# Quasispecies spatial model in a critical regime *

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**Abstract**

We study a new marked continuous contact model in $d$-dimensional space ($d \geq 3$), where the mark of any particle is interpreted as the individual genome of a bacterium. The fecundity and mortality rates are now mark dependent. Under general conditions we prove that for certain values of fecundity rates (the critical regime) this system has the one-parameter set of invariant measures parametrized by the spatial density of particles. Also we prove that non-equilibrium process starting from the marked Poisson initial state converges to one of these invariant measures.

Keywords: continuous contact model; non-equilibrium Markov process; integral equations

## 1 Introduction

In this paper we study a critical (stationary) regime in quasispecies contact model in the continuum, including the case of genome dependent mortality rates. Such models can be considered as a special case of birth-and-death processes in the continuum, see [4, 6]. The phase space of such processes is the space $\Gamma = \Gamma(R^d \times S)$ of locally finite marked configurations in $R^d$ with marks $s \in S$ from a compact metric space $S$. Our purpose here is to describe various stationary regimes and to specify relations between solutions of the Cauchy problem and these stationary regimes. A detailed analysis of non-equilibrium dynamics of the quasispecies continuous contact model and some generalizations of the model will be done in forthcoming papers.

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With biological point of view, the stochastic system under study can be considered as a model of an asexual reproduction under mutations and selections, where an individual at the point $u \in R^d$ with the genome $s \in S$ produces an offspring distributed in the coordinate space and in the genome space with the rate $\alpha(u-v)Q(s,s')$. The function $Q(s,s')$ is said to be the mutation kernel. Moreover, since mortality rates in our model can depend on genomes, then selection rules are also included in the evolution under consideration. Using results about convergence of the correlation functions as $t \to \infty$ we can identify the first correlation function of the stationary regime with a density of quasispecies in space, where the conception of quasispecies was introduced in [1], see also [8].

As in [4] we prove in this paper the existence of the stationary distributions for the marked contact model in $d$-dimensional continuous space, $d \geq 3$, see Theorems 1-2 below. We consider here the model with the inner structure of particles (genomes of bacteria). Invariant distributions form the one-parameter family parametrized by the spatial density of particles. Moreover, invariant distributions are not Poisson and the genomes of bacteria are not independent random variables. The origin of this dependence is the existence of recent common ancestors for spatially close individuals. In contrast to [4], the average spatial density for considered system is not a conserved quantity. So the asymptotic value of the density can differ from its initial value, see Remark 1.

# 2 Main results.

## 2.1 Homogeneous mortality rates

We consider a quasispecies contact model on $M = R^d \times S$, where $d \geq 3$ and $S$ is a compact metric space. A structural description of the considered process is given by a heuristic generator defined on proper class of functions (observables) $F : \Gamma \to R$ as follows:

$$(LF)(\gamma) = \sum_{x\in\gamma}(F(\gamma\backslash x) - F(\gamma)) + \kappa \int_M \sum_{y\in\gamma} a(x,y)(F(\gamma \cup x) - F(\gamma))dx, \quad (1)$$

where $dx = d\lambda d\nu$ is a product of the Lebesgue measure $\lambda$ on $R^d$ and some finite Borel measure $\nu$ on $S$ with $supp\ \nu = S$.

Here $b(x,\gamma) = \kappa \sum_{y\in\gamma} a(x,y)$ are birth rates related to the contact model, and $m(x,\gamma) \equiv 1$ are death (mortality) rates. We take $a(x,y)$ in the following form:

$$a(x,y) = \alpha(\tau(x) - \tau(y))\, Q(\sigma(x), \sigma(y)), \quad (2)$$

$\tau$ and $\sigma$ are projections of $M$ on $R^d$ and $S$ respectively, $\alpha(u) \geq 0$ is a function on $R^d$ such that

$$\int_{R^d} \alpha(u)du \ = \ 1, \tag{3}$$

$$\int_{R^d} |u|^2 \alpha(u)du \ < \ \infty, \tag{4}$$

the covariance matrix $C$

$$C_{jk} \ = \ \int_{R^d} u_j u_k \alpha(u)du \ - \ m_j m_k, \quad m_j \ = \ \int_{R^d} u_j \alpha(u)du, \tag{5}$$

is non-degenerate;

$$\hat{\alpha}(p) \ = \ \int_{R^d} e^{i(p,u)} \alpha(u)du \in L^1(R^d). \tag{6}$$

It follows in particular that $|\hat{\alpha}(p)| < 1$ for all $p \neq 0$.

We suppose that the function $Q$ on $S \times S$ is a) continuous on $S \times S$ and b) strictly positive. Then the Krein-Rutman theorem [7] implies that there are a positive number $r > 0$ and a strictly positive continuous function $q(s)$ on $S$, such that $Qq \ = \ rq$ for the integral operator

$$(Qh)(s) \ = \ \int_S Q(s, s')h(s')d\nu(s'), \tag{7}$$

and the spectrum of $Q$, except $r$, which is a discrete spectrum accumulated to 0, is contained in the open disk $\{z : \ |z| < r\} \subset \mathbb{C}$. (Here we consider the spectrum of the integral operator (7) in the Banach space of continuous functions $C(S)$). This "rest spectrum" is the spectrum of $Q$ on the subspace "biorthogonal to $q$", i.e. on the subspace of the functions $h(s)$ such that

$$\int_S h(s) \ \tilde{q}(s) \ d\nu(s) \ = \ 0.$$

Here $\tilde{q}(s)$ is the stricily positive eigenfunction of the adjoint operator $Q^\star(s, s') = Q(s', s)$. We take $\kappa_{cr} = r^{-1}$ and now including $\kappa_{cr}$ in $Q$ we shall suppose that $r = 1$, i.e. $Qq = q$. So the "renormalized critical value of $\kappa$" equals 1 and we omit $\kappa$ in (1) in what follows. We also normalize the function $q$ by the condition

$$\int_S q(s)d\nu(s) \ = \ 1. \tag{8}$$

Note that the existence problem for Markov processes in $\Gamma$ for general birth and death rates is an essentially open problem. An alternative way of studying the evolution of the system is to consider the corresponding statistical dynamics. The latter means that instead of a time evolution of configurations we consider a time evolution of initial states (distributions), i.e. solutions of the corresponding forward Kolmogorov (Fokker-Planck) equation, see ([2, 5]) for details.

We should remind basic notations and constructions to derive time evolution equations on correlation functions of the considered model. Let $\mathcal{B}(M)$ be the family of all Borel sets in $M = R^d \times S$, and $\mathcal{B}_b(M) \subset \mathcal{B}(M)$ denotes the family of all bounded sets from $\mathcal{B}(M)$. The configuration space $\Gamma(M)$ consists of all locally finite subsets of $M$:

$$\Gamma \;=\; \Gamma(M) \;=\; \{\gamma \subset M: \; |\gamma \cap \Lambda| < \infty \;\text{ for all }\; \Lambda \in \mathcal{B}_b(M)\}.$$

Together with the configuration space $\Gamma(M)$ we define the space of finite configurations

$$\Gamma_0 \;=\; \Gamma_0(M) \;=\; \bigsqcup_{n \in N \cup \{0\}} \Gamma_0^{(n)},$$

where $\Gamma_0^{(n)}$ is the space of $n$-point configurations:

$$\Gamma_0^{(n)} \;=\; \{\eta \subset M: \; |\eta| = |\tau(\eta)| = n\}.$$

We denote the set of bounded measurable functions with bounded support by $B_{bs}(\Gamma_0)$, and the set of cylinder functions on $\Gamma$ by $\mathcal{F}_{cyl}(\Gamma)$. Each $F \in \mathcal{F}_{cyl}(\Gamma)$ is characterized by the following relation: $F(\gamma) = F(\gamma_\Lambda)$ for some $\Lambda \in \mathcal{B}_b(M)$.

Next we define a mapping from $B_{bs}(\Gamma_0)$ into $\mathcal{F}_{cyl}(\Gamma)$ as follows:

$$(K\;G)(\gamma) \;=\; \sum_{\eta \subset \gamma} G(\eta), \quad \gamma \in \Gamma, \; \eta \in \Gamma_0,$$

where the summation is taken over all finite subconfigurations $\eta \in \Gamma_0$ of the infinite configuration $\gamma \in \Gamma$, see i.g. [4] for details. This mapping is called K-transform.

**Proposition 1.** *The operator $\hat{L} = K^{-1} L K$ (the image of $L$ under the K-transform) on functions $G \in B_{bs}(\Gamma_0)$ has the following form:*

$$(\hat{L}G)(\eta) \;=\; -|\tau(\eta)|G(\eta) \;+\; \int_M \sum_{y \in \eta} a(x,y) G((\eta \backslash y) \cup x) dx \;+ \tag{9}$$

$$\int_M \sum_{y \in \eta} a(x,y) G(\eta \cup x) dx.$$

The development of the formula (9) is the same as in [4].

Denote by $\mathcal{M}^1_{fm}(\Gamma)$ the set of all probability measures $\mu$ which have finite local moments of all orders, i.e.

$$\int_\Gamma |\gamma_\Lambda|^n \; \mu(d\gamma) \;<\; \infty$$

for all $\Lambda \in \mathcal{B}_b(M)$ and $n \in N$. If a measure $\mu \in \mathcal{M}^1_{fm}(\Gamma)$ is locally absolutely continuous with respect to the Poisson measure (associated with the measure $dx$), then there exists the corresponding system of the correlation functions $k_\mu^{(n)}$ of the measure $\mu$, well known in statistical physics, see e.g. [9].

As it was shown in [6] there exists a Markov process on the configuration space $\Gamma(R^d)$ of locally finite configurations in $R^d$ with the corresponding generator associated with the contact process, see [4]. In our conditions it is possible to apply arguments from [6] to construct the Markov process $X_t^\gamma$ on the space of marked configurations $\Gamma = \Gamma(M)$. Let $\{\mu_t\}_{t\geq 0} \subset \mathcal{M}^1_{fm}(\Gamma)$ be the evolution of states corresponding to such process $X_t^\gamma$ and described by the dual Kolmogorov equation with the adjoint operator $L^*$. Then the evolution of the corresponding system of correlation functions is defined by the duality equation

$$\langle \hat{L}G, k\rangle \;=\; \langle G, \hat{L}^* k\rangle, \quad G \in B_{bs}(\Gamma_0).$$

Using the representation (9) we define the operator $\hat{L}^*$ adjoint to the operator $\hat{L}$ and obtain the following system of equations for correlation functions in a recurrent form:

$$\frac{\partial k^{(n)}}{\partial t} \;=\; \hat{L}^*_n k^{(n)} \;+\; f^{(n)}, \quad n \geq 1; \quad f^{(1)} = 0, \tag{10}$$

which is the main object for study in this paper. Here $f^{(n)}$ is a function on $M^n$ defined as

$$f^{(n)}(x_1, \ldots, x_n) \;=\; \sum_{i=1}^n k^{(n-1)}(x_1, \ldots, \check{x}_i, \ldots, x_n) \sum_{j\neq i}^n a(x_i, x_j), \; n \geq 2, \tag{11}$$

$f^{(1)} \equiv 0$. The operator $\hat{L}^*_n$, $n \geq 1$, is defined as:

$$\hat{L}^*_n k^{(n)}(x_1, \ldots, x_n) \;=\; -nk^{(n)}(x_1, \ldots, x_n) +$$

$$\sum_{i=1}^n \int_M a(x_i, y) k^{(n)}(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n) dy. \tag{12}$$

We take the initial (for $t = 0$) data

$$k^{(n)}(t=0,\varrho;x_1,\dots,x_n) \;=\; \varrho^n \prod_{i=1}^{n} q(\sigma(x_i)). \tag{13}$$

corresponding to the marked Poisson point field with the intensity $\varrho$ and the distribution of marks $q(s)d\nu(s)$ meeting (8). In fact, we consider in the paper only the case when the initial measures equal to the product of Poisson measure over the space and an independent measure over the mark space, see also a general form (61) for the initial data in Remark 1 below.

Invariant measures of the contact process (if exist!) are described in terms of correlation functions $k^{(n)}$ on $M^n$ as a positive solutions of the following system:

$$\hat{L}_n^* k^{(n)} + f^{(n)} = 0, \quad n \geq 1, \quad k^{(0)} \equiv 1, \tag{14}$$

where $\hat{L}_n^*$, $f^{(n)}$ are defined as in (11) - (12).

Consider the operator $\hat{L}_n^*$ as an operator on the space

$$X_n = C\left(S^n,\ L_{inv}^{\infty}((R^n)^d)\right),$$

where $L_{inv}^{\infty}$ consists of the bounded translation invariant functions $\varphi(w_1,\dots,w_n)$ of $n$ variables:

$$\varphi(w_1+a,\dots,w_n+a) = \varphi(w_1,\dots,w_n), \quad w_i = \tau(x_i) \in R^d.$$

In this section we prove the existence of the solution $k^{(n)} \in X_n$, $n \geq 1$ of the system (14), such that $k^{(n)}$ have a specified asymptotics when $|\tau(x_i) - \tau(x_j)| \to \infty$ for all $i \neq j$. We also prove a strong convergence of the solutions of the Cauchy problem (10) - (13) to the solution of the system (14) of stationary (time-independent) equations.

**Theorem 1.** *I. Let the birth kernel $a(x,y)$ of the contact model meet conditions (2)-(7), and $\kappa_{cr} = r^{-1}$.*

*Then for any positive constant $\varrho \in R_+$ there exists a unique probability measure $\mu^\varrho$ such that its system of correlation functions $\{k_\varrho^{(n)}\}$ is translation invariant, solves (14), satisfies the following condition*

$$|k_\varrho^{(n)}(x_1,\dots,x_n) \;-\; \varrho^n \prod_{i=1}^{n} q(\sigma(x_i))| \;\to\; 0, \tag{15}$$

*when $|\tau(x_i) - \tau(x_j)| \to \infty$ for all $i \neq j$, and satisfies the following estimate*

$$k_\varrho^{(n)}(x_1,\dots,x_n) \;\leq\; D\, C^n (n!)^2 \prod_{i=1}^{n} q(\sigma(x_i)) \quad \textit{for any} \;\; x_1,\dots,x_n, \tag{16}$$

*for some positive constants $C = C(\varrho, Q, \alpha)$, $D$. Here $q(s)$ is the normalized eigenfunction of $Q$. Moreover, the first correlation function $k_\varrho^{(1)}(x)$ of $\mu^\varrho$ is exactly $\varrho\, q(s)$.*

*II. For any $n \geq 1$ the solution $k^{(n)}(t)$ of the Cauchy problem (10) - (13) converges to the solution $k_\varrho^{(n)}$ (15) of the system (14) of stationary (time-independent) equations as $t \to \infty$:*

$$\|k^{(n)}(t) \; - \; k_\varrho^{(n)}\|_{X_n} \;\to\; 0. \tag{17}$$

## 2.2 Species dependent mortality rates

Analogous results are valid in the case when mortality rates depends on $\sigma(x)$:

$$(\tilde{L}F)(\gamma) \;=\; \sum_{x\in\gamma} m(\sigma(x))\,(F(\gamma\backslash x) - F(\gamma)) + \kappa \int_M \sum_{y\in\gamma} a(x,y)(F(\gamma\cup x) - F(\gamma))dx, \tag{18}$$

with

$$a(x,y) = \alpha(\tau(x) - \tau(y))\, Q(\sigma(x), \sigma(y)),$$

$\tau$ and $\sigma$ are projections of $M$ on $R^d$ and $S$ respectively, $m(\sigma(x)) > 0$. In this case using the Krein-Rutman theorem for the integral operator $\tilde{Q}$ with the kernel

$$\tilde{Q}(s,s') \;=\; \frac{Q(s,s')}{m(s)}$$

we get the existence of the maximal eigenvalue $\tilde{r} > 0$ and the corresponding maximal positive eigenfunction $g(s) > 0$ for the operator $\tilde{Q}$.

Correlation functions for the invariant measure in this case can be constructed as a solution of the system of equations

$$\tilde{L}_n^* \tilde{k}^{(n)} + \tilde{f}^{(n)} = 0, \quad n \geq 1, \quad \tilde{k}^{(0)} \equiv 1, \tag{19}$$

where

$$\tilde{f}^{(n)}(x_1, \ldots, x_n) \;=\; \kappa \sum_{i=1}^{n} \tilde{k}^{(n-1)}(x_1, \ldots, \check{x}_i, \ldots, x_n) \sum_{j\neq i}^{n} a(x_i, x_j), \; n \geq 2,$$

$\tilde{f}^{(1)} \equiv 0,$

$$\tilde{L}_n^* \tilde{k}^{(n)}(x_1, \ldots, x_n) \;=\; -\sum_{i=1}^{n} m(\sigma(x_i))\; \tilde{k}^{(n)}(x_1, \ldots, x_n) \; +$$

$$\kappa \sum_{i=1}^{n} \int_M a(x_i, y) \tilde{k}^{(n)}(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n) dy.$$

**Theorem 2.** *Let $m(s) > 0$, $Q(s, s') > 0$, $s, s' \in S$ are continuous functions on $S$ and $S \times S$ respectively; $g(s) > 0$ is the positive eigenfunction corresponding to the maximal eigenvalue $\tilde{r} > 0$ of the integral operator*

$$(\tilde{Q}h)(x) = \int_S \tilde{Q}(s, s') h(s') d\nu(s'), \quad \text{with } \tilde{Q}(s, s') = \frac{Q(s, s')}{m(s)}.$$

*Let $\kappa_{cr} = \tilde{r}^{-1}$. Then for any positive constant $\varrho \in R_+$ there exists a unique probability measure $\tilde{\mu}^{\varrho}$ such that its system of correlation functions $\{\tilde{k}_{\varrho}^{(n)}\}$ is translation invariant, solves (19), satisfies the following condition*

$$|\tilde{k}_{\varrho}^{(n)}(x_1, \ldots, x_n) - \varrho^n \prod_{i=1}^{n} g(\sigma(x_i))| \to 0, \tag{20}$$

*when $|\tau(x_i) - \tau(x_j)| \to \infty$ for all $i \neq j$, and satisfies the following estimate*

$$\tilde{k}_{\varrho}^{(n)}(x_1, \ldots, x_n) \leq D\ C^n (n!)^2 \prod_{i=1}^{n} g(\sigma(x_i)) \quad \text{for any } x_1, \ldots, x_n,$$

*with positive constants $C$, $D$. Here $g(s)$ is the normalized eigenfunction of the operator $\tilde{Q}$. The first correlation function $\tilde{k}_{\varrho}^{(1)}(x)$ of $\tilde{\mu}^{\varrho}$ is exactly $\varrho g(s)$.*

*Moreover, the solution of the Cauchy problem for the system of equations*

$$\frac{\partial \tilde{k}^{(n)}}{\partial t} = \tilde{L}_n^* \tilde{k}^{(n)} + \tilde{f}^{(n)}, \quad n \geq 1; \quad f^{(1)} = 0,$$

*with the initial data*

$$\tilde{k}^{(n)}|_{t=0}(x_1, \ldots, x_n) = \varrho^n \prod_{i=1}^{n} g(\sigma(x_i)),$$

*converges to the system of the correlation functions $\{\tilde{k}_{\varrho}^{(n)}\}$ defined by (20).*

# 3 The proof of Theorem 1. Stationary problem.

In this section we prove the first part of Theorem 1 using the induction in $n$. For $n = 1$ in (14) we have

$$-k^{(1)}(x) + \int_M a(x, y) k^{(1)}(y) dy = 0. \tag{21}$$

As we construct a translation invariant field let us look for $k^{(1)}(x)$ in the form

$$k^{(1)}(x) \;=\; h(\sigma(x))$$

Then (21) is rewritten as

$$-h(s) + \int_S Q(s,s')h(s')d\nu(s') \;=\; 0, \tag{22}$$

which means that

$$h(s) \;=\; \varrho\, q(s),$$

or

$$k^{(1)}(x) \;=\; \varrho\, q(\sigma(x)).$$

From the normalization condition (8) it follows that $\varrho$ has the sense of the spatial density of particles.

As a warm-up let us solve the equation (14) for the special case $n = 2,\ S = \{0\},\ m(0) = q(0) = 1,\ Q(0,0) = 1$. In this case $M = R^d$. Then the equation for $k^{(2)}(x)$ is written as

$$\hat{L}_2^* k^{(2)} + f^{(2)} = 0, \tag{23}$$

with

$$f^{(2)}(x_1,x_2) \;=\; \varrho(a(x_1,x_2) + a(x_2,x_1)) = \varrho(\alpha(x_1-x_2) + \alpha(x_2-x_1)). \tag{24}$$

Thus, the operator $\hat{L}_2^* \;=\; L^{(1)} + L^{(2)}$, where

$$L^{(1)}k^{(2)}(x_1,x_2) \;=\; \int_{R^d} \alpha(x_1-y)k^{(2)}(y,x_2)dy - k^{(2)}(x_1,x_2), \tag{25}$$

and analogously

$$L^{(2)}k^{(2)}(x_1,x_2) \;=\; \int_{R^d} \alpha(x_2-y)k^{(2)}(x_1,y)dy - k^{(2)}(x_1,x_2). \tag{26}$$

Using translation invariant property we have:

$$k^{(2)}(x_1,x_2) \;=\; k^{(2)}(x_1-x_2).$$

After the Fourier transform we can rewrite (23) - (26) as

$$(\hat{\alpha}(p) + \hat{\alpha}(-p) - 2)\ \hat{k}(p) \;=\; -\varrho\ (\hat{\alpha}(p) + \hat{\alpha}(-p)). \tag{27}$$

Therefore,

$$\hat{k}(p) \;=\; \varrho\, \frac{\hat{\alpha}(p)+\hat{\alpha}(-p)}{2-\hat{\alpha}(p)-\hat{\alpha}(-p)} + A\delta(p), \tag{28}$$

where $A$ is an arbitrary constant, and we will explain later how to choose $A$ in the general case.

Expanding $\hat{\alpha}(p)$ in the Taylor series up to the second order and using the conditions (3) - (6) on the function $\alpha$ we see that $\hat{k}(p)$ has an integrable singularity $\sim |p|^{-2}$ at $p=0$ if the dimension $d \geq 3$. Thus there exist infinitely many translation invariant functions $k^{(2)}(x_1-x_2) \in L^{\infty}(R^d)$ satisfying equation (23).

Now let us turn to the general case. If for any $n>1$ we succeed to solve the equation (14) and express $k^{(n)}$ through $f^{(n)}$, then knowing the expression of $f^{(n)}$ through $k^{(n-1)}$ via (11), we get the solution of the full system (14). So we have to invert the operator $\hat{L}^*_n$, and **it is sufficient for us to do it on some class of translation invariant functions**. The precise statement will be presented later for $(\hat{L}^*_n)^{-1} f^{(n)}$, see formula (36).

Remind that

$$\hat{L}^*_n \;=\; \sum_{i=1}^{n} L^i, \tag{29}$$

where

$$L^i k^{(n)}(x_1,\ldots,x_n) \;= \tag{30}$$

$$\int_M a(x_i,y) k^{(n)}(x_1,\ldots,x_{i-1},y,x_{i+1},\ldots,x_n)dy - k^{(n)}(x_1,\ldots,x_n).$$

**Proposition 2.** *The operator $e^{t\hat{L}^*_n}$ is monotone.*

**Proof.** The monotonicity of the operator $e^{t\hat{L}^*_n}$ follows from (29) - (30):

$$e^{t\hat{L}^*_n} \;=\; \otimes_{i=1}^{n} e^{tL^i}, \quad e^{tL^i} \;=\; e^{-t} e^{tA^i},$$

and the positivity of operators

$$A^i k^{(n)} \;=\; \int_M a(x_i,y) k^{(n)}(x_1,\ldots,x_{i-1},y,x_{i+1},\ldots,x_n)dy.$$

□

First consider the restriction of $\hat{L}^*_n$ to the invariant subspace consisting of the functions of the form

$$\varphi(\tau(x_1),\ldots,\tau(x_n)) \prod_{i=1}^{n} q(\sigma(x_i)), \quad \text{where } \varphi(w_1,\ldots,w_n) \in L^{\infty}_{inv}((R^n)^d).$$

The operator $\hat{L}_n^*$ acts on these functions as

$$L_{n,max} \;=\; \sum_{i=1}^{n} L^i_{max}, \tag{31}$$

where

$$L^i_{max}\;\varphi(w_1,\ldots,w_n)\;\prod_{i=1}^{n} q(\sigma(x_i)) \;= \tag{32}$$

$$\prod_{i=1}^{n} q(\sigma(x_i))\left(\int_{R^d}\alpha(w_i-u)\varphi(w_1,\ldots,w_{i-1},u,w_{i+1},\ldots,w_n)du-\varphi(w_1,\ldots,w_n)\right).$$

This formula follows from the equality $Qq = q$. Remind that $\kappa_{cr}$ is "absorbed" in $Q$. Formula (32) means that in this case we have only spatial convolutions and no integration over $S$. In the Fourier variables the operator $L_{n,max}$ acts as a multiplication operator by the function

$$\sum_{i=1}^{n}\hat{\alpha}(p_i) \;-\; n.$$

To invert $L_{n,max}$ let us notice that if $\varphi(w_1,\ldots,w_n)$ is a translation invariant function then its Fourier transform has a form

$$\hat{\varphi}(p_1,\ldots,p_n)\;\delta(p_1+\ldots+p_n).$$

On the subspace of the "momentum space" $(p_1,\ldots,p_n)$ specified by the equation $p_1+\ldots+p_n=0$ the function $\frac{1}{\sum_{i=1}^n \hat{\alpha}(p_i)-n}$ has an integrable singularity $\sim \frac{1}{|p|^2}$ at $p=0$. This property will be crucial for inverting of the operator $L_{n,max}$ on a proper class of functions.

Next we will construct a solution of the system (14) - (12) satisfying (15) and meeting the estimate

$$k^{(n)}(x_1,\ldots,x_n)\;\le\; K_n\prod_{i=1}^{n} q(\sigma(x_i)) \tag{33}$$

where $K_n = DC^n(n!)^2$, $D,\ C$ are constants.

As follows from (11) the function $f^{(n)}$ is the sum of functions of the form

$$f(x_1,\ldots,x_n)\;=\;k^{(n-1)}(x_1,\ldots,\check{x}_i,\ldots,x_n)\;a(x_i,x_j),\quad x_i\in M. \tag{34}$$

Below we invert the operator $\hat{L}_n^*$ on the set of functions of the form (34), see (44) below.

We suppose by induction that

$$k^{(n-1)}(x_1,\dots,x_{n-1}) \;\le\; K_{n-1}\prod_{i=1}^{n-1} q(\sigma(x_i)),$$

then

$$f(x_1,\dots,x_n) \;\le\; K_{n-1}a(x_i,x_j)\prod_{l\neq i} q(\sigma(x_l)). \tag{35}$$

We put

$$v_{i,j}^{(n)} \;=\; \int_0^\infty e^{t\hat{L}_n^*} f \; dt, \tag{36}$$

where $f$ is a function of the form (34).

Since the function $q(s)$ is strictly positive on the compact $S$, the following inequality holds:

$$a(x_i,x_j) \;\le\; c\; q(\sigma(x_i))\;\alpha(\tau(x_i)-\tau(x_j)) \tag{37}$$

with a constant $c$. Then using the monotonicity, identity $Qq = q$, and inequality (37) we get from (35), (29) and (31)

$$e^{t\hat{L}_n^*} f \;\le\; K_{n-1}\, e^{t\hat{L}_n^*} a(x_i,x_j)\prod_{l\neq i} q(\sigma(x_l)) \;\le\; K_{n-1}\, e^{t(L^i+L^j)} a(x_i,x_j)\prod_{l\neq i} q(\sigma(x_l)) \;\le$$
$$cK_{n-1}e^{t(L^i_{max}+L^j_{max})}\alpha(\tau(x_i)-\tau(x_j))\;\prod_{l=1}^{n} q(\sigma(x_l)). \tag{38}$$

Using formula (32), the Fourier transform and the Fubini theorem we finally obtain from (37) - (38) the upper bound on $v_{i,j}^{(n)}$:

$$v_{i,j}^{(n)} \;=\; \int_0^\infty e^{t\hat{L}_n^*} f \; dt \;\le$$

$$cK_{n-1}\prod_{i=1}^{n} q(\sigma(x_i))\left|\int_0^\infty\int_{R^d} e^{t(\hat{\alpha}(p)+\hat{\alpha}(-p)-2)}\hat{\alpha}(p)dpdt\right| \;=\; cAK_{n-1}\prod_{i=1}^{n} q(\sigma(x_i)),$$

where

$$A = \frac{1}{(2\pi)^d}\left|\int_0^\infty\int_{R^d\setminus\{0\}} e^{t(\hat{\alpha}(p)+\hat{\alpha}(-p)-2)}\hat{\alpha}(p)dpdt\right| \le \int_{R^d}\frac{|\hat{\alpha}(p)|}{2-\hat{\alpha}(p)-\hat{\alpha}(-p)}\,dp<\infty \tag{39}$$

when $d \ge 3$.

Integrability of the function $\frac{|\hat{\alpha}(p)|}{2-\hat{\alpha}(p)-\hat{\alpha}(-p)}$, see (39), implies by the Lebesgue-Riemann lemma that the function $v_{i,j}^{(n)}$ defined by (36) satisfies the following condition:

$$v_{i,j}^{(n)}(x_1,\dots,x_n) \ \to\ 0 \quad \text{when} \ \ |\tau(x_i)-\tau(x_j)| \to \infty \tag{40}$$

for all $i \neq j$. As the function $f^{(n)}$ (11) is the sum of $n(n-1)$ similar terms we see that $k^{(n)}(x_1,\dots,x_n)$ given by

$$k^{(n)} \ = \ \left(-\hat{L}_n^*\right)^{-1} f^{(n)} \ = \ \sum_{i\neq j} v_{i,j}^{(n)}$$

is bounded by the function

$$Cn^2 K_{n-1} \prod_{i=1}^{n-1} q(\sigma(x_i))$$

for some $C > 0$. Thus we get the recurrence inequality

$$K_n \ \leq \ Cn^2 K_{n-1}, \tag{41}$$

and by induction it follows that

$$K_n \ \leq \ C^n \ (n!)^2. \tag{42}$$

The general solution $k^{(n)}(x_1,\dots,x_n)$ of the system (14) has the form

$$k^{(n)}(x_1,\dots,x_n) \ = \ \int_0^\infty e^{t\hat{L}_n^*} f^{(n)}(x_1,\dots,x_n)\ dt \ + \ A_n \prod_{i=1}^{n} q(\sigma(x_i)),$$

where $A_n$ are constants. If we are looking for the set of correlation functions $k^{(n)}$ for which

$$|k^{(n)}(x_1,\dots,x_n) \ - \ \varrho^n \prod_{i=1}^{n} q(\sigma(x_i))| \ \to \ 0, \ \ \text{when} \ |\tau(x_i)-\tau(x_j)| \to \infty \tag{43}$$

for all $i \neq j$, then we put

$$k_\varrho^{(n)} \ = \ \int_0^\infty e^{t\hat{L}_n^*} f^{(n)} dt \ + \ \varrho^n \prod_{i=1}^{n} q(\sigma(x_i)). \tag{44}$$

It is clear that the last term in (44) vanishes under the action of $\hat{L}_n^*$, and (43) holds because we got (40). Denote this solution by $k_\varrho^{(n)}$.

In this case instead of (41) we have the recurrence

$$K_n \;\leq\; Cn^2 K_{n-1} \;+\; \varrho^n. \tag{45}$$

Taking $L_n = \frac{K_n}{C^n (n!)^2}$ we have

$$L_n \;\leq\; L_{n-1} \;+\; \frac{\varrho^n}{C^n (n!)^2} \;\leq\; D$$

with some positive constant $D > 0$. Thus we have

$$K_n \;\leq\; DC^n (n!)^2, \tag{46}$$

which differs from (42) only by the constant factor.

Thus we proved the existence of solutions $\{k_\varrho^{(n)}\}$ of the system (14) corresponding to the stationary problem. To verify that this system of correlation function is associated with a measure $m_\varrho$ on the configuration space, we will prove in the next section that the measure $m_\varrho$ can be constructed as a limit of an evolution of measures $\mu_\varrho^{(t)}$ associated with the solutions of the Cauchy problem (10) with corresponding initial data (13).

# 4 The proof of Theorem 1. The Cauchy problem.

In this section we find the solution of the Cauchy problem (10) - (13) and prove the convergence (17). Using Duhamel formula we have

$$k^{(n)}(t) \;=\; k^{(n)}(0) \;+\; \int_0^t e^{(t-s)\hat{L}_n^*} f^{(n)}(s)\; ds, \tag{47}$$

where $f^{(n)}(s)$ is expressed through $k^{(n-1)}(s)$ via (11). We also used there that the operator $\hat{L}_n^*$ annihilates $k^{(n)}(0)$ of the form (13).

Let us notice that $k_\varrho^{(n)}$ given by (44) have no product form (13). We have

$$k^{(n)}(t) - k_\varrho^{(n)} \;=\; \left(e^{t\hat{L}_n^*} - E\right) k_\varrho^{(n)} \;+$$

$$e^{t\hat{L}_n^*}(k^{(n)}(0) - k_\varrho^{(n)}) \;+\; \int_0^t e^{(t-s)\hat{L}_n^*} f^{(n)}(s)\; ds \;=$$

$$e^{t\hat{L}_n^*}(k^{(n)}(0) - k_\varrho^{(n)}) \;+\; \int_0^t e^{(t-s)\hat{L}_n^*} (f^{(n)}(s) - f_\varrho^{(n)})\; ds. \tag{48}$$

Here $f_\varrho^{(n)}$ are expressed in terms of $k_\varrho^{(n-1)}$ by (11), and we used that the equation $\hat{L}_n^* k_\varrho^{(n)} \; = \; -f_\varrho^{(n)}$ implies

$$\left(e^{t\hat{L}_n^*} - E\right) k_\varrho^{(n)} \;=\; -\int_0^t \frac{d}{ds} e^{(t-s)\hat{L}_n^*} k_\varrho^{(n)} ds \;=\; -\int_0^t e^{(t-s)\hat{L}_n^*} f_\varrho^{(n)} \, ds$$

We shall prove now that both terms in (48) converge to 0 in sup-norm of $X_n$.

For the first term using inversion formula (44) and (13) we have

$$e^{t\hat{L}_n^*}(k^{(n)}(0) - k_\varrho^{(n)}) \;=\; e^{t\hat{L}_n^*}(k^{(n)}(0) - v^{(n)} - k^{(n)}(0)) \;=\; -e^{t\hat{L}_n^*} v^{(n)}, \quad (49)$$

where

$$v^{(n)} \;=\; \int_0^\infty e^{s\hat{L}_n^*} f_\varrho^{(n)} \, ds. \quad (50)$$

Since

$$f_\varrho^{(n)}(x_1, \ldots, x_n) \;=\; \sum_{i,j:\; i\neq j} k_\varrho^{(n-1)}(x_1, \ldots, \check{x}_i, \ldots, x_n) \; a(x_i, x_j),$$

then

$$v^{(n)}(x_1, \ldots, x_n) \;=\; \sum_{i,j:\; i\neq j} \int_0^\infty e^{s\hat{L}_n^*} k_\varrho^{(n-1)}(x_1, \ldots, \check{x}_i, \ldots, x_n) \; a(x_i, x_j) \; ds.$$

To prove that $\|e^{t\hat{L}_n^*} v^{(n)}\|_{X_n} \to 0$ as $t \to \infty$ it is enough to prove that its Fourier transform tends to 0 in $L^1$ norm when $t \to \infty$.

Using the estimate (16) on $k_\varrho^{(n)}$ together with the inequality (37) on $a(x_i, x_j)$ we can estimate $e^{t\hat{L}_n^*} v^{(n)}$ applying the monotonicity of $e^{t\hat{L}_n^*}$ and (31) - (32):

$$\left|\left(e^{t\hat{L}_n^*} \; v^{(n)}\right)(x_1, \ldots, x_n)\right| \;\le$$

$$DC^{n-1}((n-1)!)^2 c e^{t\hat{L}_n^*} \sum_{i\neq j} \int_0^\infty e^{s\sum_{i=1}^n L^i} \prod_{i=1}^n q(\sigma(x_i)) \; \alpha(\tau(x_i) - \tau(x_j)) \; ds \;\le$$

$$DC^{n-1}((n-1)!)^2 c \prod_{i=1}^n q(\sigma(x_i)) \; \sum_{i\neq j} \int_{R^{2d}} e^{t(\hat{\alpha}(p_i)+\hat{\alpha}(p_j)-2)}$$

$$\int_0^\infty e^{s(\hat{\alpha}(p_i)+\hat{\alpha}(p_j)-2)} \; |\hat{\alpha}(p_i)| \; \delta(p_i + p_j) \; ds dp_i dp_j \;\le$$

$$DC^{n-1}(n!)^2 c \prod_{i=1}^n q(\sigma(x_i)) \int_{R^d} e^{t(\hat{\alpha}(p)+\hat{\alpha}(-p)-2)} \frac{|\hat{\alpha}(p)|}{2 - \hat{\alpha}(p) - \hat{\alpha}(-p)} \; dp.$$

Here the presence of $\delta$-function corresponds to the shift invariance. Since the function $\frac{|\hat{\alpha}(p)|}{2-\hat{\alpha}(p)-\hat{\alpha}(-p)}$ is integrable in the momentum space for $d \geq 3$ and $\hat{\alpha}(p) + \hat{\alpha}(-p) < 2$ for $p \neq 0$, then the function

$$\tilde{A}\ e^{t(\hat{\alpha}(p)+\hat{\alpha}(-p)-2)}\ \frac{|\hat{\alpha}(p)|}{2-\hat{\alpha}(p)-\hat{\alpha}(-p)}.$$

tends to 0 in $L^1$ norm (in "momentum" variables $p$) when $t \to \infty$. Consequently its inverse Fourier transform tends to 0 in $X_n$ norm (i.e. in sup-norm) when $t \to \infty$. Thus we proved that the first term in (48) tends to 0 in sup-norm when $t \to \infty$.

We consider now the second term in (48) and will prove that

$$\int_0^t e^{(t-s)\hat{L}_n^*}(f^{(n)}(s) - f_\varrho^{(n)})\ ds\ \to 0 \tag{51}$$

in sup-norm when $t \to \infty$ using induction assumption that

$$\|k^{(n-1)}(t)\ -\ k_\varrho^{(n-1)}\|_{X_{n-1}}\ \to\ 0 \quad \text{as } t \to \infty. \tag{52}$$

As the first step of induction we have

$$k^{(1)}(t,x)\ \equiv\ k_\varrho^{(1)}(x)\ =\ \varrho\ q(\sigma(x)). \tag{53}$$

Next the induction assumption (52) implies that

$$\|k^{(n-1)}(t)\|_{X_{n-1}}\ \leq\ M_{n-1} \quad \text{for all}\ \ t \geq 0 \tag{54}$$

with some positive constant depending only on $n$. Really, the operator $\hat{L}_n^*$ is bounded and the function $a(x,y)$ is bounded, hence the norm of the solution $k^{(n)}$ of the problem (10) (with any bounded for $l \leq n$ initial data) is evidently bounded on any compact time interval $[0,\tau]$. On the other hand, for any $\varepsilon > 0$ there exists $\tau$ such that for all $t > \tau$ the norm $\|k^{(n-1)}(t) - k_\varrho^{(n-1)}\| < \varepsilon$ by (52). Thus the bound (54) is proved.

From (52) it follows that

$$\|f^{(n)}(t)\ -\ f_\varrho^{(n)}\|_{X_n}\ \to 0 \quad \text{as } t \to \infty. \tag{55}$$

To estimate the integral (51) we rewrite it as a sum

$$\left(\int_0^\tau\ +\ \int_\tau^t\right) e^{s\hat{L}_n^*}(f^{(n)}(t-s) - f_\varrho^{(n)})\ ds. \tag{56}$$

Let us estimate the second integral in (56) using the monotonicity of the semigroup $e^{s\hat{L}_n^*}$:

$$\left| \int_\tau^t e^{s\hat{L}_n^*} (f^{(n)}(t-s) - f_\varrho^{(n)}) \; ds \right| \;\leq\; \int_\tau^t e^{s\hat{L}_n^*} \left( |f^{(n)}(t-s)| + |f_\varrho^{(n)}| \right) \; ds \;\leq \tag{57}$$
$$\left( M_{n-1} \;+\; \|k_\varrho^{(n-1)}\| \right) \int_\tau^t e^{s\hat{L}_n^*} \sum_{i\neq j} a(x_i, x_j) \; ds.$$

Using the inequality

$$a(x_i, x_j) \;\leq\; cq(\sigma(x_i))q(\sigma(x_j))\alpha(\tau(x_i) - \tau(x_j))$$

we conclude that it will be sufficient to estimate for any pair $i \neq j$ the following integral

$$\int_\tau^t \int_{R^d} e^{s(\hat{\alpha}(p)+\hat{\alpha}(-p)-2)} |\hat{\alpha}(p)| \; dp \; ds \;\leq\; \int_\tau^\infty \int_{R^d} e^{s(\hat{\alpha}(p)+\hat{\alpha}(-p)-2)} |\hat{\alpha}(p)| \; dp \; ds \tag{58}$$

Since the integral

$$\int_0^\infty \int_{R^d} e^{s(\hat{\alpha}(p)+\hat{\alpha}(-p)-2)} |\hat{\alpha}(p)| \; dp \; ds \;=\; \int_{R^d} \frac{|\hat{\alpha}(p)|}{2 - \hat{\alpha}(p) - \hat{\alpha}(-p)} \; dp \tag{59}$$

converges, then the integral (58) tends to 0 when $\tau \to \infty$. Consequently we can take $\tau$ in such a way that (58) is less than $\varepsilon$, and then (57) is less than $C\varepsilon$ for some $C$ and any $t > \tau$.

Finally let us estimate the first integral in (56) for a given $\tau$:

$$\int_0^\tau e^{s\hat{L}_n^*} (f^{(n)}(t-s) - f_\varrho^{(n)}) \; ds. \tag{60}$$

From (55) it follows that we can choose $t_0 > \tau$ such that for $t > t_0$ the following estimate holds

$$\|f^{(n)}(t-\tau) - f_\varrho^{(n)}\|_{X_n} \;<\; \frac{\varepsilon}{\tau}.$$

Consequently the norm of (60) is less than $\varepsilon$. Finally, for $t > t_0$ the integral in (51) is less than $(C+1)\varepsilon$ in sup-norm and convergence (51) as well as (48) to zero is proved.

Thus we proved the strong convergence (17). The final step of the proof follows the same line as in [4]. Using results from [6] we can conclude that the solution $\{k_\varrho^{(n)}(t)\}$ of the Cauchy problem (10) is a system of correlation functions corresponding to the evolution of states $\{\mu_t\}$. And the limit of $\mu_t$ as $t \to \infty$ will be a measure $\mu_\varrho$ corresponding to the limit

$$\lim_{t\to\infty} k_\varrho^{(n)}(t) \;=\; k_\varrho^{(n)},$$

where $k_\varrho^{(n)}$ is a solution (44) of the system (14).

# 5 Concluding remarks and the proof of Theorem 2.

**Remark 1.** *If instead of (13) we take the initial data in the form*

$$\bar{k}^{(n)}|_{t=0}(x_1,\ldots,x_n) \;=\; \varrho^n \prod_{i=1}^{n} h(\sigma(x_i)) \tag{61}$$

*for some $\varrho > 0$ and some normalized positive function $h(s)$, $\int_S h(s)d\nu(s) = 1$ (not necessarily the eigenfunction of $Q$ and not necessarily continuous), then we have convergence*

$$\bar{k}^{(n)}(t) \;\to\; k_{\varrho_1}^{(n)}, \quad t \to \infty,$$

*where $k_{\varrho_1}^{(n)}$ is defined by the formula (44), and*

$$\varrho_1 \;=\; \frac{\varrho\langle h,\ \tilde{q}\rangle}{\langle q,\ \tilde{q}\rangle} \;=\; \frac{\varrho \int_S h(s)\ \tilde{q}(s)\ d\nu(s)}{\int_S q(s)\ \tilde{q}(s)\ d\nu(s)}. \tag{62}$$

*Here $\tilde{q}$ is the positive eigenfunction of the adjoint operator $Q^*$.*

To prove this convergence we should make some small modifications in the above reasoning.

1) In (49) we have the additional term

$$e^{t\hat{L}_n^*}(\bar{k}^{(n)}(0) - k^{(n)}(0,\varrho_1)).$$

This term does not depend on space coordinates and equals to

$$\otimes_i e^{t(Q^i - E)}(\bar{k}^{(n)}(0) - k^{(n)}(0,\varrho_1)). \tag{63}$$

Here $Q^i$ is the operator $Q$ acting on $i$-th spin variable. From the Krein-Rutman theorem it follows that (63) tends to 0 if (62) is fulfilled.

2) The condition (53) is also violated. The first correlation function $k^{(1)}(t)$ now depends on time ( but not depends on space variables) and satisfies an equation

$$\frac{\partial k^{(1)}}{\partial t} \;=\; \hat{L}_1^*\, k^{(1)} \;=\; (Q - E)\, k^{(1)}$$

with the initial data $\bar{k}^{(1)}|_{t=0} = \varrho\, h(s)$. Then

$$k^{(1)}(t) \;\to\; \varrho_1\, q(s), \quad t \to \infty$$

by the Krein-Rutman theorem provided the density equation (62) is fulfilled.

The same approach can be applied when the initial data are mixtures of marked Poisson fields with different spatial densities and different mark distributions (provided marks are mutually independent).

**Remark 2. Law of large numbers.** *Theorem 1 implies the correlation decay*

$$|k_\varrho^{(2)}(x_1,x_2) \;-\; k_\varrho^{(1)}(x_1)k_\varrho^{(1)}(x_2)| \;\to\; 0,$$

*when* $|\tau(x_1)-\tau(x_2)| \to \infty$. *By the standart application of the Chebyshev inequality, see e.g. [9], we get the low of large numbers for the number of particles, i.e. the existence of the spatial density of particles:*

$$\frac{N(V)}{V} \;\to\; \varrho, \qquad \text{as } \; V \to \infty.$$

**The proof of Theorem 2** is completely analogous to the proof of Theorem 1, when $m(s) \equiv 1$. We only should check that

$$\tilde{L}_n^* \prod_{i=1}^{n} g(\sigma(x_i)) \;=\; 0.$$

Indeed, we have

$$\tilde{L}_n^* \;=\; \sum_{i=1}^{n} \tilde{L}^i,$$

$$\tilde{L}^i \; \prod_{j=1}^{n} g(\sigma(x_j)) \;=$$

$$\prod_{j\neq i} g(\sigma(x_j)) \left( -m(\sigma(x_i))g(\sigma(x_i)) + \kappa_{cr} \int_S Q(\sigma(x_i),s')g(s')d\nu(s') \right) \;=\; 0,$$

since $\kappa_{cr}\tilde{Q}g \;=\; g$ implies

$$\kappa_{cr} \int_S Q(s,s')g(s')d\nu(s') \;=\; m(s)\; g(s).$$

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