Stochastic Generalized Porous Media Equations with Reflection*

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Abstract

A non-negative Markovian solution is constructed for a class of stochastic generalized porous media equations with reflection. To this end, some regularity properties and a comparison theorem are proved for stochastic generalized porous media equations, which are interesting by themselves. Invariant probability measures and ergodicity of the solution are also investigated.

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1 Introduction

Let E be a locally compact separable metric space with Borel σ-field $\mathscr{B}$ and let μ be a probability measure on $(E, \mathscr{B})$. Let $(L, \mathscr{D}(L))$ be a symmetric Dirichlet operator on $L^2(\mu)$ with empty essential spectrum and regular Dirichlet form $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$. Note that we may allow the Dirichlet form to be merely quasi-regular by "local compactification" from the book [8].

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Let $\{\lambda_i\}_{i\geq 1}$ be all eigenvalues of $-L$ counting multiplicities in increasing order such that $\lambda_1 > 0$, and let $\{e_i\}_{i\geq 1}$ be the corresponding normalized eigenfunctions.

In this paper we investigate the stochastic generalized porous medium equation with reflection of the type

$$\mathrm{d}X_t = \{L\Psi(X_t) + \Phi(X_t)\}\mathrm{d}t + \sigma_t \mathrm{d}W_t + \mathrm{d}\eta_t, \quad t \in [0, T], \tag{1.1}$$

where $T > 0$ is a fixed constant, W_t is a cylindrical Brownian motion on $L^2(\mu)$ with respect to a complete filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$, $\Psi, \Phi \in C(\mathbb{R})$, and $\sigma_t \in L^2(\Omega \times [0, T] \to \mathscr{L}_{HS})$ is progressively measurable. Here $\mathscr{L}_{HS}$ is the space of all Hilbert-Schmidt linear operators on $L^2(\mu)$. We will use $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ to denote the inner product and the norm in $L^2(\mu)$, and denote the norm in $L^p(\mu)$ for $p \geq 1$ by $\| \cdot \|_p$. Moreover, let $H^1 = \mathscr{D}(\mathscr{E})$ with inner product $\langle u, v \rangle_{H^1} = \mathscr{E}(u, v)$, and let H^{-1} be the dual space of H^1 w.r.t. $L^2(\mu)$. For simplicity we also use $\langle \cdot, \cdot \rangle$ for the dualization ${}_{H^{-1}}\langle \cdot, \cdot \rangle_{H^1}$ between H^{-1} and H^1. Since the essential spectrum of L is empty, we have $\lambda_i \uparrow \infty$ as $i \uparrow \infty$ and thus, the embedding $H^1 \subset L^2(\mu)$ is compact. As usual L extends to an operator $L : H^1 \to H^{-1}$, again denoted by L, and below by L we always mean this extension.

For stochastic partial differential equations with reflection driven by space-time white noise, we refer the reader to [5], [9], [13], [14] and [15]. The situation here is drastically different from that in the above references because of the presence of the the non-linear operator $L\Psi$, in (1.1).

The motivation to study reflection problems of type (1.1) is that eventually we would like to extend our results from this paper to stochastic fast diffusion equations, where $\Psi(s) = |s|^{r-1}s$ with $r \in (0, 1)$, or to so-called "self-organized criticality" models, where Ψ is a Heaviside function or a product of this with the identity. Such type of singular stochastic porous media equations have intensively been studied in [1, 2, 3] and [12], proving in addition to well-posedness that extinction occurs with strictly positive probability. However, in contrast to the latter papers, instead of linear multiplicative noise, we would like to study the class of additive noise. This was suggested in [4] by A. Diaz-Guilera, who derived equations of type (1.1) with $\eta \equiv 0$ as models for the phenomenon of self-organized criticality, where $X_t, t \geq 0$, has the interpretation of energy. However, for all types of additive noise suggested in [4] the solution $X_t, t \geq 0$, can take (depending on $\omega \in \Omega$) arbitrarily negative values, which is somehow in contradiction of its interpretation as energy. Our "penalization" term, however, guaranties nonnegative solutions. Therefore, we suggest our equation (1.1) as a more realistic version of the one suggested in [4], where it was also pointed out that the case $\Psi(s) = |s|^{r-1}s$ for $r = 3$ is an interesting special case, since it is the simplest fulfilling all symmetry restriction suggested by physical considerations. And this case is covered by the main result in this paper. As also pointed out in [4], this polynomial case is, however, too much of a simplification and quite far from "self-organized criticality" models as $\Psi(s) = \mathbf{H}(s)$ or $\Psi(s) = s\mathbf{H}(s)$ with $\mathbf{H}$ being the Heaviside function. In contrast to the polynomial case the latter one namely exhibits extinction of solutions as shown in [1, 3] and

[12], while the polynomial case does not. Therefore, in a future paper we plan to extend our results to singular cases as $\Psi(s) = |s|^{r-1}s$ for $r \in (0,1)$, $\Psi(s) = \mathbf{H}(s)$, or $\Psi(s) = s\mathbf{H}(s)$.

Let $\mathscr{M}_c$ be the space of all locally finite measures on E, equipped with the vague topology induced by $f \in C_0(E)$, where $C_0(E)$ be the set of all continuous functions on E with compact support.

Definition 1.1. An element $u \in H^{-1}$ is called non-negative, denoted by $u \geq 0$, if ${}_{H^1}\langle f, u\rangle_{H^{-1}} \geq 0$ holds for any non-negative $f \in H^1$. For $u_1, u_2 \in H^{-1}$, we write $u_1 \geq u_2$ if $u_1 - u_2 \geq 0$. A process u_t in H^{-1} is called increasing if $u_t \geq u_s$ for $t \geq s$.

It is easy to verify that $u_1 \geq u_2$ and $u_2 \geq u_1$ if and only if $u_1 = u_2$. Thus, H^{-1} is a partially ordered space.

Definition 1.2. A pair $(X, \eta) := (X_t, \eta_t)_{t\geq 0}$ is called a solution to (1.1), if

(1) X is a non-negative, adapted process on $L^2(\mu)$, which is cádlág in H^{-1}, such that for any $T > 0$, $\Psi(X.)|_{[0,T]} \in L^2(\Omega \times [0,T] \to H^1; \mathbb{P} \times \mathrm{d}t)$ and $\Phi(X.)|_{[0,T]} \in L^2(\Omega \times [0,T] \to H^{-1}; \mathbb{P} \times \mathrm{d}t)$;

(2) $\eta = (\eta_t)_{t\in[0,T]}$ is a right-continuous, non-negative, increasing adapted process in H^{-1}, which determines a unique adapted process $\nu := (\nu_t)_{t\geq 0}$ in $\mathscr{M}_c$, right-continuous in the topology of set-wise convergence, such that

$$\int_E f(z)\nu_t(\mathrm{d}z) = {}_{H^{-1}}\langle \eta_t, f\rangle_{H^1}, \quad f \in C_0(E) \cap H^1, \quad t \geq 0. \tag{1.2}$$

Moreover, ${}_{H^1}\langle \Psi(X_t), \eta_t\rangle_{H^{-1}} = 0$, $\mathbb{P} \times \mathrm{d}t$-a.e.;

(3) $\mathbb{P}$-a.s.

$$X_t = X_0 + \int_0^t \Big\{L\Psi(X_s) + \Phi(X_s)\Big\}\mathrm{d}s + \int_0^t \sigma_s \mathrm{d}W_s + \eta_t, \quad t \geq 0$$

holds on H^{-1}.

To construct a solution to (1.1) using a natural approximation argument, we shall need the following assumptions.

(A1) For any $f \in C_0(E)$, there exists $\tilde{f} \in H^1 \cap C_0(E)$ such that for any $\varepsilon > 0$,

$$|f - f_\varepsilon| \leq \varepsilon \tilde{f}$$

holds for some $f_\varepsilon \in H^1 \cap C_0(E)$.

(A2) $\Phi : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous such that

$$\langle \Phi(u) - \Phi(v), (-L)^{-1}(u - v)\rangle \leq c\|u - v\|^2_{H^{-1}}, \quad u, v \in L^{r+1}(\mu),$$

holds for some constant $c > 0$; and $\Psi \in C^1(\mathbb{R})$ with $\Psi(0) = 0$ and there exist constants $r \geq 1, c_1' \geq 0$ and $c_1, c_2 > 0$ such that for any $s_1, s_2, s \in \mathbb{R}$,

$$(s_2 - s_1)(\Psi(s_2) - \Psi(s_1)) \geq \{c_1|s_2 - s_1|^{r+1} + c_1'|s_2 - s_1|^2\}, \ 0 \leq \Psi'(s) \leq c_2(1 + |s|^{r-1}).$$

(A3) $\{e_i\}_{i\geq 1}\subset L^{r+1}(\mu)$ and for any $T>0$ there exists a constant $C>0$ such that

$$\sup_{t\in[0,T]}\|\sigma_t\|_{\mathscr{L}_{HS}}^2+\sup_{n\geq 1}\int_{[0,T]\times E}\bigg\{\sum_{i=1}^n\Big(\sum_{k=1}^n\langle\sigma_t e_i,e_k\rangle e_k\Big)^2\bigg\}^{(1+r)/2}\mathrm{d}t\mathrm{d}\mu\leq C.$$

Here is the main result of the paper, where for a measure ν on $(E,\mathscr{B})$ and a ν-integrable or nonnegative $\mathscr{B}$-measurable function $f:E\to\mathbb{R}$ we set $\nu(f)=\int_E f\mathrm{d}\nu$.

Theorem 1.1. *Assume that* **(A1)**, **(A2)** *and* **(A3)** *hold. If either* $c_1'>0$, *or* $\Psi(s)=c|s|^{r-1}s$ *and* $\Phi(s)=c's$ *for some constants* $c>0$ *and* $c'\in\mathbb{R}$. *Then:*

(1) *For any* $X_0=x\in L^{1+r}(\Omega\to L_+^{1+r}(\mu),\mathscr{F}_0;\mathbb{P})$, (1.1) *has a solution* $(X(x),\eta(x))$ *in the sense of Definition* 1.2 *such that*

$$\mathbb{E}\sup_{t\in[0,T]}\|X_t(x)\|^2+\sup_{t\in[0,T]}\mathbb{E}\|X_t(x)\|_{1+r}^{1+r}<\infty. \tag{1.3}$$

If $r=1$ *and* Ψ *is linear, then the solution to* (1.1) *is unique.*

(2) *If* $\sigma_t=\sigma$ *is deterministic and independent of* t, *the family* $(X(x))_{x\in L_+^{1+r}(\mu)}$ *is time-homogeneous Markovian, i.e. for any* $f\in C_b(L^{1+r}(\mu))$ *and* $0<s<t\leq T$,

$$\mathbb{E}\big(f(X_t(x))\big|\sigma(X_u(x):u\leq s)\big)=(P_{t-s}f)(X_s(x)), \tag{1.4}$$

where

$$P_uf(z):=\mathbb{E}f(X_u(z)),\quad u\in[0,T],z\in L^{1+r}(\mu).$$

(3) *Let* $\sigma_t=\sigma$ *be deterministic and independent of* t, *and let* $K\in\mathbb{R}$ *be such that*

$$\langle\Phi(x)-\Phi(y),1_{\{x-y>0\}}\rangle\leq K\mu\big((x-y)^+\big),\quad x,y\in L^{1+r}(\mu), \tag{1.5}$$

then $X_t(x)$ *is* L^1*-Lipschitz continuous in* x, *i.e.* $\mathbb{P}$*-a.s.*

$$\|X_t(x)-X_t(y)\|_1\leq \mathrm{e}^{Kt}\|x-y\|_1,\quad x,y\in L^{1+r}(\mu). \tag{1.6}$$

Consequently, P_t *extends to a unique Markov Lipschitz-Feller semigroup on* $L^1(\mu)$. *If*

$$\mathscr{E}(\Psi(x),x)-\langle\Phi(x),x\rangle\geq c_1\mathscr{E}(x,x)-c_2 \tag{1.7}$$

holds for some constants $c_1,c_2>0$ *and all* $x\in\mathscr{D}(\mathscr{E})$ *such that* $\Psi(x)\in\mathscr{D}(\mathscr{E})$, *then* P_t *has an invariant probability measure* π *with* $\pi(\|\cdot\|_{H^1}^2)<\infty$ *and*

$$|P_tf(x)-\pi(f)|\leq \mathrm{Lip}_1(f)\mathrm{e}^{Kt}\int_{L^1(\mu)}\|x-y\|_1\pi(\mathrm{d}y),\quad x\in L^1(\mu),t\geq 0, \tag{1.8}$$

where $f:L^1(\mu)\to\mathbb{R}$ *is Lipschitz and* $\mathrm{Lip}_1(f)$ *is its Lipschitz constant. In particular,* P_t *converges exponentially fast to* μ *if* $K<0$.

To illustrate this result, let us consider the following simple example.

Example 1.1. Let $E \subset \mathbb{R}^d$ be a bounded open domain, and let $L = \Delta$ be the Dirichlet Laplacian on E, which has discrete spectrum with $\lambda_1 > 0$. Let Φ be a Lipschitz continuous function on $\mathbb{R}$ and let

$$\Psi(s) = \alpha_1 s|s|^{r-1} + \alpha_2 s + \alpha_3 |s|^{r'-1} s$$

for some constants $r \geq 1, r' \in (0,1), \alpha_1 > 0$, and $\alpha_2, \alpha_3 \geq 0$. Finally, let $\sigma_t = \sigma$ be deterministic and independent of t such that

$$\sigma e_i = q_i e_i, \quad i \geq 1,$$

where $\{q_i\}_{i\geq 1} \subset [0,\infty)$ such that

$$\sum_{i=1}^{\infty} i^{(r^2-1)/(2r+4)} q_i^2 < \infty. \tag{1.9}$$

Then $\Psi(0) = 0$ and **(A1)**, **(A2)** and **(A3)** hold.

Proof. **(A2)** is trivial for the specific choice of Ψ. Noting that

$$\|e_i\|_{r+1} \leq c\lambda_i^{d(r-1)/4(r+1)}, \ \lambda_i \leq c i^{2/d}, \quad i \geq 1, \tag{1.10}$$

holds for some constant $c > 0$, we have

$$\sup_{n\geq 1} \mathbb{E} \int_{[0,T]\times E} \Big\{ \sum_{i=1}^n \Big(\sum_{k=1}^n \langle \sigma_t e_i, e_k \rangle e_k \Big)^2 \Big\}^{(1+r)/2} \mathrm{d}t\mathrm{d}\mu = T \int_E \Big\{ \sum_{i=1}^{\infty} q_i^2 e_i^2 \Big\}^{(1+r)/2} \mathrm{d}\mu$$
$$\leq T \bigg(\sum_{i=1}^{\infty} q_i^2 \|e_i\|_{1+r}^{1+r} i^{-(r-1)^2/(2r+4)} \bigg) \bigg(\sum_{i=1}^{\infty} q_i^2 i^{(r^2-1)/(2r+4)} \bigg)^{(r-1)/2},$$

which is finite according to (1.9) and (1.10). Therefore, **(A3)** holds. Finally, **(A1)** is trivial in the present situation. □

To construct the desired solution to (1.1), we first present some preparations in Section 2 concerning regularity properties and a comparison theorem for stochastic generalized porous media equations, which are interesting by themselves. A complete proof of Theorem 1.1 is given in Section 3.

2 Preparations

In this section we first consider regularity of solutions to stochastic generalized porous media equations, then prove a comparison theorem and the L^1-Lipshitz continuity for the solution. Finally, to construct the "local time" η as a locally finite measure on $[0,T] \times E$, we prove a suitable new version of the Riesz-Markov representation theorem. We note that the regularity of solutions for stochastic generalized porous media equations have been investigated e.g. in [11, 6] for either linear Φ or $\Phi = 0$. But to approximate the equation with reflection, a non-linear Lipschitzian term $s \mapsto ns^-$ will be included in $\Phi^{(n)}$ (see Section 3).

2.1 Regular solution of stochastic generalized porous media equations

In this subsection we consider the following equation with multiplicative noise:

$$\mathrm{d}X_t = \{L\Psi(X_t) + \Phi(X_t)\}\mathrm{d}t + \sigma_t(X_t)\mathrm{d}W_t, \tag{2.1}$$

where $\sigma : \Omega \times [0, \infty) \times L^2(\mu) \to \mathscr{L}_{LS}$ is progressively measurable such that

(A3') $\{e_i\}_{i\geq 1} \subset L^{r+1}(\mu)$ and for any $T > 0$ there exists a constant $C > 0$ such that for any $u, v \in L^2(\mu)$,

$$\sup_{t\in[0,T]} \|\sigma_t(u) - \sigma_t(v)\|^2_{\mathscr{L}_{HS}(L^2(\mu);H^{-1})} \leq C\|u - v\|^2_{H^{-1}},$$

$$\sup_{t\in[0,T]} \|\sigma_t(u)\|^2_{\mathscr{L}_{HS}} + \sup_{n\geq 1} \int_{[0,T]\times E} \Big\{ \sum_{i=1}^{n} \Big(\sum_{k=1}^{n} \langle \sigma_t(u)e_i, e_k\rangle e_k \Big)^2 \Big\}^{\frac{1+r}{2}} \mathrm{d}t\mathrm{d}\mu \leq C.$$

Definition 2.1. A continuous adapted process $X = (X_t)_{t\geq 0}$ on H^{-1} is called a solution to (2.1), if $\mathbb{P}$-a.s.

$$X_t = X_0 + \int_0^t \big\{L\Psi(X_s) + \Phi(X_s)\big\}\mathrm{d}s + \int_0^t \sigma_s(X_s)\mathrm{d}W_s, \quad t \geq 0$$

holds on H^{-1}. The solution is called regular, if it is a right-continuous process on $L^2(\mathbf{m})$ and for any $T > 0$,

$$\mathbb{E} \sup_{t\in[0,T]} \|X_t\|^2 + \sup_{t\in[0,T]} \mathbb{E}\|X_t\|^{1+r}_{1+r} + \mathbb{E} \int_0^T \Big\{\mathscr{E}(\Psi(X_t), \Psi(X_t))\Big\}\mathrm{d}t < \infty.$$

Theorem 2.1. *Assume* **(A1)**, **(A2)** *and* **(A3')**. *For any $\mathscr{F}_0$-measurable random variable X_0 on H^{-1} with $\mathbb{E}\|X_0\|^2_{H^{-1}} < \infty$, the equation* (2.1) *has a unique solution such that*

$$\mathbb{E} \sup_{t\in[0,T]} \|X_t\|^2_{H^{-1}} + \mathbb{E} \int_0^T \|X_t\|^{1+r}_{1+r}\mathrm{d}t < \infty, \quad T > 0. \tag{2.2}$$

If moreover $\mathbb{E}\|X_0\|^{1+r}_{1+r} < \infty$, then:

(1) *When $c_1' > 0$, the solution is regular, continuous in $L^2(\mathbf{m})$, and the Itô formula*

$$\begin{aligned}\|X_t\|^2 &= \|X_0\|^2 - 2\int_0^t \mathscr{E}(X_s, \Psi(X_s))\mathrm{d}s + 2\int_0^t \langle X_s, \Phi(X_s)\rangle\mathrm{d}s \\ &\quad +2\int_0^t \langle X_s, \sigma_s(X_s)\mathrm{d}W_s\rangle + \int_0^t ||\sigma_s(X_s)||^2_{\mathscr{L}_{LS}}ds, \quad t \in [0, T]\end{aligned}$$

holds.

(2) *When* $\Psi(s) = c|s|^{r-1}s, \Phi(s) = c's$ *for some constants* $c > 0$ *and* $c' \in \mathbb{R}$*, the solution is regular.*

To prove this theorem, we first prove that there exists a unique solution in H^{-1} using a general result in [10]. Then we show that this solution is indeed regular.

Lemma 2.2. *Assume* **(A1)***,* **(A2)** *and* **(A3')***. For any* $X_0 \in L^2(\Omega \to H^{-1}, \mathscr{F}_0; \mathbb{P})$*, there exists a unique continuous adapted process* $X = (X_t)_{t\geq 0}$ *on* H^{-1} *such that* (2.2) *holds.*

Proof. It suffices to verify assumptions (K), (H1), (H2), (H3) and (H4) in [10, Theorem 2.1]. To verify these assumptions, let $V = L^{1+r}(\mu), H = H^{-1}, V^*$ be the dual space of V w.r.t. H^{-1}, and $K = L^{1+r}(\Omega \times [0,T] \times E; \mathbb{P} \times \mathrm{d}t \times \mu)$. Then assumption (K) holds for $R(x) = \|x\|_{1+r}^{1+r}, x \in V$, and $W_1(s) = W_2(s) = s^{1/(1+r)}, s \geq 0$. Next, from **(A2)** it is easy to see that the hemicontinuity condition (H1) holds for $A(u) := L\Psi(u) + \Phi(u)$, i.e.

$$\mathbb{R} \ni \lambda \ni\mapsto \langle \Psi(u + \lambda v), w\rangle + \langle \Phi(u + \lambda v), (-L)^{-1}w\rangle, \quad u, v, w \in V$$

is continuous, where L is understood as its unique extension given by [10, Lemma 3.3] for $L_{N^*} = L^{\frac{r+1}{r}}(\mu)$. Moreover, letting l_0 be the Lipschitz constant of Φ, we obtain from **(A2)** and **(A3')** that

$$\begin{aligned}
&2_{V^*}\langle A(u) - A(v), u - v\rangle_V + \|\sigma_t(u) - \sigma_t(v)\|^2_{\mathscr{L}_{HS}(L^2(\mu);H)} \\
&\leq C\|u - v\|_H^2 - 2\langle \Psi(u) - \Psi(v), u - v\rangle + 2\langle \Phi(u) - \Phi(v), (-L)^{-1}(u - v)\rangle \\
&\leq c'\|u - v\|_H^2, \quad u, v \in V
\end{aligned}$$

for some constant $c' > 0$. Thus, the weak monotonicity condition (H2) holds. Again by **(A2)** and **(A3')**, we have

$$\begin{aligned}
&2_{V^*}\langle A(u), u\rangle_V + \|\sigma_t(u)\|^2_{\mathscr{L}_{HS}(L^2(\mu);H)} \\
&\leq -2\mu(u\Psi(u)) + 2\langle \Phi(u), (-L)^{-1}u\rangle + \frac{1}{\lambda_1}\|\sigma_t(u)\|^2_{\mathscr{L}_{HS}} \\
&\leq c' + c_1\|u\|_{r+1}^{r+1} + c''\|u\|_H^2
\end{aligned}$$

for some constants $c', c'' > 0$. This implies (H3) as $R(u) = \|u\|_{1+r}^{1+r}$. Finally, **(A2)** also implies that

$$|_{V^*}\langle A(v), u\rangle_V| \leq |\mu(\Psi(v)u)| + \|\Phi(v)\| \cdot \|(-L)^{-1}u\| \leq c'(1 + R(u) + R(v))$$

for some constant $c' > 0$ and $R = \|\cdot\|_{1+r}^{1+r}$. Therefore, (H4) holds. □

Now, let $\mathbb{E}\|X_0\|_{1+r}^{1+r} < \infty$. To prove that the unique solution X to (2.1) is a regular solution in the sense of Definition 2.1, we make use of the Galerkin approximations. For any $n \geq 1$, let $H_n = \mathrm{span}\{e_1, \cdots, e_n\}$. Since $\{e_i\}_{i\geq 1} \subset L^{1+r}(\mu)$, the orthogonal projection $\mathscr{P}_n : L^2(\mu) \to H_n$ can be extended to $L^{(1+r)/r}(\mu)$ as

$$\mathscr{P}_n u = \sum_{i=1}^n \mu(ue_i)e_i, \quad u \in L^{(1+r)/r}(\mu).$$

Let $\Psi_n(u) = \mathscr{P}_n\Psi(u), \Phi_n(u) = \mathscr{P}_n\Phi(u)$ for $u \in L^{1+r}(\mu)$, and let $\sigma_t^n = \mathscr{P}_n\sigma_t, W_t^n = \mathscr{P}_n W_t$. Finally, let X_0^n be the L^{1+r}-best approximation of X_0 in H_n; that is, X_0^n is the unique $\mathscr{F}_0$-measurable random variable in H_n such that

$$\|X_0 - X_0^n\|_{r+1} = \inf_{u \in L^{1+r}(\mu)\cap H_n} \|X_0 - u\|_{1+r}.$$

We have $\|X_0^n\|_{1+r} \le 2\|X_0\|_{1+r}$ and $X_0^n \to X_0$ in $L^{1+r}(\mu)$ $\mathbb{P}$-a.s. as $n \to \infty$, see [7, Theorems 5,6,8] and [6, §0].

For each $n \ge 1$, let $X^n = (X_t^n)_{t\in[0,T]}$ be the unique solution to the following finite-dimensional SDE with initial data X_0^n:

$$\mathrm{d}X_t^n = \{L\Psi_n(X_t^n) + \Phi_n(X_t^n)\}\mathrm{d}t + \sigma_t^n(X_t^n)\mathrm{d}W_t^n.$$

Note that since H_n is an invariant space for L, we have $L\Psi_n(u) \in H_n$ for $u \in L^{1+r}(\mu)$.

Lemma 2.3. *Assume* **(A2)** *and* **(A3')** *and let* $\mathbb{E}\|X_0\|_{1+r}^{1+r} < \infty$. *Then for any* $T > 0$ *there exists a constant* $C > 0$ *independent of* c_1' *such that for any* $n \ge 1$,

$$\mathbb{E}\sup_{t\in[0,T]}\|X_t^n\|^2 + \sup_{t\in[0,T]}\mathbb{E}\|X_t^n\|_{1+r}^{1+r} + \mathbb{E}\int_0^T \Big\{\mathscr{E}(\Psi_n(X_t^n), \Psi_n(X_t^n)) + c_1'\mathscr{E}(X_t^n, X_t^n)\Big\}\mathrm{d}t \le C.$$

Proof. Let

$$F(u) = \int_E \mathrm{d}\mu \int_0^u \Psi(s)\mathrm{d}s, \quad u \in H_n.$$

Then $F \in C^2(H_n)$. By the Itô formula, we have

$$\begin{aligned}(2.3)\qquad &\mathrm{d}F(X_t^n) = \langle \Psi(X_t^n), \sigma_t^n(X_t^n)\mathrm{d}W_t^n\rangle \\ &+ \Big\{\langle \Psi_n(X_t^n), L\Psi_n(X_t^n) + \Phi_n(X_t^n)\rangle + \frac{1}{2}\sum_{i=1}^n \int_E \Psi'(X_t^n)(\sigma_t^n(X_t^n)e_i)^2\mathrm{d}\mu\Big\}\mathrm{d}t.\end{aligned}$$

By **(A2)** and **(A3')**, there exists a constant $C_1 > 1$ independent of n such that for any $u \in H_n$,

$$\begin{aligned}&\frac{1}{C_1}\|u\|_{1+r}^{1+r} - C_1 \le F(u) \le C_1 + C_1\|u\|_{1+r}^{1+r}\\ &|\langle \Phi_n(u), \Psi_n(u)\rangle| \le \|\Phi(u)\| \cdot \|\Psi(u)\| \le C_1 + C_1\|u\|_{1+r}^{1+r}\end{aligned}$$

and

$$\begin{aligned}&\mathbb{E}\sum_{i=1}^n \int_E \psi'(X_t^n)(\sigma_t^n(X_t^n)e_i)^2\mathrm{d}\mu \\ &\le \mathbb{E}\bigg(\|\Psi'(X_t^n)\|_{\frac{1+r}{r-1}}\Big\|\sum_{i=1}^n\Big(\sum_{k=1}^n \langle \sigma_t(X_t^n)e_i, e_k\rangle e_k\Big)^2\Big\|_{\frac{1+r}{2}}\bigg) \le C_1 + C_1\mathbb{E}\|X_t^n\|_{1+r}^{1+r}.\end{aligned}$$

Combining this with (2.3) we obtain

$$\mathbb{E}\|X_t^n\|_{1+r}^{1+r} \le C_2 + C_2\int_0^t \mathbb{E}\|X_s^n\|_{1+r}^{1+r}\mathrm{d}s - C_3\mathbb{E}\int_0^t \mathscr{E}(\Psi_n(X_s^n),\Psi(X_s^n))\mathrm{d}s, \quad s\in[0,T]$$

for some constants $C_2, C_3 > 0$ independent of n. This implies

$$\sup_{t\in[0,T]}\mathbb{E}\|X_t^n\|_{1+r}^{1+r} + \mathbb{E}\int_0^T \mathscr{E}(\Psi_n(X_t^n),\Psi_n(X_t^n))\mathrm{d}t \le C$$

for some constant $C>0$ independent of n.

Next, by the Itô formula,

$$\mathrm{d}\|X_t^n\|^2 = \big\{2\langle X_t^n, L\Psi_n(X_t^n)+\Phi_n(X_t^n)\rangle + \|\sigma_t^n(X_t^n)\|_{\mathscr{L}_{HS}}^2\big\}\mathrm{d}t + 2\langle X_t^n, \sigma_t^n(X_t^n)\mathrm{d}W_t^n\rangle. \tag{2.4}$$

Since due to **(A2)** $\Psi'(s)\ge c_1'$, we have

$$\langle X_t^n, L\Psi_n(X_t^n)\rangle = \langle X_t^n, L\Psi_n(X_t^n)\rangle \le -c_1'\mathscr{E}(X_t^n, X_t^n).$$

Moreover, by **(A3')** and the Burkholder-Davies inequality for $p=2$ we have

$$\mathbb{E}\sup_{s\in[0,t]}\bigg|\int_0^s \langle X_a^n, \sigma_a^n(X_a^n)\mathrm{d}W_a^n\rangle\bigg|^2 \le 4\,\mathbb{E}\int_0^t \|\sigma_s(X_s^n)\|_{\mathscr{L}_{HS}}^2\|X_s^n\|^2\mathrm{d}s \le 4c\,\mathbb{E}\int_0^t \|X_s^n\|^2\mathrm{d}s.$$

Combining this with (2.4) we conclude that $h_n(t) := \mathbb{E}\sup_{s\in[0,t]}\|X_s^n\|^2$ satisfies

$$h_n(t) \le C_4 + C_4\int_0^t h_n(s)\mathrm{d}s - 2c_1'\int_0^t \mathscr{E}(X_s^n, X_s^n)\mathrm{d}s, \quad t\in[0,T]$$

for some constant $C_4>0$ independent of n. Therefore,

$$\mathbb{E}\sup_{t\in[0,T]}\|X_t^n\|^2 + c_1'\mathbb{E}\int_0^T \mathscr{E}(X_s^n, X_s^n)\mathrm{d}s \le C$$

for some constant $C>0$ independent of n and c_1'. □

Proof of Theorem 2.1. To see that the unique solution X from Lemma 2.2 is a regular solution, let us recall the construction of X given in the proof of [10, Theorem 2.1]. By Lemma 2.3 and **(A3')**, there exists a subsequence $n_k\to\infty$, an adapted $X\in L^\infty([0,T]\to L^{1+r}(\mathbb{P}\times\mu))$, an adapted $A\in L^2(\Omega\times[0,T]\to H^{-1};\mathbb{P}\times\mathrm{d}t)$, and some element $Z\in L^2(\Omega\times[0,T]\to \mathscr{L}_{HS};\mathbb{P}\times\mathrm{d}t)$ such that

(i) $X^{n_k}\to X$ $*$-weakly in $L^\infty([0,T]\to L^{1+r}(\mathbb{P}\times\mu))$.

(ii) $L\Psi_{n_k}(X^{n_k})+\Phi_{n_k}(X^{n_k})\to A$ weakly in $L^2(\Omega\times[0,T]\to H^{-1};\mathbb{P}\times\mathrm{d}t)$.

(iii) $\sigma(X^{n_k}) \to Z$ weakly in $L^2(\Omega \times [0,T] \to \mathscr{L}_{HS}; \mathbb{P} \times \mathrm{d}t)$.

Since these convergence properties are stronger than those used in the proof of [10, Theorem 2.1] for $p=2$ and the spaces K, V, V^*, H given in the proof of Lemma 2.2, the arguments in the proof of [10, Theorem 2.1] imply that $Z = \sigma_\cdot(X_\cdot)$ and $A = L\Psi(X) + \Phi(X)$, $\mathbb{P} \times \mathrm{d}t$-a.e., and

$$X_t = \int_0^t A_s \mathrm{d}s + \int_0^t Z_s \mathrm{d}W_s, \quad t \in [0,T]. \tag{2.5}$$

We are now able to prove the desired regularity properties as follows.

(a) Since $A \in L^2(\Omega \times [0,T] \to H^{-1}; \mathbb{P} \times \mathrm{d}t)$ and $A = L\Psi(X) + \Phi(X)$ $\mathbb{P} \times \mathrm{d}t$-a.e., we have $L\Psi(X) + \Phi(X) \in L^2(\Omega \times [0,T] \to H^{-1}; \mathbb{P} \times \mathrm{d}t)$. Moreover, since Φ is Lipschitz continuous and

$$\|\cdot\|_{1+r} \geq \|\cdot\| \geq \frac{1}{\sqrt{\lambda_1}} \|\cdot\|_{H^{-1}},$$

Lemma 2.2 implies that $\Phi(X) \in L^2(\Omega \times [0,T] \to H^{-1}; \mathbb{P} \times \mathrm{d}t)$. Therefore, $L\Psi(X) \in L^2(\Omega \times [0,T] \to H^{-1}; \mathbb{P} \times \mathrm{d}t)$, that is,

$$\mathbb{E} \int_0^T \mathscr{E}(\Psi(X_t), \Psi(X_t)) \mathrm{d}t = \mathbb{E} \int_0^T \|L\Psi(X_t)\|_{H^{-1}}^2 < \infty. \tag{2.6}$$

Since **(A2)** implies $\Psi' \geq c_1$ so that $\mathscr{E}(X_s, \Psi(X_s)) \geq c_1 \mathscr{E}(X_s, X_s)$, it follows from (2.6) that

$$c_1' \mathbb{E} \int_0^T \mathscr{E}(X_t, X_t) \mathrm{d}t < \infty.$$

(b) When $\Psi(s) = c|s|^{r-1}s$ and $\Phi(s) = c's$ for $c > 0$, the right continuity of the solution in $L^2(\mathbf{m})$ and $\mathbb{E} \sup_{t \in [0,T]} \|X_t\|^2 < \infty$ are ensured by [11, Theorem 1.2(4)]. Let $c_1' > 0$, so that $X \in L^2(\Omega \times [0,T] \to H^1; \mathbb{P} \times \mathrm{d}t)$. To see that X_t is continuous in $L^2(\mu)$, we make use of [10, Theorem A.2]. Let now $K = L^2(\Omega \times [0,T] \to H^1; \mathbb{P} \times \mathrm{d}t), H = L^2(\mu), V = H^1$ and $V^* = H^{-1}$. Then the condition (K) in [10] holds for $R(u) = \mathscr{E}(u,u) = \|u\|_{H^1}^2$ and $W_1(s) = W_2(s) = \sqrt{s}, s \geq 0$. Since $A \in K^* := L^2(\Omega \times [0,T] \to H^{-1}; \mathbb{P} \times \mathrm{d}t)$ and $Z \in J := L^2(\Omega \times [0,T] \to \mathscr{L}_{HS}; \mathbb{P} \times \mathrm{d}t)$ (see (2.8) in [10]), according to [10, Theorem A.2], (2.5) implies that X_t is continuous in $H (= L^2(\mu))$ such that $\mathbb{E} \sup_{t \in [0,T]} \|X_t\|^2 < \infty$ and that the Itô formula

$$\|X_t\|^2 = \|X_0\|^2 + \int_0^t \Big\{ 2_{H^1}\langle X_s, A_s \rangle_{H^{-1}} + \|\sigma_s(X_s)\|_{\mathscr{L}_{HS}}^2 \Big\} \mathrm{d}s + 2 \int_0^t \langle \sigma_s(X_s) \mathrm{d}W_s, X_s \rangle, \quad t \in [0,T],$$

holds. This coincides with the desired Itô formula since $A = L\Psi(X) + \Phi(X)$ $\mathbb{P} \times \mathrm{d}t$-a.e.

(c) It remains to show that $\sup_{t \in [0,T]} \mathbb{E}\|X_t\|_{1+r}^{1+r} < \infty$ for cases (1) and (2). Since $X \in L^\infty([0,T] \to L^{1+r}(\mathbb{P} \times \mu))$, there exists a constant $C > 0$ such that $\mathbb{E}\|X_t\|_{1+r}^{1+r} \leq C$ holds dt-a.e. Since X_t is right-continuous in $L^2(\mu)$, this and the Fatou lemma imply that for any $t\in[0,T]$,

$$\begin{aligned}\mathbb{E}\|X_t\|_{r+1}^{r+1} &= \mathbb{E}\sup_u|\mu(X_tu)|^{1+r} = \mathbb{E}\sup_u\liminf_{s\downarrow t}|\mu(X_su)|^{1+r}\\ &\le \liminf_{s\downarrow t}\mathbb{E}\sup_u|\mu(X_su)|^{1+r} \le \liminf_{s\downarrow t}\mathbb{E}\|X_s\|_{1+r}^{1+r}\le C,\end{aligned}$$

where sup is taken over all $u\in L^2(\mu)$ with $\|u\|_{\frac{r}{1+r}}\le 1$. □

2.2 Comparison theorem and L^1-Lipschitz continuity

In this subsection we consider the following equation with additive noise:

$$\text{(2.7)}\qquad \mathrm{d}X_t = \big\{L\Psi(X_t)+\Phi(X_t)\big\}\mathrm{d}t+\sigma_t\mathrm{d}W_t,$$

where σ_t,Ψ and Φ satisfy **(A2)** and **(A3)**. Let $\tilde\Phi$ be another Lipshitz continuous function. We shall compare regular solutions to (2.7) with those to the equation

$$\text{(2.8)}\qquad \mathrm{d}\tilde X_t = \big\{L\Psi(\tilde X_t)+\tilde\Phi(\tilde X_t)\big\}\mathrm{d}t+\sigma_t\mathrm{d}W_t.$$

Theorem 2.4. *Assume* **(A2)**, **(A3)** *and let $\tilde\Phi\le\Phi$. Let X_t and $\tilde X_t$ be solutions in the sense of Definition* 2.1 *to* (2.7) *and* (2.8) *respectively. If either $c_1'>0$, or $\Psi(s)=c|s|^{r-1}s,\Phi(s)=c's$ for some $c>0$ and $c'\in\mathbb{R}$, then these solutions are regular and $\mathbb{P}$-a.s. $\tilde X_0\le X_0$ implies $\mathbb{P}$-a.s. $\tilde X_t\le X_t$ for all $t\in[0,T]$.*

Let us first explain the main idea of the proof. The regularity of the solutions follows from Theorem 2.1. To prove $\tilde X_t\le X_t$, let $h_k\in C_b^1(\mathbb{R})$ such that $h_k'\ge 0$, $0\le h_k\le 1, h_k(s)=0$ for $s\le 0$, and $h_k\to 1_{(0,\infty)}$ as $k\to\infty$. By the definition of regular solutions, $\mathbb{P}$-a.s., $\tilde X_t-X_t$ is dt-a.e. differentiable in H^{-1} with

$$\frac{\mathrm{d}}{\mathrm{d}t}(\tilde X_t-X_t) = L\big\{\Psi(\tilde X_t)-\Psi(X_t)\big\}+\tilde\Phi(\tilde X_t)-\Phi(X_t).$$

Moreover, $h_k\big(\Psi(\tilde X_t)-\Psi(X_t)\big)\in H^1$ $\mathbb{P}\times\mathrm{d}t$-a.e. Therefore, noting that Φ is Lipschitzian and $\tilde\Phi\le\Phi$, we have $\mathbb{P}\times\mathrm{d}t$-a.e.,

$$\text{(2.9)}\qquad\begin{aligned}&\int_E\Big\{h_k\big(\Psi(\tilde X_t)-\Psi(X_t)\big)\frac{\mathrm{d}}{\mathrm{d}t}(\tilde X_t-X_t)\Big\}\mathrm{d}\mu\\ &=-\int_E\mathscr{E}\big(h_k\big(\Psi(\tilde X_t)-\Psi(X_t)\big),\Psi(\tilde X_t)-\Psi(X_t)\big)\mathrm{d}\mu\\ &\qquad+\int_E h_k\big(\Psi(\tilde X_t)-\Psi(X_t)\big)\cdot\big(\tilde\Phi(\tilde X_t)-\Phi(X_t)\big)\mathrm{d}\mu\\ &\le l_0\int_E h_k\big(\Psi(\tilde X_t)-\Psi(X_t)\big)\cdot|\tilde X_t-X_t|\mathrm{d}\mu,\end{aligned}$$

where l_0 is the Lipschitz constant of Φ. By letting $k \to \infty$ we may write formally

$$\text{(2.10)} \qquad \text{“} \frac{\mathrm{d}}{\mathrm{d}t}\mu\big((\tilde{X}_t - X_t)^+\big) = \int_E \Big\{1_{\{\tilde{X}_t > X_t\}} \frac{\mathrm{d}}{\mathrm{d}t}(\tilde{X}_t - X_t)\Big\}\mathrm{d}\mu \text{ ”} \le l_0 \mu((\tilde{X}_t - X_t)^+),$$

and hence, $(\tilde{X}_t - X_t)^+ = 0$ if $\tilde{X}_0 \le X_0$ as desired. The last step is however not rigorous since $\frac{\mathrm{d}}{\mathrm{d}t}(\tilde{X}_t - X_t)$ exists only in H^{-1} so that the terms in “ ... ” do not make sense in general. To make the argument rigorous, we consider the following approximating equations for $\varepsilon \in (0,1)$:

$$\text{(2.11)} \qquad \begin{aligned} \mathrm{d}X_t^\varepsilon &= \big\{(1-\varepsilon L)^{-1} L\Psi(X_t^\varepsilon) + \Phi(X_t^\varepsilon)\big\}\mathrm{d}t + \sigma_t \mathrm{d}W_t, \quad X_0^\varepsilon = X_0, \\ \mathrm{d}\tilde{X}_t^\varepsilon &= \big\{(1-\varepsilon L)^{-1} L\Psi(\tilde{X}_t^\varepsilon) + \Phi(\tilde{X}_t^\varepsilon)\big\}\mathrm{d}t + \sigma_t \mathrm{d}W_t, \quad \tilde{X}_0^\varepsilon = \tilde{X}_0. \end{aligned}$$

Lemma 2.5. *We have*

$$\lim_{\varepsilon \to 0} \mathbb{E} \int_0^T \Big(\|X_t - X_t^\varepsilon\|_{1+r}^{1+r} + \|\tilde{X}_t - \tilde{X}_t^\varepsilon\|_{1+r}^{1+r}\Big)\mathrm{d}t = 0.$$

Proof. We only consider the limit for $X_t - X_t^\varepsilon$. Since Φ is Lipschitz continuous, there exists a constant $C_1 > 0$ independent of ε such that

$$\langle \Phi(X_t) - \Phi(X_t^\varepsilon), X_t - X_t^\varepsilon \rangle_{H^{-1}} \le C_1 \|X_t - X_t^\varepsilon\| \cdot \|X_t - X_t^\varepsilon\|_{H^{-1}}.$$

Moreover, from the proof of Lemma 2.2 we see that

$$\sup_{\varepsilon \in (0,1)} \mathbb{E} \int_0^T (\|X_t\|_{1+r}^{1+r} + \|X_t^\varepsilon\|_{1+r}^{1+r})\mathrm{d}t < \infty.$$

Combining this with the growth condition $|\Psi(s)| \le c'(1+|s|^r)$ ensured by **(A2)**, we obtain

$$\begin{aligned} &\mathbb{E} \int_0^T \big|\langle (1-(1-\varepsilon L)^{-1})(X_t - X_t^\varepsilon), \Psi(X_t^\varepsilon)\rangle\big|\mathrm{d}t \\ &\le \mathbb{E} \int_0^T \|\Psi(X_t^\varepsilon)\| \cdot \|X_t - X_t^\varepsilon\|\mathrm{d}t \le C \end{aligned}$$

for a constant $C > 0$ independent of $\varepsilon \in (0,1)$. Therefore, by **(A2)** and the Itô formula,

$$\begin{aligned} \mathbb{E}\|X_t - X_t^\varepsilon\|_{H^{-1}}^2 &\le -2c_1 \int_0^t \mathbb{E}\|X_s - X_s^\varepsilon\|_{1+r}^{1+r}\mathrm{d}s + C_1 \int_0^t \mathbb{E}\|X_s - X_s^\varepsilon\|_{H^{-1}}^2 \mathrm{d}s \\ &\quad + 2\varepsilon \int_0^t \mathbb{E}\Big(|\langle (1-(1-\varepsilon L)^{-1})(X_s - X_s^\varepsilon), \Psi(X_s^\varepsilon)\rangle| + |\langle (1-\varepsilon L)^{-1}(X_s - X_s^\varepsilon), X_s^\varepsilon\rangle|\Big)\mathrm{d}s \\ &\le C_2 \int_0^t \mathbb{E}\|X_s - X_s^\varepsilon\|_{H^{-1}}^2 \mathrm{d}s - 2c_1 \int_0^t \mathbb{E}\|X_s - X_s^\varepsilon\|_{1+r}^{1+r}\mathrm{d}s + 2C_2\varepsilon, \quad t \in [0,T] \end{aligned}$$

holds for some constants $C_1, C_2 > 0$. This implies $\lim_{\varepsilon \to 0} \mathbb{E} \int_0^T \|X_t - X_t^\varepsilon\|_{1+r}^{1+r}\mathrm{d}t = 0$. □

Proof of Theorem 2.4. Since $(1-\varepsilon L)^{-1}L$ is a bounded operator for any $\varepsilon > 0$, the associated Dirichlet space and its dual space w.r.t. $L^2(\mu)$ coincide with $L^2(\mu)$. So, by Definition 2.1, $\frac{\mathrm{d}(\tilde{X_t}^{\varepsilon}-X_t^{\varepsilon})}{\mathrm{d}t}$ exists in $L^2(\Omega\times[0,T]\to L^2(\mathbf{m});\mathbb{P}\times\mathrm{d}t)$. We aim to prove, instead of (2.10), that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mu\big((\tilde{X}_t^{\varepsilon}-X_t^{\varepsilon})^+\big)=\int_E\Big\{1_{\{\tilde{X}_t>X_t\}}\frac{\mathrm{d}}{\mathrm{d}t}(\tilde{X}_t^{\varepsilon}-X_t^{\varepsilon})\Big\}\mathrm{d}\mu\le l_0\mu(\tilde{X}_t^{\varepsilon}-X_t^{\varepsilon})^+), \tag{2.12}$$

which implies $\tilde{X}_t^{\varepsilon}\le X_t^{\varepsilon}$ for all $t\in[0,T]$ since $\tilde{X}_0^{\varepsilon}\le X_0^{\varepsilon}$. Firstly, replacing $(\Psi,X_t,\tilde{X}_t)$ in (2.9) by $(\Psi,X_t^{\varepsilon},\tilde{X}_t^{\varepsilon})$ and letting $k\to\infty$, we obtain the inequality in (2.12). To verify the equality in (2.12), we note that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\tilde{X}_t^{\varepsilon}-X_t^{\varepsilon})=(1-\varepsilon L)^{-1}L\big(\Psi(\tilde{X}_t^{\varepsilon})-\Psi(X_t^{\varepsilon})\big)+\tilde{\Phi}(\tilde{X}_t)-\Phi(X_t^{\varepsilon})$$

and $(1-\varepsilon L)^{-1}L=\frac{1}{\varepsilon}(1-\varepsilon L)^{-1}-\frac{1}{\varepsilon}$ on $L^1(\mu)$ imply

$$\begin{aligned}&\sup_{0\le s<t\le T}\left|\frac{(\tilde{X}_t^{\varepsilon}-X_t^{\varepsilon})^+-(\tilde{X}_s^{\varepsilon}-X_s)^+}{t-s}\right|\\&\le\frac{1}{\varepsilon}\big(1+(1-\varepsilon L)^{-1}\big)\sup_{r\in[0,T]}\Big\{|\Psi(\tilde{X}_r^{\varepsilon})|+|\Psi(X_r^{\varepsilon})|+|\tilde{\Phi}(\tilde{X}_r^{\varepsilon})|+|\Phi(X_r^{\varepsilon})|\Big\}.\end{aligned}$$

By the contraction property of $(1-\varepsilon L)^{-1}$ on $L^1(\mu)$, Lemma 2.2, and the growth conditions on $\Psi,\Phi,\tilde{\Phi}$, we see that the upper bound is in $L^1(\mu)$. Therefore, the equality in (2.12) follows from the dominated convergence theorem with $s\to t$.

Now, by $\tilde{X}_t^{\varepsilon}\le X_t^{\varepsilon}$, we have

$$\mathbb{E}\int_0^T\mu\big((\tilde{X}_t^{\varepsilon}-X_t^{\varepsilon})^+\big)\mathrm{d}t=0,\quad\varepsilon\in(0,1).$$

Letting $\varepsilon\to0$ and using Lemma 2.5, we arrive at

$$\mathbb{E}\int_0^T\mu\big((\tilde{X}_t-X_t)^+\big)\mathrm{d}t=0.$$

Therefore, $\tilde{X}_t\le X_t$ holds $\mathbb{P}\times\mathrm{d}t\times\mu$. Since due to Theorem 2.1 X_t and $\tilde{X}_t$ are right-continuous in $L^2(\mu)$, we conclude that $\mathbb{P}$-a.s., $\tilde{X}_t\le X_t$ in $L^2(\mu)$ holds for all $t\in[0,T]$. □

Next, we have the following L^1-Lipschitz continuity w.r.t. initial data of the solutions.

Theorem 2.6. *Assume* **(A2)**,**(A3)** *and* (1.5). *We have*

$$\|X_t(x)-X_t(y)\|_1\le\mathrm{e}^{Kt}\|x-y\|_1,\quad x,y\in L^{1+r}(\mu).$$

Proof. Let $X_t^\varepsilon(x)$ be as in (2.11) for $X_0 = x \in L^{1+r}(\mu)$. Repeating the proof of Theorem 2.4 with $(\tilde{X}^\varepsilon, X^\varepsilon)$ replaced by $(X^\varepsilon(x), X^\varepsilon(y))$, we obtain

$$\mathrm{d}\|(X_t^\varepsilon(x)-X_t^\varepsilon(y))^+\|_1 = \langle 1_{\{X_t^\varepsilon(x)-X_t^\varepsilon(y)>0\}}, \mathrm{d}(X_t^\varepsilon(x)-X_t^\varepsilon(y))\rangle \mathrm{d}t \le K\|(X_t^\varepsilon(x)-X_t^\varepsilon(y))^+\|_1 \mathrm{d}t.$$

Then

$$\|(X_t^\varepsilon(x) - X_t^\varepsilon(y))^+\|_1 \le \mathrm{e}^{Kt}\|x-y\|_1.$$

The same holds by switching x and y so that

$$\|X_t^\varepsilon(x) - X_t^\varepsilon(y)\|_1 \le \mathrm{e}^{Kt}\|x-y\|_1, \quad t \ge 0.$$

Since due to Lemma 2.5 there exists a sequence $\varepsilon_n \downarrow 0$ such that for any $T > 0$

$$\lim_{n\to\infty} \int_0^T \big(\|X_t^{\varepsilon_n}(x) - X_t(x)\|_{1+r}^{1+r} + \|X_t^{\varepsilon_n}(y) - X_t(y)\|_{1+r}^{1+r}\big)\mathrm{d}t = 0,$$

this implies that $\|X_t(x) - X_t(y)\|_1 \le \mathrm{e}^{Kt}\|x-y\|_1$ holds $\mathrm{d}t$-a.e. Then the proof is finished by the continuity of the solutions. □

2.3 Riesz-Markov representation theorem

Let $\tilde{E}$ be a locally compact separable metric space so that $\sigma(C_0(\tilde{E})) = \tilde{\mathscr{B}}$ (the Borel σ-field on $\tilde{E}$), and let $\tilde{\mathscr{C}} \subset C_0(\tilde{E})$ be a subspace such that the following assumption holds:

(A) for any $f \in C_0(\tilde{E})$, there exists $\tilde{f} \in \tilde{\mathscr{C}}$ such that for any $\varepsilon > 0$, there exists $f_\varepsilon \in \tilde{\mathscr{C}}$ such that $|f - f_\varepsilon| \le \varepsilon \tilde{f}$.

Let $C_0^+(\tilde{E})$ and $\tilde{\mathscr{C}}^+$ denote the classes of non-negative elements in $C_0(\tilde{E})$ and $\tilde{\mathscr{C}}$ respectively.

Theorem 2.7. *Assume* **(A)**. *For any positive linear functional* $\Lambda : \tilde{\mathscr{C}} \to \mathbb{R}$, *there exists a unique measure* μ *on* $\tilde{E}$ *such that*

$$\mu(f) := \int_{\tilde{E}} f \mathrm{d}\mu = \Lambda(f), \quad f \in \tilde{\mathscr{C}}. \tag{2.13}$$

Proof. (a) The uniqueness. Let μ and $\tilde{\mu}$ be two measures satisfying (2.13), then for any $f \in C_0(\tilde{E})$, and for $\tilde{f}$ and f_ε in **(A)**, we have $f_\varepsilon + \varepsilon f \in \tilde{\mathscr{C}}$ so that

$$\mu(f) \le \mu(f_\varepsilon + \varepsilon\tilde{f}) = \tilde{\mu}(f_\varepsilon + \varepsilon\tilde{f}) \le \tilde{\mu}(f) + 2\varepsilon\Lambda(\tilde{f}).$$

Letting $\varepsilon \to 0$ we obtain $\mu(f) \le \tilde{\mu}(f)$. Similarly, $\tilde{\mu}(f) \le \mu(f)$. Therefore, $\mu = \tilde{\mu}$.

(b) The existence. For any $f \in C_0^+(\tilde{E})$, let

$$\bar{\Lambda}(f) = \sup\{\Lambda(g) : g \le f, g \in \tilde{\mathscr{C}}\}.$$

Since $0 \in \tilde{\mathscr{C}}$, we have $\bar{\Lambda}(f) \geq 0$ for $f \in C_0^+(\tilde{E})$. Next, it is easy to see that $\bar{\Lambda}$ is increasing monotone and $\bar{\Lambda} = \Lambda$ holds on $\tilde{\mathscr{C}}^+$. Moreover, by **(A)**, for $f \in C_0^+(\tilde{E})$ there exists $\tilde{f}, g \in \tilde{\mathscr{C}}$ such that $|f - g| \leq \tilde{f}$. Then $\tilde{f} + g \in \tilde{\mathscr{C}}^+$ so that

$$\Lambda(f) \leq \bar{\Lambda}(\tilde{f} + g) = \Lambda(\tilde{f} + g) < \infty.$$

Therefore, letting $\bar{\Lambda}(f) = \bar{\Lambda}(f^+) - \bar{\Lambda}(f^-)$, we extend $\bar{\Lambda}$ to a finite positive functional on $C_0(\tilde{E})$ such that $\bar{\Lambda} = \Lambda$ holds on $\tilde{\mathscr{C}}$. Then it suffices to show that

$$\Lambda(f + g) = \Lambda(f) + \bar{\Lambda}(g), \quad f, g \in C_0(\tilde{E}). \tag{2.14}$$

Indeed, it is trivial to see that $\bar{\Lambda}(cf) = c\bar{\Lambda}(f)$ for $f \in C_0(\tilde{E})$ and $c \in \mathbb{R}$. Then (2.14) implies that $\Lambda : C_0(\tilde{E}) \to \mathbb{R}$ is a positive linear functional. By the Riesz-Markov representation theorem, there exists a unique locally finite measure μ on $\tilde{E}$ such that

$$\mu(f) = \bar{\Lambda}(f), \quad f \in C_0(\tilde{E}).$$

Since $\Lambda(f) = \bar{\Lambda}(f)$ holds for $f \in \tilde{\mathscr{C}}$, this implies (2.13).

Now, let $f, g \in C_0(\tilde{E})$. By **(A)**, there exist $\tilde{f}, \tilde{g} \in \tilde{\mathscr{C}}$ such that for any $\varepsilon > 0$ there exist $f_\varepsilon, g_\varepsilon \in \tilde{\mathscr{C}}$ such that $|f - f_\varepsilon| \leq \varepsilon\tilde{f}, |g - g_\varepsilon| \leq \varepsilon\tilde{g}$. We have

$$\begin{aligned}
&\bar{\Lambda}(f + g) \leq \bar{\Lambda}(f_\varepsilon + g_\varepsilon + \varepsilon\tilde{f} + \varepsilon\tilde{g}) \\
&= \Lambda(f_\varepsilon - \varepsilon\tilde{f}) + \Lambda(g_\varepsilon - \varepsilon\tilde{g}) + 2\varepsilon\Lambda(\tilde{f} + \tilde{g}) \leq \bar{\Lambda}(f) + \bar{\Lambda}(g) + 2\varepsilon\Lambda(\tilde{f} + \tilde{g}),
\end{aligned}$$

and conversely,

$$\begin{aligned}
&\bar{\Lambda}(f) + \bar{\Lambda}(g) \leq \Lambda(f_\varepsilon + \varepsilon\tilde{f}) + \Lambda(g_\varepsilon + \varepsilon\tilde{g}) \\
&= \Lambda(f_\varepsilon + g_\varepsilon - \varepsilon\tilde{f} - \varepsilon\tilde{g}) + 2\varepsilon\Lambda(\tilde{f} + \tilde{g}) \leq \bar{\Lambda}(f + g) + 2\varepsilon\Lambda(\tilde{f} + \tilde{g}).
\end{aligned}$$

Letting $\varepsilon \to 0$ we prove (2.14). □

3 Proof of Theorem 1.1

For $n \geq 1$, let

$$\Phi^{(n)}(s) = \Phi(s) + ns^-, \quad s \in \mathbb{R}.$$

Consider the following penalized equation:

$$\mathrm{d}X_t^{(n)} = L\Psi(X_t^{(n)})dt + \Phi^{(n)}(X_t^{(n)})\mathrm{d}t + \sigma_t \mathrm{d}W_t, \quad X_0^{(n)} = X_0. \tag{3.1}$$

Let

$$\nu_t^{(n)}(\mathrm{d}z) = n\bigg(\int_0^t (X_s^{(n)}(z))^- \mathrm{d}s\bigg)\mu(\mathrm{d}z), \quad n \geq 1.$$

By Theorem 2.1, for each n, this equation has a unique regular solution in the sense of Definition 2.1. We will show that $(X_t, \nu_t) = \lim_{n\to\infty}(X_t^{(n)}, \nu^{(n)})$ exists and gives rise to a solution to equation (1.1) via (1.2).

3.1 Construction and properties of X

By Theorem 2.1 and Theorem 2.4, $\{X^{(n)}\}_{n\geq 1}$ is an increasing sequence of continuous adapted processes in $L^2(\mu)$ such that

$$X^{(n)}, \Psi(X^{(n)}) \in L^2(\Omega\times[0,T]\to H^1;\mathbb{P}\times \mathrm{d}t),\quad X^{(n+1)}\geq X^{(n)},\quad n\geq 1. \tag{3.2}$$

Let

$$X=\lim_{n\to\infty} X^{(n)}.$$

Lemma 3.1. *$X^{(n)}\to X$ in $L^2(\Omega\times[0,T]\times E;\mathbb{P}\times \mathrm{d}t\times\mu)$ and*

$$\mathbb{E}\sup_{t\in[0,T]}\|X_t\|^2<\infty. \tag{3.3}$$

Consequently, $X_t^{(n)}\to X_t$ holds in $L^2(\Omega\times E;\mathbb{P}\times\mu)$ for all $t\in[0,T]$.

Proof. By the Itô formula in Theorem 2.1 and using **(A2)**, **(A3)**, we obtain

$$\begin{aligned}\mathrm{d}\|X_t^{(n)}\|^2 &\leq \big\{2\langle\Phi(X_t^{(n)},X_t^{(n)}\rangle+2n\langle(X_t^{(n)})^-,X_t^{(n)}\rangle\\ &\qquad+\|\sigma_t\|^2_{\mathscr{L}_{LS}}\big\}\mathrm{d}t+2\langle\sigma_t\mathrm{d}W_t,X_t^{(n)}\rangle\\ &\leq\big\{C_1+C_1\|X_t^{(n)}\|^2\big\}\mathrm{d}t+2\langle\sigma_t\mathrm{d}W_t,X_t^{(n)}\rangle\end{aligned}$$

for some constant $C_1>0$ independent of n. As shown in the second part in the proof of Lemma 2.3, this implies

$$\mathbb{E}\sup_{t\in[0,T]}\|X_t^{(n)}\|^2\leq C \tag{3.4}$$

for some constant independent of n. Noting that

$$\|(X_t^{(n)})^+\|\uparrow\|X_t^+\|,\quad \|X_t^-\|\leq\|(X_t^{(1)})^-\|,$$

this implies

$$\begin{aligned}\mathbb{E}\sup_{t\in[0,T]}\|X_t\|^2&\leq\mathbb{E}\sup_{t\in[0,T]}\sup_{n\geq1}\|(X_t^{(n)})^+\|+\mathbb{E}\sup_{t\in[0,T]}\|X_t^{(1)}\|^2\\ &=\lim_{n\to\infty}\mathbb{E}\sup_{t\in[0,T]}\|(X_t^{(n)})^+\|+\mathbb{E}\sup_{t\in[0,T]}\|X_t^{(1)}\|^2\leq 2C.\end{aligned}$$

Since $X^{(n)}\uparrow X$, $\mathbb{P}\times\mathrm{d}t\times\mu$-a.e. and $|X^{(n)}|\leq X^++(X^{(1)})^-$, by the dominated convergence theorem we conclude that $X^{(n)}\to X$ in $L^2(\Omega\times[0,T]\times E;\mathbb{P}\times\mathrm{d}t\times\mu)$. □

Lemma 3.2. *$\Psi(X^{(n)})\to\Psi(X)$ weakly in $L^2(\Omega\times[0,T]\to H^1;\mathbb{P}\times\mathrm{d}t)$ and*

$$\sup_{t\in[0,T]}\mathbb{E}\|X_t\|_{1+r}^{1+r}+\mathbb{E}\int_0^T\mathscr{E}(\Psi(X_t),\Psi(X_t))\mathrm{d}t<\infty.$$

Moreover, $X\geq 0$, $\mathbb{P}\times\mathrm{d}t$-a.e.

Proof. For $m \geq 1$, let $\phi_m \in C_b^\infty(\mathbb{R})$ such that $0 \leq \phi_m' \leq 2$ and

$$\phi_m(s) = \begin{cases} s, & \text{if } |s| \leq m, \\ m+1, & \text{if } s \geq m+1, \\ -m-1, & \text{if } s \leq -m-1. \end{cases}$$

Define

$$F_m(u) = \int_E \mathrm{d}\mu \int_0^u \Psi \circ \phi_m(s)\mathrm{d}s, \quad u \in L^2(\mu).$$

Then $F_m \in C_b^2(L^2(\mu))$ with

$$\begin{aligned} \partial_{v_1} F_m(u) &= \int_E \Psi(\phi_m(u))v_1 \mathrm{d}\mu, \\ \partial_{v_1}\partial_{v_2} F(u) &= \int_E \Psi' \circ \phi_m(u)\phi_m'(u)v_1 v_2 \mathrm{d}\mu, \quad u, v_1, v_2 \in L^2(\mu). \end{aligned}$$

Since due to Theorem 2.1 we have $X^{(n)}, \Psi(X^{(n)}) \in L^2(\Omega \times [0,T] \to H^1)$, by the Itô formula we obtain

$$\begin{aligned} \mathrm{d}F_m(X_t^{(n)}) = &\Big\{ \langle \Phi(X_t^{(n)}), \Psi \circ \phi_m(X_t^{(n)})\rangle - \mathscr{E}\big(\Psi(X_t^{(n)}), \Psi \circ \phi_m(X_t^{(n)})\big) \\ &+ n\langle (X_t^{(n)})^-, \Psi \circ \phi_m(X_t^{(n)})\rangle + \frac{1}{2}\sum_{i=1}^\infty \int_E (\sigma_t e_i)^2 \Psi' \circ \phi_m(X_t^{(n)})\phi_m'(X_t^{(n)})\mathrm{d}\mu \Big\}\mathrm{d}t \\ &+ \langle \sigma_t \mathrm{d}W_t, \Psi \circ \phi_m(X_t^{(n)})\rangle. \end{aligned} \tag{3.5}$$

Since $\Psi(0) = 0, \phi_m(s) \geq 0$ for $s \leq 0$, and $\Psi' \geq 0$ imply $s^- \Psi \circ \phi_m(s) \leq 0$, we have

$$\langle (X_t^{(n)})^-, \Psi \circ \phi_m(X_t^{(n)})\rangle \leq 0.$$

Combining this with (3.5) and the property of ϕ_m, we obtain

$$\begin{aligned} F_m(X_{t_2}^{(n)}) - F_m(X_{t_1}^{(n)}) \leq &\int_{t_1}^{t_2} \Big\{ \langle \Phi(X_t^{(n)}), \Psi \circ \phi_m(X_t^{(n)})\rangle - \mathscr{E}\big(\Psi(X_t^{(n)}), \Psi \circ \phi_m(X_t^{(n)})\big)\Big\}\mathrm{d}t \\ + C_1 \sum_{i=1}^\infty \int_{t_1}^{t_2} \mu\big((\sigma_t e_i)^2(1 + |X_t^{(n)}|^{r-1})\big)\mathrm{d}t &+ \int_{t_1}^{t_2} \langle \sigma_t \mathrm{d}W_t, \Psi \circ \phi_m(X_t^{(n)})\rangle \end{aligned}$$

for some constant $C_1 > 0$ independent of m, n, and all $0 \leq t_1 \leq t_2 \leq T$. Letting $m \to \infty$ we arrive at

$$\begin{aligned} \mathrm{d}F(X_t^{(n)}) \leq &\Big\{ \langle\langle \Phi(X_t^{(n)}), \Psi(X_t^{(n)})\rangle - \mathscr{E}\big(\Psi(X_t^{(n)}), \Psi(X_t^{(n)})\big) \\ &+ C_1 \sum_{i=1}^\infty \int_{t_1}^{t_2} \mu\big((\sigma_t e_i)^2(1 + |X_t^{(n)}|^{r-1}\big)\Big\}\mathrm{d}t + \langle \sigma_t \mathrm{d}W_t, \Psi(X_t^{(n)})\rangle, \end{aligned}$$

where F is defined as in the proof of Lemma 2.3. Therefore, by repeating the proof of Lemma 2.3, we obtain

$$\sup_{t\in[0,T]} \mathbb{E}\|X_t^{(n)}\|_{1+r}^{1+r} + \mathbb{E}\int_0^T \mathscr{E}(\Psi(X_t^{(n)}), \Psi(X_t^{(n)}))\mathrm{d}t \le C \tag{3.6}$$

for some constant $C > 0$ independent of n. Since $X^{(n)} \uparrow X$ and Ψ is increasing and continuous, we have $\Psi(X^{(n)}) \uparrow \Psi(X)$. Therefore, as in the proof of Lemma 3.1 we see that (3.6) implies that $\Psi(X^{(n)}) \to \Psi(X)$ in $L^2(\Omega \times [0,T] \times \mathbb{E}; \mathbb{P} \times \mathrm{d}t \times \mu)$ and (at least a subsequence thereof) weakly in $L^2(\Omega \times [0,T] \to H^1)$, as well as

$$\sup_{t\in[0,T]} \mathbb{E}\|X_t\|_{1+r}^{1+r} + \mathbb{E}\int_0^T \mathscr{E}(\Psi(X_t), \Psi(X_t))\mathrm{d}t < \infty.$$

Finally, we prove that $X \ge 0$. To this end, let $u \in H^1$. Since $\Psi(X^{(n)}) \to \Psi(X)$ weakly in $L^2(\Omega \times [0,T] \to H^1)$, $X^{(n)} \to X$ in $L^2(\Omega \times [0,T] \times \mathbb{E}; \mathbb{P} \times \mathrm{d}t \times \mu)$, and Φ is Lipschitz continuous, we have, in $L^1(\mathbb{P})$

$$\begin{aligned}
&\int_0^T \langle X_t^-, u\rangle \mathrm{d}t = \lim_{n\to\infty} \int_{[0,T]\times E} u(X_t^{(n)})^- \mathrm{d}t\mathrm{d}\mu \\
&= \lim_{n\to\infty} \frac{1}{n}\bigg\{ \langle u, X_T - X_0\rangle - \int_0^T \Big\{\langle \Phi(X_t^{(n)}), u\rangle - \mathscr{E}(\Psi(X_t^{(n)}), u)\Big\}\mathrm{d}t - \int_0^T \langle \sigma_t \mathrm{d}W_t, u\rangle \bigg\} \\
&= 0.
\end{aligned}$$

Since H^1 is dense in $L^2(\mu)$, this implies that $\int_0^T X_t^- \mathrm{d}t = 0$ in $\mathbb{P} \times \mu$-a.e. Therefore, $X \ge 0, \mathbb{P} \times \mathrm{d}t \times \mu$-a.e. □

3.2 Construction and properties of η

Lemma 3.3. *As $n \to \infty$, $\nu_t^{(n)}$ converges vaguely to some locally bounded random measure $\bar{\nu}_t$ on E such that $\mathbb{P}$-a.s.*

$$\begin{aligned}
\bar{\nu}_t(f) =& \langle f, X_t - X_0\rangle - \Big\langle f, \int_0^t \sigma_s \mathrm{d}W_s \Big\rangle \\
&+ \int_0^t \Big\{\mathscr{E}(f, \Psi(X_s)) - \langle f, \Phi(X_s)\rangle\Big\}\mathrm{d}s, \quad f \in H^1 \cap C_0(E).
\end{aligned} \tag{3.7}$$

Consequently, $\bar{\nu}_t$ is an adapted increasing process on $\mathscr{M}_c$.

Proof. By (3.1) and noting that $\{\Psi(X^{(n)})\}_{n\ge1}$ is a bounded sequence in $L^2(\Omega \times [0,T] \to H^1; \mathbb{P} \times \mathrm{d}t)$, we have $\mathbb{P}$-a.s.

$$\begin{aligned}
\nu_t^{(n)}(f) &= n\int_E f(z)\mu(\mathrm{d}z)\int_0^t (X_s^{(n)}(z))^- \mathrm{d}s \\
&= \langle f, X_t^{(n)} - X_0\rangle - \Big\langle f, \int_0^t \sigma_s \mathrm{d}W_s\Big\rangle + \int_0^t \Big\{\mathscr{E}(f, \Psi(X_s^{(n)})) - \langle f, \Phi(X_s^{(n)})\rangle\Big\}\mathrm{d}s
\end{aligned}$$

for all $f \in H^1, t \in [0,T]$. According to Lemmas 3.1 and 3.2, selecting a subsequence if necessary, we conclude that $\mathbb{P}$-a.s.

$$
\begin{aligned}
\Lambda_t(f) &:= \lim_{n\to\infty} \nu_t^{(n)}(f) \\
&= \langle f, X_t - X_0\rangle - \left\langle f, \int_0^t \sigma_s \mathrm{d}W_s \right\rangle + \int_0^t \Big\{ \mathscr{E}(f, \Psi(X_s)) - \langle f, \Phi(X_s)\rangle \Big\} \mathrm{d}s
\end{aligned}
\tag{3.8}
$$

exists for all $f \in H^1$. Since $\nu_t^{(n)} \geq 0$, this implies that $\mathbb{P}$-a.s. $\Lambda_t : H^1 \cap C_0(E) \to \mathbb{R}$ is a positive linear functional. By **(A1)** and Theorem 2.7, $\mathbb{P}$-a.s. there exists a unique locally bounded measure $\bar{\nu}_t$ on E such that

$$
\bar{\nu}_t(f) := \int_E f(z)\bar{\nu}_t(\mathrm{d}z) = \Lambda_t(f), \quad f \in H^1 \cap C_0(E).
$$

Next, to see that $\nu_t^{(n)} \to \bar{\nu}_t$ vaguely, we first note that (3.8) and (3.7) imply

$$
\lim_{n\to\infty} \nu_t^{(n)}(f) = \bar{\nu}_t(f), \quad f \in H^1 \cap C_0(E). \tag{3.9}
$$

Now, let $f \in C_0(E)$. By **(A1)**, there exists $\tilde{f} \in H^1 \cap C_0(E)$ such that for any $\varepsilon > 0$, $|f - f_\varepsilon| \leq \varepsilon \tilde{f}$ holds for some $f_\varepsilon \in H^1 \cap C_0(E)$. Then

$$
\limsup_{n\to\infty} |\nu_t^{(n)}(f) - \bar{\nu}_t(f)| \leq \limsup_{n\to\infty} \big\{ |\nu_t^{(n)}(f_\varepsilon) - \bar{\nu}_t(f_\varepsilon)| + \varepsilon(\nu_t^{(n)}(\tilde{f}) + \bar{\nu}_t(\tilde{f})) \big\} = 2\varepsilon \bar{\nu}_t(\tilde{f}).
$$

Letting $\varepsilon \to 0$ we conclude that $\lim_{n\to\infty} \nu_t^{(n)}(f) = \bar{\nu}_t(f)$. Since $\{\bar{\nu}_t^{(n)}\}_{n\geq 1}$ are locally finite measures, $\bar{\nu}_t : C_0(E) \to \mathbb{R}$ is a non-negative linear functional and thus is realized by a locally finite measure according to the Riesz-Markov representation theorem, denoted again by $\bar{\nu}_t$. Finally, since $\nu_t^{(n)}$ is increasing in t, so is $\bar{\nu}_t$. □

To construct η, we observe from **(A3)**, the Lipschitz continuity of Φ and Lemma 2.5 that, (3.7) provides a bounded linear functional $\bar{\nu}_t : C_0(E) \cap H^1 \to \mathbb{R}$. Since the Dirichlet form is regular, $C_0(E) \cap H^1$ is dense in H^1, it can $\mathbb{P}$-a.s. be uniquely extended to an element $\eta_t \in H^{-1}$ such that $\mathbb{P}$-a.s.

$$
\begin{aligned}
{}_{H^{-1}}\langle \eta_t, f\rangle_{H^1} =& \langle f, X_t - X_0\rangle - \left\langle f, \int_0^t \sigma_s \mathrm{d}W_s \right\rangle \\
&+ \int_0^t \Big\{ \mathscr{E}(f, \Psi(X_s)) - \langle f, \Phi(X_s)\rangle \Big\} \mathrm{d}s, \quad f \in H^1.
\end{aligned}
\tag{3.10}
$$

Proposition 3.4. *X_t is weakly cádlág in $L^2(\mu)$, η_t is increasing and cádlág in H^{-1} and* (1.2) *holds.*

Proof. (3.7) and (3.10) imply (1.2) and that η_t is an increasing process in H^{-1}. In particular,

$$X_t = X_0 + \int_0^t \Big\{L\Psi(X_s) + \Phi(X_s)\Big\}\mathrm{d}s + \int_0^t \sigma_s \mathrm{d}W_s + \eta_t, \quad t \geq 0$$

holds in H^{-1}. Since the integral parts are continuous in H^{-1}, it remains to show that X_t is weakly cádlág in $L^2(\mu)$ and hence cádlág in H^{-1} as $\sup_{t\in[0,T]} \|X_t\|^2 < \infty$ for $T > 0$.

Since $\bar{\nu}_t$ is increasing in t,

$$\bar{\nu}_{t+} := \lim_{\varepsilon\downarrow 0} \bar{\nu}_{t+\varepsilon} \geq \bar{\nu}_t, \quad t \geq 0.$$

Then, it is easy to see from (3.7) and **(A1)** that X_t has weak left and right limits in $L^2(\mu)$ and its weak right limit X_{t+} satisfies

$$\langle X_{t+} - X_t, f\rangle = (\bar{\nu}_{t+} - \bar{\nu}_t)(f), \quad f \in C_0(E). \tag{3.11}$$

Since $\bar{\nu}_{t+} \geq \bar{\nu}_t$, this in particular implies that $X_{t+} \geq X_t$.

On the other hand, by Itô's formula and **(A2)**,

$$\|X_t^{(n)}\|^2 - \|X_s^{(n)}\|^2 \leq 2\int_s^t \langle \Phi(X_r^{(n)}), X_r^{(n)}\rangle \mathrm{d}r + \int_s^t \|\sigma_r\|_{HS}^2 \mathrm{d}r + 2\int_s^t \langle X_r^{(n)}, \sigma_r \mathrm{d}W_r\rangle, \quad 0 \leq s \leq t.$$

Since $X^{(n)} \uparrow X$, by (3.4), **(A2)**, **(A3)**, and letting $n \uparrow \infty$, we obtain

$$\|X_t\|^2 - \|X_s\|^2 \leq 2\int_s^t \langle \Phi(X_r), X_r\rangle \mathrm{d}r + \int_s^t \|\sigma_r\|_{HS}^2 \mathrm{d}r + 2\int_s^t \langle X_r, \sigma_r \mathrm{d}W_r\rangle, \quad 0 \leq s \leq t.$$

Therefore,

$$\|X_{t+}\|^2 \leq \liminf_{\varepsilon\downarrow 0} \|X_{(t+\varepsilon)\wedge T}\|^2 \leq \|X_t\|^2.$$

Combining this with $X_{t+} \geq X_t$, we conclude that $X_{t+} = X_t$, that is; X_t is weakly right continuous in $L^2(\mu)$. □

3.3 Proof of Theorem 1.1

(a) Existence. By Lemmas 3.1, 3.2, Proposition 3.4 and (3.10), it remains to show that ${}_{H^1}\langle \Psi(X_t), \eta_t\rangle_{H^{-1}} = 0, \mathrm{d}t$-a.e. Since $\Psi \in C^1$ with $\Psi(0) = 0$ and $\Psi' \geq 0$, by $X^{(n)} \uparrow X \geq 0$ and (3.2) we conclude that (up to a subsequence)

$$\Psi(X^{(n)^+}) = \Psi(X^{(n)})^+ \to \Psi(X)^+ = \Psi(X)$$

weakly $L^2([0,T] \to H^1; \mathrm{d}t)$. So,

$$\begin{aligned}
\int_0^T {}_{H^1}\langle \Psi(X_t), \eta_t\rangle_{H^{-1}} \mathrm{d}t &= \lim_{n\to\infty} \int_0^T {}_{H^1}\langle \Psi((X_t^{(n)})^+), \eta_t\rangle_{H^{-1}} \mathrm{d}t \\
&= \lim_{n\to\infty} \lim_{m\to\infty} \int_0^T \Psi((X_t^{(n)})^+)(z) \bar{\nu}_t^{(m)}(\mathrm{d}z) \\
&= \lim_{n\to\infty} \lim_{m\to\infty} m \int_0^T \Psi((X_t^{(n)})^+)(X_t^{(m)})^- \mathrm{d}t\mathrm{d}\mu = 0, \quad T > 0.
\end{aligned}$$

Since ${}_{H^1}\langle \Psi(X), \eta\rangle_{H^{-1}} \geq 0$, we prove that ${}_{H^1}\langle \Psi(X_t), \eta_t\rangle_{H^{-1}} = 0, \mathrm{d}t$-a.e.

(b) The Markov property. For simplicity, we set $X_t = X_t(x)$. Let $0 \leq s_1 < s_2 < \cdots < s_m \leq s < t$, and let $g \in C_b((H^{-1})^m)$. It remains to prove

$$\mathbb{E}\big\{f(X_t)g(X_{s_1}, \cdots, X_{s_m})\big\} = \mathbb{E}\big\{g(X_{s_1}, \cdots, X_{s_m})P_{t-s}f(X_s)\big\} \tag{3.12}$$

for any bounded and Lipschitz continuous f in $L^1(\mu)$. By the Markov property of $X^{(n)}$ we have

$$\mathbb{E}\big\{f(X_t^{(n)})g(X_{s_1}^{(n)}, \cdots, X_{s_m}^{(n)})\big\} = \mathbb{E}\big\{g(X_{s_1}^{(n)}, \cdots, X_{s_m}^{(n)})P_{t-s}^{(n)}f(X_{t-s}^{(n)})\big\},$$

where $P_{t-s}^{(n)}f(x) := \mathbb{E}f(X_t^{(n)}(x))$. Since $X^{(n)} \uparrow X$, and due to $\langle \Phi(x) - \Phi(y) + nx^- - ny^-, x - y\rangle \leq l_0(x-y)^+$ and Theorem 2.7, $P_{t-s}^{(n)}f$ is continuous in $L^1(\mu)$ (hence also $L^2(\mu)$) uniformly w.r.t. $n \geq 1$, by letting $n \to \infty$ we obtain (3.12).

(c) The L^1-Lipschitz continuity and consequences. Since (1.5) implies

$$\langle \Phi(x) - \Phi(y) + nx^- - ny^-, x - y\rangle \leq K(x-y)^+,$$

by applying Theorem 2.7 to $X^{(n)}$ and letting $n \to \infty$, we prove (1.6). Then, for any Lipschitz continuous function f on $L^1(\mu)$, and any $x \in L^1(\mu)$,

$$P_t f(x) := \lim_{n\to\infty} P_t f(x_n), \quad x_n \to x \text{ in } L^1(\mu), \text{ and } \{x_n\}_{n\geq 1} \subset L^{1+r}(\mu)$$

is well defined and provides a Markov Lipschitz-Feller semigroup on $L^1(\mu)$. Moreover, by (1.7) and Itô's formula we have

$$\frac{1}{T}\int_0^T \|X_t(0)\|_{H^1}^2 \mathrm{d}t \leq C, \quad T > 0,$$

for some constant $C > 0$. Since $\|\cdot\|_{H^1}^2$ is a compact function in $L^1(\mu)$, this implies that P_t has an invariant probability measure π with $\pi(\|\cdot\|_{H^1}^2) < \infty$. Finally, (1.8) follows from (1.6).

(d) Uniqueness. Let $\Psi(s) = cs$ for some constant $c > 0$, and let $(\tilde{X}, \tilde{\eta})$ be another solution. We have

$$\mathrm{d}\|X_t - \tilde{X}_t\|^2 = 2\big\{\langle X_t - \tilde{X}_t, \Phi(X_t) - \Phi(\tilde{X}_t)\rangle - c\mathscr{E}(X_t - \tilde{X}_t, X_t - \tilde{X}_t)\big\}\mathrm{d}t + 2{}_{H^1}\langle X_t - \tilde{X}_t, \mathrm{d}(\eta_t - \tilde{\eta}_t)\rangle_{H^{-1}}.$$

Since $c > 0$, Φ is Lipschitzian, $\mathrm{d}\eta_t, \mathrm{d}\tilde{\eta}_t \geq 0$ and

$${}_{H^1}\langle X_t, \mathrm{d}\eta_t\rangle_{H^{-1}} = {}_{H^1}\langle \tilde{X}_t, \mathrm{d}\tilde{\eta}_t\rangle_{H^{-1}} = 0,$$

this implies that

$$\mathrm{d}\|X_t - \tilde{X}_t\|^2 \leq 2l_0\|X_t - \tilde{X}_t\|^2 \mathrm{d}t.$$

Therefore, $X_t = \tilde{X}_t$ holds for $t \in [0, T]$ provided $X_0 = \tilde{X}_0$.

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