THE MICROSCOPIC DYNAMICS OF A SPATIAL ECOLOGICAL MODEL

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ABSTRACT. The evolution of states of a spatial ecological model is studied. The model describes an infinite population of point entities placed in $\mathbb{R}^d$ which reproduce themselves at distant points (disperse) and die with rate that includes a competition term. The system's states are probability measures on the space of configurations of entities, and their evolution is described by means of a hierarchical chain of equations for the corresponding correlation functions derived from the Fokker-Planck equation for measures. Under natural conditions imposed on the model parameters it is proved that the correlation functions evolve in a scale of Banach spaces in such a way that each correlation function corresponds to a unique sub-Poissonian state. Some further properties of the evolution of states constructed in this way are also described.

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1. Introduction

1.1. **Posing.** In this paper we continue, cf. [7, 8, 9, 18], studying the model introduced in [4, 5, 19]. It describes an infinite evolving population of identical point entities (particles) distributed over $\mathbb{R}^d$, $d \geq 1$, which reproduce themselves and die, also due to competition. Such individual-based models are used to study large ecological communities (e.g., of perennial plants), see [22] and [20, page 1311]. As is now commonly adopted, see, e.g., [4, 5, 22], the appropriate mathematical context for studying models of this kind is provided by the theory of random point fields on $\mathbb{R}^d$ in which populations are modeled as point configurations constituting the set

$$\Gamma = \{\gamma \subset \mathbb{R}^d : |\gamma \cap \Lambda| < \infty \text{ for any compact } \Lambda \subset \mathbb{R}^d \}, \tag{1.1}$$

where $|\cdot|$ denotes cardinality. It is equipped with a measurability structure that allows one to consider probability measures on Γ as states of the system. To characterize such states one employs *observables* – appropriate functions $F : \Gamma \to \mathbb{R}$. Their evolution is obtained from the Kolmogorov equation

$$\frac{d}{dt} F_t = L F_t, \qquad F_t|_{t=0} = F_0, \qquad t > 0, \tag{1.2}$$

where the 'generator' L specifies the model. The states' evolution is then obtained from the Fokker–Planck equation

$$\frac{d}{dt} \mu_t = L^* \mu_t, \qquad \mu_t|_{t=0} = \mu_0, \tag{1.3}$$

related to that in (1.2) by the duality

$$\int_\Gamma F_0 d\mu_t = \int_\Gamma F_t d\mu_0.$$

The generator of the model studied in this paper is

$$\begin{aligned} (LF)(\gamma) \quad &= \quad \sum_{x\in\gamma} \left[m + E^-(x, \gamma \setminus x) \right] \left[F(\gamma \setminus x) - F(\gamma) \right] \\ & \quad + \int_{\mathbb{R}^d} E^+(y, \gamma) \left[F(\gamma \cup y) - F(\gamma) \right] dy, \end{aligned} \tag{1.4}$$

where

$$E^{\pm}(x,\gamma) := \sum_{y\in\gamma} a^{\pm}(x-y). \tag{1.5}$$

The first summand in (1.4) corresponds to the death of the particle located at x occurring independently at rate $m \geq 0$ (intrinsic mortality) and under the influence of the other particles in γ – at rate $E^{-}(x,\gamma\setminus x) \geq 0$ (competition). The second term in (1.4) describes the birth of a particle at $y \in \mathbb{R}^d$ occurring at rate $E^{+}(y,\gamma) \geq 0$. In the sequel, we call a^{-} and a^{+} *competition* and *dispersal kernels*, respectively. This model plays a significant role in the mathematical theory of large ecological systems, see, e.g., [22] for a detailed discussion and the references on this matter. The problem of constructing spatial birth and death processes in infinite volume was first studied by R. A. Holley and D. W. Stroock in their pioneering work [12], where a spacial case of nearest neighbor interactions on the real line was considered. For more general versions of continuum birth-and-death systems, the few results known by this time were obtained under substantial restrictions on the birth and death rates. This relates to the construction of a Markov process in [11], as well as to the result obtained in [8] in the statistical approach (see below). In the present work, we make an essential step forward in studying the model as in (1.4) assuming only that the kernels $a^{\pm}$ satisfy some rather mild condition. The version of (1.4) with $a^{-} \equiv 0$ is the *continuum contact model* studied in [16, 17].

The set of finite configurations Γ_0 is a measurable subset of Γ. If μ is such that $\mu(\Gamma_0) = 1$, then the considered system is finite in this state. If μ_0 in (1.3) has such a property, the evolution $\mu_0 \mapsto \mu_t$ can be obtained directly from (1.3), see [18]. In this case $\mu_t(\Gamma_0) = 1$ for all $t > 0$. States of infinite systems are mostly such that $\mu(\Gamma_0) = 0$, which makes direct solving (1.3) with an *arbitrary* initial state μ_0 rather unaccessible for the method existing at this time, cf. [15]. In this work we continue following the statistical approach, cf. [3, 7, 8, 9, 15], in which the evolution of states is described as that of the corresponding correlation functions. To briefly explain its essence let us consider the set of all compactly supported continuous functions $\theta : \mathbb{R}^d \to (-1, 0]$. For a probability measure μ on Γ its *Bogoliubov* functional [10, 14] is defined as

$$B_\mu(\theta) = \int_\Gamma \prod_{x\in\gamma} (1+\theta(x))\mu(d\gamma), \tag{1.6}$$

with θ running through the mentioned set of functions. For $\pi_\varkappa$ – the homogeneous Poisson measure with intensity $\varkappa > 0$, (1.6) takes the form

$$B_{\pi_\varkappa}(\theta) = \exp\left(\varkappa \int_{\mathbb{R}^d} \theta(x)dx\right).$$

In state $\pi_\varkappa$, the particles are independently distributed over $\mathbb{R}^d$ with density $\varkappa$. The set of *sub-Poissonian* states $\mathcal{P}_{\rm sP}$ is then defined as that containing all the states μ for which B_μ can be continued, as a function of θ, to an exponential type entire function on $L^1(\mathbb{R}^d)$. This exactly means that B_μ can be written down in the form

$$B_\mu(\theta) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{(\mathbb{R}^d)^n} k_\mu^{(n)}(x_1, \ldots, x_n)\theta(x_1) \cdots \theta(x_n) dx_1 \cdots dx_n, \qquad (1.7)$$

where $k_\mu^{(n)}$ is the n-th order correlation function corresponding to μ. It is a symmetric element of $L^\infty((\mathbb{R}^d)^n)$ for which

$$\|k_\mu^{(n)}\|_{L^\infty((\mathbb{R}^d)^n)} \leq C \exp(\alpha n), \qquad n \in \mathbb{N}_0, \qquad (1.8)$$

with some $C > 0$ and $\alpha \in \mathbb{R}$. This guarantees that B_μ is of exponential type. One can also consider a wider class of states, $\mathcal{P}_{\rm anal}$, by imposing the condition that B_μ can be continued to a function on $L^1(\mathbb{R}^d)$ analytic in some neighborhood of the origin, see [14]. In that case, the estimate corresponding to (1.8) will contain $n!Ce^{\alpha n}$ in its right-hand side. States $\mu \in \mathcal{P}_{\rm anal}$ are characterized by strong correlations corresponding to 'clustering'. In the contact model the clustering does take place, see [16, 17] and especially [7, Eq. (3.5), page 303]. Namely, in this model for each $t > 0$ and $n \in \mathbb{N}$ the correlation functions satisfy the following estimates

$$\text{const} \cdot n! c_t^n \leq k_t^{(n)}(x_1, \ldots, x_n) \leq \text{const} \cdot n! C_t^n,$$

where the left-hand inequality holds if all x_i belong to a ball of sufficiently small radius. If the mortality rate m is big enough, then $C_t \to 0$ as $t \to +\infty$. That is, in the continuum contact model the clustering persists even if the population asymptotically dies out. With this regard, a paramount question about the model (1.4) is whether the competition contained in L can suppress clustering. In short, the answer given in this work is in affirmative provided the competition and dispersal kernels satisfy a certain natural condition. They do satisfy if a^- is strictly positive in some vicinity of the origin, and a^+ has finite range.

1.2. **Presenting the result.** In this work, for the model described in (1.4) and (1.5) we obtain the following results:

- (i) Under the condition on the kernels $a^\pm$ formulated in Assumption 1 we prove in Theorem 3.3 that the correlation functions evolve $k_{\mu_0}^{(n)} \mapsto k_t^{(n)}$ in a scale of Banach spaces in such a way that each $k_t^{(n)}$ is the correlation function of a unique sub-Poissonian measure μ_t.
- (ii) We give examples of the kernels $a^\pm$ which satisfy Assumption 1. These examples include kernels of finite range – both short and long dispersals (Proposition 3.7), and also Gaussian kernels (Propositions 3.8).

(iii) For the whole range of values of the intrinsic mortality rate m, in Theorem 3.4 we obtain the following bounds for the correlation functions holding for all $t \geq 0$:

$$(i) \quad 0 \leq k_t^{(n)}(x_1, \ldots, x_n) \leq C_\delta^n \exp\big(n(\langle a^+\rangle - \delta)t\big), \quad 0 \leq m \leq \langle a^+\rangle,$$

$$(ii) \quad 0 \leq k_t^{(n)}(x_1, \ldots, x_n) \leq C_\varepsilon^n e^{-\varepsilon t}, \quad m > \langle a^+\rangle,$$

where $\langle a^+\rangle$ is the L^1-norm of a^+, C_δ and C_ε are appropriate positive constants, whereas $\delta < m$ and $\varepsilon \in (\langle a^+\rangle, m)$ take any value in the mentioned sets. By (1.7) these estimates give upper bounds for the type of B_{μ_t}. We describe also the pure death case where $\langle a^+\rangle = 0$.

More detailed comments and comparison with the previous results on this model are given in subsection 3.3 below.

2. The Basic Notions

A detailed description of various aspects of the mathematical framework of this paper can be found in [1, 3, 7, 8, 9, 13, 16, 17, 21]. Here we present only some of its aspects and indicate in which of the mentioned papers further details can be found. By $\mathcal{B}(\mathbb{R}^d)$ and $\mathcal{B}_{\rm b}(\mathbb{R}^d)$ we denote the set of all Borel and all bounded Borel subsets of $\mathbb{R}^d$, respectively.

2.1. **The configuration spaces.**

The space Γ defined in (1.1) is endowed with the weakest topology that makes continuous all the maps

$$\Gamma \ni \gamma \mapsto \sum_{x\in\gamma} f(x), \quad f \in C_0(\mathbb{R}^d).$$

Here $C_0(\mathbb{R}^d)$ stands for the set of all continuous compactly supported functions $f : \mathbb{R}^d \to \mathbb{R}$. The mentioned topology on Γ admits a metrization which turns it into a complete and separable metric (Polish) space. By $\mathcal{B}(\Gamma)$ we denote the corresponding Borel σ-field. For $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, the set of n-particle configurations in $\mathbb{R}^d$ is

$$\Gamma^{(0)} = \{\emptyset\}, \qquad \Gamma^{(n)} = \{\eta \subset X : |\eta| = n\}, \quad n \in \mathbb{N}.$$

For $n \geq 1$, $\Gamma^{(n)}$ can be identified with the symmetrization of the set

$$\Big\{(x_1, \ldots, x_n) \in \big(\mathbb{R}^d\big)^n : x_i \neq x_j, \ \text{for } i \neq j\Big\},$$

which allows one to introduce the topology on $\Gamma^{(n)}$ related to the Euclidean topology of $\mathbb{R}^d$ and hence the corresponding Borel σ-field $\mathcal{B}(\Gamma^{(n)})$. The set of finite configurations

$$\Gamma_0 := \bigsqcup_{n\in\mathbb{N}_0} \Gamma^{(n)}$$

is endowed with the topology of the disjoint union and with the corresponding Borel σ-field $\mathcal{B}(\Gamma_0)$. It is a measurable subset of Γ. However, the topology just mentioned and that induced on Γ_0 from Γ do not coincide.

For $\Lambda \in \mathcal{B}_{\rm b}(\mathbb{R}^d)$, the set $\Gamma_\Lambda := \{\gamma \in \Gamma : \gamma \subset \Lambda\}$ is a Borel subset of Γ_0. We equip Γ_Λ with the topology induced by that of Γ_0. Let $\mathcal{B}(\Gamma_\Lambda)$ be the corresponding Borel σ-field. It can be proved, see [21, Lemma 1.1 and Proposition 1.3], that

$$\mathcal{B}(\Gamma_\Lambda) = \{\Gamma_\Lambda \cap \Upsilon : \Upsilon \in \mathcal{B}(\Gamma)\}.$$

It is known [1, page 451] that $\mathcal{B}(\Gamma)$ is the smallest σ-field of subsets of Γ such that all the projections

$$\Gamma \ni \gamma \mapsto p_\Lambda(\gamma) = \gamma_\Lambda := \gamma \cap \Lambda, \qquad \Lambda \in \mathcal{B}_{\rm b}(\mathbb{R}^d), \tag{2.1}$$

are $\mathcal{B}(\Gamma)/\mathcal{B}(\Gamma_\Lambda)$ measurable. This means that $(\Gamma, \mathcal{B}(\Gamma))$ is the projective limit of the measurable spaces $(\Gamma_\Lambda, \mathcal{B}(\Gamma_\Lambda))$, $\Lambda \in \mathcal{B}_{\rm b}(\mathbb{R}^d)$.

Remark 2.1. From the latter discussion it follows that $\Gamma_0 \in \mathcal{B}(\Gamma)$ and

$$\mathcal{B}(\Gamma_0) = \{A \cap \Gamma_0 : A \in \mathcal{B}(\Gamma)\}. \tag{2.2}$$

Hence, a probability measure μ on $\mathcal{B}(\Gamma)$ with the property $\mu(\Gamma_0) = 1$ can be considered also as a measure on $\mathcal{B}(\Gamma_0)$.

2.2. **Functions and measures on configuration spaces.** A Borel set $\Upsilon \subset \Gamma$ is said to be bounded if the following holds

$$\Upsilon \subset \bigcup_{n=0}^{N} \Gamma_\Lambda^{(n)},$$

for some $\Lambda \in \mathcal{B}_{\rm b}(\mathbb{R}^d)$ and $N \in \mathbb{N}$. In view of (2.2), each bounded set is in $\mathcal{B}(\Gamma_0)$. A function $G : \Gamma_0 \to \mathbb{R}$ is measurable if and only if, for each $n \in \mathbb{N}$, there exists a symmetric Borel function $G^{(n)} : (\mathbb{R}^d)^n \to \mathbb{R}$ such that

$$G(\eta) = G^{(n)}(x_1, \dots, x_n), \quad \text{for} \quad \eta = \{x_1, \dots, x_n\}. \tag{2.3}$$

Definition 2.2. A bounded measurable function $G : \Gamma_0 \to \mathbb{R}$ is said to have bounded support if: (a) $G(\eta) = 0$ whenever $\eta \cap \Lambda^c \neq \emptyset$ for some $\Lambda \in \mathcal{B}_{\rm b}(\mathbb{R}^d)$, $\Lambda^c := \mathbb{R}^d \setminus \Lambda$; (b) $G^{(n)} \equiv 0$ whenever $n > N$ for some $N \in \mathbb{N}$. The set of all such functions is denoted by $B_{\rm bs}(\Gamma_0)$. For a given $G \in B_{\rm bs}(\Gamma_0)$, by $N(G)$ we denote the smallest N with the property as in (b).

A map $F : \Gamma \to \mathbb{R}$ is called cylinder function if there exist $\Lambda \in \mathcal{B}_{\rm b}(\mathbb{R}^d)$ and a measurable $G : \Gamma_\Lambda \to \mathbb{R}$ such that, cf. (2.1), $F(\gamma) = G(\gamma_\Lambda)$ for all $\gamma \in \Gamma$. Clearly, such an F is measurable. By $\mathcal{F}_{\rm cyl}(\Gamma)$ we denote the set of all cylinder functions. For $\gamma \in \Gamma$, by writing $\eta \Subset \gamma$ we mean that $\eta \subset \gamma$ and η is finite, i.e., $\eta \in \Gamma_0$. For $G \in B_{\rm bs}(\Gamma_0)$, we set

$$(KG)(\gamma) = \sum_{\eta \Subset \gamma} G(\eta), \qquad \gamma \in \Gamma. \tag{2.4}$$

By [13] we know that K maps $B_{\rm bs}(\Gamma_0)$ onto $\mathcal{F}_{\rm cyl}(\Gamma)$ and is invertible. The Lebesgue-Poisson measure λ on $\mathcal{B}(\Gamma_0)$ is defined by the relation

$$\int_{\Gamma_0} G(\eta)\lambda(d\eta) = G(\emptyset) + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{(\mathbb{R}^d)^n} G^{(n)}(x_1, \ldots, x_n) dx_1 \cdots dx_n, \quad (2.5)$$

which has to hold for all $G \in B_{\rm bs}(\Gamma_0)$, cf. (2.3). Note that $B_{\rm bs}(\Gamma_0)$ is a measure defining class. Clearly, $\lambda(\Upsilon) < \infty$ for each bounded $\Upsilon \in \mathcal{B}(\Gamma_0)$. with the help of (2.5), we rewrite (1.7) in the following form

$$B_\mu(\theta) = \int_{\Gamma_0} k_\mu(\eta) \left(\prod_{x\in\eta} \theta(x) \right) \lambda(d\eta). \quad (2.6)$$

In the sequel, by saying that something holds for all η we assume that is holds for λ-almost all $\eta \in \Gamma_0$. This relates also to (2.3).

Let $\mathcal{P}(\Gamma)$, resp. $\mathcal{P}(\Gamma_0)$, stand for the set of all probability measures on $\mathcal{B}(\Gamma)$, resp. $\mathcal{B}(\Gamma_0)$. Note that $\mathcal{P}(\Gamma_0)$ can be considered as a subset of $\mathcal{P}(\Gamma)$, see Remark 2.1. For a given $\mu \in \mathcal{P}(\Gamma)$, the *projection* μ^Λ is defined as

$$\mu^\Lambda(A) = \mu(p_\Lambda^{-1}(A)), \qquad A \in \mathcal{B}(\Gamma_\Lambda), \quad (2.7)$$

where $p_\Lambda^{-1}(A) := \{\gamma \in \Gamma : p_\Lambda(\gamma) \in A\}$, see (2.1). The projections of the Lebesgue-Poisson measure λ are defined in the same way.

Recall that $\mathcal{P}_{\rm anal}$ (resp. $\mathcal{P}_{\rm sP}$) denotes the set of all those $\mu \in \mathcal{P}(\Gamma)$ for each of which B_μ defined in (1.6), or (2.6), admits continuation to a function on $L^1(\mathbb{R}^d)$ analytic in some neighborhood of zero (resp. exponential type entire function). The elements of $\mathcal{P}_{\rm sP}$ are called sub-Poissonian states. One can show [13, Proposition 4.14] that for each $\Lambda \in \mathcal{B}_{\rm b}(\mathbb{R}^d)$ and $\mu \in \mathcal{P}_{\rm sP}$, μ^Λ is absolutely continuous with respect to λ^Λ. The Radon-Nikodym derivative

$$R_\mu^\Lambda(\eta) = \frac{d\mu^\Lambda}{d\lambda^\Lambda}(\eta), \qquad \eta \in \Gamma_\Lambda, \quad (2.8)$$

and the correlation function k_μ satisfy

$$k_\mu(\eta) = \int_{\Gamma_\Lambda} R_\mu^\Lambda(\eta \cup \xi)\lambda^\Lambda(d\xi), \qquad \eta \in \Gamma_\Lambda, \quad (2.9)$$

which holds for all $\Lambda \in \mathcal{B}_{\rm b}(\mathbb{R}^d)$. Note that (2.9) relates R_μ^Λ with the restriction of k_μ to Γ_Λ. The fact that these are restrictions of one and the same function $k_\mu : \Gamma_0 \to \mathbb{R}$ corresponds to the Kolmogorov consistency of the family $\{\mu^\Lambda\}_{\Lambda\in\mathcal{B}_{\rm b}(\mathbb{R}^d)}$.

By (2.4), (2.7), and (2.9) we get

$$\int_\Gamma (KG)(\gamma)\mu(d\gamma) = \langle\!\langle G, k_\mu \rangle\!\rangle,$$

which holds for each $G \in B_{\rm bs}(\Gamma_0)$ and $\mu \in \mathcal{P}_{\rm sP}$. Here and in the sequel we use the notation

$$\langle\!\langle G, k \rangle\!\rangle = \int_{\Gamma_0} G(\eta)k(\eta)\lambda(d\eta), \quad (2.10)$$

Define

$$B^\star_{\rm bs}(\Gamma_0) = \{G \in B_{\rm bs}(\Gamma_0) : KG \not\equiv 0, \ (KG)(\gamma) \geq 0 \text{ for all } \gamma \in \Gamma\}. \qquad (2.11)$$

By [13, Theorems 6.1 and 6.2 and Remark 6.3] we know that the following holds.

Proposition 2.3. *Assume that a measurable function $k : \Gamma_0 \to \mathbb{R}$ has the following properties:*

$$(i) \quad \langle\!\langle G, k \rangle\!\rangle \geq 0 \qquad \text{for all} \quad G \in B^\star_{\rm bs}(\Gamma_0), \qquad (2.12)$$

$$(ii) \quad k(\emptyset) = 1, \qquad (iii) \quad k(\eta) \leq \prod_{x\in\eta} C(x), \qquad \eta \in \Gamma_0,$$

for some bounded measurable function $C : \mathbb{R}^d \to \mathbb{R}_+$. Then there exists a unique $\mu \in \mathcal{P}_{\rm sP}$ for which k is the correlation function.

Finally, we mention the convention

$$\sum_{a\in\emptyset} \phi_a := 0, \qquad \prod_{a\in\emptyset} \psi_a := 1$$

which we use in the sequel and the integration rule, see, e.g., [7],

$$\int_{\Gamma_0} \sum_{\xi\subset\eta} H(\xi, \eta\setminus\xi, \eta)\lambda(d\eta) = \int_{\Gamma_0}\int_{\Gamma_0} H(\xi, \eta, \eta\cup\xi)\lambda(d\xi)\lambda(d\eta), \qquad (2.13)$$

valid for appropriate functions H.

2.3. **Spaces of functions.** For each $\mu \in \mathcal{P}_{\rm sP}$, the correlation function satisfies the bound (1.8) in view of which we introduce the following Banach spaces. For $\alpha \in \mathbb{R}$, we set

$$\|k\|_\alpha = \operatorname*{ess\,sup}_{\eta\in\Gamma_0} |k(\eta)| \exp(-\alpha|\eta|). \qquad (2.14)$$

It is a norm, that can also be written as follows. As in (2.3), each $k : \Gamma_0 \to \mathbb{R}$ is defined by its restrictions to $\Gamma^{(n)}$. Let $k^{(n)} : (\mathbb{R}^d)^n \to \mathbb{R}$ be a symmetric Borel function such that $k^{(n)}(x_1, \dots, x_n) = k(\eta)$ for $\eta = \{x_1, \dots, x_n\}$. We then assume that $k^{(n)} \in L^\infty((\mathbb{R}^d)^n)$, $n \in \mathbb{N}$, cf. (1.8), and define

$$\|k\|_\alpha = \sup_{n\in\mathbb{N}_0} e^{-\alpha n} \nu_n(k), \quad \nu_n(k) := \|k^{(n)}\|_{L^\infty((\mathbb{R}^d)^n)}, \qquad (2.15)$$

that yields the same norm as in (2.14). Obviously,

$$\mathcal{K}_\alpha := \{k : \Gamma_0 \to \mathbb{R} : \|k\|_\alpha < \infty\}, \qquad (2.16)$$

is a Banach space. For $\alpha' < \alpha''$, we have $\|k\|_{\alpha''} \leq \|k\|_{\alpha'}$. Hence,

$$\mathcal{K}_{\alpha'} \hookrightarrow \mathcal{K}_{\alpha''}, \qquad \text{for } \alpha' < \alpha''. \qquad (2.17)$$

Here and in the sequel, by $X \hookrightarrow Y$ we mean a continuous embedding of these two Banach spaces. For $\alpha \in \mathbb{R}$, we define, cf. (2.11) and (2.10),

$$\mathcal{K}^\star_\alpha = \{k \in \mathcal{K}_\alpha : \forall G \in B^\star_{\rm bs}(\Gamma_0) \ \langle\!\langle G, k \rangle\!\rangle \geq 0\}. \qquad (2.18)$$

It is a subset of the cone

$$\mathcal{K}_\alpha^+ = \{k \in \mathcal{K}_\alpha : k(\eta) \geq 0 \ \text{ for } \ \text{a.a.} \ \ \eta \in \Gamma_0\}. \tag{2.19}$$

By Proposition 2.3 we have that each $k \in \mathcal{K}_\alpha^\star$ with the property $k(\emptyset) = 1$ is the correlation function of a unique $\mu \in \mathcal{P}_{\mathrm{sP}}$. We also put

$$\mathcal{K}_\infty = \bigcup_{\alpha \in \mathbb{R}} \mathcal{K}_\alpha, \tag{2.20}$$

and equip this set with the inductive topology. Finally, we define

$$\mathcal{K}_\infty^\star = \bigcup_{\alpha \in \mathbb{R}} \mathcal{K}_\alpha^\star.$$

3. The Model and the Results

3.1. **The model.** As was already mentioned, the model is specified by the expression given in (1.4). Regarding the kernels in (1.5) we suppose that

$$a^\pm \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d), \qquad a^\pm(x) = a^\pm(-x) \geq 0, \tag{3.1}$$

and thus define

$$\langle a^\pm \rangle = \int_{\mathbb{R}^d} a^\pm(x) dx, \qquad \|a^\pm\| = \operatorname*{ess\,sup}_{x \in \mathbb{R}^d} a^\pm(x), \tag{3.2}$$

and

$$E^\pm(\eta) = \sum_{x \in \eta} E^\pm(x, \eta \setminus x) = \sum_{x \in \eta} \sum_{y \in \eta \setminus x} a^\pm(x - y), \quad \eta \in \Gamma_0. \tag{3.3}$$

We also denote

$$E(\eta) = \sum_{x \in \eta} \big(m + E^-(x, \eta \setminus x) \big) = m|\eta| + E^-(\eta), \tag{3.4}$$

where m is the same as in (1.4).

In addition to the standard assumptions (3.1) we shall use the following

Assumption 1 (*(b, ϑ)-assumption*). There exist $\vartheta > 0$ and $b \geq 0$ such that the functions introduced in (3.3) satisfy

$$b|\eta| + E^-(\eta) \geq \vartheta E^+(\eta), \qquad \eta \in \Gamma_0. \tag{3.5}$$

Note that the case of point-wise domination

$$a^-(x) \geq \vartheta a^+(x), \qquad x \in \mathbb{R}^d, \tag{3.6}$$

cf. [9, Eq. (3.11)], corresponds to (3.5) with $b = 0$. In subsection 3.4 below we give examples of the kernels $a^\pm$ which satisfy (3.5). To exclude the trivial case of $a^+ = a^- = 0$ we also assume that

$$\langle a^- \rangle > 0.$$

3.2. **The results.**

3.2.1. *The operators.* In view of the relationship between states and correlation functions discussed in subsection 2.3, we describe the system's dynamics in the following way. First we obtain the evolution $k_{\mu_0} \mapsto k_t$ by proving the existence of a unique solution of the Cauchy problem of the following type

$$\frac{dk_t}{dt} = L^\Delta k_t, \qquad k_t|_{t=0} = k_{\mu_0}, \tag{3.7}$$

where the action of L^Δ is calculated from (1.4). Thereafter, we show that each k_t has the property $k_t(\emptyset) = 1$ and lies in $\mathcal{K}^\star_\alpha$ for some $\alpha \in \mathbb{R}$. Hence, it is the correlation function of a unique $\mu_t \in \mathcal{P}_{\rm sP}$. This yields in turn the evolution $\mu_0 \mapsto \mu_t$.

The describe the action of L^Δ in a systematic way we write it the following form, see [7, 9],

$$L^\Delta = A^\Delta + B^\Delta, \tag{3.8}$$

where

$$A^\Delta = A^\Delta_1 + A^\Delta_2, \tag{3.9}$$

$$(A^\Delta_1 k)(\eta) = -E(\eta)k(\eta), \qquad (A^\Delta_2 k)(\eta) = \sum_{x\in\eta} E^+(x, \eta\setminus x) k(\eta \setminus x),$$

see also (3.3), (3.4), and

$$\begin{aligned} B^\Delta &= B^\Delta_1 + B^\Delta_2, \qquad (3.10) \\ (B^\Delta_1 k)(\eta) &= -\int_{\mathbb{R}^d} E^-(y,\eta) k(\eta\cup y) dy, \\ (B^\Delta_2 k)(\eta) &= \int_{\mathbb{R}^d} \sum_{x\in\eta} a^+(x-y) k(\eta\setminus x \cup y) dy. \end{aligned}$$

The key idea of the method that we use to study (3.7) is to employ the scale of spaces (2.16) in which A^Δ and B^Δ act as bounded operators from $\mathcal{K}_{\alpha'}$, to any $\mathcal{K}_\alpha$ with $\alpha > \alpha'$, cf. (2.17). For such α and α', by (2.14) and (2.15) we have, see (3.9),

$$\begin{aligned} \|A^\Delta_1 k\|_\alpha &\leq \|k\|_{\alpha'} \operatorname*{ess\,sup}_{\eta\in\Gamma_0} E(\eta) \exp\left(-(\alpha-\alpha')|\eta|\right), \\ \|A^\Delta_2 k\|_\alpha &\leq \operatorname*{ess\,sup}_{\eta\in\Gamma_0} e^{-\alpha|\eta|} \sum_{x\in\eta} E^+(x,\eta\setminus x)|k(\eta\setminus x)| \\ &\leq \|k\|_{\alpha'} e^{-\alpha'} \operatorname*{ess\,sup}_{\eta\in\Gamma_0} E^+(\eta) \exp\left(-(\alpha-\alpha')|\eta|\right), \end{aligned}$$

which by (2.15) and (3.2) yields

$$\begin{aligned} \|A_1^\Delta k\|_\alpha &\leq \|k\|_{\alpha'} \left(\frac{m}{e(\alpha-\alpha')} + \frac{4\|a^-\|}{e^2(\alpha-\alpha')^2} \right) \\ \|A_2^\Delta k\|_\alpha &\leq \|k\|_{\alpha'} e^{-\alpha'} \frac{4\|a^+\|}{e^2(\alpha-\alpha')^2}, \end{aligned} \tag{3.11}$$

where we have used the estimate

$$n^p e^{-\sigma n} \leq \left(\frac{p}{e\sigma}\right)^p, \quad p \geq 1, \ \sigma > 0, \ n \in \mathbb{N}. \tag{3.12}$$

In a similar way, we obtain from (3.10) the following estimate, see (3.2),

$$\|B^\Delta k\|_\alpha \leq \|k\|_{\alpha'} \frac{\langle a^+ \rangle + \langle a^- \rangle e^{\alpha'}}{e(\alpha-\alpha')}. \tag{3.13}$$

Thus, by means of (3.8) – (3.10), and then by (3.11) and (3.13), for each α, $\alpha' \in \mathbb{R}$, $\alpha' < \alpha$, one can define a continuous operator

$$L^\Delta_{\alpha\alpha'} : \mathcal{K}_{\alpha'} \to \mathcal{K}_\alpha. \tag{3.14}$$

Let $\mathcal{L}(\mathcal{K}_{\alpha'}, \mathcal{K}_\alpha)$ stand for the set of all bounded linear operators $\mathcal{K}_{\alpha'} \to \mathcal{K}_\alpha$. The operator norm of $L^\Delta_{\alpha\alpha'}$ can be estimated by means of the above formulas. Thus, the family $\{L^\Delta_{\alpha\alpha'}\}_{\alpha,\alpha'}$ determines a bounded linear map $L^\Delta : \mathcal{K}_\infty \to \mathcal{K}_\infty$. Along with these continuous operators, in each $\mathcal{K}_\alpha$, $\alpha \in \mathbb{R}$, we define an unbounded operator, L^Δ_α, with domain

$$\mathcal{D}^\Delta_\alpha = \{k \in \mathcal{K}_\alpha : L^\Delta k \in \mathcal{K}_\alpha\} \supset \mathcal{K}_{\alpha'}, \tag{3.15}$$

which holds for each $\alpha' < \alpha$, see (3.11), (3.13), and (3.8). The operators such introduced are related to each other in the following way:

$$\forall k \in \mathcal{K}_{\alpha'} \qquad L^\Delta_{\alpha\alpha'} k = L^\Delta_\alpha k, \tag{3.16}$$

which holds for each $\alpha' < \alpha$.

3.2.2. *The statements.* Now we can make precise which equations we are going to solve. One possibility is to consider (3.7) in a given Banach space, $\mathcal{K}_\alpha$.

Definition 3.1. Given $\alpha \in \mathbb{R}$ and $T \in (0, +\infty]$, by a solution of the Cauchy problem

$$\frac{d}{dt} k_t = L^\Delta_\alpha k_t, \qquad k_t|_{t=0} = k_0 \in \mathcal{D}^\Delta_\alpha, \tag{3.17}$$

in $\mathcal{K}_\alpha$ we mean a continuous map $[0,T) \ni t \mapsto k_t \in \mathcal{D}_\alpha$, continuously differentiable in $\mathcal{K}_\alpha$ on $[0,T)$ and such that (3.17) is satisfied for all $t \in [0,T)$.

Another possibility is to define (3.7) in the locally convex space (2.20).

Definition 3.2. For a given $T_* \in (0, +\infty]$, by a solution of the Cauchy problem (3.7) in $\mathcal{K}_\infty$ with a given $k_0 \in \mathcal{K}_\infty$ we mean a map $[0, T_*) \ni t \mapsto k_t \in \mathcal{K}_\infty$, continuously differentiable on $[0, T_*)$ and such that (3.7) is satisfied for all $t \in [0, T_*)$.

Note that T_* in Definition 3.2 is such that for each $T < T_*$, there exist $\alpha_0, \alpha \in \mathbb{R}$, $\alpha_0 < \alpha$, for which the mentioned k_t is a solution as in Definition 3.1 with $k_0 \in \mathcal{K}_{\alpha_0}$. Our main results are contained in the following two statements.

Theorem 3.3. *Let (b, ϑ)-assumption (3.5) hold true, and μ_0 be an arbitrarily sub-Poissonian state. Then the problem (3.7) with $k_0 = k_{\mu_0}$ has a unique solution $k_t \in \mathcal{K}^\star_\infty$ on the time interval $[0, +\infty)$, which has the property $k_t(\emptyset) = 1$. Therefore, for each $t \geq 0$ there exists a unique sub-Poissonian measure μ_t such that $k_t = k_{\mu_t}$.*

The next statement describes the solutions in more detail.

Theorem 3.4. *Let (b, ϑ)-assumption (3.5) hold true with $b > 0$ (resp. $b = 0$), and let α_0 be such that $k_{\mu_0} \in \mathcal{K}_{\alpha_0}$. Then the solution k_t as in Theorem 3.3, corresponding to this k_{μ_0}, for all $t \geq 0$, satisfies the following estimates.*

(i) Case $\langle a^+ \rangle > 0$ and $m \in [0, \langle a^+ \rangle]$: for each $\delta < m$ (resp. $\delta \leq m$) there exists a positive C_δ such that $\log C_\delta \geq \alpha_0$ and

$$k_t(\eta) \leq C_\delta^{|\eta|} \exp\left[(\langle a^+ \rangle - \delta)|\eta| t\right], \qquad \eta \in \Gamma_0. \tag{3.18}$$

(ii) Case $\langle a^+ \rangle > 0$ and $m > \langle a^+ \rangle$: for each $\varepsilon \in (0, m - \langle a^+ \rangle)$, there exists a positive C_ε such that $\log C_\varepsilon \geq \alpha_0$ and

$$k_t(\eta) \leq C_\varepsilon^{|\eta|} \exp(-\varepsilon t), \qquad \eta \neq \emptyset. \tag{3.19}$$

(iii) Case $\langle a^+ \rangle = 0$:

$$k_t(\eta) \leq k_0(\eta) \exp\left[-E(\eta) t\right], \qquad \eta \in \Gamma_0. \tag{3.20}$$

If $m = 0$ and $a^-(x) = \vartheta a^+(x)$, then

$$k_t(\eta) = \vartheta^{-|\eta|}, \qquad t \geq 0, \tag{3.21}$$

is a stationary solution.

The next statement relates the solution described in Theorems 3.3 and 3.4 with the problem (3.17), see Definition 3.2.

Corollary 3.5. *In case (i) of Theorem 3.4, for each $T > 0$, k_t solves (3.17) in $\mathcal{K}_{\alpha_T}$ on the time interval $[0, T)$, where*

$$\alpha_T = \log C_\delta + \big(\langle a^+ \rangle - \delta\big) T. \tag{3.22}$$

In case (ii) (resp. (iii)), k_t solves (3.17) in $\mathcal{K}_\alpha$, $\alpha = \log C_\varepsilon$ (resp. any $\alpha > \alpha_0$) on the time interval $[0, +\infty)$.

3.3. **Comments and comparison.**

3.3.1. *On the basic assumption.* By means of the function

$$\phi_\vartheta(x) = a^-(x) - \vartheta a^+(x) \tag{3.23}$$

one can rewrite (3.5) in the following form

$$\sum_{x\in\eta}\sum_{y\in\eta\setminus x} \phi_\vartheta(x-y) \geq -b|\eta|, \qquad \eta \in \Gamma_0.$$

This resembles the stability condition (with stability constant $b \geq 0$) for the interaction potential ϕ_ϑ used in the statistical mechanics of continuum systems of interacting particles, see [25, Chapter 3]. Below we employ some techniques developed therein to prove that important classes of the kernels $a^\pm$ have this property, see Propositions 3.7 and 3.8.

The (b, ϑ) assumption holds with $b = 0$ if and only if (3.6) does. In this case, the dispersal kernel a^+ decays faster than the competition kernel a^- (short dispersal). It can be characterized as the possibility for each daughter-entity to kill her mother-entity, or to be killed by her. In the previous works on this model [7, 8, 9] the results were based on this short dispersal condition. The novelty of the result of Proposition 3.7 is that it covers also the case of long dispersal where the range of a^+ is finite but can be bigger than that of a^-. Noteworthy, by Proposition 3.7 it follows that the interaction potential Φ used in [24] is stable, which was unknown to the authors of that paper, cf. [24, page 146]. Proposition 3.8 describes Gaussian kernels, for which the basic assumption is valid also for both long and short dispersals. In this paper, we restricted our attention to the classes of kernels described in Propositions 3.7 and 3.8. Extensions beyond this classes can be made by means of the corresponding methods of the statistical mechanics of interacting particle systems.

3.3.2. *On the results.* An important feature of the results of Theorems 3.3 and 3.4 is that the intrinsic mortality rate $m \geq 0$ can be arbitrary. Theorem 3.3 gives a general existence of the evolution $\mu_0 \mapsto \mu_t$, $t > 0$, in the class of sub-Poissonian states through the evolution of the corresponding correlation functions. Its 'ecological' outcome is that the competition in the form as in (1.4), (1.5) excludes clustering provided the kernels satisfy (3.5). A complete characterization of the evolution $k_0 \mapsto k_t$ is then given in Theorem 3.4. By means of it this evolution is 'localized' in the spaces $\mathcal{K}_\alpha$ in Corollary 3.5. According to Theorem 3.4, for $m < \langle a^+\rangle$, or $m \leq \langle a^+\rangle$ and $b > 0$ in (3.24), the evolution described in Theorem 3.3 takes place in an increasing scale $\{\mathcal{K}_{\alpha_T}\}_{T\geq 0}$ of the Banach spaces introduced in (2.14) – (2.17), cf. (3.22). If $m > \langle a^+\rangle$, the evolution holds in one and the same space, see Corollary 3.5. The only difference between the cases of $b > 0$ and $b = 0$ is that one can take $\delta = m$ in the latter case. This yields different results for $m = \langle a^+\rangle$, where the evolution takes place in the same space $\mathcal{K}_\alpha$ with $\alpha = \log C_m$. Note also that for $m = 0$, one should take $\delta < 0$. For $m > \langle a^+\rangle$, it follows from (3.19) that the population dies out: for $\langle a^+\rangle > 0$, the following holds

$$k^{(n)}_{\mu_t}(x_1,\ldots,x_n) \leq e^{-\varepsilon t} k^{(n)}_{\mu_0}(x_1,\ldots,x_n), \quad t>0,$$

for some $\varepsilon \in (0, m - \langle a^+\rangle)$, almost all $(x_1,\ldots,x_n)$, and each $n\in\mathbb{N}$. For $m>0$ and $\langle a^+\rangle = 0$, by (3.20) we get

$$k^{(n)}_{\mu_t}(x_1,\ldots,x_n) \leq \exp\left(-nmt\right) k^{(n)}_{\mu_0}(x_1,\ldots,x_n), \quad t>0.$$

This means that $k^{(n)}_{\mu_t}(x_1,\ldots,x_n) \to 0$ as $n\to+\infty$ for sufficiently big $t>0$. This phenomenon does not follow from (3.19). Finally, we mention that (3.21) corresponds to a special case of (3.6) and $m=b=0$.

3.3.3. *Comparison.* Here we compare Theorems 3.3 and 3.4 with the corresponding results obtained for this model in [7, 8] (where it was called BDLP model), and in [9]. Note that these are the only works where the infinite particle version of the model considered here was studied. In [7, 8], the model was supposed to satisfy the conditions, see [8, Eqs. (3.38) and (3.39)], which in the present notations can be formulated as follows: (a) (3.6) holds with a given $\vartheta>0$; (b) $m > 16\langle a^-\rangle/\vartheta$ holding with the same ϑ. Under these conditions the global evolution $k_0\mapsto k_t$ was obtained in $\mathcal{K}_\alpha$ with some $\alpha\in\mathbb{R}$ by means of a C_0-semigroup. No information was available on whether k_t is a correlation function and hence on the sign of k_t. In [9], the restrictions were reduced just to (3.6). Then the evolution $k_0\mapsto k_t$ was obtained in a scale of Banach spaces $\mathcal{K}_\alpha$ as in Theorem 3.3, but on a bounded time interval. Also in [9], no information was available on whether k_t is a correlation function. Until this our work no results on the extinction as in (3.19) and on the case of $a^+\equiv 0$ were known.

3.4. **Kernels satisfying the basic assumption.** Our aim now is to show that the assumption (3.5) can be satisfied in the most of 'realistic' models. We begin, however, by establishing an important property of the kernels satisfying (3.5). To this end we rewrite (3.5) in the form

$$\Phi_\vartheta(\eta) := \sum_{x\in\eta}\sum_{y\in\eta\setminus x}\left[a^-(x-y) - \vartheta a^+(x-y)\right] \geq -b|\eta|, \qquad \eta\in\Gamma_0. \quad (3.24)$$

Proposition 3.6. *Assume that (3.24) holds with some $\vartheta_0>0$ and $b_0>0$. Then for each $\vartheta<\vartheta_0$, it also holds with $b=b_0\vartheta/\vartheta_0$.*

Proof. For $\vartheta\in(0,\vartheta_0]$, we have

$$\Phi_\vartheta(\eta) = \frac{\vartheta}{\vartheta_0}\left[\left(\frac{\vartheta_0}{\vartheta}-1\right)E^-(\eta) + \Phi_{\vartheta_0}(\eta)\right] \geq -\frac{\vartheta}{\vartheta_0} b_0|\eta|,$$

which yields the proof. □

In the following two propositions we give examples of the kernels with the property (3.5). In the first one, we assume that the dispersal kernel has finite range, which is quite natural in many applications. The competition kernel in turn is assumed to be just nontrivial.

Proposition 3.7. *In addition to (3.1) and (3.2) assume that the kernels $a^{\pm}$ have the following properties:*

(a) there exist positive c^- and r such that $a^-(x) \geq c^-$ for $|x| < r$;
(b) there exist positive c^+ and R such that $a^+(x) \leq c^+$ for $|x| < R$ and $a^+(x) = 0$ for $|x| \geq R$.

Then for each $b > 0$, there exists $\vartheta > 0$ such that (3.24) holds for these b and ϑ.

Proof. For $r \geq R$, (3.24) holds with $b = 0$ and $\vartheta = c^-/c^+$. Thus, it remains to consider the case $r < R$.

For $|\eta| = 0$ and $|\eta| = 1$, (3.24) trivially holds with each $b > 0$ and $\vartheta > 0$. For $|\eta| = 2$, (3.24) holds whenever $\vartheta \leq b/c^+$. For $|\eta| > 2$, we apply an induction in $|\eta|$, similarly as it was done in [2]. For $x \in \eta$, we define

$$\xi_x^- = \{y \in \eta : |y - x| < r\}, \quad \xi_x^+ = \{y \in \eta : r \leq |y - x| < R\}.$$

Set

$$U_\vartheta(\eta) = \Phi_\vartheta(\eta) + b|\eta| = b|\eta| + E^-(\eta) - \vartheta E^+(\eta).$$

Then the next estimate holds true for each $x \in \eta$:

$$\begin{aligned} U_\vartheta(x, \eta \setminus x) &:= U_\vartheta(\eta) - U_\vartheta(\eta \setminus x) \\ &= b + 2E^-(x, \eta \setminus x) - 2\vartheta E^+(x, \eta \setminus x) \\ &\geq b + 2(c^- - \vartheta c^+)|\xi_x^-| - 2\vartheta c^+ |\xi_x^+|. \end{aligned} \tag{3.25}$$

Given $n > 2$ and positive ϑ and b, assume that $U_\vartheta(\eta) \geq 0$ for each $|\vartheta| = n-1$. Then to make the inductive step by means of (3.25) we have to show that, for each η such that $|\eta| = n$, there exists $x \in \eta$ such that $U_\vartheta(x, \eta \setminus x) \geq 0$. Set

$$\bar{n} = |\xi_x^-| = \max_{y \in \eta} |\xi_y^-|, \qquad x \in \eta. \tag{3.26}$$

If $\bar{n} = 0$, then η is such that $|y - z| \geq r$ for each distinct $y, z \in \eta$. In this case, the balls $B_z := \{y \in \mathbb{R}^d : |y - z| < r/2\}$, $z \in \eta$, do not overlap. Then $|\xi_x^+| \leq \Xi(d, r, R) - 1 \leq \Delta(d)(1 + 2R/r)^d - 1$, where $\Xi(d, r, R)$ is the maximum number of rigid spheres of radius $r/2$ packed in a ball of radius $R + r/2$, and $\Delta(d)$ is the density of the densest packing of equal rigid spheres in $\mathbb{R}^d$, see e.g. [6, Chapter 1]. We apply this in (3.25) and get that $U_\vartheta(x, \eta \setminus x) \geq 0$ whenever $\vartheta \leq b/2c^+(\Xi(d, r, R) - 1)$. For $\bar{n} > 0$, let x be as in (3.26). Choose $y_1, \ldots, y_s$ in ξ_x^+ such that the balls B_x and B_{y_i}, $i = 1, \ldots, s$, realize the densest possible packing of the ball of radius $R + r/2$ centered at x. Then $s \leq \Xi(d, r, R) - 1$ and, for each $y \in \xi_x^+$, one finds i such that $|y - y_i| < r$. Otherwise B_y would not overlap each B_{y_i}, and thus the mentioned packing is not the densest one. Therefore, the balls $C_i := \{z \in \mathbb{R}^d : |z - y_i| < r\}$, $i = 1, \ldots, s$, cover ξ_x^+. By (3.26) each C_i contains $\bar{n} + 1$ elements at most. This yields

$$|\xi_x^+| \leq (\bar{n} + 1)(\Xi(d, r, R) - 1).$$

Now we apply this in (3.25) and obtain that $U_\vartheta(x, \eta \setminus x) \geq 0$ for

$$\vartheta = \min\left\{\frac{c^-}{c^+\Xi(d,r,R)}; \frac{b}{2c^+(\Xi(d,r,R)-1)}\right\}.$$

Thus, the inductive step can be done, which yields the proof. □

As an example of kernels with infinite range we consider the Gaussian kernels

$$a^{\pm}(x) = \frac{c_\pm}{(2\pi\sigma_\pm^2)^{d/2}} \exp\left(-\frac{1}{2\sigma_\pm^2}|x|^2\right), \tag{3.27}$$

where $c_\pm > 0$ and $\sigma_\pm > 0$ are parameters.

Proposition 3.8. *Let $a^\pm$ be as in (3.27). Then for each $b > 0$, there exists ϑ such that (3.5) holds for these ϑ and b.*

Proof. For $\sigma_- \geq \sigma_+$, we have $a^-(x) \geq \vartheta a^+(x)$ for all x and

$$\vartheta \leq \left(\frac{\sigma_+ c_-^{1/d}}{\sigma_- c_+^{1/d}}\right)^d.$$

Then (3.24), and thus (3.5), hold for such ϑ and all $b \geq 0$. For $\sigma_- < \sigma_+$, we can write, see (3.23),

$$\phi_\vartheta(x) = \int_{\mathbb{R}^d} \hat{\phi}_\vartheta(k) \exp(ik\cdot x) dk,$$

where

$$\hat{\phi}_\vartheta(k) = c_- \exp\left(-\frac{1}{2}\sigma_-^2|k|^2\right)\left[1 - \vartheta\frac{c_+}{c_-}\exp\left(-\frac{1}{2}(\sigma_+^2 - \sigma_-^2)|k|^2\right)\right].$$

For $\vartheta_0 = c_-/c_+$, we have that $\hat{\phi}_{\vartheta_0}(k) \geq 0$ for all $k \in \mathbb{R}^d$. Then ϕ_{ϑ_0} is positive definite in the sense of [25, Section 3.2]. This means that it is the Fourier transform of a positive finite measure on $\mathbb{R}^d$, and hence by the Bochner theorem it follows that

$$\sum_{x,y\in\eta} \phi_{\vartheta_0}(x-y) = \phi_{\vartheta_0}(0)|\eta| + \Phi_{\vartheta_0}(\eta) \geq 0.$$

Thus, Φ_{ϑ_0} satisfies (3.24) with stability constant $b_0 = \phi_{\vartheta_0}(0)$. Then we apply Proposition 3.6 and obtain that (3.24) holds for

$$\vartheta = \frac{(2\pi\sigma_-^2)^{d/2} b}{\sigma_+\left(1 - \left(\frac{\sigma_-}{\sigma_+}\right)^d\right)}$$

which completes the proof.

□

4. Evolution of Correlation Functions and States

In this section we prove Theorems 3.3 and 3.4 assuming the validity of Lemma 4.9. In the next section we prove this lemma. The proof of Theorem 3.3 is based on the construction of two families of bounded operators performed in subsection 4.2. By means of one of them we obtain the solution of the problem (3.17) on a bounded time interval, similarly as it was done in [9]. Next, assuming that Lemma 4.9 holds true, and hence $k_t \geq 0$, by means of the second family of operators we compare k_t in Lemmas 4.10 with especially constructed functions and thereby prove both Theorems 3.3 and 3.4. We begin by constructing auxiliary semigroups used to get the results of subsection 4.2.

4.1. **Auxiliary semigroups.** For a given $\alpha \in \mathbb{R}$, the space predual to $\mathcal{K}_\alpha$, defined in (2.16), is

$$\mathcal{G}_\alpha := L^1(\Gamma_0, e^{\alpha|\cdot|}d\lambda), \tag{4.1}$$

in which the norm is, cf. (2.5),

$$\begin{aligned} |G|_\alpha &= \int_{\Gamma_0} |G(\eta)| \exp(\alpha|\eta|)\lambda(d\eta) \\ &= \sum_{n=0}^{\infty} \frac{e^{\alpha n}}{n!} \|G^{(n)}\|_{L^1((\mathbb{R}^d)^n)}. \end{aligned} \tag{4.2}$$

Clearly, $|G|_{\alpha'} \leq |G|_\alpha$ for $\alpha' < \alpha$, which yields

$$\mathcal{G}_\alpha \hookrightarrow \mathcal{G}_{\alpha'}, \qquad \text{for} \quad \alpha' < \alpha, \tag{4.3}$$

cf. (2.17). One can show that this embedding is also dense.

Recall that by $m \geq 0$ we denote the mortality rate, see (1.4). For $b \geq 0$ as in (3.5) we set

$$E_b(\eta) = (b + m)|\eta| + E^-(\eta) = b|\eta| + E(\eta). \tag{4.4}$$

Here $E^-(\eta)$ and $E(\eta)$ are as in (3.3) and (3.4), respectively. For the same b, let the action of A_b on functions $G : \Gamma_0 \to \mathbb{R}$ be as follows

$$A_b = A_{1,b} + A_2 \tag{4.5}$$

$$(A_{1,b}G)(\eta) = -E_b(\eta)G(\eta),$$

$$(A_2 G)(\eta) = \int_{\mathbb{R}^d} E^+(y, \eta)G(\eta \cup y)dy.$$

Our aim now is to define A_b as a closed unbounded operator in $\mathcal{G}_\alpha$ the domain of which contains $\mathcal{G}_{\alpha'}$ for any $\alpha' > \alpha$. Let $\mathcal{G}_\alpha^+$ denote the set of all those $G \in \mathcal{G}_\alpha$ for which $G(\eta) \geq 0$ for λ-almost all $\eta \in \Gamma_0$. Set

$$\mathcal{D}_\alpha = \{G \in \mathcal{G}_\alpha : E_b(\cdot)G(\cdot) \in \mathcal{G}_\alpha\}, \tag{4.6}$$

where $E_b(\eta)$ is as in (4.4). For each $\alpha' > \alpha$, $\mathcal{D}_\alpha$ contains $\mathcal{G}_{\alpha'}$ and hence is dense in $\mathcal{G}_\alpha$, see (4.3). Then the first summand in A_b turns into a closed and densely defined operator $(A_{1,b}, \mathcal{D}_\alpha)$ in $\mathcal{G}_\alpha$ such that $-A_{1,b}G \in \mathcal{G}^+_\alpha$ for each $G \in \mathcal{D}^+_\alpha := \mathcal{D}_\alpha \cap \mathcal{G}^+_\alpha$. By (2.13) and (3.5) one gets

$$\begin{aligned} |A_2 G|_\alpha &\leq \int_{\Gamma_0}\int_{\mathbb{R}^d} E^+(y,\eta)|G(\eta\cup y)|e^{\alpha|\eta|}dy\lambda(d\eta) \qquad (4.7)\\ &= e^{-\alpha}\int_{\Gamma_0}|G(\eta)|e^{\alpha|\eta|}\left(\sum_{x\in\eta}E^+(x,\eta\setminus x)\right)\lambda(d\eta)\\ &= e^{-\alpha}|E^+(\cdot)G(\cdot)|_\alpha \leq (e^{-\alpha}/\vartheta)|A_{1,b}G|_\alpha. \end{aligned}$$

Then for $\alpha > -\log\vartheta$, we have that $e^{-\alpha}/\vartheta < 1$, and hence A_2 is $A_{1,b}$-bounded. This means that $(A_b, \mathcal{D}_\alpha)$ is closed and densely defined in $\mathcal{G}_\alpha$, see (4.5).

In the proof of Lemma 4.2 below we employ the perturbation theory for positive semigroups of operators in ordered Banach spaces developed in [26]. Prior to stating the lemma we present the relevant fragments of this theory in spaces of integrable functions. Let E be a measurable space with a σ-finite measure ν, and $X := L^1(E\to\mathbb{R}, d\nu)$ be the Banach space of ν-integrable real-valued functions on X with norm $\|\cdot\|$. Let X^+ be the cone in X consisting of all ν-a.e. nonnegative functions on E. Clearly, $\|f+g\| = \|f\| + \|g\|$ for any $f, g \in X^+$, and this cone is generating, that is, $X = X^+ - X^+$. Recall that a C_0-semigroup $\{S(t)\}_{t\geq 0}$ of bounded linear operators on X is called *positive* if $S(t)f \in X^+$ for all $f \in X^+$. A positive semigroup is called *substochastic* (resp. *stochastic*) if $\|S(t)f\| \leq \|f\|$ (resp. $\|S(t)f\| = \|f\|$) for all $f \in X^+$. Let $(A_0, D(A_0))$ be the generator of a positive C_0 -semigroup $\{S_0(t)\}_{t\geq 0}$ on X. Set $D^+(A_0) = D(A_0)\cap X^+$. Then $D(A_0)$ is dense in X, and $D^+(A_0)$ is dense in X^+. Let $P : D(A_0) \to X$ be a positive linear operator, i.e., $Pf \in X^+$ for all $f \in D^+(A_0)$. The next statement is an adaptation of Theorem 2.2 in [26].

Proposition 4.1. *Suppose that for any $f \in D^+(A_0)$, the following holds*

$$\int_E \big((A_0+P)f\big)(x)\,\nu(dx) \leq 0. \qquad (4.8)$$

Then for all $r\in[0,1)$, the operator $\big(A_0 + rP, D(A_0)\big)$ is the generator of a substochastic C_0-semigroup in X.

Lemma 4.2. *For each $\alpha > -\log\vartheta$, the operator $(A_b, \mathcal{D}_\alpha)$ is the generator of a substochastic semigroup $\{S(t)\}_{t\geq 0}$ in $\mathcal{G}_\alpha$.*

Proof. We apply Proposition 4.1 with $E = \Gamma_0$, $X = \mathcal{G}_\alpha$ as in (4.1), and $A_0 = A_{1,b}$. For $r > 0$ and A_2 as in (4.5), we set $P = r^{-1}A_2$. For such A_0

and P, and for $G \in \mathcal{D}_\alpha^+$, the left-hand side of (4.8) takes the form, cf. (4.7),

$$
\begin{aligned}
& - \int_{\Gamma_0} E_b(\eta) G(\eta) \exp(\alpha|\eta|)\lambda(d\eta) \\
& + r^{-1} \int_{\Gamma_0} \int_{\mathbb{R}^d} E^+(y,\eta) G(\eta \cup y) \exp(\alpha|\eta|) dy \lambda(d\eta) \\
= & \int_{\Gamma_0} \big(-E_b(\eta) + r^{-1} e^{-\alpha} E^+(\eta) \big) G(\eta) \exp(\alpha|\eta|)\lambda(d\eta).
\end{aligned}
$$

For a fixed $\alpha > -\log \vartheta$, pick $r \in (0,1)$ such that $r^{-1}(e^{-\alpha}/\vartheta) < 1$. Then, for such α and r, we have

$$
\int_{\Gamma_0} \big(-E_b(\eta) + r^{-1} e^{-\alpha} E^+(\eta) \big) G(\eta) \exp(\alpha|\eta|)\lambda(d\eta) \leq 0, \tag{4.9}
$$

which holds in view of (3.5). Since $r^{-1}A_2$ is a positive operator, by Proposition 4.1 we have that $A_b = A_{1,b} + A_2 = A_{1,b} + r(r^{-1}A_2)$ generates a substochastic semigroup $\{S(t)\}_{t\geq 0}$ in $\mathcal{G}_\alpha$. □

Now we turn to constructing the semigroup 'sun-dual' to that mentioned in Lemma 4.2. Let A_b^* be the adjoint of $(A_b, \mathcal{D}_\alpha)$ in $\mathcal{K}_\alpha$ with domain, cf. (3.13),

$$
\mathrm{Dom}(A_b^*) = \big\{ k \in \mathcal{K}_\alpha : \exists \tilde{k} \in \mathcal{K}_\alpha \;\; \forall G \in \mathcal{D}_\alpha \;\; \langle\!\langle A_b G, k \rangle\!\rangle = \langle\!\langle G, \tilde{k} \rangle\!\rangle \big\}.
$$

For each $k \in \mathrm{Dom}(A_b^*)$, the action of A_b^* on k is described in (3.9) with E replaced by E_b, see (4.4). By (3.11) we then get $\mathcal{K}_{\alpha'} \subset \mathrm{Dom}(A_b^*)$ for each $\alpha' < \alpha$. Let $\mathcal{Q}_\alpha$ stand for the closure of $\mathrm{Dom}(A_b^*)$ in $\|\cdot\|_\alpha$. Then

$$
\mathcal{Q}_\alpha := \overline{\mathrm{Dom}(A_b^*)} \supset \mathrm{Dom}(A_b^*) \supset \mathcal{K}_{\alpha'}, \qquad \text{for any } \alpha' < \alpha. \tag{4.10}
$$

Note that $\mathcal{Q}_\alpha$ is a proper subset of $\mathcal{K}_\alpha$. For each $t \geq 0$, the adjoint $S^*(t)$ of $S(t)$ is a bounded operator in $\mathcal{K}_\alpha$. However, the semigroup $\{S^*(t)\}_{t\geq 0}$ is not strongly continuous. For $t > 0$, let $S_\alpha^\odot(t)$ denote the restriction of $S^*(t)$ to $\mathcal{Q}_\alpha$. Since $\{S(t)\}_{t\geq 0}$ is the semigroup of contractions, for $k \in \mathcal{Q}_\alpha$ and all $t \geq 0$, we have that

$$
\|S_\alpha^\odot(t)k\|_\alpha = \|S^*(t)k\|_\alpha \leq \|k\|_\alpha. \tag{4.11}
$$

Proposition 4.3. *For every $\alpha' < \alpha$ and any $k \in \mathcal{K}_{\alpha'}$, the map*

$$
[0, +\infty) \ni t \mapsto S_\alpha^\odot(t)k \in \mathcal{K}_\alpha
$$

is continuous.

Proof. By [23, Theorem 10.4, page 39], the collection $\{S_\alpha^\odot(t)\}_{t\geq 0}$ constitutes a C_0-semigroup on $\mathcal{Q}_\alpha$ the generator of which, $A_\alpha^\odot$, is the part of A_b^* in $\mathcal{Q}_\alpha$. That is, $A_\alpha^\odot$ is the restriction of A_b^* to the set

$$
\mathrm{Dom}(A_\alpha^\odot) := \{k \in \mathrm{Dom}(A_b^*) : A_b^* k \in \mathcal{Q}_\alpha\},
$$

cf. [23, Definition 10.3, page 39]. The continuity in question follows by the C_0-property of the semigroup $\{S_\alpha^\odot(t)\}_{t\geq 0}$ and (4.10). □

By (3.11) it follows that

$$\mathrm{Dom}(A^{\odot}_{\alpha''}) \supset \mathcal{K}_{\alpha'}, \qquad \alpha' < \alpha'', \tag{4.12}$$

and hence, see [23, Theorem 2.4, page 4],

$$S^{\odot}_{\alpha''}(t)k \in \mathrm{Dom}(A^{\odot}_{\alpha''}), \tag{4.13}$$

and

$$\frac{d}{dt} S^{\odot}_{\alpha''}(t)k = A^{\odot}_{\alpha''} S^{\odot}_{\alpha''}(t)k, \tag{4.14}$$

which holds for all $\alpha'' \in (\alpha', \alpha]$ and $k \in \mathcal{K}_{\alpha'}$.

4.2. **The main operators.** For E_b as in (4.4), we set

$$\begin{aligned} A^{\Delta}_b &= A^{\Delta}_{1,b} + A^{\Delta}_2, \\ (A^{\Delta}_{1,b}k)(\eta) &= -E_b(\eta)k(\eta), \end{aligned} \tag{4.15}$$

and A^{Δ}_2 being as in (3.9). We also set

$$\begin{aligned} B^{\Delta}_b &= B^{\Delta}_1 + B^{\Delta}_{2,b}, \\ (B^{\Delta}_{2,b}k)(\eta) &= (B^{\Delta}_2 k)(\eta) + b|\eta|k(\eta). \end{aligned} \tag{4.16}$$

Here B^{Δ}_1 and B^{Δ}_2 are as in (3.10). Note that

$$L^{\Delta} = A^{\Delta} + B^{\Delta} = A^{\Delta}_b + B^{\Delta}_b. \tag{4.17}$$

The expressions given in (4.15) and (4.16) can be used to define the corresponding continuous maps acting from $\mathcal{K}_{\alpha'}$ to $\mathcal{K}_{\alpha}$, $\alpha' < \alpha$, cf. (3.14), and hence the elements of $\mathcal{L}(\mathcal{K}_{\alpha'}, \mathcal{K}_{\alpha})$ the norms of which are estimated by means of the analogies of (3.11) and (3.13). For these operators, we use notations $(B^{\Delta}_b)_{\alpha\alpha'}$ and $(B^{\Delta}_{2,b})_{\alpha\alpha'}$. Then $\|(B^{\Delta}_b)_{\alpha\alpha'}\|$ will stand for the operator norm, and thus (3.13) can be rewritten in the form

$$\|(B^{\Delta}_b)_{\alpha\alpha'}\| \le \frac{\langle a^+\rangle + b + \langle a^-\rangle e^{\alpha'}}{e(\alpha - \alpha')}. \tag{4.18}$$

For fixed $\alpha > \alpha' > -\log\vartheta$, we construct continuous operators $Q_{\alpha\alpha'}(t;\mathbb{B}) : \mathcal{K}_{\alpha'} \to \mathcal{K}_{\alpha}$, $t > 0$, which will be used to obtain the solution k_t as in Theorem 3.3 and to study its properties. Here $\mathbb{B}$ will be taken in the following two versions: (a) $\mathbb{B} = B^{\Delta}_b$; (b) $\mathbb{B} = B^{\Delta}_{2,b}$, see (4.16). In both cases, for each $\alpha_1, \alpha_2 \in [\alpha', \alpha]$ such that $\alpha_1 < \alpha_2$, cf. (4.18), the following holds

$$\|\mathbb{B}_{\alpha_2\alpha_1}\| \le \frac{\beta(\alpha_2;\mathbb{B})}{e(\alpha_2 - \alpha_1)}, \tag{4.19}$$

with

$$\begin{aligned} \beta(\alpha_2; B^{\Delta}_b) &= \langle a^+\rangle + b + \langle a^-\rangle e^{\alpha_2}, \\ \beta(\alpha_2; B^{\Delta}_{2,b}) &= \langle a^+\rangle + b. \end{aligned} \tag{4.20}$$

For $t > 0$ and α_1, α_2 as above, let $\Sigma_{\alpha_2\alpha_1}(t) : \mathcal{K}_{\alpha_1} \to \mathcal{K}_{\alpha_2}$ be the restriction of $S^{\odot}_{\alpha_2}(t)$ to $\mathcal{K}_{\alpha_1}$, cf. (4.12) and (4.13). Note that the embedding $\mathcal{K}_{\alpha_1} \hookrightarrow \mathcal{K}_{\alpha_2}$. can be written as $\Sigma_{\alpha_2\alpha_1}(0)$, and hence

$$\Sigma_{\alpha_2\alpha_1}(t) = \Sigma_{\alpha_2\alpha_1}(0) S^{\odot}_{\alpha_1}(t). \tag{4.21}$$

Also, for each $\alpha_3 > \alpha_2$, we have

$$\Sigma_{\alpha_3\alpha_1}(t) = \Sigma_{\alpha_3\alpha_2}(0)\Sigma_{\alpha_2\alpha_1}(t) := \Sigma_{\alpha_2\alpha_1}(t), \qquad t \geq 0. \tag{4.22}$$

Here and in the sequel, we omit writing embedding operators if no confusing arises. In view of (4.11), it follows that

$$\|\Sigma_{\alpha_2\alpha_1}(t)\| \leq 1. \tag{4.23}$$

Remark 4.4. By Lemma 4.2 we have that

$$\forall k \in \mathcal{K}^+_{\alpha_1} \qquad \Sigma_{\alpha_2\alpha_1}(t)k \in \mathcal{K}^+_{\alpha_2}, \quad t \geq 0,$$

see (2.19). Also $(B^{\Delta}_{2,b})_{\alpha_2\alpha_1}$, but not $(B^{\Delta}_{b})_{\alpha_2\alpha_1}$, has the same positivity property.

Set, cf. (4.20),

$$T(\alpha_2, \alpha_1; \mathbb{B}) = \frac{\alpha_2 - \alpha_1}{\beta(\alpha_2; \mathbb{B})}, \qquad \alpha_2 > \alpha_1, \tag{4.24}$$

and then

$$\mathcal{A}(B) = \{(\alpha_1, \alpha_2, t) : -\log \vartheta < \alpha_1 < \alpha_2, \quad 0 \leq t < T(\alpha_2, \alpha_1; \mathbb{B})\}. \tag{4.25}$$

Lemma 4.5. *For each of the two choices of $\mathbb{B}$, see (4.20), there exists the corresponding family of linear maps, $\{Q_{\alpha_2\alpha_1}(t; \mathbb{B}) : (\alpha_1, \alpha_2, t) \in \mathcal{A}(\mathbb{B})\}$, each element of which has the following properties:*

(i) $Q_{\alpha_2\alpha_1}(t; \mathbb{B}) \in \mathcal{L}(\mathcal{K}_{\alpha_1}, \mathcal{K}_{\alpha_2})$;

(ii) the map $[0, T(\alpha_2, \alpha_1; \mathbb{B})) \ni t \mapsto Q_{\alpha_2\alpha_1}(t; \mathbb{B}) \in \mathcal{L}(\mathcal{K}_{\alpha_1}, \mathcal{K}_{\alpha_2})$ *is continuous;*

(iii) the operator norm of $Q_{\alpha_2\alpha_1}(t; \mathbb{B}) \in \mathcal{L}(\mathcal{K}_{\alpha_1}, \mathcal{K}_{\alpha_2})$ *satisfies*

$$\|Q_{\alpha_2\alpha_1}(t; \mathbb{B})\| \leq \frac{T(\alpha_2, \alpha_1; \mathbb{B})}{T(\alpha_2, \alpha_1; \mathbb{B}) - t}; \tag{4.26}$$

(iv) for each $\alpha_3 \in (\alpha_1, \alpha_2)$ *and* $t < T(\alpha_3, \alpha_1; \mathbb{B})$*, the following holds*

$$\frac{d}{dt} Q_{\alpha_2\alpha_1}(t; \mathbb{B}) = \big((A^{\Delta}_{b})_{\alpha_2\alpha_3} + \mathbb{B}_{\alpha_2\alpha_3}\big) Q_{\alpha_3\alpha_1}(t; \mathbb{B}). \tag{4.27}$$

The proof of this lemma is based on the following construction. For $l \in \mathbb{N}$ and $t > 0$, we set

$$\mathcal{T}_l := \{(t, t_1, \ldots, t_l) : 0 \leq t_l \leq \cdots \leq t_1 \leq t\}, \tag{4.28}$$

take $\alpha \in (\alpha_1, \alpha_2]$, and then take $\delta < \alpha - \alpha_1$. Next we divide the interval $[\alpha_1, \alpha]$ into subintervals with endpoints α^s, $s = 0, \ldots, 2l+1$, as follows. Set $\alpha^0 = \alpha_1$, $\alpha^{2l+1} = \alpha$, and

$$\begin{aligned} \alpha^{2s} &= \alpha_1 + \frac{s}{l+1}\delta + s\epsilon, \qquad \epsilon = (\alpha - \alpha_1 - \delta)/l, \qquad (4.29) \\ \alpha^{2s+1} &= \alpha_1 + \frac{s+1}{l+1}\delta + s\epsilon, \qquad s = 0, 1, \ldots, l. \end{aligned}$$

Thereafter, for $(t, t_1, \ldots, t_l) \in \mathcal{T}_l$ we define

$$\Pi^{(l)}_{\alpha\alpha_1}(t, t_1, \ldots, t_l; \mathbb{B}) = \Sigma_{\alpha\alpha^{2l}}(t - t_1)\mathbb{B}_{\alpha^{2l}\alpha^{2l-1}} \times \cdots \times \qquad (4.30)$$

$$\times \Sigma_{\alpha^{2s+1}\alpha^{2s}}(t_{l-s} - t_{l-s+1})\mathbb{B}_{\alpha^{2s}\alpha^{2s-1}} \cdots \Sigma_{\alpha^3\alpha^2}(t_{l-1} - t_l)\mathbb{B}_{\alpha^2\alpha^1}\Sigma_{\alpha^1\alpha_1}(t_l).$$

Proposition 4.6. *For both choices of* $\mathbb{B}$ *and each* $l \in \mathbb{N}$*, the operators defined in (4.30) have the following properties:*

(i) for each $(t, t_1, \ldots, t_l) \in \mathcal{T}_l$*,* $\Pi^{(l)}_{\alpha\alpha_1}(t, t_1, \ldots, t_l; \mathbb{B}) \in \mathcal{L}(\mathcal{K}_{\alpha_1}, \mathcal{K}_\alpha)$*, and the map*

$$\mathcal{T}_l \ni (t, t_1, \ldots, t_l) \mapsto \Pi^{(l)}_{\alpha\alpha_1}(t, t_1, \ldots, t_l; \mathbb{B}) \in \mathcal{L}(\mathcal{K}_{\alpha_1}, \mathcal{K}_\alpha)$$

is continuous;

(ii) for fixed $t_1, t_2, \ldots, t_l$*, and each* $\varepsilon > 0$*, the map*

$$(t_1, t_1 + \varepsilon) \ni t \mapsto \Pi^{(l)}_{\alpha\alpha_1}(t, t_1, \ldots, t_l; \mathbb{B}) \in \mathcal{L}(\mathcal{K}_{\alpha_1}, \mathcal{K}_{\alpha_2})$$

is continuously differentiable and for each $\alpha' \in (\alpha_1, \alpha)$ *the following holds*

$$\frac{d}{dt}\Pi^{(l)}_{\alpha\alpha_1}(t, t_1, \ldots, t_l; \mathbb{B}) = (A^\Delta_b)_{\alpha\alpha'}\Pi^{(l)}_{\alpha'\alpha_1}(t, t_1, \ldots, t_l; \mathbb{B}). \qquad (4.31)$$

Proof. The first part of claim *(i)* follows by (4.30), (4.19), and (4.23). To prove the second part we apply Proposition 4.3 and (4.21), and then (4.19), (4.20). By (4.12), (4.14), and (4.22), and the fact that

$$A^\odot_{\alpha'}k = (A^\Delta_b)_{\alpha'\alpha}k, \qquad \text{for } \ k \in \mathcal{K}_\alpha,$$

one gets

$$\frac{d}{dt}\Sigma_{\alpha'\alpha_{2l}}(t) = (A^\Delta_b)_{\alpha'\alpha}\Sigma_{\alpha\alpha_{2l}}(t), \qquad \alpha' > \alpha, \qquad (4.32)$$

which then yields (4.31). □

Proof of Lemma 4.5. Take any $T < T(\alpha_2, \alpha_1; \mathbb{B})$ and then pick $\alpha \in (\alpha_1, \alpha_2]$ and a positive $\delta < \alpha - \alpha_1$ such that

$$T < T_\delta := \frac{\alpha - \alpha_1 - \delta}{\beta(\alpha_2; \mathbb{B})}.$$

For this δ, take $\Pi^{(l)}_{\alpha\alpha_1}$ as in (4.30), and then for $n \in \mathbb{N}$ set

$$Q^{(n)}_{\alpha\alpha_1}(t;\mathbb{B}) = \Sigma_{\alpha\alpha_1}(t) + \sum_{l=1}^{n} \int_0^t \int_0^{t_1} \cdots \int_0^{t_{l-1}} \Pi^{(l)}_{\alpha\alpha_1}(t,t_1,\ldots,t_l;\mathbb{B}) dt_l \cdots dt_1. \tag{4.33}$$

By (4.23), (4.19), and (4.29) we have from (4.30) that

$$\|\Pi^{(l)}_{\alpha\alpha_1}(t,t_1,\ldots,t_l;\mathbb{B})\| \leq \left(\frac{l}{eT_\delta}\right)^l, \tag{4.34}$$

holding for all $l = 1,\ldots,n$. This yields

$$\begin{aligned} &\|Q^{(n)}_{\alpha\alpha_1}(t;\mathbb{B}) - Q^{(n-1)}_{\alpha\alpha_1}(t;\mathbb{B})\| \\ &\leq \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} \|\Pi^{(n)}_{\alpha\alpha_1}(t,t_1,\ldots,t_l;\mathbb{B})\| dt_n \cdots dt_1 \\ &\leq \frac{1}{n!}\left(\frac{n}{e}\right)^n \left(\frac{T}{T_\delta}\right)^n, \end{aligned} \tag{4.35}$$

which implies $\forall t \in [0,T]$ $Q^{(n)}_{\alpha\alpha_1}(t;\mathbb{B}) \to Q_{\alpha\alpha_1}(t;\mathbb{B}) \in \mathcal{L}(\mathcal{K}_{\alpha_1},\mathcal{K}_\alpha)$ as $n \to +\infty$, that proves claim *(i)* of the lemma. The proof of claim *(ii)* follows by the fact that the mentioned above convergence is uniform on $[0,T]$. The estimate (4.26) readily follows from that in (4.34). Now by (4.30) and (4.32) we obtain

$$\frac{d}{dt}Q^{(n)}_{\alpha_2\alpha_1}(t;\mathbb{B}) = \left(A^\Delta_b\right)_{\alpha_2\alpha} Q^{(n)}_{\alpha\alpha_1}(t;\mathbb{B}) + B_{\alpha_2\alpha}Q^{(n-1)}_{\alpha\alpha_1}(t;\mathbb{B}), \quad n \in \mathbb{N}.$$

Then the continuous differentiability of the limit and (4.27) follow by standard arguments. □

Remark 4.7. By (4.30), (4.33), and Lemma 4.5 we have that

$$\forall k \in \mathcal{K}^+_{\alpha_1} \qquad Q_{\alpha_2\alpha_1}(t;B^\Delta_{2,b})k \in \mathcal{K}^+_{\alpha_2}, \quad t \in [0,T(\alpha_2,\alpha_1;B^\Delta_2)). \tag{4.36}$$

At the same time, $Q_{\alpha_2\alpha_1}(t;B^\Delta_b)$ is not positive, see (3.10) and Remark 4.4.

4.3. **The proof of Theorem 3.3.** First we prove that the problem (3.17) has a unique solution on a bounded time interval.

Lemma 4.8. *For each $\alpha_2 > \alpha_1 > -\log\vartheta$, the problem (3.17) with $k_0 \in \mathcal{K}_{\alpha_1}$ has a unique solution $k_t \in \mathcal{K}_{\alpha_2}$ on the time interval $[0,T(\alpha_2,\alpha_1,B^\Delta_b))$. The solution has the property: $k_t(\emptyset) = 1$ for all $t \in [0,T(\alpha_2,\alpha_1,B^\Delta_b))$.*

Proof. For each $t \in [0,T(\alpha_2,\alpha_1,B^\Delta_b))$, one finds $\alpha \in (\alpha_1,\alpha_2)$ such that also $t \in [0,T(\alpha,\alpha_1,B^\Delta_b))$. Then by claim *(i)* of Lemma 4.5 and (3.15)

$$k_t := Q_{\alpha\alpha_1}(t;B^\Delta_b)k_0 \tag{4.37}$$

lies in $\mathcal{D}^{\Delta}_{\alpha_2}$. By (4.27) the derivative of $k_t \in \mathcal{K}_{\alpha_2}$ is

$$\frac{d}{dt}k_t = \big((A^{\Delta}_b)_{\alpha_2\alpha} + (B^{\Delta}_b)_{\alpha_2\alpha}\big)k_t = L^{\Delta}_{\alpha_2\alpha}k_t.$$

Hence, k_t is a solution of (3.17), see (3.16). Moreover, $k_t(\emptyset) = 1$ since $k_0(\emptyset) = 1$, see (2.12), and

$$\left(\frac{d}{dt}k_t\right)(\emptyset) = (L^{\Delta}_{\alpha}k_t)(\emptyset) = 0,$$

see (3.8) – (3.10). To prove the stated uniqueness assume that $\tilde{k}_t \in \mathcal{D}^{\Delta}_{\alpha_2}$ is another solution of (3.17) with the same initial condition. Then for each $\alpha_3 > \alpha_2$, $v_t := k_t - \tilde{k}_t$ is a solution of (3.17) in $\mathcal{K}_{\alpha_3}$ with the zero initial condition. Here we assume that t and α_3 are such that $t < T(\alpha_3, \alpha_1; B^{\Delta}_b)$. Clearly, v_t also solves (3.17) in $\mathcal{K}_{\alpha_2}$. Thus, it can be written down in the following form

$$v_t = \int_0^t \Sigma_{\alpha_3\alpha}(t-s)\left(B^{\Delta}_b\right)_{\alpha\alpha_2} v_s ds, \tag{4.38}$$

where v_t on the left-hand side (resp. v_s on the right-hand side) is considered as an element of $\mathcal{K}_{\alpha_3}$ (resp. $\mathcal{K}_{\alpha_2}$) and $\alpha \in (\alpha_2, \alpha_3)$. Indeed, one obtains (4.38) by integrating the equation, see (4.17),

$$\frac{d}{dt}v_t = L^{\Delta}_{\alpha_3\alpha_2}v_t = \big(\left(A^{\Delta}_b\right)_{\alpha_3\alpha_2} + \left(B^{\Delta}_b\right)_{\alpha_3\alpha_2}\big)v_t, \qquad v_0 = 0,$$

in which the second summand is considered as a nonhomogeneous term, see (4.32). Let us show that for all $t < T(\alpha_2, \alpha_1; B^{\Delta}_b))$, $v_t = 0$ as an element of $\mathcal{K}_{\alpha_2}$. In view of the embedding $\mathcal{K}_{\alpha_2} \hookrightarrow \mathcal{K}_{\alpha_3}$, cf. (2.17), this will follow from the fact that $v_t = 0$ as an element of $\mathcal{K}_{\alpha_3}$. For a given $n \in \mathbb{N}$, we set $\epsilon = (\alpha_3 - \alpha_2)/2n$ and $\alpha^l = \alpha_2 + l\epsilon$, $l = 0, \dots, 2n$. Then we repeatedly apply (4.38) and obtain

$$\begin{aligned} v_t &= \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} \Sigma_{\alpha_3\alpha^{2n-1}}(t-t_1)(B^{\Delta}_b)_{\alpha^{2n-1}\alpha^{2n-2}} \times \cdots \times \\ &\times \ \Sigma_{\alpha^2\alpha^1}(t_{n-1}-t_n)(B^{\Delta}_b)_{\alpha^1\alpha_2} v_{t_n} dt_n \cdots dt_1. \end{aligned}$$

Similarly as in (4.34) we then get from the latter, see (4.19), (4.20), and (4.23),

$$\begin{aligned} \|v_t\|_{\alpha_3} &\leq \frac{t^n}{n!}\prod_{l=1}^n \|(B^{\Delta}_b)_{\alpha^{2l-1}\alpha^{2l-2}}\| \sup_{s\in[0,t]}\|v_s\|_{\alpha_2} \\ &\leq \frac{1}{n!}\left(\frac{n}{e}\right)^n \left(\frac{2t\beta(\alpha_3; B^{\Delta}_b)}{\alpha_3 - \alpha_2}\right)^n \sup_{s\in[0,t]}\|v_s\|_{\alpha_2}. \end{aligned} \tag{4.39}$$

The latter implies that $v_t = 0$ for $t < (\alpha_3 - \alpha_2)/2\beta(\alpha_3; B^{\Delta}_b)$. To prove that $v_t = 0$ for all t of interest one has to repeat the above procedure appropriate number of times. □

To make the next step we need the following result, the proof of which will be done in Section 5 below.

Lemma 4.9. *[Identification Lemma] For each $\alpha_2 > \alpha_1 > -\log\vartheta$, there exists $\tau(\alpha_2, \alpha_1) \in (0, T(\alpha_2, \alpha_1; B_b^\Delta))$ such that $Q_{\alpha_2\alpha_1}(t; B_b^\Delta) : \mathcal{K}_{\alpha_1}^\star \to \mathcal{K}_{\alpha_2}^\star$ for each $t \in [0, \tau(\alpha_2, \alpha_1)]$, see (2.18) and Lemma 4.5.*

In other words, Lemma 4.9 states that for $t \in [0, \tau(\alpha_2, \alpha_1)]$, the solution k_t as in Lemma 4.8 is the correlation function of a unique sub-Poissonian μ_t whenever $k_0 = k_{\mu_0}$ for some $\mu_0 \in \mathcal{P}_{\rm sP}$.

To complete the proof of Theorem 4.2 we need the following result. Recall that $\mathcal{K}_\alpha^\star \subset \mathcal{K}_\alpha^+$, $\alpha \in \mathbb{R}$, see (2.19).

Lemma 4.10. *Let α_2, α_1, and $\tau(\alpha_2, \alpha_1)$ be as in Lemma 4.9. Then there exists positive $\tau_1(\alpha_2, \alpha_1) \le \tau(\alpha_2, \alpha_1)$ such that, for each $t \in [0, \tau_1(\alpha_2, \alpha_1)]$ and arbitrary $k_0 \in \mathcal{K}_{\alpha_1}^\star$ the following holds, cf. (4.20) and Remark 4.4,*

$$0 \le \left(Q_{\alpha_2\alpha_1}(t; B_b^\Delta)k_0\right)(\eta) \le \left(Q_{\alpha_2\alpha_1}(t; B_{2,b}^\Delta)k_0\right)(\eta), \qquad \eta \in \Gamma_0. \tag{4.40}$$

Proof. The left-hand inequality in (4.40) follows directly by Lemma 4.9. By Lemma 4.8 k_t as in (4.37) solves (3.17) in $\mathcal{K}_{\alpha_2}$. Set

$$L_2^\Delta = A^\Delta + B_2^\Delta = A_b^\Delta + B_{2,b}^\Delta,$$

where A^Δ, B_2^Δ and A_b^Δ, $B_{2,b}^\Delta$ are as in (3.9), (3.10) and (4.15), (4.16), respectively. By this we also introduce $((L_2^\Delta)_\alpha, \mathcal{D}_\alpha^\Delta)$ and $(L_2^\Delta)_{\alpha\alpha'}$ as in subsection 3.2. Then by claims *(i)* and *(iv)* of Lemma 4.5 we have that

$$u_t := Q_{\alpha\alpha_1}(t; B_{2,b}^\Delta)k_0, \qquad \alpha \in (\alpha_1, \alpha_2), \tag{4.41}$$

solves the problem

$$\frac{d}{dt}u_t = (L_2^\Delta)_{\alpha_2}u_t, \qquad u_0 = k_0, \tag{4.42}$$

on the time interval $[0, T(\alpha_2, \alpha_1; B_{2,b}^\Delta))$. Note that

$$T(\alpha_2, \alpha_1; B_b^\Delta) \le T(\alpha_2, \alpha_1; B_{2,b}^\Delta),$$

see (4.20) and (4.24). Take $\alpha, \alpha' \in (\alpha_1, \alpha_2)$, $\alpha' < \alpha$, and pick positive $\tau_1 \le \tau(\alpha_2, \alpha_1)$ such that

$$\tau_1 = \tau_1(\alpha_2, \alpha_1) < \min\{T(\alpha_2, \alpha; B_b^\Delta); T(\alpha', \alpha_1; B_{2,b}^\Delta)\}.$$

By (4.42) the difference $u_t - k_t \in \mathcal{K}_{\alpha_2}$ can be written down in the form

$$u_t - k_t = \int_0^t Q_{\alpha_2\alpha}(t-s; B_{2,b}^\Delta)\left(-B_1^\Delta\right)_{\alpha\alpha'} k_s ds, \tag{4.43}$$

where $t \le \tau_1$ and the operator $(-B_1^\Delta)_{\alpha\alpha'}$ is positive with respect to the cone (2.19), see (3.10) and (3.13). In (4.43), $k_s \in \mathcal{K}_{\alpha'}$ and $Q_{\alpha_2\alpha}(t-s; B_{2,b}^\Delta) \in \mathcal{L}(\mathcal{K}_\alpha, \mathcal{K}_{\alpha_2})$ for all $s \in [0, \tau_1]$. Since $Q_{\alpha_2\alpha}(t-s; B_{2,b}^\Delta)$ is also positive, see Remark 4.4, and $k_s \in \mathcal{K}_{\alpha'}^\star \subset \mathcal{K}_{\alpha'}^+$ (by (4.37) and Lemma 4.9), we have $u_t - k_t \in \mathcal{K}_{\alpha_2}^+$ for $t \le \tau_1(\alpha_2, \alpha_1)$, which yields (4.40). □

Corollary 4.11. *Let α_2, α_1, and $\tau_1(\alpha_2,\alpha_1)$ be as in Lemma 4.10. Then the following holds for all $t \leq \tau_1(\alpha_2,\alpha_1)$*

$$\|k_t\|_{\alpha_2} = \left\|Q_{\alpha_2\alpha_1}(t;B_b^\Delta)k_0\right\|_{\alpha_2} \leq \frac{(\alpha_2-\alpha_1)\|k_0\|_{\alpha_1}}{\alpha_2-\alpha_1-t(\langle a^+\rangle+b)}. \tag{4.44}$$

Proof. Apply (4.40) and then (4.20) and (4.24). □

Proof of Theorem 3.3. Let $\alpha_0 > -\log\vartheta$ be such that $k_{\mu_0} \in \mathcal{K}_{\alpha_0}$, cf, (2.17). Then by Lemma 4.8 we have that for each $\alpha_1 > \alpha_0$ and $\alpha \in (\alpha_0,\alpha_1)$,

$$k_t := Q_{\alpha\alpha_0}(t;B_b^\Delta)k_0 \in \mathcal{K}_\alpha^\star, \qquad t \leq \tau_1(\alpha_1,\alpha_0),$$

solves (3.17) in $\mathcal{K}_{\alpha_1}$. Its continuation to an arbitrary $t>0$ readily follows by (4.44). □

4.4. **The proof of Theorem 3.4.**

4.4.1. *Case $\langle a^+\rangle > 0$ and $m \in [0,\langle a^+\rangle]$.* The proof will be done by picking the corresponding bounds for u_t defined in (4.41) with $k_0 = k_{\mu_0} \in \mathcal{K}_{\alpha_0}^\star$. Recall that, for $\alpha_1 > \alpha_0$, $u_t \in \mathcal{K}_{\alpha_1}$ for $t < T(\alpha_1,\alpha_0;B_{2,b}^\Delta)$. For a given $\delta \leq m$, let us choose the value of C_δ. The first condition is that

$$C_\delta^{|\eta|} \geq k_0(\eta). \tag{4.45}$$

Next, if (3.5) holds with a given $\vartheta > 0$ and $b=0$, we take any $\delta \leq m$ and $C_\delta \geq 1/\vartheta$ such that also (4.45) holds. If (3.5) holds with $b>0$, we take any $\delta < m$ and then $C_\delta \geq b/(m-\delta)\vartheta$ such that also (4.45) holds. In all this cases, by Proposition 3.6 we have that

$$E^-(\eta) - \frac{1}{C_\delta}E^+(\eta) \geq -(m-\delta)|\eta|, \qquad \eta\in\Gamma_0. \tag{4.46}$$

Let $r_t(\eta)$ denote the right-hand side of (3.18). For $\alpha_1 > \alpha_0$, we take $\alpha,\alpha' \in (\alpha_0,\alpha_1)$, $\alpha' < \alpha$ and then consider

$$\begin{aligned} v_t &:= Q_{\alpha_1\alpha_0}(t;B_{2,b}^\Delta)r_0 \\ &= r_t + \int_0^t Q_{\alpha_1\alpha}(t-s;B_{2,b}^\Delta)D_{\alpha\alpha'}r_s ds, \end{aligned} \tag{4.47}$$

where

$$t \leq \tau_2 := \min\left\{\frac{\alpha'-\alpha_0}{\langle a^+\rangle-\delta}; T(\alpha_1,\alpha;B_{2,b}^\Delta)\right\}. \tag{4.48}$$

The operator D in (4.47) is

$$\begin{aligned} (D_{\alpha\alpha'}r_s)(\eta) &= \Bigg[-m|\eta| - E^-(\eta) + \frac{1}{C_\delta}\exp\Big(-(\langle a^+\rangle-\delta)s\Big)E^+(\eta) \\ &+ \delta|\eta|\Bigg]r_s(\eta) \leq 0, \qquad \eta\in\Gamma_0. \end{aligned} \tag{4.49}$$

The latter inequality holds for all $s \in [0, \tau_2]$, see (4.46), and all $m \in [0, \langle a^+ \rangle]$ and $\delta < m$. Then by (4.36) we obtain from (4.41), the first line of (4.47), and (4.45) that

$$u_t(\eta) \leq v_t(\eta), \qquad t < T(\alpha_1, \alpha_0; B^{\Delta}_{2,b}).$$

Then by the second line of (4.47) and (4.49) we get that for $t \leq \tau_2$, see (4.48), the following holds

$$u_t(\eta) \leq v_t(\eta) \leq r_t(\eta), \qquad \eta \in \Gamma_0.$$

The continuation of the latter inequality to bigger values of t is straightforward. This completes the proof for this case.

4.4.2. *Case $\langle a^+ \rangle > 0$ and $m > \langle a^+ \rangle$.* Take $\varepsilon \in (0, m - \langle a^+ \rangle)$ and then set

$$\vartheta_\varepsilon = \vartheta \left(1 - \frac{\varepsilon + 2\langle a^+ \rangle}{2m} \right).$$

Thereafter, choose $C_\varepsilon \geq 1/\vartheta_\varepsilon$ such that

$$C_\varepsilon^{|\eta|} \geq k_0(\eta), \qquad \eta \in \Gamma_0.$$

Then, cf. (4.46),

$$E^-(\eta) - \frac{1}{C_\varepsilon} E^+(\eta) \geq -(m - \langle a^+ \rangle - \varepsilon/2)|\eta|, \qquad \eta \in \Gamma_0. \tag{4.50}$$

Let now r_t stand for the right-hand side of (3.19). Then the second line of (4.47) holds with $D_{\alpha\alpha'}$ replaced by $D^\varepsilon_{\alpha\alpha'}$. By definition the latter is such that: (a) $(D^\varepsilon_{\alpha\alpha'} r_s)(\emptyset) = 0$;

$$\text{(b)} \quad (D^\varepsilon_{\alpha\alpha'} r_s)(\{x\}) = -(m - \langle a^+ \rangle - \varepsilon) r_s(\{x\}) \leq 0,$$

and, for $|\eta| \geq 2$, see (4.50),

$$\begin{aligned} \text{(c)} \quad (D^\varepsilon_{\alpha\alpha'} r_s)(\eta) &= \left[\varepsilon - m|\eta| - E^-(\eta) + \frac{1}{C_\varepsilon} E^+(\eta) + \langle a^+ \rangle |\eta| \right] r_s(\eta) \\ &\leq \varepsilon \left(1 - |\eta|/2\right) r_s(\eta) \leq 0. \end{aligned}$$

This yields (3.19) and thus completes the proof for this case.

4.4.3. *The remaining cases.* For $\langle a^+ \rangle = 0$ and $t > 0$, we set

$$\left(Q^{(0)}_{\alpha\alpha'}(t) u \right)(\eta) = \exp\left[-tE(\eta)\right] u(\eta), \tag{4.51}$$

where $\alpha' < \alpha$ and $u \in \mathcal{K}_{\alpha'}$. Then, cf. Lemma 4.5, $Q^{(0)}_{\alpha\alpha'}(t) : \mathcal{K}_{\alpha'} \to \mathcal{K}_\alpha$ continuously, and the map

$$[0, +\infty) \ni t \mapsto Q^{(0)}_{\alpha\alpha'}(t) \in \mathcal{L}(\mathcal{K}_{\alpha'}, \mathcal{K}_\alpha)$$

is continuous and such that, cf. (4.27),

$$\frac{d}{dt} Q^{(0)}_{\alpha''\alpha'}(t) = (A^{\Delta}_1)_{\alpha''\alpha} Q^{(0)}_{\alpha\alpha'}(t), \qquad \alpha'' > \alpha, \tag{4.52}$$

where $(A_1^\Delta)_{\alpha''\alpha}$ is defined in (3.9) and (3.11). Now we set $u_t = Q^{(0)}_{\alpha\alpha_0}(t)k_{\mu_0}$ and obtain from (4.51) and (4.52), similarly as in (4.43),

$$u_t - k_t = \int_0^t Q^{(0)}_{\alpha\alpha_1}(t)\left(-B_1^\Delta\right)_{\alpha_1\alpha_2} k_s ds \geq 0,$$

which yields (3.20).

To prove that $r_t(\eta) := \vartheta^{-|\eta|}$, $t \geq 0$, is a stationary solution we set

$$k_t = Q_{\alpha\alpha_0}(t; B_b^\Delta) r_0,$$

where $\alpha_0 > -\log\vartheta$ and $\alpha > \alpha_0$. Then the following holds, cf. (4.43),

$$k_t = r_t + \int_0^t Q_{\alpha\alpha_2}(t-s; B_b^\Delta) L^\Delta_{\alpha_2\alpha_1} r_s ds,$$

where $\alpha_1 < \alpha_2$ are taken from (α_0, α). For the case considered, we have

$$L^\Delta_{\alpha_2\alpha_1} r_s = L^\Delta_{\alpha_2\alpha_1} r_0 = 0,$$

which completes the proof for this case.

5. The Proof of the Identification Lemma

To prove Lemma 4.9 we use Proposition 2.3. Note that the solution mentioned in Lemma 4.8 already has properties (ii) and (iii) of (2.12), cf. (2.14). Thus, it remains to prove that also (i) holds. We do this as follows. First, we approximate the evolution $k_0 \mapsto k_t$ established in Lemma 4.8 by evolutions $k_{0,\mathrm{app}} \mapsto k_{t,\mathrm{app}}$ such that $k_{t,\mathrm{app}}$ has property (i). Then we prove that for each $G \in B^\star_{\mathrm{bs}}(\Gamma_0)$, $\langle\!\langle G, k_{t,\mathrm{app}}\rangle\!\rangle \to \langle\!\langle G, k_t\rangle\!\rangle$ as the approximations are eliminated. The limiting transition is based on the representation $\langle\!\langle G, k_{t,\mathrm{app}}\rangle\!\rangle = \langle\!\langle G_t, k_{0,\mathrm{app}}\rangle\!\rangle$ in which we use the so called predual evolution $G \mapsto G_t$. Then we just show that $\langle\!\langle G_t, k_{0,\mathrm{app}}\rangle\!\rangle \to \langle\!\langle G_t, k_0\rangle\!\rangle$.

5.1. **The predual evolution.** The aim of this subsection is to construct the evolution $B_{\mathrm{loc}}(\Gamma_0) \ni G_0 \mapsto G_t \in \mathcal{G}_{\alpha_1}$, see (4.1) and (4.2), such that, for each $\alpha > \alpha_1$ and $k_0 \in \mathcal{K}_{\alpha_1}$, the following holds, cf. (4.37),

$$\langle\!\langle G_0, Q_{\alpha\alpha_1}(t; B_b^\Delta)k_0\rangle\!\rangle = \langle\!\langle G_t, k_0\rangle\!\rangle, \tag{5.1}$$

where $b \geq 0$ and B_b^Δ are as in (3.5) and (4.16), respectively. Let us define the action of B_b on appropriate $G : \Gamma_0 \to \mathbb{R}$ via the duality

$$\langle\!\langle G, B_b^\Delta k\rangle\!\rangle = \langle\!\langle B_b G, k\rangle\!\rangle.$$

Similarly as in (4.16) we then get

$$(B_b G)(\eta) = b|\eta|G(\eta) + \int_{\mathbb{R}^d} \sum_{x\in\eta} a^+(x-y) G(\eta \setminus x \cup y) dy \tag{5.2}$$

$$- \sum_{x\in\eta} E^-(x, \eta\setminus x) G(\eta\setminus x).$$

For $\alpha_2 > \alpha_1$, let $(B_b)_{\alpha_1\alpha_2}$ be the bounded linear operator from $\mathcal{G}_{\alpha_2}$ to $\mathcal{G}_{\alpha_1}$ the action of which is defined in (5.2). As in estimating the norm of B_b^Δ in (4.18) one then gets

$$\|(B_b)_{\alpha_1\alpha_2}\| \leq \frac{\langle a^+\rangle + b + \langle a^-\rangle e^{\alpha_2}}{e(\alpha_2 - \alpha_1)}. \tag{5.3}$$

For the same α_2 and α_1, let $S_{\alpha_1\alpha_2}(t)$ be the restriction to $\mathcal{G}_{\alpha_2}$ of the corresponding element of the semigroup mentioned in Lemma 4.2. Then $S_{\alpha_1\alpha_2}(t)$ acts as a bounded contraction from $\mathcal{G}_{\alpha_2}$ to $\mathcal{G}_{\alpha_1}$.

Now for a given $l \in \mathbb{N}$ and α, α_1 as in (5.1), let δ and α^s, $s = 0, \dots, 2l+1$, be as in (4.29). Then for $t > 0$ and $(t, t_1, \dots, t_l) \in \mathcal{T}_l$, see (4.28), we define, cf. (4.30),

$$\Omega^{(l)}_{\alpha_1\alpha}(t, t_1, \dots, t_l) = S_{\alpha_1\alpha^1}(t_l)(B_b)_{\alpha^1\alpha^2} S_{\alpha^2\alpha^3}(t_{l-1} - t_l) \times \cdots \times$$
$$\times (B_b)_{\alpha^{2s-1}\alpha^{2s}} S_{\alpha^{2s}\alpha^{2s+1}}(t_{l-s} - t_{l-s+1}) \cdots (B_b)_{\alpha^{2l-1}\alpha^{2l}} S_{\alpha^{2l}\alpha}(t - t_1).$$

As in Proposition 4.6, one shows that the map

$$\mathcal{T}_l \ni (t, t_1, \dots, t_l) \mapsto \Omega^{(l)}_{\alpha_1\alpha}(t, t_1, \dots, t_l) \in \mathcal{L}(\mathcal{G}_\alpha, \mathcal{G}_{\alpha_1})$$

is continuous. Define

$$H^{(n)}_{\alpha_1\alpha}(t) = S_{\alpha_1\alpha}(t) + \sum_{l=1}^{n} \int_0^t \int_0^{t_1} \cdots \int_0^{t_{l-1}} \Omega^{(l)}_{\alpha_1\alpha}(t, t_1, \dots, t_l) dt_l \cdots dt_1. \tag{5.4}$$

Lemma 5.1. *For each $T \in (0, T(\alpha, \alpha_1; B_b^\Delta))$, see (4.24) and (4.20), the sequence of operators defined in (5.4) converges in $\mathcal{L}(\mathcal{G}_\alpha, \mathcal{G}_{\alpha_1})$ to a certain $H_{\alpha_1\alpha}(t)$ uniformly on $[0, T]$, and for each $G_0 \in \mathcal{G}_\alpha$ and $k_0 \in \mathcal{K}_{\alpha_1}$ the following holds*

$$\langle\!\langle G_0, Q_{\alpha\alpha_1}(t; B_b^\Delta)k_0 \rangle\!\rangle = \langle\!\langle H_{\alpha_1\alpha}(t)G_0, k_0 \rangle\!\rangle, \qquad t \in [0, T]. \tag{5.5}$$

Proof. For the operators defined in (5.4), similarly as in (4.35) we get the following estimate

$$\|H^{(n)}_{\alpha_1\alpha}(t) - H^{(n-1)}_{\alpha_1\alpha}(t)\| \leq \frac{1}{n!}\left(\frac{n}{e}\right)^n \left(\frac{T}{T_\delta}\right)^n,$$

which yields the convergence stated in the lemma. By direct inspection one gets that

$$\langle\!\langle G_0, Q^{(n)}_{\alpha\alpha_1}(t; B^\Delta)k_0 \rangle\!\rangle = \langle\!\langle H^{(n)}_{\alpha_1\alpha}(t)G_0, k_0 \rangle\!\rangle,$$

see (4.33). Then (5.5) is obtained from the latter in the limit $n \to +\infty$. Similarly as in (4.26), for the limiting operator the following estimate holds

$$\|H_{\alpha_1\alpha}(t)\| \leq \frac{T(\alpha, \alpha_1; B_b^\Delta)}{T(\alpha, \alpha_1; B_b^\Delta) - t}. \tag{5.6}$$

□

5.2. **An auxiliary model.** The approximations mentioned at the beginning of this section employ also an auxiliary model, which we introduce and study now. For this model, we construct three kinds of evolutions. The first one is $k_0 \mapsto k_t \in \mathcal{K}_\alpha$ obtained as in Lemma 4.8. Another evolution $q_0 \mapsto q_t \in \mathcal{G}_\omega$ is constructed in such a way that q_t is positive definite in the sense that $\langle\langle G, q_t \rangle\rangle \geq 0$ for all $G \in B^\star_{\rm bs}(\Gamma_0)$. These evolutions, however, take place in different spaces. To relate them to each other we construct one more evolution, $u_0 \mapsto u_t$, which takes place in the intersection of the mentioned Banach spaces. The aim is to show that $k_t = u_t = q_t$ and thereby to get the desired property of k_t. Thereafter, we prove the convergence mentioned above.

5.2.1. *The model.* The function

$$\varphi_\sigma(x) = \exp\left(-\sigma|x|^2\right), \qquad \sigma > 0, \quad x \in \mathbb{R}^d, \tag{5.7}$$

has the following evident properties

$$\bar{\varphi}_\sigma := \int_{\mathbb{R}^d} \varphi(x)dx < \infty, \qquad \varphi_\sigma(x) \leq 1, \quad x \in \mathbb{R}^d. \tag{5.8}$$

The model we need is characterized by L as in (1.4) with $E^+(x, \eta)$, cf. (1.5), replaced by

$$E^+_\sigma(x, \eta) = \varphi_\sigma(x) E^+_\sigma(x, \eta) = \varphi_\sigma(x) \sum_{y \in \eta} a^+(x - y). \tag{5.9}$$

5.2.2. *The evolution in* $\mathcal{K}_\alpha$. For the new model (with E^+_σ as in (5.9)), the operator $L^{\Delta,\sigma}$ corresponding to L^Δ takes the form, cf. (3.8) – (3.10) and (4.15) – (4.17),

$$L^{\Delta,\sigma} = A^{\Delta,\sigma} + B^{\Delta,\sigma} = A^{\Delta,\sigma}_b + B^{\Delta,\sigma}_b. \tag{5.10}$$

Here

$$\begin{aligned} A^{\Delta,\sigma} &= A^\Delta_1 + A^{\Delta,\sigma}_2, \qquad A^{\Delta,\sigma}_b = A^\Delta_{1,b} + A^{\Delta,\sigma}_2, \\ B^{\Delta,\sigma} &= B^\Delta_1 + B^{\Delta,\sigma}_2, \qquad B^{\Delta,\sigma}_b = B^\Delta_1 + B^{\Delta,\sigma}_{2,b}, \end{aligned} \tag{5.11}$$

where A^Δ_1, B^Δ_1, and $A^\Delta_{1,b}$ are the same as in (3.9), (3.10), and (4.15), respectively, and

$$\begin{aligned} \left(A^{\Delta,\sigma}_2 k\right)(\eta) &= \sum_{x \in \eta} \varphi_\sigma(x) E^+(x, \eta \setminus x) k(\eta \setminus x), \\ \left(B^{\Delta,\sigma}_2 k\right)(\eta) &= b|\eta| k(\eta) + \int_{\mathbb{R}^d} \sum_{x \in \eta} \varphi_\sigma(x) a^+(x - y) k(\eta \setminus x \cup y) dy. \end{aligned} \tag{5.12}$$

Note that these $A^{\Delta,\sigma}_b$ and $B^{\Delta,\sigma}_b$ define the corresponding bounded operators acting from $\mathcal{K}_{\alpha'}$ to $\mathcal{K}_\alpha$ for each real $\alpha > \alpha'$. As in (3.15) we then set

$$\mathcal{D}^{\Delta,\sigma}_\alpha = \{k \in \mathcal{K}_\alpha : L^{\Delta,\sigma} k \in \mathcal{K}_\alpha\}, \tag{5.13}$$

and thus define the corresponding operator $(L^{\Delta,\sigma}_\alpha, \mathcal{D}^{\Delta,\sigma}_\alpha)$. Along with (3.17) we also consider

$$\frac{d}{dt}k_t = L^{\Delta,\sigma}_\alpha k_t, \qquad k_t|_{t=0} = k_0 \in \mathcal{D}^{\Delta,\sigma}_\alpha. \tag{5.14}$$

By the literal repetition of the construction used in the proof of Lemma 4.5 one obtains the operators $Q^\sigma_{\alpha\alpha'}(t; B^{\Delta,\sigma}_b)$, $(\alpha, \alpha', t) \in \mathcal{A}(B^\Delta_b)$, see (4.25), the norm of which satisfies, cf. (4.26),

$$\|Q^\sigma_{\alpha\alpha'}(t; B^{\Delta,\sigma}_b)\| \leq \frac{T(\alpha, \alpha'; B^\Delta_b)}{T(\alpha, \alpha'; B^\Delta_b) - t}, \tag{5.15}$$

which is uniform in σ.

Lemma 5.2. *Let α_1 and α_2 be as in Lemma 4.8. Then for a given $k_0 \in \mathcal{K}_{\alpha_1}$, the unique solution of (5.14) in $\mathcal{K}_{\alpha_2}$ is given by*

$$k_t = Q^\sigma_{\alpha\alpha_1}(t; B^{\Delta,\sigma}_b)k_0, \qquad \alpha \in (\alpha_1, \alpha_2), \quad t < T(\alpha_2, \alpha_1; B^\Delta_b). \tag{5.16}$$

Proof. Repeat the proof of Lemma 4.8. □

5.2.3. *The evolution in $\mathcal{U}_{\sigma,\alpha}$.* For φ_σ as in (5.7) we set

$$e(\varphi_\sigma; \eta) = \prod_{x\in\eta} \varphi_\sigma(x), \qquad \eta \in \Gamma_0,$$

and introduce the following Banach space. For $u : \Gamma_0 \to \mathbb{R}$, we define the norm, cf. (2.14),

$$\|u\|_{\sigma,\alpha} = \operatorname*{ess\,sup}_{\eta\in\Gamma_0} \frac{|u(\eta)| \exp(-\alpha|\eta|)}{e(\varphi_\sigma; \eta)}. \tag{5.17}$$

Thereafter, set

$$\mathcal{U}_{\sigma,\alpha} = \{u : \Gamma_0 \to \mathbb{R} : \|u\|_{\sigma,\alpha} < \infty\}.$$

By (5.7) and (2.14) we have that

$$\|u\|_\alpha \leq \|u\|_{\sigma,\alpha}, \qquad u \in \mathcal{U}_{\sigma,\alpha},$$

which yields $\mathcal{U}_{\sigma,\alpha} \hookrightarrow \mathcal{K}_\alpha$. Moreover, as in (2.17) we also have that $\mathcal{U}_{\sigma,\alpha'} \hookrightarrow \mathcal{U}_{\sigma,\alpha}$ for each real $\alpha > \alpha'$.

Now let us define the operator $L^{\Delta,\sigma}_{\alpha,u}$ in $\mathcal{U}_{\sigma,\alpha}$ the action of which is described in (5.10) – (5.12) and the domain is, cf. (5.13),

$$\mathcal{D}^{\Delta,\sigma}_{\alpha,u} = \{u \in \mathcal{U}_{\sigma,\alpha} : L^{\Delta,\sigma}u \in \mathcal{U}_{\sigma,\alpha}\}. \tag{5.18}$$

Then we consider

$$\frac{d}{dt}u_t = L^{\Delta,\sigma}_{\alpha,u} u_t, \qquad u_t|_{t=0} = u_0 \in \mathcal{D}^{\Delta,\sigma}_{\alpha,u}. \tag{5.19}$$

Note that $\mathcal{U}_{\sigma,\alpha''} \subset \mathcal{D}(L^{\Delta,\sigma}_{\alpha,u})$ for each $\alpha'' < \alpha$, and

$$(L^{\Delta,\sigma}_{\alpha,u}, \mathcal{D}^{\Delta,\sigma}_{\alpha,u}) \subset (L^{\Delta,\sigma}_\alpha, \mathcal{D}^{\Delta,\sigma}_\alpha). \tag{5.20}$$

Our aim now is to prove that the problem (5.19) with $u_0 \in \mathcal{U}_{\sigma,\alpha_1}$ has a unique solution in $\mathcal{U}_{\sigma,\alpha_2}$, where $\alpha_1 < \alpha_2$ are as in Lemma 4.8. To this end we first construct the semigroup analogous to that obtained in Lemma 4.2. Thus, in the predual space $\mathcal{G}_{\sigma,\alpha}$ equipped with the norm, cf. (4.2),

$$|G|_{\sigma,\alpha} := \int_{\Gamma_0} |G(\eta)| \exp(\alpha|\eta|) e(\varphi_\sigma;\eta)\lambda(d\eta)$$

we define the action of A_b^σ as follows, cf. (4.5),

$$\begin{aligned} A_b^\sigma &= A_{1,b} + A_2^\sigma \\ (A_2^\sigma G)(\eta) &= \int_{\mathbb{R}^d} \varphi_\sigma(y) E^+(y,\eta) G(\eta\cup y) dy, \end{aligned}$$

and $A_{1,b}$ acts as in (4.5). Then we have, cf. (4.7),

$$\begin{aligned} &|A_2^\sigma G|_{\sigma,\alpha} \\ &\leq \int_{\Gamma_0} \left(\int_{\mathbb{R}^d} \varphi_\sigma(y) E^+(y,\eta) |G(\eta\cup y)| dy \right) \exp(\alpha|\eta|) e(\varphi_\sigma;\eta)\lambda(d\eta) \\ &= \int_{\Gamma_0} e^{-\alpha} \left(\sum_{x\in\eta} E^+(x,\eta\setminus x) \right) |G(\eta)| \exp(\alpha|\eta|) e(\varphi_\sigma;\eta)\lambda(d\eta) \\ &\leq (e^{-\alpha}/\vartheta) |A_{1,b} G|_{\sigma,\alpha}. \end{aligned}$$

Now the existence of the substochastic semigroup $\{S_{\sigma,\alpha}(t)\}_{t\geq 0}$ generated by $(A_b^\sigma, \mathcal{D}_{\sigma,\alpha})$ follows as in Lemma 4.2. Here, cf. (4.6),

$$\mathcal{D}_{\sigma,\alpha} := \{G \in \mathcal{G}_{\sigma,\alpha} : E_b(\cdot) G \in \mathcal{G}_{\sigma,\alpha}\}.$$

Let $S^{\odot}_{\sigma,\alpha}(t)$ be the sun-dual to $S_{\sigma,\alpha}(t)$, cf. (4.11). Then for each $\alpha' < \alpha$ and any $u \in \mathcal{U}_{\sigma,\alpha'}$, the map

$$[0,+\infty) \ni t \mapsto S^{\odot}_{\sigma,\alpha}(t) u \in \mathcal{U}_{\sigma,\alpha}$$

is continuous, see Proposition 4.3. For real $\alpha' < \alpha$ and $t > 0$, let $\Sigma^{\sigma,u}_{\alpha\alpha'}(t)$ be the restriction of $S^{\odot}_{\sigma,\alpha}(t)$ to $\mathcal{U}_{\sigma,\alpha'}$. Then the map

$$[0,+\infty) \ni t \mapsto \Sigma^{\sigma,u}_{\alpha\alpha'}(t) \in \mathcal{L}(\mathcal{U}_{\sigma,\alpha'}, \mathcal{U}_{\sigma,\alpha})$$

is continuous and such that, cf. (4.23),

$$\|\Sigma^{\sigma,u}_{\alpha\alpha'}(t)\| \leq 1, \qquad t \geq 0. \tag{5.21}$$

Now we define $(B_b^{\Delta,\sigma})_{\alpha\alpha'}$ which acts from $\mathcal{U}_{\sigma,\alpha'}$ to $\mathcal{U}_{\sigma,\alpha}$ according to (5.11) and (5.12). Then its norm satisfies

$$\|(B_b^{\Delta,\sigma})_{\alpha\alpha'}\| \leq \frac{\langle a^+\rangle + b + \langle a^-\rangle e^\alpha}{e(\alpha-\alpha')}. \tag{5.22}$$

In proving this we take into account that $\varphi_\sigma(x) \leq 1$ and repeat the arguments used in obtaining (4.18).

For real $\alpha_2 > \alpha_1 > -\log\vartheta$, we take $\alpha \in (\alpha_1, \alpha_2]$ and then pick $\delta < \alpha - \alpha_1$ as in the proof of Lemma 4.5. Next, for $l \in \mathbb{N}$ we divide $[\alpha_1, \alpha]$ into subintervals according to (4.29) and take $(t, t_1, \ldots, t_l) \in \mathcal{T}_l$, see (4.28). Then define, cf. (4.30),

$$\Pi^{l,\sigma}_{\alpha\alpha_1}(t, t_1, \ldots, t_l) = \Sigma^{\sigma,u}_{\alpha\alpha^{2l}}(t - t_1)(B^{\Delta,\sigma}_b)_{\alpha^{2l}\alpha^{2l-1}}$$
$$\times \Sigma^{\sigma,u}_{\alpha^{2l-1}\alpha^{2l-2}}(t_1 - t_2) \times \cdots \times \Sigma^{\sigma,u}_{\alpha^3\alpha^2}(t_{l-1} - t_l)(B^{\Delta,\sigma}_b)_{\alpha^2\alpha^1}\Sigma^{\sigma,u}_{\alpha^1\alpha_1}(t_l).$$

Thereafter, for $n \in \mathbb{N}$ we set, cf. (4.33),

$$U^{(n)}_{\alpha\alpha_1}(t) = \Sigma^{\sigma,u}_{\alpha\alpha_1}(t)$$
$$+ \sum_{l=1}^{n} \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} \Pi^{l,\sigma}_{\alpha\alpha_1}(t, t_1, \ldots, t_l) dt_l \cdots dt_1.$$

By means of (5.21) and (5.22) we then prove that the sequence $\{U^{(n)}_{\alpha\alpha_1}(t)\}_{n\in\mathbb{N}}$ converges in $\mathcal{L}(\mathcal{U}_{\sigma,\alpha_1}, \mathcal{U}_{\sigma,\alpha})$, uniformly on $[0, T]$, $T < T(\alpha, \alpha_1; B^{\Delta}_b)$, see (4.24) and (4.20). The limit $U_{\alpha\alpha_1}(t) \in \mathcal{L}(\mathcal{U}_{\sigma,\alpha_1}, \mathcal{U}_{\sigma,\alpha})$ has the property, cf. (4.27),

$$\frac{d}{dt} U_{\alpha_2\alpha_1}(t) = \left((A^{\Delta,\sigma}_b)_{\alpha_2\alpha} + (B^{\Delta,\sigma}_b)_{\alpha_2\alpha} \right) U_{\alpha\alpha_1}(t),$$

where $(A^{\Delta,\sigma}_b)_{\alpha_2\alpha} \in \mathcal{L}(\mathcal{U}_{\sigma,\alpha}, \mathcal{U}_{\sigma,\alpha_2})$ is defined in (5.11) and (5.12), analogously to (5.22). Note that

$$\forall u \in \mathcal{U}_{\sigma,\alpha} \quad L^{\Delta,\sigma}_{\alpha_2,u} u = \left((A^{\Delta,\sigma}_b)_{\alpha_2\alpha} + (B^{\Delta,\sigma}_b)_{\alpha_2\alpha} \right) u, \tag{5.23}$$

see (5.18). Now we can state the following analog of Lemma 4.8.

Lemma 5.3. *Let $\alpha_2 > \alpha_1 > -\log\vartheta$ be as in Lemma 4.8. Then the problem (5.19) with $u_0 \in \mathcal{U}_{\sigma,\alpha_1}$ has a unique solution $u_t \in \mathcal{U}_{\sigma,\alpha_2}$ on the time interval $[0, T(\alpha_2, \alpha_1; B^{\Delta}_b))$.*

Proof. Fix $T < T(\alpha_2, \alpha_1; B^{\Delta}_b)$ and find $\alpha \in (\alpha_1, \alpha_2)$ such that also $T < T(\alpha', \alpha_1; B^{\Delta}_b)$. Then, cf. (4.37),

$$u_t := U_{\alpha\alpha_1}(t) u_0 \tag{5.24}$$

is the solution in question, which can be checked by means of (5.23). Its uniqueness can be proved by the literal repetition of the corresponding arguments used in the proof of Lemma 4.8. □

Corollary 5.4. *Let k_t be the solution of the problem (5.14) with $k_0 \in \mathcal{U}_{\sigma,\alpha_1}$ mentioned in Lemma 5.2. Then k_t coincides with the solution mentioned in Lemma 5.3.*

Proof. Since $(L^{\Delta,\sigma}_{\alpha}, \mathcal{D}^{\Delta,\sigma}_{\alpha})$ is an extension of $(L^{\Delta,\sigma}_{\alpha,u}, \mathcal{D}^{\Delta,\sigma}_{\alpha,u})$, see (5.20), and the embedding $\mathcal{U}_{\sigma,\alpha} \hookrightarrow \mathcal{K}_\alpha$ is continuous, the solution as in (5.24) with $u_0 = k_0$ satisfies also (5.14), and hence coincides with k_t in view of the uniqueness stated in Lemma 5.2. □

5.2.4. *The evolution in $\mathcal{G}_\omega$.* We recall that the space $\mathcal{G}_\alpha$ was introduced in (4.1), (4.2), where we used it as a predual space to $\mathcal{K}_\alpha$. Now we employ $\mathcal{G}_\alpha$ to get the positive definiteness mentioned at the beginning of this subsection. Here, however, we write $\mathcal{G}_\omega$ to show that we use it not as a predual space.

Let $L^{\Delta,\sigma}$ be as in (5.10). For $\omega \in \mathbb{R}$, we set, cf. (5.13) and (5.18),

$$\mathcal{D}_\omega^{\Delta,\sigma} = \{q \in \mathcal{G}_\omega : L^{\Delta,\sigma} q \in \mathcal{G}_\omega\}.$$

Then we define the corresponding operator $(L_\omega^{\Delta,\sigma}, \mathcal{D}_\omega^{\Delta,\sigma})$ and consider the following Cauchy problem

$$\frac{d}{dt} q_t = L_\omega^{\Delta,\sigma} q_t, \qquad q_t|_{t=0} = q_0 \in \mathcal{D}_\omega^{\Delta,\sigma}. \tag{5.25}$$

As above, one can show that $\mathcal{G}_{\omega'} \subset \mathcal{D}_\omega^{\Delta,\sigma}$ for each $\omega' > \omega$. By (5.17) and (4.2) for $u \in \mathcal{U}_{\sigma,\alpha}$ we have

$$\begin{aligned} |u|_\omega &\leq \|u\|_{\sigma,\alpha} \int_{\Gamma_0} \exp((\omega+\alpha)|\eta|) e(\varphi_\sigma; \eta) \lambda(d\eta) \\ &\leq \|u\|_{\sigma,\alpha} \exp\left(\bar{\varphi}_\sigma e^{\omega+\alpha}\right), \end{aligned} \tag{5.26}$$

see also (5.8). Hence $\mathcal{U}_{\sigma,\alpha} \hookrightarrow \mathcal{G}_\omega$ for each ω and α. Like in (5.20) we then get

$$(L_{\alpha,u}^{\Delta,\sigma}, \mathcal{D}_{\alpha,u}^{\Delta,\sigma}) \subset (L_\omega^{\Delta,\sigma}, \mathcal{D}_\omega^{\Delta,\sigma}). \tag{5.27}$$

Lemma 5.5. *Assume that the problem (5.25) with $\omega > 0$ and $q_0 \in \mathcal{G}_{\omega'}$, $\omega' > \omega$, has a solution, $q_t \in \mathcal{G}_\omega$, on some time interval $[0, T(\omega', \omega))$. Then this solution is unique.*

Proof. Set

$$w_t(\eta) = (-1)^{|\eta|} q_t(\eta),$$

which is an isometry on $\mathcal{G}_\omega$. Then q_t solves (5.25) if and only if w_t solves the following equation

$$\begin{aligned} \frac{d}{dt} w_t(\eta) &= -E(\eta) w_t(\eta) + \int_{\mathbb{R}^d} E^-(y,\eta) w_t(\eta \cup y) dy \\ &- \sum_{x \in \eta} \varphi_\sigma(x) E^+(x, \eta \setminus x) w_t(\eta \setminus x) \\ &+ \int_{\mathbb{R}^d} \sum_{x \in \eta} \varphi_\sigma(x) a^+(x-y) w_t(x \setminus x \cup y) dy. \end{aligned} \tag{5.28}$$

Set

$$\mathcal{D}_\omega = \{w \in \mathcal{G}_\omega : E(\cdot) w \in \mathcal{G}_\omega\}.$$

By Proposition 4.1 we prove that the operator defined by the first two summands in (5.28) with domain $\mathcal{D}_\omega$ generates a substochastic semigroup,

$\{V_\omega(t)\}_{t\geq 0}$, acting in $\mathcal{G}_\omega$. Indeed, in this case the condition analogous to that in (4.8) takes the form, cf. (4.9),

$$
\begin{aligned}
&- \int_{\Gamma_0} E(\eta) w(\eta) \exp(\omega|\eta|)\lambda(d\eta) \\
&\qquad\qquad + r^{-1} e^{-\omega} \int_{\Gamma_0} E^{-}(\eta) w(\eta) \exp(\omega|\eta|)\lambda(d\eta) \leq 0,
\end{aligned}
$$

which certainly holds for each $\omega > 0$ and an appropriate $r < 1$. For each $\omega'' \in (0, \omega)$, we have that $\mathcal{G}_\omega \hookrightarrow \mathcal{G}_{\omega''}$, and the second two summands in (5.28) define a bounded operator, $W_{\omega''\omega} : \mathcal{G}_\omega \to \mathcal{G}_{\omega''}$, the norm of which can be estimated as follows, cf. (5.3),

$$
\|W_{\omega''\omega}\| \leq \frac{(e^\omega + 1)\langle a^+ \rangle}{e(\omega - \omega'')}. \tag{5.29}
$$

Assume now that (5.28) has two solutions corresponding to the same initial condition $w_0(\eta) = (-1)^{|\eta|} q_0(\eta)$. Let v_t be their difference. Then it solves the following equation, cf. (4.38),

$$
v_t = \int_0^t V_{\omega''}(t - s) W_{\omega''\omega} v_s ds, \tag{5.30}
$$

where v_t on the left-hand side is considered as an element of $\mathcal{G}_{\omega''}$ and $t > 0$ will be chosen later. Now for a given $n \in \mathbb{N}$, we set $\epsilon = (\omega - \omega'')/n$ and then $\omega^l := \omega - l\epsilon$, $l = 0, \ldots, n$. Thereafter, we iterate (5.30) and get

$$
\begin{aligned}
v_t &= \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} V_{\omega''}(t - t_1) W_{\omega''\omega^{n-1}} V_{\omega^{n-1}}(t_1 - t_2) \times \cdots \times \\
&\times W_{\omega^2\omega^1} V_{\omega^1}(t_{n-1} - t_n) W_{\omega^1\omega} v_{t_n} dt_n \cdots dt_1.
\end{aligned}
$$

Similarly as in (4.39), by (5.29) this yields the following estimate

$$
|v_t|_{\omega''} \leq \frac{1}{n!} \left(\frac{n}{e}\right)^n \left(\frac{t\langle a^+ \rangle (e^\omega + 1)}{\omega - \omega''}\right)^n \sup_{s\in[0,t]} |v_s|_\omega.
$$

The latter implies that $v_t = 0$ for $t < (\omega - \omega'')/\langle a^+ \rangle (e^\omega + 1)$. To prove that $v_t = 0$ for all t of interest one has to repeat the above procedure appropriate number of times. □

Recall that each $\mathcal{U}_{\sigma,\alpha}$ is continuously embedded into each $\mathcal{G}_\omega$, see (5.26).

Corollary 5.6. *For each $\omega > 0$, the problem (5.25) with $q_0 \in \mathcal{U}_{\sigma,\alpha_0}$ has a unique solution q_t which coincides with the solution $u_t \in \mathcal{U}_{\sigma,\alpha}$ mentioned in Lemma 5.3.*

Proof. By (5.27) u_t is a solution of (5.25). Its uniqueness follows by Lemma 5.5. □

5.3. **Local evolution.** In this subsection we pass to the so called local evolution of states of the auxiliary model (5.10), (5.11). For this evolution, the corresponding 'correlation function' $q_t \in \mathcal{G}_\omega$ has the positive definiteness in question. Then we apply Corollaries 5.4 and 5.6 to get the same for the evolution in $\mathcal{K}_\alpha$. Thereafter, we pass to the limit and get the proof of Lemma 4.9.

5.3.1. *The evolution of densities.* In view of (2.2), each state with the property $\mu(\Gamma_0) = 1$ can be redefined as a probability measure on $\mathcal{B}(\Gamma_0)$, cf. Remark 2.1. Then the Fokker-Planck equation (1.3) can be studied directly, see [18, Eq. (2.8)]. Its solvability is described in [18, Theorem 2.2], which, in particular, states that the solution is absolutely continuous with respect to the Lebesgue-Poisson measure λ if μ_0 has the same property. In view of this we write the corresponding problem for the density

$$R_t := \frac{d\mu_t}{d\lambda}, \tag{5.31}$$

see also [18, Eq. (2.16)], and obtain

$$\frac{d}{dt} R_t(\eta) = (L^{\dagger,\sigma} R_t)(\eta), \quad R_t|_{t=0} = R_0, \tag{5.32}$$

where

$$\begin{aligned} (L^{\dagger,\sigma} R)(\eta) \quad &:= \quad -\Psi_\sigma(\eta) R(\eta) + \sum_{x\in\eta} \varphi_\sigma(x) E^+(x, \eta\setminus x) R_t(\eta \setminus x) \qquad (5.33) \\ &+ \quad \int_{\mathbb{R}^d} \left(m + E^-(x,\eta)\right) R_t(\eta\cup x) dx, \end{aligned}$$

and

$$\Psi_\sigma(\eta) = E(\eta) + \int_{\mathbb{R}^d} \varphi_\sigma(x) E^+(x,\eta) dx.$$

We solve (5.32) in the Banach spaces $\mathcal{G}_0 = L^1(\Gamma_0, d\lambda)$, cf. (4.1). For $n \in \mathbb{N}$ we denote by $\mathcal{G}_{0,n}$ the subset of $\mathcal{G}_0$ consisting of all those $R : \Gamma_0 \to \mathbb{R}$ for which

$$\int_{\mathbb{R}^d} |\eta|^n \, |R(\eta)| \, \lambda(d\eta) < \infty.$$

Let also $\mathcal{G}_\omega^+$ stand for the cone of positive elements of $\mathcal{G}_\omega$. Set

$$\mathcal{D}_0 = \{R \in \mathcal{G}_0 : \Psi_\sigma R \in \mathcal{G}_0\}. \tag{5.34}$$

Then the relevant part of [18, Theorem 2.2] can be formulated as follows.

Proposition 5.7. *The closure in $\mathcal{G}_0$ of the operator $(L^{\dagger,\sigma}, \mathcal{D}_0)$ defined in (5.33) and (5.34) generates a stochastic semigroup $\{S^{\dagger,\sigma}(t)\}_{t\geq 0} := S^{\dagger,\sigma}$ of bounded operators in $\mathcal{G}_0$, which leaves invariant each $\mathcal{G}_{0,n}$, $n \in \mathbb{N}$. Moreover, for each $\beta' > 0$ and $\beta \in (0, \beta')$, $R \in \mathcal{G}_{\beta'}^+$ implies $S^{\dagger,\sigma}(t) R \in \mathcal{G}_\beta^+$ holding for all $t < T(\beta', \beta)$, where $T(\beta',\beta) = +\infty$ for $\langle a^+\rangle = 0$, and*

$$T(\beta', \beta) = (\beta' - \beta) e^{-\beta'} / \langle a^+ \rangle, \qquad \text{for } \langle a^+\rangle > 0. \tag{5.35}$$

Let now μ_0 be the initial state as in Theorem 3.3. Then for each $\Lambda \in \mathcal{B}_{\rm b}(\mathbb{R}^d)$, the projection μ^Λ is absolutely continuous with respect to λ^Λ, see (2.7). For this μ_0, and for $\Lambda \in \mathcal{B}_{\rm b}(\mathbb{R}^d)$ and $N \in \mathbb{N}$, we set, see (5.31),

$$R_0^\Lambda(\eta) = \frac{d\mu^\Lambda}{d\lambda^\Lambda}(\eta)\mathbb{I}_{\Gamma_\Lambda}(\eta), \qquad R_0^{\Lambda,N}(\eta) = R_0^\Lambda(\eta) I_N(\eta), \quad \eta \in \Gamma_0. \tag{5.36}$$

Here I_N and $\mathbb{I}_{\Gamma_\Lambda}$ are the indicator functions of the sets $\{\eta \in \Gamma_0 : |\eta| \leq N\}$, $N \in \mathbb{N}$, and Γ_Λ, respectively. Clearly,

$$\forall \beta > 0 \qquad R_0^{\Lambda,N} \in \mathcal{G}_\beta^+. \tag{5.37}$$

Set

$$R_t^{\Lambda,N} = S^{\dagger,\sigma}(t) R_0^{\Lambda,N}, \quad t > 0, \tag{5.38}$$

where $S^{\dagger,\sigma}$ is the semigroup as in Proposition 5.7. Then also $R_t^{\Lambda,N} \in \mathcal{G}_0^+$ for all $t > 0$.

For some $G \in B_{\rm bs}(\Gamma_0)$, let us consider $F = KG$, cf. (2.4). Since $G(\xi) = 0$ for all ξ such that $|\xi| > N(G)$, see Definition 2.2, we have $F \in \mathcal{F}_{\rm cyl}(\Gamma)$ and

$$|F(\gamma)| \leq (1 + |\gamma|)^{N(G)} C(G), \qquad \gamma \in \Gamma_0,$$

for some $C(G) > 0$. By Proposition 5.7 we then have from the latter

$$\left| \langle\!\langle KG, R_t^{\Lambda,N} \rangle\!\rangle \right| < \infty. \tag{5.39}$$

5.3.2. *The evolution of local correlation functions.* For a given $\mu \in \mathcal{P}_{\rm sP}$, the correlation function k_μ and the local densities R_μ^Λ, $\Lambda \in \mathcal{B}_{\rm b}(\mathbb{R}^d)$, see (2.8), are related to each other by (2.9). In the first formula of (5.36) we extend R_0^Λ to the whole Γ_0. Then the corresponding integral as in (2.9) coincides with k_{μ_0} only on Γ_Λ. The truncation made in the second formula in (5.36) diminishes R_0^Λ. Its aim is to satisfy (5.37). Thus, with a certain abuse of the terminology we call

$$q_0^{\Lambda,N}(\eta) = \int_{\Gamma_0} R_0^{\Lambda,N}(\eta \cup \xi)\lambda(d\xi) \tag{5.40}$$

local correlation function. The evolution $q_0^{\Lambda,N} \mapsto q_t^{\Lambda,N}$ can be obtained from (5.38) by setting

$$q_t^{\Lambda,N}(\eta) = \int_{\Gamma_0} R_t^{\Lambda,N}(\eta \cup \xi)\lambda(d\xi), \qquad t \geq 0. \tag{5.41}$$

However, so far this can only be used in a weak sense based on (5.39). Note that for $G \in B_{\rm bs}^\star(\Gamma_0)$, cf. (2.11), we have

$$\langle\!\langle G, q_t^{\Lambda,N} \rangle\!\rangle = \langle\!\langle KG, R_t^{\Lambda,N} \rangle\!\rangle \geq 0, \tag{5.42}$$

since $R_t^{\Lambda,N} \in \mathcal{G}_0^+$. To place the evolution $q_0^{\Lambda,N} \mapsto q_t^{\Lambda,N}$ into an appropriate Banach space we use the concluding part of Proposition 5.7 and the following fact

$$\int_{\Gamma_0} e^{\omega|\eta|} q_t^{\Lambda,N}(\eta)\lambda(d\eta) = \int_{\Gamma_0} (1 + e^\omega)^{|\eta|} R_t^{\Lambda,N}(\eta)\lambda(d\eta), \tag{5.43}$$

that can be obtained by (2.13). Since $R_0^{\Lambda,N} \in \mathcal{G}_{\beta'}$ for any $\beta' > 0$, see (5.37), we can take $\beta' = \beta + 1$ which maximizes $T(\beta', \beta)$ given in (5.35). Then for each $\beta > 0$, we have that

$$R_t^{\Lambda,N} \in \mathcal{G}_\beta, \qquad \text{for } \ t < \tau(\beta) := \frac{e^{-\beta}}{e\langle a^+\rangle}. \tag{5.44}$$

Hence, $q_t^{\Lambda,N} \in \mathcal{G}_\omega$ whenever $R_t^{\Lambda,N} \in \mathcal{G}_\beta$ with β such that $e^\beta = 1 + e^\omega$, cf. (5.43). Moreover, for such ω and β the right-hand side of (5.41) defines a continuous map from $\mathcal{G}_\beta$ to $\mathcal{G}_\omega$.

Lemma 5.8. *Given $\omega_1 > 0$ and $\omega_2 > \omega_1$, let β_i be such that $e^{\beta_i} = e^{\omega_i} + 1$, $i = 1, 2$. Then $q_t^{\Lambda,N}$ is continuously differentiable in $\mathcal{G}_{\omega_1}$ on $[0, \tau(\beta_2))$ and the following holds*

$$\frac{d}{dt} q_t^{\Lambda,N} = L_{\omega_1}^{\Delta,\sigma} q_t^{\Lambda,N}. \tag{5.45}$$

Proof. By the mentioned continuity of the map in (5.41) the continuous differentiability of $q_t^{\Lambda,N}$ follows from the corresponding property of $R_t^{\Lambda,N} \in \mathcal{G}_{\beta_2}$, which it has in view of (5.38). Then the following holds

$$\left(\frac{d}{dt} q_t^{\Lambda,N}\right)(\eta) = \int_{\Gamma_0} \left(L_{\beta_1}^{\dagger,\sigma} R_t^{\Lambda,N}\right)(\eta \cup \xi)\lambda(d\xi) \tag{5.46}$$

Where $L_{\beta_1}^{\dagger,\sigma}$ is the trace of $L^{\dagger,\sigma}$ in $\mathcal{G}_{\beta_1}$. We define the action of $\widehat{L}^\sigma = A^\sigma + B^\sigma$ in such a way that

$$\langle\!\langle \widehat{L}^\sigma G, k\rangle\!\rangle = \langle\!\langle G, L^{\Delta,\sigma} k\rangle\!\rangle,$$

where that $L^{\Delta,\sigma}$ acts as in (5.10) and (5.11). Then A^σ acts as in (4.5) with $E^+(y,\eta)$ replaced by $\varphi_\sigma(y)E^+(y,\eta)$, and B^σ acts as in (5.2) with $a^+(x-y)$ multiplied by $\varphi_\sigma(x)$. Let $G : \Gamma_0 \to \mathbb{R}$ be bounded and continuous. Then for some $C > 0$ we have, see (2.4),

$$|\widehat{L}^\sigma G(\eta)| \le |\eta|^2 C \sup_{\eta\in\Gamma_0} |G(\eta)|, \quad |K(\widehat{L}^\sigma G)(\eta)| \le |\eta|^2 2^{|\eta|} C \sup_{\eta\in\Gamma_0} |G(\eta)|,$$

and hence we can calculate the integrals below

$$\langle\!\langle \widehat{L}^\sigma G, q_t^{\Lambda,N}\rangle\!\rangle = \langle\!\langle G, L_{\omega_1}^{\Delta,\sigma} q_t^{\Lambda,N}\rangle\!\rangle, \tag{5.47}$$

where ω_1 and $q_t^{\Lambda,N}$ are as in (5.46). On the other hand, by (5.46) we have

$$\langle\!\langle \widehat{L}^\sigma G, q_t^{\Lambda,N}\rangle\!\rangle = \langle\!\langle K\widehat{L}^\sigma G, R_t^{\Lambda,N}\rangle\!\rangle \tag{5.48}$$

$$= \langle\!\langle KG, L_{\beta_1}^{\dagger,\sigma} R_t^{\Lambda,N}\rangle\!\rangle = \langle\!\langle G, \frac{d}{dt} q_t^{\Lambda,N}\rangle\!\rangle.$$

Since (5.47) and (5.48) hold true for all bounded continuous functions, we have that the expression on both sides of (5.45) are equal to each other, which completes the proof. □

Corollary 5.9. *Let $k_t^{\Lambda,N} \in \mathcal{K}_{\alpha_2}$ be the solution of the problem (5.14) with $k_0^{\Lambda,N} = q_0^{\Lambda,N} \in \mathcal{K}_{\alpha_1}$, see Lemma 5.2. Then for each $G \in B_{\rm bs}^{\star}(\Gamma_0)$ and*

$$t < \min\{T(\alpha_2, \alpha_1; B^{\Delta}); 1/e\langle a^+\rangle\},$$

see (5.44), we have that

$$\langle\!\langle G, k_t^{\Lambda,N}\rangle\!\rangle \geq 0. \tag{5.49}$$

Proof. By (5.36) and (5.40) we have that $q_0^{\Lambda,N} \in \mathcal{U}_{\sigma,\alpha_1}$ (this is the reason to consider such local evolutions). Let then u_t be the solution as in Lemma 5.3 with this initial condition. Then by Corollaries 5.4 and 5.6 it follows that $k_t^{\Lambda,N} = u_t = q_t^{\Lambda,N}$ for the mentioned values of t. Thus, the validity of (5.49) follows by (5.42). □

5.4. **Taking the limits.** Note that (5.49) holds for

$$k_t^{\Lambda,N} = Q^{\sigma}_{\alpha\alpha_1}(t; B_b^{\Delta,\sigma}) q_0^{\Lambda,N},$$

with $\alpha \in (\alpha_1, \alpha_2)$ dependent on t, see (5.16). In this subsection, we first pass in (5.49) to the limit $\sigma \downarrow 0$, then we get rid of the locality imposed in (5.36).

Lemma 5.10. *Let k_t and k_t^{σ} be the solutions of the problems (3.17) and (5.14), respectively, with $k_t|_{t=0} = k_t^{\sigma}|_{t=0} = k_0 \in \mathcal{K}_{\alpha_0}$, $\alpha_0 > -\log\vartheta$. Then for each $\alpha > \alpha_0$ there exists $\widetilde{T} = \widetilde{T}(\alpha, \alpha_0) < T(\alpha, \alpha_0; B_b^{\Delta})$ such that for each $G \in B_{\rm bs}(\Gamma_0)$ and $t \in [0, \widetilde{T}]$ the following holds*

$$\lim_{\sigma\downarrow 0} \langle\!\langle G, k_t^{\sigma}\rangle\!\rangle = \langle\!\langle G, k_t\rangle\!\rangle. \tag{5.50}$$

Proof. Take $\alpha_2 \in (\alpha_0, \alpha)$ and $\alpha_1 \in (\alpha_0, \alpha_2)$. Thereafter, take

$$\widetilde{T} < \min\left\{T(\alpha_1, \alpha_0; B_b^{\Delta}); T(\alpha, \alpha_2; B_b^{\Delta})\right\}. \tag{5.51}$$

For $t \leq \widetilde{T}$, by (4.27), (4.37), (5.11), and (5.16) we then have that the following holds, see (4.37) and (5.24),

$$\begin{aligned} Q_{\alpha\alpha_0}(t; B_b^{\Delta}) k_0 &= Q^{\sigma}_{\alpha\alpha_0}(t) k_0 + M_{\sigma}(t) + N_{\sigma}(t), \\ M_{\sigma}(t) &:= \int_0^t Q_{\alpha\alpha_2}(t-s; B_b^{\Delta}) \left((A_2^{\Delta})_{\alpha_2\alpha_1} - (A_2^{\Delta,\sigma})_{\alpha_2\alpha_1} \right) k_s^{\sigma} ds \\ N_{\sigma}(t) &:= \int_0^t Q_{\alpha\alpha_2}(t-s; B_b^{\Delta}) \left((B_{2,b}^{\Delta})_{\alpha_2\alpha_1} - (B_{2,b}^{\Delta,\sigma})_{\alpha_2\alpha_1} \right) k_s^{\sigma} ds, \end{aligned}$$

where

$$k_s^{\sigma} = Q^{\sigma}_{\alpha_1\alpha_0}(s; B_b^{\Delta}) k_0. \tag{5.52}$$

Then

$$\langle\!\langle G, k_t\rangle\!\rangle - \langle\!\langle G, k_t^{\sigma}\rangle\!\rangle = \langle\!\langle G, M_{\sigma}(t)\rangle\!\rangle + \langle\!\langle G, N_{\sigma}(t)\rangle\!\rangle. \tag{5.53}$$

By (5.5) we get

$$\langle\!\langle G, M_\sigma(t)\rangle\!\rangle = \int_0^t \langle\!\langle G, Q_{\alpha\alpha_2}(t-s; B_b^\Delta) v_s\rangle\!\rangle ds \tag{5.54}$$

$$= \int_0^t \langle\!\langle H_{\alpha_2\alpha}(t-s) G, v_s\rangle\!\rangle ds = \int_0^t \langle\!\langle G_{t-s}, v_s\rangle\!\rangle ds,$$

where

$$\langle\!\langle G_{t-s}, v_s\rangle\!\rangle \tag{5.55}$$

$$= \int_{\Gamma_0} G_{t-s}(\eta) \sum_{x\in\eta} (1-\varphi_\sigma(x))\, E^+(x, \eta\setminus x) k_s^\sigma(\eta\setminus x)\lambda(d\eta)$$

$$= \int_{\Gamma_0}\int_{\mathbb{R}^d} G_{t-s}(\eta\cup x)\,(1-\varphi_\sigma(x))\, E^+(x,\eta) k_s^\sigma(\eta) dx \lambda(d\eta),$$

where the latter line was obtained by means of (2.13). Note that $k_s^\sigma \in \mathcal{K}_{\alpha_1}$ and $G_{t-s}\in\mathcal{G}_{\alpha_2}$ for $s\leq t\leq \widetilde{T}$, see (5.51). We use this fact to prove that

$$g_s(x) := \int_{\Gamma_0} \frac{1}{|\eta|+1} \left|G_s(\eta\cup x)\right| e^{\alpha_2|\eta|}\lambda(d\eta)$$

lies in $L^1(\mathbb{R}^d)$ for each $s\in[0,t]$. Indeed, by (2.13) and (4.2) we get

$$\|g_s\|_{L^1(\mathbb{R}^d)} \leq e^{-\alpha_2}|G_s|_{\alpha_2} \leq C_1 < \infty, \tag{5.56}$$

where

$$C_1 := e^{-\alpha_2}\max_{s\in[0,\widetilde{T}]} |G_s|_{\alpha_2} \leq \frac{e^{-\alpha_0} T(\alpha,\alpha_2; B_b^\Delta)|G|_\alpha}{T(\alpha,\alpha_2;B_b^\Delta)-\widetilde{T}}, \tag{5.57}$$

see (5.6). By (5.15) and (5.52) we also get

$$\max_{s\in[0,\widetilde{T}]} \|k_s^\sigma\|_{\alpha_2} \leq \frac{T(\alpha_1,\alpha_0;B_b^\Delta)\|k_0\|_{\alpha_0}}{T(\alpha_1,\alpha_0;B_b^\Delta)-\widetilde{T}} =: C_2 < \infty, \tag{5.58}$$

see (5.51). Now we use (5.55), (5.56), (5.58) and obtain by (3.2) and (3.12) that the following holds

$$|\langle\!\langle G, M_\sigma(t)\rangle\!\rangle| \leq \varkappa(\alpha_2-\alpha_1)\|a^+\| e^{\alpha_1} C_2 \tag{5.59}$$

$$\times \int_0^{\widetilde{T}}\int_{\mathbb{R}^d} g_s(x)(1-\varphi_\sigma(x)) dx ds,$$

where

$$\varkappa(\beta) := \frac{1}{e\beta} + \left(\frac{2}{e\beta}\right)^2.$$

By (5.56) and (5.57) we conclude that the integrand in the right-hand side of (5.59) is bounded by C_1. By the Lebesgue dominated convergence theorem this yields RHS(5.59) $\to 0$ as $\sigma\downarrow 0$. In the same way one proves that also

$$|\langle\!\langle G, N_\sigma(t)\rangle\!\rangle| \to 0, \qquad \sigma\downarrow 0,$$

which yields (5.50), see (5.53). □

Below by a cofinal sequence $\{\Lambda_n\}_{n\in\mathbb{N}} \subset \mathcal{B}_{\rm b}(\mathbb{R}^d)$ we mean a sequence such that: $\Lambda_n \subset \Lambda_{n+1}$ for all n, and each $x \in \mathbb{R}$ belongs to a certain Λ_n.

Lemma 5.11. *Let $\{\Lambda_n\}_{n\in\mathbb{N}}$ be a cofinal sequence and $q_0^{\Lambda,N}$ be as in (5.40). Let also α_1 and α_2 be as in Lemma 4.9. Then for each $t \in [0, T(\alpha_2,\alpha_1; B_b^\Delta))$ and $G \in B_{\rm bs}(\Gamma_0)$, the following holds*

$$\lim_{n\to+\infty}\lim_{N\to+\infty} \langle\!\langle G, Q_{\alpha_2\alpha_1}(t; B_b^\Delta) q_0^{\Lambda_n,N}\rangle\!\rangle = \langle\!\langle G, Q_{\alpha_2\alpha_1}(t; B_b^\Delta) k_{\mu_0}\rangle\!\rangle. \tag{5.60}$$

Proof. As in (5.54), we prove (5.60) by showing that

$$\begin{aligned} & \lim_{n\to+\infty}\lim_{N\to+\infty} \langle\!\langle G, Q_{\alpha_2\alpha_1}(t; B_b^\Delta) q_0^{\Lambda_n,N}\rangle\!\rangle \qquad (5.61) \\ & = \lim_{n\to+\infty}\lim_{N\to+\infty} \langle\!\langle H_{\alpha_1\alpha_2}(t) G, q_0^{\Lambda_n,N}\rangle\!\rangle = \langle\!\langle H_{\alpha_1\alpha_2}(t) G, k_{\mu_0}\rangle\!\rangle. \end{aligned}$$

Since $G_t := H_{\alpha_1\alpha_2}(t)G$ lies in $\mathcal{G}_{\alpha_1}$, the proof of (5.61) can be done by the repetition of arguments used in the proof of the analogous result in [3, Appendix]. □

5.5. **The proof of Lemma 4.9.** Let α_1 and α_2 be as in Lemma 4.9 and $\{\Lambda_n\}_{n\in\mathbb{N}}$ be a cofinal sequence. Take $k_{\mu_0} \in \mathcal{K}_{\alpha_1}$ and then produce $q_0^{\Lambda_n,N}$, $n \in \mathbb{N}$, by employing (5.36) and (5.40). Let $T(\alpha_2,\alpha_1) < T(\alpha_2,\alpha_1; B_b^\Delta)$ be such that (5.49) holds with

$$k_t^{\Lambda_n,N} = Q^\sigma(\alpha_2,\alpha_1; B_b^\Delta) q_0^{\Lambda_n,N}, \qquad t \le T(\alpha_2,\alpha_1).$$

Note that $T(\alpha_2,\alpha_1)$ is independent of Λ_n and N, see Corollary 5.9. By Lemma 5.11 we then have that

$$\langle\!\langle G, Q^\sigma_{\alpha_2\alpha_1}(t; B_b^\Delta) k_{\mu_0}\rangle\!\rangle \ge 0.$$

Now we apply Lemma 5.10 and obtain

$$\langle\!\langle G, Q_{\alpha_2\alpha_1}(t; B_b^\Delta) k_{\mu_0}\rangle\!\rangle \ge 0,$$

which holds for

$$t \le \tau(\alpha_2,\alpha_1) := \min\left\{T(\alpha_2,\alpha_1); \widetilde{T}(\alpha_2,\alpha_1)\right\},$$

which completes the proof.

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References

[1] Albeverio, S., Kondratiev, Yu. G., Röckner, M.: Analysis and geometry on configuration spaces. J. Func. Anal. **154**, 444–500 (1998)

[2] Angelescu, N., Nenciu, G., Protopopescu, V.: On stable potentials. Comm. Math. Phys. **22**, 162–165 (1971)

[3] Berns, Ch., Kondratiev, Yu., Kozitsky, Yu., Kutovyi, O.: Kawasaki dynamics in continuum: Micro- and mesoscopic descriptions. J. Dynam. Differential Equations. **25**, 1027–1056 (2013)

[4] Bolker, B. M., Pacala, S. W.: Using moment equations to understand stochastically driven spatial pattern formation in ecological systems. Theoret. Population Biol. **52**, 179–197 (1997)

[5] Bolker, B. M., Pacala, S. W., Neuhauser, C.: Spatial dynamics in model plant communities: What do we really know? The American Naturalist. **162**, 135–148 (2003)

[6] Conway, J. H., Sloan, N. J. A.: Sphere Packing, Lattices and Groups. Third edition. With additional contributions by E. Bannai, R. E. Borcherds, J. Leech, S. P. Norton, A. M. Odlyzko, R. A. Parker, L. Queen and B. B. Venkov. Grundlehren der Mathematischen Wissenschaften, 290. Springer-Verlag, New York, 1999

[7] Finkelshtein, D. L., Kondratiev, Yu. G., Kutovyi, O.: Individual based model with competition in spatial ecology. SIAM J. Math. Anal. **41**, 297–317 (2009)

[8] Finkelshtein, D. L., Kondratiev, Yu. G., Kutovyi, O.: Semigroup approach to birth-and-death stochastic dynamics in continuum. J. Funct. Anal. **262**, 1274-1308 (2012)

[9] Finkelshtein, D. L., Kondratiev, Yu. G., Kozitsky, Yu., Kutovyi, O.: The statistical dynamics of a spatial logistic model and the related kinetic equation. Math. Models Methods Appl. Sci. **25**, 343–370 (2015)

[10] Finkelshtein, D. L., Kondratiev, Yu. G., Oliveira, M. J.: Markov evolutions and hierarchical equations in the continuum. I. One-component systems. J. Evol. Equ. **9**, 197–233 (2009)

[11] Garcia, N. L., Kurtz, T. G.: Spatial birth and death processes as solutions of stochastic equations. ALEA Lat. Am. J. Probab. Math. Stat. **1**, 281–303 (2006)

[12] Holley, R. A., Stroock, D. W.: Nearest neighbor birth and death processes on the real line. Acta Math. **140**, 103–154 (1978)

[13] Kondratiev, Yu., Kuna, T.: Harmonic analysis on configuration space. I. General theory. Infin. Dimens. Anal. Quantum Probab. Relat. Top. **5**, 201–233 (2002)

[14] Kondratiev, Yu., Kuna, T., Oliveira, M. J.: Holomorphic Bogoliubov functionals for interacting particle systems in continuum. J. Funct. Anal. **238**, 375–404 (2006)

[15] Kondratiev, Yu., Kutoviy, O., Minlos, R.: On non-equilibrium stochastic dynamics for interacting particle systems in continuum, J. Funct. Anal. **255**, 200-227 (2008)

[16] Kondratiev, Yu., Kutovyi, O., Pirogov S.: Correlation functions and invariant measures in continuous contact model. Infin. Dimens. Anal. Quantum Probab. Relat. Top. **11** 231–258 (2008)

[17] Kondratiev, Yu. G., Skorokhod, A. V.: On contact models in continuum. Infin. Dimens. Anal. Quantum Probab. Relat. Top. **9** 187–198 (2006)

[18] Kozitsky, Yu.: Dynamics of spatial logistic model: finite systems. In: Banasiak, J., Bobrowski, A., Lachowicz, M. (eds.) Semigroups of Operators – Theory and Applications: Będlewo, Poland, October 2013. Springer Proceedings in Mathematics & Statistics 113, Springer 2015, pp. 197–211

[19] Murrell, D. J., Dieckmann, U., Law, R.: On moment closures for population dynamics in contunuous space. J. Theoret. Biol. **229**, 421–432 (2004)

[20] Neuhauser, C.: Mathematical challenges in spatial ecology. Notices of AMS **48** (11), 1304–1314 (2001)

[21] Obata, N.: Configuration space and unitary representations of the group of diffeomorphisms. RIMS Kôkyûroku **615**, 129–153 (1987)

[22] Ovaskainen, O., Finkelshtein, D., Kutovyi, O., Kornel, S., Bolker, B., Kondratiev, Yu.: A general mathematical framwork for the analysis of spatio-temporal point processes. Theor. Ecol. **7**, 101–113 (2014)

[23] Pazy, A.: Semigroups of Linear Operators and Applications to Partial Differential Equations. Applied Mathematical Sciences, 44. Springer-Verlag, New York, 1983

[24] Pechersky, E. A., Zhukov, Yu.: Uniqueness of Gibbs state for non- ideal gas in R^d: The case of pair potentials. J. Stat. Phys. **97**, 145–172 (1999)

[25] Ruelle, D.: Statistical Mechanics: Rigorous Results. W. A. Benjamin, Inc., 1969

[26] Thieme, H. R., Voigt, J.: Stochastic semigroups: their construction by perturbation and approximation, *Positivity IV—Theory and Applications,* eds. M. R. Weber and J. Voigt (Tech. Univ. Dresden, Dresden, 2006), pp. 135–146

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