\eta(t)|\le K\eta^{1-\delta}(t),\quad |\nabla\zeta(x)|\le K\zeta^{1-\delta}(x)$$

for some number $K>0$ and every $(x,t)\in U\times[s_1,s_2]$. Note that K depends only on U, U', s and s_1. Applying Lemma 3.2 we obtain

$$\Bigl(\int_{s_1}^{s_2}\int_U|\varrho^m\psi|^{2(d+2)/d}\,dx\,dt\Bigr)^{d/(d+2)}\le$$

$$\le 32C(d)m^2(1+\lambda^{-1})\int_s^{s_2}\int_U\bigl[|\psi||\psi_t|+\|A\||\nabla\psi|^2+|\sqrt{A^{-1}}B|^2\psi^2+c^+\psi^2\bigr]\varrho^{2m}\,dx\,dt.$$

Using Hölder's inequality with exponents α and α', we estimate the integral in the right side of the last inequality by the following expression:

$$K^2\Bigl(\int_{s_1}^{s_2}\int_U\bigl(1+\|A\|+|\sqrt{A^{-1}}B|^2+c^+\bigr)^{\alpha'}\varrho^{2m}\,dx\,dt\Bigr)^{1/\alpha'}\Bigl(\int_{s_1}^{s_2}\int_U|\varrho^m\psi|^{2(d+2)/d}\,dx\,dt\Bigr)^{1/\alpha}.$$

Applying the inequality $xy\le\varepsilon x^{\alpha}+C(\alpha,\varepsilon)y^{\alpha'}$ with sufficiently small $\varepsilon>0$, we obtain our assertion. □

Theorem 3.4. (L^∞-estimates) *Let $\gamma>(d+2)/2$. Let U and U' be balls in $\mathbb{R}^d$ with $\overline{U'}\subset U$. Let also $[s_1,s_2]\subset(0,T)$. Then, for every $s\in(s_1,s_2)$, there exists a number $C>0$ depending on U, U', s, s_1, d and γ such that*

$$\|\varrho\|_{L^\infty(U'\times[s,s_2])}\le C(1+\lambda^{-1})^{\gamma}\int_{s_1}^{s_2}\int_U\bigl[1+\|A\|^{\gamma}+|c^+|^{\gamma}+|\sqrt{A^{-1}}B|^{2\gamma}\bigr]\varrho\,dx\,dt,$$

where $\lambda=\lambda(U,[s_1,s_2])$, $\|A\|$ are defined as above.

Proof. If $\varrho\equiv 0$ on $U\times[s_1,s_2]$, then the assertion is trivial. Let us consider the case where $\varrho\not\equiv 0$. Multiplying the solution ϱ by the number

$$(1+\lambda^{-1})^{-\gamma}\Bigl(\int_{s_1}^{s_2}\int_U\bigl[1+\|A\|^{\gamma}+|c^+|^{\gamma}+|\sqrt{A^{-1}}B|^{2\gamma}\bigr]\varrho\,dx\,dt\Bigr)^{-1},$$

we can assume that

$$(1+\lambda^{-1})^\gamma \int_{s_1}^{s_2}\int_U \bigl[1+\|A\|^\gamma+|c^+|^\gamma+|\sqrt{A^{-1}}B|^{2\gamma}\bigr]\varrho\,dx\,dt=1.$$

In this case in order to prove the theorem it is enough to find a number C depending only on U, U', s, s_1, s_2, d and γ such that

$$\|\varrho\|_{L^\infty(U'\times[s,s_2])}\le C.$$

Let $U=U(x_0,R)$, $U'=U(x_0,R')$ and $R'<R$. Set $R_n=R'+(R-R')2^{-n}$, $s_n=s-(s-s_1)2^{-n}$ and $U_n=U(x_0,R_n)$. Let us consider the following system of increasing domains:

$$Q_n=U_n\times[s_n,s_2],\quad Q_0=U\times[s_1,s_2].$$

For each n we fix a function $\psi_n\in C_0^\infty(\mathbb{R}^d\times(0,T))$ in the same way as in the proof of Theorem 3.3, that is, $\psi(x,t)=1$ if $(x,t)\in Q_{n+1}$, $0\le\psi\le 1$, the support of ψ is contained in $U_n\times(s_n,T)$ and $|\partial_t\psi_n(x,t)|+|\nabla\psi_n(x,t)|\le K^n$ for all $(x,t)\in\mathbb{R}^d$ and some number $K>1$ depending only on the numbers s, s_1, R, R'.

Applying Lemma 3.2 and Hölder's inequality with exponents γ and γ', we obtain

$$\Bigl(\int_{Q_n}|\varrho^m\psi_n|^{2(d+2)/d}\,dx\,dt\Bigr)^{d/(d+2)}\le 32m^2C(d,s)K^{2n}\Bigl(\int_{Q_n}\varrho^{(2m-1)\gamma'+1}\,dx\,dt\Bigr)^{1/\gamma'}.$$

Set

$$p_{n+1}=\beta p_n+(\gamma'-1)\gamma'^{-1},\quad p_1=\gamma'+1,\quad \beta=(d+2)d^{-1}\gamma'^{-1}.$$

Note that $\beta^{n-1}p_1\le p_n\le\beta^{n-1}(p_1+1)$. Taking $m=p_{n+1}d/(2d+4)$, we obtain

$$\|\varrho\|_{L^{p_{n+1}}(Q_{n+1})}\le C^{n\beta^{-n}}\|\varrho\|_{L^{p_n}(Q_n)}^{p_n/(p_n+\gamma'-1)},$$

where the number C depends only on K, d, and γ. Finally, note that $\sum_n n\beta^{-n}<\infty$ and according to Theorem 3.3 the norm $\|\varrho\|_{L^{p_1}(Q_1)}$ is estimated by a number depending only on the numbers p_1, d, s, s_1, U, and U_1. □

Remark 3.5. (i) Note that the constant C in Theorem 3.3 and Theorem 3.3 does not depend on s_2.

(ii) If $c\le 0$, then all the inequalities above will be true without the coefficient c in the right-hand side.

Corollary 3.6. *Let $\gamma>(d+2)/2$, $\kappa>0$ and $t_0\in(0,T)$. Then there exists a number $C>0$ depending only on κ, t_0, d and γ such that for all $(x,t)\in\mathbb{R}^d\times(t_0,T)$*

$$\varrho(x,t)\le C(1+\lambda^{-1}(x,t))^\gamma\int_{t_0/2}^t\int_{U(x,\kappa)}(1+\|A\|^\gamma+|c^+|^\gamma+|\sqrt{A^{-1}}B|^{2\gamma})\varrho\,dy\,d\tau,$$

where

$$\lambda(x,t)=\inf\bigl\{(A(y,\tau)\xi,\xi)\colon\ |\xi|=1,\quad (y,\tau)\in U(x,\kappa)\times[t_0/2,t]\bigr\}.$$

In particular, if $\mu_t(dx)=\varrho(x,t)\,dx$ is a subprobability measure for almost all $t\in(0,T)$, the functions $\|A\|^\gamma$, $|c^+|^\gamma$, $|B|^{2\gamma}$ are in $L^1(\mathbb{R}^d\times(t_0/2,T),\mu)$ and the function $\|A\|^{-1}$ is uniformly bounded, then $\varrho\in L^\infty(\mathbb{R}^d\times(t_0,T))$.

Proof. Let us shift the point x to 0 and apply Theorem 3.4 with the balls $U=U(x,\kappa)$ and $U'=U(x,\kappa/2)$ and points $s_1=t_0/2$, $s=t_0$, $s_2=t$. □

Corollary 3.7. *Let $\gamma>(d+2)/2$ and $\Theta\in(0,1)$. Then there exists a number $C>0$ depending only on γ, d and Θ such that for all $(x,t)\in\mathbb{R}^d\times(0,T)$*

$$\varrho(x,t)\le C(1+\lambda^{-1}(x,t))^\gamma t^{-(d+2)/2}\int_{\Theta t}^t\int_{U(x,\sqrt{t})}(1+\|A\|^\gamma+t^{2\gamma}|c^+|^\gamma+t^{2\gamma}|\sqrt{A^{-1}}B|^{2\gamma})\varrho\,dy\,d\tau,$$

where

$$\lambda(x,t) = \inf\big\{(A(y,\tau)\xi,\xi)\colon\ |\xi|=1, \quad (y,\tau)\in U(x,\sqrt{t})\times[\Theta t,t]\big\}.$$

In particular, if $\mu_t(dx)=\varrho(x,t)\,dx$ is a subprobability measure for almost all $t\in(0,T)$, the functions $\|A\|^\gamma$, $|c^+|^\gamma$, $|B|^{2\gamma}$ are in $L^1(\mathbb{R}^d\times(0,T),\mu)$ and the function $\|A\|^{-1}$ is uniformly bounded, then there exists a number $\widetilde{C}>0$ such that

$$\varrho(x,t)\le \widetilde{C}t^{-d/2} \quad \textit{for all } (x,t)\in\mathbb{R}^d\times(0,T).$$

Proof. In order to prove the estimate at a point (x_0,t_0) it suffices to change variables $x\mapsto(x-x_0)/\sqrt{t_0}$ and $t\mapsto t/t_0$ and apply Theorem 3.4 with the balls $U=U(0,1)$ and $U'=U(0,1/2)$ and points $s_1=\Theta$, $s=(1+\Theta)/2$, $s_2=1$. □

Corollary 3.8. *Let $\Phi\in C^{2,1}(\mathbb{R}^d\times(0,T))$ and $\Phi>0$. Set*

$$\widetilde{c}=c+\big(\partial_t\Phi+\operatorname{div}(A\nabla\Phi)+B\nabla\Phi\big)\Phi^{-1}, \quad \widetilde{B}=B+\Phi^{-1}A\nabla\Phi.$$

Let $\gamma>(d+2)/2$ and $\Theta\in(0,1)$. Then there exists a number $C>0$ depending only on γ, d and Θ such that for all $(x,t)\in\mathbb{R}^d\times(0,T)$

$$\varrho(x,t)\le C\Phi(x,t)^{-1}(1+\lambda^{-1}(x,t))^\gamma\times \\ \times t^{-(d+2)/2}\int_{\Theta t}^t\int_{U(x,\sqrt{t})}(1+\|A\|^\gamma+t^{2\gamma}|\widetilde{c}^+|^\gamma+t^{2\gamma}|\sqrt{A^{-1}}\widetilde{B}|^{2\gamma})\Phi\varrho\,dy\,d\tau,$$

where λ is defined in the previous corollary. In particular, if

$$\sup_{t\in(0,T)}\int_{\mathbb{R}^d}\Phi(x,t)\varrho(x,t)\,dx<\infty,$$

the functions $\|A\|^\gamma\Phi$, $|\widetilde{c}^+|^\gamma\Phi$, $|\widetilde{B}|^{2\gamma}\Phi$ are in $L^1(\mathbb{R}^d\times(0,T),\mu)$ and the function $\|A\|^{-1}$ is uniformly bounded, then there exists a number $\widetilde{C}>0$ such that

$$\varrho(x,t)\le\widetilde{C}t^{-d/2}\Phi(x,t)^{-1} \quad \textit{for all } (x,t)\in\mathbb{R}^d\times(0,T).$$

Proof. It suffices to observe that the function $\Phi\varrho$ satisfies equation (3.1) with the new coefficients $\widetilde{c}$ and $\widetilde{B}$. □

Let us now consider two typical examples. We shall assume that $c\le 0$ and that $\mu_t(dx)=\varrho(x,t)\,dx$ is a subprobability solution of the Cauchy problem for equation (1.1) with the initial condition ν such that $|c|\in L^1(\mu)$ and

$$\mu_t(\mathbb{R}^d)\le\nu(\mathbb{R}^d)+\int_0^t\int_{\mathbb{R}^d}c(x,s)\,\mu_s(dx)\,ds.$$

We obtain upper estimates of ϱ in several different situations.

Example 3.9. Let $\alpha>0$, $r>2$ and $k>r$. Assume that $c\le 0$ and

$$\alpha r|x|^{r-2}\operatorname{trace}A(x,t)+ \\ +\alpha r(r-2)|x|^{r-4}(A(x,t)x,x)+\alpha^2r^2|x|^{2r-4}(A(x,t)x,x)+ \\ +\alpha r|x|^{r-2}(b(x,t),x)+c(x,t)\le C-C|x|^k$$

for some $C>0$ and all $(x,t)\in\mathbb{R}^d\times(0,T)$. Suppose also that for all $(x,t)\in\mathbb{R}^d\times(0,T)$ we have

$$C_1\exp(-\kappa_1|x|^{r-\delta})\le\|A(x,t)\|\le C_2\exp(\kappa_2|x|^{r-\delta}),$$

and

$$|b^i(x,t)|+|\partial_{x_j}a^{ij}(x,t)|\le C_3\exp(\kappa_3|x|^{r-\delta})$$

with some positive numbers C_1, C_2, C_3, κ_1, κ_2, κ_3 and $\delta \in (0,r)$. Let $\alpha' \in (0,\alpha)$. Then the density ϱ satisfies the inequality

$$\varrho(x,t) \le C_4 \exp(-\alpha'|x|^r)\exp(C_5 t^{-\frac{r}{k-r}})$$

for all $(x,t) \in \mathbb{R}^d \times (0,T)$ and some positive numbers C_4 and C_5.

Proof. According to Example 2.6 we have

$$\int_{\mathbb{R}^d} \exp\bigl(\alpha|x|^r\bigr)d\mu_t \le \gamma_1 \exp\bigl(\gamma_2 t^{-\frac{r}{k-r}}\bigr)$$

for almost every $t \in (0,T)$ and some numbers γ_1 and γ_2. Set $\Phi(x) = \exp(\alpha'|x|^r)$. Note that $\widetilde{c}^{+} \le \gamma_3$ and

$$(1 + \|A\|^\gamma + t^{2\gamma}|\sqrt{A^{-1}}\widetilde{B}|^{2\gamma})\Phi \le \gamma_4 \exp(\alpha|x|^r)$$

for all $(x,t) \in \mathbb{R}^d \times (0,T)$ and some number γ_3. Now the desired estimates follow from Corollary 3.8. □

Example 3.10. Let $r > 2$, $k > 2$, $\gamma > d+2$, $\alpha > 0$ and $\beta > r/(k-2)$. Assume that

$$\begin{aligned}\alpha r \operatorname{trace} A(x,t) + \alpha r(r-2)|x|^{-2}(A(x,t)x,x) + \\ + \alpha r(b(x,t),x) + |x|^2 c(x,t) + \alpha^2 r^2 |x|^{r-2}(A(x,t)x,x) \le C - C|x|^k,\end{aligned}$$

where $C > 0$. Suppose also that for all $(x,t) \in \mathbb{R}^d \times (0,T)$ we have

$$C_1(1+|x|^{\frac{m}{\gamma}})^{-1} \le \|A(x,t)\| \le C_2(1+|x|^{\frac{m}{\gamma}})$$

and

$$|b^i(x,t)|^{2\gamma} + |\partial_{x_j} a^{ij}(x,t)|^{2\gamma} \le C_3(1+|x|^m)$$

with some positive numbers C_1, C_2, C_3 and $m \ge \gamma \max\{r-1, r\beta^{-1}\}$. Let $\alpha' \in (0,\alpha)$. Then the density ϱ satisfies the inequality

$$\varrho(x,t) \le C_4 t^{-\frac{8m\beta + rd - 4\gamma r}{2r}} \exp(-\alpha' t^\beta |x|^r)$$

for all $(x,t) \in \mathbb{R}^d \times (0,T)$ and some positive numbers C_4 and C_5.

Proof. According to Example 2.7 we have

$$\int_{\mathbb{R}^d} \exp(\alpha t^\beta |x|^r)\, d\mu_t \le \gamma_1$$

for all $t \in (0,T)$ and some number γ_1. Note that for every $p \ge 1$ and $\varepsilon > 0$ one has

$$|x|^p \le \gamma_2 t^{-\frac{\beta p}{r}} \exp(\varepsilon t^\beta |x|^r),$$

so we can apply Corollary 3.8 with $\Phi(x,t) = \exp(\alpha t^\beta |x|^r)$. □

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