STOCHASTIC VARIATIONAL INEQUALITIES AND REGULARITY FOR DEGENERATE STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT. The regularity and characterization of solutions to degenerate, quasilinear SPDE is studied. Our results are two-fold: First, we prove regularity results for solutions to certain degenerate, quasilinear SPDE driven by Lipschitz continuous noise. In particular, this provides a characterization of solutions to such SPDE in terms of (generalized) strong solutions. Second, for the one-dimensional stochastic mean curvature flow with normal noise we adapt the notion of stochastic variational inequalities to provide a characterization of solutions previously obtained in a limiting sense only. This solves a problem left open in [ESvR12] and sharpens regularity properties obtained in [ESvRS12].

1. INTRODUCTION

The study of degenerate SPDE has attracted much interest in recent years. As a model example let us consider the following stochastic p-Laplace type SPDE

$$(1.1) \qquad dX_t = \mathrm{div}\phi(\nabla X_t)dt + B(X_t)dW_t$$
$$X_0 = x_0$$

with zero Dirichlet boundary conditions on bounded, smooth domains $\mathcal{O} \subseteq \mathbb{R}^d$, for some monotone $\phi \in C^1(\mathbb{R}^d; \mathbb{R}^d)$ satisfying the coercivity property

$$\phi(\xi) \cdot \xi \geq c|\xi|^p \quad \forall \xi \in \mathbb{R}^d$$

for some $c > 0$. In the following let W be a trace-class Wiener process on $L^2(\mathcal{O})$. In the case $p > 1$ a variational approach to such SPDE (under some further assumptions) has been developed in [RRW07] for initial conditions $x_0 \in L^2(\mathcal{O})$ based on the coercivity property

$$(1.2) \qquad _{(W^{1,p})^*}\langle \mathrm{div}\phi(\nabla v), v\rangle_{W^{1,p}} \geq c\|v\|_{W^{1,p}}^p \quad \forall v \in H^2(\mathcal{O}),$$

for some $c > 0$. In the degenerate case $p = 1$ these methods do not apply anymore, since the reflexivity of the energy space $W^{1,p}(\mathcal{O})$ is lost. In particular, this difficulty appears for the stochastic mean curvature flow in one spatial dimension

$$(1.3) \qquad dX_t = \frac{\partial_x^2 X_t}{1 + (\partial_x X_t)^2}dt + B(X_t)dW_t$$
$$= \partial_x \arctan(\partial_x X_t) + B(X_t)dW_t, \quad \text{on } \mathcal{O} = (0,1)$$

and the stochastic total variation flow

$$(1.4) \qquad dX_t \in \mathrm{div}\left(\frac{\nabla X_t}{|\nabla X_t|}\right)dt + B(X_t)dW_t.$$

Restricting to more regular initial data (i.e. $x_0 \in H_0^1(\mathcal{O})$), in [ESvR12] an alternative, variational approach, applicable to the stochastic mean curvature flow (1.3) has been developed, based on the coercivity property

$$(1.5) \qquad (\mathrm{div}\phi(\nabla v), v)_{H_0^1} \geq 0 \quad \forall v \in H^2(\mathcal{O}).$$

This approach was subsequently generalized in [GT13] to multi-valued SPDE including the stochastic total variation flow (1.4). The restriction to regular initial data ($x_0 \in H_0^1(\mathcal{O})$) is crucial to this approach, since it allows to work with solutions taking values in $H_0^1(\mathcal{O})$. For general initial data $x_0 \in L^2(\mathcal{O})$ solutions to (1.3), (1.4) could be constructed in [ESvR12, GT13] in a limiting sense only. That is, it has been shown that for each sequence $x_0^n \to x$ in $L^2(\mathcal{O})$ with $x_0^n \in H_0^1(\mathcal{O})$ the corresponding variational solutions X^n converge to a limit X independent of the chosen approximating sequence x_0^n. However, no characterization of X in terms of a (generalized) solution to the corresponding SPDE could be given. In particular, this problem remained unsolved for the stochastic mean curvature flow with normal noise

$$(1.6) \qquad dX_t = \partial_x\left(\arctan(\partial_x X_t)\right)dt + \alpha\sqrt{1 + |\partial_x X_t|^2} \circ d\beta_t,$$

on $\mathcal{O} = (0,1)$ with periodic boundary conditions. We quote from [ESvR12]: [..] *in view of the poor regularity of the operator A [A(v) = ∂_x (arctan($\partial_x v$))], a more explicit characterization of the $L^2([0,1])$-valued process $\hat{u}_t^x$ by some SPDE or even just an associated Kolmogorov operator on smooth finitely based test functions does not seem to be available.* For background and motivation of the stochastic mean curvature flow with normal noise we refer to [ESvR12, FLP14]. A numerical treatment may be found in [FLP14], higher dimensional results in [SY04, LS98b, LS98a, LS00, DLN01].

Recently, for the special case of the total variation flow with linear multiplicative noise

$$(1.7) \qquad dX_t \in \mathrm{div}\left(\frac{\nabla X_t}{|\nabla X_t|}\right)dt + \sum_{k=1}^{\infty} f_k X_t d\beta_t^k,$$

with $f_k : \mathcal{O} \to \mathbb{R}$, the problem of characterizing solutions for general initial data $x_0 \in L^2(\mathcal{O})$ has been solved in [BR13] by introducing the concept of stochastic variational inequalities (SVI), a notion first developed in [BDPR09] for (1.7) with additive noise. It is shown in [BR13] that the limiting solutions to (1.7) obtained in [GT13] can be uniquely characterized as SVI solutions to (1.7). For more general

SPDE of the type (1.1), e.g. the stochastic mean curvature flow, the problem of characterizing solutions for general initial data remained open.

The latter problem is solved in the current paper. Our results are two-fold: First, we prove that in certain situations more regularity of solutions to degenerate SPDE than previously expected can be proved. In these cases, the concept of SVI solutions is not necessary to characterize solutions for general initial data, since we may work with (analytically) strong solutions instead (cf. Definition 2.4 below). This extends regularity results for degenerate, quasilinear SPDE developed in [Ges12] and applies to degenerate p-Laplace type equations

$$(1.8) \qquad dX_t = \operatorname{div}(\phi(\nabla X_t))\, dt + \sum_{k=1}^{\infty} g_k(\cdot, X_t) d\beta_t^k$$

with $\phi \in C^1(\mathbb{R}^d; \mathbb{R}^d)$ satisfying appropriate conditions (cf. (2.2) below). Among other examples, this includes the stochastic mean curvature flow with vertical noise (cf. [ESvR12]), i.e. (1.8) with $\phi = \arctan$, which significantly sharpens the regularity results obtained in [ESvRS12]. More precisely, for general initial data $x_0 \in L^2(\mathcal{O})$ we prove

$$(1.9) \qquad \operatorname{div}(\phi(\nabla X_t)) \in L^2([\tau, T] \times \Omega; L^2(\mathcal{O}))$$

for each $\tau > 0$, which allows to characterize X as a generalized strong solution to (1.8) (cf. Definition 2.4 below). Note that (1.9) entails a regularizing effect with respect to the initial condition, i.e. while $x_0 \in L^2(\mathcal{O})$ we observe that X takes values in the domain of $\operatorname{div}\phi(\nabla \cdot)$, $dt \otimes d\mathbb{P}$-almost everywhere.

For the stochastic mean curvature flow with normal noise (1.6) additional difficulties appear, due to the irregularity of the noise. Informally rewriting (1.6) in Itô form yields

$$dX_t = \frac{\alpha^2}{2} \partial_x^2 X_t dt + (1 - \frac{\alpha^2}{2}) \frac{\partial_x^2 X_t}{1 + (\partial_x X_t)^2} dt + \alpha \sqrt{1 + (\partial_x X_t)^2} d\beta_t$$

$$= \frac{\alpha^2}{2} \partial_x^2 X_t dt + (1 - \frac{\alpha^2}{2}) \partial_x \arctan(\partial_x X_t) dt + \alpha \sqrt{1 + (\partial_x X_t)^2} d\beta_t.$$

Again, we prove new regularity results for X of the type

$$\partial_x \arctan(\partial_x X_t) \in L^2([\tau, T] \times \Omega; L^2(\mathcal{O})) \quad \forall \tau > 0.$$

In contrast to (1.8), this improved regularity does not yield the existence of generalized strong solutions due to the additional term $\frac{\alpha^2}{2} \partial_x^2 X_t$. We resolve this issue by introducing a notion of SVI solutions to (1.6) and by proving the existence and uniqueness of SVI solutions for each initial condition $x_0 \in L^2(\mathcal{O})$. The results thus parallel those of [BR13] for the case of the stochastic total variation flow (1.4). However, in contrast to [BR13] our method does not rely on a transformation into a random PDE, which leads to the restriction to linear multiplicative noise (cf. (1.7)) in [BR13]. We would also like to point out a difference in the role played by SVI solutions in the case of (1.4) and (1.6): The necessity to work with SVI solutions in [BR13] is grounded in the singularity of the multi-valued sign function $\operatorname{Sgn}(\xi) = \frac{\xi}{|\xi|}$. More precisely, if we replace Sgn by a smooth function ϕ in (1.4) we are in the setting of (1.8) and generalized strong solutions exist. In contrast to this, the difficulties arising for (1.6) are due to the irregularity of the noise, rather than the irregularity of ϕ. Nonetheless, in both cases SVI solutions provide the means to uniquely characterize solutions for general initial data.

1.1. **Notation.** In the following let $\mathcal{O} \subseteq \mathbb{R}^d$ be a bounded set with smooth boundary. For a Hilbert space H we define $C_{l.b.}^k(H)$ to be the set of all continuous functions on H with k continuous derivatives that are locally bounded on H. We

will work with the usual Lebesgue and Sobolev spaces $L^p(\mathcal{O})$, $W^{k,p}(\mathcal{O})$ writing L^p, $H^{k,p}$ for simplicity. We further set $H^k(\mathcal{O}) = W^{k,2}(\mathcal{O})$. For a function $(x,r) \mapsto g(x,r) \in C^1(\bar{\mathcal{O}} \times \mathbb{R})$ we define the partial gradient $\nabla_x g(x,r) := (\partial_{x_i} g)_{i=1}^d (x,r)$ while for a function $v \in C^1(\mathcal{O})$ we let $\nabla g(\cdot, v) = (\nabla_x g)(\cdot, v) + (\partial_r g)(\cdot, v)\nabla v$ be the total gradient. In the proofs, as usual, constants may change from line to line.

2. Stochastic Parabolic quasilinear problems for linear growth functionals

We consider SPDE of the form

$$(2.1) \qquad dX_t = \operatorname{div}(\phi(\nabla X_t))\, dt + \sum_{k=1}^{\infty} g^k(\cdot, X_t) d\beta_t^k,$$

$$X_0 = x_0$$

with zero Dirichlet boundary conditions on bounded, convex, smooth domains $\mathcal{O} \subseteq \mathbb{R}^d$ with $d \leq 6$. Here, β^k are independent Brownian motions and

(B) $g^k \in C^1(\bar{\mathcal{O}} \times \mathbb{R})$ with $g^k(x,0) = 0$ for all $x \in \partial\mathcal{O}$ and

$$\sum_{k=1}^{\infty} \mu_k < \infty,$$

where

$$\mu_k := \left\| \partial_r g^k \right\|_{C^0(\bar{\mathcal{O}} \times \mathbb{R})}^2 + \left\| \frac{|\nabla_x g^k(x,r)|}{1+|r|} \right\|_{C^0(\bar{\mathcal{O}} \times \mathbb{R})}^2.$$

In particular, this includes additive noise, i.e. $g^k \in C_0^1(\bar{\mathcal{O}})$ and linear multiplicative noise, i.e. $g^k(x,r) = \varphi^k(x)r$ for $\varphi^k \in C^1(\bar{\mathcal{O}})$. We assume $\phi = \nabla\psi$ for some non-negative, convex, radial $\psi \in C^2(\mathbb{R}^d; \mathbb{R})$ with $\psi(0) = 0$, $\phi, D\phi$ being Lipschitz and

$$(2.2) \qquad \begin{array}{ccccc} c|\xi| - C & \leq & \psi(\xi) & \leq & C(1+|\xi|) \\ c\psi(\xi) - C & \leq & \phi(\xi) \cdot \xi & & \\ & & |D\phi(\xi)||\xi|^2 & \leq & C(1+\psi(\xi)), \end{array}$$

for all $\xi \in \mathbb{R}^d$ and some constants $c > 0$, $C \geq 0$. Then the recession function ψ^0 defined by

$$\psi^0(\xi) := \lim_{t \downarrow 0} \psi\left(\frac{\xi}{t}\right) t$$

exists and is finite since ψ is of linear growth and $t \mapsto \psi\left(\frac{\xi}{t}\right) t$ is non-increasing.

Example 2.1. Typical examples of ψ are

i. Mean curvature flow in one dimension (cf. [ESvR12, ESvRS12]):

$$\psi(\xi) = \xi \arctan(\xi) - \frac{1}{2} \log(\xi^2 + 1)$$

$$\dot{\psi}(\xi) = \phi(\xi) := \arctan(\xi)$$

$$\dot{\phi}(\xi) = \frac{1}{1+\xi^2}, \quad \xi \in \mathbb{R}.$$

ii. Minimal surface/image denoising (cf. [GT13, BR13, GR92, KOJ05]):

$$\psi(\xi) = \sqrt{\varepsilon + |\xi|^2}$$

$$\nabla\psi(\xi) = \phi(\xi) = \frac{\xi}{\sqrt{\varepsilon + |\xi|^2}}$$

$$D\phi(\xi) = \frac{1}{\sqrt{\varepsilon + |\xi|^2}}\left(-\frac{\xi \otimes \xi}{\varepsilon + |\xi|^2} + Id\right), \quad \xi \in \mathbb{R}^d, \varepsilon > 0, d \geq 1.$$

We note that $\psi(\xi) = \sqrt{\varepsilon + |\xi|^2}$ may be considered as a smooth approximation to the total variation functional $\psi(\xi) = |\xi|$. In [GR92] a general class of nonlinearities ψ has been considered with regard to application in image restoration (cf. also [ROF92]).

Remark 2.2. The same methods as developed in this section may be applied to nonlinearities arising in generalized Newtonian fluids:

$$\psi(\xi) = \frac{1}{p}(1 + |\xi|^2)^{\frac{p}{2}}$$

$$\nabla\psi(\xi) = \phi(\xi) = (1 + |\xi|^2)^{\frac{p-2}{2}}\xi$$

$$D\phi(\xi) = (1 + |\xi|^2)^{\frac{p-2}{2}}\left((p-2)\frac{\xi \otimes \xi}{1 + |\xi|^2} + Id\right), \quad \xi \in \mathbb{R}^d,$$

with $p \in (1, 2)$. For simplicity we restrict to linear growth functionals satisfying (2.2).

In this section we will work with the Hilbert spaces $H = L^2(\mathcal{O})$ and $V = H_0^1(\mathcal{O})$. For $v \in H$ we set $B(v)(h) := \sum_{k=1}^{\infty} g^k(\cdot, v)(e_k, h)_2$, where $e_i \in H$ is an orthonormal basis of H. Then $B : H \to L_2(H, H)$ is of linear growth, i.e.

$$\|B(v)\|_{L_2(H,H)}^2 = \sum_{k=1}^{\infty} \|g^k(\cdot, v)\|_H^2$$

$$\leq C(1 + \|v\|_H^2)\sum_{k=1}^{\infty} \mu_k,$$

for all $v \in H$. Similarly one shows that $H \ni v \mapsto B(v) \in L_2(H, H)$ is Lipschitz. Following [Anz85] we define

$$\varphi(v) := \begin{cases} \int_{\mathcal{O}} \psi(Dv)dx + \int_{\partial\mathcal{O}} \psi^0(\nu(x)v(x))H^{d-1}(dx) & \text{if } v \in L^2 \cap BV \\ +\infty & \text{if } v \in L^2 \setminus BV, \end{cases}$$

where $\int_{\mathcal{O}} \psi(\mu)dx$ for a bounded Radon measure μ with Lebesgue decomposition $\mu = \mu^a + \mu^s$ is defined as in [Anz85], i.e.

$$\int_{\mathcal{O}} \psi(\mu)dx = \int_{\mathcal{O}} \psi(\mu^a)dx + \int_{\mathcal{O}} \psi^0\left(\frac{d\mu}{d|\mu|}\right)d|\mu|^s$$

and ν is the outer normal on $\partial\mathcal{O}$. For $u \in BV$ we consider the Lebesgue decomposition $Du = D^a u + D^s u$ where $D^a u$ denotes the absolutely continuous part of Du with respect to the Lebesgue measure with density ∇u. Obviously, φ is convex and φ restricted to $W_0^{1,1}$ is continuous. Furthermore it follows from [Anz83, Fact 3.5] that φ is the lower-semicontinuous hull on L^2 of $\varphi_{|W_0^{1,1} \cap L^2}$. In the sequel $\partial\varphi := \partial_H \varphi$ denotes the subgradient of φ on H.

Remark 2.3. Under certain additional assumptions on ψ (cf. [ACM02] for details) the following characterization of the subgradient of φ has been given in [ACM02]:

We have $(u, v) \in \partial\varphi$ iff $u \in L^2 \cap BV$, $v \in L^2$, $\phi(\nabla u) \in X(\mathcal{O}) = \{z \in L^\infty(\mathcal{O}; \mathbb{R}^d) : \mathrm{div}(z) \in L^1(\mathcal{O})\}$ and

$$v = -\mathrm{div}\phi(\nabla u),$$

$$\phi(\nabla u) \cdot D^s u = \psi^0(D^s u),$$

$$[\phi(\nabla u), \nu](x) \in -\mathrm{Sgn}(u(x))\psi^0(\nu(x)), \quad H^{d-1} - \text{a.e.},$$

where $[z, \nu]$ denotes the weak trace for $z \in X(\mathcal{O})$.

Our arguments will, however, not rely on this identification of the subgradient. We note that $\varphi_{|H_0^1}$ is Gateaux-differentiable with derivative

$$D\varphi_{|H_0^1}(u)(v) = \int_{\mathcal{O}} \phi(\nabla u)\nabla v \, dx.$$

Using that φ is the lower-semicontinuous hull of $\varphi_{|H_0^1}$ this implies: If $u \in H_0^1$ then

(2.3) $$\partial_{H_0^1}\varphi(u) = \mathrm{div}\phi(\nabla u) \in H^{-1},$$

where $\partial_{H_0^1}\varphi : H_0^1 \to H^{-1}$ denotes the subgradient of $\varphi_{|H_0^1}$. If, in addition, $\mathrm{div}\phi(\nabla u) \in L^2$ then

$$\partial\varphi(u) = -\mathrm{div}\phi(\nabla u).$$

Hence, we may rewrite (2.1) in the *relaxed* form

(2.4) $$dX_t \in -\partial\varphi(X_t)dt + B(X_t)dW_t,$$

$$X_0 = x_0.$$

In [ESvR12, GT13] a variational framework for regular initial conditions $x_0 \in H_0^1$ has been developed, while for general initial data $x_0 \in H$ solutions to (2.1) could only be constructed in a limiting sense. In the following we will introduce stronger notions of solutions to (2.1) based on the subgradient formulation (2.4). The main result will be the proof of existence of solutions in this stronger sense. This includes the proof of higher regularity of solutions.

Definition 2.4. Let $x_0 \in L^2(\Omega; H)$. An H-continuous, $\mathcal{F}_t$-adapted process $X \in L^2(\Omega; C([0, T]; H))$ for which there exists a selection $\eta \in -\partial\varphi(X)$, $dt \otimes d\mathbb{P}$-a.e. is said to be a

i. strong solution to (2.1) if

$$\eta \in L^2([0, T] \times \Omega; H)$$

and $\mathbb{P}$-a.s.

$$X_t = x_0 + \int_0^t \eta_r dr + \int_0^t B(X_r)dW_r, \quad \forall t \in [0, T],$$

as an equation in H.

ii. generalized strong solution to (2.1) if

$$\eta \in L^2([\tau, T] \times \Omega; H), \quad \forall \tau > 0$$

and $\mathbb{P}$-a.s.

$$X_t = X_\tau + \int_\tau^t \eta_r dr + \int_\tau^t B(X_r)dW_r, \quad \forall t \in [\tau, T],$$

for all $\tau > 0$, as an equation in H.

We prove the existence of strong solutions to (2.1) for initial conditions $x_0 \in L^2(\Omega; H)$ satisfying $\mathbb{E}\varphi(x_0) < \infty$. Moreover, we will prove regularizing properties with respect to the initial condition due to the subgradient structure of the drift. This allows to characterize solutions for initial conditions $x_0 \in L^2(\Omega; H)$, which before were constructed in a limiting sense only.

Theorem 2.5. *Let $x_0 \in L^2(\Omega; H)$.*

    i. There is a unique generalized strong solution (X, η) to (2.1) and (X, η) satisfies

$$\mathbb{E}t\varphi(X_t) + \mathbb{E} \int_0^t r\|\eta_r\|_H^2 dr \leq C \left(\mathbb{E}\|x_0\|_H^2 + 1\right) \quad \forall t \in [0, T].$$

    ii. If $\mathbb{E}\varphi(x_0) < \infty$. Then, there is a unique strong solution (X, η) to (2.1) satisfying

$$(2.5) \qquad \mathbb{E}\varphi(X_t) + \mathbb{E} \int_0^t \|\eta_r\|_H^2 dr \leq \mathbb{E}\varphi(x_0) + C \quad \forall t \in [0, T].$$

The (generalized) strong solution X coincides with the limit solution constructed in [ESvR12].

The proof of Theorem 2.5 proceeds in several steps. In order to justify our calculations we will consider a vanishing viscosity approximation to (2.1). Let us first assume $x_0 \in L^2(\Omega; H_0^1)$. We will remove this restriction in the end of the proof. We consider the following non-degenerate approximation:

$$(2.6) \qquad dX_t^\varepsilon \in \varepsilon \Delta X_t^\varepsilon dt + \mathrm{div}\phi(\nabla X_t^\varepsilon)dt + B(X_t^\varepsilon)dW_t,$$
$$X_0^\varepsilon = x_0.$$

For $v \in H$ we define

$$\varphi^\varepsilon(v) := \begin{cases} \frac{\varepsilon}{2} \int_{\mathcal{O}} |\nabla v|^2 dx + \int_{\mathcal{O}} \psi(\nabla v)dx, & \text{for } v \in H_0^1 \\ +\infty, & \text{otherwise.} \end{cases}$$

Note that $\varphi^\varepsilon \in C^2(H_0^1 \cap H^2)$ with Lipschitz continuous derivatives given by

$$D\varphi^\varepsilon(v)(h) = \varepsilon \int_{\mathcal{O}} \nabla v \cdot \nabla h dx + \int_{\mathcal{O}} \phi(\nabla v) \cdot \nabla h dx,$$

$$D^2\varphi^\varepsilon(v)(g, h) = \varepsilon \int_{\mathcal{O}} \nabla h \cdot \nabla g dx + \int_{\mathcal{O}} \nabla h \cdot D\phi(\nabla v)\nabla g dx.$$

To check the claimed continuity we note that

$$D\varphi^\varepsilon(v)(h) - D\varphi^\varepsilon(w)(h) = \varepsilon \int_{\mathcal{O}} (\nabla v - \nabla w) \cdot \nabla h dx + \int_{\mathcal{O}} (\phi(\nabla v) - \phi(\nabla w)) \cdot \nabla h dx$$
$$\leq \varepsilon \|v - w\|_{H_0^1} \|h\|_{H_0^1} + \|\phi(\nabla v) - \phi(\nabla w)\|_2 \|h\|_{H_0^1}$$
$$\lesssim (\varepsilon + 1)\|v - w\|_{H_0^1} \|h\|_{H_0^1}$$

and

$$D^2\varphi^\varepsilon(v)(g, h) - D^2\varphi^\varepsilon(w)(g, h) = \int_{\mathcal{O}} \nabla h \cdot (D\phi(\nabla v) - D\phi(\nabla w)) \nabla g dx$$
$$(2.7) \qquad\qquad \leq \|\nabla h\|_3 \|\nabla g\|_3 \|D\phi(\nabla v) - D\phi(\nabla w)\|_3$$
$$\lesssim \|\nabla h\|_3 \|\nabla g\|_3 \|\nabla v - \nabla w\|_3$$
$$\lesssim \|h\|_{H_0^1 \cap H^2} \|g\|_{H_0^1 \cap H^2} \|v - w\|_{H_0^1 \cap H^2},$$

where we use the Sobolev embedding $H^1 \hookrightarrow L^3$ due to $d \leq 6$. Hence, $\varphi^\varepsilon \in C^2(H_0^1 \cap H^2)$ with Lipschitz continuous derivatives. Moreover, φ^ε is a convex, lower-semicontinuous function on H_0^1 with (single-valued) subgradient given by

$$A^\varepsilon(v) := -\partial_{H_0^1}\varphi^\varepsilon(v) = \varepsilon\Delta v + \mathrm{div}\phi(\nabla v) \in H^{-1}, \quad \text{for } v \in H_0^1.$$

By [PR07] we know that there is a unique, variational solution $X^\varepsilon \in L^2(\Omega; C([0, T]; H)) \cap L^2([0, T] \times \Omega; H_0^1)$ to (2.6) satisfying the estimate

$$\mathbb{E} \sup_{t \in [0, T]} \|X_t^\varepsilon\|_H^2 \leq C\mathbb{E}\|x_0\|_H^2.$$

We will now prove that in fact X^ε is a strong solution in the following sense:

Lemma 2.6. *For each $\varepsilon > 0$ we have $X^\varepsilon \in L^2([0,T] \times \Omega; H^2 \cap H_0^1)$ and*

$$\mathbb{E} \sup_{t \in [0,T]} e^{-Ct} \|X_t^\varepsilon\|_{H_0^1}^2 + 4\varepsilon \int_0^t \mathbb{E} e^{-Cr} \|X_r^\varepsilon\|_{H^2}^2 dr \leq C \left(\mathbb{E}\|x_0\|_{H_0^1}^2 + 1 \right),$$

for some constant C independent of $\varepsilon > 0$.

Proof. In the following we let $(e_i)_{i=1}^\infty$ be an orthonormal basis of eigenvectors of $-\Delta$ on L^2. We further let $P^n : H \to \text{span}\{e_1, \ldots, e_n\}$ be the orthogonal projection onto the span of the first n eigenvectors. We recall that the unique variational solution X^ε to (2.6) is constructed in [PR07] as a (weak) limit in $L^2([0,T] \times \Omega; H_0^1)$ of the solutions to the following Galerkin approximation

$$dX_t^n = \varepsilon P^n \Delta X_t^n dt + P^n \text{div}\phi(\nabla X_t^n) dt + P^n B(X_t^n) dW_t^n,$$
$$X_0^n = P^n x_0.$$

Itô's formula then yields

$$\|X_t^n\|_{H_0^1}^2 = \|P^n x_0\|_{H_0^1}^2 + 2 \int_0^t (X_r^n, \varepsilon P^n \Delta X_r^n + P^n \text{div}\phi(\nabla X_t^n))_{H_0^1} dr$$

$$+ 2 \int_0^t (X_r^n, P^n B(X_r^n) dW_r^n)_{H_0^1} dr + \int_0^t \|P^n B(X_r^n)\|_{L_2(H, H_0^1)}^2 dr$$

$$= \|P^n x_0\|_{H_0^1}^2 - 2\varepsilon \int_0^t \|\Delta X_r^n\|_2^2 dr + 2 \int_0^t (X_r^n, P^n \text{div}\phi(\nabla X_t^n))_{H_0^1} dr$$

$$+ 2 \int_0^t (X_r^n, P^n B(X_r^n) dW_r^n)_{H_0^1} dr + \int_0^t \|P^n B(X_r^n)\|_{L_2(H, H_0^1)}^2 dr.$$

For $v \in H_0^1$ smooth we note

$$(v, \text{div}\phi(\nabla v))_{H_0^1} = (-\Delta v, \text{div}\phi(\nabla v))_2$$
$$= \lim_{n \to \infty} (T^n v, \text{div}\phi(\nabla v))_2$$
$$= -\lim_{n \to \infty} n(J^{\frac{1}{n}} v - v, \text{div}\phi(\nabla v))_2,$$

where $J^{\frac{1}{n}} := (1 - \frac{1}{n}\Delta)^{-1}$ is the resolvent and $T^n = n(1 - J^{\frac{1}{n}})$ is the Yosida-approximation of $-\Delta$ on L^2. Since $\text{div}\phi(\nabla v) = -\partial\varphi(v)$ we obtain

$$(v, \text{div}\phi(\nabla v))_{H_0^1} \leq \lim_{n \to \infty} n(\varphi(J^{\frac{1}{n}} v) - \varphi(v)).$$

We note that

$$\varphi(J^{\frac{1}{n}} v) = \int_{\mathcal{O}} \psi(\nabla J^{\frac{1}{n}} v) dx \leq \int_{\mathcal{O}} \psi(\nabla v) dx = \varphi(v)$$

due to [BR13, Proposition 8.2] (using that ψ is radial) and thus (choosing $v = X_r^n$)

$$(X_r^n, \text{div}\phi(\nabla X_r^n))_{H_0^1} \leq 0.$$

Using this,

$$\|B(v)\|_{L_2(H, H_0^1)}^2 = \sum_{k=1}^\infty \|g^k(x, v)\|_{H_0^1}^2$$

$$= \sum_{k=1}^\infty \|\nabla_x g^k(x, v) + \partial_r g^k(x, v)\nabla v\|_{L^2}^2$$

$$\leq C(1 + \|v\|_{H_0^1}^2) \sum_{k=1}^\infty \mu_k \quad \forall v \in H_0^1$$

and the Burkholder-Davis-Gundy inequality yields

$$
(2.8) \qquad \frac{1}{2}\mathbb{E}\sup_{t\in[0,T]} e^{-Ct}\|X_t^n\|_{H_0^1}^2 \leq \mathbb{E}\|x_0\|_{H_0^1}^2 - 2\varepsilon\mathbb{E}\int_0^T e^{-Cr}\|\Delta X_r^n\|_2^2 dr + C,
$$

for some $C > 0$ large enough. Hence, X^n is uniformly bounded in $L^2([0,T]\times\Omega;H^2)$ and $L^2(\Omega;L^\infty([0,T];H_0^1))$ and we may extract a weakly (weak* resp.) convergent subsequence (for simplicity we stick with the notation X^n). Therefore, we have

$$
X^n \rightharpoonup X, \quad \text{in } L^2([0,T]\times\Omega;H^2),
$$
$$
X^n \rightharpoonup^* X, \quad \text{in } L^2(\Omega;L^\infty([0,T];H_0^1)),
$$

for $n \to \infty$. By weak lower semicontinuity of the norms we may pass to the limit in (2.8) which yields the claim. $\qquad\square$

Lemma 2.7. *For each $\varepsilon > 0$ we have $\varphi^\varepsilon(X^\varepsilon) \in L^1([0,T]\times\Omega)$ with*

$$
\sup_{t\in[0,T]}\mathbb{E}\|X_t^\varepsilon\|_H^2 + \mathbb{E}\int_0^T e^{-Cr}\varphi^\varepsilon(X_r^\varepsilon)dr \leq C\left(\mathbb{E}\|x_0\|_H^2 + 1\right),
$$

for some constant C independent of $\varepsilon > 0$.

Proof. Note that, using (2.2)

$$
(2.9) \qquad
\begin{aligned}
{}_{V^*}\langle A^\varepsilon(v), v\rangle_V &= -\int_{\mathcal{O}}\left(\varepsilon|\nabla v|^2 + \phi(\nabla v)\cdot\nabla v\right)dx \\
&\leq -\int_{\mathcal{O}}\left(\varepsilon|\nabla v|^2 + c\psi(\nabla v) + C\right)dx \\
&\leq -c\varphi^\varepsilon(v) + C,
\end{aligned}
$$

for all $v \in V$. By Itô's formula we have

$$
\mathbb{E}e^{-Kt}\|X_t^\varepsilon\|_H^2
$$
$$
=\mathbb{E}\|x_0^\varepsilon\|_H^2 + 2\mathbb{E}\int_0^t e^{-Kr}\,{}_{V^*}\langle A^\varepsilon(X_r^\varepsilon), X_r^\varepsilon\rangle_V + e^{-Kr}\|B(X_r^\varepsilon)\|_{L_2(H,H)}^2 dr
$$
$$
- K\int_0^t e^{-Kr}\|X_r^\varepsilon\|_H^2 dr
$$
$$
\leq\mathbb{E}\|x_0^\varepsilon\|_H^2 - 2\mathbb{E}\int_0^t ce^{-Kr}\varphi^\varepsilon(X_r^\varepsilon) + Ce^{-Kr}\|X_r^\varepsilon\|_H^2 dr - K\int_0^t e^{-Kr}\|X_r^\varepsilon\|_H^2 dr + C.
$$

Choosing K large enough yields the claim. $\qquad\square$

Based on the strong solution property of X^ε we derive the key estimate in the following

Lemma 2.8. *Let $x_0 \in L^2(\Omega;H_0^1)$. For all $\varepsilon > 0$ we have*

$$
(2.10) \qquad \mathbb{E}t\varphi^\varepsilon(X_t^\varepsilon) + \mathbb{E}\int_0^t r\|\varepsilon\Delta X_r^\varepsilon + \operatorname{div}\phi(\nabla X_r^\varepsilon)\|_H^2 dr
$$
$$
\leq \varepsilon C\left(\mathbb{E}\|x_0\|_{H_0^1}^2 + 1\right) + C\left(\mathbb{E}\|x_0\|_H^2 + 1\right).
$$

and

$$
(2.11) \qquad \mathbb{E}\varphi^\varepsilon(X_t^\varepsilon) + \mathbb{E}\int_0^t \|\varepsilon\Delta X_r^\varepsilon + \operatorname{div}\phi(\nabla X_r^\varepsilon)\|_H^2 dr \leq \mathbb{E}\varphi^\varepsilon(x_0) + C,
$$

for some constant $C > 0$.

Proof. We first prove (2.10): Let J^λ be the resolvent of $-\Delta$ on L^2, i.e. $J^\lambda :=$ $(1 - \lambda\Delta)^{-1}$. We define $\varphi^{\varepsilon,\lambda} := \varphi^\varepsilon \circ J^\lambda$. Since $J^\lambda : H \to H^2 \cap H_0^1$ is a linear, continuous operator we have $\varphi^{\varepsilon,\lambda} \in C^2(H)$ with Lipschitz continuous derivatives (cf. (2.7)) given by

$$
\begin{aligned}
D\varphi^{\varepsilon,\lambda}(v)(h) &= D\varphi^\varepsilon(J^\lambda v)(J^\lambda h) \\
&= \int_{\mathcal{O}} \left(\varepsilon(\nabla J^\lambda v) \cdot (\nabla J^\lambda h) + \phi(\nabla J^\lambda v) \cdot \nabla J^\lambda h \right) dx \\
&= -\left(\varepsilon\Delta J^\lambda v + \mathrm{div}\phi(\nabla J^\lambda v), J^\lambda h \right)_H, \\
D^2\varphi^{\varepsilon,\lambda}(v)(g,h) &= D^2\varphi^\varepsilon(J^\lambda v)(J^\lambda g, J^\lambda h) \\
&= \int_{\mathcal{O}} \varepsilon(\nabla J^\lambda h) \cdot (\nabla J^\lambda g) dx + \int_{\mathcal{O}} (\nabla J^\lambda h) \cdot D\phi(\nabla J^\lambda v)(\nabla J^\lambda g) dx.
\end{aligned}
$$

For (2.10): We apply Itô's formula to $t\varphi^{\varepsilon,\lambda}(X_t^\varepsilon)$ to get:

$$
\begin{aligned}
&\mathbb{E}t\varphi^{\varepsilon,\lambda}(X_t^\varepsilon) \\
&= \mathbb{E}\int_0^t r(D\varphi^{\varepsilon,\lambda}(X_r^\varepsilon), \varepsilon\Delta X_r^\varepsilon + \mathrm{div}\phi(\nabla X_r^\varepsilon))_H dr \\
&\quad + \frac{1}{2}\mathbb{E}\int_0^t rTr[D^2\varphi^{\varepsilon,\lambda}(X_r^\varepsilon)B(X_r^\varepsilon)B^*(X_r^\varepsilon)]dr + \mathbb{E}\int_0^t \varphi^{\varepsilon,\lambda}(X_r^\varepsilon)dr \\
&= -\mathbb{E}\int_0^t r(\varepsilon\Delta J^\lambda(X_r^\varepsilon) + \mathrm{div}\phi(\nabla J^\lambda X_r^\varepsilon), \varepsilon J^\lambda\Delta X_r^\varepsilon + J^\lambda\mathrm{div}\phi(\nabla X_r^\varepsilon))_H dr \\
&\quad + \frac{\varepsilon}{2}\sum_{k=1}^\infty \mathbb{E}\int_0^t r\int_{\mathcal{O}} |\nabla J^\lambda g^k(x, X_r^\varepsilon(x))|^2 dx dr \\
&\quad + \frac{1}{2}\sum_{k=1}^\infty \mathbb{E}\int_0^t r\int_{\mathcal{O}} (\nabla J^\lambda g^k(x, X_r^\varepsilon(x))) \cdot D\phi(\nabla J^\lambda X_r^\varepsilon(x))(\nabla J^\lambda g^k(x, X_r^\varepsilon(x))) dx dr \\
&\quad + \mathbb{E}\int_0^t \varphi^{\varepsilon,\lambda}(X_r^\varepsilon)dr.
\end{aligned}
$$

We first note that

$$
\begin{aligned}
\int_{\mathcal{O}} |\nabla J^\lambda g^k(x, X_r^\varepsilon(x))|^2 dx &\leq \int_{\mathcal{O}} |\nabla g^k(x, X_r^\varepsilon(x))|^2 dx \\
&\leq C\int_{\mathcal{O}} |\nabla_x g^k(x, X_r^\varepsilon(x)) + \partial_r g^k(x, X_r^\varepsilon(x))\nabla X_r^\varepsilon(x)|^2 dx \\
&\leq C\mu_k(1 + \|X_r^\varepsilon\|_{H_0^1}^2).
\end{aligned}
$$

Moreover,

$$
|(\nabla J^\lambda g^k(\cdot, v)) \cdot D\phi(\nabla J^\lambda v)(\nabla J^\lambda g^k(\cdot, v))| \leq |\nabla J^\lambda g^k(\cdot, v)|^2 |D\phi|(\nabla J^\lambda v).
$$

We note $J^\lambda v \to v$ in $H_0^1 \cap H^2$ for $v \in H_0^1 \cap H^2$ and thus $\nabla J^\lambda v \to \nabla v$ in H^1 for $\lambda \to 0$. Since $D\phi$ is Lipschitz we have $|D\phi(\nabla J^\lambda v) - D\phi(\nabla v)| \to 0$ in $L^2([0,T] \times \Omega; H^1)$ for all $v \in L^2([0,T] \times \Omega; H_0^1 \cap H^2)$ for $\lambda \to 0$. Moreover, $|\nabla J^\lambda g^k(\cdot, v) - \nabla g^k(\cdot, v)| \to 0$

in $L^2([0, T] \times \Omega; H)$ for $\lambda \to 0$. Hence, (using $H^1 \hookrightarrow L^3$, $D\phi \in C_b^0$ and (2.2))

$$\lim_\lambda \mathbb{E} \int_0^t r \int_{\mathcal{O}} (\nabla J^\lambda g^k(\cdot, v_r)) \cdot D\phi(\nabla J^\lambda v_r)(\nabla J^\lambda g^k(\cdot, v_r)) dx dr$$

$$\leq \mathbb{E} \int_0^t r \int_{\mathcal{O}} |\nabla g^k(\cdot, v_r))|^2 |D\phi|(\nabla v_r) dx dr$$

$$\leq C\mathbb{E} \int_0^t r \int_{\mathcal{O}} \left(|\nabla_x g^k(\cdot, v_r)|^2 + |\partial_r g^k(\cdot, v_r)\nabla v_r|^2 \right) |D\phi|(\nabla v_r) dx dr$$

$$\leq C\mu_k \mathbb{E} \int_0^t r \int_{\mathcal{O}} (1 + |v_r|^2 + |\nabla v_r|^2)|D\phi|(\nabla v_r) dx dr$$

$$\leq C\mu_k \mathbb{E} \int_0^t r \int_{\mathcal{O}} (1 + |v_r|^2 + \psi(\nabla v_r)) dx dr$$

$$\leq C\mu_k \left(1 + \mathbb{E} \int_0^t r\|v_r\|_H^2 dr + \mathbb{E} \int_0^t r\varphi^\varepsilon(v_r) dr \right),$$

for all $v \in L^2([0, T] \times \Omega; H_0^1 \cap H^2)$. Moreover,

$$\left| \mathbb{E} \int_0^t r \int_{\mathcal{O}} (\nabla J^\lambda g^k(\cdot, v_r)) \cdot D\phi(\nabla J^\lambda(v_r))(\nabla J^\lambda g^k(\cdot, v)) dx dr \right|$$

$$\leq C\mu_k \mathbb{E} \int_0^t r(1 + \|v_r\|_{H_0^1}^2) dr$$

and the right hand side is summable in k. Hence, dominated convergence applies and we obtain

$$\lim_{\lambda \to 0} \frac{\varepsilon}{2} \sum_{k=1}^\infty \mathbb{E} \int_0^t r \int_{\mathcal{O}} |\nabla J^\lambda g^k(x, X_r^\varepsilon(x))|^2 dx dr$$

$$+ \lim_{\lambda \to 0} \frac{1}{2} \sum_{k=1}^\infty \mathbb{E} \int_0^t r \int_{\mathcal{O}} (\nabla J^\lambda g^k(x, X_r^\varepsilon(x))) \cdot D\phi(\nabla J^\lambda(X_r^\varepsilon(x)))(\nabla J^\lambda g^k(x, X_r^\varepsilon(x))) dx dr$$

$$\leq C \left(1 + \frac{\varepsilon}{2} \mathbb{E} \int_0^t r(1 + \|X_r^\varepsilon\|_{H_0^1}^2) dr + \mathbb{E} \int_0^t r\|X_r^\varepsilon\|_H^2 dr + \mathbb{E} \int_0^t r\varphi^\varepsilon(X_r^\varepsilon) dr \right).$$

Since $X^\varepsilon \in L^2([0, T] \times \Omega; H^2 \cap H_0^1)$ we have (using dominated convergence) $\nabla J^\lambda X^\varepsilon \to \nabla X^\varepsilon$ in $L^2([0, T] \times \Omega; H^1)$ for $\lambda \to 0$. Since $\phi, D\phi$ are Lipschitz this implies $\phi(\nabla J^\lambda X^\varepsilon) \to \phi(\nabla X^\varepsilon)$ for $\lambda \to 0$ in $L^2([0, T] \times \Omega; H^1)$. Moreover, $\Delta J^\lambda X^\varepsilon = J^\lambda \Delta X^\varepsilon \to \Delta X^\varepsilon$ and $J^\lambda \mathrm{div}\phi(\nabla X_r^\varepsilon) \to \mathrm{div}\phi(\nabla X_r^\varepsilon)$ in $L^2([0, T] \times \Omega; H)$. Hence, we obtain

$$\lim_{\lambda \to 0} -\mathbb{E} \int_0^t r(\varepsilon \Delta J^\lambda(X_r^\varepsilon) + \mathrm{div}\phi(\nabla J^\lambda X_r^\varepsilon), \varepsilon J^\lambda \Delta X_r^\varepsilon + J^\lambda \mathrm{div}\phi(\nabla X_r^\varepsilon))_H dr$$

$$= -\mathbb{E} \int_0^t r\|\varepsilon \Delta X_r^\varepsilon + \mathrm{div}\phi(\nabla X_r^\varepsilon)\|_H^2 dr.$$

Since $\varphi^{\varepsilon,\lambda}(X_r^\varepsilon) = \varphi^\varepsilon(J_\lambda X_r^\varepsilon)$, $J_\lambda X^\varepsilon \to X^\varepsilon$ in $L^2([0, T] \times \Omega; H^2 \cap H_0^1)$, $J_\lambda X_t^\varepsilon \to X_t^\varepsilon$ in $L^2(\Omega; H_0^1)$ for all $t \in [0, T]$ and since φ^ε is continuous on H_0^1 and

$$|\varphi^{\varepsilon,\lambda}(X_t^\varepsilon)| \leq C(1 + \|X_t^\varepsilon\|_{H_0^1}^2),$$

by Lebesgue's dominated convergence theorem we get

$$\lim_{\lambda \to 0} \mathbb{E} t\varphi^{\varepsilon,\lambda}(X_t^\varepsilon) = \mathbb{E} t\varphi^\varepsilon(X_t^\varepsilon)$$

and

$$\lim_{\lambda \to 0} \mathbb{E} \int_0^t \varphi^{\varepsilon,\lambda}(X_r^\varepsilon) dr = \mathbb{E} \int_0^t \varphi^\varepsilon(X_r^\varepsilon) dr.$$

Putting these estimates together yields

$$\mathbb{E}t\varphi^\varepsilon(X_t^\varepsilon) \le -\mathbb{E}\int_0^t r\|\varepsilon\Delta X_r^\varepsilon + \mathrm{div}\phi(\nabla X_r^\varepsilon)\|_H^2 dr$$

$$+ C\left(1 + \varepsilon\mathbb{E}\int_0^t r(1 + \|X_r^\varepsilon\|_{H_0^1}^2)dr + \mathbb{E}\int_0^t r\|X_r^\varepsilon\|_H^2 dr + \mathbb{E}\int_0^t r\varphi^\varepsilon(X_r^\varepsilon)dr\right)$$

$$+ \mathbb{E}\int_0^t \varphi^\varepsilon(X_r^\varepsilon)dr.$$

By Lemma 2.6 and Lemma 2.7 we conclude

$$\mathbb{E}t\varphi^\varepsilon(X_t^\varepsilon) + \mathbb{E}\int_0^t r\|\varepsilon\Delta X_r^\varepsilon + \mathrm{div}\phi(\nabla X_r^\varepsilon)\|_H^2 dr$$

$$\le \varepsilon C\left(\mathbb{E}\|x_0\|_{H_0^1}^2 + 1\right) + C\left(\mathbb{E}\|x_0\|_H^2 + 1\right).$$

To prove (2.11) we proceed as above but applying Itô's formula for $\varphi^{\varepsilon,\lambda}(X_t^\varepsilon)$ instead of $t\varphi^{\varepsilon,\lambda}(X_t^\varepsilon)$. $\qquad\square$

Proof of Theorem 2.5: **Step 1:** $x_0 \in L^2(\Omega; H_0^1)$

For $\varepsilon_1, \varepsilon_2 > 0$ let $X^{\varepsilon_1}, X^{\varepsilon_2}$ be two solutions to (2.6) with initial conditions $x_0^1, x_0^2 \in L^2(\Omega; H_0^1)$ respectively. Itô's formula implies

$$e^{-Kt}\|X_t^{\varepsilon_1} - X_t^{\varepsilon_2}\|_H^2$$

$$= \|x_0^1 - x_0^2\|_H^2$$

$$+ \int_0^t 2e^{-Kr}{}_{V^*}\langle \varepsilon_1\Delta X_r^{\varepsilon_1} + \mathrm{div}\phi(\nabla X_r^{\varepsilon_1}) - (\varepsilon_2\Delta X_r^{\varepsilon_2} + \mathrm{div}\phi(\nabla X_r^{\varepsilon_2})), X_r^{\varepsilon_1} - X_r^{\varepsilon_2}\rangle_V dr$$

$$+ \int_0^t 2e^{-Kr}(X_r^{\varepsilon_1} - X_r^{\varepsilon_2}, B(X_r^{\varepsilon_1}) - B(X_r^{\varepsilon_2}))_H dW$$

$$+ \int_0^t e^{-Kr}\|B(X_r^{\varepsilon_1}) - B(X_r^{\varepsilon_2})\|_{L_2(U,H)}^2 dr - K\mathbb{E}\int_0^t e^{-Kr}\|X_r^{\varepsilon_1} - X_r^{\varepsilon_2}\|_H^2 dr.$$

Since

$$_{V^*}\langle \mathrm{div}\phi(\nabla X_r^{\varepsilon_1}) - \mathrm{div}\phi(\nabla X_r^{\varepsilon_2}), X_r^{\varepsilon_1} - X_r^{\varepsilon_2}\rangle_V \le 0$$

and

$$_{V^*}\langle \varepsilon_1\Delta X_r^{\varepsilon_1} - \varepsilon_2\Delta X_r^{\varepsilon_1}, X_r^{\varepsilon_1} - X_r^{\varepsilon_2}\rangle_V \le 2(\varepsilon_1 + \varepsilon_2)(\|X_r^{\varepsilon_1}\|_{H_0^1}^2 + \|X_r^{\varepsilon_2}\|_{H_0^1}^2),$$

we obtain

$$e^{-Kt}\|X_t^{\varepsilon_1} - X_t^{\varepsilon_2}\|_H^2$$

$$\le \|x_0^1 - x_0^2\|_H^2 + 4(\varepsilon_1 + \varepsilon_2)\int_0^t e^{-Kr}(\|X_r^{\varepsilon_1}\|_{H_0^1}^2 + \|X_r^{\varepsilon_2}\|_{H_0^1}^2)dr$$

$$+ \int_0^t 2e^{-Kr}(X_r^{\varepsilon_1} - X_r^{\varepsilon_2}, B(X_r^{\varepsilon_1}) - B(X_r^{\varepsilon_2}))_H dW$$

$$+ \int_0^t e^{-Kr}\|B(X_r^{\varepsilon_1}) - B(X_r^{\varepsilon_2})\|_{L_2(U,H)}^2 dr - K\mathbb{E}\int_0^t e^{-Kr}\|X_r^{\varepsilon_1} - X_r^{\varepsilon_2}\|_H^2 dr.$$

Using the Burkholder-Davis-Gundy inequality, Lemma 2.6 and choosing K large enough implies

$$(2.12) \qquad \mathbb{E}\sup_{t\in[0,T]}\|X_t^{\varepsilon_1} - X_t^{\varepsilon_2}\|_H^2 \le C\mathbb{E}\|x_0^1 - x_0^2\|_H^2$$

$$+ (\varepsilon_1 + \varepsilon_2)C\left(\mathbb{E}\|x_0^1\|_{H_0^1}^2 + \mathbb{E}\|x_0^2\|_{H_0^1}^2 + 1\right).$$

Now considering X^ε to be a solution to (2.6) with initial condition $x_0 \in L^2(\Omega; H_0^1)$ for all $\varepsilon > 0$ yields

$$X^\varepsilon \to X \quad \text{in } L^2(\Omega; C([0,T]; H)),$$

for $\varepsilon \to 0$. Due to Lemma 2.6 we have

$$\mathbb{E} \sup_{t \in [0,T]} e^{-Ct} \|X_t\|_{H_0^1}^2 \le C \left(\mathbb{E}\|x_0\|_{H_0^1}^2 + 1 \right).$$

For two initial conditions $x_0^1, x_0^2 \in H_0^1$ and respective limits X^1, X^2, (2.12) then yields

(2.13) $$\mathbb{E} \sup_{t \in [0,T]} \|X_t^1 - X_t^2\|_H^2 \le C\mathbb{E}\|x_0^1 - x_0^2\|_H^2.$$

It remains to identify X as a strong solution to (2.1). By Lemma 2.8 we have $\varepsilon \Delta X^\varepsilon + \mathrm{div}\phi(\nabla X^\varepsilon)$ uniformly bounded in $L^2([0,T] \times \Omega; H)$. Hence, there is an $\eta \in L^2([0,T] \times \Omega; H)$ and we can choose a sequence ε_n such that

$$\varepsilon_n \Delta X^{\varepsilon_n} + \mathrm{div}\phi(\nabla X^{\varepsilon_n}) \rightharpoonup \eta, \quad \text{in } L^2([0,T] \times \Omega; H).$$

We now aim to identify $\eta \in -\partial\varphi(X)$, $dt \otimes d\mathbb{P}$-almost everywhere. By the subgradient property

$$(\partial\varphi^\varepsilon(X_t^\varepsilon), z - X_t^\varepsilon)_2 + \varphi^\varepsilon(X_t^\varepsilon) - \varphi^\varepsilon(z) \le 0$$

for all $z \in L^2$. Since $X_t^\varepsilon \in H_0^1 \cap H^2$ we have $\partial\varphi^\varepsilon(X_t^\varepsilon) = -\varepsilon\Delta X^\varepsilon - \mathrm{div}\phi(\nabla X^\varepsilon) \in L^2$, $dt \otimes d\mathbb{P}$-almost everywhere. Integration yields

$$\mathbb{E} \int_0^T \theta \left[-(\varepsilon\Delta X^\varepsilon + \mathrm{div}\phi(\nabla X^\varepsilon), z - X^\varepsilon)_2 + \varphi^\varepsilon(X^\varepsilon) - \varphi^\varepsilon(z) \right] dt \le 0$$

for all $z \in L^2$ and all non-negative $\theta \in L^\infty([0,T] \times \Omega)$. Taking the limit yields

$$\mathbb{E} \int_0^T \theta \left[(-\eta, z - X)_2 + \varphi(X) - \varphi(z) \right] dt \le 0$$

and thus

$$(-\eta, z - X)_2 + \varphi(X) - \varphi(z) \le 0,$$

for all $z \in L^2(\mathcal{O})$, $dt \otimes d\mathbb{P}$-almost everywhere. Thus, $\eta \in -\partial\varphi(X)$, $dt \otimes d\mathbb{P}$-almost everywhere. Since $\eta \in H$ and $X \in H_0^1$ $dt \otimes d\mathbb{P}$-a.e. by (2.3) we have $\eta = \mathrm{div}\phi(\nabla X)$. It is now easy to deduce that X is a strong solution to (2.1) and X satisfies

$$\mathbb{E}t\varphi(X_t) + \mathbb{E} \int_0^t r\|\eta_r\|_H^2 dr \le C \left(\mathbb{E}\|x_0\|_H^2 + 1 \right).$$

and

(2.14) $$\mathbb{E}\varphi(X_t) + \mathbb{E} \int_0^t \|\eta_r\|_H^2 dr \le \mathbb{E}\varphi(x_0) + C,$$

for some constant $C > 0$ (independent of x_0).

Step 2: $x_0 \in L^2(\Omega; H)$ with $\mathbb{E}\varphi(x_0) < \infty$

By [BR13, Proposition 8.2], for $v \in H_0^1$ we have

$$\varphi(J^\lambda v) = \int_\mathcal{O} \psi(\nabla J^\lambda v) dx$$
$$\le \int_\mathcal{O} \psi(\nabla v) dx$$
$$= \varphi(v),$$

where $J^\lambda := (1 - \lambda\Delta)^{-1}$. Since φ is the lower-semicontinuous hull of $\varphi_{|W_0^{1,1} \cap L^2}$ and thus of $\varphi_{|H_0^1}$, for every $v \in H$ there is a sequence $v^n \in H_0^1$ with $v^n \to v$ in H and $\varphi(v^n) \to \varphi(v)$. Hence,

$$\varphi(J^\lambda v) \leq \varphi(v),$$

for all $v \in H$. We set $x_0^n := J^{\frac{1}{n}} x_0$ and obtain

$$(2.15) \qquad \mathbb{E}\varphi(x_0^n) + \mathbb{E}\|x_0^n\|_2^2 \leq \mathbb{E}\varphi(x_0) + \mathbb{E}\|x_0\|_2^2 < \infty.$$

For $n, m > 0$ let X^n, X^m be two solutions to (2.1) with $x_0 = x_0^n, x_0^m$ respectively as constructed in Step 1. From (2.13) we obtain

$$(2.16) \qquad \mathbb{E}\sup_{t \in [0,T]} \|X_t^n - X_t^m\|_H^2 \leq C\mathbb{E}\|x_0^n - x_0^m\|_H^2$$

and thus

$$X^n \to X \quad \text{in } L^2(\Omega; C([0,T]; H)),$$

for $n \to \infty$. Moreover, we have the uniform estimates

$$(2.17) \qquad \mathbb{E}t\varphi(X_t^n) + \mathbb{E}\int_0^t r\|\eta_r^n\|_H^2 dr \leq C\left(\mathbb{E}\|x_0^n\|_H^2 + 1\right)$$
$$\leq C\left(\mathbb{E}\|x_0\|_H^2 + 1\right)$$

and

$$(2.18) \qquad \mathbb{E}\varphi(X_t^n) + \mathbb{E}\int_0^t \|\eta_r^n\|_H^2 dr \leq \mathbb{E}\varphi(x_0^n) + C,$$
$$\leq \mathbb{E}\varphi(x_0) + C.$$

This allows the extraction of a subsequence and an $\eta \in L^2([0,T] \times \Omega; H)$ such that

$$\eta^n \rightharpoonup \eta, \quad \text{in } L^2([0,T] \times \Omega; H).$$

We may identify $\eta \in -\partial\varphi(X)$ as in Step 1: Since $\eta^n \in -\partial\varphi(X^n)$, $dt \otimes d\mathbb{P}$-a.e.

$$(-\eta^n, z - X^n)_2 + \varphi(X^n) - \varphi(z) \leq 0$$

for all $z \in L^2$. Integrating against a non-negative testfunction $\theta \in L^\infty([0,T] \times \Omega)$ and taking the limit $n \to \infty$ yields

$$\mathbb{E}\int_0^T \theta\left[(-\eta, z - X)_2 + \varphi(X) - \varphi(z)\right] dt \leq 0$$

and thus $\eta \in -\partial\varphi(X)$, $dt \otimes d\mathbb{P}$-almost everywhere. Note that $-\partial\varphi(v) = \text{div}\phi(\nabla v)$ is known only for $v \in H_0^1$ which does not apply do X in general. Taking the limit $n \to \infty$ in (2.17), (2.18) yields the claim.

Step 3: $x_0 \in L^2(\Omega; H)$

Let $x_0^n \in L^2(\Omega; H_0^1)$ with $x_0^n \to x_0$ in $L^2(\Omega; H)$, $\mathbb{E}\|x_0^n\|_H^2 \leq \mathbb{E}\|x_0\|_H^2$ and let X^n be the corresponding strong solutions constructed in step one. By (2.13) we have

$$\mathbb{E}\sup_{t \in [0,T]} \|X^n - X^m\|_H^2 \leq C\mathbb{E}\|x_n - x_m\|_H^2.$$

Hence, $X^n \to X$ in $L^2(\Omega; C([0,T]; H))$. Moreover,

$$\mathbb{E}t\varphi(X_t^n) + \mathbb{E}\int_0^t r\|\eta_r^n\|_H^2 dr \leq C\left(\mathbb{E}\|x_0\|_H^2 + 1\right).$$

Hence, there is a map η with $\eta \in L^2([\tau, T] \times \Omega; H)$ such that

$$\eta^n \rightharpoonup \eta, \quad \text{in } L^2([\tau, T] \times \Omega; H),$$

for all $\tau > 0$. We may then prove $\eta \in -\partial\varphi(X)$ as in Step 1. Hence, X is a generalized strong solution satisfying

$$\mathbb{E}t\varphi(X_t) + \mathbb{E}\int_0^t r\|\eta_r\|_H^2 dr \leq C\left(\mathbb{E}\|x_0\|_H^2 + 1\right).$$

$\square$

3. MEAN CURVATURE FLOW WITH (PERIODIC) HOMOGENEOUS NORMAL NOISE

In this section we consider the SPDE

$$(3.1) \qquad dX_t = \frac{\partial_x^2 X_t}{1 + (\partial_x X_t)^2}dt + \alpha\sqrt{1 + (\partial_x X_t)^2}\circ d\beta_t$$
$$X_0 = x_0,$$

with periodic boundary conditions on $\mathcal{O} = (0,1)$ (i.e. $d = 1$), β being a standard real-valued Brownian motion and $\alpha \leq \sqrt{2}$. Informally rewriting the Stratonovich formulation of (3.1) in Itô form as in [ESvR12] leads to the SPDE

$$(3.2) \qquad dX_t = \frac{\alpha^2}{2}\partial_x^2 X_t dt + (1 - \frac{\alpha^2}{2})\frac{\partial_x^2 X_t}{1 + (\partial_x X_t)^2}dt + \alpha\sqrt{1 + (\partial_x X_t)^2}d\beta_t$$

with periodic boundary conditions on $\mathcal{O} = (0,1)$. Let

$$\psi(r) = (1 - \frac{\alpha^2}{2})\left(r\arctan(r) - \frac{1}{2}\log(r^2 + 1)\right)$$

$$\phi(r) = \dot\psi(r) = (1 - \frac{\alpha^2}{2})\arctan(r).$$

For $v \in L^2(0,1)$ we define $v^\perp(x) := v(1-x)$. We then set

$$\varphi(v) := \begin{cases} \int_\mathcal{O} \psi(Dv)dx + \frac{1}{2}\int_{\partial\mathcal{O}}|[v - v^\perp]|H^{d-1}(dx) & \text{if } v \in (L^2 \cap BV)(\mathcal{O}) \\ +\infty & \text{if } v \in (L^2 \setminus BV)(\mathcal{O}), \end{cases}$$

where $\int_\mathcal{O} \psi(Dv)dx$ is defined as in Section 2. Since $\mathcal{O} = (0,1)$ we have

$$\varphi(v) := \int_{(0,1)} \psi(Dv)dx + |v(1) - v(0)| \quad \text{for } v \in (L^2 \cap BV)(\mathcal{O}).$$

In the following, for $p \geq 1$ let

$$W_{per}^{1,p}(\mathcal{O}) := \{f \in W^{1,p}(\mathcal{O})|f(0) = f(1)\}$$
$$W_{per}^{2,p}(\mathcal{O}) := \{f \in (W^{2,p} \cap W_{per}^{1,p})(\mathcal{O})|\partial_x f \in W_{per}^{1,p}\}.$$

Moreover, let $H_{per}^1 = W_{per}^{1,2}$, $H_{per}^2 = W_{per}^{2,2}$, $H = L^2(\mathcal{O})$. Then φ is the lower-semicontinuous envelope on L^2 of φ restricted to $W_{per}^{1,1}$ (cf. Appendix A), i.e. of

$$\varphi_{|W_{per}^{1,1}}(v) = \int_\mathcal{O} \psi(\partial_x v)dx, \quad v \in W_{per}^{1,1} \cap L^2.$$

It is easy to see that $\varphi_{|H_{per}^1}$ is Gateaux differentiable with

$$D\varphi_{|H_{per}^1}(v)(h) = \int_\mathcal{O} \phi(\partial_x v)\partial_x h dx.$$

Since $\varphi_{|W_{per}^{1,1}}$ is continuous on $W_{per}^{1,1}$ it is easy to see that φ is the lower-semicontinuous hull of $\varphi_{|H_{per}^1}$ on L^2. This implies

$$\partial\varphi(u) = -\partial_x\phi(\partial_x u) = -(1 - \frac{\alpha^2}{2})\frac{\partial_x^2 u}{1 + (\partial_x u)^2}, \quad \text{for } u \in H_{per}^2.$$

For $v \in H^1$ we define

$$B(v) := \alpha\sqrt{1 + (\partial_x v)^2}.$$

Hence, (3.1) may be rewritten in the form

$$(3.3) \qquad dX_t \in \frac{\alpha^2}{2} \partial_x^2 X_t dt - \partial \varphi(X_t) dt + B(X_t) d\beta_t,$$
$$X_0 = x_0.$$

Due to the irregularity of the diffusion coefficients B it does not seem possible to establish the existence of (generalized) strong solution as considered in Section 2. Instead, we introduce a notion of stochastic variational inequalities for (3.3).

For regular initial data $x_0 \in H_{per}^1$, the existence and uniqueness of variational solutions to (3.1) has been shown in [ESvR12] (cf. also [GT13] for multivalued generalizations). For general initial conditions $x_0 \in L^2$ solutions have been constructed in [ESvR12] in a limiting sense. We now define what we mean by a solution to (3.1):

Definition 3.1. Let $x_0 \in L^2(\Omega; H)$. An $\mathcal{F}_t$-adapted process $X \in C([0, T]; L^2(\Omega; H))$ is said to be an SVI solution to (3.1) if there is an $\eta \in L^2([\tau, T] \times \Omega; H)$, $\forall \tau > 0$ such that

    i. [Regularity]

$$\varphi(X) \in L^1([0, T] \times \Omega).$$

    ii. [Subgradient property]

$$\eta \in -\partial \varphi(X), \quad dt \otimes d\mathbb{P} - a.e..$$

    iii. [Stochastic variational inequality] For each $\mathcal{F}_t$-progressively measurable process $G \in L^2([0, T] \times \Omega; H)$ and each $\mathcal{F}_t$-adapted H-valued process Z with $\mathbb{P}$-a.s. continuous sample paths such that $Z \in L^2([0, T] \times \Omega; H_{per}^2)$ and solving the equation

$$Z_t - Z_0 = \int_0^t G_s ds + \int_0^t Z_s dW_s, \quad \forall t \in [0, T]$$

    we have

$$(3.4) \qquad \mathbb{E}\|X_t - Z_t\|_H^2 \le \mathbb{E}\|X_\tau - Z_\tau\|_H^2 + 2 \int_\tau^t (\eta_r - G_r, X_r - Z_r)_2 dr$$
$$+ \alpha^2 \mathbb{E} \int_\tau^t (\partial_x^2 Z_r, X_r - Z_r)_2 dr, \quad \forall \tau > 0.$$

Remark 3.2. If (X, η) is a generalized strong solution (defined analogously to Definition 2.4) to (3.1) satisfying $\varphi(X) \in L^1([0, T] \times \Omega)$ then (X, η) is an SVI solution to (3.1).

Proof. Definition 3.1 (i),(ii) are satisfied by assumption. For (iii): Let $Z \in L^2([0, T] \times \Omega; H_{per}^2)$ be a solution to

$$dZ_t = G_t dt + \alpha \sqrt{1 + (\partial_x Z_t)^2} d\beta_t$$
$$= G_t dt + B(Z_t) d\beta_t$$

for some $G \in L^2([0, T] \times \Omega; H)$. Then Itô's formula implies

$$\mathbb{E}\|X_t - Z_t\|_H^2 = \mathbb{E}\|X_\tau - Z_\tau\|_H^2 + \alpha^2 \mathbb{E} \int_\tau^t (\partial_x^2 X_r, X_r - Z_r)_2 dr$$
$$+ 2\mathbb{E} \int_\tau^t (\eta_r - G_r, X_r - Z_r)_2 dr$$
$$+ \alpha^2 \mathbb{E} \int_\tau^t \|\sqrt{1 + (\partial_x X_r)^2} - \sqrt{1 + (\partial_x Z_r)^2}\|_2^2 dr \quad \forall \tau > 0.$$

We note that

$$\alpha^2 \| \sqrt{1 + (\partial_x X_r)^2} - \sqrt{1 + (\partial_x Z_r)^2} \|_2^2$$
$$\leq \alpha^2 \| \partial_x X_r - \partial_x Z_r \|_2^2$$
$$= -\alpha^2 (\partial_x^2 X_r, X_r - Z_r)_2 + \alpha^2 (\partial_x^2 Z_r, X_r - Z_r)_2, \quad dt \otimes d\mathbb{P} - \text{a.e.}$$

and thus

$$\mathbb{E}\|X_t - Z_t\|_H^2 \leq \mathbb{E}\|X_\tau - Z_\tau\|_H^2 + 2\mathbb{E}\int_\tau^t (\eta_r - G_r, X_r - Z_r)_2 dr$$
$$+ \alpha^2 \mathbb{E}\int_\tau^t (\partial_x^2 Z_r, X_r - Z_r)_2 dr.$$

In conclusion, each strong solution to (3.1) is an SVI solution to (3.1). □

The main result of the current section is the proof of well-posedness of (3.1) in the sense of Definition 3.1:

Theorem 3.3. *Let $x_0 \in L^2(\Omega; H)$. Then there is a unique SVI solution (X, η) to (3.1) in the sense of Definition 3.1 satisfying*

$$\mathbb{E}t\varphi(X_t) + \mathbb{E}\int_0^t r\|\eta_r\|_H^2 dr \leq C \left(\mathbb{E}\|x_0\|_H^2 + 1 \right).$$

In addition, if $\mathbb{E}\varphi(x_0) < \infty$ then

$$\mathbb{E}\varphi(X_t) + \mathbb{E}\int_0^t \|\eta_r\|_H^2 dr \leq \mathbb{E}\varphi(x_0) + C.$$

In particular, $\eta \in L^2([0, T] \times \Omega; H)$ and we may take $\tau = 0$ in (3.4).

For two SVI solutions (X, η), (Y, ζ) with initial conditions $x_0, y_0 \in L^2(\Omega; H)$ we have

$$\mathbb{E}\|X_t - Y_t\|_H^2 \leq \mathbb{E}\|x_0 - y_0\|_H^2, \quad \forall t \geq 0.$$

For notational convenience we introduce the following semi-norm on H_{per}^1

$$\|v\|_{H_{per,0}^1} := \|\partial_x v\|_2.$$

We note

$$(3.5) \quad \|B(v)\|_{L_2(\mathbb{R};H)}^2 = \alpha^2 \left\| \sqrt{1 + (\partial_x v)^2} \right\|_H^2 = \alpha^2 \int_{\mathcal{O}} 1 + (\partial_x v)^2 dx, \quad \forall v \in H^1.$$

and

$$\|B(v)\|_{L_2(\mathbb{R};H_{per,0}^1)}^2 = \alpha^2 \left\| \partial_x \sqrt{1 + (\partial_x v)^2} \right\|_H^2$$
$$(3.6) \qquad\qquad = \alpha^2 \int_{\mathcal{O}} \frac{(\partial_x v)^2 (\partial_x^2 v)^2}{1 + (\partial_x v)^2} dx$$
$$\leq \alpha^2 \int_{\mathcal{O}} (\partial_x^2 v)^2 dx, \quad \forall v \in H^2.$$

Moreover,

$$(3.7) \quad \|B(v) - B(w)\|_{L_2(\mathbb{R};H)}^2 = \alpha^2 \left\| \sqrt{1 + (\partial_x v)^2} - \sqrt{1 + (\partial_x w)^2} \right\|_H^2$$
$$\leq \alpha^2 \int_{\mathcal{O}} (\partial_x v - \partial_x w)^2 dx, \quad \forall v, w \in H^1.$$

Some parts of the proof of Theorem 3.3 are analogous to the proof of Theorem 2.5. In this case we will restrict to short comments on the required modifications.

The proof proceeds by considering vanishing viscosity approximations and regularizations in the initial condition. We shall first consider the case $x_0 \in L^2(\Omega; H^1_{per})$ and

$$(3.8) \qquad dX^\varepsilon_t = \varepsilon \partial^2_x X^\varepsilon_t dt + (1 - \frac{\alpha^2}{2}) \frac{\partial^2_x X^\varepsilon_t}{1 + (\partial_x X^\varepsilon_t)^2} dt + \alpha \sqrt{1 + (\partial_x X^\varepsilon_t)^2} \circ d\beta_t$$

$$X^\varepsilon_0 = x_0,$$

for $\varepsilon \geq 0$. Correspondingly, we set

$$\varphi^\varepsilon(v) = \frac{\varepsilon}{2} \int |\partial_x v|^2 dx + \int \psi(\partial_x v) dx, \quad \text{for } v \in H^1_{per}.$$

The variational formulation of (3.8) is based on the Gelfand triple

$$H^1_{per} \hookrightarrow L^2 \hookrightarrow (H^1_{per})^*$$

and the variational operator

$$_{(H^1_{per})^*}\langle A^\varepsilon(v), w \rangle_{H^1_{per}} := -\varepsilon \int_{\mathcal{O}} \partial_x v \partial_x w \, dx - \int_{\mathcal{O}} \phi(\partial_x v) \partial_x w \, dx, \quad \text{for } v, w \in H^1_{per}.$$

By [PR07] there is a unique solution to (3.8) in the sense of a variational solution $X^\varepsilon \in L^2(\Omega; C([0,T]; H)) \cap L^2([0,T] \times \Omega; H^1_{per})$ to

$$(3.9) \qquad dX^\varepsilon_t = \varepsilon \partial^2_x X^\varepsilon_t dt + \frac{\alpha^2}{2} \partial^2_x X^\varepsilon_t dt + (1 - \frac{\alpha^2}{2}) \frac{\partial^2_x X^\varepsilon_t}{1 + (\partial_x X^\varepsilon_t)^2} dt$$

$$+ \alpha \sqrt{1 + (\partial_x X^\varepsilon_t)^2} d\beta_t.$$

Lemma 3.4. *For each $\varepsilon > 0$ we have $X^\varepsilon \in L^2([0,T] \times \Omega; H^2_{per})$ and*

$$\mathbb{E} \sup_{t \in [0,T]} e^{-Ct} \|X^\varepsilon_t\|^2_{H^1_{per}} + 2\varepsilon \int_0^t \mathbb{E} e^{-Cr} \|\partial^2_x X^\varepsilon_r\|^2_2 dr \leq C \left(\mathbb{E} \|x_0\|^2_{H^1_{per}} + 1 \right),$$

for some constant C independent of $\varepsilon > 0$.

Proof. As in Lemma 2.6 we may argue via Galerkin approximations X^n, where $(e_i)^\infty_{i=1}$ now is an orthonormal basis of the periodic Laplacian ∂^2_x on $L^2(\mathcal{O})$. First note

$$\frac{\alpha^2}{2} \partial^2_x v + (1 - \frac{\alpha^2}{2}) \frac{\partial^2_x v}{1 + (\partial_x v)^2} = \frac{\alpha^2}{2} \frac{\partial^2_x v (1 + (\partial_x v)^2)}{1 + (\partial_x v)^2} + (1 - \frac{\alpha^2}{2}) \frac{\partial^2_x v}{1 + (\partial_x v)^2}$$

$$= \frac{\alpha^2}{2} \frac{\partial^2_x v (\partial_x v)^2}{1 + (\partial_x v)^2} + \frac{\partial^2_x v}{1 + (\partial_x v)^2}.$$

Hence,

$$2(-\partial \varphi^\varepsilon(v), v)_{H^1_{per,0}} + \|B(v)\|^2_{H^1_{per,0}}$$

$$= - (2\varepsilon \partial^2_x v + \alpha^2 \partial^2_x v + (2 - \alpha^2) \frac{\partial^2_x v}{1 + (\partial_x v)^2}, \partial^2_x v)_2 + \|\partial_x \alpha \sqrt{1 + (\partial_x v)^2}\|^2_2$$

$$= - (2\varepsilon \partial^2_x v + \alpha^2 \frac{\partial^2_x v (\partial_x v)^2}{1 + (\partial_x v)^2} + 2 \frac{\partial^2_x v}{1 + (\partial_x v)^2}, \partial^2_x v)_2 + \|\partial_x \alpha \sqrt{1 + (\partial_x v)^2}\|^2_2$$

$$= - 2\varepsilon \int_{\mathcal{O}} (\partial^2_x v)^2 dx - 2 \int_{\mathcal{O}} \frac{(\partial^2_x v)^2}{1 + (\partial_x v)^2} dx - \alpha^2 \int_{\mathcal{O}} \frac{(\partial_x v)^2 (\partial^2_x v)^2}{1 + (\partial_x v)^2} dx + \alpha^2 \int_{\mathcal{O}} \frac{(\partial_x v)^2 (\partial^2_x v)^2}{1 + (\partial_x v)^2} dx$$

$$= - 2\varepsilon \int_{\mathcal{O}} (\partial^2_x v)^2 dx - 2 \int_{\mathcal{O}} \frac{(\partial^2_x v)^2}{1 + (\partial_x v)^2} dx$$

$$\leq - 2\varepsilon \int_{\mathcal{O}} (\partial^2_x v)^2 dx,$$

for all $v \in H^2_{per}$. Itô's formula thus implies

$$\|X^n_t\|^2_{H^1_{per,0}} = \|P^n x_0\|^2_{H^1_{per,0}}$$
$$+ 2\int_0^t (\varepsilon P^n \partial^2_x X^n_r + P^n \alpha^2 \partial^2_x X^n_r + (2 - \alpha^2)P^n \frac{\partial^2_x X^n_r}{1 + (\partial_x X^n_r)^2}, X^n_r)_{H^1_{per,0}} dr$$
$$+ 2\int_0^t (P^n B(X^n_r), X^n_r)_{H^1_{per,0}} dW_r + \int_0^t \|P^n B(X^n_r)\|^2_{L_2(\mathbb{R}, H^1_{per,0})} dr$$
$$\leq \|x_0\|^2_{H^1_{per,0}} - 2\varepsilon \int_0^t \|\partial^2_x X^n_r\|^2_2 dr + \int_0^t (B(X^n_r), X^n_r)_{H^1_{per,0}} dW_r.$$

Observing

$$\|X^n_t\|^2_H = \|P^n x_0\|^2_H$$
$$+ 2\int_0^t (\varepsilon P^n \partial^2_x X^n_r + P^n \alpha^2 \partial^2_x X^n_r + (2 - \alpha^2)P^n \frac{\partial^2_x X^n_r}{1 + (\partial_x X^n_r)^2}, X^n_r)_H dr$$
$$+ 2\int_0^t (P^n B(X^n_r), X^n_r)_H dW_r + \int_0^t \|P^n B(X^n_r)\|^2_{L_2(\mathbb{R};H)} dr$$
$$\leq \|x_0\|^2_H + \int_0^t (B(X^n_r), X^n_r)_H dW_r,$$

the proof may be completed as in Lemma 2.6. $\qquad\square$

Lemma 3.5. *For each $\varepsilon > 0$ we have $\varphi^\varepsilon(X^\varepsilon) \in L^1([0, T] \times \Omega)$ with*

$$\mathbb{E} \int_0^T e^{-Cr} \varphi^\varepsilon(X^\varepsilon_r) dr \leq C \left(\mathbb{E}\|x_0\|^2_H + 1 \right),$$

for some constant C independent of $\varepsilon > 0$.

Proof. By Itô's formula and (2.9) we have

$$\mathbb{E}e^{-Kt}\|X^\varepsilon_t\|^2_H = \mathbb{E}\|x_0\|^2_H + 2\mathbb{E} \int_0^t e^{-Kr}(\frac{\alpha^2}{2}\partial^2_x X^\varepsilon_r + A^\varepsilon(X^\varepsilon_r), X^\varepsilon_r)_H dr$$
$$+ 2\mathbb{E} \int_0^t e^{-Kr}\|B(X^\varepsilon_r)\|^2_{L_2(\mathbb{R},H)} dr - K \int_0^t e^{-Kr}\|X^\varepsilon_r\|^2_H dr$$
$$\leq \mathbb{E}\|x_0\|^2_H - 2\mathbb{E} \int_0^t e^{-Kr}\varphi^\varepsilon(X^\varepsilon_r) + Ce^{-Kr}\|X^\varepsilon_r\|^2_H dr$$
$$- K \int_0^t e^{-Kr}\|X^\varepsilon_r\|^2_H dr.$$

Choosing K large enough yields the claim. $\qquad\square$

Lemma 3.6. *Let $x_0 \in L^2(\Omega; H^1_{per})$. For all $\varepsilon > 0$ we have*

$$(3.10) \qquad \mathbb{E}t\varphi^\varepsilon(X^\varepsilon_t) + \mathbb{E} \int_0^t r\|\varepsilon\partial^2_x X^\varepsilon_r + \partial_x\phi(\partial_x X^\varepsilon_r)\|^2_H dr \leq C \left(\mathbb{E}\|x_0\|^2_H + 1 \right)$$

and

$$(3.11) \qquad \mathbb{E}\varphi^\varepsilon(X^\varepsilon_t) + \mathbb{E} \int_0^t \|\varepsilon\partial^2_x X^\varepsilon_r + \partial_x\phi(\partial_x X^\varepsilon_r)\|^2_H dr \leq \mathbb{E}\varphi^\varepsilon(x_0) + C,$$

for some constant $C > 0$.

Proof. We first prove (3.10): Let J^λ be the resolvent of $-\partial_x^2$ on $L^2(\mathcal{O})$ with domain $\mathcal{D}(-\partial_x^2) = H_{per}^2(\mathcal{O})$. As in the proof of Lemma 2.8 we obtain

$$\mathbb{E}t\varphi^{\varepsilon,\lambda}(X_t^\varepsilon)$$

$$= -\mathbb{E}\int_0^t r(\varepsilon\partial_x^2 J^\lambda X_r^\varepsilon + \partial_x\phi(\partial_x J^\lambda X_r^\varepsilon), \varepsilon J^\lambda \partial_x^2 X_r^\varepsilon + J^\lambda \partial_x\phi(\partial_x X_r^\varepsilon))_H dr$$

(3.12)
$$- \mathbb{E}\int_0^t r(\varepsilon\partial_x^2 J^\lambda X_r^\varepsilon + \partial_x\phi(\partial_x J^\lambda X_r^\varepsilon), \frac{\alpha^2}{2} J^\lambda \partial_x^2 X_r^\varepsilon)_H dr$$

$$+ \frac{\varepsilon\alpha^2}{2}\int_0^t r\int_\mathcal{O} \left(\partial_x J^\lambda\sqrt{1+(\partial_x X_r^\varepsilon)^2}\right)^2 dxdr$$

$$+ \frac{\alpha^2}{2}\int_0^t r\int_\mathcal{O} \dot{\phi}(\partial_x J^\lambda X_r^\varepsilon)\left(\partial_x J^\lambda\sqrt{1+(\partial_x X_r^\varepsilon)^2}\right)^2 dxdr$$

$$+ \mathbb{E}\int_0^t \varphi^{\varepsilon,\lambda}(X_r^\varepsilon)dr.$$

We first note that

$$\int_\mathcal{O}(\partial_x J^\lambda\sqrt{1+(\partial_x v)^2})^2 dx$$

$$\leq \int_\mathcal{O}(\partial_x\sqrt{1+(\partial_x v)^2})^2 dx$$

$$= \int_\mathcal{O}\frac{(\partial_x v)^2(\partial_x^2 v)^2}{1+(\partial_x v)^2}dx$$

$$\leq \|\partial_x^2 v\|_2^2,$$

for all $v \in H_{per}^2$. Moreover,

$$\dot{\phi}(\partial_x J^\lambda(v))(\partial_x J^\lambda\sqrt{1+(\partial_x v)^2})^2$$

$$= (1-\frac{\alpha^2}{2})\frac{(\partial_x J^\lambda\sqrt{1+(\partial_x v)^2})^2}{1+(\partial_x J^\lambda v)^2}$$

$$\leq (1-\frac{\alpha^2}{2})(\partial_x J^\lambda\sqrt{1+(\partial_x v)^2})^2,$$

for all $v \in H_{per}^2$. Since $r \mapsto \sqrt{1+r^2}$ is Lipschitz we have

$$\sqrt{1+(\partial_x v)^2} \in H_{per}^1$$

for $v \in H_{per}^2$ and thus (cf. [MR92, Theorem 2.13])

$$J^\lambda\sqrt{1+(\partial_x v)^2} \to \sqrt{1+(\partial_x v)^2} \quad \text{in } H_{per}^1,$$

for $\lambda \to 0$. Moreover, $J^\lambda v \to v$ in H_{per}^2 for $v \in H_{per}^2$. Thus, $\partial_x J^\lambda v \to \partial_x v$ in $H_{per}^1 \subseteq L^\infty$. Since $\dot{\phi}$ is Lipschitz we have $\dot{\phi}(\partial_x J^\lambda v) \to \dot{\phi}(\partial_x v)$ in $L^2([0,T]\times\Omega; L^\infty)$ for all $v \in L^2([0,T]\times\Omega; H_{per}^2)$. Hence,

$$\lim_\lambda \mathbb{E}\int_0^t r\int_\mathcal{O} \dot{\phi}(\partial_x J^\lambda v_r)\left(\partial_x J^\lambda\sqrt{1+(\partial_x v_r)^2}\right)^2 dxdr$$

$$= \mathbb{E}\int_0^t r\int_\mathcal{O}\frac{(\partial_x\sqrt{1+(\partial_x v_r)^2})^2}{1+(\partial_x v_r)^2}dxdr$$

$$= \mathbb{E}\int_0^t r\int_\mathcal{O}\frac{(\partial_x v_r)^2(\partial_x^2 v_r)^2}{(1+(\partial_x v_r)^2)^2}dxdr$$

$$\leq \mathbb{E}\int_0^t r\int_\mathcal{O}\frac{(\partial_x^2 v_r)^2}{1+(\partial_x v_r)^2}dxdr,$$

for all $v \in L^2([0,T] \times \Omega; H^2_{per})$. Taking the limit $\lambda \to 0$ in the first term on the right hand side of (3.12) can be justified as in Lemma 2.8 . This yields

$$\mathbb{E}t\varphi^\varepsilon(X^\varepsilon_t) \leq - \mathbb{E}\int_0^t r\|\varepsilon\partial_x^2 X^\varepsilon_r + \partial_x\phi(\partial_x X^\varepsilon_r)\|^2_H dr$$
$$- \frac{\varepsilon\alpha^2}{2}\mathbb{E}\int_0^t r\|\partial_x^2 X^\varepsilon_r\|^2_H dr - \frac{\alpha^2}{2}\mathbb{E}\int_0^t r\int_\mathcal{O} \frac{(\partial_x^2 X^\varepsilon_r)^2}{1+(\partial_x X^\varepsilon_r)^2}dxdr$$
$$+ \frac{\varepsilon\alpha^2}{2}\mathbb{E}\int_0^t r\|\partial_x^2 X^\varepsilon_r\|^2_H dr + \frac{\alpha^2}{2}\mathbb{E}\int_0^t r\int_\mathcal{O} \frac{(\partial_x^2 X^\varepsilon_r)^2}{1+(\partial_x X^\varepsilon_r)^2}dxdr$$
$$+ \mathbb{E}\int_0^t \varphi^\varepsilon(X^\varepsilon_r)dr$$
$$= - \mathbb{E}\int_0^t r\|\varepsilon\partial_x^2 X^\varepsilon_r + \partial_x\phi(\partial_x X^\varepsilon_r)\|^2_H dr + \mathbb{E}\int_0^t \varphi^\varepsilon(X^\varepsilon_r)dr.$$

By Lemma 3.5 we conclude

$$\mathbb{E}t\varphi^\varepsilon(X^\varepsilon_t) + \mathbb{E}\int_0^t r\|\varepsilon\partial_x^2 X^\varepsilon_r + \partial_x\phi(\partial_x X^\varepsilon_r)\|^2_H dr \leq C\left(\mathbb{E}\|x_0\|^2_H + 1\right).$$

To prove (3.11) we proceed as above but applying Itô's formula for $\varphi^{\varepsilon,\lambda}(X^\varepsilon_t)$ instead of $t\varphi^{\varepsilon,\lambda}(X^\varepsilon_t)$. $\qquad\square$

Proof of Theorem 3.3. **Step 1:** Existence

We start with the construction via an approximation of the initial condition. Let $x_0^n \in L^2(\Omega; H^1_{per})$ with $x_0^n \to x$ in $L^2(\Omega; H)$. By Lemma 3.4 there are strong solutions $X^{\varepsilon,n}$ to

$$dX_t^{\varepsilon,n} = \varepsilon\partial_x^2 X_t^{\varepsilon,n}dt + \frac{\alpha^2}{2}\partial_x^2 X_t^{\varepsilon,n}dt - \partial\varphi(X_t^{\varepsilon,n})dt + B(X_t^{\varepsilon,n})d\beta_t$$
$$X_t^{\varepsilon,n} = x_0^n,$$

satisfying

$$\mathbb{E}\sup_{t\in[0,T]}\|X_t^{\varepsilon,n}\|^2_{H^1_{per}} + \varepsilon\mathbb{E}\int_0^T \|\partial_x^2 X_r^{\varepsilon,n}\|^2_2 dr \leq C(\mathbb{E}\|x_0^n\|^2_{H^1_{per}} + 1).$$

We will first prove convergence of $X^{\varepsilon,n}$ for $\varepsilon \to 0$. For $\varepsilon_1, \varepsilon_2 > 0$ let $X^{\varepsilon_1}, X^{\varepsilon_2}$ be two solutions to (3.9) with initial conditions $x_0^1, x_0^2 \in L^2(\Omega; H^1_{per})$. Itô's formula, Lemma 3.4 and (3.7) imply

$$\mathbb{E}\|X_t^{\varepsilon_1} - X_t^{\varepsilon_2}\|^2_H$$
$$= \mathbb{E}\|x_0^1 - x_0^2\|^2_H$$
$$+ \mathbb{E}\int_0^t 2(\varepsilon_1\partial_x^2 X_r^{\varepsilon_1} - \varepsilon_2\partial_x^2 X_r^{\varepsilon_2}, X_r^{\varepsilon_1} - X_r^{\varepsilon_2})_2 dr$$
$$+ \frac{\alpha^2}{2}\mathbb{E}\int_0^t 2(\partial_x^2 X_r^{\varepsilon_1} - \partial_x^2 X_r^{\varepsilon_1}, X_r^{\varepsilon_1} - X_r^{\varepsilon_2})_2 dr$$
(3.13) $\quad + \mathbb{E}\int_0^t 2(\partial_x\phi(\partial_x X_r^{\varepsilon_1}) - \partial_x\phi(\partial_x X_r^{\varepsilon_2}), X_r^{\varepsilon_1} - X_r^{\varepsilon_2})_2 dr$
$$+ \mathbb{E}\int_0^t \|B(X_r^{\varepsilon_1}) - B(X_r^{\varepsilon_2})\|^2_{L_2(\mathbb{R},H)} dr$$
$$\leq \mathbb{E}\|x_0^1 - x_0^2\|^2_H + 4(\varepsilon_1 + \varepsilon_2)\mathbb{E}\int_0^t \|X_r^{\varepsilon_1}\|^2_{H^1_{per}} + \|X_r^{\varepsilon_2}\|^2_{H^1_{per}} dr$$
$$- \alpha^2\mathbb{E}\int_0^t \int_\mathcal{O} (\partial_x X_r^{\varepsilon_1} - \partial_x X_r^{\varepsilon_2})^2 dxdr + \alpha^2\mathbb{E}\int_0^t \int_\mathcal{O} (\partial_x X_r^{\varepsilon_1} - \partial_x X_r^{\varepsilon_2})^2 dxdr$$

$$\leq \mathbb{E}\|x_0^1 - x_0^2\|_H^2 + (\varepsilon_1 + \varepsilon_2)C\left(\mathbb{E}\|x_0^1\|_{H_{per}^1}^2 + \mathbb{E}\|x_0^2\|_{H_{per}^1}^2 + 1\right).$$

Hence,

$$X^{\varepsilon,n} \to X^n \quad \text{in } C([0,T]; L^2(\Omega; H)), \quad \text{for } \varepsilon \to 0$$

for some $\mathcal{F}_t$-adapted X^n. Due to (3.13) we obtain

$$\mathbb{E}\|X_t^n - X_t^m\|_H^2 \leq \mathbb{E}\|x_0^n - x_0^m\|_H^2$$

and thus

$$X^n \to X \quad \text{in } C([0,T]; L^2(\Omega; H)), \quad \text{for } n \to \infty$$

for some $\mathcal{F}_t$-adapted X. We shall now prove that X is an SVI solution to (3.1). By Lemma 3.6 we have

$$(3.14) \qquad \mathbb{E}t\varphi^\varepsilon(X_t^{\varepsilon,n}) + \mathbb{E}\int_0^t r\|\varepsilon\partial_x^2 X^{\varepsilon,n} + \partial_x\phi(\partial_x X^{\varepsilon,n})\|_H^2 dr \leq C\left(\mathbb{E}\|x_0^n\|_H^2 + 1\right)$$

and thus there is a function η and a sequence $\varepsilon_m \to 0$ such that for each $\tau > 0$

$$\varepsilon_m \partial_x^2 X^{\varepsilon_m,n} + \partial_x\phi(\partial_x X^{\varepsilon_m,n}) \rightharpoonup \eta^n, \quad \text{in } L^2([\tau,T] \times \Omega; H).$$

We can prove $\eta^n \in -\partial\varphi(X^n)$ as in the proof of Theorem 2.5. Taking the limit in (3.14) yields

$$\mathbb{E}t\varphi(X_t^n) + \mathbb{E}\int_0^t r\|\eta^n\|_H^2 dr \leq C\left(\mathbb{E}\|x_0^n\|_H^2 + 1\right).$$

We can now argue as above to obtain the existence of an $\eta \in -\partial\varphi(X)$ and a subsequence of η^n (again denoted by η^n for simplicity) such that

$$\eta^n \rightharpoonup \eta, \quad \text{in } L^2([\tau,T] \times \Omega; H),$$

for all $\tau > 0$. As in Remark 3.2 we have

$$\mathbb{E}\|X_t^{\varepsilon,n} - Z_t\|_H^2 \leq \mathbb{E}\|X_\tau^{\varepsilon,n} - Z_\tau\|_H^2 + 2\mathbb{E}\int_\tau^t (-\partial\varphi(X_r^{\varepsilon,n}) - G_r, X_r^{\varepsilon,n} - Z_r)_2 dr$$

$$+ \alpha^2\mathbb{E}\int_\tau^t (\partial_x^2 Z_r, X_r^{\varepsilon,n} - Z_r)_2 dr + 2\varepsilon\mathbb{E}\int_\tau^t (\partial_x^2 X_r^{\varepsilon,n}, X_r^{\varepsilon,n} - Z_r)_2 dr,$$

for all $t \geq \tau > 0$. Using Lemma 3.4 we observe

$$\varepsilon\mathbb{E}\int_\tau^t (\partial_x^2 X_r^{\varepsilon,n}, X_r^{\varepsilon,n} - Z_r)_2 dr \leq \varepsilon^{\frac{4}{3}}\mathbb{E}\int_\tau^t \|\partial_x^2 X_r^{\varepsilon,n}\|_2^2 dr + \varepsilon^{\frac{2}{3}}\mathbb{E}\int_\tau^t \|X_r^{\varepsilon,n} - Z_r\|_2^2 dr$$

$$\leq \varepsilon^{\frac{1}{3}}C(\mathbb{E}\|x_0^n\|_{H_{per}^1}^2 + 1) + \varepsilon^{\frac{2}{3}}\mathbb{E}\int_\tau^t \|X_r^{\varepsilon,n} - Z_r\|_2^2 dr.$$

We obtain

$$\mathbb{E}\|X_t^{\varepsilon,n} - Z_t\|_H^2 \leq \mathbb{E}\|X_\tau^{\varepsilon,n} - Z_\tau\|_H^2 + 2\mathbb{E}\int_\tau^t (-\partial\varphi(X_r^{\varepsilon,n}) - G_r, X_r^{\varepsilon,n} - Z_r)_2 dr$$

$$+ \alpha^2\mathbb{E}\int_\tau^t (\partial_x^2 Z_r, X_r^{\varepsilon,n} - Z_r)_2 dr$$

$$+ 2\varepsilon^{\frac{1}{3}}C(\mathbb{E}\|x_0^n\|_{H_{per}^1}^2 + 1) + 2\varepsilon^{\frac{2}{3}}\mathbb{E}\int_\tau^t \|X_r^{\varepsilon,n} - Z_r\|_2^2 dr.$$

Now we take $\varepsilon \to 0$ to obtain

$$\mathbb{E}\|X_t^n - Z_t\|_H^2 \leq \mathbb{E}\|X_\tau^n - Z_\tau\|_H^2 + 2\mathbb{E}\int_\tau^t (\eta_r^n - G_r, X_r^n - Z_r)_2 dr$$

$$+ \alpha^2\mathbb{E}\int_\tau^t (\partial_x^2 Z_r, X_r^n - Z_r)_2 dr.$$

Taking $n \to 0$ yields the claim.

Step 2: Uniqueness

Let X be an SVI solution to (3.1) with initial condition $x_0 \in L^2(\Omega; H)$. Further let $y_0 \in L^2(\Omega; H)$ and $y_0^n \in L^2(\Omega; H_{per}^1)$ with $y_0^n \to y$ in $L^2(\Omega; H)$. Due to Lemma 3.6 there are strong solutions $Y^{\varepsilon,n}$ to

$$dY_t^{\varepsilon,n} = \varepsilon \partial_x^2 Y_t^{\varepsilon,n} dt + \frac{\alpha^2}{2} \partial_x^2 Y_t^{\varepsilon,n} dt + (1 - \frac{\alpha^2}{2}) \frac{\partial_x^2 Y_t^{\varepsilon,n}}{1 + (\partial_x Y_t^{\varepsilon,n})^2} dt$$
$$+ \alpha \sqrt{1 + (\partial_x Y_t^{\varepsilon,n})^2} d\beta_t.$$
$$Y_0^{\varepsilon,n} = y_0^n,$$

satisfying

$$\mathbb{E} \sup_{t \in [0,T]} \|Y_t^{\varepsilon,n}\|_{H_{per}^1}^2 + \varepsilon \mathbb{E} \int_0^T \|\partial_x^2 Y_r^{\varepsilon,n}\|_2^2 dr \leq C(\mathbb{E}\|x_0^n\|_{H_{per}^1}^2 + 1).$$

Using the variational inequality with $Z \equiv Y^{\varepsilon,n}$ and

$$G = \varepsilon \partial_x^2 Y^{\varepsilon,n} + \frac{\alpha^2}{2} \partial_x^2 Y^{\varepsilon,n} + \partial_x \phi(\partial_x Y^{\varepsilon,n})$$

we obtain

$$\mathbb{E}\|X_t - Y_t^{\varepsilon,n}\|_H^2 \leq \mathbb{E}\|X_\tau - Y_\tau^{\varepsilon,n}\|_H^2$$
$$+ 2\mathbb{E} \int_\tau^t (\eta_r - \varepsilon \partial_x^2 Y_r^{\varepsilon,n} - \frac{\alpha^2}{2} \partial_x^2 Y_r^{\varepsilon,n} - \partial_x \phi(\partial_x Y_r^{\varepsilon,n}), X_r - Y_r^{\varepsilon,n})_2 dr$$
$$+ \alpha^2 \mathbb{E} \int_\tau^t (\partial_x^2 Y_r^{\varepsilon,n}, X_r - Y_r^{\varepsilon,n})_2 dr$$
$$= \mathbb{E}\|X_\tau - Y_\tau^{\varepsilon,n}\|_H^2$$
$$+ 2\mathbb{E} \int_\tau^t (\eta_r - \partial_x \phi(\partial_x Y_r^{\varepsilon,n}), X_r - Y_r^{\varepsilon,n})_2 dr$$
$$- \varepsilon 2\mathbb{E} \int_\tau^t (\partial_x^2 Y_r^{\varepsilon,n}, X_r - Y_r^{\varepsilon,n})_2 dr,$$

for all $t \geq \tau > 0$. Since $-\partial_x \phi(\partial_x Y^{\varepsilon,n}) = \partial \varphi(Y^{\varepsilon,n})$ and $\eta \in -\partial \varphi(X)$, $dt \otimes d\mathbb{P}$-a.e. we have

$$(\eta - \partial_x \phi(\partial_x Y^{\varepsilon,n}), X - Y^{\varepsilon,n})_2 = -(-\eta - \partial \varphi(Y^{\varepsilon,n}), X - Y^{\varepsilon,n})_2$$
$$\leq 0, \quad dt \otimes d\mathbb{P} - \text{a.e.}.$$

Hence,

$$\mathbb{E}\|X_t - Y_t^{\varepsilon,n}\|_H^2 \leq \mathbb{E}\|X_\tau - Y_\tau^{\varepsilon,n}\|_H^2 - 2\varepsilon \mathbb{E} \int_0^t (\partial_x^2 Y_r^{\varepsilon,n} dt, X_r - Y_r^{\varepsilon,n})_2 dr.$$

We further note

$$\varepsilon(\partial_x^2 Y_r^{\varepsilon,n}, X_r - Y_r^{\varepsilon,n}) \leq \varepsilon \|\partial_x^2 Y_r^{\varepsilon,n}\|_2 \|X_r - Y_r^{\varepsilon,n}\|_2$$
$$\leq \varepsilon^{\frac{4}{3}} \|\partial_x^2 Y_r^{\varepsilon,n}\|_2^2 + \varepsilon^{\frac{2}{3}} \|X_r - Y_r^{\varepsilon,n}\|_2^2.$$

Due to Lemma 3.6 this implies

$$2\varepsilon \mathbb{E} \int_0^t (\partial_x^2 Y_r^{\varepsilon,n}, X_r - Y_r^{\varepsilon,n})_2 dr \leq \varepsilon^{\frac{1}{4}} C(\mathbb{E}\|y_0^n\|_{H_{per}^1}^2 + 1) + 2\varepsilon^{\frac{2}{3}} \mathbb{E} \int_0^t \|X_r - Y_r^{\varepsilon,n}\|_2^2 dr.$$

Thus

$$\mathbb{E}\|X_t - Y_t^{\varepsilon,n}\|_H^2 \leq \mathbb{E}\|X_\tau - Y_\tau^{\varepsilon,n}\|_H^2$$
$$+ \varepsilon^{\frac{1}{4}} C(\mathbb{E}\|y_0^n\|_{H_{per}^1}^2 + 1) + 2\varepsilon^{\frac{2}{3}} \mathbb{E} \int_0^t \|X_r - Y_r^{\varepsilon,n}\|_2^2 dr.$$

For $n \in N$ arbitrary, fixed we have seen in step one

$$Y^{\varepsilon,n} \to Y^n \quad \text{in } C([0,T]; L^2(\Omega; H)) \quad \text{for } \varepsilon \to 0$$

and

$$Y^n \to Y \quad \text{in } C([0,T]; L^2(\Omega; H)), \quad \text{for } n \to \infty,$$

where Y is an SVI solution to (3.1) with initial condition y_0. We obtain

$$\sup_{t \in [0,T]} \mathbb{E}\|X_t - Y_t\|_H^2 \le \mathbb{E}\|X_\tau - Y_\tau\|_H^2.$$

Now letting $\tau \to 0$ yields

$$\sup_{t \in [0,T]} \mathbb{E}\|X_t - Y_t\|_H^2 \le \mathbb{E}\|x_0 - y_0\|_H^2.$$

In particular, choosing $y_0 = x_0$ implies uniqueness of SVI solutions.

Step 3: $x_0 \in L^2(\Omega; H)$ with $\mathbb{E}\varphi(x_0) < \infty$

As in (2.15) we may choose the approximations $x_0^n \in L^2(\Omega; H_0^1) \subseteq L^2(\Omega; H_{per}^1)$ of x_0 considered in step one such that

$$\mathbb{E}\varphi(x_0^n) + \mathbb{E}\|x_0^n\|_2^2 \le \mathbb{E}\varphi(x_0) + \mathbb{E}\|x_0\|_2^2 < \infty.$$

By Lemma 3.6 we then have

$$\mathbb{E}\varphi(X_t^{\varepsilon,n}) + \mathbb{E}\int_0^t \|\partial_x \phi(\partial_x X_r^{\varepsilon,n})\|_H^2 dr \le \mathbb{E}\varphi(x_0^n) + C,$$
$$\le \mathbb{E}\varphi(x_0) + C$$

and we follow the same arguments as in Step 1 to pass to the limit.   □

Appendix A. Relaxation of a linear growth functional with periodic boundary conditions

In this section we will prove that the functional

$$\varphi(v) := \begin{cases} \int_{\mathcal{O}} \psi(Dv)dx + \frac{1}{2}\int_{\partial \mathcal{O}} |[v - v^\perp]|H^{d-1}(dx) & \text{if } v \in L^2 \cap BV \\ +\infty & \text{if } v \in L^2 \setminus BV, \end{cases}$$

where $\int_{\mathcal{O}} \psi(Dv)dx$ is defined as in Section 2 is the lower-semicontinuous hull on L^2 of its restriction to $W_{per}^{1,1}(\mathcal{O})$, where $\mathcal{O} = (0,1)$ (i.e. $d = 1$). The arguments closely follow those from [Anz83, Fact 3.3] for the case of (inhomogeneous) Dirichlet boundary conditions.

Lemma A.1. *For all $u \in BV \cap L^2$ there exists a sequence of functions $u_j \in C^1 \cap W_{per}^{1,1} \cap L^2$ such that*

$$u_j \to u \quad \text{in } L^2$$

and

$$\varphi(u_j) \to \varphi(u).$$

Proof. Let $v_j \in C^1 \cap W^{1,1} \cap L^2$ be a sequence of functions satisfying

$$v_j \to u \quad \text{in } L^2$$

$$\int_{\mathcal{O}} |Dv_j|dx \to \int_{\mathcal{O}} |Du|dx \quad \text{for } j \to \infty$$

and $v_j = u$ on $\partial\mathcal{O}$ (cf. [ABM06, Theorem 10.1.2 and Remark 10.2.1]). We further define cut-off functions $w_j \in C^1 \cap W^{1,1} \cap L^2$

$$w_j|_{\partial\mathcal{O}} = \frac{u^\perp - u}{2}|_{\partial\mathcal{O}},$$

$$w_j(x) = 0, \quad \forall dist(x, \partial\mathcal{O}) > \frac{1}{j},$$

$$\int_\mathcal{O} |Dw_j| dx \le \int_{\partial\mathcal{O}} |\frac{u^\perp - u}{2}| H^{d-1}(dx) + \frac{1}{j},$$

$$\int_\mathcal{O} |w_j|^2 dx \le \frac{C}{j}.$$

Let $u_j = v_j + w_j$. Then $u_j(1) = u_j(0) = \frac{u(0)+u(1)}{2}$, in particular $u_j \in C^1 \cap W^{1,1}_{per} \cap L^2$. Moreover,

$$u_j \to u, \quad \text{in } L^2$$

$$\int_\mathcal{O} \sqrt{1 + |Du_j|^2} dx \to \int_\mathcal{O} \sqrt{1 + |Du|^2} dx + \int_{\partial\mathcal{O}} |\frac{u^\perp - u}{2}| H^{d-1}(dx).$$

We then complete the proof precisely as in [Anz83, Fact 3.3]. $\qquad\square$

Lemma A.2. *For every $u \in L^2$ and every sequence $u_j \in BV \cap L^2$ with $u_j \to u$ in L^2 we have*

$$\liminf_j \varphi(u_j) \ge \varphi(u).$$

Proof. The proof is the same as for [Anz83, Fact 3.4]. $\qquad\square$

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