On the uniqueness of solutions to continuity equations

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Abstract

We obtain sufficient conditions for the uniqueness of solutions to the Cauchy problem for the continuity equation in classes of measures that need not be absolutely continuous.

Keywords: Continuity equation, Cauchy problem, Uniqueness

Introduction

In this paper we study the uniqueness problem for the continuity equation

$$\partial_t \mu_t + \operatorname{div}(b\mu_t) = 0$$

with respect to measures on $\mathbb{R}^d$. We consider solutions given by families of locally bounded Borel measures $(\mu_t)_{t\in[0,T]}$. A precise definition is given below.

There is a vast literature devoted to uniqueness and existence problems for the Cauchy problem for such equations. An important problem is to specify a class of measures μ_t in which, under reasonable assumptions about coefficients and initial data, there is a unique solution to the Cauchy problem. Certainly, if the coefficients are sufficiently regular, say, Lipschitzian or satisfy the Osgod type condition, then we can take the whole class of bounded measures μ_t (see, e.g., [4]). According to a well-known result of Ambrosio [3] (see also [29] for equations with a potential term) on representations of nonnegative bounded solutions by means of averaging with respect to measures concentrated on solutions to the corresponding ordinary equation $\dot{x} = b(x,t)$, any uniqueness condition for the ordinary equation guarantees uniqueness in the class of nonnegative bounded measures. However, in the class of signed measures there is no such representation.

In the case of non smooth coefficients a class convenient in many respects is the class of measures absolutely continuous with respect to Lebesgue measure, which is quite natural, in particular, taking into account existence results. A study of this class initiated by Cruzeiro [17], [18] and DiPerna and Lions [24] was continued by many researchers. A large number of papers are devoted to the so-called Lagrangian flows and their generalizations (see [2] and [3]). However, this class of absolutely continuous measures is rather narrow, in particular, it does not enable one to deal with singular initial data (and is essentially oriented towards vector fields having at least some minimal regularity such as the existence of their divergence or being BV). In addition, this class has no universal analogs in infinite dimensions. There are several papers (see [17], [32], [10], [34], [2], [20], and [27]) concerned with the infinite-dimensional case and using as reference measures certain special measures (all reducing to the absolute continuity with respect to Lebesgue measure when the infinite-dimensional state space is replaced by $\mathbb{R}^d$) such as Gaussian measures, convex measures, and differentiable measures, which becomes rather restrictive in infinite dimensions in spite of importance of such classes of measures in applications (e.g., when they are Gibbs measures) . The recent paper [7] develops continuity equations in metric measure spaces, but again only considering solutions absolutely continuous with respect to the underlying fixed measure.

Thus, it is natural to look for other classes of measures, apart from absolutely continuous measures, in which the existence and uniqueness of solutions hold in the case of non-smooth coefficients. In this paper we consider the finite-dimensional case; it turns out that even in the one-dimensional case in the present framework new results can be obtained.

The main result in this direction obtained in our paper can be briefly formulated as follows: uniqueness holds in a certain class of measures with respect to which the given vector field b can be suitably

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approximated by smooth vector fields (thus, the uniqueness class may depend on b). A precise formulation is given below (see Theorems 1.1 and 1.4), but we observe that this result is consistent with typical methods of constructing solutions when b is approximated by smooth fields b_k and the solution is obtained as a limit point of the sequence of solutions μ_t^k for b_k. As in many existing papers, our conditions admit discontinuous fields, but the hypotheses are mostly incomparable (with the already cited papers and, e.g., [12], [13], [16], [21], [22], [23], and [25]).

As an application we obtain some new results for the continuity equation with a merely continuous coefficient b. In particular, we substantially improve the recent result from [15], where the uniqueness is proved in dimension one for absolutely continuous solutions under the assumptions that b is continuous and nonnegative, the trajectories of $\dot{x} = b(x)$ do not blow up in finite time and the set of zeros $Z = \{x\colon b(x) = 0\}$ consists of a finite union of points and closed intervals. For example, we prove the following assertion: if $b \in C(\mathbb{R})$ and $0 \le b(x) \le C + C|x|$ (which is a constructive condition to exclude a blowup), then the uniqueness holds in the class of all locally bounded (possibly signed) measures μ on $\mathbb{R}^d \times [0,T]$ given by families of locally bounded measures $(\mu_t)_{t\in[0,T]}$ such that

$$|\mu_t|(\partial Z) = 0 \quad \text{for almost all} \quad t \in [0,T],$$

where ∂Z is the boundary of the set $Z = \{x\colon\ b(x) = 0\}$. So there are no structure restrictions on the zero set of b and no assumption that μ has a density. Moreover, we consider the multidimensional case. Finally, we prove an existence result which produces solutions with our uniqueness properties.

1. Main result

Let us consider the Cauchy problem

$$\partial_t\mu + \mathrm{div}(b\mu) = 0, \quad \mu|_{t=0} = \nu, \tag{1.1}$$

where ν is a locally bounded Borel measure on $\mathbb{R}^d$, i.e., a real function on the class of all bounded Borel sets in $\mathbb{R}^d$ that is countably additive on the class of Borel subsets of every compact set. Locally bounded Borel measures on $\mathbb{R}^d \times [0,T]$ are defined similarly. In particular, for every compact set $K \subset \mathbb{R}^d \times [0,T]$, the total variation of a locally bounded Borel measure μ on K (denoted by $|\mu|(K)$) is finite. A Borel measure is bounded if it has a finite total variation.

We say that a locally bounded Borel measure μ on $\mathbb{R}^d \times [0,T]$ is given by a family of Borel locally bounded measures $(\mu_t)_{t\in[0,T]}$ on $\mathbb{R}^d$ if, for every bounded Borel set $B \subset \mathbb{R}^d$, the mapping $t \mapsto \mu_t(B)$ is measurable, $|\mu_t|(K) \in L^1[0,T]$ for every compact $K \subset \mathbb{R}^d$ and

$$\int_0^T \int_{\mathbb{R}^d} u(x,t)\,\mu(dxdt) = \int_0^T \int_{\mathbb{R}^d} u(x,t)\,\mu_t(dx)\,dt \quad \forall\, u \in C_0^\infty(\mathbb{R}^d \times (0,T)).$$

Clearly, the previous equality extends to bounded Borel measurable functions u that vanish for x outside a ball. In the considered framework there is no difference between measures on $\mathbb{R}^d \times [0,T]$ and on $\mathbb{R}^d \times (0,T)$, because we study only measures of the above form represented by families of measures on $\mathbb{R}^d$ via Lebesgue measure. Not every measure on $\mathbb{R}^d \times [0,T]$ has this property, of course.

A Borel locally bounded measure μ on $\mathbb{R}^d \times [0,T]$ given by a family of locally bounded measures $(\mu_t)_{t\in[0,T]}$ is a solution to the Cauchy problem (1.1) if $b \in L^1(|\mu|, U \times [0,T])$ for every ball $U \subset \mathbb{R}^d$ and, for every function u in the class $C^{1,1}(\mathbb{R}^d \times [0,T))$ (consisting of functions that are continuous on $\mathbb{R}^d \times [0,T)$ along with their first order derivatives in t and x) such that $u(x,t) \equiv 0$ if $|x| > R$ for some R, one has

$$\int_{\mathbb{R}^d} u(x,t)\,\mu_t(dx) = \int_{\mathbb{R}^d} u(x,0)\,\nu(dx) + \int_0^t \int_{\mathbb{R}^d} \Big[\partial_t u + \langle b, \nabla u\rangle\Big]\,\mu_s(dx)\,ds \tag{1.2}$$

for almost every $t \in [0,T]$. Note that the integrals exist for almost all t.

Let us observe that it is possible to pick a common full measure set $S \subset [0,T]$ such that (1.2) will hold for each $t \in S$ for all admissible functions u. Indeed, such a set exists for a countable collection of functions u, so it is sufficient to choose this countable collection in such a way that for any function u in the indicated class there is a sequence $\{u_n\}$ in this collection that is uniformly bounded along with the derivatives in t and x and $u_n \to u$, $\partial_t u_n \to \partial_t u$, $\nabla u_n \to \nabla u$ pointwise.

Theorem 1.1. *Let $\mu^1 = \mu^1(dx)\,dt$ and $\mu^2 = \mu^2(dx)\,dt$ be two solutions to the Cauchy problem* (1.1)*. Assume that the measure $\mu = \mu_t(dx)\,dt$, where $\mu_t = \mu_t^1 - \mu_t^2$, satisfies the following conditions: for some number $C_1 \geq 0$ and for every ball $U \subset \mathbb{R}^d$ one can find a number $C_2 \geq 0$, a sequence of vector fields $b_k \in C^\infty(\mathbb{R}^d \times \mathbb{R}^1, \mathbb{R}^d)$ and a sequence of positive functions $V_k \in C^1(\mathbb{R}^d)$ (all depending on the considered measure μ, the constant C_1 and the ball U) such that*

(i) $|b_k(x,t)| \leq C_1 + C_1|x|$,

(ii) $\inf_k \inf_U V_k(x) > 0$ *and*

$$\langle b_k(x,t), \nabla V_k(x)\rangle \leq \Big(C_2 - 2\max_{|\xi|=1}\langle \mathcal{B}_k(x,t)\xi, \xi\rangle\Big) V_k(x)$$

for every k and $(x,t) \in U \times [0,T]$, where $\mathcal{B}_k = (\partial_{x_i} b_k^j)_{i,j\leq d}$,

(iii) $\lim\limits_{k\to\infty} \|(b_k - b)\sqrt{V_k}\|_{L^1(|\mu_t|\,dt,\, U\times[0,T])} = 0$.

Then $\mu = 0$, i.e., $\mu^1 = \mu^2$.

Note that the equality $\mu^1 = \mu^2$ is equivalent to the equality $\mu_t^1 = \mu_t^2$ for almost every $t \in [0,T]$; changing μ_t for t in a measure zero set we do not change the solution. However, under broad assumptions about b, one can find a version of $t \mapsto \mu_t$ that is continuous in the sense of generalized functions (or even weakly continuous in the case of probability measures), and then such versions are uniquely defined. Moreover, in that case (1.2) holds poitwise and, as $t \to 0$, the measures μ_t converge to the initial measure ν in the respective sense (as distributions or weakly). This can be done by taking in (1.2) functions u independent of t (see [8, Lemma 2.1]).

So in order to find a uniqueness class for the Cauchy problem (1.1) one should find approximations of b satisfying conditions (i), (ii) and describe measures satisfying condition (iii).

Let us consider the standard example of an ordinary differential equation on the real line without uniqueness and see what kind of uniqueness for the associated continuity equation is offered by our theorem.

Example 1.2. Let $d = 1$ and $b(x) = \sqrt{|x|}$. Set $b_k(x) = (x^2 + k^{-2})^{1/4}$. It is obvious that

$$|b(x) - b_k(x)| \leq k^{-1/2}.$$

Let us calculate $b_k'(x)$:

$$b_k'(x) = \frac{x}{2(x^2 + k^{-2})^{3/4}}.$$

Finally, we take

$$V_k(x) = \frac{1}{b_k(x)^2} = \frac{1}{(x^2 + k^{-2})^{1/2}}.$$

Clearly, (ii) is fulfilled. Moreover, we have

$$g_k(x) := |b(x) - b_k(x)|\sqrt{V_k(x)} = |b(x) - b_k(x)| b_k(x)^{-1}.$$

Note that $g_k(x) = 1$ if $b(x) = 0$, i.e., $x = 0$. If $x \neq 0$, then

$$g_k(x) \leq \frac{k^{-1/2}}{|x|^{1/2}} \to 0 \quad \text{if} \quad k \to \infty.$$

Thus $|g_k| \leq 1$ and $g_k \to I_{\{0\}}$, where $I_{\{0\}}$ is the indicator of the set $\{b = 0\} = \{0\}$. Let $\mu = \mu_t\,dt$ be a locally bounded measure on $\mathbb{R}^1 \times [0,T]$. According to the Lebesgue dominated convergence theorem, one has

$$\lim_{k\to\infty} \int_0^T \int_U |b - b_k|\sqrt{V_k}\, d|\mu_t|\, dt = \int_0^T |\mu_t|(\{x \in U\colon\ b(x) = 0\})\, dt$$

for every interval U. Hence the uniqueness holds in the class of all locally bounded (possibly, signed) solutions $\mu = \mu_t\,dt$ such that $|\mu_t|(\{0\}) = 0$ for almost all $t \in [0,T]$.

It is interesting that this result is sharp: there exist two solutions $\mu_t^1 \equiv \delta_0$ and $\mu_t^2 = \delta_{t^2/4}$ to the Cauchy problem with $b(x) = \sqrt{|x|}$ and $\nu = \delta_0$. Note that only μ_t^2 belongs to our uniqueness class and that this solution is not absolutely continuous.

The proof of Theorem 1.1 is based on the maximum principle and Holmgren's principle.

Lemma 1.3. *Suppose that* $h \in C^\infty(\mathbb{R}^d \times \mathbb{R}^1, \mathbb{R}^d)$, $|h(x,t)| \le C_1 + C_1|x|$ *for some number* $C_1 > 0$ *and for all* $(x,t) \in \mathbb{R}^d \times [0,T]$. *Assume also that for some positive function* $V \in C^1(\mathbb{R}^d)$, *and number* $C_2 > 0$ *one has*

$$\langle h(x,t), \nabla V(x)\rangle \le \Bigl(C_2 - 2\sup_{|\xi|=1}\langle \mathcal{H}(x,t)\xi,\xi\rangle\Bigr)V(x),$$

for all $(x,t) \in \mathbb{R}^d \times [-1,T]$, *where* $\mathcal{H}(x,t) = (\partial_{x_i} h^j(x,t))_{1\le i,j\le d}$.

Then, for any $s \in (0,T)$, *the Cauchy problem*

$$\partial_t f + \langle h, \nabla f\rangle = 0, \quad f|_{t=s} = \psi,$$

where $\psi \in C_0^\infty(\mathbb{R}^d)$, *has a smooth solution* f *on* $\mathbb{R}^d \times (-1,s]$ *such that*

$$|f(x,t)| \le \max_y |\psi(y)|, \quad |\nabla f(x,t)|^2 \le V(x)e^{C_2(s-t)}\max_y \frac{|\nabla\psi(y)|^2}{V(y)}, \quad t \in [0,s].$$

Moreover, $f(x,t) = 0$ *if* $|x| > R$ *for some number* $R = R(s,\psi,C_1) > 0$.

Proof. The solution f is given by the equality $f(x_0,t_0) = \psi(x(s))$, where $x(\cdot)$ is a solution to the ordinary equation $\dot{x}(t) = h(x(t),t)$, $x(t_0) = x_0$. Note that

$$\frac{d}{dt}|x(t)|^2 = 2\langle h(x(t),t), x(t)\rangle \ge -C_1 - 3C_1|x(t)|^2.$$

Hence

$$|x(t)|^2 \ge |x_0|^2 e^{3C_1(t_0-t)} + C_1(t_0-t)e^{-3C_1 t} \ge |x_0|^2 e^{-3C_1 s} - C_1 s.$$

Assume that $\psi(x) = 0$ if $|x| > r$. Then there exists a number $R = R(s,r,C_1)$ such that

$$|x_0|^2 e^{-3C_1 s} - C_1 s \ge r \quad \text{if} \quad |x_0| \ge R.$$

Therefore, $|x(s)| \ge r$ and $f(x_0,t_0) = 0$.

Let us prove the announced gradient estimate. Set $u = 2^{-1}\sum_{k=1}^d |\partial_{x_k} f|^2$. Differentiating the equation $\partial_t f + \langle h, \nabla f\rangle = 0$ with respect to x_k and multiplying by $\partial_{x_k} f$ we find that

$$\partial_t u + \langle h, \nabla u\rangle + \langle \mathcal{H}\nabla f, \nabla f\rangle = 0.$$

Since $\langle \mathcal{H}\nabla f, \nabla f\rangle \le 2u\sup_{|\xi|=1}\langle \mathcal{H}(x,t)\xi,\xi\rangle$, we have

$$\partial_t u + \langle h, \nabla u\rangle + 2u\sup_{|\xi|=1}\langle \mathcal{H}\xi,\xi\rangle \ge 0.$$

Set $u = wV$. Then

$$\partial_t w + \langle h, \nabla w\rangle + qw \ge 0,$$

where

$$q = 2\sup_{|\xi|=1}\langle \mathcal{H}\xi,\xi\rangle + \langle h, \nabla V\rangle V^{-1} \le C_2.$$

Note that $w(x,s) \le \max_y |\nabla\psi(y)|^2/V(y)$. Then the maximum principle (see [33, Theorem 3.1.1]) yields that $w(x,t) \le e^{C_2(s-t)}\max_y |\nabla\psi(y)|^2/V(y)$, which completes the proof. □

Proof of Theorem 1.1. Let us fix $\psi \in C_0^\infty(\mathbb{R}^d)$ and s in the full measure set $S \subset (0,T)$ mentioned before the theorem (such that (1.2) holds for all admissible u and each $t \in S$). Let $R = R(s,\psi,C_1)$ be a number from Lemma 1.3. According to the hypotheses of the theorem, for the ball $\{x\colon |x| < 2R\}$ there exist sequences b_k and V_k satisfying all conditions (i)–(iii).

According to Lemma 1.3 applied with $h = b_k$ there exists a smooth solution f to the Cauchy problem

$$\partial_t f + \langle b_k, \nabla f\rangle = 0, \quad f|_{t=s} = \psi$$

satisfying the estimate

$$|f(x,t)| \le \max_y |\psi(y)|, \quad |\nabla f(x,t)|^2 \le V_k(x)e^{C_2(s-t)}\max_y \frac{|\nabla\psi(y)|^2}{2V_k(y)}, \quad (x,t) \in \mathbb{R}^d \times [0,s].$$

Moreover, $f(x,t) = 0$ if $|x| > R$. Certainly, f depends on several parameters (k, s, ψ, etc.), which is suppressed in our notation.

Let U be a ball containing the support of ψ. By our assumptions, $C(U)=\inf_k \inf_U V_k(x)>0$, hence it follows that

$$|\nabla f(x,t)|\le (2C(U))^{-1}e^{C_2(s-t)/2}\sqrt{V_k(x)}\max|\nabla\psi|.$$

Substituting the function $u=f$ in (1.2) for the solution $\mu_t=\mu_t^1-\mu_t^2$, we arrive at the following equality:

$$\int_{\mathbb{R}^d}\psi\,d\mu_s=\int_0^s\int_{\mathbb{R}^d}\langle b-b_k,\nabla f\rangle\,d\mu_t\,dt.$$

Hence

$$\int_{\mathbb{R}^d}\psi\,d\mu_s\le \widetilde{C}\int_0^s\int_{|x|<2R}|b-b_k|\sqrt{V_k(x)}\,d|\mu_t|\,dt,$$

where $\widetilde{C}=(2C(U))^{-1}e^{C_2T/2}\max|\nabla\psi|$ does not depend on k. Letting $k\to\infty$, we conclude that

$$\int_{\mathbb{R}^d}\psi\,d\mu_s\le 0.$$

Recall that ψ was an arbitrary function in $C_0^\infty(\mathbb{R}^d)$. Then $\mu_s=\mu_s^1-\mu_s^2=0$ for almost all s. □

We also give a result without restrictions on the growth of b_k, but with some additional conditions on the solution.

Theorem 1.4. *Let $\mu^1=\mu^1(dx)\,dt$ and $\mu^2=\mu^2(dx)\,dt$ be two solutions to the Cauchy problem* (1.1). *Assume that the measure $\mu=\mu_t(dx)\,dt$, where $\mu_t=\mu_t^1-\mu_t^2$, satisfies the following condition*:

(i) $\displaystyle\lim_{N\to\infty}N^{-1}\int_0^T\int_{N<|x|<2N}|b(x,t)|\,|\mu_t|(dx)\,dt=0,$

(ii) *for every ball $U\subset\mathbb{R}^d$, one can find numbers $C>0$ and $\delta>0$, a sequence of vector fields $b_k\in C^\infty(U\times\mathbb{R}^1,\mathbb{R}^d)$ and a sequence of positive functions $V_k\in C^1(U)$* (*all depending on the considered measure μ and the ball U*) *such that*

$$\inf_k\inf_U V_k(x)>0,$$

$$\langle b_k(x,t),\nabla V_k(x)\rangle\le\Bigl(C-\delta|b_k(x,t)|^2-2\max_{|\xi|=1}\langle\mathcal{B}_k(x,t)\xi,\xi\rangle\Bigr)V_k(x)$$

for every k and all $(x,t)\in U\times[0,T]$, where $\mathcal{B}_k=(\partial_{x_i}b_k^j)_{i,j\le d}$,

(iii) $\displaystyle\lim_{k\to\infty}\|(b_k-b)\sqrt{V_k}\|_{L^1(|\mu_t|\,dt,\,U\times[0,T])}=0.$

Then $\mu=0$, i.e., $\mu^1=\mu^2$.

Remark 1.5. Note that if the sequence of functions $|b_k|$ on $U\times[0,T]$ is uniformly bounded, by making δ smaller and C larger we can restate the second estimate in (i) as

$$\langle b_k(x,t),\nabla V_k(x)\rangle\le\Bigl(C-2\max_{|\xi|=1}\langle\mathcal{B}_k(x,t)\xi,\xi\rangle\Bigr)V_k(x).$$

It is worth noting that V_k looks like a so-called Lyapunov function, but there is an important difference: its sublevel sets need not be compact.

As above, the proof of Theorem 1.4 is based on the maximum principle and Holmgren's principle.

Let $\zeta\in C_0^\infty(\mathbb{R}^d)$ be such that $0\le\zeta\le 1$ and there exists a number $C(\zeta)>0$ such that $|\nabla\zeta(x)|^2\zeta^{-1}(x)\le C(\zeta)$ for every x. Let U be a ball containing the support of ζ.

We need an analog of Lemma 1.3.

Lemma 1.6. *Suppose that $h\in C^\infty(\mathbb{R}^d\times\mathbb{R}^1,\mathbb{R}^d)$ and that for some positive function $V\in C^1(\mathbb{R}^d)$ and numbers $C>0$, $\delta>0$ one has*

$$\langle h(x,t),\nabla V(x)\rangle\le\Bigl(C-2\sup_{|\xi|=1}\langle\mathcal{H}(x,t)\xi,\xi\rangle-\delta|h(x,t)|^2\Bigr)V(x),$$

for all $(x,t)\in U\times[0,T]$, where $\mathcal{H}(x,t)=(\partial_{x_i}h^j(x,t))_{1\le i,j\le d}$.

Then, for any $s\in(0,T)$, the Cauchy problem

$$\partial_t f+\zeta\langle h,\nabla f\rangle=0,\quad f|_{t=s}=\psi,$$

where $\psi \in C_0^\infty(\mathbb{R}^d)$, has a smooth solution f such that

$$|f(x,t)| \le \max_y |\psi(y)|, \quad |\nabla f(x,t)|^2 \le V(x) e^{M(s-t)} \max_y \frac{|\nabla\psi(y)|^2}{V(y)}, \quad t\in[0,s],$$

where $M = C + \delta^{-1}C(\zeta)$.

Proof. The reasoning is similar to that of Lemma 1.3, but we explain the necessary changes. The existence of a smooth bounded solution with bounded derivatives is well known. Set $u = 2^{-1}\sum_{k=1}^d |\partial_{x_k} f|^2$. Differentiating the equation $\partial_t f + \zeta\langle h, \nabla f\rangle = 0$ with respect to x_k and multiplying by $\partial_{x_k} f$ we obtain

$$\partial_t u + \zeta\langle h, \nabla u\rangle + \zeta\langle \mathcal{H}\nabla f, \nabla f\rangle + \langle \nabla\zeta, \nabla f\rangle\langle h, \nabla f\rangle = 0.$$

Note that

$$\langle \mathcal{H}\nabla f, \nabla f\rangle \le 2u \sup_{|\xi|=1}\langle \mathcal{H}(x,t)\xi,\xi\rangle, \quad \langle\nabla\zeta,\nabla f\rangle\langle h,\nabla f\rangle \le 2|\nabla\zeta||h|u.$$

Since

$$2|\nabla\zeta||h| \le \delta^{-1}\frac{|\nabla\zeta|^2}{\zeta_M} + \delta\zeta|h|^2 \le \delta^{-1}C(\zeta) + \delta\zeta|h|^2,$$

we have

$$\partial_t u + \zeta\langle h, \nabla u\rangle + u\Bigl(2\zeta \sup_{|\xi|=1}\langle\mathcal{H}\xi,\xi\rangle + \delta\zeta|h|^2 + \delta^{-1}C(\zeta)\Bigr) \ge 0.$$

Set $u = wV$. Then

$$\partial_t w + \zeta\langle h, \nabla w\rangle + qw \ge 0,$$

where

$$q = \zeta\Bigl(2\sup_{|\xi|=1}\langle\mathcal{H}\xi,\xi\rangle + \delta|h|^2 + \langle h, \nabla V\rangle V^{-1}\Bigr) + \delta^{-1}C(\zeta).$$

By our assumptions $q \le C + \delta^{-1}C(\zeta) = M$. Note that $w(x,s) \le \max_x |\nabla\psi(x)|^2/V(x)$. Then again the maximum principle (see [33, Theorem 3.1.1]) yields that $w(x,t) \le e^{M(s-t)}\max_y |\nabla\psi(y)|^2/V(y)$, which completes the proof. □

Proof of Theorem 1.4. Let us fix a function $\varphi \in C_0^\infty(\mathbb{R}^d)$ such that $0 \le \varphi \le 1$, $\varphi(x) = 1$ if $|x| < 1$ and $\varphi(x) = 0$ if $|x| > 2$. Assume also that $|\nabla\varphi(x)|^2\varphi(x)^{-1} \le C(\varphi)$ for every x. Set $\varphi_{4N}(x) = \varphi(x/N)$. For every ball $U_{4N} = \{x\colon |x| < 4N\}$, there exist numbers $C > 0$, $\delta > 0$ and sequences $\{b_k\}$, $\{V_k\}$ such that all conditions (ii) and (iii) are fulfilled on U_{4N}. Let us fix $\psi \in C_0^\infty(\mathbb{R}^d)$ and $s \in (0,T)$ in the same full measure set S as in the proof of Theorem 1.1. According to Lemma 1.6 there exists a smooth solution f to the Cauchy problem

$$\partial_t f + \varphi_{2N}\langle b_k, \nabla f\rangle = 0, \quad f|_{t=s} = \psi$$

satisfying the estimate

$$|f(x,t)| \le \max_y |\psi(y)|, \quad |\nabla f(x,t)|^2 \le V_k(x)e^{M(s-t)}\max_y \frac{|\nabla\psi(y)|^2}{2V_k(y)},$$

where $M = C + \delta^{-1}C(\varphi_{2N})$. Certainly, f depends on several parameters (k, s, N, ψ, etc.), which is suppressed in our notation.

Let U be a ball containing the support of ψ. By our assumptions, $C(U) = \inf_k \inf_U V_k(x) > 0$, hence it follows that

$$|\nabla f(x,t)| \le (2C(U))^{-1}e^{M(s-t)/2}\sqrt{V_k(x)}\max|\nabla\psi|.$$

Substituting the function $u = f\varphi_N$ in (1.2) for the solution $\mu_t = \mu_t^1 - \mu_t^2$ and noting that $\varphi_{2N}(x) = 1$ if $\varphi_N(x) \ne 0$, we arrive at the following equality:

$$\int_{\mathbb{R}^d}\psi\,d\mu_s = \int_0^s\int_{\mathbb{R}^d}\Bigl[\varphi_N\langle b - b_k, \nabla f\rangle + \langle b, \nabla\varphi_N\rangle f\Bigr]\,d\mu_t\,dt.$$

Then we obtain

$$\int_{\mathbb{R}^d}\psi\,d\mu_s \le \int_0^s\int_{|x|<2N}\bigl[\widetilde{C}\varphi_N|b-b_k|\sqrt{V_k(x)} + |b||\nabla\varphi_N|\bigr]\,d|\mu_t|\,dt,$$

where $\widetilde{C} = (2C(U))^{-1}e^{MT/2}\max|\nabla\psi|$ does not depend on k. Letting $k \to \infty$, we conclude that

$$\int_{\mathbb{R}^d} \psi\, d\mu_s \le \int_0^s \int_{|x|<2N} |b||\nabla\varphi_N|\, d|\mu_t|\, dt.$$

Since $|\nabla\varphi_N| \le N^{-1}\max|\nabla\varphi|$ and $\nabla\varphi_N|$ vanishes outside of the set $\{N < |x| < 2N\}$, we have

$$\int_{\mathbb{R}^d} \psi\, d\mu_s \le N^{-1}\max|\nabla\varphi| \int_0^s \int_{N<|x|<2N} |b|\, d|\mu_t|\, dt.$$

Letting $N \to \infty$, we obtain that

$$\int_{\mathbb{R}^d} \psi\, d\mu_s \le 0,$$

which means that $\mu_s = \mu_s^1 - \mu_s^2 = 0$ for almost all s. ☐

Thus, the main difficulty is to construct the required approximations.

2. Corollaries and examples

In this section we obtain some corollaries of the main result and consider some examples. Our first corollary generalizes a result of [15] by omitting restrictions on the structure of the boundary of the zero set of the field.

Corollary 2.1. *Let $d = 1$, $b \in C(\mathbb{R}^1)$ and $0 \le b(x) \le C + C|x|$ for every $x \in \mathbb{R}^1$ and some number $C > 0$. Then the corresponding Cauchy problem* (1.1) *has at most one solution in the class of all locally bounded* (*possibly, signed*) *measures μ on $\mathbb{R}^d \times [0,T]$ given by families of locally bounded measures $(\mu_t)_{t\in[0,T]}$ such that*

$$|\mu_t|(\partial Z) = 0 \quad \text{for almost all} \quad t \in [0,T], \tag{2.1}$$

where ∂Z is the boundary of the set $Z = \{x\colon\ b(x) = 0\}$. In particular, the latter holds for absolutely continuous μ_t provided that $b^{-1}(0)$ has Lebesgue measure zero.

Proof. Set $Z^0 = \{x\colon\ b(x) = 0\} \setminus \partial Z$ and suppose that Z^0 is not empty. Obviously, the measure $\nu_0 = \nu|_{Z^0}$ is a stationary solution to the equation $\partial_t\mu_t + \operatorname{div}(b\mu_t) = 0$ and, for every solution μ_t, we have $\mu_t = \nu_0$ on Z^0, because the term with $\operatorname{div}(b\mu)$ vanishes in the domain Z^0. Replacing μ_t by $\mu_t - \nu_0$ we can trivially prove the uniqueness of solutions μ_t such that $\mu_t(Z^0) = 0$.

We now prove that for all locally bounded measures $\mu = \mu_t\, dt$ such that $|\mu_t|(\partial Z) = 0$ for almost all $t \in [0,T]$ there exist appropriate b_k and V_k such that all conditions (i)–(iii) in Theorem 1.1 are fulfilled.

Let us fix an interval $[-N,N]$. In order to apply Theorem 1.1, one has to find suitable approximations for b on $[-N,N]$. Let ω be the modulus of continuity of b on $[-1-N, N+1]$. If $\omega(\delta) = 0$ for some $\delta > 0$, then $b = const$ and one can take $b_k = b$ and $V_k = 1$. Let $\omega(\delta) > 0$ if $\delta > 0$. Let $k \ge 1$, $\varrho_{1/k}(x) = k\varrho(kx)$, where $\varrho \in C_0^\infty(\mathbb{R}^1)$, $\varrho \ge 0$ and $\|\varrho\|_{L^1} = 1$. We use the following approximations:

$$b_k = b * \varrho_{1/k} + \omega(k^{-1}), \quad V_k = b_k^{-2}.$$

Let us verify conditions (ii) and (iii) (since (i) is obviously true for such b_k with $C_1 = 2C$). Since $b_k(x) \le \max_{[-1-N,N+1]} b(x) = C_N$ on $[-N,N]$, we have $\inf_k \inf_{[-N,N]} V_k \ge (4C_N)^{-2} > 0$. Moreover, $b_k V_k' = -2b_k' V_k$. Thus, condition (ii) in Theorem 1.1 is fulfilled.

Set $g_k(x) := |b - b_k|\sqrt{V_k} = |b - b_k| b_k^{-1}$. Note that if $b(x) = 0$, then $g_k(x) = 1$. If $b(x) \ne 0$, then $g_k(x) \to 0$. Finally,

$$g_k = \frac{|b - b_k|}{b_k} \le \frac{2\omega(k^{-1})}{b * \varrho_{1/k} + \omega(k^{-1})} \le 2.$$

This yields the equality

$$\lim_{k\to\infty} \int_0^T \int_{-N}^N |b - b_k|\sqrt{V_k}\, d|\mu_t|\, dt = \int_0^T \int_{-N}^N \chi_{\{b=0\}}(x)\, |\mu_t|(dx)\, dt = 0.$$

So condition (iii) is fulfilled. ☐

Remark 2.2. The case of nonpositive b can be reduced to the considered situation: it suffices to replace x by $-x$ and $b(x)$ by $-b(-x)$. By the way, there is no direct generalization of the last corollary to the case of a signed drift b. Indeed, it fails even for $b(x) = x^{1/3}$. But we can assert the uniqueness for solutions concentrated on the set where $b \geq 0$ (or $b \leq 0$) and satisfying condition (2.1). This is a trivial observation because such a solution satisfies the equation with $b^+ = b \wedge 0$ and we can apply Corollary 2.1. There is a more interesting case where one can extract a negative part of b with some better regularity.

Corollary 2.3. *Let $d = 1$. Assume that $b \in C(\mathbb{R}^1)$, $|b(x)| \leq C + C|x|$ and $b = g + f$, where $f \geq 0$, g is a Lipschitzian function with Lipschitz constant Λ, and*

$$g(x) < 0 \Longrightarrow f(x) = 0. \tag{2.2}$$

Then the corresponding Cauchy problem (1.1) has at most one solution in the class of all locally bounded measures μ given by families of locally bounded measures $(\mu_t)_{t\in[0,T]}$ such that

$$|\mu_t|(\partial Z_f) = 0 \quad \text{for almost all} \quad t \in [0, T],$$

where ∂Z_f is the boundary of the set $Z_f = \{x\colon\ f(x) = 0\}$.

Proof. As in the previous corollary, we prove that all conditions (i)–(iii) in Theorem 1.1 are fulfilled for all locally bounded measures $\mu = \mu_t\,dt$ such that $|\mu_t|(\partial Z_f) = 0$ for almost all $t \in [0, T]$.

There exists a 1-Lipschitzian function h such that $h \geq 0$, $h(x) = 0$ if $f(x) > 0$, and the set of zeros of $h+f$ coincides with ∂Z_f. Let $\widetilde{g} = g-h$ and $\widetilde{f} = f+h$. As above, we fix an interval $[-N, N]$ and find suitable approximations of b on $[-N, N]$. Let ω be the modulus of continuity of f on $[-1-N, N+1]$. Let $k \geq 1$, $\varrho_{1/k}(x) = k\varrho(kx)$, where $\varrho \in C_0^\infty(\mathbb{R}^1)$, $\varrho \geq 0$, $\varrho(x) = 0$ if $|x| < 1$, and $\|\varrho\|_{L^1} = 1$. Set

$$\varepsilon_k = 8\Lambda k^{-1/2},$$

$$\widetilde{f}_k = (f - 2\omega(3/k))^+ + h,\ \ b_k = (\widetilde{g} + \widetilde{f}_k) * \varrho_{1/k} + \varepsilon_k,\ \ V_k = (|\widetilde{g}| * \varrho_{1/k} + \widetilde{f}_k * \varrho_{1/k} + \varepsilon_k)^{-2}.$$

Let us verify condition (ii). We shall prove that

$$b_k V_k' \leq -2(-C + b_k')V_k \tag{2.3}$$

for some $C > 0$. Since g and $|g|$ are Lipschitzian functions, we can replace b_k' in the latter inequality by the $|\widetilde{g}| * \varrho'_{1/k} + \widetilde{f}_k * \varrho'_{1/k}$. Note that

$$b_k V_k' = -2V_k \frac{(\widetilde{g} * \varrho_{1/k} + \widetilde{f}_k * \varrho_{1/k} + \varepsilon_k)(|\widetilde{g}| * \varrho'_{1/k} + \widetilde{f}_k * \varrho'_{1/k})}{|\widetilde{g}| * \varrho_{1/k} + \widetilde{f}_k * \varrho_{1/k} + \varepsilon_k}.$$

In order to prove (2.3) it is enough to show that

$$\frac{((\widetilde{g} - |\widetilde{g}|) * \varrho_{1/k})(|\widetilde{g}| * \varrho'_{1/k} + \widetilde{f}_k * \varrho'_{1/k})}{|\widetilde{g}| * \varrho_{1/k} + \widetilde{f}_k * \varrho_{1/k} + \varepsilon_k} \geq -\gamma$$

for some $\gamma > 0$. If $(\widetilde{g} - |\widetilde{g}|) * \varrho_{1/k}(x) \neq 0$, then there is a point z in the interval $(x - k^{-1}, x + k^{-1})$ such that $\widetilde{g}(z) = g(z) - h(z) < 0$. If $g(z) < 0$ or $h(z) > 0$, then $f(z) = 0$ and $(f - 2\omega(3/k))^+(x) = 0$ on the interval $(z - 3k^{-1}, z + 3k^{-1})$. In particular, $\widetilde{f}_k = h$ on $(x - k^{-1}, x + k^{-1})$ and

$$|\widetilde{f}_k * \varrho'_{1/k}(x)| = |h * \varrho'_{1/k}(x)| \leq 1.$$

We use here that h is a 1-Lipschitzian function. Moreover, for every $y \in (x - k^{-1}, x + k^{-1})$ we have

$$|\widetilde{g}(y)| \geq |\widetilde{g}(z)| - 2\Lambda k^{-1} \geq 2^{-1}(|\widetilde{g}| - \widetilde{g})(y) - 4\Lambda k^{-1}.$$

Thus, $|\widetilde{g}| * \varrho_{1/k}(x) \geq 2^{-1}(|\widetilde{g}| - \widetilde{g}) * \varrho_{1/k}(x) - 4\Lambda k^{-1}$ and

$$|\widetilde{g}| * \varrho_{1/k}(x) + \widetilde{f}_k * \varrho_{1/k}(x) + \varepsilon_k \geq 2^{-1}(|\widetilde{g}| - \widetilde{g}) * \varrho_{1/k}(x),$$

because $\varepsilon_k - 4\Lambda k^{-1} > 0$. Therefore, we have

$$\left|\frac{((\widetilde{g} - |\widetilde{g}|) * \varrho_{1/k})(|\widetilde{g}| * \varrho'_{1/k} + \widetilde{f}_k * \varrho'_{1/k})}{|\widetilde{g}| * \varrho_{1/k} + \widetilde{f}_k * \varrho_{1/k} + \varepsilon_k}\right| \leq 2(\Lambda + 1).$$

Thus, condition (ii) is fulfilled. Let us verify condition (iii). We have

$$|b - b_k|\sqrt{V_k} = \frac{|\widetilde{g} * \varrho_{1/k} - \widetilde{g} + \widetilde{f}_k * \varrho_{1/k} - \widetilde{f} + \varepsilon_k|}{|\widetilde{g}| * \varrho_{1/k} + \widetilde{f}_k * \varrho_{1/k} + \varepsilon_k}.$$

If $\widetilde{f}(x) = 0$, then $\widetilde{f}_k * \varrho_{1/k}(x) = 0$ and

$$-\varepsilon_k^{-1}\Lambda k^{-1} + 1 \le |b(x) - b_k(x)|\sqrt{V_k(x)} \le \varepsilon_k^{-1}\Lambda k^{-1} + 1.$$

Since $\varepsilon_k^{-1}\Lambda k^{-1} \to 0$, we have $|b(x) - b_k(x)|\sqrt{V_k(x)} \to 1$. If $\widetilde{f}(x) \neq 0$, then $|b - b_k|\sqrt{V_k} \to 0$. Thus, we obtain

$$\lim_{k\to\infty} \int_0^T \int_{-N}^N |b - b_k|\sqrt{V_k}\, d|\mu_t|\, dt = \int_0^T \int_{-N}^N \chi_{\{\widetilde{f}=0\}}(x)\, |\mu_t|(dx)\, dt = 0$$

and condition (iii) is fulfilled as well. □

Remark 2.4. (i) In the same way one can prove the above result under the assumptions that $b = g + f$, where

$$(g(x + y) - g(x))y \le C|y|^2$$

and f is an increasing nonnegative function that is constant on the set $\{x\colon\ g(x) < 0\}$.

(ii) The same assertion is true in the case, where $b = g - f$, g is a Lipschitzian function and f satisfies all conditions of the last corollary.

(iii) According to Corollaries 2.1 and 2.3 the uniqueness holds in the class of solutions with the following property: "you must go if you can".

Let us now consider the case $d \ge 2$. First we mention a known example of uniqueness (see, e.g., [4]), in which it is easy to check our hypotheses.

Example 2.5. Assume that $b \in C(\mathbb{R}^d \times \mathbb{R}^1, \mathbb{R}^d)$ and there exist numbers C_1, C_2 and C_3 such that

$$|b(x, t)| \le C_1 + C_2|x|, \quad \langle b(x + \xi, t) - b(x, t), \xi\rangle \le C_3|\xi|^2$$

for all $(x, t) \in \mathbb{R}^d \times [0, T]$ and every $\xi \in \mathbb{R}^d$. Then there exists at most one solution to (1.1) in the class of all locally bounded measures μ given by families of locally bounded measures $(\mu_t)_{t\in[0,T]}$.

Proof. All hypotheses of Theorem 1.1 are fulfilled with the functions $V_k = 1$ and $b_k = b * \varrho_{1/k}$, where $\varrho_{1/k} = k^d\varrho(kx)$ and $\varrho \in C_0^\infty(\mathbb{R}^{d+1})$, $\varrho \ge 0$, $\|\varrho\|_{L^1} = 1$. □

Let us consider a more specific situation, where

$$b(x) = -\beta(|x|^2)x$$

is a radially symmetric vector field.

Example 2.6. Assume that $\beta \ge 0$ is bounded and continuous on $[0, +\infty)$ and that for every interval $[0, N]$ there exists a number $\Lambda_N > 0$ such that $s \mapsto \beta(s) - \Lambda_N s$ is a decreasing function on $[0, N]$. Then the Cauchy problem (1.1) with $b(x) = -\beta(|x|^2)x$ has at most one solution in the class of all locally bounded measures μ given by families of locally bounded measures $(\mu_t)_{t\in[0,T]}$ such that

$$|\mu_t|(\partial Z \setminus \{0\}) = 0 \quad \text{for almost all} \quad t \in [0, T],$$

where ∂Z is the boundary of the set $Z = \{x\colon\ \beta(|x|^2) = 0\}$.

Proof. Let us fix a natural number N. Let ω be the modulus of continuity of β on $[0, N + 1]$. If $\omega(\delta) = 0$ for some $\delta > 0$, then $\beta = \mathit{const}$ on $[0, N]$ and one can take $b_k = b$ and $V_k = 1$. Let us consider the case where $\omega(\delta) > 0$ if $\delta > 0$. Applying the reasoning from the proof of Corollary 2.1, we arrive at the case where $\mu_t = 0$ on the set $\{x\colon\ \beta(|x|^2) = 0\} \setminus \partial Z$. Let

$$b_k(x) = -(\beta_k(|x|^2) + \omega(k^{-1}))x, \quad V_k(x) = (\beta_k(|x|^2) + \omega(k^{-1}))^{-2},$$

where $\beta_k(z) = \beta * \varrho_{1/k}$. Then

$$\langle b_k(x), \nabla V_k(x)\rangle = 4|x|^2\beta_k'(|x|^2)V_k(x)$$

and

$$\sum_{i,j}\partial_{x_i}b_k^j(x)h_ih_j = -2\beta_k'(|x|^2)\sum_{i,j}x_ih_ix_jh_j - \big(\beta_k(|x|^2)+\omega(k^{-1})\big)|h|^2$$
$$\le -2(\beta_k'(|x|^2)\wedge 0)|x|^2|h|^2.$$

Note also that $\beta_k'(z)\le\Lambda_{N+1}$ if $z\in[0,N]$. Hence $4|x|^2\beta_k'(|x|^2)\le 4N\Lambda_{N+1}$ and

$$\langle b_k(x),\nabla V_k(x)\rangle \le \Big(4N\Lambda_{N+1}+4(\beta_k'(|x|^2)\wedge 0)|x|^2\Big)V_k(x)$$

if $|x|\le\sqrt{N}$. Thus, condition (ii) in the definition of $\mathcal{M}_b$ required in Theorem 1.1 holds. Let us verify condition (iii). We have

$$g_k(x)=|b_k(x)-b(x)|\sqrt{V_k(x)}=\frac{|\beta_k(|x|^2)-\beta(|x|^2)+\omega(k^{-1})||x|}{\beta_k(|x|^2)+\omega(k^{-1})}.$$

Note that if $x\ne 0$ and $\beta(|x|^2)=0$, then $g_k(x)=1$. If $\beta(|x|^2)\ne 0$, then $g_k(x)\to 0$. Since $g_k(x)\le 2|x|$, we obtain the equality

$$\lim_{k\to\infty}\int_0^T\int_{|x|\le\sqrt{N}}g_k(x)\,|\mu_t|(dx)\,dt=\int_0^T\int_{|x|\le\sqrt{N}}\chi_{\{\beta(|x|^2)=0\}}\,|\mu_t|(dx)\,dt=0.$$

Thus, condition (iii) is fulfilled as well. □

Remark 2.7. In the same way one can consider a more general situation:

$$b(x)=-\beta(W(x))\nabla W(x),$$

where W is a smooth bounded function on $\mathbb{R}^d$ with bounded derivatives and β is a nonnegative continuous function on $\mathbb{R}^1$ and such that $s\mapsto\beta(s)-\Lambda s$ is a decreasing function for some number $\Lambda>0$. Then the uniqueness holds in the class of all bounded measures μ given by families of bounded measures $(\mu_t)_{t\in[0,T]}$ such that

$$|\mu_t|(\partial Z\setminus\mathcal{W})=0\quad\text{for almost all}\quad t\in[0,T],$$

where ∂Z is the boundary of the set $Z=\{x\colon\ \beta(W(x))=0\}$ and $\mathcal{W}=\{x\colon\ |\nabla W(x)|=0\}$.

Indeed, we can take

$$b_k(x)=-(\beta_k(W(x))+\omega(k^{-1}))\nabla W(x)\quad\text{and}\quad V_k(x)=(\beta_k(W(x))+\omega(k^{-1}))^{-2},$$

where $\beta_k(z)=\beta*\varrho_{1/k}$, and repeat the reasoning from the proof of the previous example.

Finally, let us discuss the existence of solutions possessing our uniqueness properties.

We observe that not every approximation of b yields a solution from the considered uniqueness class. Indeed, let $b(0)=0$ and take the approximations by smooth functions b_k such that $b_k(0)=0$. Then for $\nu=\delta_0$ we obtain the unique solution $\mu_t^k=\delta_0$ for every k. So, the limit measure of such μ_t^k is again δ_0.

Proposition 2.8. *Assume that $b\colon\ \mathbb{R}^d\times[0,T]\to\mathbb{R}^d$ is continuous. Suppose that there exist a sequence of vector fields $b_k\in C^\infty(\mathbb{R}^d\times\mathbb{R}^1,\mathbb{R}^d)$ and a sequence of positive functions $V_k\in C^1(\mathbb{R}^d)$ such that*

(i) *there exist numbers C_1, C_2 and C_3 such that $V_k\le C_1V_m+C_2$ for every $k\le m$ and*

$$\langle b_k(x,t),\nabla V_k(x)\rangle\le C_3V_k(x)\quad\forall\,(x,t)\in\mathbb{R}^d\times[0,T],$$

(ii) *the function $(1+|x|)^{-1}|b_k(x,t)|$ is bounded for every k and $b_k\to b$ uniformly on $U\times[0,T]$ for every ball U.*

Then, for every initial nonnegative finite measure ν such that $\sup_k\|V_k\|_{L^1(\nu)}<\infty$, there exists a family of nonnegative finite measures $(\mu_t)_{t\in[0,T]}$ solving the Cauchy problem (1.1) with the following property: for every ball U one has

$$\lim_{k\to\infty}\int_0^T\int_U|b-b_k|\sqrt{V_k}\,d\mu_t\,dt=0. \tag{2.4}$$

Proof. Since b_k is a smooth vector field of linear growth, there exists a nonnegative finite solution $(\mu_t^k)_{t\in[0,T]}$ to the Cauchy problem (1.1) with b_k. Moreover, $\mu_t^k(\mathbb{R}^d) \le \nu(\mathbb{R}^d)$. Using the standard compactness arguments and the diagonal procedure (see [30] for details in a similar situation), one can find a subsequence $\{k_l\}$ such that on every compact set in $\mathbb{R}^d$ the sequence $\{\mu_t^{k_l}\}$ converges weakly to some solution μ_t for every $t \in [0,T]$.

Let $\psi_N(x) = \psi(x/N)$. We have

$$\int_{\mathbb{R}^d} V_{k_l}\psi_N\, d\mu_t^{k_l} = \int_{\mathbb{R}^d} V_{k_l}\psi_N\, d\nu + \int_0^t \int_{\mathbb{R}^d} \bigl[\langle b_{k_l}, \nabla\psi_N\rangle V_{k_l} + \psi_N\langle b_{k_l}, \nabla V_{k_l}\rangle\bigr]\, d\mu_s^{k_l}\, ds.$$

Using conditions (i) and (ii) and letting $N \to \infty$, we arrive at the equality

$$\int_{\mathbb{R}^d} V_{k_l}\, d\mu_t^{k_l} = \int_{\mathbb{R}^d} V_{k_l}\, d\nu + C_3 \int_0^t \int_{\mathbb{R}^d} V_{k_l}\, d\mu_s^{k_l}\, ds.$$

Applying Gronwall's inequality we obtain that

$$\int_{\mathbb{R}^d} V_{k_l}\, d\mu_t^{k_l} \le e^{C_3 t} \int_{\mathbb{R}^d} V_{k_l}\, d\nu.$$

Since $V_k \le C_1 V_m + C_2$ for every k, m such that $k \le m$, we obtain the estimate

$$\int_{\mathbb{R}^d} V_k\, d\mu_t \le \lim_{l\to\infty} \int_{\mathbb{R}^d} V_k\, d\mu_t^{k_l} \le C_2\nu(\mathbb{R}^d) + C_1 \lim_{l\to\infty} \int_{\mathbb{R}^d} V_{k_l}\, d\mu_t^{k_l}.$$

Hence

$$\int_{\mathbb{R}^d} V_k\, d\mu_t \le C_2\nu(\mathbb{R}^d) + C_1 e^{C_3 t} \sup_k \int_{\mathbb{R}^d} V_k\, d\nu.$$

Finally, we have

$$\int_0^T \int_U |b - b_k|\sqrt{V_k}\, d\mu_t\, dt \le \Bigl(\int_0^T \int_U |b - b_k|^2\, d\mu_t\, dt\Bigr)^{1/2} \Bigl(\int_0^T \int_U V_k\, d\mu_t\, dt\Bigr)^{1/2},$$

where the first factor tends to zero and the second one is uniformly bounded. □

Example 2.9. Let us illustrate the previous proposition in the situation of Corollary 2.1. Assume that b on $\mathbb{R}^1$ satisfies the following additional condition:

$$(b(x+y) - b(x))y \ge -C|y|^2$$

for all x, y and some $C > 0$. Then all conditions of Proposition 2.8 are fulfilled with the approximations $b_k = b * \varrho_{1/k} + 3\omega(1/k)$ and $V_k = b_k^{-2}$. Indeed, one has

$$b_k' \ge -C \quad \text{and} \quad b_k V_k' = -2b_k' V_k \le 2CV_k.$$

Let $k > m$. Note that the sequence of numbers $\omega(k^{-1})$ is decreasing and

$$b * \varrho_{1/k}(x) + 3\omega(1/k) \ge b(x) + 2\omega(1/k) \ge b(x) + 2\omega(1/m) \ge b * \varrho_{1/m}(x) + w(1/m).$$

Hence we have $V_k \le 9V_m$. Assume that

$$\int_{\mathbb{R}^1\setminus Z^0} \frac{1}{b^2}\, d\nu < \infty.$$

As above we replace ν by $\nu - \nu_0$, where $\nu_0 = \nu|_{Z^0}$ is a stationary solution. Then

$$\sup_k \int_{\mathbb{R}^1} V_k\, d\nu \le \int_{\mathbb{R}^1\setminus Z^0} \frac{1}{(b + 2\omega(1/k))^2}\, d\nu = \int_{\mathbb{R}^1\setminus Z^0} \frac{1}{b^2}\, d\nu < \infty.$$

By Proposition 2.8 there exists a solution $(\mu_t)_{t\in[0,T]}$ such that equality (2.4) holds. Moreover,

$$\int_{\mathbb{R}^1\setminus Z^0} \frac{1}{b^2}\, d\mu_t < \infty$$

and $\mu_t(\partial Z) = 0$ for almost all $t \in [0,T]$.

Remark 2.10. There is another way of constructing a solution to the Cauchy problem (1.1). Given $\varepsilon > 0$, let us consider the Cauchy problem

$$\partial_t \mu^\varepsilon - \varepsilon \Delta \mu^\varepsilon + \mathrm{div}(b\mu^\varepsilon) = 0, \quad \mu^\varepsilon|_{t=0} = \nu.$$

Under suitable conditions on b, there exists a limit point μ of the solutions μ^ε. To describe the set of all limit points μ is a well-known problem. Assume that $d = 1$, $b \geq 0$, b has a single zero $b(0) = 0$ and $1/b$ is Lebesgue integrable near the origin. Let $\nu = \delta_0$. It turns out (see [28]) that the measure μ is given by a family of measures $\mu_t = \delta_{x_t}$, where x_t is the upper extreme solution of the Cauchy problem $\dot{x} = b(x)$, $x(0) = 0$. Note that the upper extreme solution does not stay at $x = 0$ and $\mu_t(\{0\}) = 0$ if $t > 0$. Thus, we obtain a solution from our uniqueness class.

Note also that there are many results on existence of solutions to continuity equations based on Lyapunov functions conditions (see, e.g., [8, Corollary 3.4]), but our purpose here is to ensure existence of solutions in our uniqueness classes. In a separate paper the infinite-dimensional case will be considered; some related results are obtained in our forthcoming paper [9].

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