

A moment problem for random discrete measures

Abbreviated Title: Random discrete measures

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Abstract

Let X be a locally compact Polish space. A random measure on X is a probability measure on the space $\mathbb{M}(X)$ of all (nonnegative) Radon measures on X . For $n \in \mathbb{N}$, the n -th moment of a random measure μ is a Radon measure $M^{(n)}$ on X^n which satisfies

$$\int_{\mathbb{M}(X)} \int_{X^n} f^{(n)} d\eta^{\otimes n} d\mu(\eta) = \int_{X^n} f^{(n)} dM^{(n)}$$

for each measurable, bounded, compactly supported function $f^{(n)} : X^n \rightarrow \mathbb{R}$. In this paper, we are interested in moments of a random discrete measure. Denote by $\mathbb{K}(X)$ the cone of all (nonnegative) Radon measures η on X which are of the form $\eta = \sum_i s_i \delta_{x_i}$. Here, for each i , $s_i > 0$ and δ_{x_i} is the Dirac measure at $x_i \in X$. Note that the set $\{x_i\}$ is not necessarily locally finite in X , but can even be dense in X . A random discrete measure μ is a random measure on X which satisfies $\mu(\mathbb{K}(X)) = 1$, i.e., μ is a probability measure on $\mathbb{K}(X)$. The main result of this paper is a theorem that states a necessary and sufficient condition for a random measure μ to be a random discrete measure. This condition is formulated solely in terms of moments $M^{(n)}$ of the random measure μ .

1 Preliminaries and formulation of the problems

Let X be a locally compact Polish space, and let $\mathcal{B}(X)$ denote the Borel σ -algebra on it. For example, X can be the Euclidean space \mathbb{R}^d , $d \in \mathbb{N}$. Let $\mathbb{M}(X)$ denote the space of all (nonnegative) Radon measures on $(X, \mathcal{B}(X))$. The space $\mathbb{M}(X)$ is equipped with the vague topology, i.e., the coarsest topology making all mappings

$$\mathbb{M}(X) \ni \eta \mapsto \langle \eta, f \rangle := \int_X f(x) d\eta(x), \quad f \in C_0(X),$$

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continuous. Here $C_0(X)$ is the space all continuous functions on X with compact support. Let $\mathcal{B}(\mathbb{M}(X))$ denote the Borel σ -algebra on $\mathbb{M}(X)$. A *random measure on X* is a probability measure on $(\mathbb{M}(X), \mathcal{B}(\mathbb{M}(X)))$, see e.g. [6, 7, 10].

An important characteristic of a random measure is its moment sequence. We say that a random measure μ has finite moments if, for each $n \in \mathbb{N}$ and any $f_1, \dots, f_n \in C_0(X)$, we have

$$\int_{\mathbb{M}(X)} |\langle \eta, f_1 \rangle \cdots \langle \eta, f_n \rangle| d\mu(\eta) < \infty.$$

Then, the n -th moment of μ is the functional

$$C_0(X)^n \ni (f_1, \dots, f_n) \mapsto \int_{\mathbb{M}(X)} \langle \eta, f_1 \rangle \cdots \langle \eta, f_n \rangle d\mu(\eta).$$

For each $n \in \mathbb{N}$, we equip the space $C_0(X^n)$ of all continuous, compactly supported functions on X^n with a natural topology of uniform convergence on compact sets from X^n . Clearly, for each $f^{(n)} \in C_0(X^n)$, the function

$$\mathbb{M}(X) \ni \eta \mapsto \langle \eta^{\otimes n}, f^{(n)} \rangle := \int_{X^n} f^{(n)}(x_1, \dots, x_n) d\eta(x_1) \cdots d\eta(x_n)$$

is measurable. By the dominated convergence theorem, the n -th moment of the random measure μ can be extended, by linearity and continuity, to a continuous functional

$$C_0(X^n) \ni f^{(n)} \mapsto M^{(n)}(f^{(n)}) := \int_{\mathbb{M}(X)} \langle \eta^{\otimes n}, f^{(n)} \rangle d\mu(\eta). \quad (1)$$

By the Riesz representation theorem, the dual space of $C_0(X^n)$ can be identified with the space of all signed Radon measures on X^n . For each $f^{(n)} \in C_0(X^n)$ such that $f^{(n)} \geq 0$, we clearly have $\langle \eta^{\otimes n}, f^{(n)} \rangle \geq 0$ for all $\eta \in \mathbb{M}(X)$, hence $M^{(n)}(f^{(n)}) \geq 0$. Therefore, each moment functional $M^{(n)}$ can be identified with a nonnegative Radon measure on X^n , i.e., an element of $\mathbb{M}(X^n)$. We also set $M^{(0)} := \int_{\mathbb{M}(X)} d\mu(\eta) = 1$. The $(M^{(n)})_{n=0}^\infty$ is called the *moment sequence of the random measure μ* .

As follows from (1), for each $n \geq 2$,

$$M^{(n)}(f^{(n)}) = M^{(n)}(\text{Sym}_n f^{(n)}), \quad f^{(n)} \in C_0(X^n), \quad (2)$$

where $\text{Sym}_n f^{(n)}$ denotes the symmetrization of the function $f^{(n)}$:

$$\text{Sym}_n f^{(n)} := \sum_{\sigma \in \mathfrak{S}_n} \frac{1}{n!} f(x_{\sigma(1)}, \dots, x_{\sigma(n)}), \quad (3)$$

\mathfrak{S}_n being the group of all permutations of $1, \dots, n$. Hence, $M^{(n)}$ is a symmetric measure on X^n , i.e., the measure $M^{(n)}$ remains invariant under the natural action of permutations $\sigma \in \mathfrak{S}_n$ on X^n .

In this paper, we will be interested in the so-called random discrete measures. *The cone of (nonnegative) discrete Radon measure on X* is defined as

$$\mathbb{K}(X) := \left\{ \eta = \sum_i s_i \delta_{x_i} \in \mathbb{M}(X) \mid s_i > 0, x_i \in X \right\}.$$

Here, δ_{x_i} is the Dirac measure with mass at x_i , the atoms x_i are assumed to be distinct, and their total number is at most countable. By convention, the cone $\mathbb{K}(X)$ contains the null mass $\eta = 0$, which is represented by the sum over an empty set of indices i . One refers to the points x_i as *positions*, and to the s_i as *weights*. For $\eta = \sum_i s_i \delta_{x_i} \in \mathbb{K}(X)$ we denote $\tau(\eta) := \{x_i\}$. Note that the closure of $\mathbb{K}(X)$ in the vague topology coincides with $\mathbb{M}(X)$. As shown in [9], $\mathbb{K}(X) \in \mathcal{B}(\mathbb{M}(X))$. A *random discrete measure on X* is a probability measure on $(\mathbb{K}(X), \mathcal{B}(\mathbb{K}(X)))$, where $\mathcal{B}(\mathbb{K}(X))$ is the trace σ -algebra of $\mathcal{B}(\mathbb{M}(X))$ on $\mathbb{K}(X)$. Equivalently, a random discrete measure μ is a random measure which satisfies $\mu(\mathbb{K}(X)) = 1$.

In most interesting examples of random discrete measures, the set of positions, $\tau(\eta)$, is almost surely a countable dense subset of X . We note that a study of countable dense random subsets of X leads to “situations in which probabilistic statements about such sets can be uninformative” [11], see also [2]. It is the presence of the weights s_i in random discrete measures that makes a real difference.

Further on we will need the notion of a point process. The *configuration space over X* is defined as the set of all locally finite subsets of X :

$$\Gamma(X) := \{\gamma \subset X \mid |\gamma \cap \Lambda| < \infty \text{ for each compact } \Lambda \subset X\}.$$

Here, $|\gamma \cap \Lambda|$ denotes the number of points in the set $\gamma \cap \Lambda$. One usually identifies a configuration $\gamma = \{x_i\} \in \Gamma(X)$ with a Radon measure $\gamma = \sum_i \delta_{x_i}$. Thus, we get the inclusions $\Gamma(X) \subset \mathbb{K}(X) \subset \mathbb{M}(X)$. We denote by $\mathcal{B}(\Gamma(X))$ the trace σ -algebra of $\mathcal{B}(\mathbb{M}(X))$ on $\Gamma(X)$. A *point process in X* is a probability measure on $(\Gamma(X), \mathcal{B}(\Gamma(X)))$. Equivalently, a point process μ is a random measure which satisfies $\mu(\Gamma(X)) = 1$.

A point process is often characterized by its correlation measure. Let us recall the latter notion. Let $\Gamma_0(X)$ denote the space of all finite configurations in X :

$$\Gamma_0(X) := \{\gamma \subset X \mid |\gamma| < \infty\}.$$

Note that $\Gamma_0(X) = \bigcup_{n=0}^{\infty} \Gamma^{(n)}(X)$, where $\Gamma^{(n)}(X)$ is the space of all n -point configurations (subsets) in X . Clearly, $\Gamma_0(X) \subset \Gamma(X)$, and we denote by $\mathcal{B}(\Gamma_0(X))$ the trace σ -algebra of $\mathcal{B}(\Gamma(X))$ on $\Gamma_0(X)$. The σ -algebra $\mathcal{B}(\Gamma_0(X))$ admits the following description: for each $n \in \mathbb{N}$, $\Gamma^{(n)}(X) \in \mathcal{B}(\Gamma_0(X))$ and the restriction of $\mathcal{B}(\Gamma_0(X))$ to $\Gamma^{(n)}(X)$ coincides (under a natural isomorphism) with the collection of all symmetric (i.e., invariant under the action of $\sigma \in \mathfrak{S}_n$) Borel-measurable subsets of \tilde{X}^n , where

$$\tilde{X}^n := \{(x_1, \dots, x_n) \in X^n \mid x_i \neq x_j \text{ if } i \neq j\}.$$

Let now μ be a point process in X , i.e., a probability measure on $(\Gamma(X), \mathcal{B}(\Gamma(X)))$. The *correlation measure of μ* is defined as the (unique) measure ρ on $(\Gamma_0(X), \mathcal{B}(\Gamma_0(X)))$ which satisfies

$$\int_{\Gamma(X)} \sum_{\lambda \in \gamma} G(\lambda) d\mu(\gamma) = \int_{\Gamma_0(X)} G(\lambda) d\rho(\lambda) \quad (4)$$

for each measurable function $G : \Gamma_0(X) \rightarrow [0, \infty]$. In formula (4), the summation $\sum_{\lambda \in \gamma}$ is over all finite subsets λ of γ . Under a very mild condition on the correlation measure ρ , it uniquely identifies the point processes μ , see [13].

For example, let ζ be a Radon, non-atomic measure on X . The Lebesgue–Poisson measure L_ζ is defined as the measure on $(\Gamma_0(X), \mathcal{B}(\Gamma_0(X)))$ which satisfies

$$\int_{\Gamma_0(X)} G(\lambda) dL_\zeta(\lambda) = G(\emptyset) + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\tilde{X}^n} G(\{x_1, \dots, x_n\}) d\zeta(x_1) \cdots d\zeta(x_n)$$

for each measurable function $G : \Gamma_0(X) \rightarrow [0, \infty]$. Then the Poisson point process in X with intensity ζ can be characterized as the unique point process in X whose correlation measure is L_ζ .

Let us now briefly mention the problems we are going to discuss in this paper.

Denote $\mathbb{R}_+^* := (0, \infty)$, the upper index $*$ in \mathbb{R}_+^* denoting that point 0 is removed from the set $\mathbb{R}_+ = [0, \infty)$. We introduce a logarithmic metric on \mathbb{R}_+^* : for $a, b \in \mathbb{R}_+^*$, $\text{dist}(a, b) := \left| \ln\left(\frac{a}{b}\right) \right|$. Then \mathbb{R}_+^* becomes a locally compact Polish space, and any set of the form $[a, b]$, with $0 < a < b < \infty$, is compact. Thus, $Y := X \times \mathbb{R}_+^*$ is also a locally compact Polish space, and we can consider the configuration space over Y , i.e., $\Gamma(Y)$.

Let μ be a random discrete measure on X . It is often convenient to interpret μ as a point process in Y . More precisely, take any discrete Radon measure $\eta = \sum_i s_i \delta_{x_i} \in \mathbb{K}(X)$ and set

$$\mathcal{E}\eta := \{(x_i, s_i)\}.$$

As easily seen $\mathcal{E}\eta \in \Gamma(Y)$. Furthermore, it can be shown that the mapping $\mathcal{E} : \mathbb{K}(X) \rightarrow \Gamma(Y)$ is measurable. (Note, however, that the range of the mapping \mathcal{E} is not the whole space $\Gamma(Y)$.) We denote $\nu := \mathcal{E}(\mu)$, i.e., the pushforward of μ under \mathcal{E} . Thus, ν is a point process in Y . Thus, one can study the random discrete measure μ through the point process ν .

For example, the remarkable Gamma measure, see e.g. [8, 16, 18], is the random discrete measure μ on $X = \mathbb{R}^d$ for which $\mathcal{E}(\mu) = \nu$ is the Poisson point process in $\mathbb{R}^d \times \mathbb{R}_+^*$ with intensity measure $dx s^{-1} e^{-s} ds$.

Assume now that we know the moment sequence $(M^{(n)})_{n=0}^{\infty}$ of the random discrete measure μ . The first problem we are going to solve in this paper is how to recover the correlation measure of the point process ν from the moment sequence $(M^{(n)})_{n=0}^{\infty}$. A solution to this problem is given in Section 2. Our approach is significantly influenced by the paper of Rota and Wallstrom [15], which combines ideas of probability theory

and combinatorics. Additionally, to find the correlation measure of ν , one has to solve a sequence of finite-dimensional moment problems.

The second problem can be formulated as follows: Assume that μ is a random measure on X , whose moment sequence $(M^{(n)})_{n=0}^{\infty}$ is known. Give a necessary and sufficient condition, in terms of the moments $(M^{(n)})_{n=0}^{\infty}$, for μ to be a random discrete measure, i.e., for the random measure μ to be concentrated on $\mathbb{K}(X)$. A solution to this problem is given in Section 3. The main idea of our approach is that, in order that μ be a random discrete measure, the correlation measure of a corresponding point process ν on Y must exist.

2 Recovering the correlation measure of ν

Recall that a partition of a nonempty set Z is any finite collection $\pi = \{A_1, \dots, A_k\}$, where A_1, \dots, A_k are mutually disjoint subsets of Z such that $Z = \bigcup_{i=1}^k A_i$. The sets A_1, \dots, A_k are called blocks of the partition π .

For each $n \in \mathbb{N}$, denote by $\Pi(n)$ the set of all partitions of the set $\{1, 2, \dots, n\}$. For each partition $\pi = \{A_1, \dots, A_k\} \in \Pi(n)$, we denote by $X_{\pi}^{(n)}$ the subset of X^n which consists of all $(x_1, \dots, x_n) \in X^n$ such that, for any $1 \leq i < j \leq n$, $x_i = x_j$ if and only if i and j belong to the same block of the partition π , say A_l . For example, for the so-called zero partition $\hat{0} = \{\{1\}, \{2\}, \dots, \{n\}\}$, the set $X_{\hat{0}}^{(n)}$ consists of all points $(x_1, \dots, x_n) \in X^n$ whose all coordinates are different. For the so-called one partition $\hat{1} = \{\{1, 2, \dots, n\}\}$, the set $X_{\hat{1}}^{(n)}$ consists of all points $(x_1, \dots, x_n) \in X^n$ such that $x_1 = x_2 = \dots = x_n$. Clearly, the sets $X_{\pi}^{(n)}$ with π running over $\Pi(n)$ form a partition of X^n .

Let $m^{(n)}$ be any nonnegative Radon measure on X^n , i.e., $m^{(n)} \in \mathbb{M}(X^n)$. For each partition $\pi \in \Pi(n)$, we denote by $m_{\pi}^{(n)}$ the restriction of the measure $m^{(n)}$ to the set $X_{\pi}^{(n)}$. Note that we may also treat $m_{\pi}^{(n)}$ as a measure on X^n by setting

$$m_{\pi}^{(n)}(X^n \setminus X_{\pi}^{(n)}) := 0.$$

Then we get

$$m^{(n)} = \sum_{\pi \in \Pi(n)} m_{\pi}^{(n)}.$$

Let us fix a partition $\pi = \{A_1, A_2, \dots, A_k\} \in \Pi(n)$ and assume that the blocks of this partition are enumerated so that

$$\min A_1 < \min A_2 < \dots < \min A_k.$$

We denote $|\pi| := k$, the number of blocks in the partition π . We construct a measurable, bijective mapping

$$B_{\pi} : X_{\pi}^{(n)} \rightarrow X_{\hat{0}}^{(k)}$$

as follows. For any $(x_1, \dots, x_n) \in X_\pi^{(n)}$, we set

$$B_\pi(x_1, \dots, x_n) = (y_1, \dots, y_k),$$

where, for $i = 1, 2, \dots, k$, $y_i = x_j$ with $j \in A_i$. (Note that, if $\pi = \widehat{0}$, then B_π is just the identity mapping.) We denote by $B_\pi(m_\pi^{(n)})$ the pushforward of the measure $m_\pi^{(n)}$ under B_π .

Let us now additionally assume that the initial measure $m^{(n)}$ is symmetric, i.e., the measure $m^{(n)}$ remains invariant under the natural action of permutations $\sigma \in \mathfrak{S}_n$ on X^n . For a partition π as in the above paragraph, we set, for each $l = 1, 2, \dots, k$, $i_l := |A_l|$, the number of elements of the block A_l . Note that $i_1 + i_2 + \dots + i_k = n$. Since $m^{(n)}$ is symmetric, it is clear that the measure $B_\pi(m_\pi^{(n)})$ is completely identified by the numbers i_1, \dots, i_k . That is, if $\pi' = \{A'_1, \dots, A'_k\}$ is another partition from $\Pi(n)$, for which

$$\min A'_1 < \min A'_2 < \dots < \min A'_k$$

and $|A'_l| = i_l$, $l = 1, \dots, k$, then $B_\pi(m_\pi^{(n)}) = B_{\pi'}(m_{\pi'}^{(n)})$. Hence, we will denote

$$m_{i_1, \dots, i_k} := B_\pi(m_\pi^{(n)}), \quad (5)$$

and we may assume that, in formula (5), the partition $\pi = \{A_1, \dots, A_k\}$ is given by

$$A_1 = \{1, \dots, i_1\}, \quad A_2 = \{i_1 + 1, \dots, i_1 + i_2\}, \quad A_3 = \{i_1 + i_2 + 1, \dots, i_1 + i_2 + i_3\}, \dots \quad (6)$$

Note that, since $m^{(n)}$ is a Radon measure on X^n , each measure m_{i_1, \dots, i_k} is a Radon measure on $X_{\widehat{0}}^{(k)}$, i.e., for each $\Delta \in \mathcal{B}_c(X_{\widehat{0}}^{(k)})$, we have $m_{i_1, \dots, i_k}(\Delta) < \infty$. Here $\mathcal{B}_c(X_{\widehat{0}}^{(k)})$ denotes the collection of all sets $\Delta \in \mathcal{B}(X_{\widehat{0}}^{(k)})$ which have a compact closure in X^k , and $\mathcal{B}(X_{\widehat{0}}^{(k)})$ is the trace σ -algebra of $\mathcal{B}(X^k)$ on $X_{\widehat{0}}^{(k)}$. Thus, a given sequence of symmetric Radon measures $m^{(n)}$ on X^n , $n \in \mathbb{N}$, uniquely identifies a sequence of Radon measures m_{i_1, \dots, i_k} on $X_{\widehat{0}}^{(k)}$, where $i_1, \dots, i_k \in \mathbb{N}$, $k \in \mathbb{N}$. As easily seen the inverse implication is also true, i.e., any sequence of Radon measures m_{i_1, \dots, i_k} on $X_{\widehat{0}}^{(k)}$, with $i_1, \dots, i_k \in \mathbb{N}$ and $k \in \mathbb{N}$ uniquely identifies a sequence of symmetric Radon measures $m^{(n)}$ on X^n , $n \in \mathbb{N}$.

Let now μ be a random discrete measure on X which has finite moments, and let $(M^{(n)})_{n=0}^\infty$ be its moment sequence. So, below we will deal with the measures M_{i_1, \dots, i_k} derived from the the moment sequence $(M^{(n)})_{n=0}^\infty$.

It is clear that a result we wish to derive can only hold under an appropriate estimate on the growth of the measures $M^{(n)}$. Below we will assume that the following condition is satisfied:

(C1) For each $\Lambda \in \mathcal{B}_c(X)$, there exists a constant $C_\Lambda > 0$ such that

$$M^{(n)}(\Lambda^n) \leq C_\Lambda^n n!, \quad n \in \mathbb{N}. \quad (7)$$

Here $\mathcal{B}_c(X)$ denotes the collection of all sets from $\mathcal{B}(X)$ which have compact closure

Consider the locally compact Polish space $Y = X \times \mathbb{R}_+^*$ (see Section 1), and consider the configuration space $\Gamma(Y)$. Denote by $\Gamma_p(Y)$ the set of so-called *pinpointing configurations* in Y . By definition, $\Gamma_p(Y)$ consists of all configurations $\gamma \in \Gamma(Y)$ such that if $(x_1, s_1), (x_2, s_2) \in \gamma$ and $(x_1, s_1) \neq (x_2, s_2)$, then $x_1 \neq x_2$. Thus, a configuration $\gamma \in \Gamma_p(Y)$ can not contain two points (x, s_1) and (x, s_2) with $s_1 \neq s_2$. For each $\gamma \in \Gamma_p(Y)$ and $\Lambda \in \mathcal{B}_c(X)$, we define a *local mass* by

$$\mathfrak{M}_\Lambda(\gamma) := \int_Y \chi_\Lambda(x) s \, d\gamma(x, s) = \sum_{(x,s) \in \gamma} \chi_\Lambda(x) s \in [0, \infty]. \quad (8)$$

Here χ_Λ denotes the indicator function of the set Λ . The set of *pinpointing configurations with finite local mass* is then defined by

$$\Gamma_f(Y) := \{\gamma \in \Gamma_p(Y) \mid \mathfrak{M}_\Lambda(\gamma) < \infty \text{ for each compact } \Lambda \subset X\}. \quad (9)$$

As easily seen, $\Gamma_f(Y) \in \mathcal{B}(\Gamma(Y))$ and we denote by $\mathcal{B}(\Gamma_f(Y))$ the trace σ -algebra of $\mathcal{B}(\Gamma(Y))$ on $\Gamma_f(Y)$.

We construct a bijective mapping $\mathcal{E} : \mathbb{K}(X) \rightarrow \Gamma_f(Y)$ by setting, for each $\eta = \sum_i s_i \delta_{x_i} \in \mathbb{K}(X)$, $\mathcal{E}\eta := \{(x_i, s_i)\}$. By [9, Theorem 6.2], we have

$$\mathcal{B}(\Gamma_f(Y)) = \{\mathcal{E}A \mid A \in \mathcal{B}(\mathbb{K}(X))\}.$$

Hence, both \mathcal{E} and its inverse \mathcal{E}^{-1} are measurable mappings.

We denote by $\nu := \mathcal{E}(\mu)$ the pushforward of the measure μ under the mapping \mathcal{E} . Thus ν is a probability measure on $\Gamma_p(Y)$, in particular, it is a point process in Y .

Let ρ denote the correlation measure of the point process ν . In particular, ρ is a measure on $\Gamma_0(Y)$. For each $n \in \mathbb{N}$, we denote by $\rho^{(n)}$ the restriction of the measure ρ to $\Gamma^{(n)}(Y)$. The measure $\rho^{(n)}$ can be identified with the symmetric measure on $Y_0^{(n)}$ which satisfies

$$\begin{aligned} & \int_{\Gamma_f(Y)} \sum_{\{(x_1, s_1), \dots, (x_n, s_n)\} \subset \gamma} f^{(n)}(x_1, s_1, \dots, x_n, s_n) \, d\nu(\gamma) \\ &= \int_{Y_0^{(n)}} f^{(n)}(x_1, s_1, \dots, x_n, s_n) \, d\rho^{(n)}(x_1, s_1, \dots, x_n, s_n) \end{aligned} \quad (10)$$

for each symmetric measurable function $f^{(n)} : Y_0^{(n)} \rightarrow [0, \infty]$. Since $\nu(\Gamma_p(Y)) = 1$, the measure $\rho^{(n)}$ is concentrated on the smaller set

$$Y_n := \{(x_1, s_1, \dots, x_n, s_n) \in Y^n \mid (x_1, \dots, x_n) \in X_0^{(n)}\}. \quad (11)$$

Note that $Y_1 = Y$.

Theorem 1. Let μ be a random discrete measure on X which has finite moments. Let $(M^{(n)})_{n=0}^{\infty}$ be the moment sequence of μ , and assume that condition (C1) is satisfied.

(i) For each $n \in \mathbb{N}$ and $\Delta \in \mathcal{B}_c(X_{\hat{0}}^{(n)})$, there exists a unique finite measure $\xi_{\Delta}^{(n)}$ on $(\mathbb{R}_+^*)^n$ which satisfies

$$\int_{(\mathbb{R}_+^*)^n} s_1^{i_1} \cdots s_n^{i_n} d\xi_{\Delta}^{(n)}(s_1, \dots, s_n) = \frac{1}{n!} M_{i_1+1, \dots, i_n+1}(\Delta), \quad (i_1, \dots, i_n) \in \mathbb{Z}_+^n. \quad (12)$$

Here $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$.

(ii) For each $n \in \mathbb{N}$, there exists a unique measure $\xi^{(n)}$ on Y_n which satisfies

$$\xi_{\Delta}^{(n)}(A) = \int_{Y_n} \chi_{\Delta}(x_1, \dots, x_n) \chi_A(s_1, \dots, s_n) d\xi^{(n)}(x_1, s_1, \dots, x_n, s_n). \quad (13)$$

for all $\Delta \in \mathcal{B}_c(X_{\hat{0}}^{(n)})$ and $A \in \mathcal{B}((\mathbb{R}_+^*)^n)$.

(iii) For each $n \in \mathbb{N}$, let $\rho^{(n)}$ be the measure on Y_n given by

$$d\rho^{(n)}(x_1, s_1, \dots, x_n, s_n) := (s_1 \cdots s_n)^{-1} d\xi^{(n)}(x_1, s_1, \dots, x_n, s_n). \quad (14)$$

Then $\rho^{(n)}$ is the restriction of the correlation measure ρ of the point process $\nu = \mathcal{E}(\mu)$ to $\Gamma^{(n)}(X)$.

Remark 2. Note that, by the definition of a correlation measure, one always has $\rho(\emptyset) = 1$. Thus, Theorem 1 gives a three-step way of recovering the correlation measure ρ of the point process $\nu = \mathcal{E}(\mu)$ from the moment sequence $(M^{(n)})_{n=0}^{\infty}$.

Proof. We start the proof with the following

Lemma 3. Assume that, for each $n \in \mathbb{N}$, $m^{(n)}$ is a symmetric measure on X^n . Assume that, for each $\Lambda \in \mathcal{B}_c(X)$, there exists a constant $C_{\Lambda} > 0$ such that $m^{(n)}(\Lambda^n) \leq C_{\Lambda}^n n!$ for all $n \in \mathbb{N}$. Then, for any $i_1, \dots, i_n \in \mathbb{N}$, $n \in \mathbb{N}$, and $\Lambda \in \mathcal{B}_c(X)$,

$$\frac{1}{n!} m_{i_1, \dots, i_n}(\Lambda_{\hat{0}}^{(n)}) \leq i_1! \cdots i_n! C_{\Lambda}^{i_1 + \dots + i_n}.$$

Proof. Fix any $i_1, \dots, i_n \in \mathbb{N}$ and $\Lambda \in \mathcal{B}_c(X)$. Let $\pi = \{A_1, \dots, A_n\} \in \Pi(i_1 + \dots + i_n)$ be as in (6). By the construction of the measure m_{i_1, \dots, i_n} , we get

$$\begin{aligned} & m_{i_1, \dots, i_n}(\Lambda_{\hat{0}}^{(n)}) \\ &= \int_{X_{\pi}^{(i_1 + \dots + i_n)}} \chi_{\Lambda^n}(x_1, x_{i_1+1}, \dots, x_{i_1 + \dots + i_{n-1} + 1}) dm^{(i_1 + \dots + i_n)}(x_1, \dots, x_{i_1 + \dots + i_n}) \end{aligned}$$

$$\begin{aligned}
&= \int_{X^{i_1+\dots+i_n}} \chi_{\Lambda^{i_1+\dots+i_n} \cap X_\pi^{(i_1+\dots+i_n)}}(x_1, \dots, x_{i_1+\dots+i_n}) dm^{(i_1+\dots+i_n)}(x_1, \dots, x_{i_1+\dots+i_n}) \\
&= \int_{X^{i_1+\dots+i_n}} \chi_{\Lambda_\pi^{(i_1+\dots+i_n)}} dm^{(i_1+\dots+i_n)} \\
&= \int_{X^{i_1+\dots+i_n}} \text{Sym}_{i_1+\dots+i_n} \chi_{\Lambda_\pi^{(i_1+\dots+i_n)}} dm^{(i_1+\dots+i_n)}. \tag{15}
\end{aligned}$$

Let $\psi \in \Pi(i_1 + \dots + i_n)$ be a partition having exactly n blocks:

$$\psi = \{B_1, \dots, B_n\},$$

where the blocks B_1, \dots, B_n are enumerated so that $\min B_1 < \min B_2 < \dots < \min B_n$. Set $j_l := |B_l|$, $l = 1, \dots, n$. Denote by Ψ_{i_1, \dots, i_n} the set of all such partitions ψ which satisfy

$$(i_1, \dots, i_n) = (j_{\sigma(1)}, \dots, j_{\sigma(n)})$$

for some permutation $\sigma \in \mathfrak{S}_n$. An easy combinatoric argument shows that the number of all partitions in Ψ_{i_1, \dots, i_n} is equal to

$$N_{i_1, \dots, i_n} = \frac{(i_1 + \dots + i_n)!}{i_1! \dots i_n! r_1! r_2! r_3! \dots}.$$

Here for $l = 1, 2, 3, \dots$, r_l denotes the number of coordinates in the vector (i_1, i_2, \dots, i_n) which are equal l . In particular,

$$r_1 + r_2 + r_3 + \dots = n,$$

which implies

$$r_1! r_2! r_3! \dots \leq n!.$$

Therefore,

$$N_{i_1, \dots, i_n} \geq \frac{(i_1 + \dots + i_n)!}{i_1! \dots i_n! n!}. \tag{16}$$

For each $\psi \in \Psi_{i_1, \dots, i_n}$,

$$\text{Sym}_{i_1+\dots+i_n} \chi_{\Lambda_\psi^{(i_1+\dots+i_n)}} = \text{Sym}_{i_1+\dots+i_n} \chi_{\Lambda_\pi^{(i_1+\dots+i_n)}}.$$

Hence, by (15) and (16),

$$\begin{aligned}
&\frac{1}{n!} m_{i_1, \dots, i_n}(\Lambda_{\hat{0}}^{(n)}) \\
&= \frac{1}{n! N_{i_1, \dots, i_n}} \sum_{\psi \in \Psi_{i_1, \dots, i_n}} \int_{X^{i_1+\dots+i_n}} \chi_{\Lambda_\psi^{(i_1+\dots+i_n)}} dm^{(i_1+\dots+i_n)}
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{i_1! \cdots i_n!}{(i_1 + \cdots + i_n)!} \int_{X^{i_1 + \cdots + i_n}} \sum_{\psi \in \Psi_{i_1, \dots, i_n}} \chi_{\Lambda_\psi^{(i_1 + \cdots + i_n)}} dm^{(i_1 + \cdots + i_n)} \\
&\leq \frac{i_1! \cdots i_n!}{(i_1 + \cdots + i_n)!} m^{(i_1 + \cdots + i_n)}(\Lambda^{i_1 + \cdots + i_n}) \\
&\leq i_1! \cdots i_n! C_\Lambda^{i_1 + \cdots + i_n}.
\end{aligned}$$

□

To prove statements (i)–(iii) of the theorem, let us first carry out some considerations. Note that, for each $n \in \mathbb{N}$ and each measurable function $f^{(n)} : X^n \rightarrow [0, \infty]$, the functional

$$\mathbb{K}(X) \ni \eta \mapsto \langle \eta^{\otimes n}, f^{(n)} \rangle \in [0, \infty]$$

is measurable and

$$\int_{\mathbb{K}(X)} \langle \eta^{\otimes n}, f^{(n)} \rangle d\mu(\eta) = \int_{X^n} f^{(n)} dM^{(n)}. \quad (17)$$

As easily seen, equality (10) can be extended to the class of all measurable (not necessarily symmetric) functions $f^{(n)} : Y_n \rightarrow [0, \infty]$ as follows:

$$\begin{aligned}
&\int_{\Gamma_f(Y)} \frac{1}{n!} \sum_{\substack{(x_1, s_1), \dots, (x_n, s_n) \in \gamma \\ x_1, \dots, x_n \text{ different}}} f^{(n)}(x_1, s_1, \dots, x_n, s_n) d\nu(\gamma) \\
&= \int_{Y_n} f^{(n)}(x_1, s_1, \dots, x_n, s_n) d\rho^{(n)}(x_1, s_1, \dots, x_n, s_n).
\end{aligned} \quad (18)$$

If we extend the function $f^{(n)}$ by zero to the whole space Y^n , we can rewrite (18) in the equivalent form:

$$\begin{aligned}
&\int_{\Gamma_f(Y)} \frac{1}{n!} \sum_{(x_1, s_1), \dots, (x_n, s_n) \in \gamma} f^{(n)}(x_1, s_1, \dots, x_n, s_n) d\nu(\gamma) \\
&= \int_{Y_n} f^{(n)}(x_1, s_1, \dots, x_n, s_n) d\rho^{(n)}(x_1, s_1, \dots, x_n, s_n).
\end{aligned} \quad (19)$$

In particular, for any measurable function $g^{(n)} : X^n \rightarrow [0, \infty]$ which vanishes outside $X_{\hat{0}}^{(n)}$ and any $i_1, \dots, i_n \in \mathbb{N}$, we get

$$\begin{aligned}
&\int_{\Gamma_f(Y)} \frac{1}{n!} \sum_{(x_1, s_1), \dots, (x_n, s_n) \in \gamma} g^{(n)}(x_1, \dots, x_n) s_1^{i_1} \cdots s_n^{i_n} d\nu(\gamma) \\
&= \int_{Y_n} g^{(n)}(x_1, \dots, x_n) s_1^{i_1} \cdots s_n^{i_n} d\rho^{(n)}(x_1, s_1, \dots, x_n, s_n).
\end{aligned} \quad (20)$$

For simplicity of notation, we will write below

$$\delta(x_1, \dots, x_n) := \chi_{X_1^{(n)}}(x_1, \dots, x_n), \quad (x_1, \dots, x_n) \in X^n.$$

Thus, $\delta(x_1, \dots, x_n)$ is equal to 1 if $x_1 = x_2 = \dots = x_n$, and is equal to zero otherwise. For $i_1, \dots, i_n \in \mathbb{N}$, we define a function $\mathcal{I}_{i_1, \dots, i_n} : X^{i_1 + \dots + i_n} \rightarrow \{0, 1\}$ by setting

$$\begin{aligned} & \mathcal{I}_{i_1, \dots, i_n}(x_1, \dots, x_{i_1 + \dots + i_n}) \\ & := \delta(x_1, \dots, x_{i_1}) \delta(x_{i_1+1}, \dots, x_{i_1+i_2}) \cdots \delta(x_{i_1+\dots+i_{n-1}+1}, \dots, x_{i_1+\dots+i_n}). \end{aligned}$$

For a measurable function $g^{(n)} : X^n \rightarrow [0, \infty)$ which vanishes outside $X_0^{(n)}$, we define a measurable function $\mathcal{R}_{i_1, \dots, i_n} g^{(n)} : X^{i_1 + \dots + i_n} \rightarrow [0, \infty]$ by

$$\begin{aligned} & (\mathcal{R}_{i_1, \dots, i_n} g^{(n)})(x_1, \dots, x_{i_1 + \dots + i_n}) \\ & := g^{(n)}(x_1, x_{i_1+1}, x_{i_1+i_2+1}, \dots, x_{i_1+\dots+i_{n-1}+1}) \mathcal{I}_{i_1, \dots, i_n}(x_1, \dots, x_{i_1+\dots+i_n}). \end{aligned} \quad (21)$$

Note that the function $\mathcal{R}_{i_1, \dots, i_n} g^{(n)}$ vanishes outside the set $X_\pi^{(i_1 + \dots + i_n)}$, where $\pi = \{A_1, \dots, A_n\}$ with the sets A_1, \dots, A_n being as in (6). For each $\eta \in \mathbb{K}(X)$,

$$\begin{aligned} & \langle \eta^{\otimes(i_1 + \dots + i_n)}, \mathcal{R}_{i_1, \dots, i_n} g^{(n)} \rangle \\ & = \sum_{x_1, \dots, x_{i_1 + \dots + i_n} \in \tau(\eta)} (\mathcal{R}_{i_1, \dots, i_n} g^{(n)})(x_1, \dots, x_{i_1 + \dots + i_n}) s_1 \cdots s_{i_1 + \dots + i_n} \\ & = \sum_{x_1, \dots, x_n \in \tau(\eta)} g^{(n)}(x_1, \dots, x_n) s_1^{i_1} \cdots s_n^{i_n}. \end{aligned} \quad (22)$$

By (20), (22), and the definition of the measure ν , we get

$$\begin{aligned} & \frac{1}{n!} \int_{\mathbb{K}(X)} \langle \eta^{\otimes(i_1 + \dots + i_n)}, \mathcal{R}_{i_1, \dots, i_n} g^{(n)} \rangle d\mu(\eta) \\ & = \int_{Y_n} g^{(n)}(x_1, \dots, x_n) s_1^{i_1} \cdots s_n^{i_n} d\rho^{(n)}(x_1, s_1, \dots, x_n, s_n). \end{aligned}$$

Hence, by (17),

$$\begin{aligned} & \int_{Y_n} g^{(n)}(x_1, \dots, x_n) s_1^{i_1} \cdots s_n^{i_n} d\rho^{(n)}(x_1, s_1, \dots, x_n, s_n) \\ & = \frac{1}{n!} \int_{X^{i_1 + \dots + i_n}} \mathcal{R}_{i_1, \dots, i_n} g^{(n)} dM^{(i_1 + \dots + i_n)} \\ & = \frac{1}{n!} \int_{X_\pi^{(i_1 + \dots + i_n)}} \mathcal{R}_{i_1, \dots, i_n} g^{(n)} dM^{(i_1 + \dots + i_n)}, \end{aligned}$$

where the partition π is as above. From here we conclude,

$$\begin{aligned} & \int_{Y_n} g^{(n)}(x_1, \dots, x_n) s_1^{i_1} \dots s_n^{i_n} d\rho^{(n)}(x_1, s_1, \dots, x_n, s_n). \\ &= \frac{1}{n!} \int_{X_0^{(n)}} g^{(n)}(x_1, \dots, x_n) dM_{i_1, \dots, i_n}(x_1, \dots, x_n). \end{aligned} \quad (23)$$

We define a symmetric measure $\xi^{(n)}$ on Y_n by setting

$$d\xi^{(n)}(x_1, s_1, \dots, x_n, s_n) := s_1 \cdots s_n d\rho^{(n)}(x_1, s_1, \dots, x_n, s_n). \quad (24)$$

Then, equality (23) can be rewritten as follows:

$$\begin{aligned} & \int_{Y_n} g^{(n)}(x_1, \dots, x_n) s_1^{i_1} \dots s_n^{i_n} d\xi^{(n)}(x_1, s_1, \dots, x_n, s_n). \\ &= \frac{1}{n!} \int_{X_0^{(n)}} g^{(n)}(x_1, \dots, x_n) dM_{i_1+1, \dots, i_n+1}(x_1, \dots, x_n), \quad (i_1, \dots, i_n) \in \mathbb{Z}_+^n. \end{aligned} \quad (25)$$

For any $\Delta \in \mathcal{B}_c(X_0^{(n)})$, let $\xi_\Delta^{(n)}$ be the finite measure on $(\mathbb{R}_+^*)^n$ which satisfies (13). Denote

$$\xi_{\mathbf{i}}^\Delta = \xi_{i_1, \dots, i_n}^\Delta := \frac{1}{n!} M_{i_1+1, \dots, i_n+1}(\Delta), \quad \mathbf{i} = (i_1, \dots, i_n) \in \mathbb{Z}_+^n. \quad (26)$$

Then, by (25) and (26),

$$\xi_{\mathbf{i}}^\Delta = \int_{(\mathbb{R}_+^*)^n} s_1^{i_1} \cdots s_n^{i_n} d\xi_\Delta^{(n)}(s_1, \dots, s_n), \quad \mathbf{i} = (i_1, \dots, i_n) \in \mathbb{Z}_+^n. \quad (27)$$

Thus, $(\xi_{\mathbf{i}}^\Delta)_{\mathbf{i} \in \mathbb{Z}_+^n}$ is the moment sequence of the finite measure $\xi_\Delta^{(n)}$.

Choose any $\Lambda \in \mathcal{B}_c(X)$ such that $\Delta \subset \Lambda_0^{(n)}$. By formulas (7), (26) and Lemma 3,

$$\begin{aligned} \xi_{i_1, \dots, i_n}^\Delta &\leq \frac{1}{n!} M_{i_1+1, \dots, i_n+1}(\Lambda_0^{(n)}) \\ &\leq (i_1 + 1)! \cdots (i_n + 1)! C_\Lambda^{i_1 + \cdots + i_n + n} \\ &\leq (i_1 + \cdots + i_n + n)! C_\Lambda^{i_1 + \cdots + i_n + n}, \quad (i_1, \dots, i_n) \in \mathbb{Z}_+^n. \end{aligned} \quad (28)$$

We are now ready to finish the proof of the theorem. Since $(\xi_{\mathbf{i}}^\Delta)_{\mathbf{i} \in \mathbb{Z}_+^n}$ is the moment sequence of the finite measure $\xi_\Delta^{(n)}$ on $(\mathbb{R}_+^*)^n$, and since this moment sequence satisfies estimate (28), we conclude from e.g. [4, Chapter 5, Subsec. 2.1, Examples 2.1, 2.2] that the moment sequence $(\xi_{\mathbf{i}}^\Delta)_{\mathbf{i} \in \mathbb{Z}_+^n}$ uniquely identifies the measure $\xi_\Delta^{(n)}$. Hence, statement (i) holds. Next, equality (13) evidently holds. Note also that the values of the measure $\xi^{(n)}$ on the sets of the form

$$\{(x_1, s_1, \dots, x_n, s_n) \in Y_n \mid (x_1, \dots, x_n) \in \Delta, (s_1, \dots, s_n) \in A\}$$

where $\Delta \in \mathcal{B}_c(X_0^{(n)})$ and $A \in \mathcal{B}((\mathbb{R}_+^*)^n)$, completely identify the measure $\xi^{(n)}$ on Y^n . Thus, statement (ii) holds. Finally, statement (iii) trivially follows from (24). \square

3 A characterization of random discrete measure in terms of moments

In this section, we assume that μ is a random measure on X which has finite moments. Let $(M^{(n)})_{n=0}^{\infty}$ be its moment sequence. We assume that condition (C1) is satisfied. Additionally, we will assume that the following condition holds:

(C2) For each $\Lambda \in \mathcal{B}_c(X)$, there exists a constant $C'_\Lambda > 0$ such that

$$M^{(n)}(\Lambda_0^{(n)}) \leq (C'_\Lambda)^n n!, \quad n \in \mathbb{N}, \quad (29)$$

and for any sequence $\{\Lambda_k\}_{k=1}^{\infty} \in \mathcal{B}_c(X)$ such that $\Lambda_k \downarrow \emptyset$, we have $C'_{\Lambda_k} \rightarrow 0$ as $k \rightarrow \infty$.

Remark 4. Assumption (C2) is usually satisfied by a measure μ being concentrated on the cone $\mathbb{K}(X)$. In the latter case, by the proof of Theorem 1, we have

$$\begin{aligned} M^{(n)}(\Lambda_0^{(n)}) &= n! \xi^{(n)}(Y_n \cap (\Lambda \times \mathbb{R}_+^*))^n \\ &= n! \int_{Y_n \cap (\Lambda \times \mathbb{R}_+^*)^n} s_1 \cdots s_n d\rho^{(n)}(x_1, s_1, \dots, x_n, s_n), \end{aligned}$$

so that estimate (29) becomes

$$\int_{Y_n \cap (\Lambda \times \mathbb{R}_+^*)^n} s_1 \cdots s_n d\rho^{(n)}(x_1, s_1, \dots, x_n, s_n) \leq (C'_\Lambda)^n.$$

For example, in the case of the Gamma measure (see Section 1), we have

$$\int_{Y_n \cap (\Lambda \times \mathbb{R}_+^*)^n} s_1 \cdots s_n d\rho^{(n)}(x_1, s_1, \dots, x_n, s_n) = \frac{1}{n!} \left(\int_{\Lambda} dx \right)^n,$$

so condition (C2) is trivially satisfied.

Note also that one should not expect that the constant C_Λ in estimate (7) becomes small as set Λ shrinks to an empty set. This, for example, is not even true in the case of the Gamma measure.

We fix a sequence $(\Lambda_l)_{l=1}^{\infty}$ of compact subsets of X such that $\Lambda_1 \subset \Lambda_2 \subset \Lambda_3 \subset \cdots$ and $\bigcup_{l=1}^{\infty} \Lambda_l = X$. For example, in the case $X = \mathbb{R}^d$, one may choose $\Lambda_l = [-l, l]^d$.

Theorem 5. *Let μ be a random measure on X , i.e., a probability measure on $(\mathbb{M}(X), \mathcal{B}(\mathbb{M}(X)))$. Assume that μ has finite moments, and let $(M^{(n)})_{n=0}^{\infty}$ be its moment sequence. Further assume that conditions (C1) and (C2) are satisfied. Then μ is a random discrete measure, i.e., $\mu(\mathbb{K}(X)) = 1$ if and only if the moment sequence $(M^{(n)})_{n=0}^{\infty}$ satisfies the following conditions:*

(i) For any $n \in \mathbb{N}$, $\Delta \in \mathcal{B}_c(X_0^{(n)})$, and $\mathbf{i} = (i_1, \dots, i_n) \in \mathbb{Z}_+^n$, let $\xi_{\mathbf{i}}^\Delta = \xi_{i_1, \dots, i_n}^\Delta$ be defined by (26). Then the sequence $(\xi_{\mathbf{i}}^\Delta)_{\mathbf{i} \in \mathbb{Z}_+^n}$ is positive definite, i.e., for any finite sequence of complex numbers indexed by elements of \mathbb{Z}_+^n , $(z_{\mathbf{i}})_{\mathbf{i} \in \mathbb{Z}_+^n, |\mathbf{i}| \leq N}$, we have

$$\sum_{\substack{\mathbf{i}, \mathbf{j} \in \mathbb{Z}_+^n \\ \max\{|\mathbf{i}|, |\mathbf{j}|\} \leq N}} \xi_{\mathbf{i}+\mathbf{j}}^\Delta z_{\mathbf{i}} \bar{z}_{\mathbf{j}} \geq 0.$$

Here $|\mathbf{i}| := \max\{i_1, \dots, i_n\}$ and $N \in \mathbb{N}$.

(ii) For each $\Delta \in \mathcal{B}_c(X_0^{(n)})$ of the form $\Delta = (\Lambda_l)_0^{(n)}$ with $l \in \mathbb{N}$, set

$$r_i^\Delta := \xi_{i,0,0,\dots,0}^\Delta, \quad i \in \mathbb{Z}_+. \quad (30)$$

Then, for any finite sequence of complex numbers, $(z_n)_{n=0}^N$, we have

$$\sum_{i,j=0}^N r_{i+j+1}^\Delta z_i \bar{z}_j \geq 0, \quad (31)$$

and furthermore

$$\sum_{k=1}^{\infty} (D_{k-1}^\Delta D_k^\Delta)^{-1/2} \det \begin{bmatrix} r_1^\Delta & r_2^\Delta & \dots & r_k^\Delta \\ r_2^\Delta & r_3^\Delta & \dots & r_{k+1}^\Delta \\ \vdots & \vdots & \vdots & \vdots \\ r_k^\Delta & r_{k+1}^\Delta & \dots & r_{2k-1}^\Delta \end{bmatrix} = \infty, \quad (32)$$

where

$$D_k := \det \begin{bmatrix} r_0^\Delta & r_1^\Delta & \dots & r_k^\Delta \\ r_1^\Delta & r_2^\Delta & \dots & r_{k+1}^\Delta \\ \vdots & \vdots & \vdots & \vdots \\ r_k^\Delta & r_{k+1}^\Delta & \dots & r_{2k}^\Delta \end{bmatrix}, \quad k \in \mathbb{Z}_+.$$

Proof. Assume that $\mu(\mathbb{K}(X)) = 1$ and let us show that conditions (i) and (ii) are satisfied. Let $\Delta \in \mathcal{B}_c(X_0^{(n)})$. It follows from the proof of Theorem 1 (see in particular formula (27)) that the sequence $(\xi_{\mathbf{i}}^\Delta)_{\mathbf{i} \in \mathbb{Z}_+^n}$ is the moment sequence of the finite measure $\xi_\Delta^{(n)}$. Hence, condition (i) is indeed satisfied (see e.g. [4, Chapter 5, Subsec. 2.1]).

Next, let $\Delta \in \mathcal{B}(X_0^{(n)})$ be of the form $\Delta = (\Lambda_l)_0^{(n)}$. Clearly, $(r_i^\Delta)_{i=0}^\infty$ is the moment sequence of the first coordinate projection of the measure $\xi_\Delta^{(n)}$, which we denote by $P_1 \xi_\Delta^{(n)}$. The measure $P_1 \xi_\Delta^{(n)}$ is concentrated on $[0, \infty)$, hence (31) follows (see e.g. [1, Chapter 2, Subsec. 6.5]). By (7), (13), (25), and Lemma 3,

$$r_i^\Delta = \int_{(\mathbb{R}_+^*)^n} s_1^i d\xi_\Delta^{(n)}(s_1, \dots, s_n)$$

$$\begin{aligned}
&= \int_{Y_n} \chi_\Delta(x_1, \dots, x_n) s_1^i d\xi^{(n)}(x_1, s_1, \dots, x_n, s_n) \\
&= \frac{1}{n!} \int_{X_0^{(n)}} \chi_\Delta(x_1, \dots, x_n) dM_{i+1,1,1,\dots,1}(x_1, \dots, x_n) \\
&= \frac{1}{n!} M_{i+1,1,1,\dots,1}((\Lambda_l)_0^{(n)}) \\
&\leq (i+1)! C_\Lambda^{n+i}, \quad i \in \mathbb{Z}_+. \tag{33}
\end{aligned}$$

Hence, by the Carleman criterion (see e.g. [1]), the measure $P_1 \xi_\Delta^{(n)}$ is the unique measure on \mathbb{R} which has moments $(r_i^\Delta)_{i=0}^\infty$. Therefore, by [1, formula (4) in Chapter I, Sect.1; Chater II, Subsec. 4.1; Theorem 2.5.3], (32) follows from the fact that the measure $P_1 \xi_\Delta^{(n)}$ has no atom at point 0. Thus, condition (ii) is satisfied.

Remark 6. Note that, in this part of the proof, we have not used condition (C2).

Let us now prove the inverse statement. So, we assume that $(M^{(n)})_{n=0}^\infty$ is the moment sequence of a probability measure μ on $(\mathbb{M}(X), \mathcal{B}(\mathbb{M}(X)))$. We assume that conditions (i), (ii) are satisfied, and we have to prove that $\mu(\mathbb{K}(X)) = 1$.

Fix any $n \in \mathbb{N}$ and $\Delta \in \mathcal{B}_c(X_0^{(n)})$. Choose any $\Lambda \in \mathcal{B}_c(X)$ such that $\Delta \subset \Lambda_0^{(n)}$. By (7), (26) and Lemma 3,

$$\begin{aligned}
\xi_{i_1, \dots, i_n}^\Delta &= \frac{1}{n!} M_{i_1+1, \dots, i_n+1}(\Delta) \\
&\leq \frac{1}{n!} M_{i_1+1, \dots, i_n+1}(\Lambda_0^{(n)}) \\
&\leq (i_1+1)! \cdots (i_n+1)! C_\Lambda^{i_1+\dots+i_n+n} \\
&\leq (i_1+\dots+i_n+n)! C_\Lambda^{i_1+\dots+i_n+n}, \quad (i_1, \dots, i_n) \in \mathbb{Z}_+^n. \tag{34}
\end{aligned}$$

Furthermore, by condition (i), the sequence $(\xi_{\mathbf{i}}^\Delta)_{\mathbf{i} \in \mathbb{Z}_+^n}$ is positive definite. Hence, using e.g. [4, Chapter 5, Subsec. 2.1, Examples 2.1, 2.2], we conclude that there exists a unique measure $\xi_\Delta^{(n)}$ on \mathbb{R}^n such that $(\xi_{\mathbf{i}}^\Delta)_{\mathbf{i} \in \mathbb{Z}_+^n}$ is its moment sequence, i.e.,

$$\xi_{\mathbf{i}}^\Delta = \int_{\mathbb{R}^n} s_1^{i_1} \cdots s_n^{i_n} d\xi_\Delta^{(n)}(s_1, \dots, s_n), \quad \mathbf{i} = (i_1, \dots, i_n) \in \mathbb{Z}_+^n. \tag{35}$$

Lemma 7. *Let $n \in \mathbb{N}$. Let $\{\Delta_k\}_{k=1}^\infty$ be a sequence of disjoint sets from $\mathcal{B}_c(X_0^{(n)})$. Denote $\Delta := \bigcup_{k=1}^\infty \Delta_k$ and assume that $\Delta \in \mathcal{B}_c(X_0^{(n)})$. We then have*

$$\sum_{k=1}^\infty \xi_{\Delta_k}^{(n)} = \xi_\Delta^{(n)}. \tag{36}$$

Proof. Consider the measure

$$\psi_{\Delta}^{(n)} := \sum_{k=1}^{\infty} \xi_{\Delta_k}^{(n)}.$$

Fix any $i_1, \dots, i_n \in \mathbb{Z}_+$. Since $M^{(i_1+\dots+i_n)}$ is a measure, we easily get

$$\begin{aligned} \int_{\mathbb{R}^n} s_1^{i_1} \cdots s_n^{i_n} d\psi_{\Delta}^{(n)}(s_1, \dots, s_n) &= \sum_{k=1}^{\infty} \int_{\mathbb{R}^n} s_1^{i_1} \cdots s_n^{i_n} d\xi_{\Delta_k}^{(n)}(s_1, \dots, s_n) \\ &= \sum_{k=1}^{\infty} \frac{1}{n!} M_{i_1+1, \dots, i_n+1}(\Delta_k) \\ &= \frac{1}{n!} M_{i_1+1, \dots, i_n+1}(\Delta) \\ &= \int_{\mathbb{R}^n} s_1^{i_1} \cdots s_n^{i_n} d\xi_{\Delta}^{(n)}(s_1, \dots, s_n). \end{aligned}$$

Hence, the measures $\psi_{\Delta}^{(n)}$ and $\xi_{\Delta}^{(n)}$ have the same moments. But the measure $\xi_{\Delta}^{(n)}$ is uniquely identified by its moments, so $\psi_{\Delta}^{(n)} = \xi_{\Delta}^{(n)}$. \square

Fix any $l \in \mathbb{N}$ and set $\Delta = (\Lambda_l)_{\hat{0}}^{(n)}$. By (30) and (35),

$$r_i^{\Delta} = \int_{\mathbb{R}^n} s_1^i d\xi_{\Delta}^{(n)}(s_1, \dots, s_n), \quad i \in \mathbb{Z}_+.$$

Thus, the numbers $(r_i^{\Delta})_{i=0}^{\infty}$ form the moment sequence of the first coordinate projection of the measure $\xi_{\Delta}^{(n)}$, which we denote, as above, by $P_1\xi_{\Delta}^{(n)}$. As easily follows from (34) and the Carleman criterion, the measure $P_1\xi_{\Delta}^{(n)}$ is uniquely identified by its moment sequence. Then, by (31), the measure $P_1\xi_{\Delta}^{(n)}$ is concentrated on $\mathbb{R}_+ = [0, \infty)$, and by (32), $(P_1\xi_{\Delta}^{(n)})(\{0\}) = 0$, see [1]. Therefore, the measure $P_1\xi_{\Delta}^{(n)}$ is concentrated on \mathbb{R}_+^* . Evidently, for any $(i_1, \dots, i_n) \in \mathbb{Z}_+^n$ and any $\sigma \in \mathfrak{S}_n$, we get

$$\begin{aligned} \int_{\mathbb{R}^n} s_{\sigma(1)}^{i_1} \cdots s_{\sigma(n)}^{i_n} d\xi_{\Delta}^{(n)}(s_1, \dots, s_n) &= \int_{\mathbb{R}^n} s_1^{i_{\sigma^{-1}(1)}} \cdots s_n^{i_{\sigma^{-1}(n)}} d\xi_{\Delta}^{(n)}(s_1, \dots, s_n) \\ &= \int_{\mathbb{R}^n} s_1^{i_1} \cdots s_n^{i_n} d\xi_{\Delta}^{(n)}(s_1, \dots, s_n). \end{aligned}$$

Hence, the measure $\xi_{\Delta}^{(n)}$ is symmetric on \mathbb{R}^n . Therefore, for each $j = 1, \dots, n$, the j -th coordinate projection of $\xi_{\Delta}^{(n)}$ is concentrated on \mathbb{R}_+^* . This implies that the measure $\xi_{\Delta}^{(n)}$ is concentrated on $(\mathbb{R}_+^*)^n$. In view of Lemma 7, we easily conclude that the latter statement holds, in fact, for each set $\Delta \in \mathcal{B}_c(X_{\hat{0}}^{(n)})$.

Lemma 8. For each $n \in \mathbb{N}$, there exists a unique measure $\xi^{(n)}$ on Y_n which satisfies (13) for all $\Delta \in \mathcal{B}_c(X_0^{(n)})$ and $A \in \mathcal{B}((\mathbb{R}_+^*)^n)$.

Proof. For each $\Delta \in \mathcal{B}_c(X^n)$, we define a measure $\xi_\Delta^{(n)}$ on $(\mathbb{R}_+^*)^n$ by

$$\xi_\Delta^{(n)} := \xi_{\Delta \cap X_0^{(n)}}^{(n)}.$$

(Note that $\Delta \cap X_0^{(n)} \in \mathcal{B}_c(X_0^{(n)})$.) Clearly, the statement of Lemma 7 remains true when the sets Δ_k , $k \in \mathbb{N}$, and Δ belong to $\mathcal{B}_c(X^n)$. So, it suffices to prove that there exists a unique measure $\xi^{(n)}$ on Y^n which satisfies

$$\xi_\Delta^{(n)}(A) = \int_{Y^n} \chi_\Delta(x_1, \dots, x_n) \chi_A(s_1, \dots, s_n) d\xi^{(n)}(x_1, s_1, \dots, x_n, s_n)$$

for all $\Delta \in \mathcal{B}_c(X^n)$ and $A \in \mathcal{B}((\mathbb{R}_+^*)^n)$. By changing the order of the variables, we will equivalently prove that there exists a unique measure $\xi^{(n)}$ on $X^n \times (\mathbb{R}_+^*)^n$ which satisfies

$$\xi^{(n)}(\Delta \times A) = \xi_\Delta^{(n)}(A), \quad \Delta \in \mathcal{B}_c(X^n), \quad A \in \mathcal{B}((\mathbb{R}_+^*)^n). \quad (37)$$

Our proof of this fact is a modification of the proof of [3, 4.4 Theorem]. We denote by $\mathcal{R}^{(n)}$ the ring of subsets of $X^n \times (\mathbb{R}_+^*)^n$ which are finite, disjoint unions of sets of the form $\Delta \times A$, where $\Delta \in \mathcal{B}_c(X^n)$ and $A \in \mathcal{B}((\mathbb{R}_+^*)^n)$. We define a content $\xi^{(n)}$ on $\mathcal{R}^{(n)}$ through formula (37).

By [3, Sections 3 and 5], to prove the lemma it suffices to prove the following statement: *Let $(F_k)_{k=1}^\infty$ be a sequence of sets from $\mathcal{R}^{(n)}$ such that $F_1 \supset F_2 \supset F_3 \supset \dots$ and $\bigcap_{k=1}^\infty F_k = \emptyset$. Then $\lim_{k \rightarrow \infty} \xi^{(n)}(F_k) = 0$.*

Hence, it suffices to prove the following statement: *Let $(F_k)_{k=1}^\infty$ be a sequence of sets from $\mathcal{R}^{(n)}$ such that $F_1 \supset F_2 \supset F_3 \supset \dots$. Assume that*

$$\delta := \lim_{k \rightarrow \infty} \xi^{(n)}(F_k) = \inf_{k \in \mathbb{N}} \xi^{(n)}(F_k) > 0. \quad (38)$$

Then $\bigcap_{k=1}^\infty F_k \neq \emptyset$.

We state that, for each $k \in \mathbb{N}$, there exists a set $G_k \in \mathcal{R}^{(n)}$ such that G_k is a compact set in $X^n \times (\mathbb{R}_+^*)^n$, $G_k \subset F_k$, and

$$\xi^{(n)}(F_k) - \xi^{(n)}(G_k) \leq 2^{-k} \delta. \quad (39)$$

Indeed, in order to prove (39), it suffices to show that, for any $\Delta \in \mathcal{B}_c(X)$, $A \in \mathcal{B}((\mathbb{R}_+^*)^n)$, and $\varepsilon > 0$, there exist a compact set $\Delta' \subset \Delta$ and a compact set $A' \subset A$ such that

$$\xi^{(n)}(\Delta \times A) - \xi^{(n)}(\Delta' \times A') \leq \varepsilon. \quad (40)$$

Let $\overline{\Delta}$ denote the closure of Δ in X . Note that $\overline{\Delta}$ is a compact set in X^n , hence $\xi_{\overline{\Delta}}^{(n)}((\mathbb{R}_+^*)^n) < \infty$. Denote by $\mathcal{B}(\overline{\Delta})$ the trace σ -algebra of $\mathcal{B}(X^n)$ on $\overline{\Delta}$. By Lemma 7, the mapping

$$\mathcal{B}(\overline{\Delta}) \ni \Psi \mapsto \xi_{\Psi}^{(n)}(A) = \xi^{(n)}(\Psi \times A)$$

is a finite measure. By e.g. [3, 26.2 Lemma], this measure is regular. Hence, there exists a compact set $\Delta' \subset \Delta$ such that

$$\xi_{\Delta}^{(n)}(A) - \xi_{\Delta'}^{(n)}(A) = \xi^{(n)}(\Delta \times A) - \xi^{(n)}(\Delta' \times A) \leq \frac{\varepsilon}{2}. \quad (41)$$

Next, $\xi_{\Delta'}^{(n)}$ is a finite measure on $(\mathbb{R}_+^*)^n$. Hence, it is regular, too. Thus, there exists a compact set $A' \subset A$ such that

$$\xi_{\Delta'}^{(n)}(A) - \xi_{\Delta'}^{(n)}(A') = \xi^{(n)}(\Delta' \times A) - \xi^{(n)}(\Delta' \times A') \leq \frac{\varepsilon}{2}. \quad (42)$$

Formulas (41) and (42) imply (40).

Next, analogously to the proof of [3, 4.4 Lemma], we conclude from (39) by induction that

$$\xi^{(n)}(H_k) \geq \xi^{(n)}(F_k) - (1 - 2^{-k})\delta, \quad (43)$$

where $H_k := G_1 \cap G_2 \cap \dots \cap G_k$. By (38) and (43), we get $\xi^{(n)}(H_k) \geq 2^{-k}\delta$. Hence, $H_k \neq \emptyset$. Since H_k are compact sets and $H_1 \supset H_2 \supset H_3 \supset \dots$, we therefore conclude that $\bigcap_{k=1}^{\infty} H_k \neq \emptyset$, see e.g. [17, p. 118]. But $\bigcap_{k=1}^{\infty} H_k \subset \bigcap_{k=1}^{\infty} F_k$, so that $\bigcap_{k=1}^{\infty} F_k \neq \emptyset$. \square

We define the measures $\rho^{(n)}$ on Y_n by setting

$$d\rho^{(n)}(x_1, s_1, \dots, x_n, s_n) := (s_1 \cdots s_n)^{-1} d\xi^{(n)}(x_1, s_1, \dots, x_n, s_n), \quad n \in \mathbb{N}. \quad (44)$$

Note that $\rho^{(n)}$ is a symmetric measure on Y_n . We next define a measure ρ on $(\Gamma_0(Y), \mathcal{B}(\Gamma_0(Y)))$ which satisfies $\rho(\Gamma^{(0)}(Y)) = 1$, and for each $n \in \mathbb{N}$, the restriction of the measure ρ to $\Gamma^{(n)}(Y)$ can be identified with $\rho^{(n)}$, i.e., for each measurable function $G : \Gamma_0(Y) \rightarrow [0, \infty]$

$$\int_{\Gamma^{(n)}(Y)} G(\lambda) d\rho(\lambda) = \int_{Y_n} G(\{x_1, s_1, \dots, x_n, s_n\}) d\rho^{(n)}(x_1, s_1, \dots, x_n, s_n).$$

Lemma 9. *There exists a unique point process ν in Y whose correlation measure is ρ .*

Proof. We divide the proof of this lemma into several steps.

Step 1. By [14, Corollary 1] and its proof (see also [5, 12]), to prove the lemma it suffices to show that the conditions (LB) and (PD) below are satisfied.

(LB) *Local bound:* For any $\Lambda \in \mathcal{B}_c(X)$ and $A \in \mathcal{B}_c(\mathbb{R}_+^*)$, there exists a constant $\text{const}_{\Lambda, A} > 0$ such that

$$\rho^{(n)}((\Lambda \times A)^n \cap Y_n) \leq \text{const}_{\Lambda, A}^n, \quad n \in \mathbb{N},$$

and for any sequence $\Lambda_k \in \mathcal{B}_c(X)$ such that $\Lambda_k \downarrow \emptyset$ and $A \in \mathcal{B}_c(\mathbb{R}_+^*)$, we have $\text{const}_{\Lambda_k, A} \rightarrow 0$ as $k \rightarrow \infty$.

To formulate condition (PD) we first need to give some definitions. For any measurable functions $G_1, G_2 : \Gamma_0(Y) \rightarrow \mathbb{R}$, we define their \star -product as the measurable function $G_1 \star G_2 : \Gamma_0(Y) \rightarrow \mathbb{R}$ given by

$$G_1 \star G_2(\lambda) := \sum_{\substack{\lambda_1 \subset \lambda, \lambda_2 \subset \lambda \\ \lambda_1 \cup \lambda_2 = \lambda}} G_1(\lambda_1) G_2(\lambda_2), \quad \lambda \in \Gamma_0(Y). \quad (45)$$

We denote by \mathcal{S} the class of all functions $G : \Gamma_0(Y) \rightarrow \mathbb{R}$ which satisfy the following assumptions:

- (i) There exists $N \in \mathbb{N}$ such that $G^{(n)} := G \upharpoonright \Gamma^{(n)}(Y) = 0$ for all $n > N$.
- (ii) For each $n = 1, \dots, N$, the function $G^{(n)} := G \upharpoonright \Gamma^{(n)}(Y)$ can be identified with a finite linear combination of functions of the form

$$\text{Sym}_n(\chi_{B_1} \otimes \cdots \otimes \chi_{B_n}),$$

where for $i = 1, \dots, n$ $B_i = \Lambda_i \times A_i$ with $\Lambda_i \in \mathcal{B}_c(X)$ and $A_i \in \mathcal{B}_c(\mathbb{R}_+^*)$, Sym_n denotes the operator of symmetrization of a function, and

$$(\chi_{B_1} \otimes \cdots \otimes \chi_{B_n})(x_1, s_1, \dots, x_n, s_n) := \chi_{B_1}(x_1, s_1) \cdots \chi_{B_n}(x_n, s_n),$$

with $(x_1, s_1), \dots, (x_n, s_n) \in Y$ and $(x_i, s_i) \neq (x_j, s_j)$ if $i \neq j$.

It is evident that each function $G \in \mathcal{S}$ is bounded and integrable with respect to the measure ρ , and for any $G_1, G_2 \in \mathcal{S}$, we have $G_1 \star G_2 \in \mathcal{S}$.

(PD) *\star -positive definiteness:* For each $G \in \mathcal{S}$, we have

$$\int_{\Gamma_0(Y)} G \star G d\rho \geq 0. \quad (46)$$

Remark 10. For a function $G \in \mathcal{S}$, denote

$$(KG)(\gamma) := \sum_{\lambda \in \gamma} G(\lambda), \quad \gamma \in \Gamma(Y).$$

Then, according to Section 1 (see, in particular, formula (4)), if ρ is the correlation measure of a point process μ in Y , then

$$\int_{\Gamma_0(Y)} G d\rho = \int_{\Gamma(Y)} KG d\mu.$$

Furthermore, an easy calculation shows that, for any $G_1, G_2 \in \mathcal{S}$, we have

$$K(G_1 \star G_2) = KG_1 \cdot KG_2.$$

Hence, in this case, formula (46) becomes

$$\int_{\Gamma(Y)} (KG)^2 d\mu \geq 0.$$

Step 2. Let $A \in \mathcal{B}_c(\mathbb{R}_+^*)$. By (44), there exists a constant C , depending only on A , such that

$$\rho^{(n)}((\Lambda \times A)^n \cap Y_n) \leq C \xi^{(n)}((\Lambda \times \mathbb{R}_+^*)^n \cap Y_n) \quad (47)$$

for each $\Lambda \in \mathcal{B}_c(X)$. By (26), (35), Lemma 8, and condition (C2),

$$\begin{aligned} \xi^{(n)}((\Lambda \times \mathbb{R}_+^*)^n \cap Y_n) &= \xi_{\Lambda_0^{(n)}}^{(n)}((\mathbb{R}_+^*)^n) \\ &= \xi_{0, \dots, 0}^{\Lambda_0^{(n)}} \\ &= \frac{1}{n!} M_{1, \dots, 1}(\Lambda_0^{(n)}) \\ &\leq (C'_\Lambda)^n. \end{aligned} \quad (48)$$

Condition (LB) now follows from (47) and (48).

Step 3. We denote

$$\Phi(Y) := \bigcup_{n=0}^{\infty} \Phi^{(n)}(Y),$$

where the set $\Phi^{(0)}(Y)$ contains just one element, and for $n \in \mathbb{N}$, $\Phi^{(n)}(Y) := Y_n$. We define a σ -algebra $\mathcal{B}(\Phi(Y))$ on $\Phi(Y)$ so that, for each $n = 0, 1, 2, \dots$, $\Phi^{(n)}(Y) \in \mathcal{B}(\Phi(Y))$ and for each $n \in \mathbb{N}$, the restriction of $\mathcal{B}(\Phi(Y))$ to $\Phi^{(n)}(Y)$ coincides with $\mathcal{B}(Y_n)$. We can equivalently treat ρ as a measure on $\Phi(Y)$, so that $\rho(\Phi^{(0)}(Y)) = 1$ and, for $n \in \mathbb{N}$, the restriction of ρ to $\Phi^{(n)}(Y)$ is $\rho^{(n)}$. We call a function $G : \Phi(Y) \rightarrow \mathbb{R}$ symmetric if, for each $n \in \mathbb{N}$, the restriction of G to $\Phi^{(n)}(Y)$ is a symmetric function. Clearly, each function G on $\Gamma_0(Y)$ determines a symmetric function on $\Phi(Y)$, for which we preserve the notation G . Furthermore, for an integrable function G , we then have $\int_{\Gamma_0(Y)} G d\rho = \int_{\Phi(Y)} G d\rho$.

Let $m, n \in \mathbb{N}$. Denote by $\text{Pair}(m, n)$ the set of all possible collections of pairs of numbers $\varkappa = \{(\alpha_i, \beta_i)\}_{i=1}^k$ such that $\alpha_i \in \{1, \dots, m\}$ and $\beta_i \in \{m+1, \dots, m+n\}$.

We also set $|\varkappa| := k$. We assume than an empty collection belongs to $\text{Pair}(m, n)$, for which $|\varkappa| = 0$.

Let $G_1^{(m)} : Y_m \rightarrow \mathbb{R}$, $G_2^{(n)} : Y_n \rightarrow \mathbb{R}$, and let $\varkappa = \{(\alpha_i, \beta_i)\} \in \text{Pair}(m, n)$. We define a function $(G_1^{(m)} \otimes G_2^{(n)})_{\varkappa} : Y_{m+n-k} \rightarrow \mathbb{R}$ as follows. Assume that, in \varkappa ,

$$\beta_1 < \beta_2 < \cdots < \beta_k.$$

Take the function

$$(G_1^{(m)} \otimes G_2^{(n)})(y_1, \dots, y_{m+n}) = G_1^{(m)}(y_1, \dots, y_m) G_2^{(n)}(y_{m+1}, \dots, y_{m+n}).$$

For each $i \in \{1, \dots, k\}$, replace the variable y_{β_i} with y_{α_i} . After this, replace the variables y_j with $j \in \{m+1, \dots, m+n\} \setminus \{\beta_1, \dots, \beta_k\}$ with the variables $y_{m+1}, y_{m+2}, \dots, y_{m+n-k}$, respectively. Here, $y_l := (x_l, s_l)$.

For example, for $m = 3$, $n = 4$, $\varkappa = \{(3, 5), (2, 6)\}$, we have

$$(G_1^{(3)} \otimes G_2^{(4)})_{\varkappa}(y_1, y_2, y_3, y_4, y_5) = G_1^{(3)}(y_1, y_2, y_3) G_2^{(4)}(y_4, y_3, y_2, y_5), \quad (y_1, y_2, y_3, y_4, y_5) \in Y_5.$$

Let us interpret $G_1^{(m)} : Y_m \rightarrow \mathbb{R}$ and $G_2^{(n)} : Y_n \rightarrow \mathbb{R}$ as functions defined on $\Phi(Y)$ which vanish outside $\Phi^{(m)}(Y)$ and $\Phi^{(n)}(Y)$, respectively. We then define a function

$$G_1^{(m)} \diamond G_2^{(n)} : \Phi(Y) \rightarrow \mathbb{R}$$

by

$$G_1^{(m)} \diamond G_2^{(n)} := \sum_{\varkappa \in \text{Pair}(m, n)} \frac{(m+n-|\varkappa|)!}{m!n!} (G_1^{(m)} \otimes G_2^{(n)})_{\varkappa}. \quad (49)$$

In the above formula, each $(G_1^{(m)} \otimes G_2^{(n)})_{\varkappa}$ is also treated as a function on $\Phi(Y)$.

Note that a function $G_1^{(0)} : \Phi^{(0)}(Y) \rightarrow \mathbb{R}$ is just a real number. We set, for each function $G_2 : \Phi(Y) \rightarrow \mathbb{R}$,

$$G_1^{(0)} \diamond G_2 = G_2 \diamond G_1^{(0)} := G_1^{(0)} \cdot G_2. \quad (50)$$

Extending formulas (49), (50) by linearity, we identify, for any functions $G_1, G_2 : \Phi(Y) \rightarrow \mathbb{R}$, their \diamond -product $G_1 \diamond G_2$ as a function on $\Phi(Y)$.

Step 4. Claim. Assume that G_1 and G_2 are symmetric functions on $\Phi(Y)$ which vanish outside the set $\bigcup_{n=0}^N \Phi^{(n)}(Y)$ for some $N \in \mathbb{N}$. Then

$$\int_{\Phi(Y)} G_1 \star G_2 d\rho = \int_{\Phi(Y)} G_1 \diamond G_2 d\rho,$$

provided the integrals in the above formulas make sense.

To prove the claim, it suffices to consider the case where $G_1 = G_1^{(m)} : Y_m \rightarrow \mathbb{R}$, $G_2 = G_2^{(n)} : Y_n \rightarrow \mathbb{R}$, and $m, n \in \mathbb{N}$. Using (45), we have

$$\begin{aligned} & \int_{\Phi(Y)} G_1^{(m)} \star G_2^{(n)} d\rho \\ &= \sum_{k=0}^{m \wedge n} \sum_{\substack{(\theta_1, \theta_2, \theta_3) \in \mathcal{P}_3(m+n-k) \\ |\theta_1|=m-k, |\theta_2|=k, |\theta_3|=n-k}} \int_{Y_{m+n-k}} G_1^{(m)}(y_{\theta_1}, y_{\theta_2}) G_2^{(n)}(y_{\theta_2}, y_{\theta_3}) d\rho^{(m+n-k)}(y_1, \dots, y_{m+n-k}). \end{aligned}$$

Here $\mathcal{P}_3(m+n-k)$ denotes the set of all ordered partitions $(\theta_1, \theta_2, \theta_3)$ of the set $\{1, \dots, m+n-k\}$ into three parts, $|\theta_i|$ denotes the number of elements in block θ_i , and, for block $\theta_i = \{r_1, r_2, \dots, r_{|\theta_i|}\}$, y_{θ_i} denotes $y_{r_1}, y_{r_2}, \dots, y_{r_{|\theta_i|}}$. Evidently, the set $\mathcal{P}_3(m+n-k)$ contains $\frac{(m+n-k)!}{(m-k)!(n-k)!k!}$ elements $(\theta_1, \theta_2, \theta_3)$ such that $|\theta_1| = m-k$, $|\theta_2| = k$, $|\theta_3| = n-k$. Hence

$$\begin{aligned} \int_{\Phi(Y)} G_1^{(m)} \star G_2^{(n)} d\rho &= \sum_{k=0}^{m \wedge n} \frac{(m+n-k)!}{(m-k)!(n-k)!k!} \\ &\quad \times \int_{Y_{m+n-k}} G_1^{(m)}(x_1, \dots, x_m) G_2^{(n)}(x_{m-k+1}, \dots, x_{m+n-k}) d\rho^{(m+n-k)}(x_1, \dots, x_{m+n-k}). \end{aligned} \tag{51}$$

On the other hand, by (49),

$$\int_{\Phi(Y)} G_1^{(m)} \diamond G_2^{(n)} d\rho = \sum_{k=0}^{m \wedge n} \frac{(m+n-k)!}{m!n!} \sum_{\substack{\varkappa \in \text{Pair}(m,n) \\ |\varkappa|=k}} \int_{Y_{m+n-k}} (G_1^{(m)} \otimes G_2^{(n)})_{\varkappa} d\rho^{(m+n-k)}.$$

An easy combinatoric argument shows that there are

$$\frac{m!}{(m-k)!k!} \times \frac{n!}{(n-k)!k!} \times k! = \frac{m!n!}{(m-k)!(n-k)!k!}$$

elements $\varkappa \in \text{Pair}(m, n)$ such that $|\varkappa| = k$. Hence

$$\begin{aligned} \int_{\Phi(Y)} G_1^{(m)} \diamond G_2^{(n)} d\rho &= \sum_{k=0}^{m \wedge n} \frac{(m+n-k)!}{m!n!} \times \frac{m!n!}{(m-k)!(n-k)!k!} \\ &\quad \times \int_{Y_{m+n-k}} G_1^{(m)}(x_1, \dots, x_m) G_2^{(n)}(x_{m-k+1}, \dots, x_{m+n-k}) d\rho^{(m+n-k)}(x_1, \dots, x_{m+n-k}). \end{aligned} \tag{52}$$

By (51) and (52) the claim follows.

Step 5. We denote

$$\Psi(X) := \bigcup_{n=0}^{\infty} \Psi^{(n)}(X),$$

where the set $\Psi^{(0)}(X)$ contains one element, and for $n \in \mathbb{N}$, $\Psi^{(n)}(X) := X^n$. Analogously to $\mathcal{B}(\Phi(Y))$, we define a σ -algebra $\mathcal{B}(\Psi(X))$. We next define a measure M on $(\Psi(X), \mathcal{B}(\Psi(X)))$ so that $M(\Psi^{(0)}(X)) := M^{(0)} = 1$ and, for $n \in \mathbb{N}$, the restriction of M to $\Psi^{(n)}(X)$ is $M^{(n)}$. For any functions $F_1^{(m)}$ and $F_2^{(n)}$ on $\Psi^{(m)}(X)$ and $\Psi^{(n)}(X)$, respectively, their tensor product $F_1^{(m)} \otimes F_2^{(n)}$ is a function on $\Psi^{(m+n)}(X)$. (In the case where either m or n is equal to zero, the tensor product becomes a usual product.) Extending the tensor product by linearity, we define, for any functions F_1 and F_2 on $\Psi(X)$, their tensor product $F_1 \otimes F_2$ as a function on $\Psi(X)$.

We next note that the measure M on $\Psi(X)$ is \otimes -positive definite. More precisely, assume that a function F on $\Psi(X)$ vanishes outside a set $\bigcup_{n=0}^N \Psi^{(n)}(X)$ for some $N \in \mathbb{N}$. Assume that the function $F \otimes F$ is integrable with respect to M . Then, it immediately follows from (17) that

$$\int_{\Psi(X)} F \otimes F dM \geq 0. \quad (53)$$

Step 6. Let a function $g^{(n)} : X_0^{(n)} \rightarrow \mathbb{R}$ be bounded, measurable, and having support from $\mathcal{B}_c(X_0^{(n)})$. For $i_1, \dots, i_n \in \mathbb{N}$, we set

$$G^{(n)}(x_1, s_1, \dots, x_n, s_n) := g^{(n)}(x_1, \dots, x_n) s_1^{i_1} \cdots s_n^{i_n}, \quad (x_1, s_1, \dots, x_n, s_n) \in Y_n. \quad (54)$$

We extend the function $g^{(n)}$ by zero to the whole space X^n . We define a function $\mathcal{R}_{i_1, \dots, i_n} g^{(n)} : X^{i_1 + \dots + i_n} \rightarrow \mathbb{R}$ by using formula (21). We denote

$$\mathcal{K}G^{(n)} := \frac{1}{n!} \mathcal{R}_{i_1, \dots, i_n} g^{(n)}. \quad (55)$$

We denote by \mathcal{Q} the class of all functions on $\Phi(Y)$ which are finite sums of functions as in (54). Extending \mathcal{K} by linearity, we define, for each $G \in \mathcal{Q}$, $\mathcal{K}G$ as a function on $\Psi(X)$.

Let $\Delta \in \mathcal{B}_c(X_0^{(n)})$, let $g^{(n)} = \chi_\Delta$, and let $G^{(n)}$ be given by (54). By Lemma 8 and formulas (26), (35), (44), and (55),

$$\begin{aligned} \int_{Y_n} G^{(n)} d\rho^{(n)} &= \int_{Y_n} \chi_\Delta(x_1, \dots, x_n) s_1^{i_1} \cdots s_n^{i_n} d\rho^{(n)}(x_1, s_1, \dots, x_n, s_n) \\ &= \int_{Y_n} \chi_\Delta(x_1, \dots, x_n) s_1^{i_1-1} \cdots s_n^{i_n-1} d\xi^{(n)}(x_1, s_1, \dots, x_n, s_n) \\ &= \int_{(\mathbb{R}_+^*)^n} s_1^{i_1-1} \cdots s_n^{i_n-1} d\xi_\Delta^{(n)}(s_1, \dots, s_n) \end{aligned}$$

$$\begin{aligned}
&= \xi_{i_1-1, \dots, i_n-1}^\Delta \\
&= \frac{1}{n!} M_{i_1, \dots, i_n}(\Delta) \\
&= \int_{X^{i_1+\dots+i_n}} \frac{1}{n!} \mathcal{R}_{i_1, \dots, i_n} \chi_\Delta dM^{(i_1+\dots+i_n)} \\
&= \int_{\Psi(X)} \mathcal{K}G^{(n)} dM.
\end{aligned}$$

From here it easily follows by approximation that, for each $G \in \mathcal{Q}$, we have

$$\int_{\Phi(Y)} G d\rho = \int_{\Psi(X)} \mathcal{K}G dM. \quad (56)$$

Step 7. Let functions $g_1^{(m)} : X_{\hat{0}}^{(m)} \rightarrow \mathbb{R}$ and $g_2^{(n)} : X_{\hat{0}}^{(n)} \rightarrow \mathbb{R}$ be bounded, measurable, and having support from $\mathcal{B}_c(X_{\hat{0}}^{(m)})$ and $\mathcal{B}_c(X_{\hat{0}}^{(n)})$, respectively. Let $i_1, \dots, i_m, j_1, \dots, j_n \in \mathbb{N}$. Let

$$\begin{aligned}
G_1^{(m)}(x_1, s_1, \dots, x_m, s_m) &:= g_1^{(m)}(x_1, \dots, x_m) s_1^{i_1} \cdots s_m^{i_m}, \quad (x_1, s_1, \dots, x_m, s_m) \in Y_m \\
G_2^{(n)}(x_1, s_1, \dots, x_n, s_n) &:= g_2^{(n)}(x_1, \dots, x_n) s_1^{j_1} \cdots s_n^{j_n}, \quad (x_1, s_1, \dots, x_n, s_n) \in Y_n.
\end{aligned}$$

Then, by (21) and Step 6,

$$\begin{aligned}
&(\mathcal{K}G_1^{(m)} \otimes \mathcal{K}G_2^{(n)})(x_1, \dots, x_{i_1+\dots+i_m+j_1+\dots+j_n}) \\
&= \frac{1}{m! n!} (\mathcal{R}_{i_1, \dots, i_m} g_1^{(m)} \otimes \mathcal{R}_{j_1, \dots, j_n} g_2^{(n)})(x_1, \dots, x_{i_1+\dots+i_m+j_1+\dots+j_n}) \\
&= \frac{1}{m! n!} g_1^{(m)}(x_1, x_{i_1+1}, \dots, x_{i_1+\dots+i_m-1+1}) \\
&\times g_2^{(n)}(x_{i_1+\dots+i_m+1}, x_{i_1+\dots+i_m+j_1+1}, \dots, x_{i_1+\dots+i_m+j_1+\dots+j_n-1+1}) \\
&\times \mathcal{I}_{i_1, \dots, i_m}(x_1, \dots, x_{i_1+\dots+i_m}) \mathcal{I}_{j_1, \dots, j_n}(x_{i_1+\dots+i_m+1}, \dots, x_{i_1+\dots+i_m+j_1+\dots+j_n}). \quad (57)
\end{aligned}$$

By (49), (54)–(57) and recalling that the measure M is symmetric on each $\Psi^{(k)}(X)$,

$$\int_{\Psi(X)} \mathcal{K}G_1^{(m)} \otimes \mathcal{K}G_2^{(n)} dM = \int_{\Psi(X)} \mathcal{K}(G_1^{(m)} \diamond G_2^{(n)}) dM.$$

Hence, for any $G_1, G_2 \in \mathcal{Q}$,

$$\int_{\Psi(X)} \mathcal{K}G_1 \otimes \mathcal{K}G_2 dM = \int_{\Psi(X)} \mathcal{K}(G_1 \diamond G_2) dM. \quad (58)$$

(Note that $G_1 \diamond G_2 \in \mathcal{Q}$.) Hence, by (53) and (58), for each $G \in \mathcal{Q}$

$$\int_{\Psi(X)} \mathcal{K}(G \diamond G) dM \geq 0.$$

Therefore, by (56), for each $G \in \mathcal{Q}$,

$$\int_{\Phi(Y)} G \diamond G d\rho \geq 0.$$

Step 8. Fix any $\Lambda \in \mathcal{B}_c(X)$. For each $i \in \mathbb{N}$, denote $\Delta_i := \Lambda_0^{(i)}$. Fix any $n, N \in \mathbb{N}$ such that $n \leq N$. We define a measure $\zeta_{n,N}$ on $(\mathbb{R}_+^*)^n$ as follows:

$$\zeta_{n,N} := \sum_{i=n}^{2N} P_n \xi_{\Delta_i}^{(i)}. \quad (59)$$

Here $P_n \xi_{\Delta_i}^{(i)}$ denotes the projection of the (symmetric) measure $\xi_{\Delta_i}^{(i)}$ onto its first n coordinates. Note that $\zeta_{n,N}$ is a symmetric measure on $(\mathbb{R}_+^*)^n$. We next define a measure $Z_{n,N}$ on $(\mathbb{R}_+^*)^n$ by

$$dZ_{n,N}(s_1, \dots, s_n) := d\zeta_{n,N}(s_1, \dots, s_n) \sum_{A \in \mathcal{P}(n)} \prod_{j \in A} s_j. \quad (60)$$

Here $\mathcal{P}(n)$ denotes the power set of $\{1, \dots, n\}$ and $\prod_{j \in \emptyset} := 1$. Clearly, $Z_{n,N}$ is also a symmetric measure. By (35), (59), and (60), the moments of the measure $Z_{n,N}$ are given by

$$\int_{(\mathbb{R}_+^*)^n} s_1^{i_1} \cdots s_n^{i_n} dZ_{n,N}(s_1, \dots, s_n) = \sum_{i=n}^{2N} \sum_{A \in \mathcal{P}(n)} \xi_{i_1 + \chi_A(1), \dots, i_n + \chi_A(n), 0, \dots, 0}^{\Delta_i}, \quad (i_1, \dots, i_n) \in \mathbb{Z}_+^n.$$

Hence, by (34),

$$\int_{(\mathbb{R}_+^*)^n} s_1^{i_1} \cdots s_n^{i_n} dZ_{n,N}(s_1, \dots, s_n) \leq (2N - n - 1) 2^n (i_1 + \cdots + i_n + n + 2N)! C_\Lambda^{i_1 + \cdots + i_n + n + 2N}. \quad (61)$$

By (61) and [4, Chapter 5, Subsec. 2.1, Examples 2.1, 2.2], the set of polynomials is dense in $L^2((\mathbb{R}_+^*)^n, dZ_{n,N})$.

Let us fix a function $G : \Phi(Y) \rightarrow \mathbb{R}$ of the form

$$G = \sum_{j=1}^J G_j^{(n_j)}, \quad (62)$$

where each function $G_j^{(n_j)} : \Phi^{(n_j)} \rightarrow \mathbb{R}$ is of the form

$$G_j^{(n_j)}(x_1, s_1, \dots, x_{n_j} s_{n_j}) = g_j^{(n_j)}(x_1, \dots, x_{n_j}) f_j^{(n_j)}(s_1, \dots, s_{n_j}) s_1 \cdots s_{n_j}, \quad (63)$$

unless $n_j = 0$. Here the functions $g_j^{(n_j)}$ and $f_j^{(n_j)}$ are measurable and bounded, the support of $g_j^{(n_j)}$ is a subset of Δ_{n_j} , and all $n_j \leq N$. For each $j = 1, \dots, J$, we clearly have $f_j^{(n_j)} \in L^2((\mathbb{R}_+^*)^{n_j}, Z_{n_j, N})$. Hence, there exists a sequence of polynomials $(p_{j,k}^{(n_j)})_{k=1}^\infty$ such that

$$p_{j,k}^{(n_j)} \rightarrow f_j^{(n_j)} \text{ in } L^2((\mathbb{R}_+^*)^{n_j}, Z_{n_j, N}) \text{ as } k \rightarrow \infty. \quad (64)$$

Set $G_k := \sum_{j=1}^J G_{j,k}^{(n_j)}$, where

$$G_{j,k}^{(n_j)} := g_j^{(n_j)}(x_1, \dots, x_{n_j}) p_{j,k}^{(n_j)}(s_1, \dots, s_{n_j}) s_1 \cdots s_{n_j}.$$

We then have $G_k \in \mathcal{Q}$ for each $k \in \mathbb{N}$. By Step 7,

$$\int_{\Phi(Y)} G_k \diamond G_k d\rho \geq 0, \quad k \in \mathbb{N}. \quad (65)$$

Claim. We have

$$\int_{\Phi(Y)} G_k \diamond G_k d\rho \rightarrow \int_{\Phi(Y)} G \diamond G d\rho \text{ as } k \rightarrow \infty. \quad (66)$$

To prove the claim, it suffices to fix any $i, j \in \{1, \dots, J\}$ with $n_i + n_j \geq 1$ and any $\varkappa \in \text{Pair}(n_i, n_j)$ with $|\varkappa| = l$, and prove that

$$\int_{Y_{n_i+n_j-l}} (G_{i,k}^{(n_i)} \otimes G_{j,k}^{(n_j)})_{\varkappa} d\rho^{(n_i+n_j-l)} \rightarrow \int_{Y_{n_i+n_j-l}} (G_i^{(n_i)} \otimes G_j^{(n_j)})_{\varkappa} d\rho^{(n_i+n_j-l)} \text{ as } k \rightarrow \infty. \quad (67)$$

For simplicity of notation, let us assume that \varkappa is of the form

$$\{(n_i - l + 1, n_i + 1), (n_i - l + 2, n_i + 2), (n_i - l + 3, n_i + 3) \dots, (n_i, n_i + l)\}.$$

Then

$$\begin{aligned} & \int_{Y_{n_i+n_j-l}} (G_{i,k}^{(n_i)} \otimes G_{j,k}^{(n_j)})_{\varkappa} d\rho^{(n_i+n_j-l)} \\ &= \int_{Y_{n_i+n_j-l}} g_i^{(n_i)}(x_1, \dots, x_{n_i}) p_{i,k}^{(n_i)}(s_1, \dots, s_{n_i}) \\ & \quad \times g_j^{(n_j)}(x_{n_i-l+1}, x_{n_i-l+2}, \dots, x_{n_i+n_j-l}) p_{j,k}^{(n_j)}(s_{n_i-l+1}, s_{n_i-l+2}, \dots, s_{n_i+n_j-l}) \\ & \quad \times s_{n_i-l+1} s_{n_i-l+2} \cdots s_{n_i} d\xi^{(n_i+n_j-l)}(x_1, s_1, \dots, x_{n_i+n_j-l}, s_{n_i+n_j-l}). \end{aligned} \quad (68)$$

Hence, there exists $C > 0$ such that

$$\left| \int_{Y_{n_i+n_j-l}} (G_{i,k}^{(n_i)} \otimes G_{j,k}^{(n_j)})_{\varkappa} d\rho^{(n_i+n_j-l)} - \int_{Y_{n_i+n_j-l}} (G_i^{(n_i)} \otimes G_j^{(n_j)})_{\varkappa} d\rho^{(n_i+n_j-l)} \right|$$

$$\begin{aligned}
&\leq \int_{Y_{n_i+n_j-l}} |g_i^{(n_i)}(x_1, \dots, x_{n_i}) g_j^{(n_j)}(x_{n_i-l+1}, x_{n_i-l+2}, \dots, x_{n_i+n_j-l})| \\
&\quad \times |p_{i,k}^{(n_i)}(s_1, \dots, s_{n_i}) - p_i^{(n_i)}(s_1, \dots, s_{n_i})| \\
&\quad \times |p_{j,k}^{(n_j)}(s_{n_i-l+1}, s_{n_i-l+2}, \dots, s_{n_i+n_j-l})| \\
&\quad \times s_{n_i-l+1} s_{n_i-l+2} \cdots s_{n_i} d\xi^{(n_i+n_j-l)}(x_1, s_1, \dots, x_{n_i+n_j-l}, s_{n_i+n_j-l}) \\
&\leq C \int_{Y_{n_i+n_j-l}} \chi_{\Lambda_0^{(n_i+n_j-l)}}(x_1, \dots, x_{n_i+n_j-l}) \\
&\quad \times |p_{i,k}^{(n_i)}(s_1, \dots, s_{n_i}) - p_i^{(n_i)}(s_1, \dots, s_{n_i})| \\
&\quad \times |p_{j,k}^{(n_j)}(s_{n_i-l+1}, s_{n_i-l+2}, \dots, s_{n_i+n_j-l})| \\
&\quad \times s_{n_i-l+1} s_{n_i-l+2} \cdots s_{n_i} d\xi^{(n_i+n_j-l)}(x_1, s_1, \dots, x_{n_i+n_j-l}, s_{n_i+n_j-l}) \\
&\leq C \left(\int_{Y_{n_i+n_j-l}} \chi_{\Lambda_0^{(n_i+n_j-l)}}(x_1, \dots, x_{n_i+n_j-l}) \right. \\
&\quad \times |p_{i,k}^{(n_i)}(s_1, \dots, s_{n_i}) - p_i^{(n_i)}(s_1, \dots, s_{n_i})|^2 \\
&\quad \times s_{n_i-l+1} s_{n_i-l+2} \cdots s_{n_i} d\xi^{(n_i+n_j-l)}(x_1, s_1, \dots, x_{n_i+n_j-l}, s_{n_i+n_j-l}) \left. \right)^{1/2} \\
&\quad \times \left(\int_{Y_{n_i+n_j-l}} \chi_{\Lambda_0^{(n_i+n_j-l)}}(x_1, \dots, x_{n_i+n_j-l}) \right. \\
&\quad \times |p_{j,k}^{(n_j)}(s_{n_i-l+1}, s_{n_i-l+2}, \dots, s_{n_i+n_j-l})|^2 \\
&\quad \times s_{n_i-l+1} s_{n_i-l+2} \cdots s_{n_i} d\xi^{(n_i+n_j-l)}(x_1, s_1, \dots, x_{n_i+n_j-l}, s_{n_i+n_j-l}) \left. \right)^{1/2} \\
&\leq C \|p_{i,k}^{(n_i)} - p_i^{(n_i)}\|_{L^2((\mathbb{R}_+^*)^{n_i}, dZ_{n_i, N})} \|p_{j,k}^{(n_j)}\|_{L^2((\mathbb{R}_+^*)^{n_j}, dZ_{n_j, N})} \rightarrow 0 \text{ as } k \rightarrow \infty, \quad (69)
\end{aligned}$$

where we used the Cauchy inequality and (64). Analogously,

$$\left| \int_{Y_{n_i+n_j-l}} (G_i^{(n_i)} \otimes G_{j,k}^{(n_j)})_{\neq} d\rho^{(n_i+n_j-l)} - \int_{Y_{n_i+n_j-l}} (G_i^{(n_i)} \otimes G_j^{(n_j)})_{\neq} d\rho^{(n_i+n_j-l)} \right| \rightarrow 0 \text{ as } k \rightarrow \infty. \quad (70)$$

By (69) and (70), formula (67) follows.

Step 9. By Steps 7 and 8, for each function $G : \Phi(Y) \rightarrow \mathbb{R}$ as in formulas (62), (63), we have

$$\int_{\Phi(Y)} G \diamond G d\rho \geq 0. \quad (71)$$

As a special case, formula (71) holds for each function $G \in \mathcal{S}$ (recall Step 1). Now, by Step 4, we conclude that condition (PD) is satisfied. \square

Since the correlation measure ρ of the point process ν from Lemma 9 is concentrated on $\Phi(Y)$, it immediately follows from the proof of [14, Corollary 1] that the point process ν is concentrated on $\Gamma_p(Y)$, the set of pinpointing configurations in Y . Recall formula (8). For each $\Lambda \in \mathcal{B}_c(X)$,

$$\begin{aligned} \int_{\Gamma_p(Y)} \mathfrak{M}_\Lambda d\nu &= \int_{\Gamma_p(Y)} \sum_{(x,s) \in \gamma} \chi_\Lambda(x)s d\nu(\gamma) \\ &= \int_Y \chi_\Lambda(x)s d\rho^{(1)}(x,s) \\ &= \int_Y \chi_\Lambda(x) d\xi^{(1)}(x,s) < \infty. \end{aligned} \tag{72}$$

Hence, $\mathfrak{M}_\Lambda < \infty$ ν -a.s. From here, it follows that $\mathfrak{M}_\Lambda < \infty$ ν -a.s. Therefore $\nu(\Gamma_f(Y)) = 1$, see (9). Recall the bijective mapping $\mathcal{E} : \mathbb{K}(X) \rightarrow \Gamma_f(Y)$. As we already discussed in Section 2, the inverse mapping \mathcal{E}^{-1} is measurable. So we can define a probability measure μ' on $\mathbb{K}(X)$ as the pushforward of ν under \mathcal{E}^{-1} . Thus, to finish the proof of the theorem, it suffices to show that $\mu = \mu'$.

Let $\Lambda \in \mathcal{B}_c(X)$. Recall that, for any $i_1, \dots, i_k \in \mathbb{N}$, $k \in \mathbb{N}$,

$$\int_{Y_k} \chi_{\Lambda_0^{(k)}}(x_1, \dots, x_k) s_1^{i_1} \cdots s_k^{i_k} d\rho^{(k)}(x_1, s_1, \dots, x_k, s_k) < \infty.$$

Hence, using the definition of a correlation measure, we easily see that, for each $n \in \mathbb{N}$,

$$\int_{\Gamma_f(Y)} \left(\sum_{(x,s) \in \gamma} \chi_\Lambda(x)s \right)^n d\nu(\gamma) < \infty$$

(compare with (72)). Therefore, for each $n \in \mathbb{N}$,

$$\int_{\mathbb{K}(X)} \eta(\Lambda)^n d\mu'(\eta) < \infty.$$

Here $\eta(\Lambda) := \langle \eta, \chi_\Lambda \rangle$, i.e., the η -measure of Λ . Hence, μ' has finite moments. We denote by $(M_{\mu'}^{(n)})_{n=0}^\infty$ the moment sequence of the point process μ' . By Theorem 1 and the construction of the measure ρ , it follows that

$$M'_{i_1, \dots, i_n} = M_{i_1, \dots, i_n}, \quad i_1, \dots, i_n \in \mathbb{N}, \quad n \in \mathbb{N}, \tag{73}$$

where the measures M'_{i_1, \dots, i_n} are defined analogously to M_{i_1, \dots, i_n} , by starting with the moment sequence $(M_{\mu'}^{(n)})_{n=0}^\infty$, rather than $(M^{(n)})_{n=0}^\infty$. By virtue of (73), the moment sequence $(M_{\mu'}^{(n)})_{n=0}^\infty$ coincides with the moment sequence $(M^{(n)})_{n=0}^\infty$.

Now, fix any sets $\Lambda_1, \dots, \Lambda_n \in \mathcal{B}_c(X)$. For any $i_1, \dots, i_n \in \mathbb{Z}_+$, we get

$$\begin{aligned}
& \int_{\mathbb{K}(X)} \eta(\Lambda_1)^{i_1} \cdots \eta(\Lambda_n)^{i_n} d\mu'(\eta) = \int_{\mathbb{M}(X)} \eta(\Lambda_1)^{i_1} \cdots \eta(\Lambda_n)^{i_n} d\mu(\eta) \\
& = \int_{X^{i_1+\cdots+i_n}} (\chi_{\Lambda_1}^{\otimes i_1} \otimes \cdots \otimes \chi_{\Lambda_n}^{\otimes i_n})(x_1, \dots, x_{i_1+\cdots+i_n}) dM^{(i_1+\cdots+i_n)}(x_1, \dots, x_{i_1+\cdots+i_n}). \quad (74)
\end{aligned}$$

By (C1), (74), and the Carleman criterion, the joint distribution of the random variables $\eta(\Lambda_1), \dots, \eta(\Lambda_n)$ under μ' coincides with the joint distribution of the random variables $\eta(\Lambda_1), \dots, \eta(\Lambda_n)$ under μ . But it is well known (see e.g. [10]) that $\mathcal{B}(\mathbb{M}(X))$ coincides with the minimal σ -algebra on $\mathbb{M}(X)$ with respect to which each function $\eta \mapsto \eta(\Lambda)$ with $\Lambda \in \mathcal{B}_c(X)$, is measurable. Therefore, we indeed get the equality $\mu = \mu'$. \square

As a consequence of our results, we also obtain a characterization of point processes in terms of their moments.

Corollary 11. *Let μ be a random measure on X , i.e., a probability measure on $(\mathbb{M}(X), \mathcal{B}(\mathbb{M}(X)))$. Assume that μ has finite moments, and let $(M^{(n)})_{n=0}^\infty$ be its moment sequence. Further assume that conditions (C1) and (C2) are satisfied. Then μ is a point process, i.e., $\mu(\Gamma(X)) = 1$, if and only if, for any $n \in \mathbb{N}$ and any $i_1, \dots, i_n \in \mathbb{N}$, we have $M_{i_1, \dots, i_n} = M_{1, \dots, 1}$, i.e., for each $\Delta \in \mathcal{B}(X_{\hat{0}}^{(n)})$,*

$$M_{i_1, \dots, i_n}(\Delta) = M^{(n)}(\Delta), \quad i_1, \dots, i_n \in \mathbb{N}. \quad (75)$$

In the latter case, the correlation measure ρ of μ is given by

$$\rho^{(n)}(\Delta) = \frac{1}{n!} M^{(n)}(\Delta), \quad \Delta \in \mathcal{B}(X_{\hat{0}}^{(n)}), \quad (76)$$

where $\rho^{(n)}$ is the restriction of ρ to $\Gamma^{(n)}(X)$, $\rho^{(n)}$ being identified with a measure on $X_{\hat{0}}^{(n)}$.

Proof. Assume that μ is a point process in X . Hence, μ is a random discrete measure on X . The corresponding point process $\nu = \mathcal{E}(\mu)$ is concentrated on

$$\Gamma(X \times \{1\}) = \{ \{(x, 1)\}_{x \in \gamma} \mid \gamma \in \Gamma(X) \}.$$

Hence, $\Gamma(X \times \{1\})$ can naturally be identified with $\Gamma(X)$, and under this identification we get $\mu = \nu$. Furthermore, the correlation measure ρ of μ coincides with the correlation measure of ν , provided we have identified $\Gamma_0(X)$ with $\Gamma_0(X \times \{1\})$. Now, formulas (75), (76) follow from Theorem 1.

Next, assume that μ is a random measure which satisfies (75). Hence, for any $n \in \mathbb{N}$ and $\Delta \in \mathcal{B}_c(X_{\hat{0}}^{(n)})$, we get

$$\xi_{i_1, \dots, i_n}^\Delta = \xi_{1, \dots, 1}^\Delta, \quad i_1, \dots, i_n \in \mathbb{Z}_+.$$

Hence, conditions (i) and (ii) Theorem 5 are satisfied, and so μ is a random discrete measure. By (12) and (75), for each $n \in \mathbb{N}$ and $\Delta \in \mathcal{B}_c(X_0^{(n)})$, the measure $\xi_\Delta^{(n)}$ on $(\mathbb{R}_+^*)^n$ is concentrated at one point, $(1, \dots, 1)$. Hence, by (13) and (14), the measure $\rho^{(n)}$ is concentrated on the set

$$\{(x_1, 1, \dots, x_n, 1) \mid (x_1, \dots, x_n) \in X_0^{(n)}\}.$$

Therefore, the point process $\nu = \mathcal{E}(\mu)$ is concentrated on $\Gamma(X \times \{1\})$. Hence, μ is a point process in X . \square

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