Strong uniqueness for stochastic evolution equations with unbounded measurable drift term

G. Da Prato [*]

Scuola Normale Superiore

Piazza dei Cavalieri 7, 56126 Pisa, Italy

F. Flandoli [†]

Dipartimento di Matematica Applicata "U. Dini"

Università di Pisa, Italy

E. Priola [‡]

Dipartimento di Matematica, Università di Torino

via Carlo Alberto 10, Torino, Italy

M. Röckner [§]

Faculty of Mathematics, Bielefeld University

33501 Bielefeld, Germany

January 10, 2014

Abstract We consider stochastic evolution equations in Hilbert spaces with merely measurable and locally bounded drift term B and cylindrical Wiener noise. We prove pathwise (hence strong) uniqueness in the class of global solutions. This paper extends our previous paper [5] which generalized Veretennikov's fundamental result to infinite dimensions assuming boundedness of the drift term. As in [5] pathwise uniqueness holds for a large class, but not for every initial condition. We also include an application of our result to prove existence of strong solutions when the drift B is only measurable, locally bounded and grows more than linearly.

Key words: Pathwise uniqueness, stochastic PDEs, locally bounded measurable drift term, strong mild solutions

Mathematics Subject Classification 35R60, 60H15

[*]E-mail: g.daprato@sns.it

[†]E-mail: flandoli@dma.unipi.it

[‡]E-mail: enrico.priola@unito.it

[§]Research supported by the DFG through IRTG 1132 and CRC 701 and the I. Newton Institute, Cambridge, UK.

E-mail: roeckner@mathematik.uni-bielefeld.de

1 Introduction

We consider the following abstract stochastic differential equation in a separable Hilbert space H

$$dX_t = (AX_t + B(X_t))dt + dW_t, \quad t \geq 0, \qquad X_0 = x \in H, \tag{1}$$

where $A : D(A) \subset H \to H$ is self-adjoint, negative definite and such that $(-A)^{-1+\delta}$, for some $\delta \in (0, 1)$, is of trace class, $B : H \to H$ and $W = (W_t)$ is a cylindrical Wiener process. In [5] under the assumption that B is Borel measurable and (globally) bounded we prove pathwise uniqueness of solutions to (1). A natural generalization is to extend it to the case where we only assume that B is *Borel measurable and locally bounded* (i.e., bounded on balls):

$$B \in B_{b,loc}(H, H). \tag{2}$$

In this paper we prove that assuming (2) pathwise uniqueness holds for μ-a.e. initial condition x in the class of global mild solutions to (1). Here μ denotes the Gaussian measure which is invariant for the Ornstein-Uhlenbeck process $Z = (Z_t)$ which solves (1) when $B = 0$ (see Section 1.1 for more details).

In other words if for some initial condition $x \in H$, μ-a.e., there exists a solution for (1) on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ with a cylindrical $(\mathcal{F}_t)$-Wiener process W then our main result shows that this solution is pathwise unique. This is in particular the case when

$$B \text{ is measurable and at most of linear growth} \tag{3}$$

(i.e., B is measurable and there exists $a, b \geq 0$ such that $|B(x)| \leq a + b|x|$, $x \in H$), because then existence of weak mild solutions is well-known (see Chapter 10 in [6], [11, 14] and also Appendix in [5]). Moreover, under condition (3), the unique law of any mild solution X^x is equivalent to the law of the Ornstein-Uhlenbeck process starting at x (corresponding to $B = 0$).

By our main result, using a generalization of the Yamada-Watanabe theorem (see [18, 21]), one deduces that under (3) equation (1) has a unique strong mild solution, for μ-a.e. $x \in H$, generalizing A. Veretennikov's seminal result [26] in the case $H = \mathbb{R}^d$ (see also [9, 10, 15, 16, 17, 25, 27, 28]) to infinite dimensions.

In Section 4 we generalize this by relaxing assumption (3). We prove existence of strong mild solutions, starting from μ-a.e. initial condition $x \in H$, when $B \in B_{b,loc}(H, H)$ and moreover there exist $C > 0$, $p > 0$, such that

$$\langle B(y + z), y \rangle \leq C\big(|y|^2 + e^{p|z|} + 1\big) \tag{4}$$

for all $y, z \in H$ (see also Remark 17). Finally in Section 4.3 we show a possible extension of our result by considering local mild solutions.

In order to prove *pathwise uniqueness* for (1) we will consider bounded truncated drifts like

$$B_N = B\, 1_{B(0,N)}, \quad N \geq 1, \tag{5}$$

where $B(0, N)$ is the open ball of center 0 and of radius N and $1_{B(0,N)}$ is the indicator function of $B(0, N)$, by performing a suitable stopping time argument. This argument is not straightforward since it must be also used in combination with the Ito-Tanaka trick from [5] (see also [8, 12, 13, 23, 26]).

In addition in this paper we also simplify some arguments used in [5] in the case of $B \in B_b(H, H)$ (see, in particular, Lemma 8). Before stating our main result precisely, let us recall the following definition (cf. [21] and [18]).

Definition 1. *Let $x \in H$.*
(a) We call weak mild solution to (1) a tuple $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}, W, X)$, where $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ is a filtered probability space on which it is defined a cylindrical $(\mathcal{F}_t)$-Wiener process W and a continuous $(\mathcal{F}_t)$-adapted H-valued process $X = (X_t) = (X_t)_{t \geq 0}$ such that, $\mathbb{P}$-a.s.,

$$X_t = e^{tA}x + \int_0^t e^{(t-s)A} B\left(X_s\right) ds + \int_0^t e^{(t-s)A} dW_s, \quad t \geq 0. \tag{6}$$

(b) A weak mild solution X which is $(\bar{\mathcal{F}}_t^W)$-adapted (here $(\bar{\mathcal{F}}_t^W)$ denotes the completed natural filtration of the cylindrical process W) is called strong mild solution.

We will often use stopping times

$$\tau_N^X = \inf\{t \geq 0 \ : \ X_t \notin B(0, N)\} \tag{7}$$

($\tau_N^X = +\infty$ if the set is empty), $N \geq 1$.

In our main result we consider two mild solutions X and Y, having the same initial condition $x \in H$, and solving the same equation (1) but with possibly different drift terms, respectively B and $B' \in B_{b,loc}(H, H)$, i.e.,

$$dX_t = (AX_t + B(X_t))dt + dW_t, \quad X_0 = x, \tag{8}$$

$$dY_t = (AY_t + B'(Y_t))dt + dW_t, \quad Y_0 = x. \tag{9}$$

Theorem 1. *Assume Hypothesis 1 (see Section 1.1) and let μ be the centered Gaussian measure on H with covariance $Q = -\frac{1}{2}A^{-1}$.*

Then for μ-a.e. $x \in H$, if X and Y are two weak mild solutions, respectively of (8) *and* (9)*, defined on the same filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ with the same cylindrical Wiener process W, and if, for some $N \geq 1$,*

$$B(x) = B'(x), \quad x \in B(0, N), \tag{10}$$

then, $\mathbb{P}$-a.s.,

$$X_{t \wedge \tau_N^X \wedge \tau_N^Y} = Y_{t \wedge \wedge \tau_N^X \wedge \tau_N^Y}, \quad t \geq 0, \tag{11}$$

and so $\tau_N^X = \tau_N^Y$, $\mathbb{P}$-a.s..

Above we restrict to W which are cylindrical with respect to the eigenbasis of A (see Section 1.1 for details). Clearly if $B = B'$ the result implies that, $\mathbb{P}$-a.s.,

$$X_t = Y_t, \quad t \geq 0. \tag{12}$$

Indeed using that $\tau_N^X \uparrow +\infty$ and $\tau_N^Y \uparrow +\infty$ as $N \to \infty$ (because X and Y are both global solutions) we deduce easily (12) from (11).

The proof of Theorem 1, performed in Section 3, uses a truncation argument and regularity results for elliptic equations in Hilbert spaces involving truncated drift terms B_N (cf. (5)). Such regularity results are given in Section 2, where we also establish an Itô type formula involving $u(X_t)$ with u in some Sobolev space associated to μ (see Theorem 10). In comparison with [5] to prove such an Itô type formula we use a new analytic lemma (see Lemma 8).

There are several other quite essential differences in comparison with [5] in our proof. We refer to Remarks 9 and 12 for details.

1.1 Assumptions and preliminaries

As in [5] we are given a real separable Hilbert space H and denote its norm and inner product by $|\cdot|$ and $\langle \cdot, \cdot \rangle$ respectively. We follow [4, 6, 7, 22], and assume

Hypothesis 1 $A : D(A) \subset H \to H$ is a negative definite self-adjoint operator and $(-A)^{-1+\delta}$, for some $\delta \in (0, 1)$, is of trace class.

Remark 2. Our uniqueness result continues to hold under the following more general assumption: $A : D(A) \subset H \to H$ is self-adjoint and there exists $\omega \in \mathbb{R}$ such that $(A - \omega)$ is negative definite and $(\omega - A)^{-1+\delta}$, for some $\delta \in (0, 1)$, is of trace class. Indeed if we write equation (1) in the form

$$dX_t = (AX_t - \omega X_t)dt \; + \; (\omega X_t + B(X_t))dt + dW_t, \qquad X_0 = x \in H,$$

then the linear operator $(A - \omega I)$ verifies Hypothesis 1 and the drift $\omega I + B$ continues to satisfy (2).

Since A^{-1} is compact, there exists an orthonormal basis (e_k) in H and a sequence of positive numbers (λ_k) such that

$$Ae_k = -\lambda_k e_k, \quad k \in \mathbb{N}. \tag{13}$$

Recall that A generates an analytic semigroup e^{tA} on H such that $e^{tA}e_k = e^{-\lambda_k t}e_k$.

From now on until and including Section 3 we fix $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$, W, X and Y as in the assertion of Theorem 1. As said before we will consider a cylindrical Wiener process W_t with respect to the previous basis (e_k). The process W_t is formally given by "$W_t = \sum_{k \geq 1} \beta_k(t)e_k$" where $\beta_k(t)$ are independent one dimensional Wiener processes (see [6] for more details).

By R_t we denote the Ornstein-Uhlenbeck semigroup in $B_b(H)$ (the Banach space of Borel and bounded real functions endowed with the essential supremum norm $\|\cdot\|_0$) defined as

$$R_t\varphi(x) = \int_H \varphi(y) N(e^{tA}x, Q_t)(dy), \quad \varphi \in B_b(H), \tag{14}$$

where $N(e^{tA}x, Q_t)$ is the Gaussian measure in H of mean $e^{tA}x$ and covariance operator Q_t given by,

$$Q_t = -\frac{1}{2}\, A^{-1}(I - e^{2tA}), \quad t \geq 0. \tag{15}$$

We note that R_t has the unique invariant measure $\mu := N(0, Q)$ where $Q = -\frac{1}{2}\, A^{-1}$. Moreover, since under the previous assumptions, the Ornstein-Uhlenbeck semigroup is strong Feller and irreducible we have by Doob's theorem that, for any $t > 0$, $x \in H$, the measures $N(e^{tA}x, Q_t)$ and μ are equivalent (see [7]). On the other hand, our assumption that $(-A)^{-1+\delta}$ is trace class guarantees that the OU process

$$Z_t = Z(t, x) = e^{tA}x + \int_0^t e^{(t-s)A} dW_s \tag{16}$$

has a continuous H-valued version.

If H and K are separable Hilbert spaces, the Banach space $L^p(H, \mu;\, K)$, $p \geq 1$, is defined to consist of equivalent classes of measurable functions $f : H \to K$ such that $\int_H |f|_K^p\, \mu(dx) < +\infty$ (if $K = \mathbb{R}$ we set $L^p(H, \mu; \mathbb{R}) = L^p(H, \mu)$). We also use the notation $L^p(\mu)$ instead of $L^p(H, \mu, K)$ when no confusion may arise.

The semigroup R_t can be uniquely extended to a strongly continuous semigroup of contractions on $L^p(H, \mu)$, $p \geq 1$, which we still denote by R_t, whereas we denote by L_p (or L when no confusion may arise) its infinitesimal generator, which is defined on smooth functions φ as

$$L\varphi(x) = \frac{1}{2}\mathrm{Tr}(D^2\varphi(x)) + \langle Ax, D\varphi(x)\rangle,$$

where $D\varphi(x)$ and $D^2\varphi(x)$ denote respectively the first and second Fréchet derivatives of φ at $x \in H$. For Banach spaces E and F we denote by $C_b^k(E, F)$, $k \geq 1$, the Banach space of all functions $f : E \to F$ which are bounded and Fréchet differentiable on E up to order $k \geq 1$ with all derivatives bounded and continuous. We also set $C_b^k(E, \mathbb{R}) = C_b^k(E)$.

According to [7], for any $\varphi \in B_b(H)$ and any $t > 0$ one has $R_t\varphi \in C_b^\infty(H) = \cap_{k\geq 1} C_b^k(H)$. Moreover,

$$\langle DR_t\varphi(x), h\rangle = \int_H \langle \Lambda_t h, Q_t^{-\frac{1}{2}} y\rangle \varphi(e^{tA}x + y) N(0, Q_t)(dy), \ \ h \in H, \tag{17}$$

where Q_t is as defined in (15),

$$\Lambda_t = Q_t^{-1/2} e^{tA} = \sqrt{2}\, (-A)^{1/2} e^{tA} (I - e^{2tA})^{-1/2} \tag{18}$$

and $y \mapsto \langle \Lambda_t h, Q_t^{-\frac{1}{2}} y\rangle$ is a centered Gaussian random variable under $\mu_t = N(0, Q_t)$ with variance $|\Lambda_t h|^2$ for any $t > 0$ (cf. Theorem 6.2.2 in [6]). Since

$$\Lambda_t e_k = \sqrt{2}\, (\lambda_k)^{1/2} e^{-t\lambda_k} (1 - e^{-2t\lambda_k})^{-1/2} e_k,$$

we see that there exists $C_0' > 0$ such that $\|\Lambda_t\| \le C_0'\, t^{-\frac{1}{2}}$.

In the sequel $\|\cdot\|$ always denotes the *Hilbert-Schmidt norm*; on the other hand $\|\cdot\|_{\mathcal{L}}$ indicates the *operator norm.* By (17) we deduce

$$\sup_{x \in H} |DR_t\varphi(x)| = \|DR_t\varphi\|_0 \le C_0\, t^{-\frac{1}{2}} \|\varphi\|_0, \quad t > 0, \tag{19}$$

which by taking the Laplace transform yields

$$\|D\,(\lambda - L_2)^{-1}\varphi\|_0 \le \frac{C_0}{\lambda^{\frac{1}{2}}}\, \|\varphi\|_0, \quad \lambda > 0.$$

Similarly, we find

$$\|DR_t\varphi\|_{L^2(\mu)} \le C_0 t^{-\frac{1}{2}}\, \|\varphi\|_{L^2(\mu)}$$

and

$$\|D\,(\lambda - L_2)^{-1}\varphi\|_{L^2(\mu)} \le \frac{C_0}{\lambda^{\frac{1}{2}}}\, \|\varphi\|_{L^2(\mu)}.$$

Recall that the Sobolev space $W^{2,p}(H,\mu)$, $p \ge 1$, is defined in Section 3 of [3] as the completion of a suitable set of smooth functions endowed with the Sobolev norm (see also Section 9.2 in [6] for the case $p = 2$ and [24]). Under our above assumptions, the following result can be found in Section 10.2.1 of [7].

Theorem 3. *Let $\lambda > 0$, $f \in L^2(H,\mu)$ and let $\varphi \in D(L_2)$ be the solution of the equation*

$$\lambda\varphi - L_2\varphi = f.$$

Then $\varphi \in W^{2,2}(H,\mu)$, $(-A)^{1/2}D\varphi \in L^2(H,\mu;H)$ and there exists a constant $C(\lambda)$ such that

$$\|\varphi\|_{L^2(\mu)} + \Big(\int_H \|D^2\varphi(x)\|^2\, \mu(dx)\Big)^{1/2} + \|(-A)^{1/2}D\varphi\|_{L^2(\mu)} \le C\|f\|_{L^2(\mu)}.$$

The following extension to $L^p(\mu)$, $p > 1$, can be found in Section 3 of [3] (see also [2, 20]).

Theorem 4. *Let $\lambda > 0$, $f \in L^p(H,\mu)$ and let $\varphi \in D(L_p)$ be the solution of the equation*

$$\lambda\varphi - L_p\varphi = f.$$

Then $\varphi \in W^{2,p}(H,\mu)$, $(-A)^{1/2}D\varphi \in L^p(H,\mu;H)$ and there exists a constant $C = C(\lambda, p)$ such that

$$\|\varphi\|_{L^p(\mu)} + \Big(\int_H \|D^2\varphi(x)\|^p\, \mu(dx)\Big)^{1/p} + \|(-A)^{1/2}D\varphi\|_{L^p(\mu)} \le C\|f\|_{L^p(\mu)}.$$

2 Analytic results and an Itô type formula

2.1 Existence and uniqueness for the Kolmogorov equation when B is bounded

We are here concerned with the equation

$$\lambda u - L_2 u - \langle B, Du\rangle = f, \tag{20}$$

where $\lambda > 0$, $f \in B_b(H)$ and $B \in B_b(H, H)$ (i.e., $B : H \to H$ is Borel and bounded).

Remark 5. Since the corresponding Dirichlet form

$$\mathcal{E}(u, v) := \int_H \langle Du, Dv\rangle d\mu - \int_H \langle B, Du\rangle v \, d\mu + \lambda \int_H uv \, d\mu,$$

u, $v \in W^{1,2}(\mu)$, is weakly sectorial for λ big enough, it follows by Chap. I and Subsection 3e in Chap. II of [19] that (20) has a unique solution in $D(L_2)$. However, we need more regularity for u.

We recall a result from [5].

Proposition 6. *Let $\lambda \geq \lambda_0$, where*

$$\lambda_0 := 4\|B\|_0^2 C_0^2. \tag{21}$$

Then there is a unique solution $u \in D(L_2)$ of (20) given by

$$u = u_\lambda = (\lambda - L_2)^{-1}(I - T_\lambda)^{-1} f,$$

where

$$T_\lambda \varphi := \langle B, D(\lambda - L_2)^{-1}\varphi\rangle. \tag{22}$$

Moreover, $u \in C_b^1(H)$ with

$$\|u\|_0 \leq 2\|f\|_0, \quad \|Du\|_0 \leq \frac{2C_{1,0}}{\lambda^{\frac{1}{2}}} \|f\|_0, \tag{23}$$

and, for any $p \geq 2$, $u \in W^{2,p}(H, \mu)$ and, for some $C = C(\lambda, p, \|B\|_0)$,

$$\int_H \|D^2 u(x)\|^p \, \mu(dx) \leq C \int_H |f(x)|^p \, \mu(dx). \tag{24}$$

2.2 Approximations

We are given two sequences $(f_n) \subset B_b(H)$ and $(B_n) \subset B_b(H, H)$ such that

$$\begin{aligned} &(i) \quad f_n(x) \to f(x), \quad B_n(x) \to B(x) \quad \mu\text{-a.e..} \\ &(ii) \ \|f_n\|_0 \le M, \quad \|B_n\|_0 \le M. \end{aligned} \tag{25}$$

The following result has been proved in [5].

Proposition 7. *Let $\lambda \ge \lambda_0$, where λ_0 is defined in (21). Then the equation*

$$\lambda u_n - L u_n - \langle B_n, D u_n \rangle = f_n, \tag{26}$$

has a unique solution $u_n \in C_b^1(H) \cap D(L_2)$ given by

$$u_n = (\lambda - L)^{-1}(I - T_{n,\lambda})^{-1} f_n,$$

where

$$T_{n,\lambda}\varphi := \langle B_n, D(\lambda - L_2)^{-1}\varphi \rangle.$$

Moreover,

$$\|u_n\|_0 \le 2M, \quad \|Du_n\|_0 \le \frac{2C_{1,0}}{\lambda^{\frac{1}{2}}} M, \quad n \ge 1. \tag{27}$$

Finally, we have $u_n \to u$, and $Du_n \to Du$, in $L^2(\mu)$, where u is the solution to (20).

Next we prove a new result. The idea behind the result is that if (f_n) satisfies (25) then, for any $x \in H$ (not only μ-a.e.), $t > 0$,

$$R_t f_n(x) \to R_t f(x) \tag{28}$$

as $n \to \infty$, due to the fact that, for any $x \in H$, the law of the OU process $Z(t, x)$ at time $t > 0$ is absolutely continuous with respect to μ.

Lemma 8. *Consider the situation of Proposition 7. Then we have:*

$$u_n(x) \to u(x), \quad Du_n(x) \to Du(x), \tag{29}$$

for any $x \in H$.

Proof. By a standard argument, possibly passing to a subsequence, we may assume that $Du_n(x) \to Du(x)$, μ-a.e.. It follows that for any x, μ-a.e.,

$$f_n(x) + \langle B_n(x), Du_n(x) \rangle \to f(x) + \langle B(x), Du(x) \rangle,$$

as $n \to \infty$. We write, for any $\lambda \ge \lambda_0$, $x \in H$,

$$u_n(x) = \int_0^\infty e^{-\lambda t} R_t \big(f_n + \langle B_n, Du_n \rangle \big)(x) dt.$$

By the argument used in (28) we can apply the dominated convergence theorem and obtain that $u_n(x) \to u(x)$, as $n \to \infty$, $x \in H$.

Concerning Du_n, using (19), we obtain, for any $x \in H$, $h \in H$,

$$\langle Du_n(x), h\rangle = \int_0^\infty e^{-\lambda t}\langle DR_t\big(f_n + \langle B_n, Du_n\rangle\big)(x), h\rangle dt.$$

Setting $g_n = f_n + \langle B_n, Du_n\rangle$, $g = f + \langle B, Du\rangle$, note that

$$\langle DR_t g_n(x), e_k\rangle = \int_H \langle \Lambda_t e_k, Q_t^{-\frac{1}{2}} y\rangle\, g_n(e^{tA}x + y) N(0, Q_t)(dy), \quad k \geq 1,\ x \in H,\ t > 0.$$

It follows that

$$|\langle DR_t(g_n - g)(x), e_k\rangle|^2 \leq |\Lambda_t e_k|^2 \int_H |g(e^{tA}x + y) - g_n(e^{tA}x + y)|^2 N(0, Q_t)(dy),$$

and so

$$|DR_t(g_n - g)(x)|^2 \leq \|\Lambda_t\|^2 \int_H |g(e^{tA}x + y) - g_n(e^{tA}x + y)|^2 N(0, Q_t)(dy).$$

Now we get $|DR_t(g_n - g)(x)|^2 \to 0$ as $n \to \infty$, for any $x \in H$, $t > 0$, by the same argument used in (28).

Using again the dominated convergence theorem we obtain easily that $Du_n(x) \to Du(x)$, as $n \to \infty$, for any $x \in H$. □

2.3 Modified mild formulation

Recall the notation

$$B_N = B\, 1_{B(0,N)}, \quad N \geq 1, \tag{30}$$

where $B(0, N)$ is the open ball of radius N (hence $B_N \in B_b(H, H)$, $N \geq 1$).

For any $i \in \mathbb{N}$ we denote the i^{th} component of B by $B^{(i)}$, i.e.,

$$B^{(i)}(x) := \langle B(x), e_i\rangle, \quad x \in H.$$

Then for $\lambda \geq 4\|B_N\|_0^2 C_0^2$ we consider the solution $u_N^{(i)}$ of the equation

$$\lambda u_N^{(i)} - L u_N^{(i)} - \langle B_N, D u_N^{(i)}\rangle = B_N^{(i)}, \quad \mu\text{ -a.e.} \tag{31}$$

We recall that by Proposition 6, $u_N^{(i)} \in C_b^1(H)$ and, for any $p \geq 2$, $u_N^{(i)} \in W^{2,p}(H, \mu)$.

The next result is a kind of "local version" of Theorem 7 in [5]. In contrast to Theorem 1 the result holds for any initial condition $x \in H$.

Remark 9. Compared with the proof of Theorem 7 in [5], here we will use Lemma 8 which allows to simplify some arguments of [5] and it is also needed to justify the approximation procedure (see in particular (45)). We also mention that differently with respect to [5] in Step 3 of the proof we need to construct a suitable auxiliary process $\hat{X}^N = (\hat{X}_t^N)$ (see (47)) in order to apply the Girsanov theorem and get the assertion.

Theorem 10. *Let $X = (X_t)$ be a weak mild solution of equation* (1) *defined on some filtered probability space with a cylindrical $(\mathcal{F}_t)$-Wiener process W. Consider the stopping time*

$$\tau_N^X = \inf\{t \geq 0 \ : \ X_t \notin B(0,N)\}$$

Let $u_N^{(i)}$ be the solution of (31) *and set $X_t^{(i)} = \langle X_t, e_i \rangle$. For any $t > 0$ we have* $\mathbb{P}$-a.s. *on the event $\{t \leq \tau_N^X\}$*

$$\begin{aligned} X_t^{(i)} =& e^{-\lambda_i t}(\langle x, e_i \rangle + u_N^{(i)}(x)) - u_N^{(i)}(X_t) \\ &+ (\lambda + \lambda_i) \int_0^t e^{-\lambda_i (t-s)} u_N^{(i)}(X_s) ds \\ &+ \int_0^t e^{-\lambda_i (t-s)} (d\langle W_s, e_i \rangle + \langle D u_N^{(i)}(X_s), dW_s \rangle). \end{aligned} \tag{32}$$

Proof. We fix $t > 0$, $N \geq 1$ and $i \geq 1$.

Step 1 Approximation of B_N and u_N.

Set

$$B_{N,n}(x) = \int_H B_N(e^{\frac{1}{n}A}x + y) N(0, Q_{\frac{1}{n}})(dy), \quad x \in H. \tag{33}$$

Then $B_{N,n}$ is of C^∞ class and all its derivatives are bounded. Moreover, $\|B_{N,n}\|_0 \leq \|B_N\|_0$, $n \geq 1$. It is easy to see that, possibly passing to a subsequence,

$$B_{N,n} \to B_N, \qquad \mu - a.e.. \tag{34}$$

(indeed $B_{N,n} \to B_N$ in $L^2(H, \mu; H)$; this result can be first checked for continuous and bounded B). Now we denote by $u_{N,n}^{(i)}$ the solution of the equation

$$\lambda u_{N,n}^{(i)} - L u_{N,n}^{(i)} - \langle B_{N,n}, D u_{N,n}^{(i)} \rangle = B_{N,n}^{(i)}, \tag{35}$$

where $B_{N,n}^{(i)} = \langle B_{N,n}, e_i \rangle$. By Lemma 8 we have, possibly passing to a subsequence, for any $x \in H$,

$$\begin{aligned} &\lim_{n\to\infty} u_{N,n}^{(i)}(x) = u_N^{(i)}(x), \ \ \lim_{n\to\infty} D u_{N,n}^{(i)}(x) = D u_N^{(i)}(x), \\ &\qquad\qquad\qquad \sup_{n \geq 1} \|u_{N,n}^{(i)}\|_{C_b^1(H)} = C_{i,N} < \infty, \end{aligned} \tag{36}$$

where $u_N^{(i)}$ is the solution of (31).

Step 2 Approximation of X_t.

For any $m \geq i$ we set $X_m = (X_{m,t})$, $X_{m,t} := \pi_m X_t$, where $\pi_m = \sum_{j=1}^m e_j \otimes e_j$. Then we have

$$X_{m,t} = \pi_m x + \int_0^t A_m X_s ds + \int_0^t \pi_m B(X_s) ds + \pi_m W_t, \tag{37}$$

where $A_m = \pi_m A$.

Now we denote by $u^{(i)}_{N,n,m}$ the solution of the equation

$$\lambda u^{(i)}_{N,n,m} - L u^{(i)}_{N,n,m} - \langle \pi_m B_{N,n} \circ \pi_m, D u^{(i)}_{N,n,m} \rangle = B^{(i)}_{N,n} \circ \pi_m, \tag{38}$$

where $(B_{N,n} \circ \pi_m)(x) = B_{N,n}(\pi_m x)$, $x \in H$. Since only a finite number of variables is involved, we have, equivalently,

$$\lambda u^{(i)}_{N,n,m} - L^{(m)} u^{(i)}_{N,n,m} - \langle \pi_m B_{N,n} \circ \pi_m, D u^{(i)}_{N,n,m} \rangle = B^{(i)}_{N,n} \circ \pi_m,$$

with

$$L^{(m)} \varphi(x) = \frac{1}{2} \text{ Tr } [\pi_m D^2 \varphi(x)] + \langle A_m x, D\varphi(x) \rangle. \tag{39}$$

Moreover, since $u^{(i)}_{N,n,m}$ depends only on the first m variables, we have

$$u^{(i)}_{N,n,m}(\pi_m y) = u^{(i)}_{N,n,m}(y), \quad y \in H. \tag{40}$$

Applying the finite-dimensional Itô formula to $u^{(i)}_{N,n,m}(X_{m,t}) = u^{(i)}_{N,n,m}(X_t)$ with the stopping time τ_N^X yields

$$\begin{aligned} &u^{(i)}_{N,n,m}(X_{m,t\wedge\tau_N^X}) - u^{(i)}_{N,n,m}(\pi_m x) = \\ &\int_0^{t\wedge\tau_N^X} \Big(\frac{1}{2} \text{ Tr } [D^2 u^{(i)}_{N,n,m}(X_{m,s})] + \langle D u^{(i)}_{N,n,m}(X_{m,s}), A_m X_s + \pi_m B(X_s) \rangle \Big) ds \\ &+ \int_0^{t\wedge\tau_N^X} \langle D u^{(i)}_{n,m}(X_{m,s}), \pi_m dW_s \rangle. \end{aligned} \tag{41}$$

On the other hand, by (38) we have

$$\begin{aligned} &\lambda u^{(i)}_{N,n,m}(X_{m,t}) - \tfrac{1}{2} \text{ Tr } [D^2 u^{(i)}_{N,n,m}(X_{m,t})] \\ &- \langle D u^{(i)}_{N,n,m}(X_{m,t}), A_m X_{m,t} + \pi_m B_{N,n}(X_{m,t}) \rangle = B^{(i)}_{N,n}(X_{m,t}). \end{aligned}$$

Let us fix $r \in]0, t]$. This will be useful in Step 3 of the proof to apply the Girsanov theorem (see in particular (49)).

Comparing with (41) and using (40) we find

$$u^{(i)}_{N,n,m}(X_{t\wedge\tau_N^X}) - u^{(i)}_{N,n,m}(X_{r\wedge\tau_N^X}) \tag{42}$$

$$= \lambda \int_{r\wedge\tau_N^X}^{t\wedge\tau_N^X} u_{N,n,m}^{(i)}(X_s)ds - \int_{r\wedge\tau_N^X}^{t\wedge\tau_N^X} B_{N,n}^{(i)}(X_{m,s})ds$$

$$+ \int_{r\wedge\tau_N^X}^{t\wedge\tau_N^X} \langle Du_{N,n,m}^{(i)}(X_s), \pi_m(B(X_s) - B_{N,n}(X_{m,s}))\rangle ds + \int_{r\wedge\tau_N^X}^{t\wedge\tau_N^X} \langle Du_{N,n,m}^{(i)}(X_s), dW_s\rangle.$$

Possibly passing to a subsequence, and taking the limit in probability as $m \to \infty$ (with respect to $\mathbb{P}$), we arrive at

$$u_{N,n}^{(i)}(X_{t\wedge\tau_N^X}) - u_{N,n}^{(i)}(X_{r\wedge\tau_N^X}) \tag{43}$$

$$= \lambda \int_{r\wedge\tau_N^X}^{t\wedge\tau_N^X} u_{N,n}^{(i)}(X_s)ds - \int_{r\wedge\tau_N^X}^{t\wedge\tau_N^X} B_{N,n}^{(i)}(X_s)ds$$

$$+ \int_{r\wedge\tau_N^X}^{t\wedge\tau_N^X} \langle Du_{N,n}^{(i)}(X_s), (B(X_s) - B_{N,n}(X_s))\rangle ds + \int_{r\wedge\tau_N^X}^{t\wedge\tau_N^X} \langle Du_{N,n}^{(i)}(X_s), dW_s\rangle.$$

Let us justify this assertion.

First note that in equation (38) we have the drift term $\pi_m B_{N,n} \circ \pi_m$ which converges pointwise to $B_{N,n}$ and $B_{N,n}^{(i)} \circ \pi_m$ which converges pointwise to $B_{N,n}^{(i)}$ as $m \to \infty$. Since such functions are also uniformly bounded, we can apply Proposition 7 and Lemma 8 and obtain that, possibly passing to a subsequence (recall that n is fixed),

$$\lim_{m\to\infty} u_{N,n,m}^{(i)}(x) = u_{N,n}^{(i)}(x), \quad \lim_{m\to\infty} Du_{N,n,m}^{(i)}(x) = Du_{N,n}^{(i)}(x), \quad x \in H, \tag{44}$$

$$\sup_{m\geq 1} \|u_{N,n,m}^{(i)}\|_{C_b^1(H)} = C_i^N < \infty.$$

We only consider convergence of the two most involved terms in (42).

We first treat convergence in $L^2(\Omega)$ of the stochastic integral. Recall that

$$\int_0^{t\wedge\tau_N^X} \langle Du_{N,n}^{(i)}(X_s), dW_s\rangle = \int_0^t 1_{\{s\leq t\wedge\tau_N^X\}} \langle Du_{N,n}^{(i)}(X_s), dW_s\rangle;$$

by the isometry formula and (44) we get

$$\mathbb{E}\Big| \int_0^{t\wedge\tau_N^X} \langle Du_{N,n,m}^{(i)}(X_s) - Du_{N,n}^{(i)}(X_s), dW_s\rangle \Big|^2 \to 0 \tag{45}$$

as $m \to \infty$. Note that we have used that $\lim_{m\to\infty} Du_{N,n,m}^{(i)}(x) = Du_{N,n}^{(i)}(x)$, for any $x \in H$ (not only for μ-a.e $x \in H$). In a similar way we get

$$\mathbb{E}\Big| \int_0^{r\wedge\tau_N^X} \langle Du_{N,n,m}^{(i)}(X_s) - Du_{N,n}^{(i)}(X_s), dW_s\rangle \Big|^2 \to 0$$

as $m \to \infty$. This shows that $\int_{r\wedge\tau_N^X}^{t\wedge\tau_N^X} \langle Du_{N,n,m}^{(i)}(X_s), dW_s\rangle$ converges to $\int_{r\wedge\tau_N^X}^{t\wedge\tau_N^X} \langle Du_{N,n}^{(i)}(X_s), dW_s\rangle$ in $L^2(\Omega)$ as $m \to \infty$. To show that, $\mathbb{P}$-a.s.,

$$\lim_{m\to\infty} \int_{r\wedge\tau_N^X}^{t\wedge\tau_N^X} \Big| \langle Du_{N,n,m}^{(i)}(X_s), \pi_m(B(X_s) - B_{N,n}(X_{m,s}))\rangle - \langle Du_{N,n}^{(i)}(X_s), (B(X_s) - B_{N,n}(X_s))\rangle \Big| ds = 0, \quad (46)$$

it is enough to prove that $\lim_{m\to\infty} H_m + K_m = 0$, where

$$H_m = \int_0^{t\wedge\tau_N^X} \big| \langle Du_{N,n,m}^{(i)}(X_s) - Du_{N,n}^{(i)}(X_s), \pi_m(B(X_s) - B_{N,n}(X_{m,s}))\rangle \big| ds$$

and

$$K_m = \int_0^{t\wedge\tau_N^X} \big| \langle Du_{N,n}^{(i)}(X_s), [\pi_m B(X_s) - B(X_s)] + [B_{N,n}(X_s) - \pi_m B_{N,n}(X_{m,s})]\rangle \big| ds.$$

By using (44) we easily get the assertion.

Step 3 A convergence result involving stopping times.

In order to pass to the limit in probability as $n \to \infty$ in (43) we recall formula (36) and argue as before. The only difficult term is

$$\int_{r\wedge\tau_N^X}^{t\wedge\tau_N^X} \langle Du_{N,n}^{(i)}(X_s), (B(X_s) - B_{N,n}(X_s))\rangle ds = J_n + I_n,$$

$$\text{where } J_n = \int_{r\wedge\tau_N^X}^{t\wedge\tau_N^X} \langle Du_{N,n}^{(i)}(X_s) - Du_N^{(i)}(X_s), (B(X_s) - B_{N,n}(X_s))\rangle ds,$$

$$I_n = \int_{r\wedge\tau_N^X}^{t\wedge\tau_N^X} \langle Du_N^{(i)}(X_s), (B_N(X_s) - B_{N,n}(X_s))\rangle ds$$

(using that $s \le t \wedge \tau_N^X$). As for J_n we have

$$J_n = \int_{r\wedge\tau_N^X}^{t\wedge\tau_N^X} \langle Du_{N,n}^{(i)}(X_s) - Du_N^{(i)}(X_s), (B_N(X_s) - B_{N,n}(X_s))\rangle ds,$$

and so $|J_n| \le 2\|B_N\|_0 \int_0^t |Du_{N,n}^{(i)}(X_s) - Du_N^{(i)}(X_s)| ds \to 0$, $\mathbb{P}$-a.s., as $n \to \infty$, by Lemma 8.

Let us consider I_n. We define an auxiliary process $\hat{X}^N = (\hat{X}_t^N)$ as follows:

$$\hat{X}_t^N := e^{tA}x + \int_0^t e^{(t-s)A} B_N(X_{s\wedge\tau_N^X}) ds + \int_0^t e^{(t-s)A} dW_s, \quad t \ge 0, \quad (47)$$

Note that $X_{s\wedge\tau_N^X} = \hat{X}^N_{s\wedge\tau_N^X}$, $s \geq 0$, so that

$$|I_n| = \Big|\int_{r\wedge\tau_N^X}^{t\wedge\tau_N^X} \langle Du_N^{(i)}(\hat{X}_s^N), (B_N(\hat{X}_s^N) - B_{N,n}(\hat{X}_s^N))\rangle ds\Big|$$

$$\leq \|D^{(i)}u_N\|_0 \int_r^t |B_N(\hat{X}_s^N) - B_{N,n}(\hat{X}_s^N)|ds. \tag{48}$$

Now we use the Girsanov theorem (see e.g. Appendix in [5]). Let $T > 0$. Since

$$\hat{X}_t^N := e^{tA}x + \int_0^t e^{(t-s)A}\hat{B}_s^N ds + \int_0^t e^{(t-s)A}dW_s, \quad t \geq 0,$$

where $\hat{B}_s^N = B_N(\hat{X}^N_{s\wedge\tau_N^X})$, $s \geq 0$, is an adapted and bounded process, we have that

$$\tilde{W}_t^N := W_t + \int_0^t \hat{B}_s^N ds$$

is a cylindrical Wiener process on $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \widetilde{\mathbb{P}}_N\right)$ where $\frac{d\widetilde{\mathbb{P}}_N}{d\mathbb{P}}\Big|_{\mathcal{F}_T} = \rho_N$,

$$\rho_N = \exp\left(-\int_0^T \hat{B}_s^N dW_s - \frac{1}{2}\int_0^T |\hat{B}_s^N|^2 ds\right).$$

Hence $\hat{X}_t^N = e^{tA}x + \int_0^t e^{(t-s)A}d\tilde{W}_s^N$ is an OU process on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \widetilde{\mathbb{P}}_N)$.

Moreover, we know that the law of $(\hat{X}_t^N)_{t\in[0,T]}$ on $C([0,T];H)$ is equivalent to the law of the OU process $Z(t,x)$ given in (16). In particular, all their transition probabilities are equivalent. Now under our assumptions the law of $Z(t,x)$ is equivalent to μ for all $t > 0$ and $x \in H$ (see Theorem 11.3 in [6]).

Let us come back to (48). Using that the law $\pi_t^N(x,\cdot)$ of $\hat{X}_t^N$ is absolutely continuous with respect to μ, we obtain

$$\mathbb{E}\int_r^t |B_N(\hat{X}_s^N) - B_{N,n}(\hat{X}_s^N)|ds = \int_r^t ds \int_H \big|B_N(y) - B_{N,n}(y)\big| \frac{d\pi_s^N(x,\cdot)}{d\mu}(y)\mu(dy), \tag{49}$$

which tends to 0, as $n \to \infty$, by the dominated convergence theorem. Hence we have found that $I_n \to 0$ in $L^1(\Omega, \mathbb{P})$.

Up to now we have

$$u_N^{(i)}(X_{t\wedge\tau_N^X}) - u_N^{(i)}(X_{r\wedge\tau_N^X})$$
$$= \lambda\int_{r\wedge\tau_N^X}^{t\wedge\tau_N^X} u_N^{(i)}(X_s)ds - \int_{r\wedge\tau_N^X}^{t\wedge\tau_N^X} B^{(i)}(X_s)ds + \int_{r\wedge\tau_N^X}^{t\wedge\tau_N^X}\langle Du_N^{(i)}(X_s), dW_s\rangle.$$

Passing to the limit as $r \to 0^+$, since the trajectories of X are continuous, we finally get

$$\begin{aligned} & u_N^{(i)}(X_{t\wedge\tau_N^X}) - u_N^{(i)}(x) \qquad (50) \\ &= \lambda \int_0^{t\wedge\tau_N^X} u_N^{(i)}(X_s)ds - \int_0^{t\wedge\tau_N^X} B^{(i)}(X_s)ds + \int_0^{t\wedge\tau_N^X} \langle Du_N^{(i)}(X_s), dW_s\rangle. \end{aligned}$$

Step 4 The final formula.

By (1) we deduce

$$dX_t^{(i)} = -\lambda_i X_t^{(i)} dt + B^{(i)}(X_t)dt + dW_t^{(i)}.$$

Inserting the expression for $B^{(i)}(X_t)$, which we get from this identity, into (50), we obtain

$$\begin{aligned} & u_N^{(i)}(X_{t\wedge\tau_N^X}) - u_N^{(i)}(x) \\ &= -X^i_{t\wedge\tau_N^X} + x^i + \lambda \int_0^{t\wedge\tau_N^X} u_N^{(i)}(X_s)ds - \lambda_i \int_0^{t\wedge\tau_N^X} X_s^{(i)} ds + \int_0^{t\wedge\tau_N^X} \langle Du_N^{(i)}(X_s), dW_s\rangle \\ & \quad + W^i_{t\wedge\tau_N^X}. \end{aligned}$$

By the variation of constants formula this is equivalent to

$$\begin{aligned} X^{(i)}_{t\wedge\tau_N^X} &= e^{-\lambda_i t\wedge\tau_N^X}\langle x, e_i\rangle + \lambda \int_0^{t\wedge\tau_N^X} e^{-\lambda_i(t\wedge\tau_N^X - s)} u_N^{(i)}(X_s)ds \\ & - \int_0^{t\wedge\tau_N^X} e^{-\lambda_i(t\wedge\tau_N^X - s)} du_N^{(i)}(X_s) + e^{-\lambda_i(t\wedge\tau_N^X)} \int_0^{t\wedge\tau_N^X} e^{\lambda_i s}[dW_s^{(i)} + \langle Du_N^{(i)}(X_s), dW_s\rangle]. \end{aligned}$$

This identity yields (32) on $\{t \le \tau_N^X\}$. □

The next lemma is similar to Lemma 9 in [5] and shows that $u_N(x) = \sum_{k\ge1} u_N^{(k)}(x)e_k$ (where $u_N^{(k)}$ is as in (31)) is a well defined function which belongs to $C_b^1(H,H)$.

Lemma 11. *For λ sufficiently large, i.e., $\lambda \ge \tilde\lambda$, with $\tilde\lambda = \tilde\lambda(A, \|B_N\|_0) > 0$ there exists a unique $u_N = u_{\lambda,N} \in C_b^1(H,H)$ which solves*

$$u_N(x) = \int_0^\infty e^{-\lambda t} R_t\big(Du_N(\cdot)B_N(\cdot) + B_N(\cdot)\big)(x)dt, \quad x \in H,$$

where R_t is the OU semigroup defined as in (14) and acting on H-valued functions. Moreover, we have the following assertions.

(i) For any $h \in H$, $Du_N(\cdot)[h] \in C_b(H,H)$ and $\|Du_N(\cdot)[h]\|_0 \le C_{0,\lambda,N}|h|$;

(ii) for any $k \ge 1$, $\langle u_N(\cdot), e_k\rangle = u_N^{(k)}$, where $u_N^{(k)}$ is the solution defined in (31);

(iii) There exists $c_3 = c_3(A, \|B_N\|_0) > 0$ such that, for any $\lambda \ge \tilde\lambda$, $u = u_\lambda$ satisfies

$$\|Du_N\|_0 \le \frac{c_3}{\sqrt{\lambda}}. \qquad (51)$$

3 Proof of Theorem 1

Let $X = (X_t)$ and $Y = (Y_t)$ be two weak mild solutions (see (8) and (9)) defined on the same filtered probability space (solutions with respect to the same cylindrical Wiener process W) starting at $x \in H$.

In the first part of the proof we will adapt the proof of Theorem 1 in [5] by introducing additional stopping times. The main difference with respect to [5] will appear in Proposition 13 which is needed to finish the proof.

For the time being, x is not specified. In Proposition 13 a restriction on x will emerge.

Note that by our hypothesis

$$B_N = B 1_{B(0,N)} = B' 1_{B(0,N)} = B'_N. \tag{52}$$

It follows that Kolmogorov equation (31) written with respect to the truncated drift B_N (with $B_N^{(i)}$ in the right-hand side) or with respect to B'_N (with $B_N^{'(i)}$ in the right-hand side) is the same and gives the same solution $u_{N,\lambda}^{(i)}$.

It follows that both X and Y satisfy (32) on the event $\{t \le \tau_N^X \wedge \tau_N^Y\}$.

Now we consider

$$u_N = u_{N,\lambda} : H \to H$$

be such that $u_N(x) = \sum_{i \ge 1} u_N^{(i)}(x) e_i$, $x \in H$, where $u_N^{(i)} = u_{N,\lambda}^{(i)}$ solve (31) for some λ large enough possibly depending on N.

Let us fix $T > 0$. By (32), taking into account (52), we have, for $t \in [0, T \wedge \tau_N^X \wedge \tau_N^Y]$, $\mathbb{P}$-a.s.,

$$\begin{aligned} X_t - Y_t = u_N(Y_t) - u_N(X_t) + (\lambda - A) \int_0^t e^{(t-s)A} \left(u_N(X_s) - u_N(Y_s) \right) ds \\ + \int_0^t e^{(t-s)A} (Du_N(X_s) - Du_N(Y_s)) dW_s. \end{aligned}$$

Here and in the sequel we will drop the λ-dependence of u_N to simplify notation. However, at the end we will fix a value of λ large enough. By (51) we may assume that $\|Du_N\|_0 \le 1/2$.

It follows that for $t \in [0, T \wedge \tau_N^X \wedge \tau_N^Y]$,

$$\begin{aligned} |X_t - Y_t| \le \frac{1}{2} |X_t - Y_t| + \Big| (\lambda - A) \int_0^t e^{(t-s)A} \left(u_N(X_s) - u_N(Y_s) \right) ds \Big| \\ + \Big| \int_0^t e^{(t-s)A} (Du_N(X_s) - Du_N(Y_s)) dW_s \Big|. \end{aligned}$$

Let η be a stopping time to be specified later and set

$$\tau = \eta \wedge T \wedge \tau_N^X \wedge \tau_N^Y. \tag{53}$$

Using that $1_{[0,\tau]}(t) = 1_{[0,\tau]}(t)\cdot\, 1_{[0,\tau]}(s)$, $0 \le s \le t \le T$, we have (cf. page 187 in [6])

$$1_{[0,\tau]}(t)\,|X_t - Y_t| \le C\, 1_{[0,\tau]}(t)\Big|\,(\lambda - A)\int_0^t e^{(t-s)A}\,(u_N\,(X_s) - u_N\,(Y_s))\,ds\Big|$$
$$+ C\,\Big|1_{[0,\tau]}(t)\int_0^t e^{(t-s)A}\,(Du_N\,(X_s) - Du_N\,(Y_s))\;1_{[0,\tau]}(s)\,dW_s\Big|,$$

where by C we denote any constant which may depend on the assumptions on A, B_N and T.

Writing $1_{[0,\tau]}(s)\,X_s = \tilde{X}_s$ and $1_{[0,\tau]}(s)\,Y_s = \tilde{Y}_s$, and, using the Burkholder-Davis-Gundy inequality with $q > 2$ which will be determined below, we obtain (recall that $\|\cdot\|$ is the Hilbert-Schmidt norm, cf. Chapter 4 in [6]) with $C = C_q$,

$$\mathbb{E}\left[\left|\tilde{X}_t - \tilde{Y}_t\right|^q\right] \le C\,\mathbb{E}\left[e^{\lambda qt}\left|(\lambda - A)\int_0^t e^{(t-s)A}e^{-\lambda s}(u_N(X_s) - u_N(Y_s))1_{[0,\tau]}(s)ds\right|^q\right]$$
$$+ C\mathbb{E}\left[\left(\int_0^t 1_{[0,\tau]}(s)\,\left\|e^{(t-s)A}\,(Du_N\,(X_s) - Du_N\,(Y_s))\,\right\|^2 ds\right)^{q/2}\right].$$

In the sequel we also introduce a parameter $\theta > 0$ and C_θ will denote suitable constants such that $C_\theta \to 0$ as $\theta \to +\infty$ (the constants may change from line to line). Similarly, we will indicate by $C(\lambda)$ suitable constants (possibly depending on N) such that $C(\lambda) \to 0$ as $\lambda \to +\infty$.

From the previous inequality we deduce, multiplying by $e^{-q\theta t}$, for any $\theta > 0$,

$$\mathbb{E}\left[e^{-q\theta t}\left|\widetilde{X}_t - \widetilde{Y}_t\right|^q\right] \tag{54}$$
$$\le C\mathbb{E}\left[\left|(\lambda - A)\int_0^t e^{-\theta(t-s)}e^{(t-s)A}\,(u_N\,(X_s) - u_N\,(Y_s))\,e^{-\theta s}1_{[0,\tau]}\,(s)\,ds\right|^q\right]$$
$$+ C\mathbb{E}\left[\left(\int_0^t e^{-2\theta(t-s)}\,\left\|e^{(t-s)A}\,(Du_N\,(X_s) - Du_N\,(Y_s))\right\|^2 e^{-2\theta s}1_{[0,\tau]}\,(s)\,ds\right)^{q/2}\right].$$

Now proceeding as in the proof of Theorem 7 of [5] we arrive at

$$\int_0^T \mathbb{E}\left[e^{-q\theta t}\left|\widetilde{X}_t - \widetilde{Y}_t\right|^q\right] dt \tag{55}$$
$$\le C(\lambda)\int_0^T e^{-q\theta s}\mathbb{E}|\tilde{X}_s - \tilde{Y}_s|^q ds + \tilde{C}_\theta\mathbb{E}\Big[\Lambda_T\int_0^T e^{-q\theta s}|\tilde{X}_s - \tilde{Y}_s|^q ds\Big]$$

provided that $q \in (4, \infty)$, $\gamma = q/2$, $\theta \ge \lambda$ and

$$\Lambda_T := \int_0^T\int_0^t 1_{[0,\tau]}(s)\int_0^1\Big(\sum_{n\ge1}\frac{1}{\lambda_n^{1-\delta}}\|D^2u_N^{(n)}(Z_s^r)\|^2\Big)^\gamma dr\;ds\;dt,$$
$$\text{where } Z_t^r = rX_t + (1-r)Y_t, \tag{56}$$

and $D^2u_N^{(n)}(x)$ is defined for $\mu-$a.e. $x \in H$. The existence of $D^2u_N^{(n)} \in L^p(\mu)$, $p \geq 2$, follows from Proposition 6 applied to equation (31) (see also Lemma 23 in [5]).

Since

$$\Lambda_T \leq T \cdot \int_0^{T\wedge\tau} \int_0^1 \Big(\sum_{n\geq 1} \frac{1}{\lambda_n^{1-\delta}} \|D^2 u_N^{(n)}(Z_s^r)\|^2 \Big)^\gamma dr\ ds$$

it is natural to define, for any $R > 0$, the stopping time

$$\bar{\tau}_R^{x,N} = \inf \left\{ t \geq 0\ :\ \int_0^t \int_0^1 \Big(\sum_{n\geq 1} \frac{1}{\lambda_n^{1-\delta}} \|D^2 u_N^{(n)}(Z_s^r)\|^2 \Big)^\gamma dr\ ds \geq R \right\} \wedge T.$$

Take $\eta = \bar{\tau}_R^{x,N}$ in the previous expressions so that

$$\tau = \bar{\tau}_R^{x,N} \wedge \tau_N^X \wedge \tau_N^Y.$$

We get from (55),

$$\begin{aligned} &\int_0^T e^{-q\theta t}\mathbb{E}|\tilde{X}_t - \tilde{Y}_t|^q dt \\ &\leq C(\lambda) \int_0^T e^{-q\theta s}\mathbb{E}|\tilde{X}_s - \tilde{Y}_s|^q ds + \tilde{C}_\theta R \int_0^T e^{-q\theta s}\mathbb{E}|\tilde{X}_s - \tilde{Y}_s|^q ds. \end{aligned}$$

Now we fix λ large enough such that $C(\lambda) < 1/2$. For sufficiently large $\theta = \theta_R \geq \lambda$, *depending on* R and N, we have $\tilde{C}_\theta R < 1/2$ and so

$$\mathbb{E}\Big[\int_0^T e^{-q\theta_R\, t} 1_{[0,\tau]}(t)\, |X_t - Y_t|^q dt \Big] = \mathbb{E}\Big[\int_0^\tau e^{-q\theta_R\, t}\, |X_t - Y_t|^q dt \Big] = 0.$$

In other words, for every $R > 0$, $N \geq 1$, $\mathbb{P}$-a.s., $X = Y$ on $\big[0, \bar{\tau}_R^{x,N} \wedge \tau_N^X \wedge \tau_N^Y\big]$ (identically in t, not only a.e. in t, since X and Y are continuous processes).

If we prove that

$$\lim_{R\to\infty} \bar{\tau}_R^{x,N} = T \wedge \tau_N^X \wedge \tau_N^Y, \quad \mathbb{P} - a.s., \tag{57}$$

then we obtain that $X = Y$ on $[0, T \wedge \tau_N^X \wedge \tau_N^Y]$ and this finishes the proof.

The crucial assertion (57) follows by the next proposition.

Remark 12. Assertion (57) is a "local version" of Proposition 10 in [5]. Similarly to (47) also in the next proof we have to find an auxiliary process (see (58)) which allows to apply the Girsanov theorem.

Proposition 13. *Let* $N \geq 1$ *and* $T > 0$ *and suppose* X *and* Y *as in Theorem 1. For* μ*-a.e.* $x \in H$*, we have* $\mathbb{P}\big(S^x_{T\wedge\tau_N^X\wedge\tau_N^Y} < \infty\big) = 1$*, where*

$$S_t^x = S_t^{x,N} = \int_0^t \int_0^1 \left(\sum_{n\geq 1} \frac{1}{\lambda_n^{1-\delta}} \|D^2 u_N^{(n)}(Z_s^r)\|^2 \right)^\gamma dr\ ds, \quad t \in [0,T],$$

with $\gamma = q/2$ *(*$u_N(x) = \sum_{i\geq 1} u_N^{(i)}(x) e_i$*,* $x \in H$*, where* $u_N^{(i)} = u_{N,\lambda}^{(i)}$ *solve* (31)*).*

Proof. To prove the assertion we will show that, for μ-a.e. $x \in H$,

$$\mathbb{E}[S^x_{T\wedge\tau^X_N\wedge\tau^Y_N}] < +\infty.$$

In the first part of the proof, $x \in H$ is given, without restriction. Let us consider stopped processes

$$X^N_t = X_{t\wedge\tau^X_N\wedge\tau^Y_N}, \quad Y^N_t = Y_{t\wedge\tau^X_N\wedge\tau^Y_N},$$

and then we define an auxiliary process $(Z^{r,N}_t)_{t\in[0,T]}$ as follows

$$Z^{r,N}_t := e^{tA}x + \int_0^t e^{(t-s)A}\bar{B}^{r,N}_s ds + \int_0^t e^{(t-s)A} dW_s \tag{58}$$

where (recall (10))

$$\bar{B}^{r,N}_s := [rB(X^N_s) + (1-r)B(Y^N_s)], \quad r \in [0,1], \ \ s \in [0,T].$$

Comparing Z^r (see (56)) and $Z^{r,N}$ we see that $Z^r_{s\wedge\tau^X_N\wedge\tau^Y_N} = Z^{r,N}_{s\wedge\tau^X_N\wedge\tau^Y_N}$, $s \in [0,T]$, $r \in [0,1]$. Hence we have to prove

$$\mathbb{E}[S^x_{T\wedge\tau^X_N\wedge\tau^Y_N}] = \mathbb{E}\int_0^{T\wedge\tau^X_N\wedge\tau^Y_N} \int_0^1 \Big(\sum_{n\geq 1}\frac{1}{\lambda_n^{1-\delta}}\|D^2u^{(n)}_N(Z^{r,N}_s)\|^2\Big)^\gamma dr\, ds < \infty.$$

This follows if we can show that

$$\mathbb{E}\int_0^T \int_0^1 \Big(\sum_{n\geq 1}\frac{1}{\lambda_n^{1-\delta}}\|D^2u^{(n)}_N(Z^{r,N}_s)\|^2\Big)^\gamma dr\, ds < \infty. \tag{59}$$

We fix $N \geq 1$. To verify (59) we can follow the proof of Proposition 10 in [5]. We only indicate the small changes which are needed.

Define

$$\rho_{r,N} = \exp\left(-\int_0^t \bar{B}^{r,N}_s dW_s - \frac{1}{2}\int_0^t |\bar{B}^{r,N}_s|^2 ds\right).$$

We have, since $|\bar{B}^r_s| \leq \|B_N\|_0$, $r \in [0,1]$, $s \geq 0$, $\mathbb{P}$-a.s.,

$$\mathbb{E}\left[\exp\left(k\int_0^T |\bar{B}^{r,N}_s|^2 ds\right)\right] \leq C_k < \infty, \tag{60}$$

for all $k \in \mathbb{R}$, independently of x and r, simply because B_N is bounded. Hence an infinite dimensional version of Girsanov's Theorem with respect to a cylindrical Wiener process (the proof of which is included in the Appendix of [5]) applies and gives us that

$$\tilde{W}^N_t := W_t + \int_0^t \bar{B}^{r,N}_s ds$$

is a cylindrical Wiener process on $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \widetilde{\mathbb{P}}_{r,N}\right)$ where $\frac{d\widetilde{\mathbb{P}}_{r,N}}{d\mathbb{P}}\Big|_{\mathcal{F}_T} = \rho_{r,N}$. Hence

$$Z_t^{r,N} = e^{tA}x + \int_0^t e^{(t-s)A} d\tilde{W}_s^N$$

is an OU process on $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \widetilde{\mathbb{P}}_{r,N}\right)$. Continuing as in the proof of Proposition 10 in [5] with $u^{(n)}$ replaced by $u_N^{(n)}$, ρ_r replaced by $\rho_{r,N}$, Z_s^r replaced by $Z_s^{r,N}$, $\bar{B}_s$ replaced by $\bar{B}_s^{r,N}$, we see that (59) holds if we prove that

$$\int_0^T \mathbb{E}\Big[\Big(\sum_{n\geq 1} \frac{1}{\lambda_n^{1-\delta}} \|D^2 u_N^{(n)}(e^{sA}x + W_A(s))\|^2\Big)^{2\gamma}\Big] ds < \infty, \tag{61}$$

where $W_A(t) = \int_0^t e^{(t-s)A} dW(s), \quad t \geq 0$. If μ_s^x denotes the law of $e^{sA}x + W_A(s)$, we have to prove that

$$\int_0^T \int_H \Big(\sum_{n\geq 1} \frac{1}{\lambda_n^{1-\delta}} \|D^2 u_N^{(n)}(y)\|^2\Big)^{2\gamma} \mu_s^x(dy) ds < \infty. \tag{62}$$

This can be checked as in the mentioned proof (see in particular Steps 3 and 4 in that proof) *only for μ-a.e. $x \in H$*; one has to replace B in the proof in [5] with our B_N. □

Remark 14. As is easily checked in Theorem 1 the ball $B(0, N)$ can be replaced by any open bounded set in H.

Remark 15. According to Remark 11 in [5] our Theorem 1 provides an alternative approach to Veretennikov's uniqueness result in finite dimension. In this respect first note that Theorem 1 when $H = \mathbb{R}^d$ does not require to start from μ-a.e. initial conditions x, but works for any initial $x \in \mathbb{R}^d$.

Note that in finite dimension an SDE like $dX_t = b(X_t)dt + dW_t$ with b Borel and bounded is equivalent to $dX_t = -X_t dt + (b(X_t) + X_t)dt + dW_t$ which is in the form (1) with $A = -I$, and with a drift term $B(x) = b(x) + x$ which is completely covered by Theorem 1.

Recall that in this alternative approach to Veretennikov's result, basically the elliptic L^p-estimates with respect to Lebesgue measure used in [26] are replaced by elliptic $L^p(\mu)$-estimates.

4 Existence of strong mild solutions

Here we will use our Theorem 1 to prove existence of strong mild solutions when B grows more than linearly. We will construct such solutions for μ-a.e. initial $x \in H$.

According to Chapter 1 in [21] (see also [18]) if $x \in H$ we say that equation (1) has a (global) *strong mild solution* if, for every filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ on which there is defined a cylindrical $(\mathcal{F}_t)$-Wiener process W there exists an H-valued continuous $(\mathcal{F}_t)$-adapted process $X = (X_t) = (X_t)_{t\geq 0}$ such that

$$(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}, W, X)$$

is a weak mild solution.

Theorem 16. *Let us consider equation* (1) *and assume Hypothesis 1 and $B \in B_{b,loc}(H, H)$. Moreover, suppose that there exist $C > 0$, $p > 0$, such that*

$$\langle B(y+z), y\rangle \leq C\big(|y|^2 + e^{p|z|} + 1\big), \quad y,\, z \in H. \tag{63}$$

Then, for μ-a.e. $x \in H$ (where $\mu = N(0, -\frac{1}{2}A^{-1})$), equation (1) *has a strong mild solution. Moreover, this solution is pathwise unique.*

Remark 17. Condition (63) is a bit stronger than the classical one: $\langle B(y), y\rangle \leq C\big(|y|^2 + 1\big)$, $y \in H$, which is usually imposed in finite dimension to have non-explosion for SDEs with additive noise. We can not use such condition. Indeed for a given mild solution (X_t) we can not write the Itô formula for $|X_t|^2$ due to the fact that our noise is cylindrical.

To prove the result we will use our Theorem 1 together with a generalization of the Yamada-Watanabe theorem (see Theorem 2 in [21] and [18]) and some a-priori estimates on mild solutions (see Section 4.1).

Example 18. To introduce an example of a drift $\tilde{B}$ which satisfies the assumptions of Theorem 16, we first consider a measurable function $g : \mathbb{R} \to \mathbb{R}_+$ such that $g(s) = 0$ if $s \geq 0$ and $0 \leq g(s) \leq Ce^{q|s|}$, $s < 0$, for some $q > 0$. It is easy to check that $g(s+r)r \leq C'(1 + |r|^2 + e^{p|s|})$, $s, r \in \mathbb{R}$. We define $\tilde{B} : H \to H$,

$$\tilde{B}(x) = \sum_{k\geq 1} \frac{g(x_k)}{k^2} e_k, \quad x \in H.$$

It is not difficult to verify that $\tilde{B}$ satisfies the assumptions of the previous theorem. We can also add to our drift $\tilde{B}$ one of the singular drifts considered in Section 4 of [5]; we will still obtain an admissible drift for our theorem.

4.1 An a-priori estimate

Here we prove an a-priori estimate for mild solutions to (1) under condition (63). For this purpose let us consider the OU process

$$Z_t = Z(t, x) = e^{tA}x + \int_0^t e^{(t-s)A} dW_s$$

which under our hypotheses has a continuous H-valued version. It satisfies

$$\langle Z_t, \varphi \rangle = \int_0^t \langle Z_s, A\varphi \rangle \, ds + \langle W_t, \varphi \rangle$$

for all $\varphi \in D(A)$. By Proposition 18 in [5] we deduce, in particular, that for any $p > 0$, $T > 0$

$$K_T := \mathbb{E}\Big[\sup_{t\in[0,T]} e^{p|Z_t|} \Big] < \infty.$$

Recall that under our hypotheses a weak mild solution to (1) can be defined, equivalently, as a tuple $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}, W, X)$, where $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ is a filtered probability space on which there is defined a cylindrical $(\mathcal{F}_t)$-Wiener process W and a continuous $(\mathcal{F}_t)$-adapted H-valued process $X = (X_t) = (X_t)_{t\geq 0}$ such that, $\mathbb{P}$-a.s.,

$$\langle X_t, \varphi \rangle = \langle x, \varphi \rangle + \int_0^t (\langle X_s, A\varphi \rangle + \langle B(X_s), \varphi \rangle) ds + \langle W_t, \varphi \rangle, \quad t \geq 0,$$

for all $\varphi \in D(A)$ (cf. Chapter 6 of [6]).

Theorem 19. *Assume Hypothesis 1, $B \in B_{b,loc}(H,H)$ and condition (63). Let $X = (X_t)_{t\geq 0}$ be a weak mild solution of equation (1) with $X_0 = x \in H$.*

There exists $C_p > 0$ (possibly depending on C and p given in (63)*) such that*

$$\mathbb{E}\Big[\sup_{t\in[0,T]} |X_t|^2 \Big] \leq e^{C_p T} \left(|x|^2 + K_T + 1 \right), \quad T > 0. \tag{64}$$

Proof. The process $Y_t = X_t - Z_t$ satisfies

$$\langle Y_t, \varphi \rangle = \langle x, \varphi \rangle + \int_0^t (\langle Y_s, A\varphi \rangle + \langle B(Y_s + Z_s), \varphi \rangle) ds, \tag{65}$$

for all $\varphi \in D(A)$, and it has continuous trajectories in H. Let us consider (65) with $\varphi = e_k$ (see (13)) and set $Y_t^{(k)} = \langle Y_t, e_k \rangle$. Since $\frac{dY_t^{(k)}}{dt} \cdot Y_t^{(k)} \leq B^{(k)}(Y_t + Z_t) Y_t^{(k)}$, we find

$$\sum_{k\geq 1} \langle Y_t, e_k \rangle^2 \leq |x|^2 + 2 \int_0^t \langle B(Y_s + Z_s), Y_s \rangle \, ds.$$

Hence, by assumption (63), for $t \in [0,T]$,

$$|Y_t|^2 \leq |x|^2 + 2C \int_0^t \left(|Y_s|^2 + e^{p|Z_s|} + 1 \right) ds,$$

and therefore, by the Gronwall lemma,

$$|Y_t|^2 \leq e^{CT} \Big(|x|^2 + 2CT(\sup_{s\in[0,T]} e^{p|Z_s|} + 1) \Big).$$

This implies

$$\mathbb{E}\Big[\sup_{t\in[0,T]}|Y_t|^2\Big] \le e^{C_1 T}\Big(|x|^2 + \mathbb{E}\Big[\sup_{s\in[0,T]} e^{p|Z_s|}\Big] + 1\Big).$$

Therefore,

$$\begin{aligned}\mathbb{E}\Big[\sup_{t\in[0,T]}|X_t|^2\Big] &\le 2e^{C_1 T}\Big(|x|^2 + \mathbb{E}\Big[\sup_{s\in[0,T]} e^{p|Z_s|}\Big] + 1\Big) + 2\mathbb{E}\Big[\sup_{t\in[0,T]}|Z_t|^2\Big]\\ &\le e^{C_p T}\Big(|x|^2 + \mathbb{E}\Big[\sup_{s\in[0,T]} e^{p|Z_s|}\Big] + 1\Big).\end{aligned}$$

□

4.2 Proof of Theorem 16

By Theorem 1 we only have to prove existence of strong solution for μ-a.e. $x \in H$.

We will again consider truncated bounded drifts $B_N = B\,1_{B(0,N)}$, $N \ge 1$.

By the main result in [5] there exists a Borel set $\tilde{G} \subset H$ with $\mu(\tilde{G}) = 1$ such that for any $x \in \tilde{G}$ we have pathwise uniqueness for each stochastic equation

$$dX_t = (AX_t + B_N(X_t))dt + dW_t, \qquad X_0 = x \in H, \tag{66}$$

$N \ge 1$. Let $x \in \tilde{G}$. By the Girsanov theorem (see Appendix in [5]) there exists (a unique in law) weak mild solution $X_N = (X_N(t)) = (X_N(t))_{t\ge0}$ for each stochastic equation (66).

Therefore we can apply a generalization of the Yamada-Watanabe theorem (see Theorem 2 in [21] and [18]) to (66) when $x \in \tilde{G}$.

Let us fix any filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ on which there is defined a cylindrical $(\mathcal{F}_t)$-Wiener process W. By the Yamada-Watanabe theorem, for any $N \ge 1$, on the fixed filtered probability space above there exists a (unique) strong mild solution X_N to (66). Moreover, since

$$B_N(x) = B_{N+k}(x), \quad x \in B(0,N),$$

$k \ge 1$, we have by Theorem 1 that, $\mathbb{P}$-a.s.,

$$\tau_N := \tau_N^{X_N} = \tau_N^{X_{N+k}}, \;\; k \ge 1, \;\; N \ge 1,$$

and $X_N(t\wedge\tau_N) = X_{N+k}(t\wedge\tau_N)$, $t \ge 0$.

It is enough to construct the strong solution X to (1) on $[0,T]$ for a fixed $T > 0$. We define an H-valued stochastic process X on $\Omega' = \cup_{N\ge1}\{\tau_N > T\}$ as

$$X(t)(\omega) := X_N(t)(\omega), \quad t \in [0,T],$$

if $\omega \in \{\tau_N > T\}$ (we set $X_t(\omega) = 0$ if $\omega \notin \Omega'$, $t \in [0,T]$). Then $X(t)$ is well defined.

It is not difficult to prove that X is a strong mild solution on $[0,T]$ if we show that

$$\lim_{N\to\infty}\mathbb{P}(\tau_N > T) = 1 \tag{67}$$

(this will imply that $\mathbb{P}(\Omega') = 1$). To verify (67) we will apply Theorem 19. Note that each B_N satisfies

$$\langle B_N\,(y+z)\,,y\rangle \leq C\big(\,|y|^2 + e^{p|z|} + 1\big),\quad y,\,z\in H,$$

with the same constants C and p of (63). By Theorem 19 we obtain

$$\mathbb{E}\Big[\sup_{t\in[0,T]}|X_N(t)|^2\Big] \leq e^{C_pT}\left(|x|^2 + K_T + 1\right), \tag{68}$$

with C_p independent of $N\geq 1$. Since

$$\mathbb{P}(\tau_N \leq T) = \mathbb{P}\big(\sup_{t\in[0,T]}|X_N(t)|\geq N\big)$$

by (68) and the Chebychev inequality we easily get assertion (67) from (68) and this completes the proof.

4.3 Existence and uniqueness of local mild solutions

Finally, let us discuss a possible extension of our result to the case when the drift term B only belongs to $B_{b,loc}\,(H,H)$, without requiring hypothesis (63).

We need the concept of local solution (see, for instance, [1] for some additional facts about local solutions). Let $(\Omega,\mathcal{F},(\mathcal{F}_t),\mathbb{P})$ be a filtered probability space. A stopping time $\tau:\Omega\to[0,+\infty]$ is called *accessible* if there exists a sequence of stopping times $(\tau_n) = (\tau_n)_{n\in\mathbb{N}}$ such that $\mathbb{P}\left(\tau_n<\tau\right) = 1$ and $\mathbb{P}\left(\lim_{n\to\infty}\tau_n = \tau\right) = 1$. The previous sequence (τ_n) is called an approximating sequence of τ.

Notice that, if τ_1 and τ_2 are accessible stopping times, then also $\tau = \tau_1\wedge\tau_2$ is an accessible stopping time.

Let τ be an accessible stopping time and consider $[0,\tau)\times\Omega = \{(t,\omega)\in[0,+\infty)\times\Omega : 0\leq t<\tau(\omega)\}$. An H-valued stochastic process X defined on $[0,\tau)$ (i.e., $X:[0,\tau)\times\Omega\to H$) is called $(\mathcal{F}_t)$-adapted if $X_t(\cdot):\{t<\tau\}\to H$ is $\mathcal{F}_t$-measurable, for any $t\geq 0$ (on $\{t<\tau\}$ we consider the restricted σ-algebra $\{A\cap\{t<\tau\}\}_{A\in\mathcal{F}_t}$); moreover, it is called continuous if trajectories are continuous on $[0,\tau)$, $\mathbb{P}$-a.s.. Note that X is $(\mathcal{F}_t)$-adapted if and only if the process $\tilde{X} = (\tilde{X}_t)_{t\geq 0}$,

$$\tilde{X}_t = X_t 1_{\{\tau>t\}} + 0\cdot 1_{\{\tau\leq t\}} \tag{69}$$

is $(\mathcal{F}_t)$-adapted.

Definition 2. *Let $x \in H$. We call local weak mild solution to (1) a tuple $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}, W, X, \tau)$, where $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ is a filtered probability space on which it is defined a cylindrical $(\mathcal{F}_t)$-Wiener process W, an accessible stopping time τ and a continuous $(\mathcal{F}_t)$-adapted H-valued process X defined on $[0, \tau)$ such that, there exists an approximating sequence (τ_n) of τ for which, $\mathbb{P}$-a.s., on $\{t \leq \tau_n\}$*

$$X_t = e^{tA}x + \int_0^t e^{(t-s)A} B(X_s)\, ds + \int_0^t e^{(t-s)A} dW_s,$$

for all $n \in \mathbb{N}$ and $t \geq 0$.

A local weak mild solution X which is $(\bar{\mathcal{F}}_t^W)$-adapted (here $(\bar{\mathcal{F}}_t^W)$ denotes the completed natural filtration of the cylindrical process W) and such that τ is an $(\bar{\mathcal{F}}_t^W)$-stopping time is called a local strong mild solution.

Theorem 20. *Assuming Hypothesis 1 and $B \in B_{b,loc}(H, H)$, existence of local strong mild solutions holds for μ-a.e. initial condition $x \in H$. Moreover, for μ-a.e. $x \in H$, if X and Y are two local weak mild solutions on the same $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ with the same cylindrical Wiener process W, defined on $[0, \tau^X)$ and $[0, \tau^Y)$ respectively (τ^X and τ^Y are accessible stopping times as in Definition 2), then, $\mathbb{P}$-a.s., $X = Y$ on $[0, \tau)$, where $\tau = \tau^X \wedge \tau^Y$.*

Proof. Let us sketch some of the details of the proof.

From the first part of the proof of Theorem 16 (see also Theorem 1 in [5]), given a-priori a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ and a cylindrical $(\mathcal{F}_t)-$Wiener process W, we have the existence of a local strong mild solution X_N on $[0, \tau_N)$, for every $N \in \mathbb{N}$. Note that each τ_N is accessible since as approximating sequence we may take $\tau_{N,n} = \inf\{t \geq 0 \ : \ X_N(t) \notin B(0, N - \frac{1}{n})\} \wedge n$, for $n \geq 1$.

Thus, taking $\tau = \sup_{N \in \mathbb{N}} \tau_N$, τ is accessible and we have a local strong mild solution on $[0, \tau)$.

In order to prove uniqueness, we first note that if X is a local weak mild solution defined on $[0, \tau^X)$ and (τ_n^X) is an approximating sequence of τ^X as in the definition of solution, then assertion (32) of Theorem 10 holds, for any $t > 0$, $n \geq 1$, $\mathbb{P}$-a.s., on the event $\{t \leq \tau_{n,N}^X\}$, where $\tau_{n,N}^X$ is the stopping time

$$\tau_{n,N}^X = \inf\{t \in [0, \tau^X) \ : \ X_t \notin B(0, N)\} \wedge \tau_n^X$$

($\tau_{n,N}^X = \tau_n^X$ if the set is empty; to show that $\tau_{n,N}^X$ is a stopping time, note that $\tau_{n,N}^X = \inf\{t \geq 0 \ : \ \tilde{X}_t \notin B(0, N)\} \wedge \tau_n^X$, where $\tilde{X}_t$ is defined in (69) with τ replaced by τ^X).

Indeed, one can repeat the arguments in the proof of Theorem 10 with the same functions u_N, replacing τ_N^X with $\tau_{n,N}^X$.

Now let X and Y be two local weak mild solutions as in the second part of the theorem.

If (τ_n^X) and (τ_n^Y) are, respectively, approximating sequences of τ^X and τ^Y as in the definition of solution, then in order to prove uniqueness, it is enough to consider $\sigma_n = \tau_n^X \wedge \tau_n^Y$ and check that, $\mathbb{P}$-a.s.,

$$X = Y \quad \text{on} \quad [0, \sigma_n], \quad n \geq 1. \tag{70}$$

Let us fix $n \geq 1$. We can adapt the proof of Theorem 1, arguing on the interval $[0, \eta \wedge T \wedge \tau_{n,N}^X \wedge \tau_{n,N}^Y]$ (cf. (53)). We finally get that $X = Y$ on $[0, T \wedge \sigma_n]$, $T > 0$, and this gives the assertion. □

Acknowledgement. The authors would like to thank the referees for their useful remarks and suggestions.

References

[1] Albeverio, S., Brzezniak, Z., Wu, J.F.: Existence of global solutions and invariant measures for stochastic differential equations driven by Poisson type noise with non-Lipschitz coefficients, J. Math. Anal. Appl. 371, 309-322 (2010)

[2] Chojnowska-Michalik, A., Goldys, B.: Generalized Ornstein-Uhlenbeck Semigroups: Littlewood-Paley–Stein Inequalities and the P. A. Meyer Equivalence of Norms. J. Funct. Anal. 182, 243-279 (2001)

[3] Chojnowska-Michalik, A., Goldys, B.: Symmetric Ornstein-Uhlenbeck Semigroups and their Generator. Probab. Theory Relat. Fields 124, 459-486 (2002)

[4] Da Prato, G.: Kolmogorov equations for stochastic PDEs. Advanced Courses in Mathematics. CRM Barcelona. Birkhäuser Verlag, Basel (2004)

[5] Da Prato, G., Flandoli, F., Priola, E., Röckner, M.: Strong uniqueness for stochastic evolution equations in Hilbert spaces perturbed by a bounded measurable drift, Annals of Probability 41, 3306-3344 (2013). doi: 10.1214/12-AOP763

[6] Da Prato, G., Zabczyk, J.: Stochastic equations in infinite dimensions, Encyclopedia of Mathematics and its Applications, 44. Cambridge University Press, Cambridge (1992)

[7] Da Prato, G., Zabczyk, J.: Second Order Partial Differential Equations in Hilbert Spaces, London Math. Soc. Lecture Note Ser., vol. 293, Cambridge University Press, Cambridge (2002)

[8] Da Prato, G., Flandoli, F., Pathwise uniqueness for a class of SDEs in Hilbert spaces and applications, J. Funct. Anal. 259, 243-267 (2010)

[9] Fedrizzi, E.: thesis, Pisa (2009)

[10] Fedrizzi, E., Flandoli, F.: Pathwise uniqueness and continuous dependence for SDEs with nonregular drift, Stochastics 83, 241-257 (2011)

[11] Ferrario, B.: Uniqueness and absolute continuity for semilinear SPDEs, to appear in the Proceedings of "Seminar on Stochastic Analysis, Random Fields and Applications-VII", Ascona, Birkhäuser (2011)

[12] Flandoli, F.: Random perturbation of PDEs and fluid dynamic models, Lecture Notes in Mathematics 2015, Springer, Berlin (2011)

[13] Flandoli, F., Gubinelli, M., Priola, E.: Well-posedness of the transport equation by stochastic perturbation, Invent. Math. 180, 1-53 (2010)

[14] Gatarek, D., Goldys, B.: On solving stochastic evolution equations by the change of drift with application to optimal control, "Stochastic partial differential equations and applications" (Trento, 1990), 180-190, Pitman Res. Notes Math. Ser., 268, Longman Sci. Tech., Harlow (1992)

[15] Gyöngy, I., Krylov N. V.: Existence of strong solutions for Itô's stochastic equations via approximations, Probab. Theory Relat. Fields 105, 143-158 (1996)

[16] Gyöngy, I., Martínez, T.: On stochastic differential equations with locally unbounded drift, Czechoslovak Math. J. 51 (126), 763-783 (2001)

[17] Krylov, N. V., Röckner, M.: Strong solutions to stochastic equations with singular time dependent drift, Probab. Theory Relat. Fields 131, 154-196 (2005)

[18] Liu, W., Röckner, M.: Introduction to Stochastic Partial Differential Equations, in preparation.

[19] Ma, Z. M., Röckner, M.: Introduction to the theory of (nonsymmetric) Dirichlet forms, Universitext. Springer-Verlag, Berlin, 1992

[20] Maas, J., van Neerven, J.M.A.M.: Gradient estimates and domain identification for analytic Ornstein-Uhlenbeck operators, Nonlinear Parabolic Problems: Herbert Amann Festschrift Progress in Nonlinear Differential Equations and Their Applications, Vol. 60, pp. 463-477. Birkhäuser (2011)

[21] Ondreját, M.: Uniqueness for stochastic evolution equations in Banach spaces. Dissertationes Math. (Rozprawy Mat.) 426 (2004)

[22] Prévôt, C., Röckner, M.: A concise course on stochastic partial differential equations, Lecture Notes in Mathematics, 1905, Springer, Berlin, 2007

[23] Priola, E.: Pathwise uniqueness for singular SDEs driven by stable processes, Osaka J. Math. 49, 421-447 (2012)

[24] Schmuland, B.: Dirichlet forms with polynomial domain, Math. Japon. 37, 1015-1024 (1992)

[25] Tanaka, H., Tsuchiya, M., Watanabe, S.: Perturbation of drift-type for Lévy processes, J. Math. Kyoto Univ. 14, 73-92 (1974)

[26] Veretennikov, A. J.: Strong solutions and explicit formulas for solutions of stochastic integral equations, Mat. Sb. (N.S.) 111(153) (3), 434-452 (1980)

[27] Zhang, X.: Strong solutions of SDES with singular drift and Sobolev diffusion coefficients, Stochastic Process. Appl. 115, 1805-1818 (2005)

[28] Zvonkin, A. K.: A transformation of the phase space of a diffusion process that removes the drif, Mat. Sb. (N.S.) 93 (135), 129-149 (1974)