FINITE SPEED OF PROPAGATION FOR STOCHASTIC POROUS MEDIA EQUATIONS.

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Abstract. We prove finite speed of propagation for stochastic porous media equations perturbed by linear multiplicative space-time rough signals. Explicit and optimal estimates for the speed of propagation are given. The result applies to any continuous driving signal, thus including fractional Brownian motion for all Hurst parameters. The explicit estimates are then used to prove that the corresponding random attractor has infinite fractal dimension.

1. Introduction

In this paper we prove finite speed of propagation for solutions to stochastic porous media equations (SPME) driven by linear multiplicative space-time rough signals, i.e. to equations of the form

$$\begin{aligned} dX_t &= \Delta\left(|X_t|^m \operatorname{sgn}(X_t)\right) dt + \sum_{k=1}^{N} f_k X_t \circ dz_t^{(k)}, \text{ on } \mathcal{O}_T, \\ X(0) &= X_0, \text{ on } \mathcal{O}, \end{aligned} \tag{1.1}$$

with homogeneous Dirichlet boundary conditions on a bounded, smooth domain $\mathcal{O} \subseteq \mathbb{R}^d$, $m \in (1,\infty)$, rough driving signals $z^{(k)} \in C([0,T];\mathbb{R})$ and diffusion coefficients $f_k \in C^\infty(\bar{\mathcal{O}})$. We assume the number of signals N to be finite and high regularity of f_k for simplicity only. In fact, the proofs only require $\sum_{k=1}^\infty f_k(\xi) z_t^{(k)} \in C([0,T];C^2(\bar{\mathcal{O}}))$. The stochastic Stratonovich integral $\circ$ occurring in (1.1) is informal but may be justified by stability with respect to smooth approximations of $z^{(k)}$ and can be shown to coincide with the usual Stratonovich integral in case of $z^{(k)}$ being given as paths of continuous semimartingales (cf. [6,25] and Remark 3.5 below). Our analysis of (1.1) is based on the transformation $Y_t := e^{\mu_t} X_t$, where $\mu_t = -\sum_{k=1}^N f_k z_t^{(k)}$, which (informally) leads to the random PDE

$$\begin{aligned} \partial_t Y_t &= e^{\mu_t} \Delta \Phi(e^{-\mu_t} Y_t), \text{ on } \mathcal{O}_T \\ Y(0) &= e^{\mu_0} X_0, \text{ on } \mathcal{O}. \end{aligned} \tag{1.2}$$

In fact, we will say that X is a solution to (1.1) if $Y_t := e^{\mu_t} X_t$ is a solution to (1.2).

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Recently, a hole-filling property for SPME driven by multiplicative space-time Brownian noise has been shown in [9], which may be seen as an important step towards proving finite speed of propagation. However, no explicit control on the rate of growth of the support of the solution could be established, which made it impossible to deduce finite speed of propagation. In the present paper, we prove explicit (and locally optimal) estimates on the speed of hole-filling and thus deduce finite speed of propagation for SPME. Moreover, we will completely remove the non-degeneracy assumption on the noise as it was conjectured to be possible in [9], which makes it possible to analyze the dependence of the speed of propagation on the strength of the noise. In particular, we prove convergence to the deterministic, optimal estimates when the noise-intensity converges to zero (cf. Remark 3.8 below). In [9] restrictions on the dimension d and on the order of the nonlinearity m had to be supposed for technical reasons and it was conjectured that these could be completely removed. In the present paper we prove that this indeed is the case. As concerning finite speed of propagation, one of our main results is

Theorem 1.1 (Finite speed of propagation)**.** *Let $X \in C((0,T] \times \mathcal{O})$ be an essentially bounded, non-negative solution to the homogeneous Dirichlet problem to* (1.1) *and set $H := \|X\|_{L^\infty(\mathcal{O}_T)}$. Then, for every $s \in [0,T]$*

$$supp(X_{s+t}) \subseteq B_{\sqrt{t}\left(\frac{H^{m-1}}{C_{det}}\right)^{\frac{1}{2}}\sqrt{C_t}}(supp(X_s)), \quad \forall t \in [0, T-s],$$

where $t \mapsto C_t$ is a continuous, non-decreasing function with $C_t \to 1$ for $t \to 0$ and C_{det} is the same constant as in the corresponding deterministic result.

Our methods are purely local and thus apply without change to the homogeneous Cauchy-Dirichlet problem to (1.1) on not necessarily bounded domains $\mathcal{O} \subseteq \mathbb{R}^d$, as soon as the problem of unique existence of corresponding solutions is solved. Since up to now this problem remains open, we restrict to bounded domains for simplicity (cf., however, Remark 3.13 below).

As outlined above, the stochastic case is contained in our setup by choosing $z^{(k)}$ to be given as paths of some continuous stochastic process. Therefore, our results yield purely pathwise results for the stochastic case. Due to the explicit form of our estimates, moment estimates also immediately follow.

In the deterministic case it is well-known that the attractor corresponding to

$$dX_t = \Delta\left(|X_t|^m \mathrm{sgn}(X_t)\right)dt + \lambda X_t dt, \tag{1.3}$$

with Dirichlet boundary conditions has infinite fractal dimension iff $\lambda > 0$ (cf. [21]). Generally speaking, it highly depends on the drift of an SPDE as well as on the type of random perturbation, whether the noise has a regularizing effect on the long-time dynamics of the unperturbed system.

In [28] it has been shown that sufficiently non-degenerate additive Wiener noise stabilizes the dynamics of (1.3) in the sense that the random attractor consists of a single random point and thus is zero dimensional. Moreover, it is well-known that multiplicative Itô noise may stabilize the long-time dynamics due to the Itô correction term. For example, this has been realized in [13] in case of the Chafee-Infante equation perturbed by spatially homogeneous, linear multiplicative Itô noise. The more intriguing case of space-time, linear multiplicative Itô noise has been analyzed in [7] for fast diffusion equations (cf. also the references therein), where a regularizing effect due to the Itô correction term has been observed in [7, Theorem 3.5].

This correction term is absent in the case of linear multiplicative Stratonovich noise. In this spirit, it has been shown in [13] that spatially homogeneous, linear multiplicative Stratonovich noise does not have any regularizing effect on the long-time behavior of the Chafee-Infante equation. On the other hand, each linear PDE with non-negative, self-adjoint drift having negative trace (possibly $-\infty$) may be stabilized by linear multiplicative space-time Stratonovich noise (cf. [14]). For these reasons, it is an intriguing question, whether including linear multiplicative space-time Stratonovich noise in (1.3) stabilizes the long-time behavior, or whether the random attractor associated to

$$dX_t = \Delta\left(|X_t|^m \mathrm{sgn}(X_t)\right) dt + \lambda X_t dt + \sum_{k=1}^{N} f_k X_t \circ dz_t^{(k)}, \tag{1.4}$$

remains infinite dimensional. Based on the explicit bounds on the rate of propagation obtained in this paper, we prove lower bounds for the Kolmogorov ε-entropy of random attractors corresponding to (1.4) and thus conclude that random attractors remain infinite dimensional. More precisely, we get

Theorem 1.2 (Attracting sets are ∞-dimensional)**.** *Let $\mathcal{A}$ be a random set in $L^1(\mathcal{O})$ attracting all L^∞-bounded sets. Then the fractal dimension $d_f(\mathcal{A}(\omega))$ is infinite for all $\omega \in \Omega$.*

The SPME (1.1) with driving signals $z^{(k)}$ given as paths of independent Brownian motions $\beta^{(k)}$ has been intensively studied in the recent history (cf. e.g. [3–5, 18, 19, 27, 31, 41, 42] and references therein). The construction of a continuous random dynamical system (RDS) associated to (1.1) and the proof of existence of a corresponding random attractor has been given in [25, 26]. In case of porous media equations (PME) perturbed by additive noise, the existence of a random attractor has been shown in [11] and has subsequently been generalized to more general additive perturbations [29] and spatially rougher noise [28].

The sublinear, fast diffusion case ($m \in [0, 1)$) exhibits completely different propagation properties. In particular, finite speed of propagation does not hold for fast diffusion equations, but the positivity set of non-trivial solutions will cover the hole domain of definition after an arbitrarily small timespan (cf. [46] and references therein). On the other hand, solutions to the fast diffusion equation become extinct in finite time (cf. [46] for the deterministic case, [6, 43] for the stochastic case).

In the following let $\mathcal{O} \subseteq \mathbb{R}^d$ be a bounded domain with smooth boundary $\Sigma := \partial\mathcal{O}$. For $T > 0$ we define the space-time domain $\mathcal{O}_T := [0, T] \times \mathcal{O}$, the lateral boundary $\Sigma_T := [0, T] \times \partial\mathcal{O}$ and the parabolic boundary $\mathcal{P}_T := \Sigma_T \cup (\{T\} \times \mathcal{O})$. Let ϑ be the surface measure on Σ and ν be the outward pointing normal vector to Σ. For a set $A \subseteq \mathbb{R}^d$ and $h > 0$ we define

$$B_h(A) := \{x \in \mathbb{R}^d |\ d(x, A) := \inf_{y \in A} |x - y| < h\}.$$

By $C^0(\mathcal{O})$ we denote the space of continuous functions on $\mathcal{O}$ and by $C^{m,n}(\mathcal{O}_T)$ the space of continuous functions on $\mathcal{O}_T$ with m continuous derivatives in time and n continuous derivatives in space. $C_c^{m,n}(\mathcal{O}_T)$ is the subspace of all compactly supported functions in $C^{m,n}(\mathcal{O}_T)$. We define $C^n(\bar{\mathcal{O}})$, $C^{m,n}(\bar{\mathcal{O}}_T)$ to be the spaces obtained by restricting the functions in $C^n(\mathbb{R}^d)$, $C^{m,n}([0, T] \times \mathbb{R}^d)$ onto $\bar{\mathcal{O}}$. Let $H_0^{m,p}(\mathcal{O})$ be the usual Sobolev space of order m and exponent p with zero trace (cf. e.g. [1, Chapter 1.25]) and set $H^{-1} := \left(H_0^1(\mathcal{O})\right)^*$. For two non-empty subsets A, B of a metric space (E, d) we define $\mathrm{dist}(A, B) := \inf\{d(a, b)|\ a \in A,\ b \in B\}$. If X is a Banach space, then $L^p_{loc}((0, T]; X)$ denotes the space of all X-valued functions f such that $f \in L^p([\tau, T]; X)$ for all $\tau \in (0, T]$. As usual in probability theory we often denote the time-dependency of functions by a subscript X_t rather than by $X(t)$ in order to keep the equations at a bearable length.

1.1. **Finite speed of propagation for deterministic PME.** Let us start by recalling the finite speed of propagation properties for deterministic PME

$$\partial_t u = \Delta\Phi(u), \tag{1.5}$$

where for simplicity of notation we have set $\Phi(u) := |u|^m \mathrm{sgn}(u)$. Finite speed of propagation for deterministic PME has been known for a long time and was first proved in [38]. For a more detailed study on interfaces for the one dimensional case we refer to [45]. Our main reference for the deterministic PME and main source of inspiration for the stochastic case will be [46] where a beautiful account on the propagation and expansion properties for deterministic PME is given.

Definition 1.3. *Let $u_0 \in L^\infty(\mathcal{O})$ and $\Phi(g) \in L^2(\Sigma_T)$. A function $u \in C([0,T]; L^1(\mathcal{O}))$ with $\Phi(u) \in L^2([0,T]; H^1(\mathcal{O}))$ is said to be a solution to the (inhomogeneous) Dirichlet problem to* (1.5) *with initial condition u_0 and boundary value g if $\Phi(u) = \Phi(g)$ on Σ_T in the sense of traces and*

$$\int_{\mathcal{O}_T} u \partial_r \eta \; d\xi dr + \int_{\mathcal{O}} u_0 \eta_0 \; d\xi = \int_{\mathcal{O}_T} \nabla \Phi(u) \cdot \nabla \eta \; d\xi dr, \tag{1.6}$$

for all $\eta \in C^{1,2}(\bar{\mathcal{O}}_T)$ with $\eta_{|\mathcal{P}_T} = 0$.

Sub/supersolutions to (1.5) *are defined by replacing "$=$" by "$\geq$" ("$\leq$" resp.) in* (1.6) *and requiring the testfunctions η to be non-negative.*

We note that each essentially bounded solution u to (1.5) is a solution to (1.5) on each smooth subdomain $K \subseteq \mathcal{O}$ with initial data $u_{0|K}$ and boundary data $\Phi(g) = \Phi(u)$ in the sense of traces.

Remark 1.4.
i. There are several other notions of solutions to (1.5) *available in the literature (cf. e.g. [46, 47]). We choose to work with the one given in Definition 1.3 since it is consistent with the result on the existence of solutions to* (1.1) *(with* (1.5) *as a special case) given in Appendix A.*
ii. If we consider (1.5) *as an equation in H^{-1} then solutions in the sense of Definition 1.3 are strong solutions, i.e.*

$$\frac{d}{dt} u = \Delta \Phi(u), \quad dt\text{-a.e.}$$

as an equation in H^{-1}.
iii. As concerning comparison results, uniqueness of solutions and finite speed of propagation it is often possible to work with distributional (sub/super)solutions satisfying the initial-boundary conditions in some weak sense. For simplicity of the presentation we refrain from working with the most general notions. Possible extensions will be discussed in Remark 3.12 below.

The proof of finite speed of propagation is a direct consequence of the so-called hole-filling problem

Lemma 1.5 (Deterministic hole-filling, [46], Lemma 14.5)**.** *Let $\xi_0 \in \mathbb{R}^d$, $T, R > 0$ and $u \in C((0,T) \times B_R(\xi_0))$ be an essentially bounded, non-negative solution to* (1.5) *with vanishing initial value u_0 on $B_R(\xi_0)$ and boundary value g satisfying $H := \|g\|_{L^\infty([0,T] \times \partial B_R(\xi_0))} < \infty$. Define $C_{det} = \frac{m-1}{2dm(m-1)+4m}$ and*

$$T_{det} := R^2 \frac{C_{det}}{H^{m-1}}.$$

Then $u(t)$ vanishes in $B_{R_{det}(t)}(\xi_0)$ for all $t \in [0, T_{det} \wedge T]$, where

$$R_{det}(t) = R - \sqrt{t} \left(\frac{H^{m-1}}{C_{det}} \right)^{\frac{1}{2}}.$$

For boundary value g given as $g \equiv H$ for some $H > 0$, the bound on the rate of hole-filling from Lemma 1.5 is optimal (cf. [46, p. 339]).

From the hole-filling Lemma one may deduce

Theorem 1.6 (Deterministic finite speed of propagation, [46], Theorem 14.6)**.** *Let $u \in C((0,T)\times\mathcal{O})$ be an essentially bounded, non-negative solution to the homogeneous Dirichlet problem to* (1.5) *and set $H = \|u\|_{L^\infty(\mathcal{O}_T)}$. Then*

i. For every $s \in [0,T]$ and every $h > 0$ there is a time-span $T_h > 0$ such that

$$supp(u_{s+t}) \subseteq B_h(supp(u_s)), \quad \forall t \in [0, T_h \wedge (T-s)].$$

More precisely, T_h is given by

$$T_h := h^2 \frac{C_{det}}{H^{m-1}}.$$

ii. For every $s \in [0,T]$

$$supp(u_{s+t}) \subseteq B_{\sqrt{t}\left(\frac{H^{m-1}}{C_{det}}\right)^{\frac{1}{2}}}(supp(u_s)), \quad \forall t \in [0, T-s].$$

Proof. For each non-negative $u_0 \in L^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ there is a unique non-negative, essentially bounded solution $u_C \in C([0,T]; L^1(\mathbb{R}^d))$ to the Cauchy problem for (1.5) (cf. [46, Theorem 9.3]). This in turn is a supersolution to the Dirichlet problem on $\mathcal{O}$. Since $t \mapsto \|u_C(t)\|_{L^\infty(\mathcal{O})}$ is non-increasing, we have $\|u_C\|_{L^\infty(\mathcal{O}_T)} = \|u\|_{L^\infty(\mathcal{O}_T)}$. Without loss of generality one may thus assume that u is a solution to the Cauchy problem, which simplifies the argument since no difficulties at the boundary appear. Noticing that u in particular is an essentially bounded solution on each $B_R(\xi_0)$, the claim becomes a direct consequence of Lemma 1.5. □

2. Real-valued linear multiplicative noise

In order to get an idea of what is to be expected in the stochastically perturbed case, we start with the much simpler situation of spatially homogeneous noise. I.e. we consider the homogeneous Dirichlet problem to

$$dX_t = \Delta\Phi(X_t)dt + \sum_{k=1}^{N} f_k X_t \circ dz_t^{(k)}, \quad \text{on } \mathcal{O}_T, \tag{2.1}$$

where $f_k \in \mathbb{R}$ are $\mathbb{R}$-valued constants and $\mathcal{O} \subseteq \mathbb{R}^d$ is as before. It has been pointed out in [35] that (2.1) reduces to the deterministic PME (1.5) by rescaling and a random transformation in time if the signals $z^{(k)}$ are given as paths of continuous semimartingales. Since the bounds on the rate of propagation are known to be optimal in the deterministic case, we deduce optimal bounds for the case of spatially homogeneous perturbations (cf. also [9, Remark 3.2]).

Let $\mu_t = -\sum_{k=1}^N f_k z_t^{(k)}$ and $Y_t := e^{\mu_t} X_t$. Then (informally)

$$\partial_t Y_t = e^{-(m-1)\mu_t} \Delta\Phi(Y_t), \quad \text{on } \mathcal{O}_T. \tag{2.2}$$

Solutions to (2.1) are then defined by the reverse transformation, i.e. a function X is a solution to (2.1) with initial value $X_0 \in L^1(\mathcal{O})$ and boundary value g iff $Y_t := e^{\mu_t} X_t$ is a solution to (2.2) with initial value $Y_0 := e^{\mu_0} X_0$ and boundary value $e^{\mu} g$.

In [6, 25] it has been shown that this transformation can be made rigorous if the signals $z^{(k)}$ are given as paths of continuous semimartingales (cf. Remark 3.5 below). In addition, in case of continuous driving signals, solutions to (2.1) were obtained in [25] as limits of approximating solutions driven by smoothed signals $z^{(\delta)} \in C^\infty([0,T]; \mathbb{R}^N)$ with $z^{(\delta)} \to z$ in $C([0,T]; \mathbb{R}^N)$ (cf. Remark 3.6 below).

We set $F(t) := \int_0^t e^{-(m-1)\mu_r} dr \in C^1(\mathbb{R}_+; \mathbb{R}_+)$. Since F is strictly increasing we may define $G(t) := F^{-1}(t)$ to be the inverse of F and $u_t := Y_{G(t)}$. An informal computation suggests

$$\partial_t u_t = \Delta\Phi(u_t), \quad \text{on } \mathcal{O}_T. \tag{2.3}$$

A rigorous justification of this temporal transformation can easily be given by considering an artificial viscosity approximation, i.e. $\partial_t u^{(\varepsilon)} = \Delta\Phi(u_t^{(\varepsilon)}) + \varepsilon\Delta u_t^{(\varepsilon)}$. By local uniform continuity of $u^{(\varepsilon)}$ (cf. [20, 46]) we may pass to the limit in a pointwise manner and thus prove the claim.

Vice versa, solutions X to (2.1) can be expressed by solutions to (2.3) via:

$$X_t := e^{-\mu_t} u_{F(t)}. \tag{2.4}$$

Lemma 1.5 implies

Proposition 2.1 (Hole-filling for spatially homogeneous noise)**.** *Let $\xi_0 \in \mathbb{R}^d$, $T, R > 0$ and $X \in C((0,T) \times B_R(\xi_0))$ be an essentially bounded, non-negative solution to (2.1) with vanishing initial value X_0 on $B_R(\xi_0)$ and boundary value g satisfying $H := \|e^{\mu} g\|_{L^\infty([0,T]\times\partial B_R(\xi_0))} < \infty$. Define*

$$T_{stoch} := F^{-1}\left(R^2 \frac{C_{det}}{H^{m-1}}\right),$$

where $F(t) := \int_0^t e^{-(m-1)\mu_r} dr$.

Then X_t vanishes in $B_{R_{stoch}(t)}(\xi_0)$ for all $t \in [0, T_{stoch} \wedge T]$, where

$$R_{stoch}(t) = R - \sqrt{F(t)}\left(\frac{H^{m-1}}{C_{det}}\right)^{\frac{1}{2}}.$$

As pointed out above, the rates and constants given in Proposition 2.1 are optimal. Analogously, bounds on the rate of expansion of the support of solutions to (2.1) may be derived from (2.4) and Theorem 1.6.

Remark 2.2. *For $T \approx 0$ we have $\mu_t \approx \mu_0$ on $[0,T]$ and thus*

$$\sqrt{F(t)} = \left(\int_0^t e^{-(m-1)\mu_r} dr\right)^{\frac{1}{2}} \approx e^{-\frac{(m-1)}{2}\mu_0}\sqrt{t}, \quad \text{on } [0,T]$$

and $H = \|e^{\mu} g\|_{L^\infty([0,T]\times\partial B_R(\xi_0))} \approx e^{\mu_0}\|g\|_{L^\infty([0,T]\times\partial B_R(\xi_0))}$. Thus

$$R_{stoch}(t) \approx R - \sqrt{t}\left(\frac{\|g\|^{m-1}_{L^\infty([0,T]\times\partial B_R(\xi_0))}}{C_{det}}\right)^{\frac{1}{2}}, \quad \forall t \in [0, T_{stoch} \wedge T].$$

Consequently, we recover the deterministic rate of expansion for small times $T \approx 0$.

3. Linear multiplicative space-time noise

We now turn to the case of SPME perturbed by spatially inhomogeneous noise (1.1). Since spatially homogeneous noise is contained as a special case, the precise bounds derived in the last section will serve as optimal bounds for the inhomogeneous case. Let

$$\mu_t(\xi) := -\sum_{k=1}^N f_k(\xi) z_t^{(k)}.$$

As in the case of spatially homogeneous noise, solutions to (1.1) are defined via the transformation $Y_t := e^{\mu_t} X_t$ which (informally) leads to the transformed equation (first studied in [6, 8])

$$\begin{aligned} \partial_t Y_t &= e^{\mu_t}\Delta\Phi(e^{-\mu_t} Y_t), \text{ on } \mathcal{O}_T \\ Y(0) &= Y_0 = e^{\mu_0} X_0, \text{ on } \mathcal{O}, \end{aligned} \tag{3.1}$$

with homogeneous Dirichlet boundary conditions. Note that μ_t now depends on the spatial variable $\xi \in \mathcal{O}$. As before, this transformation can be made rigorous if the driving signals $z^{(k)}$ are given as paths of continuous semimartingales (cf. Remark 3.5 below).

Similar results and methods as presented in this section may be applied to the more general equation

$$\partial_t Y_t = \rho_1 \Delta \Phi(\rho_2 Y_t), \text{ on } \mathcal{O}_T$$
$$Y(0) = Y_0, \text{ on } \mathcal{O},$$

with $\rho_1, \rho_2 \in C^{0,2}(\mathcal{O}_T)$ and zero Dirichlet boundary conditions. For simplicity and in order to derive locally optimal estimates we restrict to equations of the form (3.1) and postpone the treatment of the more general case to the Appendix A.4.

In analogy to the deterministic case we define

Definition 3.1. *Let $Y_0 \in L^\infty(\mathcal{O})$ and $\Phi(g) \in L^2(\Sigma_T)$. A function $Y \in C([0,T]; L^1(\mathcal{O}))$ with $\Phi(e^{-\mu}Y) \in L^2([0,T]; H^1(\mathcal{O}))$ is said to be a solution to the (inhomogeneous) Dirichlet problem to* (3.1) *with initial condition Y_0 and boundary value g if $\Phi(u) = \Phi(g)$ on Σ_T in the sense of traces and*

$$\int_{\mathcal{O}_T} Y \partial_r \eta \; d\xi dr + \int_{\mathcal{O}} Y_0 \eta_0 \; d\xi = \int_{\mathcal{O}_T} \nabla \Phi(e^{-\mu} Y) \cdot \nabla (e^{\mu} \eta) \; d\xi dr, \tag{3.2}$$

for all $\eta \in C^{1,2}(\bar{\mathcal{O}}_T)$ with $\eta_{|\mathcal{P}_T} = 0$.

Sub/supersolutions to (3.1) *are defined by replacing "$=$" by "$\geq$" ("$\leq$" resp.) in* (3.2) *and requiring the testfunctions η to be non-negative.*

It is easy to see that every essentially bounded solution Y is a solution on each smooth subdomain $K \subseteq \mathcal{O}$ with initial condition $Y_{0|K}$ and boundary data $\Phi(g) = \Phi(Y)$ in the sense of traces. As concerning the chosen notion of a solution to (3.1) the same remark as in the deterministic case (Remark 1.4) applies.

As outlined in the beginning of this section, solutions to (1.1) are now defined via the transformation $Y_t = e^{\mu_t} X_t$, i.e.

Definition 3.2 (Solutions for (1.1))**.** *A function $X \in C([0,T]; L^1(\mathcal{O}))$ is said to be a solution to* (1.1) *with initial condition X_0 and boundary value g, if $Y_t = e^{\mu_t} X_t$ is a solution to* (3.1) *with initial condition $Y_0 = X_0 e^{\mu_0}$ and boundary value $g e^{\mu}$.*

Sub/super-solutions to (1.1) *are defined analogously.*

The existence and uniqueness of solutions to (1.1) has first been proven in [8,25]. In order not to divert from the main content we only state the corresponding result at this point, while a proof of a more general result may be found in the Appendix A.2.

Proposition 3.3 (Existence of solutions)**.** *Let $X_0 \in L^\infty(\mathcal{O})$. Then*

i. There is an essentially bounded solution X to the homogeneous Dirichlet problem for (1.1) *satisfying $X \in C((0,T] \times \mathcal{O})$.*

ii. If $X_0 \geq 0$ a.e. on $\mathcal{O}$, then $X \geq 0$ a.e. on $\mathcal{O}$.

The proof of finite speed of propagation will rely on local comparison to supersolutions. We now present the required comparison result for essentially bounded solutions to the inhomogeneous Dirichlet problem.

Theorem 3.4 (Comparison)**.** *Let $X^{(1)}, X^{(2)}$ be essentially bounded sub/supersolutions to* (1.1) *with initial conditions $X_0^{(1)} \leq X_0^{(2)}$ and boundary data $g^{(1)} \leq g^{(2)}$, a.e. in $\mathcal{O}$ respectively. Then,*

$$X^{(1)} \leq X^{(2)}, \quad \text{a.e. in } \mathcal{O}.$$

In particular, essentially bounded solutions to (1.1) *are unique.*

The proof of a more general version of Theorem 3.4 may be found in the Appendix A.2.

We now comment on the case of stochastic PDE of the form (1.1). There are two main arguments which justify the notation of a Stratonovich integral used in (1.1) and show in which sense stochastic perturbations are included: consistency in case of signals given by semimartingales and stability in the driving signals. Let z be an $\mathbb{R}^N$-valued, continuous semimartingale on a normal, complete, filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}_+}, \mathbb{P})$. For each $\omega \in \Omega$ we may consider (1.1) and (3.1) driven by $z \equiv z(\omega)$. According to Proposition 3.3, for each $\omega \in \Omega$ there is a solution $X(\omega)$ to (1.1) in the sense of Definition 3.2. It is easy to see that this defines an $\{\mathcal{F}_t\}$-adapted process in $L^1(\mathcal{O})$ by the construction of the solution given in Appendix A.1. Moreover, we have

Remark 3.5 (Semimartingale signals)**.** *i. Let $X_0 \in L^\infty(\mathcal{O})$. Then X satisfies*

$$\begin{aligned}\int_{\mathcal{O}} X_t\varphi \, d\xi = \int_{\mathcal{O}} X_0\varphi \, d\xi - \int_0^t \int_{\mathcal{O}} \nabla\Phi(X_r)\cdot\nabla\varphi \, d\xi dr \\ + \sum_{k=1}^N \int_0^t \left(\int_{\mathcal{O}} f_k X_r \varphi \, d\xi\right) \circ dz_r^{(k)},\end{aligned} \tag{3.3}$$

for all $\varphi \in C_0^2(\bar{\mathcal{O}})$ and all $0 \le t \le T$, $\mathbb{P}$-almost surely, where $\circ$ denotes the stochastic Stratonovich integral.

ii. If $z = W$ is given by an $\mathbb{R}^N$-valued Brownian motion, then a probabilistic theory of existence and uniqueness of variational solutions to (1.1) *is available from [32, 39, 40] and the solution is uniquely determined (up to indistinguishability) by* (3.3)*. Thus, the pathwise solutions considered in this paper coincide $\mathbb{P}$-a.s. with the solutions constructed in [32, 39, 40].*

On an informal level, the proof of Remark 3.5, (i) follows directly by applying Itô's product rule to $X_t = e^{-\mu_t}Y_t$. A rigorous proof has been provided in [6] and under less regularity assumptions in [25].

If the driving signals are given as paths of fractional Brownian motion (fBm), i.e. $z \equiv W^H(\omega)$ where W^H denotes $\mathbb{R}^N$-valued fBm with Hurst parameter $H \in (0, 1)$ then the above argument does not apply, since fBm is not a semimartingale and thus the Stratonovich integral in (3.3) is not well-defined. In this case we may justify the notation chosen in (1.1) by smooth approximations. More precisely,

Remark 3.6 (Stability)**.** *Let $z^{(\varepsilon)} \in C^\infty([0, T]; \mathbb{R}^N)$ with $z^{(\varepsilon)} \to z$ in $C([0, T]; \mathbb{R}^N)$ and let $X^{(\varepsilon)}$ be the solution to* (1.1) *driven by $z^{(\varepsilon)}$, i.e.*

$$\frac{d}{dt}X_t^{(\varepsilon)} = \Delta\Phi(X_t^{(\varepsilon)}) + \sum_{k=1}^N f_k X_t^{(\varepsilon)} \dot{z}_t^{(\delta,k)}.$$

Then

$$X_t^{(\varepsilon)} \to X_t, \quad \text{in } L^1(\mathcal{O}),$$

for all $t \in [0, T]$. In other words, the solution X depends continuously on the driving signal z with respect to the usual sup-norm on the space of continuous paths. Therefore, no stronger rough paths topologies and no iterated integrals (cf. e.g. [23, 36]) of the signals are required. This fact is due to the linear structure of the noise (cf. e.g. [15, 22]) while for more general diffusion coefficients the situation is much more involved (cf. [24, 34]).

The proof of Remark 3.6, (ii) follows from the construction of solutions to (1.1) presented in Appendix A.1, since the estimates derived there can easily be seen to be independent of $\varepsilon > 0$ which implies convergence. Note that we do not give separate meaning to the stochastic integrals appearing in (1.1), but a solution X satisfies (1.1) in a limiting sense (Remark 3.6) and in sense of a transformation (Definition 3.2). From a physical and numerical point of view, any notion of solution to (1.1) should be consistent with smooth approximations of the driving signals and thus coincide with the solutions considered in this paper.

3.1. **Finite speed of propagation.** We are going to prove bounds on the speed of propagation for (1.1), based on estimates for the rate of hole-filling as in the deterministic case. As we have seen in Section 2, the optimal bounds on the rate of collapse of balls have to depend on the driving signal. Since the perturbation now is spatially dependent, we expect worse estimates than in Proposition 2.1.

On the other hand, since $\xi \mapsto \mu_t(\xi)$ is continuous and thus $\mu_t(\xi) \approx \mu_t(\xi_0)$ on small balls $B_R(\xi_0)$, locally in space the rate of expansion should be given as in Proposition 2.1 with $\mu_r \equiv \mu_r(\xi_0)$. This line of thought leads to optimal bounds on the rate of collapse of asymptotically small balls, proven in Theorem 3.7 below.

Moreover, due to the continuity of $t \mapsto \mu_t(\xi)$ we have $\mu_t(\xi) \approx \mu_0(\xi)$ on small time intervals $[0,T]$. Therefore, we expect to recover the optimal bounds from the deterministic case at least for asymptotically small times T, which indeed is proven in Theorem 3.9 below. In case of spatially homogeneous perturbations this has been observed in Remark 2.2.

Theorem 3.7 (Hole-filling theorem for small balls)**.** *Let $\xi_0 \in \mathbb{R}^d$, $T, R > 0$ and $X \in C((0,T] \times B_R(\xi_0))$ be an essentially bounded, non-negative solution to* (1.1) *with vanishing initial value X_0 on $B_R(\xi_0)$ and boundary value g satisfying $H := \|e^\mu g\|_{L^\infty([0,T]\times\partial B_R(\xi_0))} < \infty$. Define $F(t) := \int_0^t e^{-(m-1)\mu_r(\xi_0)}dr$ and*

$$T_{stoch} := F^{-1}\left(R^2 \frac{C_{det}}{H^{m-1}} C_R\right),$$

where $R \mapsto C_R$ is a continuous, non-increasing function with $\lim_{R\downarrow 0} C_R = 1$.

Then X_t vanishes in $B_{R_{stoch}(t)}(\xi_0)$ for all $t \in [0, T_{stoch} \wedge T]$, where

$$R_{stoch}(t) = R - \sqrt{F(t)}\left(\frac{H^{m-1}}{C_{det}}\right)^{\frac{1}{2}} C_R^{-\frac{1}{2}}.$$

Note that for $R \approx 0$ we recover the optimal rate from Proposition 2.1 with $\mu_r \equiv \mu_r(\xi_0)$.

Proof. Since X is a solution to (1.1) with initial value $X_0 \equiv 0$ and boundary value g, $Y := e^\mu X$ is a solution to (3.1) with initial value $Y_0 \equiv 0$ and boundary value $e^\mu g$.

For $\xi_1 \in B_R(\xi_0)$, $\tilde{T} \in (0,T]$, $r \in (0, \operatorname{dist}(\xi_1, \partial B_R(\xi_0))]$ we construct an explicit supersolution to (3.1) in $[0,\tilde{T}] \times B_r(\xi_1)$. Let

$$W(t,\xi,\xi_1) := \tilde{C}|\xi - \xi_1|^{\frac{2}{m-1}} \left(F(\tilde{T}) - F(t)\right)^{\frac{-1}{m-1}}, \quad t \in [0,\tilde{T}), \xi \in B_r(\xi_1),$$

where $\tilde{C}$ will be chosen below (only depending on R) and F is as in Section 2 for noise frozen at ξ_0, i.e.

$$F(t) := \int_0^t e^{-(m-1)\mu_r(\xi_0)}dr.$$

We compute:

$$\partial_t W(t,\xi,\xi_1) = \frac{1}{m-1}\tilde{C}|\xi - \xi_1|^{\frac{2}{m-1}} \left(F(\tilde{T}) - F(t)\right)^{\frac{-m}{m-1}} e^{-(m-1)\mu_t(\xi_0)},$$

for all $(t,\xi) \in [0,\tilde{T}) \times B_r(\xi_1)$ and

$$\begin{aligned}
&\Delta \left(e^{-\mu_t(\xi)} W(t,\xi,\xi_1)\right)^m \\
&\leq \frac{\tilde{C}^m}{(m-1)C_{det}} \left(F(\tilde{T})-F(t)\right)^{\frac{-m}{m-1}} |\xi-\xi_1|^{\frac{2}{m-1}} \\
&\quad \left(e^{-m\mu_t(\xi)} + \frac{2(m-1)}{d(m-1)+2}|\nabla e^{-m\mu_t(\xi)}|r + (m-1)C_{det}r^2|\Delta e^{-m\mu_t(\xi)}|\right) \\
&\leq \frac{\tilde{C}^m}{(m-1)C_{det}} \left(F(\tilde{T})-F(t)\right)^{\frac{-m}{m-1}} |\xi-\xi_1|^{\frac{2}{m-1}} e^{-m\mu_t(\xi)} \\
&\quad \left(1 + \frac{2(m-1)m}{d(m-1)+2}|\nabla \mu_t(\xi)|r + m(m-1)C_{det}r^2(m|\nabla\mu_t(\xi)|^2 + \Delta|\mu_t(\xi)|)\right) \\
&\leq \frac{\tilde{C}^m}{(m-1)C_{det}} \left(F(\tilde{T})-F(t)\right)^{\frac{-m}{m-1}} |\xi-\xi_1|^{\frac{2}{m-1}} e^{-m\mu_t(\xi)} \\
&\quad \left(1 + C(d,m)R(1+R)\|\mu\|_{C^{0,2}([0,\tilde{T}]\times B_R(\xi_0))}\right),
\end{aligned}$$

for all $(t,\xi) \in [0,\tilde{T}) \times B_r(\xi_1)$. We conclude that

$$\partial_t W(t,\xi,\xi_1) \geq e^{\mu_t(\xi)} \Delta \left(e^{-\mu_t(\xi)} W(t,\xi,\xi_1)\right)^m$$

on $[0,\tilde{T}) \times B_r(\xi_1)$ if

$$e^{-(m-1)\|\mu(\xi_0)-\mu(\cdot)\|_{C^0([0,T]\times B_R(\xi_0))}} \geq \frac{\tilde{C}^{m-1}}{C_{det}}\left(1+C(d,m)R(1+R)\|\mu\|_{C^{0,2}([0,T]\times B_R(\xi_0))}\right),$$

which is satisfied if we choose $\tilde{C}^{m-1} = C_{det}C_R$ with

$$C_R := \frac{e^{-(m-1)\|\mu(\xi_0)-\mu(\cdot)\|_{C^0([0,T]\times B_R(\xi_0))}}}{1+C(d,m)R(1+R)\|\mu\|_{C^{0,2}([0,T]\times B_R(\xi_0))}}.$$

We note that $R \mapsto C_R$ is continuous, non-increasing in R and $\lim_{R\downarrow 0} C_R = 1$. In contrast, C_R does not necessarily converge to 1 for $T \to 0$. Thus, the bounds become optimal locally in space but not locally in time.

In order to derive the upper bound $Y(t,\xi) \leq W(t,\xi,\xi_1)$ on $[0,\tilde{T}) \times B_r(\xi_1)$ we need $g(t,\xi)e^{\mu_t(\xi)} \leq W(t,\xi,\xi_1)$ for a.a. $(t,\xi) \in [0,\tilde{T}) \times \partial B_r(\xi_1)$. For this it is sufficient to have

$$W(t,\xi,\xi_1) = \tilde{C}|\xi-\xi_1|^{\frac{2}{m-1}}\left(F(\tilde{T})-F(t)\right)^{\frac{-1}{m-1}} \geq H,$$

for a.a. $(t,\xi) \in [0,\tilde{T}) \times \partial B_r(\xi_1)$. This is satisfied if we choose $\tilde{T} = \tilde{T}_r$ by

$$\tilde{T}_r := F^{-1}\left(\frac{\tilde{C}^{m-1}r^2}{H^{m-1}}\right) = F^{-1}\left(r^2\frac{C_{det}}{H^{m-1}}C_R\right). \tag{3.4}$$

By Theorem 3.4 and by continuity of Y, W we conclude

$$0 \leq Y(t,\xi_1) \leq W(t,\xi_1,\xi_1) = 0, \quad \forall t \in [0,\tilde{T}_r].$$

Let $R_1 \in (0,R)$, $\xi_1 \in B_{R_1}(\xi_0)$ and $r = \mathrm{dist}(\xi_1, \partial B_R(\xi_0)) \geq R-R_1 > 0$. Resolving (3.4) for r yields

$$R(T) := R_1 = R - \sqrt{F(T)}\left(\frac{H}{\tilde{C}}\right)^{\frac{m-1}{2}} = R - \sqrt{F(T)}\left(\frac{H^{m-1}}{C_{det}}\right)^{\frac{1}{2}} C_R^{-\frac{1}{2}}.$$

Hence,

$$Y(t,\xi) = 0, \quad \forall \xi \in B_{R(t)}(\xi_0),\ t \in [0, T_{stoch} \wedge T],$$

where $T_{stoch} = T_R = F^{-1}\left(R^2\frac{C_{det}}{H^{m-1}}C_R\right)$. □

Remark 3.8. *Due to the explicit form of the estimates and the constant C_R in Theorem 3.7, the dependence of the bounds on the strength of the noise is obvious. In particular, when the noise intensity $\sum_{k=1}^N \|f_k\|_{C^2(B_R(\xi_0))}$ decreases to 0, then the bounds from Theorem 3.7 approach the corresponding deterministic, optimal ones.*

We will now derive a second bound on the rate of collapse of balls for (1.1). In contrast to Theorem 3.7 the construction of a suitable supersolution will be based on a temporal discretization, i.e. on freezing the noise at time $t = 0$.

Theorem 3.9 (Hole-filling theorem for small times)**.** *Let $\xi_0 \in \mathbb{R}^d$, $T, R > 0$ and $X \in C((0,T] \times B_R(\xi_0))$ be an essentially bounded, non-negative solution to (1.1) with vanishing initial value X_0 on $B_R(\xi_0)$ and boundary value g satisfying $H := \|g\|_{L^\infty([0,T]\times\partial B_R(\xi_0))} < \infty$. Define T_{stoch} by*

$$T_{stoch} := \sup \left\{ \tilde{T} \in [0,T] \middle| \; \tilde{T} C_{\tilde{T}} \le R^2 \frac{C_{det}}{H^{m-1}} \right\},$$

where $t \mapsto C_t$ is a continuous, non-decreasing function with $\lim_{t\downarrow 0} C_t = 1$.

Then X_t vanishes in $B_{R_{stoch}(t)}(\xi_0)$ for all $t \in [0, T_{stoch}]$, where

$$R_{stoch}(t) = R - \sqrt{t} \left(\frac{H^{m-1}}{C_{det}} \right)^{\frac{1}{2}} \sqrt{C_t}.$$

Note that for $t \approx 0$ we recover the optimal rate from the deterministic case.

Proof. The proof proceeds similarly to Theorem 3.7. Hence, let $Y := e^{\mu} X$ be a solution to (3.1) with initial value $Y_0 \equiv 0$ and boundary value $e^{\mu} g$.

For $\xi_1 \in B_R(\xi_0)$, $\tilde{T} \in (0,T]$, $r \in (0, \mathrm{dist}(\xi_1, \partial B_R(\xi_0)))$ we again construct an explicit supersolution to (3.1) in $[0,\tilde{T}) \times B_r(\xi_1)$:

$$W(t,\xi,\xi_1) := \tilde{C} e^{\mu_0(\xi)} |\xi - \xi_1|^{\frac{2}{m-1}} \left(\tilde{T} - t\right)^{\frac{-1}{m-1}}, \quad t \in [0,\tilde{T}), \xi \in B_r(\xi_1),$$

where $\tilde{C}$ will be chosen below, depending on $\tilde{T}, R$ only. We compute:

$$\partial_t W(t,\xi,\xi_1) = \frac{1}{m-1} \tilde{C} e^{\mu_0(\xi)} |\xi - \xi_1|^{\frac{2}{m-1}} \left(\tilde{T} - t\right)^{\frac{-m}{m-1}},$$

for all $(t,\xi) \in [0,\tilde{T}) \times B_r(\xi_1)$ and

$$\begin{aligned}
&\Delta \left(e^{-\mu_t(\xi)} W(t,\xi,\xi_1) \right)^m \\
&\le \frac{\tilde{C}^m}{(m-1)C_{det}} \left(\tilde{T} - t\right)^{\frac{-m}{m-1}} |\xi - \xi_1|^{\frac{2}{m-1}} \Big(e^{m(\mu_0(\xi) - \mu_t(\xi))} \\
&\quad + \frac{2(m-1)}{2 + d(m-1)} |\nabla e^{m(\mu_0(\xi)-\mu_t(\xi))}| R + (m-1) C_{det} R^2 |\Delta e^{m(\mu_0(\xi)-\mu_t(\xi))}| \Big) \\
&\le \frac{\tilde{C}^m}{(m-1)C_{det}} \left(\tilde{T} - t\right)^{\frac{-m}{m-1}} |\xi - \xi_1|^{\frac{2}{m-1}} e^{m(\mu_0(\xi)-\mu_t(\xi))} \\
&\quad \Big(1 + C(d,m) R(1+R) \|\mu_0 - \mu_\cdot\|_{C^{0,2}([0,\tilde{T}]\times B_R(\xi_0))} \Big)
\end{aligned}$$

for all $(t,\xi) \in [0,\tilde{T}) \times B_r(\xi_1)$. We conclude that

$$\partial_t W(t,\xi,\xi_1) \ge e^{\mu_t(\xi)} \Delta \left(e^{-\mu_t(\xi)} W(t,\xi,\xi_1) \right)^m$$

on $[0,\tilde{T}) \times B_r(\xi_1)$ if

$$e^{(m-1)(\mu_t(\xi)-\mu_0(\xi))} \ge \frac{\tilde{C}^{m-1}}{C_{det}} \Big(1 + C(d,m) R(1+R) \|\mu_0 - \mu_\cdot\|_{C^{0,2}([0,\tilde{T}]\times B_R(\xi_0))} \Big),$$

for all $(t,\xi) \in [0,\tilde{T}) \times B_r(\xi_1)$, which is satisfied for the choice

$$\tilde{C}^{m-1} = \frac{C_{det}}{C_{\tilde{T}}} e^{(m-1)\|\mu_\cdot - \mu_0\|_{C^0([0,\tilde{T}]\times \partial B_R(\xi_0))}}$$

with

$$C_{\tilde{T}} := \frac{1 + C(d,m)R(1+R)\|\mu_0 - \mu_\cdot\|_{C^{0,2}([0,\tilde{T}]\times B_R(\xi_0))}}{e^{-2(m-1)\|\mu_0-\mu_\cdot\|_{C^0([0,\tilde{T}]\times B_R(\xi_0))}}}.$$

In order to derive the upper bound $Y(t,\xi) \le W(t,\xi,\xi_1)$ on $[0,\tilde{T}) \times B_r(\xi_1)$ we need $g(t,\xi)e^{\mu_t}(\xi) \le W(t,\xi,\xi_1)$ for a.e. $(t,\xi) \in [0,\tilde{T}) \times \partial B_r(\xi_1)$. For this to be true it is sufficient to have

$$W(t,\xi,\xi_1) = \tilde{C} e^{\mu_0(\xi)} |\xi - \xi_1|^{\frac{2}{m-1}} \left(\tilde{T} - t\right)^{\frac{-1}{m-1}} \ge H e^{\mu_t(\xi)},$$

for a.a. $(t,\xi) \in [0,\tilde{T}) \times \partial B_r(\xi_1)$. This is satisfied if

$$\frac{\tilde{T} e^{(m-1)\|\mu_\cdot - \mu_0\|_{C^0([0,\tilde{T}]\times \partial B_R(\xi_0))}}}{\tilde{C}^{m-1}} \le r^2 \frac{1}{H^{m-1}},$$

for which in turn it is sufficient to have

$$\tilde{T} C_{\tilde{T}} \le r^2 \frac{C_{det}}{H^{m-1}}. \tag{3.5}$$

Since the left hand side is continuous in $\tilde{T}$ we may choose $\tilde{T}$ as

$$\tilde{T} := \sup\left\{ \tilde{T} \in [0,T] \middle| \, \tilde{T} C_{\tilde{T}} \le r^2 \frac{C_{det}}{H^{m-1}} \right\}.$$

Note that $\tilde{T} \mapsto C_{\tilde{T}} > 0$ is continuous, non-decreasing and

$$C_{\tilde{T}} \to \begin{cases} 1, & \text{for } \tilde{T} \to 0 \\ e^{2(m-1)\|\mu_0 - \mu_\cdot\|_{C^0([0,\tilde{T}]\times\{\xi_0\})}}, & \text{for } R \to 0, \end{cases}$$

i.e. we recover the optimal constant from the deterministic case for asymptotically small time, while locally in space the estimates will not be optimal.

Let now $R_1 \in (0,R)$, $\xi_1 \in B_{R_1}(\xi_0)$ and $r = \mathrm{dist}(\xi_1, \partial B_R(\xi_0)) \ge R - R_1 > 0$. By Theorem 3.4 and by continuity of Y, W we conclude

$$0 \le Y(t,\xi_1) \le W(t,\xi_1,\xi_1) = 0, \quad \forall t \in [0, \tilde{T}(r)].$$

Resolving (3.5) for R_1 yields

$$R(T) := R_1 = R - \sqrt{T} \left(\frac{H^{m-1}}{C_{det}}\right)^{\frac{1}{2}} \sqrt{C_T}.$$

Hence,

$$Y(t,\xi) = 0, \quad \forall \xi \in B_{R(t)}(\xi_0),\ t \in [0, T_{stoch}],$$

where

$$\begin{aligned} T_{stoch} &:= \tilde{T}(R) \\ &:= \sup\left\{ \tilde{T} \in [0,T] \middle| \, \tilde{T} C_{\tilde{T}} \le R^2 \frac{C_{det}}{H^{m-1}} \right\}. \end{aligned}$$

□

We are now ready to derive bounds on the speed of propagation for (1.1). We give two formulations of this property

Theorem 3.10 (Finite speed of propagation)**.** *Let $X \in C((0,T]\times\mathcal{O})$ be an essentially bounded, non-negative solution to the homogeneous Dirichlet problem to* (1.1) *and set $H := \|e^{\mu}X\|_{L^\infty(\mathcal{O}_T)}$. Then, for each $s\in[0,T]$ and every $h>0$ there is a time-span $T_h>0$ such that*

$$supp(X_{s+t}) \subseteq B_h(supp(X_s)), \quad \forall t\in[0,T_h\wedge(T-s)]. \tag{3.6}$$

More precisely, T_h is given by

$$T_h := F_h^{-1}\left(h^2\frac{C_{det}}{H^{m-1}}C_h\right),$$

where $F_h(t) := \int_0^t e^{-(m-1)\inf_{\xi_0\in\partial B_h(supp(X_s))}\mu_r(\xi_0)}dr$ and $h\mapsto C_h$ is a continuous, non-increasing function satisfying $\lim_{h\downarrow 0}C_h=1$. In particular,

$$\left|T_h - F_0^{-1}\left(h^2\frac{C_{det}}{H^{m-1}}\right)\right| \to 0, \quad \textit{for } h\to 0,$$

with $F_0(t)=\int_0^t e^{-(m-1)\inf_{\xi_0\in\partial\overline{supp(X_s)}}\mu_r(\xi_0)}dr$.

Proof. Without loss of generality we assume $s=0$. In order to avoid difficulties at the boundary we first replace X by a solution to (1.1) on some large ball $B_R(0)\supseteq\mathcal{O}$, where we choose $R>0$ large enough, such that the boundary $\partial B_R(0)$ becomes “invisible” for the solution on $[0,T]$: Let $h>0$, $R>0$ such that

$$\tilde{\mathcal{O}} := B_R(0) \supseteq \bar{B}_{2h}(\mathrm{supp}(X_0))\cup\mathcal{O}.$$

By Proposition 3.3 there is a unique, essentially bounded, non-negative solution $X\in C((0,T]\times\tilde{\mathcal{O}})$ to (1.1) on $\tilde{\mathcal{O}}$ with zero Dirichlet boundary conditions and initial condition $\tilde{X}_0 := X_0\mathbb{1}_{\mathcal{O}}\in L^\infty(\tilde{\mathcal{O}})$. Since $\tilde{X}$ is a supersolution to the homogeneous Dirichlet problem to (1.1) on $\mathcal{O}$, by Theorem 3.4 we have

$$X\le\tilde{X}, \quad \text{on } \mathcal{O}_T.$$

Thus, it is sufficient to prove the claim for $\tilde{X}$. Hence, without loss of generality we may assume X to be an essentially bounded solution to (1.1) and

$$\mathrm{dist}(\mathrm{supp}(X_0),\partial\mathcal{O})>2h.$$

Let $\xi_0\in\partial B_h(\mathrm{supp}(X_0))\subseteq\mathcal{O}$. Then, $X_0=0$ on $B_h(\xi_0)\subseteq\mathcal{O}$ and Theorem 3.7 (with $R=h$) implies that X_t vanishes on $B_{R_{stoch}(t)}(\xi_0)$ for all $t\in[0,T_{stoch}\wedge T]$, where $R_{stoch}(t)$ and T_{stoch} given in Theorem 3.7 depend on ξ_0 via the constant C_h and the function F. We note that C_h may be uniformly estimated by

$$C_h \ge \bar{C}_h := \frac{e^{-h\|\mu\|_{C^{0,1}([0,T]\times(\mathcal{O}\cap\mathrm{supp}(X_0)^c))}}}{\left(1+C(m)h(1+h)\|\mu\|_{C^{0,2}([0,T]\times(\mathcal{O}\cap\mathrm{supp}(X_0)^c))}\right)^{\frac{1}{m-1}}}$$

and F by

$$\bar{F}_h(t) = \int_0^t e^{-(m-1)\inf_{\xi_0\in\partial B_h(\mathrm{supp}(X_0))}\mu_r(\xi_0)}dr \ge F(t).$$

Therefore, T_{stoch} is uniformly bounded from below by

$$\bar{T}_h := \bar{F}_h^{-1}\left(h^2\frac{C_{det}}{H^{m-1}}\bar{C}_h\right),$$

and $R_{stoch}(t)$ by

$$\bar{R}_h(t) = h-\sqrt{\bar{F}_h(t)}\left(\frac{H^{m-1}}{C_{det}}\right)^{\frac{1}{2}}\bar{C}_h^{-\frac{1}{2}}.$$

Hence, X_t vanishes on $B_{\bar{R}_h(t)}(\partial B_h(\mathrm{supp}(X_0)))$ for all $t\in[0,\bar{T}_h\wedge T]$.

In particular, this implies that X is a solution to the homogeneous Dirichlet problem to (1.1) on $[0,T_h\wedge T]\times\mathcal{O}\cap B_h(\mathrm{supp}(X_0))^c$. Since $X_0\equiv 0$ on $\mathcal{O}\cap B_h(\mathrm{supp}(X_0))^c$ this implies $X_t\equiv 0$ on $\mathcal{O}\cap B_h(\mathrm{supp}(X_0))^c$ for all $t\in[0,T_h\wedge T]$. $\square$

Theorem 3.11 (Finite speed of propagation)**.** *Let $X\in C((0,T]\times\mathcal{O})$ be an essentially bounded, non-negative solution to the homogeneous Dirichlet problem to* (1.1) *and set $H:=\|X\|_{L^\infty(\mathcal{O}_T)}$. Then, for every $s\in[0,T]$*

$$supp(X_{s+t})\subseteq B_{\sqrt{t}\left(\frac{H^{m-1}}{C_{det}}\right)^{\frac12}\sqrt{C_t}}(supp(X_s)),\quad \forall t\in[0,T-s],$$

where $t\mapsto C_t$ is a continuous, non-decreasing function with $C_t\to 1$ for $t\to 0$.

Proof. We argue as for Theorem 3.10 but apply Theorem 3.9 instead of Theorem 3.7. Let $D:=\mathrm{diam}(\mathcal{O})$. We then estimate C_T uniformly by

$$C_T\le \bar C_T:=\frac{1+C(d,m)D(1+D)\|\mu_0-\mu_\cdot\|_{C^{0,2}([0,T]\times(\mathcal{O}\cap\mathrm{supp}(X_0)^c))}}{e^{-2(m-1)\|\mu_\cdot-\mu_0\|_{C^0([0,T]\times(\mathcal{O}\cap\mathrm{supp}(X_0)^c))}}}.$$

Hence, for

$$\bar T(h):=\sup\left\{\tilde T\in[0,T]\middle|\ \tilde T\bar C_{\tilde T}\le h^2\frac{C_{det}}{H^{m-1}}\right\},$$

we have $\bar T(h)\le T_{stoch}$ for all $\xi_0\in\partial(B_h(\mathrm{supp}(X_0)))$ as in the proof of Theorem 3.10 and for

$$\bar R(t):=h-\sqrt{t}\left(\frac{H^{m-1}}{C_{det}}\right)^{\frac12}\sqrt{\bar C_t},$$

we have $\bar R(t)\le R_{stoch}(t)$ for all $t\in[0,T]$. In particular, for all $\xi_0\in\partial(B_h(\mathrm{supp}(X_0)))$ we deduce

$$X_t(\xi_0)=0,\quad \forall t\in[0,\bar T(h)]$$

Arguing as for Theorem 3.10 this implies $\mathrm{supp}(X_t)\subseteq B_h(\mathrm{supp}(X_0))$ for all $t\le \bar T(h)$. Resolving for h yields

$$X_t\equiv 0 \text{ on } B_{\sqrt{t}\left(\frac{H^{m-1}}{C_{det}}\right)^{\frac12}\sqrt{\bar C_t}}(\mathrm{supp}(X_0)),$$

$\square$

Remark 3.12 (Weaker notions of solutions)**.** *The techniques used above to obtain bounds on the rate of hole-filling and on the speed of propagation rely on local comparison to explicit supersolutions. Since the comparison result Theorem 3.4 also holds for distributional subsolutions, i.e. $Y\in L^1(\mathcal{O}_T)$, $\Phi(e^{-\mu}Y)\in L^1(\mathcal{O}_T)$ satisfying*

$$\int_{\mathcal{O}_T}Y\partial_r\eta\ d\xi dr+\int_{\mathcal{O}}Y_0\eta_0\ d\xi\ge-\int_{\mathcal{O}_T}\Phi(e^{-\mu}Y)\Delta(e^{\mu}\eta)\ d\xi dr$$
$$+\int_{\Sigma_T}\Phi(e^{-\mu}g)\partial_\nu(e^{\mu}\eta)d\vartheta dr,$$

for all non-negative $\eta\in C^{1,2}(\bar{\mathcal{O}}_T)$ with $\eta_{|\mathcal{P}_T}=0$, all the results proved in this section also hold for solutions of this more general type.

Remark 3.13 (Unbounded domains $\mathcal{O}\subseteq\mathbb{R}^d$)**.** *In case of unbounded domains $\mathcal{O}\subseteq\mathbb{R}^d$ no pathwise uniqueness and existence theory (in the sense of existence of a stochastic flow) has been established for* (1.1) *so far. We note, however, that the simpler problem of constructing probabilistic solutions to* (1.1) *with $z^{(k)}$ being given as paths of Brownian motions has been solved in [41] for $d\ge 3$.*

If the support of the initial condition $X_0\in L^\infty(\mathcal{O})$ is compact and bounded away from $\partial\mathcal{O}$ then the existence of corresponding essentially bounded solutions X to the homogeneous Cauchy-Dirichlet problem on short time intervals $[0,T]$ follows *from the finite speed of propagation properties proved in this paper. The time of existence T allowed by this approach is limited due to the support $supp(X_t)$ reaching the boundary $\partial\mathcal{O}$. In particular, for the Cauchy problem no restriction on the time of existence has to be made.*

For initial conditions X_0 with compact support, also uniqueness of essentially bounded solutions may be deduced from the methods of this paper at least on short time intervals $[0,T]$. Again, for the Cauchy problem no restriction on the time interval has to be supposed.

The case of initial conditions with unbounded support, however, remains open.

4. Infinite dimensional random attractors

In this section we use the result of finite speed of propagation for SPME of the form (1.1) to prove that the random attractor associated to

$$\begin{aligned} dX_t &= \Delta\left(|X_t|^m \mathrm{sgn}(X_t)\right)dt + \lambda X_t dt + \sum_{k=1}^N f_k X_t \circ dz_t^{(k)}, \text{ on } \mathcal{O}_T, \\ X(0) &= X_0, \text{ on } \mathcal{O}, \end{aligned} \tag{4.1}$$

with homogeneous Dirichlet boundary conditions and $\lambda > 0$ has infinite fractal dimension. First, we will prove the existence of an RDS φ corresponding to (4.1) (Section 4.1), then we will provide lower bounds on the Kolmogorov ε-entropy for random attractors of φ (Section 4.2).

In the following we assume the driving signals $z^{(k)}$ to be given as paths of a stochastic process with strictly stationary increments. More precisely, let $(\Omega,\mathcal{F},\mathcal{F}_t,\mathbb{P})$ be a filtered probability space, $(z_t)_{t\in\mathbb{R}}$ be an $\mathbb{R}^N$-valued adapted stochastic process and $((\Omega,\mathcal{F},\mathbb{P}),(\theta_t)_{t\in\mathbb{R}})$ be a metric dynamical system. For notions and results from the theory of RDS and random attractors we refer to [25, Section 1.2.1] and [2, 12, 16, 17, 44]. We suppose:

($S1$) (Strictly stationary increments) For all $t,s\in\mathbb{R}$, $\omega\in\Omega$:

$$z_t(\omega) - z_s(\omega) = z_{t-s}(\theta_s\omega),$$

where we assume $z_0 = 0$ for notational convenience only.

($S2$) (Regularity) z_t has continuous paths.

($S3$) (Sublinear growth) $z_t(\omega) = o(|t|)$ for $t\to-\infty$, for all $\omega\in\Omega$.

As a concrete example we may consider fractional Brownian motion W^H with Hurst parameter $H\in(0,1)$: Let $\tilde{\Omega} = \{z\in C(\mathbb{R};\mathbb{R}^N)|z(0)=0\}$, $\theta_t(\omega) := \omega(t+\cdot)-\omega(t)$ be the Wiener shift and μ^H be the two-sided fractional Wiener measure on $\tilde{\Omega}$ (cf. [29]). Then (S1)-(S2) are satisfied for the canonical realization of two-sided fBm $z_t(\omega) := \omega(t)$. From the law of iterated logarithm for fractional Brownian motion (cf. e.g. [37, Theorem 7.2.15]) it follows that there is a θ-invariant set $\Omega\subseteq\tilde{\Omega}$ such that (S3) is satisfied.

4.1. **Generation of an RDS.**

Proving the existence of a stochastic flow of solutions to an SPDE is a difficult problem in general. So far, mainly SPDE with "simple" diffusion coefficients (additive or linear multiplicative) are well-understood in this regard, with some notable exceptions [24, 30, 34]. The usual approach to prove the existence of a stochastic flow for such SPDE relies on a transformation of the SPDE into a random PDE which may then be treated in a pathwise manner (cf. e.g. [16, 17, 28]). In particular, the uniqueness of solutions for each fixed path then yields the stochastic flow property. We will follow the same idea in order to prove that the solutions to (4.1) form an RDS.

If we set $f_{N+1} := \lambda$ and $z_t^{(N+1)} := t$, then (4.1) is of the form (1.1) and Proposition 3.3 implies the unique existence of a solution $X(\cdot,s;\omega)x$ with $X(s,s;\omega)x = x$

for each $s \in \mathbb{R}$, $x \in L^\infty(\mathcal{O})$ and driving signals $t \mapsto z_t^{(k)}(\omega)$. Recall that $X(\cdot, s; \omega)x$ is defined to be a solution to (4.1) (resp. (1.1)) if

$$Y(t, s; \omega)(e^{\mu_s(\omega)-\lambda s}x) := e^{\mu_t(\omega)-\lambda t}X(t, s; \omega)x, \quad t \in [s, \infty), \tag{4.2}$$

is a solution to (3.1) with initial condition $Y(s, s; \omega)(e^{\mu_s(\omega)-\lambda s}x) = e^{\mu_s(\omega)-\lambda s}x$. We set

$$\varphi(t - s, \theta_s\omega)x := X(t, s; \omega)x, \quad \text{for } t \geq s,\ \omega \in \Omega,\ x \in L^\infty(\mathcal{O}).$$

Using the uniqueness of solutions to (3.1) it is easy to see that φ defines an RDS on $L^\infty(\mathcal{O})$. However, continuity of $x \mapsto \varphi(t, \omega)x$ in $L^\infty(\mathcal{O})$ is not clear. On the other hand, by Proposition A.5 we have

$$\|\varphi(t, \omega)x - \varphi(t, \omega)y\|_{L^1(\mathcal{O})} \leq C\|x - y\|_{L^1(\mathcal{O})}, \quad \forall x, y \in L^\infty(\mathcal{O}).$$

Therefore, we may uniquely extend φ to a continuous RDS on $X := L^1(\mathcal{O})$.

4.2. **Lower bounds on the Kolmogorov ε-entropy.** We say that a random set $\mathcal{A}$ in X attracts all L^∞-bounded sets if

$$d(\varphi(t, \theta_{-t}\omega)K, \mathcal{A}(\omega)) \to 0, \quad \text{for } t \to \infty,$$

for all $\omega \in \Omega$ and all $K \subseteq L^\infty(\mathcal{O})$ with $\sup_{x \in K} \|x\|_{L^\infty(\mathcal{O})} < \infty$. In this section we will prove that each random set $\mathcal{A}$ attracting all L^∞-bounded sets and thus each random attractor for φ has infinite fractal dimension in $L^1(\mathcal{O})$.

The existence of a random attractor for the RDS associated to (1.1) has been shown in [25, Theorem 1.33]. Using the same methods it is possible to prove the existence of a random attractor for (4.1).

A precompact set $\mathcal{A} \subseteq X$ can be covered by a finite number of balls of radius ε for each $\varepsilon > 0$. Let $N_\varepsilon(\mathcal{A})$ be the minimal number of such balls. Then, the Kolmogorov ε-entropy of $\mathcal{A}$ is defined by

$$\mathbb{H}_\varepsilon(\mathcal{A}) := \log_2(N_\varepsilon(\mathcal{A})).$$

The fractal dimension of $\mathcal{A}$ is defined by

$$d_f(\mathcal{A}) = \limsup_{\varepsilon \to 0} \frac{\mathbb{H}_\varepsilon(\mathcal{A})}{\log_2(\frac{1}{\varepsilon})}.$$

If $\mathcal{A}$ is not pre-compact in X then we set $d_f(\mathcal{A}) = \infty$. We obtain

Theorem 4.1 (Lower bounds on the Kolmogorov ε-entropy)**.** *Let $\mathcal{A}$ be a random set in X attracting all L^∞-bounded sets. Then, the Kolmogorov ε-entropy of $\mathcal{A}$ is bounded below by*

$$\mathbb{H}_\delta(\mathcal{A}(\omega)) \geq C(\omega)\delta^{\frac{-d(m-1)}{2+d(m-1)}}, \quad \forall \omega \in \Omega,$$

where $C(\omega) > 0$ is a constant which may depend on m, d. In particular, the fractal dimension $d_f(\mathcal{A}(\omega))$ is infinite for all $\omega \in \Omega$.

Proof. The proof is inspired by [21, Theorem 4.1] and [28, Theorem 3.3]. In order to prove the lower bound on the Kolmogorov ε-entropy we consider the unstable manifold of the equilibrium point 0 defined by

$$\begin{aligned} \mathcal{M}^+(0, \omega) := \{u_0 \in X \mid \exists u : (-\infty, 0] \to X, \text{ such that } \varphi(t; \theta_{-t}\omega)u(-t) = u_0 \\ \text{for all } t \geq 0 \text{ and } \|u(t)\|_{L^\infty(\mathcal{O})} \to 0 \text{ for } t \to -\infty\}. \end{aligned}$$

Since $\mathcal{A}(\omega)$ attracts all L^∞-bounded sets we have

$$\mathcal{M}^+(0, \omega) \subseteq \mathcal{A}(\omega), \quad \forall \omega \in \Omega.$$

Therefore, it is sufficient to derive a lower bound on the Kolmogorov ε-entropy for the unstable manifold of 0.

In order to construct an element $u_0 \in \mathcal{M}^+(0,\omega)$ we need to find a function $u:(-\infty,0]\to X$ converging to 0 for $t\to-\infty$ such that

$$u_0=\varphi(t;\theta_{-t}\omega)u(-t)=X(0,-t;\omega)u(-t)=Y(0,-t;\omega)\left(e^{\mu_{-t}(\omega)+\lambda t}u(-t)\right),\ \forall t\ge 0,$$

where we used (4.2). By defining $u(-t):=e^{-\mu_{-t}(\omega)-\lambda t}v(-t)$, due to (S3) it is enough to find an L^∞-bounded function $v:(-\infty,0]\to X$ such that

$$u_0=Y(0,-t;\omega)v(-t),\quad \forall t\ge 0. \tag{4.3}$$

We note that (3.1) in case of (4.1) reads

$$\begin{aligned}&\partial_t Y(t,s;\omega)x=e^{\mu_t(\omega)-\lambda t}\Delta\Phi(e^{-\mu_t(\omega)+\lambda t}Y(t,s;\omega)x),\\ &Y(s,s;\omega)x=x,\end{aligned} \tag{4.4}$$

for a.e. $t\ge s$. For $x\in L^\infty(\mathcal{O})$ let $Y(t,s;\omega)x\in C((s,\infty)\times\mathcal{O})$ denote the corresponding essentially bounded solution to (4.4) given by Proposition A.5.

In order to find a function v satisfying (4.3), we use a time scaling to transform (4.4) from the infinite time interval $(-\infty,0]$ into a PDE on a finite time interval. Let $\delta>0$ small enough such that $(m-1)\lambda-\delta>0$ and set $\eta:=\frac{(m-1)\lambda-\delta}{m+1}$. Then (4.4) may be rewritten as

$$\partial_t Y(t,s;\omega)x=e^{\delta t}e^{\mu_t+\eta t}\Delta\Phi(e^{-\mu_t+\eta t}Y(t,s;\omega)x).$$

We define $T=\frac{1}{\delta}$ and

$$\begin{aligned}F(t)&:=\frac{e^{\delta t}}{\delta}:(-\infty,0]\mapsto(0,T],\\ G(t)&=F^{-1}(t)=\frac{\log(\delta t)}{\delta}:(0,T]\mapsto(-\infty,0].\end{aligned}$$

We note $G\in C^1(0,T]$ with $G'(t)>0$, $G(T)=0$ and $G(t)\to-\infty$ for $t\to 0$. Let $U(t,s;\omega)x:=Y(G(t),G(s);\omega)x$ for $t\ge s,\ t,s\in(0,T]$. Then $U(\cdot,s;\omega)x$ is a solution to

$$\begin{aligned}&\partial_t U(t,s;\omega)x=e^{\mu_{G(t)}+\eta G(t)}\Delta\Phi(e^{-\mu_{G(t)}+\eta G(t)}U(t,s;\omega)x),\ \text{on } [s,\infty)\times\mathcal{O}\\ &U(s,s;\omega)x=x.\end{aligned} \tag{4.5}$$

The rigorous proof of this transformation proceeds by considering a non-degenerate approximation $\Phi^{(\delta)}(r):=\Phi(r)+\delta r$ and smoothed coefficients $\mu^{(\delta)}$. In this case the transformation is a direct consequence of the classical chain-rule. One may then use local equicontinuity and uniform boundedness of the approximating solutions $Y^{(\delta)}$ to pass to the limit (cf. also Proposition A.5).

Thus, we can solve (4.5) on each interval $[\tau,T]$ with $\tau>0$. In order to construct the required function $v:(-\infty,0]\to X$ we aim to solve (4.5) on the whole interval $[0,T]$. Let $\rho_1(t):=e^{\mu_{G(t)}+\eta G(t)}$, $\rho_2(t):=e^{-\mu_{G(t)}+\eta G(t)}$. Due to condition (S3), for each $\varepsilon>0$ there is a $t_0(\varepsilon)<0$ small enough, such that

$$\|\mu_{G(t)}\|_{C^n(\mathcal{O})}\le\varepsilon\left(\sum_{k=1}^N\|f_k\|_{C^n(\mathcal{O})}\right)|G(t)|,\quad \forall t\le t_0(\varepsilon),\ n\in\mathbb{N}.$$

Choosing $\varepsilon>0$ small enough we thus obtain

$$\|\rho_1(t)\|_{C^n(\mathcal{O})}\le e^{\|\mu_{G(t)}\|_{C^0(\mathcal{O})}+\eta G(t)}P(\|\mu_{G(t)}+\eta G(t)\|_{C^n(\mathcal{O})})\to 0,\quad \text{for } t\to 0,$$

for some polynomial P. Similarly,

$$\begin{aligned}\frac{|\partial_{\xi_{i_1},\dots,\xi_{i_n}}\rho_1(t)|^2}{\rho_1(t)}&\le\frac{\rho_1(t)^2P(\|\mu_{G(t)}+\eta G(t)\|_{C^n(\mathcal{O})})}{\rho_1(t)}\\ &\le e^{\|\mu_{G(t)}\|_{C^0(\mathcal{O})}+\eta G(t)}P(\|\mu_{G(t)}+\eta G(t)\|_{C^n(\mathcal{O})})\to 0,\quad \text{for } t\to 0,\end{aligned}$$

for all $i_1, ..., i_n \in \{1, ..., d\}$. The same reasoning applies for ρ_2. In particular, $\rho_1, \rho_2 \in C^{0,n}(\bar{\mathcal{O}}_T)$ for all $n \in \mathbb{N}$. Hence, (4.5) is of the form (A.1) and Proposition A.5 implies the existence of a solution

$$U(\cdot, 0; \omega)x \in L^\infty([0,T] \times \mathcal{O}) \cap C((0,T] \times \mathcal{O})$$

with homogeneous Dirichlet boundary conditions for each initial condition $x \in L^\infty(\mathcal{O})$.

Reversing the time transformation we define

$$v(t) := U(F(t), 0; \omega)x, \quad t \in (-\infty, 0].$$

Uniqueness of essentially bounded solutions to (4.5) (Theorem A.6) implies

$$U(t,s;\omega)x = U(t,r;\omega)U(r,s;\omega)x, \quad \forall 0 \le s \le r \le t \le T.$$

Hence,

$$\begin{aligned} v(0) &= U(F(0),0;\omega)x = U(F(0),F(s);\omega)U(F(s),0;\omega)x \\ &= U(F(0),F(s);\omega)v(s) = Y(0,s;\omega)v(s), \end{aligned}$$

for all $s < 0$. Consequently, $v(0) \in \mathcal{M}^+(0,\omega)$ for each $x \in L^\infty(\mathcal{O})$.

In order to use this construction of elements $v(0) \in \mathcal{M}^+(0,\omega)$ to derive a lower bound on the Kolmogorov ε-entropy of $\mathcal{M}^+(0,\omega)$ we consider solutions to (4.5) so that the final values $v(0) = U(F(0),0;\omega)x$ are sufficiently far apart (w.r.t. the L^1-norm): For $\varepsilon > 0$ small enough we can find a finite set $R_\varepsilon = \{\xi_i\} \subseteq \mathcal{O}$ such that

$$\begin{aligned} B(\varepsilon,\xi_i) \cap B(\varepsilon,\xi_j) &= \emptyset, \quad \text{for } i \neq j, \\ |R_\varepsilon| &\ge C\varepsilon^{-d}, \\ \bar{B}(\varepsilon,\xi_i) &\subset \mathcal{O}, \quad \forall i. \end{aligned}$$

Let $x_0^i := M\mathbb{1}_{B(\frac{\varepsilon}{2},\xi_i)}$ and $M = (m\varepsilon)^{\frac{2}{m-1}}$, where $m > 0$ will be specified below. By Proposition A.5,

$$H^i := \|U^i(\cdot,0;\omega)x\|_{L^\infty(\mathcal{O}_T)} \le C\|x_0^i\|_{L^\infty(\mathcal{O})} \le C(m\varepsilon)^{\frac{2}{m-1}}.$$

Thus, the bound on the rate of expansion of the support of U^i given in Theorem A.9 becomes

$$\operatorname{supp}(U_t^i) \subseteq B_{C\varepsilon m\sqrt{C_t}\sqrt{t}}(\operatorname{supp}(x_0^i)) \subseteq B_{C\varepsilon m\sqrt{C_T}\sqrt{T}+\frac{\varepsilon}{2}}(\xi_i), \quad \forall t \in [0,T],$$

where $t \mapsto C_t$ is a continuous function. Thus, choosing m small enough yields

$$\operatorname{supp}(U_t^i) \subseteq B_\varepsilon(\xi_i), \quad \forall t \in [0,T].$$

Hence, U^i, U^j have disjoint support on $[0,T]$. Therefore, also

$$U^m(t,\xi) = \sum_{i=1}^{|R_\varepsilon|} m_i U^i(t,\xi),$$

for each $m \in \{0,1\}^{|R_\varepsilon|}$ is a solution to (4.5) with homogeneous Dirichlet boundary conditions. For $m^1 \neq m^2$ let i such that $m_i^1 \neq m_i^2$. By Proposition A.7 we observe

$$\|U^{m^1}(T) - U^{m^2}(T)\|_{L^1(\mathcal{O})} \ge \|U^i(T)\|_{L^1(\mathcal{O})} \ge e^{-CT}\|U^i(0)\|_{L^1(\mathcal{O})} \ge Ce^{-CT}\varepsilon^{\frac{2}{m-1}+d}.$$

Hence,

$$\mathbb{H}_\delta(\mathcal{A}(\omega)) \ge \mathbb{H}_\delta(\mathcal{M}^+(0,\omega)) \ge \log_2 2^{|R_\delta|} \ge C(\omega)\delta^{-\frac{d(m-1)}{2+d(m-1)}}$$

and

$$d_f(\mathcal{A}(\omega)) \ge d_f(\mathcal{M}^+(0,\omega)) = \limsup_{\delta \to 0} \frac{\mathbb{H}_\delta(\mathcal{M}^+(0,\omega))}{log_2(\frac{1}{\delta})} = \infty.$$

$\square$

Appendix A. Finite speed of propagation for more general perturbations

In Section 3.1 we proved finite speed of propagation for (1.1) via the transformed equation (3.1). The precise structure of the spatially dependent perturbing factors $e^{\mu}, e^{-\mu}$ has been used to provide explicit and locally optimal bounds on the rate of hole-filling. By disregarding the optimality of the estimates, more general perturbations may be allowed. Such an extension of the results of Section 3.1 is required in Section 4 in order to prove lower bounds for the Kolmogorov ε-entropy of the random attractor. In this section we provide some details on the proof of finite speed of propagation for more general perturbing factors. We consider the homogeneous Dirichlet problem for

$$\begin{aligned} \partial_t Y_t &= \rho_1 \Delta \Phi(\rho_2 Y_t), \text{ on } \mathcal{O}_T \\ Y(0) &= Y_0, \text{ on } \mathcal{O}, \end{aligned} \tag{A.1}$$

where $\rho_1, \rho_2 \in C^{0,2}(\bar{\mathcal{O}}_T)$ are non-negative. Solutions to (A.1) are defined similar to Definition 3.1.

Definition A.1. *Let $Y_0 \in L^\infty(\mathcal{O})$ and $\Phi(g) \in L^1(\Sigma_T)$. A function $Y \in C([0,T]; L^1(\mathcal{O}))$ with $\Phi(\rho_2 Y) \in L^1(\mathcal{O}_T) \cap L^2_{loc}((0,T]; H^1(\mathcal{O}))$ is said to be a solution to the (inhomogeneous) Dirichlet problem to* (A.1) *if*

$$\begin{aligned} \int_{\mathcal{O}_T} Y \partial_r \eta \, d\xi dr + \int_{\mathcal{O}} Y_0 \eta_0 \, d\xi = &- \int_{\mathcal{O}_T} \Phi(\rho_2 Y) \Delta(\rho_1 \eta) \, d\xi dr \\ &+ \int_{\Sigma_T} \Phi(\rho_2 g) \partial_\nu (\rho_1 \eta) d\vartheta dr, \end{aligned} \tag{A.2}$$

for all $\eta \in C^{1,2}(\bar{\mathcal{O}}_T)$ with $\eta_{|\mathcal{P}_T} = 0$.

Sub/supersolutions to (A.1) *are defined by replacing "$=$" by "$\geq$" ("$\leq$" resp.) in* (A.2) *and requiring the testfunctions η to be non-negative.*

A.1. **Existence of solutions to** (A.1)**.** Let $Y_0 \in L^\infty(\mathcal{O})$. We will only require the existence of solutions to (A.1) with homogeneous Dirichlet boundary conditions (i.e. $g \equiv 0$) and for ρ_1, ρ_2 satisfying one of the following conditions

- (A1) ρ_1, ρ_2 are strictly positive on $[0,T] \times \bar{\mathcal{O}}$,
- (A2) ρ_1, ρ_2 are strictly positive on $(0,T] \times \bar{\mathcal{O}}$ and $\|\rho_2(t)\|_{C^2(\mathcal{O})} \to 0$ for $t \to 0$.

The construction of solutions for (A.1) relies on a smooth, non-degenerate approximation of Φ. I.e. for $\delta > 0$ let

$$\Phi^{(\delta)}(r) := \Phi(r) + \delta r,$$

$\rho_1^{(\delta)}, \rho_2^{(\delta)} \in C^\infty(\mathcal{O}_T)$ be smooth, non-degenerate approximations of ρ_1, ρ_2 in $C^{0,2}(\bar{\mathcal{O}}_T)$ and let $Y_0^{(\delta)} \in C^\infty(\mathcal{O})$ be smooth approximations of Y_0 in $L^\infty(\mathcal{O})$. We consider the approximating problems

$$\begin{aligned} \partial_t Y_t^{(\delta)} &= \rho_1^{(\delta)} \Delta \left(\Phi(\rho_2^{(\delta)}) \Phi^{(\delta)}(Y_t^{(\delta)}) \right), \text{ on } \mathcal{O}_T \\ Y^{(\delta)}(0) &= Y_0^{(\delta)}, \text{ on } \mathcal{O}, \end{aligned} \tag{A.3}$$

with homogeneous Dirichlet boundary conditions. Since (A.3) is a non-degenerate, quasilinear PDE with smooth coefficients, standard results imply the unique existence of a classical solution $Y^{(\delta)}$ (cf. e.g. [33]).

The main ingredient of the construction of solutions to (A.1) is the following a-priori L^∞ bound

Lemma A.2. *Let $M := \|Y_0\|_{L^\infty(\mathcal{O})} < \infty$ and assume (A1) or (A2). Then, there are constants $C, \delta_0 = \delta_0(M) > 0$ such that*

$$\sup_{\delta\in[0,\delta_0]} \|Y^{(\delta)}\|_{C^0([0,T]\times\overline{\mathcal{O}})} \le C\|Y_0\|_{L^\infty(\mathcal{O})} < \infty.$$

Proof. Case (A1): The proof relies on a combination of explicit supersolutions to (1.5) with an interval splitting technique as it has been used in [10, 25].

In the following let $\varphi \in C^2(\mathcal{O})$ be the solution to

$$\begin{aligned} \Delta\varphi &= -1, \quad \text{on } \mathcal{O} \\ \varphi &= 1, \quad \text{on } \partial\mathcal{O}. \end{aligned}$$

By the maximum principle we have $\varphi \ge 1$.

Since $\{\rho_2^{(\delta)}\}_{\delta\in[0,1]}$ is a compact set in $C^{0,2}(\mathcal{O}_T)$ and may be chosen such that

$$\inf_{\delta\in[0,1],\ (t,\xi)\in[0,T]\times\mathcal{O}} \rho_2^{(\delta)}(t,\xi) > 0,$$

we have

$$\eta_i^{(\delta)} := \Phi\left(\frac{\rho_2^{(\delta)}}{\rho_2^{(\delta)}(\tau_i)}\right) \in C^{0,2}(\mathcal{O}_T)$$

with $\eta_i^{(\delta)}(t) \to 1$ in $C^2(\mathcal{O})$ for $t \to \tau_i$ uniformly in $\delta \in [0,1]$ and $\tau_i \in [0,T]$. Hence,

$$\Delta(\varphi\eta_i^{(\delta)}) = -\eta_i^{(\delta)} + 2\nabla\varphi\cdot\nabla\eta_i^{(\delta)} + \varphi\Delta\eta_i^{(\delta)} \le -\frac{1}{2}, \quad \forall\xi\in\mathcal{O},\ \delta\in[0,1]$$

and all $|t-\tau_i|$ small enough. We can thus choose a finite partition $0 = \tau_0 < \tau_1 < \tau_2 < ... < \tau_N = T$ of $[0,T]$ such that

$$\sup_{\delta\in[0,1]} \Delta\left(\varphi\Phi\left(\frac{\rho_2^{(\delta)}}{\rho_2^{(\delta)}(\tau_i)}\right)\right) \le -\frac{1}{2}, \quad \text{on } [\tau_i,\tau_{i+1}]\times\mathcal{O},$$

for all $i = 0, ..., N-1$.

We will prove the bound iteratively over $i = 0, ..., N-1$. Suppose the bound has been shown on $[0,\tau_i]$ for some $i \ge 0$ and let $\|Y_{\tau_i}\|_{L^\infty(\mathcal{O})} \le C_i M$. Choosing

$$K^{(i)}(t,\xi) := \varphi(\xi)^{\frac{1}{m}} \frac{\|\rho_2^{(\delta)}\|_{C^{0,2}(\mathcal{O}_T)}}{\rho_2^{(\delta)}(\tau_i,\xi)} C_i M \in C^2(\mathcal{O}_T),$$

we have $K^{(i)}(\tau_i,\xi) \ge \|Y_{\tau_i}\|_{L^\infty(\mathcal{O})}$, $\partial_t K^{(i)} = 0$ and

$$\begin{aligned} &\rho_1^{(\delta)}\Delta\left(\Phi(\rho_2^{(\delta)})\Phi^{(\delta)}(K^{(i)})\right) \\ &= \rho_1^{(\delta)}\Delta\left(\Phi(\rho_2^{(\delta)}K^{(i)})\right) + \delta\rho_1^{(\delta)}\Delta\left(\Phi(\rho_2^{(\delta)})K^{(i)}\right) \\ &\le \|\rho_2^{(\delta)}\|_{C^{0,2}(\mathcal{O}_T)} C_i M \rho_1^{(\delta)}\left(-\frac{1}{2}\|\rho_2^{(\delta)}\|_{C^{0,2}(\mathcal{O}_T)}^{m-1}(C_iM)^{m-1} + \delta\Delta\left(\frac{\Phi(\rho_2^{(\delta)})\varphi^{\frac{1}{m}}}{\rho_2^{(\delta)}(\tau_i)}\right)\right) \\ &\le 0, \end{aligned}$$

by the choice of the partition $\{\tau_i\}_{i=0,...,N}$, for all $\delta \le \delta_0(M)$ small enough.

Consequently, $K^{(i)}$ is a supersolution to (A.3) on $[\tau_i,\tau_{i+1}]\times\mathcal{O}$ and the upper bound follows since $K^{(i)}(t,\xi) \le C_{i+1}M$, with C_{i+1} depending on the data only. The derivation of the lower bound proceeds analogously.

Case (A2): We only need to prove the claim on some small interval $[0,\tau_1]$ with $\tau_1 > 0$, since case (A1) may be applied on $[\tau_1,T]$ subsequently. Let $K^{(0)}(t,\xi) := M + t$. Then $\partial_t K = 1$ and

$$\rho_1^{(\delta)}\Delta\left(\Phi(\rho_2^{(\delta)})\Phi^{(\delta)}(K)\right) = \rho_1^{(\delta)}\Delta\left(\Phi(\rho_2^{(\delta)})\right)\Phi^{(\delta)}(K) \le 1,$$

on $[0,\tau_1]$ for $\tau_1 \in (0,M)$ and $\delta > 0$ small enough. Hence, $K^{(0)}$ is a supersolution to (A.3) on $[0,\tau_1]\times\mathcal{O}$ and

$$Y^{(\delta)} \le K^{(0)} \le CM, \quad \text{ on } [0,\tau_1]\times\mathcal{O}.$$

The lower bound may be derived analogously. □

Lemma A.3. *Let $M := \|Y_0\|_{L^\infty(\mathcal{O})} < \infty$ and assume (A1). Then, there are constants $C, \delta_0 = \delta_0(M) > 0$ such that*

$$\sup_{t\in[0,T]} \|Y_t^{(\delta)}\|_{m+1}^{m+1} + \|\nabla\left(\Phi(\rho_2^{(\delta)})\Phi^\delta(Y^{(\delta)})\right)\|_{L^2(\mathcal{O}_T)} \le C < \infty.$$

Proof. Let $\Psi^{(\delta)} \in C^1(\mathbb{R})$ so that $\dot\Psi^{(\delta)} = \Phi^{(\delta)}$. We compute

$$\begin{aligned}
\partial_t \int_{\mathcal{O}} \Psi^{(\delta)}(Y_t^{(\delta)})\,d\xi &= \int_{\mathcal{O}} \Phi^{(\delta)}(Y_t^{(\delta)})\rho_1^{(\delta)}\Delta\left(\Phi(\rho_2^{(\delta)})\Phi^{(\delta)}(Y_t^{(\delta)})\right)\,d\xi\\
&= \int_{\mathcal{O}} \frac{\rho_1^{(\delta)}}{\Phi(\rho_2^{(\delta)})}\Phi(\rho_2^{(\delta)})\Phi^{(\delta)}(Y_t^{(\delta)})\Delta\left(\Phi(\rho_2^{(\delta)})\Phi^{(\delta)}(Y_t^{(\delta)})\right)\,d\xi\\
&= -\int_{\mathcal{O}} \frac{\rho_1^{(\delta)}}{\Phi(\rho_2^{(\delta)})}\left|\nabla\Phi(\rho_2^{(\delta)})\Phi^{(\delta)}(Y_t^{(\delta)})\right|^2\,d\xi\\
&\quad -\int_{\mathcal{O}} \nabla\left(\frac{\rho_1^{(\delta)}}{\Phi(\rho_2^{(\delta)})}\right)\Phi(\rho_2^{(\delta)})\Phi^{(\delta)}(Y_t^{(\delta)})\nabla\left(\Phi(\rho_2^{(\delta)})\Phi^{(\delta)}(Y_t^{(\delta)})\right)\,d\xi\\
&\le -c\int_{\mathcal{O}} \left|\nabla\Phi(\rho_2^{(\delta)})\Phi^{(\delta)}(Y_t^{(\delta)})\right|^2\,d\xi + C\int_{\mathcal{O}} |\Phi^{(\delta)}(Y_t^{(\delta)})|^2\,d\xi,
\end{aligned}$$

which yields the claim by Lemma A.2. □

Lemma A.4. *Let $M := \|Y_0^{(1)}\|_{L^\infty(\mathcal{O})} + \|Y_0^{(2)}\|_{L^\infty(\mathcal{O})} < \infty$ and assume (A1). Then, there is a constant C such that*

$$\sup_{t\in[0,T]} \|Y_t^{(1,\delta)} - Y_t^{(2,\delta)}\|_{L^1(\mathcal{O})} \le C\|Y_0^{(1,\delta)} - Y_0^{(2,\delta)}\|_{L^1(\mathcal{O})},$$

for all $\delta > 0$ small enough.

Proof. Case (A1): Since $\rho_1 \in C^2(\bar{\mathcal{O}})$ and using (A1) we may choose a $\tau_0 > 0$, independent of t_0 such that

$$\left|\left(\Phi^{(\delta)}\right)'(M)\Phi(\rho_2^{(\delta)})\Delta\left(\frac{\rho_1^{(\delta)}}{\rho_1^{(\delta)}(t_0)}\right)\right| \le 1, \quad \text{ on } [t_0, t_0+\tau_0]\times\bar{\mathcal{O}}.$$

Assume that the claim has been shown on $[0,t_0]$ and let

$$w^{(\delta)} := \Phi(\rho_2^{(\delta)})\left(\Phi^{(\delta)}(Y^{(1,\delta)}) - \Phi^{(\delta)}(Y^{(2,\delta)})\right).$$

Then (informally)

$$\begin{aligned}
&\partial_t \int_{\mathcal{O}} \left|Y_t^{(1,\delta)} - Y_t^{(2,\delta)}\right|\left(\frac{1}{\rho_1^{(\delta)}(t_0)}\right)d\xi\\
&= \int_{\mathcal{O}} \operatorname{sgn}(w^{(\delta)})\left(\frac{\rho_1^{(\delta)}}{\rho_1^{(\delta)}(t_0)}\right)\Delta w^{(\delta)} d\xi\\
&\le \left(\Phi^{(\delta)}\right)'(M)\left\|\Phi(\rho_2^{(\delta)})\Delta\left(\frac{\rho_1^{(\delta)}}{\rho_1^{(\delta)}(t_0)}\right)\right\|_{C^0(\mathcal{O})}\int_{\mathcal{O}} \left|Y_t^{(1,\delta)} - Y_t^{(2,\delta)}\right| d\xi\\
&\le \int_{\mathcal{O}} \left|Y_t^{(1,\delta)} - Y_t^{(2,\delta)}\right| d\xi, \quad \forall t \in [t_0, t_0+\tau_0].
\end{aligned}$$

The informal derivation given above may be made rigorous by considering a smooth approximation of $|\cdot|$ (for details we refer to [25]). Iteration then implies the claim.

Case (A2): As for case (A1), (informally)

$$\begin{aligned}\partial_t \int_{\mathcal{O}} \left|Y_t^{(1,\delta)} - Y_t^{(2,\delta)}\right| d\xi &= \int_{\mathcal{O}} \operatorname{sgn}(w^{(\delta)})\rho_1^{(\delta)}\Delta w^{(\delta)} d\xi \\ &\le \left(\Phi^{(\delta)}\right)'(M)\|\Phi(\rho_2^{(\delta)})\Delta\rho_1^{(\delta)}\|_{C^0(\mathcal{O})} \int_{\mathcal{O}} \left|Y_t^{(1,\delta)} - Y_t^{(2,\delta)}\right| d\xi \\ &\le \int_{\mathcal{O}} \left|Y_t^{(1,\delta)} - Y_t^{(2,\delta)}\right| d\xi, \quad \forall t \in [0,\tau],\end{aligned}$$

for all $\tau \le \tau_0(M)$ and $\delta > 0$ small enough. □

Proposition A.5 (Existence of solutions to (A.1)). *Let $Y_0 \in L^\infty(\mathcal{O})$ and assume (A1) or (A2). Then, there exists a solution $Y \in C((0,T]\times\mathcal{O})$ to* (A.1) *with Dirichlet boundary conditions satisfying $\Phi(\rho_2 Y) \in L^2_{loc}((0,T];H^1_0(\mathcal{O}))$ and*

$$\|Y\|_{L^\infty(\mathcal{O}_T)} \le C\|Y_0\|_{L^\infty(\mathcal{O})},$$

for some constant $C > 0$. For each two initial conditions $Y_0^{(1)}, Y_0^{(2)} \in L^\infty(\mathcal{O})$ the constructed solutions satisfy

$$\sup_{t\in[0,T]} \|Y_t^1 - Y_t^2\|_{L^1(\mathcal{O})} \le C\|Y_0^1 - Y_0^2\|_{L^1(\mathcal{O})}.$$

If (A1) is satisfied, then $\Phi(\rho_2 Y) \in L^2([0,T];H^1_0(\mathcal{O}))$.

Proof. Based on the uniform L^∞ estimate for the approximating solutions $Y^{(\delta)}$ derived in Lemma A.2, we obtain local equicontinuity of $Y^{(\delta)}$ in $\mathcal{O}$ by [20] (cf. also [25, Theorem 1.12] for details). I.e. $Y^{(\delta)} \in C(K)$ for each compact set $K \subseteq (0,T]\times\mathcal{O}$ with modulus of continuity independent of $\delta > 0$.

By a diagonal argument it follows that there exists a $Y \in C((0,T]\times\mathcal{O})$ with $\|Y\|_{L^\infty(\mathcal{O}_T)} \le C\|Y_0\|_{L^\infty(\mathcal{O})}$ such that $Y^\delta \to Y$ (passing to a subsequence if necessary) locally uniformly on $\mathcal{O}_T$. Lemma A.4 implies the L^1-stability.

On each strictly positive interval $[\tau,T]$ with $\tau > 0$ we may apply Lemma A.3 to conclude $\Phi(\rho_2 Y) \in L^2([\tau,T];H^1_0(\mathcal{O}))$ (if (A1) is satisfied we may choose $\tau = 0$).

By dominated convergence we obtain that Y is a solution to (A.1).

Assume now $Y_0 \in C^0(\mathcal{O})$. By [20] with details given in [25, Theorem 1.12] we have $Y \in C([0,T];L^1(\mathcal{O}))$. The L^1-stability then implies that this remains true for all $Y_0 \in L^\infty(\mathcal{O})$. □

A.2. **Comparison and uniqueness for** (A.1)**.** We now prove a comparison result for (A.1). In particular, this implies Theorem 3.4 since sub/supersolutions to (1.1) are defined in terms of solutions to (3.1) and thus it is enough to prove the comparison result for (3.1). We will assume either of

(A1') ρ_1 is strictly positive on $[0,T]\times\bar{\mathcal{O}}$,

(A2') ρ_1 is strictly positive on $(0,T]\times\bar{\mathcal{O}}$ and

$$\left\|\frac{|\nabla\rho_1(t)|^2}{\rho_1(t)}\right\|_{C^0(\mathcal{O})} + \left\|\frac{|\Delta\rho_1(t)|^2}{\rho_1(t)}\right\|_{C^0(\mathcal{O})} \to 0, \text{ for } t \to 0.$$

Theorem A.6 (Comparison). *Let $Y^{(1)}, Y^{(2)}$ be essentially bounded sub/supersolutions to* (A.1) *with initial conditions $Y_0^{(1)} \le Y_0^{(2)}$ and boundary data $g^{(1)} \le g^{(2)}$ a.e. in $\mathcal{O}$ respectively. Assume either (A1') or (A2'). Then,*

$$Y^{(1)} \le Y^{(2)}, \quad \text{a.e. in } \mathcal{O}.$$

In particular, essentially bounded solutions are unique.

Proof. The proof proceeds similar to [25, Theorem 1.3]. For the readers convenience we recall the proof. Let $Y^{(1)}$, $Y^{(2)}$ be as in the statement, $Y := Y^{(1)} - Y^{(2)}$ and $g := g^{(1)} - g^{(2)}$. Then

$$\begin{aligned}&\int_{\mathcal{O}_T} Y \partial_r \eta \; d\xi dr \\ &\geq -\int_{\mathcal{O}} (Y_0^{(1)} - Y_0^{(2)}) \eta_0 d\xi - \int_{\mathcal{O}_T} \left(\Phi(\rho_2 Y^{(1)}) - \Phi(\rho_2 Y^{(2)}) \right) \Delta(\rho_1 \eta) \; d\xi dr \\ &\quad + \int_{\Sigma_T} (\Phi(\rho_2 g^{(1)}) - \Phi(\rho_2 g^{(2)})) \partial_\nu (\rho_1 \eta) d\vartheta dr \\ &\geq -\int_{\mathcal{O}} Y_0 \eta_0 d\xi - \int_{\mathcal{O}_T} a Y \Delta(\rho_1 \eta) \; d\xi dr + \int_{\Sigma_T} (\Phi(\rho_2 g^{(1)}) - \Phi(\rho_2 g^{(2)})) \partial_\nu (\rho_1 \eta) d\vartheta dr,\end{aligned}$$

for all non-negative $\eta \in C^{1,2}(\bar{\mathcal{O}}_T)$ with $\eta = 0$ on $\mathcal{P}_T$, where

$$a_t := \begin{cases} \frac{\Phi(\rho_2(t) Y_t^{(1)}) - \Phi(\rho_2(t) Y_t^{(2)})}{Y_t^{(1)} - Y_t^{(2)}} & , \text{ for } Y_t^{(1)} \neq Y_t^{(2)} \\ 0 & , \text{ otherwise.} \end{cases}$$

Case (A1'): Let $\rho_1^{(\varepsilon)} \in C^\infty(\mathcal{O}_T)$ be a smooth approximation of ρ_1 in $C^{0,2}(\mathcal{O}_T)$, such that $\|\rho_1^{(\varepsilon)} - \rho_1\|_{C^{0,2}(\mathcal{O}_T)} \leq \varepsilon^2$. By equicontinuity of $t \mapsto \rho_1^{(\varepsilon)}(t)$ in $C^2(\mathcal{O})$ we can choose a partition $0 = \tau_0 < ... < \tau_N = T$ such that

(A.4)

$$\begin{aligned}&C_1 \|\rho_1^{(\varepsilon)}(\tau_i)\|_{C^0(\mathcal{O})} \left(\left\| \nabla \left(\frac{\rho_1^{(\varepsilon)}}{\rho_1^{(\varepsilon)}(\tau_i)} \right) \right\|^2_{C^0([\tau_i, \tau_{i+1}] \times \mathcal{O})} + \left\| \Delta \left(\frac{\rho_1^{(\varepsilon)}}{\rho_1^{(\varepsilon)}(\tau_i)} \right) \right\|^2_{C^0([\tau_i, \tau_{i+1}] \times \mathcal{O})} \right) \\ &\leq \frac{c}{4}, \quad \forall i = 0, ..., N-1, \; \varepsilon > 0,\end{aligned}$$

where $c, C_1 > 0$ are constants that will be specified below (depending on $\|a\|_{L^\infty(\mathcal{O}_T)}$ only). Let $\gamma := \max_{i=0,...,N-1} |\tau_{i+1} - \tau_i|$.

We prove $Y \leq 0$ a.e. via induction over $i = 0, ..., N-1$. Thus, assume $Y \leq 0$ on $[0, \tau_i] \times \mathcal{O}$ almost everywhere. We can modify τ_i so that (A.4) is preserved and $Y(\tau_i) \leq 0$ a.e. in $\mathcal{O}$. Define $\mathcal{O}_i := [\tau_i, \tau_{i+1}] \times \mathcal{O}$, $\Sigma_i = [\tau_i, \tau_{i+1}] \times \partial\mathcal{O}$, $\mathcal{P}_i = \Sigma_i \cup (\{T\} \times \mathcal{O})$. Then

$$\begin{aligned}\int_{\mathcal{O}_i} Y \big(\partial_r \eta + a \Delta(\rho_1 \eta) \big) \; d\xi dr \geq &-\int_{\mathcal{O}} Y_{\tau_i} \eta_{\tau_i} d\xi \\ &+ \int_{\Sigma_i} (\Phi(\rho_2 g^{(1)}) - \Phi(\rho_2 g^{(2)})) \partial_\nu (\rho_1 \eta) d\vartheta dr,\end{aligned}$$

for all non-negative $\eta \in C^{1,2}([\tau_i, \tau_{i+1}] \times \bar{\mathcal{O}})$ with $\eta = 0$ on $\mathcal{P}_i$. Since $\eta \geq 0$ on $\mathcal{O}_i$, we have $\partial_\nu(\rho_1 \eta) \leq 0$ on Σ_i and thus

$$-\int_{\mathcal{O}} Y_{\tau_i} \eta_{\tau_i} d\xi + \int_{\Sigma_i} (\Phi(\rho_2 g^{(1)}) - \Phi(\rho_2 g^{(2)})) \partial_\nu (\rho_1 \eta) d\vartheta dr \geq 0.$$

We conclude,

$$\int_{\mathcal{O}_i} Y \big(\partial_r \eta + a \Delta(\rho_1 \eta) \big) \; d\xi dr \geq 0,$$

for all non-negative $\eta \in C^{1,2}([\tau_i, \tau_{i+1}] \times \bar{\mathcal{O}})$ with $\eta = 0$ on $\mathcal{P}_i$.

For $Y_t^{(1)} \neq Y_t^{(2)}$ we have $a_t = \rho_2(t) \dot{\Phi}(\zeta_t)$ with $\zeta_t \in [\rho_2(t) Y_t^{(1)}, \rho_2(t) Y_t^{(2)}]$ and thus $\|a\|_{L^\infty(\mathcal{O}_T)} < \infty$ by essential boundedness of $Y^{(i)}$. We consider a non-degenerate, smooth approximation of a. Set $\hat{a}_\varepsilon := a \vee \varepsilon$ and let $a_{\varepsilon,\delta}$ be a smooth approximation of $\hat{a}_\varepsilon$ such that $a_{\varepsilon,\delta} \geq \varepsilon$ and $\int_{\mathcal{O}_T} |Y|^2 (\hat{a}_\varepsilon - a_{\varepsilon,\delta})^2 \; d\xi dr \leq \delta$. Then choose $a_\varepsilon = a_{\varepsilon,\varepsilon^2}$.

Let $\eta = \frac{\varphi}{\rho_1^{(\varepsilon)}(\tau_i)} \in C^{0,2}(\mathcal{O}_i)$ with φ being the classical solution to

$$\begin{aligned}\partial_t \varphi + a_\varepsilon \rho_1^{(\varepsilon)}(\tau_i)\Delta\left(\frac{\rho_1^{(\varepsilon)}}{\rho_1^{(\varepsilon)}(\tau_i)}\varphi\right) - \theta = 0, &\text{ on } \mathcal{O}_i \\ \varphi = 0, &\text{ on } [\tau_i, \tau_{i+1}] \times \partial\mathcal{O} \\ \varphi(\tau_{i+1}) = 0, &\text{ on } \mathcal{O},\end{aligned} \tag{A.5}$$

where θ is an arbitrary, non-positive, smooth testfunction and for simplicity of notation we suppress the ε-dependency of φ. Time inversion transforms (A.5) into a uniformly parabolic linear equation with smooth coefficients. Thus, unique existence of a non-negative classical solution follows from standard results (cf. e.g. [33]).

Consequently,

$$\begin{aligned}0 &\le \int_{\mathcal{O}_i} Y\big(\partial_r \eta + a\Delta(\rho_1\eta)\big)\, d\xi dr \\ &= \int_{\mathcal{O}_i} Y\big(\partial_r\eta + a_\varepsilon\Delta(\rho_1^{(\varepsilon)}\eta)\big)\, d\xi dr + \int_{\mathcal{O}_i} Y(a-a_\varepsilon)\Delta(\rho_1^{(\varepsilon)}\eta)\, d\xi dr \\ &\quad + \int_{\mathcal{O}_i} Ya\Delta((\rho_1 - \rho_1^{(\varepsilon)})\eta)\, d\xi dr \\ &= \int_{\mathcal{O}_i} \frac{1}{\rho_1^{(\varepsilon)}(\tau_i)} Y\theta\, d\xi dr + \int_{\mathcal{O}_i} Y(a-a_\varepsilon)\Delta\left(\frac{\rho_1^{(\varepsilon)}}{\rho_1^{(\varepsilon)}(\tau_i)}\varphi\right) d\xi dr \\ &\quad + \int_{\mathcal{O}_i} Ya\Delta\left(\frac{\rho_1 - \rho_1^{(\varepsilon)}}{\rho_1^{(\varepsilon)}(\tau_i)}\varphi\right) d\xi dr.\end{aligned} \tag{A.6}$$

We need to prove that the last two terms vanish for $\varepsilon \to 0$. We note

$$\begin{aligned}&\int_{\mathcal{O}_i} Y(a-a_\varepsilon)\Delta\left(\frac{\rho_1^{(\varepsilon)}}{\rho_1^{(\varepsilon)}(\tau_i)}\varphi\right) d\xi dr \\ &\quad\le C\left(\int_{\mathcal{O}_i} a_\varepsilon \Big|\Delta\left(\frac{\rho_1^{(\varepsilon)}}{\rho_1^{(\varepsilon)}(\tau_i)}\varphi\right)\Big|^2 d\xi dr\right)^{\frac12}\sqrt{\varepsilon} \\ &\quad\le C\left\|\frac{\rho_1^{(\varepsilon)}}{\rho_1^{(\varepsilon)}(\tau_i)}\right\|_{C^2(\mathcal{O}_i)}\left(\int_{\mathcal{O}_i} a_\varepsilon|\Delta\varphi|^2 + |\nabla\varphi|^2\, d\xi dr\right)^{\frac12}\sqrt{\varepsilon}\end{aligned} \tag{A.7}$$

and

$$\begin{aligned}\int_{\mathcal{O}_i} Ya\Delta\left(\frac{\rho_1 - \rho_1^{(\varepsilon)}}{\rho_1^{(\varepsilon)}(\tau_i)}\varphi\right) d\xi dr &\le C\left\|\frac{\rho_1 - \rho_1^{(\varepsilon)}}{\rho_1^{(\varepsilon)}(\tau_i)}\right\|_{H^2(\mathcal{O}_i)}\|\varphi\|_{H^2(\mathcal{O}_i)} \\ &\le C\varepsilon^2\|\varphi\|_{H^2(\mathcal{O}_i)}.\end{aligned} \tag{A.8}$$

Therefore, we first derive a bound for $\|\varphi\|_{H^2(\mathcal{O}_i)}$ with explicit control on the possible explosion for $\varepsilon \to 0$. Let $\zeta \in C^\infty(\mathbb{R})$ with $\zeta(\tau_i) = 0$, $\zeta \le 1$ on $[0,T]$ and $\dot\zeta \ge c > 0$, for some $c \le \frac{1}{4\gamma}$. Multiplying (A.5) by $\zeta\Delta\varphi$ and integrating yields

$$\begin{aligned}&\int_{\mathcal{O}_i} (\partial_r\varphi)\,\zeta\Delta\varphi\, d\xi dr \\ &\quad= \int_{\mathcal{O}_i}\left(-a_\varepsilon\rho_1^{(\varepsilon)}(\tau_i)\Delta\left(\frac{\rho_1^{(\varepsilon)}}{\rho_1^{(\varepsilon)}(\tau_i)}\varphi\right)\zeta\Delta\varphi + \theta\zeta\Delta\varphi\right) d\xi dr.\end{aligned} \tag{A.9}$$

We compute

$$
\begin{aligned}
&-\int_{\mathcal{O}_i} a_\varepsilon \rho_1^{(\varepsilon)}(\tau_i)\Delta\left(\frac{\rho_1^{(\varepsilon)}}{\rho_1^{(\varepsilon)}(\tau_i)}\varphi\right)\zeta\Delta\varphi\ d\xi dr \\
&= -\int_{\mathcal{O}_i} \zeta a_\varepsilon \rho_1^{(\varepsilon)}(\tau_i)\left(\frac{\rho_1^{(\varepsilon)}}{\rho_1^{(\varepsilon)}(\tau_i)}\right)|\Delta\varphi|^2\ d\xi dr \\
&\quad + \int_{\mathcal{O}_i} \zeta a_\varepsilon \rho_1^{(\varepsilon)}(\tau_i)\left(2\nabla\left(\frac{\rho_1^{(\varepsilon)}}{\rho_1^{(\varepsilon)}(\tau_i)}\right)\nabla\varphi + \varphi\Delta\left(\frac{\rho_1^{(\varepsilon)}}{\rho_1^{(\varepsilon)}(\tau_i)}\right)\right)\Delta\varphi\ d\xi dr \\
&\le -\frac{1}{4}\int_{\mathcal{O}_i} \zeta a_\varepsilon \rho_1^{(\varepsilon)}|\Delta\varphi|^2\ d\xi dr \\
&\quad + C_1\|\rho_1^{(\varepsilon)}(\tau_i)\|_{C^0(\mathcal{O})}\left\|\nabla\left(\frac{\rho_1^{(\varepsilon)}}{\rho_1^{(\varepsilon)}(\tau_i)}\right)\right\|^2_{C^0(\mathcal{O}_i)}\int_{\mathcal{O}_i}|\nabla\varphi|^2\ d\xi dr \\
&\quad + C_1\|\rho_1^{(\varepsilon)}(\tau_i)\|_{C^0(\mathcal{O})}\left\|\Delta\left(\frac{\rho_1^{(\varepsilon)}}{\rho_1^{(\varepsilon)}(\tau_i)}\right)\right\|^2_{C^0(\mathcal{O}_i)}\int_{\mathcal{O}_i}|\varphi|^2\ d\xi dr \\
&\le -\frac{1}{4}\int_{\mathcal{O}_i} \zeta a_\varepsilon \rho_1^{(\varepsilon)}(\tau_i)|\Delta\varphi|^2\ d\xi dr + \frac{c}{4}\int_{\mathcal{O}_i}|\nabla\varphi|^2\ d\xi dr,
\end{aligned}
$$

where we use (A.4). Using this in (A.9) together with the arbitrariness of ζ with the above properties, Fatou's Lemma and strict positivity of $\rho_1^{(\varepsilon)}(\tau_i)$ we deduce

$$
\frac{c}{2}\int_{\mathcal{O}_i}|\nabla\varphi|^2\ d\xi dr + \frac{1}{4}\int_{\mathcal{O}_i} a_\varepsilon|\Delta\varphi|^2\ d\xi dr \le C\int_{\mathcal{O}_i}|\nabla\theta|^2\ d\xi dr
$$

and $\|\varphi\|_{H^2(\mathcal{O}_i)} \le \frac{C}{\varepsilon}\int_{\mathcal{O}_i}|\nabla\theta|^2 d\xi dr$ due to $a_\varepsilon \ge \varepsilon$. For (A.7) this implies

$$
\int_{\mathcal{O}_i} Y(a-a_\varepsilon)\Delta\left(\frac{\rho_1^{(\varepsilon)}}{\rho_1^{(\varepsilon)}(\tau_i)}\varphi\right)\ d\xi dr \le C\|\theta\|^2_{H^1_0(\mathcal{O}_i)}\sqrt{\varepsilon}.
$$

for (A.8)

$$
\int_{\mathcal{O}_i} Ya\Delta\left(\frac{\rho_1-\rho_1^{(\varepsilon)}}{\rho_1^{(\varepsilon)}(\tau_i)}\varphi\right)\ d\xi dr \le C\varepsilon\|\theta\|^2_{H^1_0(\mathcal{O}_i)}.
$$

Taking $\varepsilon \to 0$ in in (A.6) thus yields

$$
0 \le \int_{\mathcal{O}_i}\frac{1}{\rho_1^{(\varepsilon)}(\tau_1)}Y\theta\ d\xi dr,
$$

for any non-positive, smooth testfunction θ. Thus $Y^{(1)} \le Y^{(2)}$ in $\mathcal{O}_i = [\tau_i,\tau_{i+1}]\times\mathcal{O}$ almost everywhere. Induction finishes the proof.

Case (A2'): It is sufficient to prove comparison for a short time-interval $[0,\tau_1]$ for some $\tau_1 > 0$, since case (A1') may be applied on $[\tau_1, T]$ subsequently. Let $0 = \tau_0 < \tau_1$. As for case (A1') we note

$$
\int_{\mathcal{O}_0} Y\big(\partial_r\eta + a\Delta(\rho_1\eta)\big)\ d\xi dr \ge 0,
$$

for all non-negative $\eta \in C^{1,2}([0,\tau_1]\times\bar{\mathcal{O}})$ with $\eta = 0$ on $\mathcal{P}_0$.

We follow the same idea of proof as in the case of (A1'). Hence, let $a^{(\varepsilon)}, \rho_1^{(\varepsilon)}$ be smooth approximations as before and φ be the classical solution to

$$\begin{aligned}\partial_t\varphi + a_\varepsilon\Delta(\rho_1^{(\varepsilon)}\varphi) - \theta = 0, &\text{ on } \mathcal{O}_0\\ \varphi = 0, &\text{ on } [0,\tau_1]\times\partial\mathcal{O}\\ \varphi(\tau_1) = 0, &\text{ on } \mathcal{O},\end{aligned}\tag{A.10}$$

where θ is an arbitrary, non-positive, smooth testfunction. As for (A.6) this yields

$$\begin{aligned}0 \le &\int_{\mathcal{O}_0} Y\theta\ d\xi dr + \int_{\mathcal{O}_0} Y(a-a_\varepsilon)\Delta(\rho_1^{(\varepsilon)}\varphi)\ d\xi dr\\ &+\int_{\mathcal{O}_0} Ya\Delta((\rho_1-\rho_1^{(\varepsilon)})\varphi)\ d\xi dr.\end{aligned}$$

We thus aim to show that the last two terms vanish for $\varepsilon\to 0$. Due to the degeneracy of $\rho_1(t)$ for $t\to 0$ care has be taken in establishing the required a-priori bound on φ. Multiplying (A.10) by $\zeta\Delta\varphi$ as before and noting

$$\Delta(\rho_1^{(\varepsilon)}\varphi) = \rho_1^{(\varepsilon)}\Delta\varphi + 2\nabla\rho_1^{(\varepsilon)}\cdot\nabla\varphi + \varphi\Delta\rho_1^{(\varepsilon)}$$

we obtain

$$\begin{aligned}&\int_{\mathcal{O}_0}(\partial_r\varphi)\,\zeta\Delta\varphi\ d\xi dr\\ &= \int_{\mathcal{O}_0}\Big(-a_\varepsilon\Delta(\rho_1^{(\varepsilon)}\varphi)\zeta\Delta\varphi + \theta\zeta\Delta\varphi\Big)\ d\xi dr\\ &\le -\frac{1}{2}\int_{\mathcal{O}_0} a_\varepsilon\rho_1^{(\varepsilon)}\zeta|\Delta\varphi|^2\ d\xi dr\\ &\quad+\left(C_1\left\|\frac{|\nabla\rho_1^{(\varepsilon)}|^2}{\rho_1^{(\varepsilon)}}\right\|_{C^0(\mathcal{O}_0)} + C_1\left\|\frac{|\Delta\rho_1^{(\varepsilon)}|^2}{\rho_1^{(\varepsilon)}}\right\|_{C^0(\mathcal{O}_0)} + \frac{c}{4}\right)\int_{\mathcal{O}_0}|\nabla\varphi|^2\ d\xi dr\\ &\quad+C\int_{\mathcal{O}_0}|\nabla\theta|^2\ d\xi dr,\end{aligned}$$

where C_1 is a constant depending only on $\|a\|_{L^\infty(\mathcal{O}_T)}$ and $c>0$ is a constant as in case (A1'). We now choose $\tau_1>0$ such that

$$C_1\left(\left\|\frac{|\nabla\rho_1^{(\varepsilon)}|^2}{\rho_1^{(\varepsilon)}}\right\|_{C^0(\mathcal{O}_0)} + \left\|\frac{|\Delta\rho_1^{(\varepsilon)}|^2}{\rho_1^{(\varepsilon)}}\right\|_{C^0(\mathcal{O}_0)}\right) \le \frac{c}{4}.$$

By the choice of τ_1 and Fatou's Lemma we get

$$\frac{c}{2}\int_{\mathcal{O}_0}|\nabla\varphi|^2\ d\xi dr + \frac{1}{2}\int_{\mathcal{O}_0} a_\varepsilon\rho_1^{(\varepsilon)}|\Delta\varphi|^2\ d\xi dr \le C\int_{\mathcal{O}_0}|\nabla\theta|^2\ d\xi dr$$

and we conclude the proof as in case (A1'). □

A.3. **Lower bound on L^1-decay for** (A.1)**.** In this section we provide a lower bound for the decay of the L^1 norm of solutions to (A.1). This estimate is required in Section 4 in order to ensure that the constructed solutions with disjoint support have a sufficiently large distance with respect to the L^1-norm.

Proposition A.7. *Let $Y\in C((0,T]\times\mathcal{O})$ be an essentially bounded, non-negative solution to the homogeneous Dirichlet problem for* (A.1) *with uniformly compact support, i.e. $K := \bigcup_{t\in[0,T]} supp(Y_t) \Subset \mathcal{O}$ is a precompact set. Then, there is a constant $C>0$ such that*

$$\|Y_t\|_{L^1(\mathcal{O})} \ge e^{-Ct\|Y\|^{m-1}_{L^\infty(\mathcal{O}_T)}}\|Y_0\|_{L^1(\mathcal{O})},\quad \forall t\in[0,T].$$

Proof. It is easy to see that if Y is a solution to the homogeneous Dirichlet problem for (A.2) then

(A.11)
$$\int_{\mathcal{O}} Y_t\varphi \, d\xi - \int_{\mathcal{O}} Y_s\varphi \, d\xi = -\int_s^t \int_{\mathcal{O}} \Phi(\rho_2 Y)\Delta(\rho_1\varphi) \, d\xi dr, \quad \forall 0 \le s < t \le T,$$

and all non-negative $\varphi \in C^2(\bar{\mathcal{O}})$ with $\varphi_{|\partial\mathcal{O}} = 0$. Let $M := \|Y\|_{L^\infty(\mathcal{O}_T)}$. We choose a test-function $\varphi \in C_c^2(\mathcal{O})$ with $\varphi \equiv 1$ on K. Then

$$\begin{aligned}
\|Y_t\|_{L^1(\mathcal{O})} &= \int_{\mathcal{O}} Y_t\varphi d\xi \\
&= \int_{\mathcal{O}} Y_s\varphi d\xi - \int_s^t \int_{\mathcal{O}} \Phi(\rho_2 Y)\Delta(\rho_1\varphi) d\xi dr \\
&= \|Y_s\|_{L^1(\mathcal{O})} - M^{m-1}\int_s^t \int_{\mathcal{O}} \Phi(\rho_2)\left(\frac{Y}{M}\right)^{m-1} |Y|\Delta\rho_1 d\xi dr \\
&\ge \|Y_s\|_{L^1(\mathcal{O})} - M^{m-1}(\|\rho_1\|_{C^{0,2}(\mathcal{O}_T)} + \|\rho_2^m\|_{C^0(\mathcal{O}_T)})\int_s^t \|Y_r\|_{L^1(\mathcal{O})} dr,
\end{aligned}$$

for all $0 \le s < t \le T$. Gronwall's Lemma finishes the proof. □

A.4. **Finite speed of propagation for** (A.1)**.** The proof of finite speed of propagation for (A.1) is very similar to Theorem 3.10 and is based on a bound for the speed of hole-filling as given in Theorem 3.9. The arguments remain the same with minor changes in the calculation. For the readers convenience we state the corresponding results in detail and give some short remarks on the proofs.

Theorem A.8. *Let $\xi_0 \in \mathbb{R}^d$, $T, R > 0$ and $Y \in C((0,T] \times B_R(\xi_0))$ be an essentially bounded, non-negative solution to* (A.1) *with vanishing initial value Y_0 on $B_R(\xi_0)$ and boundary value g satisfying $H := \|g\|_{L^\infty([0,T]\times\partial B_R(\xi_0))} < \infty$. Define T_{stoch} by*

$$T_{stoch} := \sup\left\{\tilde{T} \in [0,T] \middle| \; \tilde{T} C_{\tilde{T}} \le R^2 H^{-(m-1)}\right\},$$

where $t \mapsto C_t$ is a continuous, non-decreasing function.

Then Y_t vanishes in $B_{R_{stoch}(t)}(\xi_0)$ for all $t \in [0, T_{stoch}]$, where

$$R_{stoch}(t) = R - \sqrt{t}\sqrt{C_t} H^{\frac{m-1}{2}}.$$

Proof. As in Theorem 3.9 the proof is based on the construction of an appropriate supersolution to (A.1). For $r \in (0, R]$, $\xi_1 \in \mathbb{R}^d$, $\tilde{T} > 0$ let

$$W(t,\xi,\xi_1) := \tilde{C}|\xi - \xi_1|^{\frac{2}{m-1}} \left(\tilde{T} - t\right)^{\frac{-1}{m-1}}, \quad t \in [0,\tilde{T}), \xi \in B_r(\xi_1).$$

Direct computations yield

$$\partial_t W(t,\xi,\xi_1) \ge \rho_1 \Delta \left(\rho_2 W(t,\xi,\xi_1)\right)^m$$

on $[0,\tilde{T}) \times B_r(\xi_1)$ if

$$1 \ge C(d,m)\tilde{C}^{m-1}(1+R)^2 \|\rho_1\|_{C^0([0,\tilde{T}]\times B_R(\xi_0))} \|\rho_2\|_{C^{0,2}([0,\tilde{T}]\times B_R(\xi_0))},$$

for all $(t,\xi) \in [0,\tilde{T}) \times B_r(\xi_1)$ and some generic constant $C(d,m)$. This is satisfied for the choice

$$\tilde{C}^{m-1} = \tilde{C}_{\tilde{T}}^{m-1} := \left(C(d,m)(1+R)^2 \|\rho_1\|_{C^0([0,\tilde{T}]\times B_R(\xi_0))} \|\rho_2\|_{C^{0,2}([0,\tilde{T}]\times B_R(\xi_0))}\right)^{-1}.$$

Moreover,

$$W(t,\xi,\xi_1) = \tilde{C}|\xi - \xi_1|^{\frac{2}{m-1}} \left(\tilde{T} - t\right)^{\frac{-1}{m-1}} \ge H,$$

for a.a. $(t,\xi) \in [0,\tilde{T}) \times \partial B_r(\xi_1)$ is satisfied if

$$\tilde{T}\tilde{C}_{\tilde{T}}^{-(m-1)} \le r^2 H^{-(m-1)}$$

We conclude the proof as for Theorem 3.9. □

As in Theorem 3.11 we may now use Theorem A.8 to deduce

Theorem A.9. *Let $Y \in C((0,T] \times \mathcal{O})$ be an essentially bounded, non-negative solution to the homogeneous Dirichlet problem to* (A.1) *and set $H := \|Y\|_{L^\infty(\mathcal{O}_T)}$. Then, for every $s \in [0,T]$*

$$supp(Y_{s+t}) \subseteq B_{\sqrt{t}\sqrt{C_t}H^{\frac{m-1}{2}}}(supp(Y_s)), \quad \forall t \in [0,T-s],$$

where $t \mapsto C_t$ is a continuous, non-decreasing function.

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