Stochastic nonlinear Schrödinger equations with linear multiplicative noise: the rescaling approach

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Abstract. We prove well-posedness results for stochastic nonlinear Schrödinger equations with linear multiplicative Wiener noise including the non-conservative case. Our approach is different from the standard literature on stochastic nonlinear Schrödinger equations. By a rescaling transformation we reduce the stochastic equation to a random nonlinear Schrödinger equation with lower order terms and treat the resulting equation by a fixed point argument, based on generalizations of Strichartz estimates proved by J. Marzuola, J. Metcalfe and D. Tataru in 2008. This approach allows to improve earlier well-posedness results obtained in the conservative case by a direct approach to the stochastic Schrödinger equation. In contrast to the latter, we obtain well-posedness in the full range $[1, 1+4/d)$ of admissible exponents in the non-linear part (where d is the dimension of the underlying Euclidean space), i.e. in exactly the same range as in the deterministic case.

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1 Introduction

We are here concerned with the stochastic nonlinear Schrödinger equation

$$\begin{aligned} idX(t,\xi) &= \Delta X(t,\xi)dt - i\mu(\xi)X(t,\xi)dt + \lambda|X(t,\xi)|^{\alpha-1}X(t,\xi)dt \\ &\qquad + iX(t,\xi)dW(t,\xi),\ t\in(0,T),\ \xi\in\mathbb{R}^d, \qquad (1.1)\\ X(0) &= x \in L^2(\mathbb{R}^d) := L^2(\mathbb{R}^d;\mathbb{C}), \end{aligned}$$

where W is the Wiener process

$$W(t,\xi) = \sum_{j=1}^{N} \mu_j e_j(\xi)\beta_j(t),\ t\ge 0,\ \xi\in\mathbb{R}^d, \tag{1.2}$$

$$\mu(\xi) = \frac{1}{2}\sum_{j=1}^{N} |\mu_j|^2|e_j(\xi)|^2,\ \xi\in\mathbb{R}^d. \tag{1.3}$$

Here, $\{e_j\}_{j=1}^N \subset C^\infty(\mathbb{R}^d)\cap L^2(\mathbb{R}^d;\mathbb{R})$, $\{\mu_j\}_{j=1}^N$ are complex valued and $\{\beta_j\}_{j=1}^N$ is a family of independent real valued Brownian motions on a probability space $(\Omega,\mathcal{F},\mathbb{P})$ with normal (in particular right-continuous) filtration $(\mathcal{F}_t)_{t\ge 0}$. In addition, $\lambda\in\mathbb{R}$ and $\alpha\ge 1$.

The physical significance of (1.1) is well known. $X = X(t,\xi,\omega)$, $\xi\in\mathbb{R}^d$, $t\ge 0$, $\omega\in\Omega$, represents the quantum state at time t, while the stochastic perturbation $iXdW$ represents a stochastic continuous measurement via the quantum observables $\mu_j e_j$. A better insight in equation (1.1) can be gained from the analysis in [3], [4]. Then, an (at this stage) heuristic application of Itô's formula implies that

$$|X(t)|_{L^2}^2 = |x|_{L^2} + 2\sum_{j=1}^{N} \mathrm{Re}(\mu_j)\int_0^t \langle X(s), X(s)e_j\rangle_{L^2}\, d\beta_j(s),\ t\ge 0. \tag{1.4}$$

Applying Itô's formula to $\log|X(t)|_{L^2}^2$, we see that

$$|X(t)|_{L^2}^2 = |x|_2^2 \exp\left\{\sum_{j=1}^{N}\left[\int_0^t v_j(s)d\beta_j(s) - \frac{1}{2}\int_0^t v_j^2(s)ds\right]\right\},$$

where $v_j(t) = 2\mathrm{Re}\,\langle X(t),\mu_j e_j X(t)\rangle_{L^2}|X(t)|_{L^2}^{-2}$. Clearly, by (1.4), $t\to|X(t)|_{L^2}^2$ is a continuous martingale and so, if $|x|_{L^2}=1$,

$$\widehat{\mathbb{P}}_x^T(F) = \int_F |X(T,\omega)|_{L^2}^2 d\mathbb{P}(d\omega),\ F\in\mathcal{F}_T,$$

defines a probability law on $\{\Omega, \mathcal{F}_T\}$ (the physical probability law) and, under this law by Girsanov's theorem the continuous process

$$\widetilde{\beta}_j(t) = \beta_j(t) - \int_0^t v_j(s)ds, \ t \in [0,T], \ j = 1, ..., N, \tag{1.5}$$

are independent Gaussian processes with respect to the filtration $(\mathcal{F}_t)$ (Theorem 2.14 in [1]). Here $\widehat{P}_x^T$ is the physical probability law of the events occurring in time $[0,T]$, while $\widehat{\psi}(t,\omega) = X(t,\omega)|X(t,\omega)|_2^{-1}$ is the state of the quantum system conditioned by observation of $s \to \beta_j(s,\omega)$, $0 \le s < t$.

In the particular case (conservative case), $\mu_j = -i\widetilde{\mu}_j$, with $\widetilde{\mu}_j$ real, which is considered in [5], [6], we have $v_j(t) = 0$, $|X(t)|_{L^2} = |x|_{L^2}, \forall t$ and $\widehat{P}_x^T = \mathbb{P}|_{\mathcal{F}_T}$. Then, by (1.5), $\widetilde{\beta}_j = \beta_j$, $\forall j$, and so, in this case, the randomness is independent of the quantum system, and the measurement does not provide any information on the quantum system.

Here, we shall study existence and uniqueness of solutions to (1.1) under the following key assumption on the basis $\{e_j\}_{j=1}^N$

(H1) $e_j \in C_b^\infty(\mathbb{R}^d)$ *such that*

$$\lim_{|\xi|\to\infty} \zeta(\xi)(|e_j(\xi)| + |\nabla e_j(\xi)| + |\Delta e_j(\xi)|) = 0,$$

where $j \in \{1, ..., N\}$ *and*

$$\zeta(\xi) = \begin{cases} 1 + |\xi|^2, & \text{if } d \neq 2, \\ (1 + |\xi|^2)(\ln(1 + |\xi|^2))^2, & \text{if } d = 2. \end{cases}$$

The assumption that each e_j is smooth is made only for simplicity, in order to be able to apply results from [9] directly (see Lemma 3.3 below) on well posedness of linear Schrödinger equations with lower order terms. But, as in [14], an approximation procedure allows to weaken (H1) and to just assume that $e_j \in C_b^2(\mathbb{R}^d)$ for all $j \in \mathbb{N}$.

Under assumption (H1), one shows in Theorem 2.2 below that, for each $x \in L^2(\mathbb{R}^d)$ and $1 \le \alpha < 1 + \frac{4}{d}$, equation (1.1) has a unique global solution in a sense to be made precise later. In the critical case $\alpha = 1 + \frac{4}{d}$ there is a unique local solution (Corollary 5.2). These results improve an earlier result of A. de Bouard and A. Debussche [5] obtained by a direct approach under the more restrictive condition: $1 < \alpha < 1 + \frac{2}{d-1}$ if $d \ge 3$. (See, also, [6], [7].)

It should be mentioned, however, that the results from [5] are concerned with the stochastic equation

$$idX = \Delta X\,dt - i\mu X\,dt + \lambda|X|^{\alpha-1}X\,dt + iX\,dW, \tag{1.6}$$

where W is given by (1.2) and $\mu_j = -i\widetilde{\mu}_j$, $\widetilde{\mu}_j \in \mathbb{R}$, $1 \le j \le N$, i.e., the conservative case discussed above.

The sharper existence and uniqueness result we prove here is the same as for the deterministic Schrödinger equation (see, e.g., [13], p. 92) and it is a direct consequence of our rescaling approach which reduces the stochastic equation to a random Schrödinger equation for which pointwise for $\mathbb{P}$-a.e. $\omega \in \Omega$ estimates similar to that in the classical theory can be obtained. As a matter of fact, this is one of the main advantages of this rescaling approach: one can replace the $L^1(\Omega)$ estimates by pointwise $\mathbb{P}$-a.s. estimates. In a different context, this approach was used in [1], [2].

The main existence result, Theorem 2.2, is presented in Section 2 and proved in Sections 3 and 4. In Section 5 we briefly discuss the critical case $\alpha = 1 + \frac{4}{d}$. We conclude the paper with some final remarks in Section 6 and some calculational details in the Appendix.

2 Notations and the main result

For $1 \le p \le \infty$, we denote by $L^p(\mathbb{R}^d) = L^p$ the space of all Lebesgue p-integrable (complex valued) function on $\mathbb{R}^d$. The Hilbert space $L^2(\mathbb{R}^d)$ is endowed with the scalar product

$$\langle \cdot, \cdot \rangle_2 = \int_{\mathbb{R}^d} u(\xi)\bar{v}(\xi)d\xi;\ u, v \in L^2(\mathbb{R}^d).$$

The norm of L^p is denoted by $|\cdot|_{L^p}$. For $p \in [1,\infty]$, $p' \in [1,\infty]$ denotes the unique number such that $\frac{1}{p} + \frac{1}{p'} = 1$.

We also set $|\cdot|_{L^2} = |\cdot|_2$. By $L^q(0,T;L^p)$ we denote the space of all integrable L^p-valued functions $u : (0,T) \to L^p$ with norm

$$\|u\|_{L^q(0,T;L^p)} = \left(\int_0^T \left(\int_{\mathbb{R}^d} |u(t,\xi)|^p d\xi\right)^{\frac{q}{p}} dt\right)^{\frac{1}{q}}.$$

By $C([0,T];L^p)$ we denote the standard space of all L^p-valued continuous functions on $[0,T]$ with the sup norm in t. Finally, $H^k(\mathbb{R}^d)$, $k = 1, 2$, are the classical Sobolev spaces of complex valued functions on $\mathbb{R}^d$.

Definition 2.1 Let $\alpha \in \left[1, 1+\frac{4}{d}\right]$ and fix $T > 0$. A solution to equation (1.1) is an L^2-valued continuous $(\mathcal{F}_t)$-adapted process $X = X(t)$, $t \in [0,T]$, such that $|X|^\alpha \in L^1([0,T],(H^2(\mathbb{R}^d))')$ and it satisfies

$$\begin{aligned} X(t) &= x - \int_0^t (i\Delta X(s) + \mu X(s) + \lambda i |X(s)|^{\alpha-1} X(s)) ds \\ &\quad + \int_0^t X(s) dW(s), \ \forall t \in [0,T], \ \mathbb{P}\text{-a.s.} \end{aligned} \tag{2.1}$$

Here, the integral

$$\int_0^t X(s) dW(s) = \sum_{j=1}^N \int_0^t \mu_j e_j X(s) d\beta_j(s)$$

is taken in sense of Itô, $\Delta X \in L^2(0,T;(H^2(\mathbb{R}^d))')$ and (2.1) is understood as an equation in $(H^2(\mathbb{R}^d))' = H^{-2}(\mathbb{R}^d)$.

Theorem 2.2 is the main result.

Theorem 2.2 *Assume W is as in* (1.2) *and satisfies* (H1), $\lambda \in \mathbb{R}$ *and* $1 \le \alpha < 1 + \frac{4}{d}$, $1 \le d < \infty$. *Then, for each $x \in L^2$ and $0 < T < \infty$, there is a unique solution $X = X(t,x)$ to* (1.1) *in the sense of Definition* 2.1, *which satisfies*

$$X \in L^2(\Omega; C([0,T]; L^2)) \tag{2.2}$$

$$X \in L^q(0,T;L^{\alpha+1}), \ \mathbb{P}\text{-}a.s., \tag{2.3}$$

where $q = \frac{4(\alpha+1)}{d(\alpha-1)} \in \left(2 + \frac{4}{d}, \infty\right]$.

Moreover, for $\mathbb{P}$-a.e. $\omega \in \Omega$, the map $x \to X(\cdot, x, \omega)$ is continuous from L^2 to $C([0,T];L^2) \cap L^q(0,T;L^{\alpha+1})$, $t \to |X(t)|_2^2$ is a continuous martingale with the representation

$$|X(t)|_2^2 = |x|_2^2 + 2\int_0^t \int_{\mathbb{R}^d} \sum_{j=1}^N \mathrm{Re}(\mu_j) e_j |X(s)|^2 d\xi \, d\beta_j(s), \ t \in [0,T]. \tag{2.4}$$

By Sobolev's embedding theorem, it is easily seen that (2.3) implies also that $|X|^\alpha \in L^1(0,T;H^{-2}(\mathbb{R}^d))$, as claimed in Definition 2.1.

As explicitly stated, Theorem 2.2 is a global existence and uniqueness result for equation (1.1) in the subcritical case $1 \le \alpha < 1 + \frac{4}{d}\cdot$ The more delicate critical case $\alpha = 1 + \frac{4}{d}$ will be briefly discussed in Section 5.

3 Proof of Theorem 2.2

We apply in equation (1.1) the rescaling transformation

$$X = e^W y. \tag{3.1}$$

By a heuristic application of Itô's product formula, we see that $\mathbb{P}$-a.s.

$$dX = e^W dy + e^W y dW + \widetilde{\mu} e^W y dt$$

where $\widetilde{\mu} = \frac{1}{2} \sum\limits_{j=1}^{N} \mu_j^2 e_j^2$.

Substituting into (1.1) yields $\mathbb{P}$-a.s.

$$\begin{aligned} & i \, \frac{\partial y}{\partial t} = e^{-W} \Delta(e^W y) - (\mu + \widetilde{\mu}) iy + \lambda |e^{(\alpha-1)W}| \, |y|^{\alpha-1} y, \\ & y(0) = x \in L^2. \end{aligned} \tag{3.2}$$

Equation (3.2) can be rewritten as

$$\begin{aligned} & i \, \frac{\partial y}{\partial t} = \Delta y + \Big(\sum_{i=1}^{d} (\partial i W)^2 + \Delta W - (\mu + \widetilde{\mu}) i \Big) y + 2 \nabla W \cdot \nabla y \\ & \qquad\qquad + \lambda |e^{(\alpha-1)W}| \, |y|^{\alpha-1} y, \ t \in (0,T), \\ & y(0) = x. \end{aligned} \tag{3.3}$$

Definition 3.1 A solution to (3.3) is an L^2-valued continuous $(\mathcal{F}_t)$-adapted process $y = y(t)$, $t \in [0,T]$, such that $|y|^\alpha \in L^1([0,T]; (H^2(\mathbb{R}^d))')$ and it satisfies (3.3) $\mathbb{P}$-a.s. as an equation in $(H^2(\mathbb{R}^d))'$.

A rigorous proof of the equivalence of (2.1) and (3.3) is included in the Appendix (see Lemma A.1).

We set

$$\begin{aligned} c(t,\xi) &= \sum_{i=1}^{d} (\partial_i W)^2 + \Delta W(t,\xi) - i(\mu + \widetilde{\mu}) \\ b(t,\xi) &= 2 \nabla W(t,\xi) \end{aligned} \tag{3.4}$$

and rewrite (3.3) as

$$\begin{aligned} & i \, \frac{\partial y}{\partial t} = \Delta y + cy + b \cdot \nabla y + \lambda |e^{(\alpha-1)W}| \, |y|^{\alpha-1} y \text{ in } (0,T) \times \mathbb{R}^d, \\ & y(0,\xi) = x(\xi), \ \xi \in \mathbb{R}^d, \end{aligned} \tag{3.5}$$

for each $\omega \in \Omega$. We note that, by (H1), we have $\mathbb{P}$-a.s. that conditions (1.5), (1.6) on b, c in [14] are satisfied with $\mathbb{R}$ replaced by $[0, T]$, which will be crucially used below.

It should be said that, though (3.5) is similar to the standard nonlinear Schrödinger equation $iy_t = \Delta y + \lambda |y|^{\alpha-1} y$, its existence theory is not reducible to the latter due to the presence of lower order terms which excludes the direct use of classical Strichartz estimates.

However, we have the following existence and uniqueness result, which shall be proved in the next section.

Proposition 3.2 *Under the assumptions of Theorem 2.2, for each $x \in L^2$ and $T \in (0, \infty)$ there is a unique solution y to equation (3.5) which satisfies*

$$e^W y \in L^2(\Omega; C([0,T]; L^2)) \tag{3.6}$$

$$y \in L^q(0, T; L^{\alpha+1}), \ \mathbb{P}\text{-a.s.}, \tag{3.7}$$

where $q = \frac{4(\alpha+1)}{d(\alpha-1)} \in \left(2 + \frac{4}{d}, \infty\right]$. The mapping $x \to y(\cdot, x, \omega)$ is continuous from L^2 to $C([0,T]; L^2) \cap L^q(0, T; L^{\alpha+1})$, for $\mathbb{P}$-a.e. $\omega \in \Omega$.

The solution y to (3.5) is taken in the following *mild* sense: for $t \in [0, T]$

$$y(t) = U(t,0)x - \lambda i \int_0^t U(t,s)(|e^{(\alpha-1)W(s)}| \, |y(s)|^{\alpha-1} y(s)) ds, \tag{3.8}$$

where $U = U(t,s) \in L(L^2, L^2)$, $-\infty < s \le t < \infty$, is the evolution generated by the random time-dependent operator

$$A(t)u = -i(\Delta u + c(t)u + b(t) \cdot \nabla u), \ \forall u \in H^2(\mathbb{R}^d), \tag{3.9}$$

where Δ and ∇ are taken in the sense of distributions on $\mathbb{R}^d$. By standard arguments, it then follows in our case that y is also a solution to (3.5) (equivalently, (3.3)) in the sense of Definition 3.1.

The evolution U must satisfy $\mathbb{P}$-a.s. the equation

$$\frac{\partial}{\partial t} U(t,s)x = A(t)U(t,s)x, \ \forall x \in H^2(\mathbb{R}^d), \ t \ge s, \tag{3.10}$$

$$\frac{\partial}{\partial s} U(t,s)x = -U(t,s)A(s)x, \ \forall x \in H^2(\mathbb{R}^d). \tag{3.11}$$

We have

Lemma 3.3 *For* ($\mathbb{P}$-*almost*) *every* $\omega \in \Omega$, *the operator* $A(t)$ *generates an evolution* $U(t,s) = U(t,s,\omega)$ *in the space* L^2. *Moreover, for each* $x \in L^2$ *and* $s \in [0,T]$, *the process* $[s,T] \ni t \to U(t,s)x$ *is continuous and* $(\mathcal{F}_t)_{t \ge s}$*-adapted, hence progressively measurable with respect to the filtration* $(\mathcal{F}_t)_{t\ge s}$.

Proof. The existence of the evolution operator U generated by $A(t)$ is a direct consequence of the fact that, for ($\mathbb{P}$-almost) every $\omega \in \Omega$, the Cauchy problem

$$\begin{aligned} &\frac{dy}{dt} = A(t)y, \\ &y(s) = x, \ \ s \le t < \infty, \end{aligned} \tag{3.12}$$

for each $x \in L^2(\mathbb{R}^d)$ has a unique continuous solution $y \in C([s,T];L^2(\mathbb{R}^d))$ for all $T > s$. Indeed, by Theorem 1.1 in Doi [9] (see, also, [8]), under our assumptions on c and b, for each $x \in L^2$ and $f \in L^1(s,T;L^2)$, the Cauchy problem

$$\begin{aligned} &i\,\frac{\partial u}{\partial t} = \Delta u + cu + b\cdot\nabla u + f \ \text{ in } \ (s,T)\times\mathbb{R}^d, \\ &u(s) = x, \end{aligned} \tag{3.13}$$

has a unique solution $u \in C([s,T];L^2)$, which satisfies the estimate

$$|u(t)|_{L^2} \le C\left(|x|_{L^2} + \int_s^t |f(s)|_{L^2} ds\right), \ \ s \le t \le T. \tag{3.14}$$

The solution u to (3.13) is taken here in sense of distribution on $(0,T)\times\mathbb{R}^d$. More precisely, for each $u \in L^2((0,T)\times\mathbb{R}^d)$, $Lu = -i\Delta u - icu - ib\cdot\nabla u - if \in L^2(0,T;H^{-2}(\mathbb{R}^d))$ and so (3.13) reduces to

$$\frac{du}{dt}(t) = Lu(t), \ \ \text{a.e. } t \in (s,T), \ u(s) = x, \tag{3.15}$$

where $\frac{d}{dt}$ is taken in sense of vectorial $H^{-2}(\mathbb{R}^d)$-valued distributions on $(0,T)$. This means that $u : [0,T] \to H^{-2}(\mathbb{R}^d)$ is absolutely continuous and a.e. differentiable on $(0,T)$. Moreover, if $x \in H^\sigma(\mathbb{R}^d)$, $f \in L^1(0,T;H^\sigma)$, $\sigma \in \mathbb{R}$, then $u \in C([s,T];\ H^\sigma(\mathbb{R}^d))$. This implies the existence for (3.12) and so, of an evolution $U(t,s) \in L(L^2,L^2)$ defined by $U(t,s)x = y(t)$, $0 \le s \le t \le T$.

Moreover, since $U(t,s)H^2(\mathbb{R}^d) \subset H^2(\mathbb{R}^d)$ for every $t,s \in [0,T]$, and $U(t,s)x \in C([s,T];H^2(\mathbb{R}^d))$ for $x \in H^2(\mathbb{R}^d)$, we see by (3.15) that $t \to U(t,s)x$ is continuously differentiable for each $x \in H^2(\mathbb{R}^d)$ and is also easily seen by the continuity of b and c that $s \to U(t,s)x$ is continuous.

Since the Cauchy problem (3.12) is, by virtue of the above results, uniformly well posed, that is, $D(A(t)) \equiv H^2(\mathbb{R}^d)$ for all t and for each $x \in H^2(\mathbb{R}^d)$, the function $(t,s) \to U(t,s)x$ is continuous together with $\frac{\partial}{\partial t}U(t,s)x$ on $\{(s,t);\ 0 \le s \le t \le T\}$, it follows that besides (3.10) we have also (3.11) (see, e.g., [12], Sect. 3, Chap. II).

The second part of Lemma 3.3, that is the adaptedness of the process $t \to U(t,s)x$, follows immediately from the fact that, by (3.4), the processes $t \to c(t)$ and $t \to b(t)$ are progressively measurable with respect to the filtration $(\mathcal{F}_t)_{t\ge0}$. □

By (3.10), (3.11), it follows that, in terms of U, the solution to (3.13) can be represented in the "mild" sense below

$$u(t) = U(t,s)x - i\int_s^t U(t,r)f(r)dr,\ \ s \le t \le T.$$

By Lemma 3.3, it follows that u is progressively measurable for each $x \in L^2$ and any progressively measurable process $f : [s,T] \times \Omega \to H^{-1}(\mathbb{R}^d)$.

4 Proof of Proposition 3.2

Wee need a Strichartz type estimate for the solutions to the random non-homogeneous linear Schrödinger equation

$$\begin{aligned} &i\,\frac{\partial u}{\partial t} = \Delta u + cu + b\cdot\nabla u + f \text{ on } (0,T)\times\mathbb{R}^d,\\ &u(0) = u_0 \text{ on } \mathbb{R}^d, \end{aligned} \tag{4.1}$$

where $c = c(t,\xi),\ b = b(t,\xi)$ are defined by (3.4). Indeed, we have

Lemma 4.1 *Assume* (H1). *Then, for any $T > 0$ and $u_0 \in L^2$, $f \in L^{q_2'}(0,T;L^{p_2'})$, the solution*

$$u(t) = U(t,0)u_0 - i\int_0^t U(t,s)f(s)ds,\ \ 0 \le t \le T, \tag{4.2}$$

to equation (4.1) *satisfies the estimate*

$$\|u\|_{L^{q_1}(0,T;L^{p_1})} \le C_T(|u_0|_{L^2} + \|f\|_{L^{q_2'}(0,T;L^{p_2'})}), \tag{4.3}$$

where (p_1, q_1) *and* (p_2, q_2) *belong to the set*

$$\left\{(p,q) \in [2,\infty] \times [2,\infty] : \frac{2}{q} = \frac{d}{2} - \frac{d}{p}\right\}, \quad \text{if } \ d \neq 2, \tag{4.4}$$

respectively,

$$\left\{(p,q) \in [2,\infty) \times (2,\infty] : \frac{2}{q} = \frac{d}{2} - \frac{d}{p}\right\}, \quad \text{if } \ d = 2. \tag{4.5}$$

The process C_t*,* $t \geq 0$*, can be taken to be* $(\mathcal{F}_t)$*-progressively measurable, increasing continuous, with* $C_0 = 0$*.*

Here, any pair belonging to the set in (4.4), (4.5) respectively, is called a Strichartz pair.

Lemma 4.1 follows by the results of J. Marzuola, J. Metcalfe and D. Tataru [14] on Strichartz estimates for the linear Schrödinger operator with non-smooth and asymptotically flat coefficients, which is the case for equation (4.2) under assumption (H1). The proof is outlined in the Appendix.

The proof of Proposition 3.2 will be completed in several steps. First, one proves the existence of a local solution y to (3.5) (see Lemma 4.2). As happens in the deterministic case, the next step from a local solution to a global one is determined by the existence of an $L^\infty(0,T;L^2)$ estimate for the local solution. To this end, one proves an L^2-estimate for this solution independent of the interval $[0,\tau]$ of maximal existence (Lemma 4.3) and, finally, one extends y to a global solution of (3.5) satisfying all the requirements of Proposition 3.2.

In the following, we take $q = \frac{4(\alpha+1)}{d(\alpha-1)} \in \left(2+\frac{4}{d}, \infty\right]$.

Lemma 4.2 *Under the assumptions of Proposition* 3.2*, for each* $x \in L^2$*, there exists an increasing sequence of stopping times* τ_n *and* $\tau^*(x)$*, satisfying* $\tau^*(x) = \lim_{n\to\infty} \tau_n$*, a.s., and a solution* y *to* (3.5) *on* $[0, \tau^*(x))$ *starting from* x *such that*

$$y \in C([0,\tau_n]; L^2) \cap L^q(0,\tau_n; L^{\alpha+1}), \tag{4.6}$$

for each $n \geq 1$*. The process* $t \to y(t) \in L^2$ *is adapted to the filtration* $(\mathcal{F}_t)_{t\geq 0}$*.*

Proof. We construct the solution to (3.5) in the "mild" sense

$$y(t) = U(t,0)x - i\int_0^t U(t,s)(\lambda|e^{(\alpha-1)W(s)}|g(y(s)))ds, \; t \in [0,T], \tag{4.7}$$

where $g(y) = |y|^{\alpha-1}y,\ \forall y \in \mathbb{C}$. This solution is then also a solution in the sense of Definition 3.1. We set $\mathcal{X} = C([0,T];L^2)\cap L^q(0,T;L^{\alpha+1})$ and consider the integral operator

$$F(y)(t) = U(t,0)x - i\int_0^t U(t,s)(\lambda|e^{(\alpha-1)W(s)}|g(y(s)))ds, t \in [0,T], \quad (4.8)$$

defined on $\mathcal{X}$ and, for $\mathbb{P}$-a.e. $\omega \in \Omega$, we construct a unique local solution by Banach's fixed point theorem applied to F. Then, we extend the solution to a maximal interval $[0,\tau^*(x))$.

Step 1. By estimate (4.3), with Strichartz pairs $(\alpha+1,q)$, we have for $y \in \mathcal{X}$,

$$\begin{aligned}
\|F(y)\|_{L^q(0,T;L^{\alpha+1})} &\leq C_T|x|_2 + C_T|\lambda|\|\,|e^{(\alpha-1)W}|g(y)\|_{L^{q'}(0,T;L^{\frac{\alpha+1}{\alpha}})} \\
\|\,|e^{(\alpha-1)W}|g(y)\|_{L^{q'}(0,T;L^{\frac{\alpha+1}{\alpha}})} &\leq e^{(\alpha-1)|W|_{\infty,\infty}}\|\,|y|^\alpha\|_{L^{q'}(0,T;L^{\frac{\alpha+1}{\alpha}})} \\
&\leq e^{(\alpha-1)|W|_{\infty,\infty}}T^\theta\|y\|^\alpha_{L^q(0,T;L^{\alpha+1})},
\end{aligned}$$

where $|W|_{\infty,\infty} = \|W\|_{L^\infty(0,T;L^\infty)}$ and $\theta = 1 - \frac{d(\alpha-1)}{4} > 0$. We set $\gamma_T = e^{(\alpha-1)|W|_{\infty,\infty}}$. We have, therefore,

$$\|F(y)\|_{L^q(0,T;L^{\alpha+1})} \leq C_T\left[|x|_2+|\lambda|T^\theta\gamma_T\|y\|^\alpha_{L^q(0,T;L^{\alpha+1})}\right]. \quad (4.9)$$

By (4.3), with the Strichartz pair $(2,\infty)$ and $(\alpha+1,q)$, we also have

$$\|F(y)\|_{L^\infty(0,T;L^2)} \leq C_T\left[|x|_2 + |\lambda|\gamma_T T^\theta\|y\|^\alpha_{L^q(0,T;L^{\alpha+1})}\right]. \quad (4.10)$$

In particular, this implies that $F(\mathcal{X}) \subset \mathcal{X}$.

We note that, in (4.9), (4.10), the constant C_T, coming from the Strichartz estimate (4.3), depends on $\omega \in \Omega$. However, as mentioned in Lemma 4.1, the process $t \to C_t$ is $(\mathcal{F}_t)$-adapted.

Now, we fix $\omega \in \Omega$ and consider the operator F on the set

$$\mathcal{X}^\tau_{M_1}=\left\{y \in C([0,\tau];L^2)\cap L^q(0,\tau;L^{\alpha+1});\ \sup_{0\leq t\leq\tau}|y(t)|_2+\|y\|_{L^q(0,\tau;L^{\alpha+1})} \leq M_1\right\}$$

where $\tau = \tau(\omega) \in (0,T]$ and $M_1 = M_1(\omega) > 0$ are random variables.

For $y \in \mathcal{X}^\tau_{M_1}$, we have, by estimates (4.9), (4.10), that

$$\begin{aligned}
\|F(y)\|_{L^\infty(0,\tau;L^2)} + \|F(y)\|_{L^q(0,\tau;L^{\alpha+1})} &\leq 2C_\tau(|x|_2 + |\lambda|\gamma_\tau\tau^\theta\|y\|^\alpha_{L^q(0,\tau;L^{\alpha+1})}) \\
&\leq 2C_\tau(|x|_2 + |\lambda|\gamma_\tau\tau^\theta M_1^\alpha),
\end{aligned}$$

where $\gamma_t = \exp((\alpha-1)\|W\|_{L^\infty(0,t;L^\infty)})$. This means that $F(\mathcal{X}^\tau_{M_1}) \subset \mathcal{X}^\tau_{M_1}$, if M_1 and τ are chosen in a such way that

$$2C_\tau(|x|_2 + \alpha|\lambda|\gamma_\tau \tau^\theta M_1^\alpha) \le M_1. \tag{4.11}$$

To this end, we choose $M_1 = 3C_\tau|x|_2$, and define the real-valued continuous, $(\mathcal{F}_t)$-adapted process

$$Z_t^{(1)} := 2\cdot 3^{\alpha-1} C_t^\alpha |x|_2^{\alpha-1}\alpha|\lambda|\gamma_t t^\theta,\ t\in[0,T].$$

Then (4.11) is equivalent to $Z_\tau^{(1)} \le \frac{1}{3}\cdot$ Hence, defining the $(\mathcal{F}_t)$-stopping time

$$\tau_1 = \inf\left\{t\in[0,T] : Z_t^{(1)} > \frac{1}{3}\right\}\wedge T,$$

we have $\tau_1 > 0$ and $Z_{\tau_1}^{(1)} \le \frac{1}{3}$ and hence

$$F(\mathcal{X}^{\tau_1}_{3C_{\tau_1}|x|_2}) \subset \mathcal{X}^{\tau_1}_{3C_{\tau_1}|x|_2}.$$

Now, let us show that F is a contraction in $C([0,\tau_1];L^2)\cap L^q(0,\tau_1;L^{\alpha+1})$. The argument is standard (see, e.g., [13], p. 92, and also [10]), but we reproduce it for completeness. Arguing as in the proof of (4.9), (4.10), we get, for $y_1, y_2 \in \mathcal{X}^{\tau_1}_{3C_{\tau_1}|x|_2}$,

$$\begin{aligned}
&\|F(y_1)-F(y_2)\|_{L^q(0,\tau_1;L^{\alpha+1})} + \|F(y_1)-F(y_2)\|_{L^\infty(0,\tau_1;L^2)}\\
&\le 2C_{\tau_1}|\lambda|\gamma_{\tau_1}\left(\int_0^{\tau_1} \||y_1|^{\alpha-1}y_1 - |y_2|^{\alpha-1}y_2\|^{q'}_{L^{\frac{\alpha+1}{\alpha}}}dt\right)^{\frac{1}{q'}}\\
&\le 2C_{\tau_1}\alpha|\lambda|\tau_1^\theta\gamma_{\tau_1}(\|y_1\|^{\alpha-1}_{L^q(0,\tau_1;L^{\alpha+1})}\\
&+\|y_2\|^{\alpha-1}_{L^q(0,\tau_1;L^{\alpha+1})})\|y_1-y_2\|_{L^q(0,\tau_1;L^{\alpha+1})}\\
&\le 4C_{\tau_1}\alpha|\lambda|\gamma_{\tau_1}\tau_1^\theta M_1^{\alpha-1}\|y_1-y_2\|_{L^q(0,\tau_1;L^{\alpha+1})}\\
&= 2Z_{\tau_1}^{(1)}\|y_1-y_2\|_{L^q(0,\tau_1;L^{\alpha+1})}\\
&\le \frac{2}{3}\|y_1-y_2\|_{L^q(0,\tau_1;L^{\alpha+1})},
\end{aligned} \tag{4.12}$$

by definition of τ_1. We see, by (4.12), that F is a contraction on the space $C([0,\tau_1];L^2)\cap L^q(0,\tau_1;L^{\alpha+1})$. Hence, by Banach's fixed point theorem, we know that there exists a unique solution $y \in C(0,\tau_1;L^2)\cap L^q(0,\tau_1;L^{\alpha+1})$

satisfying $y = F(y)$ on $[0,\tau_1]$, which implies that y is a solution to (4.7) on $[0,\tau_1]$. Moreover, there exists a sequence $u_{1,m} \in \mathcal{X}$, $m \in \mathbb{N}$, such that $u_{1,m+1} = F(u_{1,m})$, $m \geq 1$, $u_{1,1} = x$ and $\lim_{m\to\infty} u_{1,m}|_{[0,\tau_1]} = y$ in $C([0,\tau_1];L^2) \cap L^q(0,\tau_1;L^{\alpha+1})$. Define $y_1(t) := y(t\wedge\tau_1)$, $t \in [0,T]$. Then

$$y_1 = \lim_{m\to\infty} u_{1,m}(\cdot\wedge\tau_1) \text{ in } C([0,T];L^2).$$

Since, obviously, each $u_{1,m}$ is $(\mathcal{F}_t)$-adapted, so is y_1.

Step 2. We shall use an induction argument to extend y_1 to y_{n+1}, which is a solution to (4.7) on a larger interval $[0,\tau_{n+1}]$. Suppose that at the n-th step we have a continuous, $(\mathcal{F}_t)$-adapted process $y_n(t)$, $t \in [0,T]$, and an $(\mathcal{F}_t)$-stopping time τ_n with $\tau_n \geq \tau_{n-1}$, such that $y_n(t) = y_n(t\wedge\tau_n)$, $t \in [0,T]$, and it satisfies (4.7) on $[0,\tau_n]$.

We define the integral operator

$$F_n(z)(t) = U(\tau_n+t,\tau_n)y_n(\tau_n) - i\int_0^t U(\tau_n+t,\tau_n+s)(\lambda|e^{(\alpha-1)W(\tau_n+s)}|g(z(s)))ds,$$
$$t \in [0,T-\tau_n],$$

and consider F_n on the set

$$\mathcal{X}_{M_{n+1}}^{\sigma_n} = \Big\{z \in C(0,\sigma_n;L^2)\cap L^q(0,\sigma_n;L^{\alpha+1}); \sup_{0\leq t\leq\sigma_n}|z(t)|_2 + \|z\|_{L^q(0,\sigma_n;L^{\alpha+1})} \leq M_{n+1}\Big\}$$

where $\sigma_n = \sigma_n(\omega)$ and $M_{n+1} = M_{n+1}(\omega)$ are random variables.

By a similar calculation, we have, for every $z \in \mathcal{X}_{M_{n+1}}^{\sigma_n}$,

$$\begin{aligned}&\|F_n(z)\|_{L^\infty(0,\sigma_n;L^2)} + \|F_n(z)\|_{L^q(0,\sigma_n;L^{\alpha+1})}\\ &\quad\leq 2C_{(\tau_n+\sigma_n)}(|y_n(\tau_n)|_2 + |\lambda|\gamma_{(\tau_n+\sigma_n)}\sigma_n^\theta\|z\|^\alpha_{L^q(0,\sigma_n;L^{\alpha+1})}),\end{aligned}$$

which implies that $F_n(\mathcal{X}_{M_{n+1}}^{\sigma_n}) \subset \mathcal{X}_{M_{n+1}}^{\sigma_n}$ and F_n is a contraction in $\mathcal{X}_{M_{n+1}}^{\sigma_n}$, if we take $M_{n+1} = 3C_{(\tau_n+\sigma_n)}|y_n(\tau_n)|_2$ and choose σ_n such that

$$2C_{(\tau_n+\sigma_n)}(|y_n(\tau_n)|_2 + \alpha|\lambda|\gamma_{(\tau_n+\sigma_n)}\sigma_n^\theta M_{n+1}^\alpha) \leq M_{n+1},$$

i.e.,

$$2\cdot 3^{\alpha-1}C^\alpha_{(\tau_n+\sigma_n)}|y_n(\tau_n)|_2^{\alpha-1}\alpha|\lambda|\gamma_{(\tau_n+\sigma_n)}\sigma_n^\theta \leq \frac{1}{3}\cdot \tag{4.13}$$

So, similarly as above, we define the real-valued continuous, $(\mathcal{F}_t)$-adapted process

$$Z_t^{(n)} := 2 \cdot 3^{\alpha-1} C^{\alpha}_{(\tau_n+t)} |y_n(\tau_n)|^{\alpha-1} \alpha|\lambda| \gamma_{(\tau_n+t)} t^{\theta}, \ t \in [0, T-\tau_n],$$

and

$$\sigma_n := \inf \left\{ t \in [0, T-\tau_n] : Z_t^{(n)} > \frac{1}{3} \right\} \wedge (T-\tau_n).$$

Then $\sigma_n > 0$ and $Z_{\sigma_n}^{(n)} \leq \frac{1}{3}$, i.e., (4.13) holds.

Set $\tau_{n+1} := \tau_n + \sigma_n$. Then τ_{n+1} is an $(\mathcal{F}_t)$-stopping time. Indeed, for $t \in [0, T]$,

$$\{\tau_n + \sigma_n < t\} = \bigcup_{\substack{q_1, q_2 \in Q_+ \\ q_1+q_2<t}} \{\tau_n < q_1, \sigma_n < q_2\},$$

where Q_+ denotes the nonnegative rational numbers.

But, by induction, τ_n is an $(\mathcal{F}_t)$-stopping time and

$$\begin{aligned} \{\tau_n < q_1, \sigma_n < q_2\} &= \bigcup_{\substack{q \in Q_+ \\ q<q_2}} \left\{ \tau_n + q_2 < q_1 + q_2, Z_q^{(n)} > \frac{1}{3} \right\} \\ &\in \mathcal{F}_{(\tau_n+q_2)\wedge(q_1+q_2)} \subset \mathcal{F}_{q_1+q_2} \subset \mathcal{F}_t, \end{aligned}$$

since $\left\{ Z_q^{(n)} > \frac{1}{3} \right\} \in \mathcal{F}_{\tau_n+q} \subset \mathcal{F}_{\tau_n+q_2}$. Since $(\mathcal{F}_t)$ is right-continuous, τ_{n+1} is thus an $(\mathcal{F}_t)$-stopping time.

Analogously to the case $n = 1$, one now shows that, by Banach's fixed point theorem, there exists a unique $z_{n+1} \in \mathcal{X}^{\sigma_n}_{M_{n+1}}$, satisfying $z_{n+1} = F_n(z_{n+1})$. We define

$$y_{n+1}(t) = \begin{cases} y_n(t), & t \in [0, \tau_n], \\ z_{n+1}((t-\tau_n) \wedge \sigma_n), & t \in (\tau_n, T]. \end{cases}$$

It follows from the definition of F in Step 1 and F_n that $y_{n+1} = F(y_{n+1})$ on $[0, \tau_{n+1}]$, which implies that y_{n+1} is a solution to (4.7) on $[0, \tau_{n+1}]$. Moreover, y_{n+1} is adapted to $(\mathcal{F}_t)$ (see Lemma A.2 in the Appendix).

Therefore, we can extend y_n to a new $(\mathcal{F}_t)$-adapted y_{n+1}, which is a continuous process in L^2 and a solution to (4.7) on $[0, \tau_{n+1}]$.

Step 3. Starting from Step 1 and reiterating the process in Step 2, we finally have a solution $y(t)$ of equation (3.5) on a maximal interval $[0, \tau^*(x))$, where $\tau^*(x) = \lim_{n\uparrow\infty} \tau_n(x)$ $(\leq T)$. This completes the proof of Lemma 4.2. $\square$

In order to get a global solution, i.e., $\tau^*(x) \geq T$ a.s. (for every fixed $T > 0$), we need an estimate of $\mathbb{E}\left[\sup_{0\leq t<\tau^*(x)} |e^{W(t)}y(t)|_2^2\right]$, which is given by Lemma 4.3 below.

Lemma 4.3 *Let y be the solution from Lemma 4.2. Then we have $\mathbb{P}$-a.s.*

$$\frac{1}{2}\,|e^{W(t)}y(t)|_2^2 = \frac{1}{2}\,|x|_2^2 + \sum_{j=1}^{N}\int_0^t\int_{\mathbb{R}^d} \mathrm{Re}(\mu_j)e_j|e^{W(s)}y(s)|^2 d\xi\, d\beta_j(s), \qquad 0 \leq t < \tau^*(x). \tag{4.14}$$

Moreover, we have, for $T > 0$,

$$\mathbb{E}\left[\sup_{0\leq t<\tau^*(x)} |e^{W(t)}y(t)|_2^2\right] \leq \widetilde{C}_T < \infty. \tag{4.15}$$

Proof. In order to obtain (4.14), we apply Itô's formula to $|e^{W(t)}y(t)|_2^2$, we first note that, for $t < \tau_n$, $y(t)$ satisfies equation (3.5) in the mild sense (4.7), thus we use the idea in the proof of Lemma A.1 to apply Itô's formula to $|e^{W(t)}y_\varepsilon(t)|_2^2$, where y_ε satisfies the approximation equation (7.4). After that, by taking $\varepsilon \to 0$ we obtain the Itô formula of $|e^{W(t)}y(t)|_2^2$ up to each stopping time τ_n, which implies the desired formula (4.15).

Now, let $\{f_j\}_{j\geq 1}$ be an orthonormal basis in L^2, $f_j \in H^2(\mathbb{R}^d)$. As in the proof of Lemma A.1, we have for each f_j and $t \leq \tau_n$

$$\begin{aligned} \left\langle f_j, e^{W(t)}y_\varepsilon(t)\right\rangle_2 &= \langle f_j, x_\varepsilon\rangle_2 + \int_0^t \left\langle f_j, -ie^{W(s)}J_\varepsilon(e^{-W(s)}\Delta(e^{W(s)}y(s)))\right\rangle_2 ds \\ &\quad + \int_0^t \left\langle f_j, -e^{W(s)}J_\varepsilon((\mu+\widetilde{\mu})y(s)) + \widetilde{\mu}e^{W(s)}y_\varepsilon(s)\right\rangle_2 ds \\ &\quad + \int_0^t \left\langle f_j, -\lambda ie^{W(s)}J_\varepsilon(|e^{(\alpha-1)W(s)}|\,|y(s)|^{\alpha-1}y(s))\right\rangle_2 ds \\ &\quad + \sum_{k=1}^{N}\int_0^t \left\langle f_j, \mu_k e_k e^{W(s)}y_\varepsilon(s)\right\rangle_2 d\beta_k(s). \end{aligned}$$

Applying the Itô product rule, we get

$$
\begin{aligned}
|\left\langle e^{W(t)}y_\varepsilon(t), f_j\right\rangle_2|^2 &= |\left\langle x_\varepsilon, f_j\right\rangle_2|^2 + 2\mathrm{Re}\int_0^t \left\langle e^{W(s)}y_\varepsilon(s), f_j\right\rangle_2 d\left\langle f_j, e^{W(s)}y_\varepsilon(s)\right\rangle \\
&+ \left\langle \left\langle e^{W(t)}y_\varepsilon(t), f_j\right\rangle_2, \left\langle f_j, e^{W(t)}y_\varepsilon(t)\right\rangle_2\right\rangle \\
&= |\left\langle x_\varepsilon, f_j\right\rangle_2|^2 \\
&+2\mathrm{Re}\int_0^t \left\langle e^{W(s)}y_\varepsilon(s), f_j\right\rangle_2 \left\langle f_j, -ie^{W(s)}J_\varepsilon(e^{-W(s)}\Delta(e^{W(s)}y(s)))\right\rangle_2 ds \\
&+2\mathrm{Re}\int_0^t \left\langle e^{W(s)}y_\varepsilon(s), f_j\right\rangle_2 \left\langle f_j, -e^{W(s)}J_\varepsilon((\mu+\widetilde{\mu})y_\varepsilon(s)) + \widetilde{\mu}e^{W(s)}y_\varepsilon(s)\right\rangle_2 ds \\
&+2\mathrm{Re}\int_0^t \left\langle e^{W(s)}y_\varepsilon(s), f_j\right\rangle_2 \left\langle f_j, -\lambda ie^{W(s)}J_\varepsilon(|e^{(\alpha-1)W(s)}|\,|y(s)|^{\alpha-1}y(s))\right\rangle_2 ds \\
&+2\sum_{k=1}^N \mathrm{Re}\int_0^t \left\langle e^{W(s)}y_\varepsilon(s), f_j\right\rangle_2 \left\langle f_j, \mu_k e_k e^{W(s)}y_\varepsilon(s)\right\rangle_2 d\beta_k(s) \\
&+\sum_{k=1}^N \int_0^t |\left\langle f_j, \mu_k e_k e^{W(s)}y_\varepsilon(s)\right\rangle_2|^2 ds,\ \ t\in[0,\tau_n].
\end{aligned}
$$

Now, summing over $j\in\mathbb{N}$ and interchanging the infinite sum with the integrals, we arrive at

$$
\begin{aligned}
\frac{1}{2}|e^{W(t)}y_\varepsilon(t)|_2^2 &= \frac{1}{2}\sum_{j=1}^\infty |\left\langle e^{W(t)}y_\varepsilon(t), f_j\right\rangle_2|^2 = \frac{1}{2}|x_\varepsilon|_2^2 \\
&+\mathrm{Re}\int_0^t \left\langle e^{W(s)}y_\varepsilon(s), -ie^{W(s)}J_\varepsilon(e^{-W(s)}\Delta(e^{W(s)}y(s)))\right\rangle_2 ds \\
&+\mathrm{Re}\int_0^t \left\langle e^{W(s)}y_\varepsilon(s), -e^{W(s)}J_\varepsilon((\mu+\widetilde{\mu})y(s)) + (\mu+\widetilde{\mu})e^{W(s)}y_\varepsilon(s)\right\rangle_2 ds \\
&+\mathrm{Re}\int_0^t \left\langle e^{W(s)}y_\varepsilon(s), -\lambda ie^{W(s)}J_\varepsilon(|e^{(\alpha-1)W(s)}|\,|y(s)|^{\alpha-1}y(s))\right\rangle_2 ds \\
&+\sum_{k=1}^N \mathrm{Re}\int_0^t \left\langle e^{W(s)}y_\varepsilon(s), \mu_k e_k e^{W(s)}y_\varepsilon(s)\right\rangle_2 d\beta_k(s),\ \ t\in[0,\tau_n].
\end{aligned}
$$

Hence, by taking $\varepsilon \to 0$, we finally obtain for $t \le \tau_n$

$$\frac{1}{2}\,|e^{W(t)}y(t)|_2^2 = \frac{1}{2}\,|x|_2^2 + \sum_{k=1}^{N} \mathrm{Re} \int_0^t \left\langle e^{W(s)}y(s), \mu_k e_k e^{W(s)} y(s) \right\rangle_2 d\beta_k(s),$$

which implies (4.14) since $\tau_n \uparrow \tau^*(x)$, a.s.

In order to get (4.15), taking into account that $\sum_{j=1}^{N} |\mu_j|^2 |e_j|_{L^\infty}^2 < \infty$, by the Burkholder–Davis–Gundy and Young's inequality, we have for $t \in [0,T]$ and all $n \in \mathbb{N}$

$$\begin{aligned}
&\mathbb{E}\left[\sup_{s\in[0,t\wedge\tau_n]} \left| \sum_{j=1}^{N} \int_0^s \int_{\mathbb{R}^d} \mathrm{Re}(\mu_j) e_j |e^{W(r)}y(r)|^2 d\xi \, d\beta_j(r) \right| \right] \\
&\quad \le C\mathbb{E}\left[\int_0^{t\wedge\tau_n} \sum_{j=1}^{N} \left(\int_{\mathbb{R}^d} \mathrm{Re}(\mu_j) e_j |e^{W(s)}y(s)|^2 d\xi \right)^2 ds \right]^{\frac{1}{2}} \\
&\quad \le C\mathbb{E}\left[\int_0^{t\wedge\tau_n} |e^{W(s)}y_\varepsilon(s)|_2^4 ds \right]^{\frac{1}{2}} \\
&\quad \le C\mathbb{E}\left[\sup_{s\in[0,t\wedge\tau_n]} |e^{W(s)}y(s)|_2 \left(\int_0^{t\wedge\tau_n} |e^{W(s)}y(s)|_2^2 ds \right)^{\frac{1}{2}} \right] \\
&\quad \le C \sqrt{\mathbb{E} \sup_{s\in[0,t\wedge\tau_n]} |e^{W(s)}y(s)|_2^2} \sqrt{\mathbb{E} \int_0^{t\wedge\tau_n} |e^{W(s)}y(s)|_2^2 ds} \\
&\quad \le \frac{1}{4}\, \mathbb{E} \sup_{s\in[0,t\wedge\tau_n]} |e^{W(s)}y(s)|_2^2 + C \int_0^t \mathbb{E} \left(\sup_{r\in[0,s\wedge\tau_n]} |e^{W(r)}y(r)|_2^2 \right) ds,
\end{aligned}$$

where C is a constant independent of n and may change from line to line. Together with (4.14), this yields

$$\mathbb{E}\left[\sup_{s\in[0,t\wedge\tau_n]} |e^{W(s)}y(s)|_2^2\right] \le 2|x|_2^2 + 4C \int_0^t \mathbb{E}\left(\sup_{r\in[0,s\wedge\tau_n]} |e^{W(r)}y(r)|_2^2 \right) ds,$$

which implies

$$\mathbb{E}\left[\sup_{t\in[0,T\wedge\tau_n]} |e^{W(t)}y(t)|_2^2\right] \le \widetilde{C}_T,$$

where $\widetilde{C}_T$ is independent of n.

Finally, taking $n \uparrow \infty$ and applying Fatou's lemma, we obtain (4.15), as claimed. $\square$

Proof of Proposition 3.2 (continued). By Lemma 4.3, we have $\mathbb{P}$-a.s.

$$\sup_{0\le s<\tau^*(x)} |e^{W(s)}y(s)|_2^2 < \infty.$$

We set $M = \sup_{0\le t<\tau^*(x)} |y(t)|_2^2$. We have, therefore,

$$\begin{aligned} M &= \sup_{0\le s<\tau^*(x)} |e^{-W(s)}e^{W(s)} \cdot y(s)|_2^2 \\ &\le e^{2|W|_{\infty,\infty}} \sup_{0\le s<\tau^*(x)} |e^{W(s)}y(s)|_2^2 < \infty, \ \mathbb{P}\text{-a.s.} \end{aligned}$$

Let us first show that $\tau^*(x) = T$ $\mathbb{P}$-a.s. We know already that $\tau^*(x) \le T$ $\mathbb{P}$-a.s. So, let $\omega \in \{\tau^*(x) < T\} \cap \{M < \infty\}$ and choose $\sigma = \sigma(\omega) \in (0, T - \tau^*(x)(\omega))$, such that

$$2 \cdot 3^{\alpha-1} C^{\alpha}_{(\tau^*(x)+\sigma)} M^{\alpha-1} \alpha |\lambda| \gamma_{(\tau^*(x)+\sigma)} \sigma^{\theta} \le \frac{1}{6}\,.$$

More precisely, define the real-valued continuous process

$$Z_t := 2 \cdot 3^{\alpha-1} C^{\alpha}_{(\tau^*(x)+t)} M^{\alpha-1} \alpha |\lambda| \gamma_{(\tau^*(x)+t)} t^{\theta}, \ t \in [0,T],$$

and

$$\sigma := \inf\left\{ t \in [0,T] : Z_t > \frac{1}{6} \right\} \wedge (T - \tau^*(x)).$$

Then σ has the desired property, and $Z_t \ge Z_t^{(n)}$ for each $n \in \mathbb{N}$, since, for every $n \ge 1$, $|y(\tau_n)|_2 \le M$, $C_{(\tau_n+\sigma)} \le C_{(\tau^*(x)+\sigma)}$ and $\gamma_{(\tau_n+\sigma)} \le \gamma_{(\tau^*(x)+\sigma)}$. By the definition of σ_n in Step 2 of Lemma 4.2, we thus have for each $n \in \mathbb{N}$, $\sigma_n(\omega) \ge \sigma(\omega)$.

Hence $\tau_{n+1}(\omega) = \tau_n(\omega) + \sigma_n(\omega) \ge \tau_n(\omega) + \sigma(\omega)$, which implies that $\tau_{n+1}(\omega) \ge \tau_1(\omega) + n\sigma(\omega)$, $n \ge 1$. Thus, after finitely many steps, $\tau_n(\omega)$ will exceed T, which contradicts the fact that $\tau_n(\omega) \le \tau^*(x)(\omega) \le T$. Therefore, we conclude that $\mathbb{P}(\tau^*(x) = T) = 1$.

Now, (3.6) follows from (4.15).

Now, let us prove (3.7). By Step 1, we have $\|y\|_{L^q(0,\tau_1;L^{\alpha+1})} \le 3C_{\tau_1}|x|_2$. Moreover, if $\tau_1 < T$, we choose $L = L(\omega) \in \mathbb{N}$ such that $\tau_L < T = \tau_{L+1}$. At the $(n+1)$–th step, since $z_{n+1} \in \mathcal{X}^{\sigma_n}_{M_{n+1}}$, we get, for $1 \le n \le L$,

$$\begin{aligned} \|y\|_{L^q(\tau_n,\tau_{n+1};L^{\alpha+1})} &= \|z_{n+1}\|_{L^q(0,\sigma_n;L^{\alpha+1})} \le M_{n+1} \\ &= 3C_{(\tau_n+\sigma_n)}|y_n(\tau_n)|_2 \le 3C_{\tau_{L+1}}\|y\|_{L^\infty(0,T;L^2)} < \infty. \end{aligned}$$

Therefore, we obtain

$$\|y\|_{L^q(0,T;L^{\alpha+1})} \le 3(L+1)C_{\tau_{L+1}}\|y\|_{L^\infty(0,T;L^2)} < \infty, \ \mathbb{P}\text{-a.s.}$$

This completes the proof of the existence in Proposition 3.2.

As regards the uniqueness, as in (4.12) we have for two solutions y_1, y_2 to (3.5)

$$\begin{aligned} &\|y_1 - y_2\|_{L^\infty(0,t;L^2)} + \|y_1 - y_2\|_{L^q(0,t;L^{\alpha+1})} \\ &\quad \le 4C_T\alpha|\lambda|\gamma_T t^\theta M^{\alpha-1}(\|y_1 - y_2\|_{L^\infty(0,t;L^2)} + \|y_1 - y_2\|_{L^q(0,t;L^{\alpha+1})}), \end{aligned}$$

where $M = |y_1|_{L^q(0,T;L^{\alpha+1})} + |y_2|_{L^q(0,T;L^{\alpha+1})} < \infty$, a.s., which implies $y_1 = y_2$ on a sufficiently small interval $(0,t)$. Then, by a standard argument, global uniqueness follows. It remains to prove the dependence with respect to the initial data $x \in L^2$.

Suppose that $x_m \to x$ in L^2. For every x_m (resp. x), we know that, for $\mathbb{P}$-a.e. $\omega \in \Omega$, there exists a unique solution y_m (resp. y) to equation (3.5) satisfying $y_m(0) = x_m$ (resp. $y(0) = x$).

First, we use a similar argument as in the proof of (3.7) to show that

$$\|y_m\|_{L^q(0,T;L^{\alpha+1})} \le M, \ \forall m \in \mathbb{N},$$

where $M = (L+1)3^{L+1}\prod_{j=1}^{L+1} C_{\tau_j} \sup_m |x_m|_2 < \infty$, a.s. with L as in the proof of (3.7) above.

In fact, by Step 1, we have

$$\|y_m\|_{L^q(0,\tau_1;L^{\alpha+1})} \le 3C_{\tau_1}|x_m|_2 \le 3C_{\tau_1}\sup_m |x_m|_2,$$

while, at the $(n+1)$–th extension step, we have, for $1 \le n \le L$,

$$\begin{aligned} &\|y_m\|_{L^q(\tau_n,\tau_{n+1};L^{\alpha+1})} \le 3C_{\tau_{n+1}}|y_m(\tau_n)|_2 \\ &\quad \le 3^{n+1}\prod_{j=1}^{n+1} C_{\tau_j}|x_m|_2 \le 3^{L+1}\prod_{j=1}^{L+1} C_{\tau_j}\sup_m |x_m|_2, \end{aligned}$$

hence

$$\|y_m\|_{L^q(0,T;L^{\alpha+1})} \le (L+1)3^{L+1} \prod_{j=1}^{L+1} C_{\tau_j} \sup_m |x_m|_2,$$

as claimed.

Then, we have, for $t, \widetilde{t} \in (0,T)$ (cf. (4.12)),

$$\begin{aligned}
&\|y_m - y\|_{L^\infty(t,t+\widetilde{t};L^2)} + \|y_m - y\|_{L^q(t,t+\widetilde{t};L^{\alpha+1})} \\
&\le 2C_T|x_m - x|_2 + 4C_T\alpha|\lambda|\gamma_T \widetilde{t}^\theta M^{\alpha-1}\|y_m - y\|_{L^q(t,t+\widetilde{t};L^{\alpha+1})},
\end{aligned}$$

where $\theta = 1 - \frac{d(\alpha-1)}{4} > 0$. If we choose $\widetilde{t}$ such that

$$4C_T\alpha|\lambda|\gamma_T \widetilde{t}^\theta M^{\alpha-1} \le \frac{1}{2},$$

we obtain

$$\|y_m - y\|_{L^\infty(t,t+\widetilde{t};L^2)} + \|y_m - y\|_{L^q(t,t+\widetilde{t};L^{\alpha+1})} \le 4C_T|x_m - x|_2.$$

Since $\widetilde{t} = \widetilde{t}(M)$ is independent of m and for $m \to \infty$ $x_m \to x$ in L^2, we get that

$$y_m \to y \text{ in } L^\infty(t, t+\widetilde{t}; L^2) \cap L^q(t, t+\widetilde{t}; L^{\alpha+1}).$$

Moreover, as $\widetilde{t} = \widetilde{t}(M)$ is also independent of t, we conclude that

$$y_m \to y \text{ in } L^\infty(0,T;L^2) \cap L^q(0,T;L^{\alpha+1}). \tag{4.16}$$

This completes the proof. □

Proof of Theorem 2.2. As noticed earlier, the existence, uniqueness and continuous dependence on initial data of the solution X follows directly by Proposition 3.2. Moreover, by (4.14) we see that $t \to \frac{1}{2}|X(t)|_2^2$ is a continuous martingale and that (2.4) holds. This completes the proof. □

5 The critical case

In the critical case, equation (3.5) and, consequently, (1.1) has a local solution only. More precisely, we have

Proposition 5.1 *Assume $\alpha = 1 + \frac{4}{d}\cdot$ Then, for each $x \in L^2$, there exists a stopping time $\tau^*(x)$ and a unique solution y to (3.5) starting from x such that $t \to y(t \wedge \tau)$ is adapted on $[0, T]$ and*

$$y \in C([0, \tau]; L^2) \cap L^\rho(0, \tau; L^\rho),\ \mathbb{P}\text{-}a.s., \tag{5.1}$$

for any stopping time $\tau < \tau^*(x)$. Here, $\rho = 2 + \frac{4}{d}\cdot$

Proof. Since the proof is essentially the same as that of Lemma 4.2, it will here be sketched only. Consider, as above, for fixed $\omega \in \Omega$, the set (see, e.g., [13], p. 97)

$$G_M^\tau = \{y \in C([0, \tau]; L^2) \cap L^\rho(0, \tau; L^\rho);\ \sup_{0 \le t \le \tau} |y(t) - U(t, 0)x|_2 + \|y\|_{L^\rho(0,\tau;L^\rho)} \le M\},$$

where U is defined by Lemma 3.3. We have

$$\begin{aligned}\sup_{0 \le t \le \tau} |F(y)(t) - U(t, 0)x|_2 &\le \sup_{0 \le t \le \tau} \left| \int_0^t U(t, s)(\lambda |e^{(\alpha-1)W(s)}| g(y(s)))ds \right|_2 \\ &\le C_\tau |\lambda| \| \, |e^{(\alpha-1)W}| g(y)\|_{L^{\rho'}(0,\tau;L^{\rho'})} \le \widetilde{C}_1 M^\alpha.\end{aligned}$$

Similarly,

$$|F(y)|_{L^\rho(0,\tau;L^\rho)} \le |U(t, 0)x|_{L^\rho(0,\tau;L^\rho)} + \widetilde{C}_1 M^\alpha,$$

where $\widetilde{C}_1$ depends on C_τ, γ_τ and λ. Then, arguing as in the proof of Lemma 4.2, it follows that, for τ and M suitably chosen $F(G_M^\tau) \subset G_M^\tau$ and F is a contraction on G_M^τ in the norm of $L^\rho(0, \tau; L^\rho)$. Hence, on $[0, \tau]$, with τ a stopping time, there is a solution y to (3.5) on $[0, \tau]$ and the process $t \to y(t \wedge \tau)$ is adapted. Arguing as in the proof of Lemma 4.2, one finds a maximal interval $[0, \tau^*(x))$ and a solution y to (3.5) on each $[0, \tau]$, $\tau < \tau^*$, $\mathbb{P}$-a.s. Since, in this case, $\tau^*(x)$ is not a function of $|x|_2^2$, as happens in the subcritical case, in general $\tau^*(x) < T$ and so the solution y is local only. □

As regards the stochastic equation (1.1), Proposition 5.1 implies, via transformation (3.1), the following local existence result.

Corollary 5.2 *Assume $\alpha = 1 + \frac{4}{d}\cdot$ Then, for each $x \in L^2$, there exists a stopping time $\tau^*(x)$ and a unique solution X to equation (1.1) such that $t \to X(t \wedge \tau)$ is adapted on $[0, T]$ and*

$$X \in C([0, \tau]; L^2) \cap L^\rho(0, \tau; L^\rho),\ \mathbb{P}\text{-}a.s., \tag{5.2}$$

for any stopping time $\tau < \tau^*(x)$.

In this case, Lemma 4.3 holds too, and so X satisfies the martingale equality

$$\frac{1}{2}\,|X(t)|_2^2 = \frac{1}{2}\,|x|_2^2 + \int_0^t \mathrm{Re}\,\langle X(s), X(s)dW(s)\rangle_2\,,\ t \in [0, \tau^*(x)). \tag{5.3}$$

6 Final remarks

1° Theorem 2.2 and Corollary 5.2 remain true for more general real Gaussian processes $W(t)$ in L^2 with $\mathrm{cov}(W(t)) = tQ$, where Q is a symmetric nonnegative operator with $\mathrm{Tr}\,Q < \infty$ with appropriate spatial assumptions so that [14] applies. We omit the details.

2° The H^1-existence theory for equation (1.1) can be treated in a similar way and leads to results comparable with that in a deterministic case, by using in the proof of Proposition 3.2 the Strichartz estimates (4.3) for $\nabla^s u$, $s = 1$. We omit the details which will be contained in a forthcoming work. (We refer to [6] for a direct approach in this case.)

7 Appendix

Proof of Lemma 4.1. Under assumption (H1), the coefficients c, b defined in (3.4) satisfy (1.4)-(1.6) in [14] on $[0,T] \times \mathbb{R}^d$. We would like to recall here that in [14] (e.g., (1.1)), the common notation $D_t = -i\partial_t$, $D_j = -i\partial_{x_j}$ is used. Then, by Theorem 1.13 in [14] and, more precisely, by estimate (1.24) (see Remark 1.17 in [14]), we have

$$\|u\|_{L^{q_1}(0,T;L^{p_1})} \le C\left(|u_0|_2 + \|f\|_{L^{q_2'}(0,T;L^{p_2'})} + \|u\|_{L^2(0,T;L^2(|\xi|\le 2R))}\right), \tag{7.1}$$

for R sufficiently large.

We are going to prove first that (4.3) holds for T sufficiently small. To this end, we note that

$$\begin{aligned}\|u\|^2_{L^2(0,T;L^2(|\xi|\le 2R))} &\le (m(B_{2R}))^{\frac{p_1-2}{p_1}} \int_0^T |u(t)|^2_{L^{p_1}}\,dt \\ &\le (m(B_{2R}))^{\frac{p_1-2}{p_1}}\, T^{\frac{q_1-2}{q_1}}\, \|u\|^2_{L^{q_1}(0,T;L^{p_1})},\end{aligned}$$

where $m(B_{2R})$ is the volume of the ball B_{2R} of radius $2R$. For simplicity, we assume that $q_1 > 2$, which is, in fact, the case in the application of Lemma 4.1 to problem (3.5). Then, for

$$0 < T = \left((2C)^{-2}(m(B_{2R}))^{-\frac{p_1-2}{p_1}}\right)^{\frac{q_1}{q_1-2}}, \tag{7.2}$$

we get by (7.1) that

$$\|u\|_{L^{q_1}(0,T;L^{p_1})} \leq 2C\left(|u_0|_2 + \|f\|_{L^{q_2'}(0,T;L^{p_2'})}\right). \tag{7.3}$$

For $q_1 = \infty,\ p_1 = 2$, we get in a similar way

$$\|u\|_{L^\infty(0,T;L^2)} \leq 2C\left(|u_0|_2 + \|f\|_{L^{q_2'}(0,T;L^{p_2'})}\right),$$

for $0 < T < (2C)^{-2}$. Reiterating (7.3) on the interval $(T, 2T)$, we get therefore

$$\begin{aligned}
\|u\|_{L^{q_1}(T,2T;L^{p_1})} &\leq 2C\left(|u(T)|_2 + \|f\|_{L^{q_2'}(T,2T;L^{p_2'})}\right) \\
&\leq 2C\left[2C(|u_0|_2 + \|f\|_{L^{q_2'}(0,T;L^{p_2'})}) + \|f\|_{L^{q_2'}(T,2T;L^{p_2'})}\right] \\
&\leq 2C\left[2C|u_0|_2 + (2C+1)\|f\|_{L^{q_2'}(0,2T;L^{p_2'})}\right], \\
&\leq 4C(C+1)\left(|u_0|_2 + \|f\|_{L^{q_2'}(0,2T;L^{p_2'})}\right)
\end{aligned}.$$

Hence

$$\|u\|_{L^{q_1}(0,2T;L^{p_1})} \leq 8C(C+1)\left(|u_0|_2 + \|f\|_{L^{q_2'}(0,2T;L^{p_2'})}\right).$$

Then, after a finite number of steps, we get estimate (4.3) on an arbitrary bounded interval, as claimed.

Furthermore, for each $t \in [0, T]$, we may take

$$\begin{aligned}
C_t &= \sup\{\|U(t,0)u_0\|_{L^{q_1}(0,t;L^{p_1})};\, |u_0|_2 \leq 1\} \\
&\quad + \sup\left\{\left\|\int_0^t U(t,s)f(s)ds\right\|_{L^{q_1}(0,t;L^{p_1})};\, \|f\|_{L^{q_2'}(0,t;L^{p_2'})} = 1\right\}.
\end{aligned}$$

Obviously, the function $t \to C_t$ is monotonically increasing, $C_0 = 0$, and it follows by (4.3) and standard arguments that it is continuous. Since by separability the sup in the definition of C_t is a sup over countably many

$u_0 \in L^2$ and $f \in L^{q_2'}(0,t;L^{p_2'}) \subset L^1(0,t;H^{-1})$ (by Sobolev embedding) and since, as seen earlier in Lemma 3.3, $t \to U(t,0)u_0$, $t \to \int_0^t U(t,s)f(s)ds$ is adapted, we conclude that $t \to C_t$ is adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$. But then, as a continuous process C_t is $(\mathcal{F}_t)$-progressively measurable, thereby completing the proof.

Lemma A.1.

(i) *Let $y = y(t)$, $t \in [0,T]$, be an L^2-valued $(\mathcal{F}_t)$-adapted process with continuous sample paths satisfying* (3.3), (3.6), (3.7). *Then, $X := e^W y$ is a solution to* (2.1).

(ii) *Suppose $X = X(t)$, $t \in [0,T]$ is an L^2-valued $(\mathcal{F}_t)$-adapted process with continuous sample paths satisfying* (2.1), (2.2) *and* (2.3). *Then, $y := e^{-W}X$ satisfies* (3.3) (*equivalently,* (3.5)).

Before going to the proof of Lemma A.1, a few remarks are in order concerning the formal calculation given at the beginning of Section 3 to link (2.1) and (3.2). In fact, it is purely heuristic since we applied the Itô product to y though it is not of bounded variation in L^2. Furthermore, taking into account that the exponential is an operator of Nemitsky type in L^2 which is not differentiable, the infinite dimensional Itô formula in L^2 is not justified. Also, when we try to apply Itô's product rule for real valued stochastic processes after evaluating the L^2-valued processes X, W, y at $\xi \in \mathbb{R}^d$, which by itself is delicate since L^2 consists of equivalence classes of functions, we run into problems since e.g. again $X(t,\xi)$, $y(t,\xi)$, $t \in [0,T]$, might not be semi-martingales.

The proof we give below is based on the stochastic Fubini theorem and uses the stochastic calculus for complex valued processes and their products in $\mathbb{C}$. (We refer to [11], Section 2, as background literature in regard to this.)

Proof of Lemma A.1. We only prove (i), since (ii) can be proved analogously. Let $\varphi \in H^2(\mathbb{R}^d)$. Then, for every $t \in [0,T]$, we have

$$\left\langle \varphi, e^{W(t)}y(t)\right\rangle_2 = \sum_{j=1}^{\infty} \left\langle \overline{e^{W(t)}}\varphi, f_j\right\rangle_2 \langle f_j, y(t)\rangle_2 \,,$$

where $\{f_j\}_{j=1}^{\infty}$ is an orthonormal basis in L^2; $f_j \in H^2(\mathbb{R}^d)$.

By Itô's formula, we have for all $\xi \in \mathbb{R}^d$, $t \in [0,T]$,

$$e^{W(t,\xi)} = 1 + \int_0^t e^{W(s,\xi)} dW(s,\xi) + \widetilde{\mu}(\xi) \int_0^t e^{W(s,\xi)} ds.$$

Fix $j \in \mathbb{N}$. Then, we have $\mathbb{P}$-a.s. for all $t \in [0,T]$,

$$\begin{aligned}\left\langle \overline{e^{W(t)}}\varphi, f_j \right\rangle_2 &= \langle \varphi, f_j \rangle_2 \\ &\quad + \sum_{k=1}^N \overline{\mu}_k \int_{\mathbb{R}^d} \varphi(\xi) e_k(\xi) \bar{f}_j(\xi) d\xi \int_0^t \overline{e^{W(s,\xi)}} d\beta_k(s) + \int_0^t \left\langle \overline{\widetilde{\mu}}\, \overline{e^{W(s)}}\varphi, f_j \right\rangle_2 ds \\ &= \langle \varphi, f_j \rangle_2 + \sum_{k=1}^N \overline{\mu}_k \int_0^t \left\langle e_k\, \overline{e^{W(s)}}\varphi, f_j \right\rangle_2 d\beta_k(s) + \int_0^t \left\langle \overline{\widetilde{\mu}}\, \overline{e^{W(s)}}\varphi, f_j \right\rangle_2 ds.\end{aligned}$$

(Here, we have used the stochastic Fubini theorem in the second equality.)

Now, we set $A_0 = i\Delta,\ D(A_0) = H^2(\mathbb{R}^d)$ and $J_\varepsilon = (I + \varepsilon A_0)^{-1}$.

Let $y_\varepsilon = J_\varepsilon(y)$. Then, $y_\varepsilon \in C([0,T], H^2(\mathbb{R}^d))$ and

$$\begin{aligned}\frac{\partial y_\varepsilon}{\partial t} &= -iJ_\varepsilon(e^{-W}\Delta(e^W y)) - J_\varepsilon((\mu + \widetilde{\mu})y) \\ &\quad -\lambda i J_\varepsilon(|e^{(\alpha-1)W}|\,|y|^{\alpha-1}y),\ t \in (0,T), \\ y_\varepsilon(0) &= J_\varepsilon(x) = x_\varepsilon.\end{aligned} \tag{7.4}$$

Since $f_j \in H^2(\mathbb{R}^d)$, for each j, $\langle f_j, y_\varepsilon(t)\rangle_2$, $t \in [0,T]$, is of bounded variation. Hence, we can apply the Itô product rule (for scalar valued processes) to obtain

$$\begin{aligned}\left\langle \overline{e^{W(t)}}\varphi, f_j \right\rangle_2 \langle f_j, y_\varepsilon(t)\rangle_2 &= \langle \varphi, f_j \rangle_2 \langle f_j, x_\varepsilon \rangle_2 \\ &\quad + i \int_0^t \left\langle \overline{e^{W(s)}}\varphi, f_j \right\rangle_2 \left\langle f_j, J_\varepsilon(e^{-W(s)}\Delta(e^{W(s)}y(s)))\right\rangle_2 ds \\ &\quad - \int_0^t \left\langle \overline{e^{W(s)}}\varphi, f_j \right\rangle_2 \langle f_j, J_\varepsilon((\mu + \widetilde{\mu})y(s))\rangle_2 ds \\ &\quad + \lambda i \int_0^t \left\langle \overline{e^{W(s)}}\varphi, f_j \right\rangle_2 \left\langle f_j, J_\varepsilon(|e^{(\alpha-1)W(s)}|\,|y(s)|^{\alpha-1}y(s))\right\rangle_2 ds \\ &\quad + \sum_{k=1}^N \overline{\mu}_k \int_0^t \langle f_j, y_\varepsilon(s)\rangle_2 \left\langle e_k\, \overline{e^{W(s)}}\varphi, f_j \right\rangle_2 d\beta_k(s) \\ &\quad + \int_0^t \langle f_j, y_\varepsilon(s)\rangle_2 \left\langle \overline{\widetilde{\mu}}\, \overline{e^{W(s)}}\varphi, f_j \right\rangle_2 ds.\end{aligned}$$

(We note that, since $J_\varepsilon(e^{-W}\Delta(e^W y)) \in C([0,T];L^2)$, the second integral in the above equality makes sense.)

Now, summing over $j \in \mathbb{N}$ and interchanging the infinite sum with the integrals, we obtain $\mathbb{P}$-a.s., for all $t \in [0,T]$,

$$\begin{aligned}
&\left\langle \varphi, e^{W(t)} y_\varepsilon(t) \right\rangle_2 = \langle \varphi, x_\varepsilon \rangle_2 + i \int_0^t \left\langle \varphi, e^{W(s)} J_\varepsilon(e^{-W(s)} \Delta(e^{W(s)} y(s))) \right\rangle_2 ds \\
&- \int_0^t \left\langle \varphi, e^{W(s)} J_\varepsilon((\mu + \widetilde{\mu}) y) \right\rangle_2 ds + \lambda i \int_0^t \left\langle \varphi, e^{W(s)} J_\varepsilon(|e^{(\alpha-1)W}|\, |y(s)|^{\alpha-1} y(s)) \right\rangle_2 ds \\
&+ \sum_{k=1}^N \int_0^t \left\langle \varphi, \mu_k e_k e^{W(s)} y_\varepsilon(s) \right\rangle_2 d\beta_k(s) + \int_0^t \left\langle \varphi, \widetilde{\mu} e^{W(s)} y_\varepsilon(s) \right\rangle_2 ds.
\end{aligned}$$

On the other hand, we have, for $\varepsilon \to 0$,

$$J_\varepsilon(f) \to f \text{ strongly in } H^k.$$

Furthermore

$$\|J_\varepsilon(f)\|_{H^k} \le \|f\|_{H^k},$$

where $f \in H^k$ and $k = 0,1,2$. Then, we may pass to the limit $\varepsilon \to 0$ in the previous equality to obtain

$$\begin{aligned}
\left\langle \varphi, e^{W(t)} y(t) \right\rangle_2 &= \langle \varphi, x \rangle_2 + i \int_0^t \left\langle \varphi, \Delta(e^{W(s)} y(s)) \right\rangle ds - \int_0^t \left\langle \varphi, \mu e^{W(s)} y(s) \right\rangle_2 ds \\
&+ \lambda i \int_0^t \left\langle \varphi, e^{W(s)} |e^{(\alpha-1)W(s)}|\, |y(s)|^{\alpha-1} y(s) \right\rangle ds \\
&+ \sum_{k=1}^N \int_0^t \left\langle \varphi, \mu_k e_k e^{W(s)} y(s) \right\rangle_2 d\beta_k(s), \ \forall t \in [0,T],
\end{aligned}$$

which implies the fact that $X(t) = e^{W(t)} y(t)$ is the solution to (2.1), as claimed. In the above equality, $\langle \cdot, \cdot \rangle$ is the pairing between L^2, H^2 and H^{-2} or, equivalently,

$$\left\langle \varphi, \Delta(e^W y) \right\rangle = \int_{\mathbb{R}^d} \Delta\varphi \overline{e^W} \bar{y}\, d\xi, \varphi \in H^2.$$

This completes the proof. □

Lemma A.2. *Let τ_{n+1} be defined as in Step 2 in the proof of Lemma 4.2. Then y_{n+1} is adapted to $(\mathcal{F}_t)$.*

Proof. We first note that, by $z_{n+1} = F_n(z_{n+1})$ and Banach's fixed point theorem, there exists a sequence $\{v_{n+1,m}\}_{m\geq 1}$, adapted to $(\mathcal{F}_{\tau_n+t})$, satisfying $v_{n+1,m+1} = F_n(v_{n+1,m})$ for $m \geq 1$, $v_{n+1,1} = y_n(\tau_n)$ and $z_{n+1} = \lim_{m\to\infty} v_{n+1,m}$ in $C([0,t];L^2) \cap L^q(0,t;L^{\alpha+1}))$, $t \in [0,\sigma_n]$. Now, we define

$$u_{n+1,m}(t) = \begin{cases} y_n(t), & t \in [0,\tau_n], \\ v_{n+1,m}(t-\tau_n), & t \in (\tau_n,\infty). \end{cases}$$

Thus, $y_{n+1} = \lim_{m\to\infty} u^{\tau_{n+1}}_{n+1,m}$, in $C([0,T];L^2)$. Below, we show that $u_{n+1,m}$ is adapted to $(\mathcal{F}_t)$. In fact, let f_j, $j \in \mathbb{N}$, be an orthonormal basis of L^2. We have, for each $a > 0$, $\{|\langle u_{n+1,m}(t), f_j\rangle_2| < a\} = J_{1,a} \cup J_{2,a}$, where $J_{1,a} = \{|\langle y_n(t), f_j\rangle_2| < a,\, t \leq \tau_n\}$ and $J_{2,a} = \{|\langle v_{n+1,m}(t-\tau_n), f_j\rangle_2| < a,\, \tau_n < t\}$. Since y_n is adapted to $(\mathcal{F}_t)$ and τ_n is an $(\mathcal{F}_t)$ stopping time, it follows that $J_{1,a} \in \mathcal{F}_t$.

By the continuity of $t \mapsto |\langle v_{n+1,m}(t-\tau_n), f_j\rangle_2|$ we see that

$$J_{2,a} = \bigcup_{\substack{q\in Q\\ q<a}} \bigcup_{h\in\mathbb{N}} \bigcap_{s\in Q} J_{q,h,s},$$

where $J_{q,h,s} = \left\{|\langle v_{n+1,m}(s), f_j\rangle_2| < q,\; t-\tau_n-\frac{1}{h} < s < t-\tau_n, \tau_n < t\right\}$.

Taking into account that $\{|\langle v_{n+1,m}(s), f_j\rangle_2| < q\} \in \mathcal{F}_{\tau_n+s}$ and $\tau_n + s < t$, we have $J_{q,h,s} \in \mathcal{F}_t$, which implies that $J_{2,a} \in \mathcal{F}_t$.

Collecting the above results, we obtain that, for any $j \in \mathbb{N}$ and $a > 0$, $\{|\langle u_{n+1,m}(t), f_j\rangle_2| < a\} \in \mathcal{F}_t$. This is enough to imply that $u_{n+1,m}$ is adapted to $(\mathcal{F}_t)$. Therefore, as the limit of $u^{\tau_{n+1}}_{n+1,m}$, y_{n+1} is also adapted to $(\mathcal{F}_t)$. This completes the proof. □

Acknowledgement. The authors are indebted to Daniel Tataru for fruitful discussions and suggestions regarding the proof of Lemma 4.1 [15].

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