Uniqueness for a class of stochastic Fokker-Planck and porous media equations.

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August 3rd 2016

Summary. The purpose of the present note consists of first showing a uniqueness result for a stochastic Fokker-Planck equation under very general assumptions. In particular, the second order coefficients may be just measurable and degenerate. We also provide a proof for uniqueness of a stochastic porous media equation in a fairly large space.

Key words: stochastic partial differential equations; infinite volume; porous media type equation; multiplicative noise; stochastic Fokker-Planck type equation.

2010 AMS-classification: 35R60; 60H15; 82C31.

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1 Introduction

We consider real functions $e^0, \dots, e^N, \dots : \mathbb{R} \to \mathbb{R}$ fulfilling Assumption 2.3 below. In particular they are H^{-1}-multipliers, see Definition 2.1.

Let $T > 0$ and $(\Omega, \mathcal{F}, P)$, be a fixed probability space. Let $(\mathcal{F}_t, t \in [0,T])$ be a filtration fulfilling the usual conditions and we suppose $\mathcal{F} = \mathcal{F}_T$. Let $\mu(t,\xi), t \in [0,T], \xi \in \mathbb{R}$, be a random field of the type

$$\mu(t,\xi) = \sum_{i=1}^{\infty} e^i(\xi) W_t^i + e^0(\xi) t, \; t \in [0,T], \xi \in \mathbb{R},$$

where $W^i, i \geq 1$ are independent continuous $(\mathcal{F}_t)$-Brownian motions on $(\Omega, \mathcal{F}, P)$, which are fixed from now on until the end of the paper. For technical reasons we will sometimes set $W_t^0 \equiv t$.

We now consider a random field a and a deterministic function ψ as follows.

Assumption 1.1. *$a : [0,T] \times \mathbb{R} \times \Omega \to \mathbb{R}_+$ is a bounded progressively measurable random field.*

Assumption 1.2. *ψ is monotone increasing, Lipschitz such that $\psi(0) = 0$.*

Let $x_0 \in \mathcal{S}'(\mathbb{R})$. We will consider the following two types of equations (1.1) and (1.2). The first one is a (linear) stochastic Fokker-Planck equation, the second one a stochastic porous media type equation, i.e.

$$\begin{cases} \partial_t z(t,\xi) = \partial^2_{\xi\xi}((az)(t,\xi)) + z(t,\xi)\partial_t \mu(t,\xi), \\ z(0, \cdot) = x_0, \end{cases} \tag{1.1}$$

and

$$\begin{cases} \partial_t X(t,\xi) &= \frac{1}{2}\partial^2_{\xi\xi}(\psi(X(t,\xi)) + X(t,\xi)\partial_t \mu(t,\xi), \\ X(0,\mathrm{d}\xi) &= x_0. \end{cases} \tag{1.2}$$

They are both to be understood in the sense of (Schwartz) distributions. Their precise sense will be given in Remark 3.2 a) and in Definition 4.1. The stochastic multiplication above is of Itô type. In this paper we confine ourselves to the case of the underlying space being $\mathbb{R}^1$.

Fokker-Planck equations have been investigated until now in the deterministic framework, i.e. when $e^i = 0, i \geq 1$. There is a huge literature about existence and uniqueness in this case, see e.g. [6] and references therein. More particularly, concerning uniqueness, in addition we draw the attention to Proposition 3.4 [5] and Theorem 3.1 of [4]. As far as we know this is the first time that a Fokker-Planck equation as

(1.1) is considered in the literature, in particular for uniqueness, except for the unpublished work by the same authors [2]. We point out that we can allow degenerate coefficients in the second order term.

Concerning porous media equations, both in the deterministic and stochastic cases, there is a huge number of contributions, especially in finite volume. As far as the infinite volume case is concerned, in the deterministic situation a good framework is the classical Benilan-Crandall approach of the seventies; in the stochastic case some recent significant contributions have been made, see [11, 12, 3] and in particular [1] and references therein. As mentioned, this paper draws however the attention on uniqueness for equations in the sense of distributions, within a large solutions class. For instance, in the deterministic case a typical result in that sense is the paper [7] of Brezis and Crandall, which establishes uniqueness in the sense of distributions in the class $(L^1 \cap L^\infty)([0,T] \times \mathbb{R}^d)$. Here we consider the equation (1.2) in the sense of distributions and we investigate uniqueness in the class of progressively measurable random fields $X : \Omega \times [0,T] \times \mathbb{R}$ such that $\int_{[0,T]\times\mathbb{R}} X^2(s,\xi)\mathrm{d}s\mathrm{d}\xi < \infty$ a.s, see Definition 4.1 and condition below (4.2). To the best of our knowledge, this constitutes a new result of uniqueness in the sense of distributions; for this we need only a.s. conditions in (4.2) and not necessarily in the expectation as it is mostly done in the standard literature.

The paper is organized as follows. After, this introduction and Section 2 devoted to preliminaries, in Section 3 the uniqueness Theorem 3.1 for an SPDE of Fokker-Planck type is formulated and proved. This, in turn, is an important ingredient for the probabilistic representation of a solution to a stochastic porous media type equation, see [2]. In the final section 4, using the same ideas as in Section 3 we prove a uniqueness result for (1.2), see Theorem 4.2.

2 Preliminaries

First we introduce some basic recurrent notations. $\mathcal{M}(\mathbb{R})$ denotes the space of signed Borel measures with finite total variation. We recall that $\mathcal{S}(\mathbb{R})$ is the space of the Schwartz fast decreasing test functions with its usual topology. $\mathcal{S}'(\mathbb{R})$ is its dual, i.e. the space of Schwartz tempered distributions. On $\mathcal{S}'(\mathbb{R})$, the map $(I-\Delta)^{\frac{s}{2}}, s \in \mathbb{R}$, is well-defined, via Fourier transform. For $s \in \mathbb{R}$, $H^s(\mathbb{R})$ denotes the classical Sobolev space consisting of all functions $f \in \mathcal{S}'(\mathbb{R})$ such that $(I-\Delta)^{\frac{s}{2}} f \in L^2(\mathbb{R})$. We introduce the norm

$$\|f\|_{H^s} := \|(I-\Delta)^{\frac{s}{2}} f\|_{L^2},$$

where $\|\cdot\|_{L^p}$ is the classical $L^p(\mathbb{R})$-norm for $1 \le p \le \infty$. In the sequel, we will often simply denote $H^{-1}(\mathbb{R})$, by H^{-1} and $L^2(\mathbb{R})$ by L^2. Furthermore, $W^{r,p}$ denotes the classical Sobolev space of order $r \in \mathbb{N}$ in $L^p(\mathbb{R})$ for $1 \le p \le \infty$.

Definition 2.1. *Given a function e belonging to $L^1_{\mathrm{loc}}(\mathbb{R}) \cap \mathcal{S}'(\mathbb{R})$, we say that it is an H^{-1}-**multiplier**, if the map $\varphi \mapsto \varphi e$ is continuous from $\mathcal{S}(\mathbb{R})$ to H^{-1} with respect to the H^{-1}-topology on both spaces. $\mathcal{C}(e)$ denotes the norm of this operator and we will call it **multiplier norm**. We remark that φe is always a well-defined Schwartz tempered distribution, whenever φ is a fast decreasing test function.*

Remark 2.2. *Let $e : \mathbb{R} \to \mathbb{R}$. If $e \in W^{1,\infty}$ (for instance if $e \in W^{2,1}$), then e is a $H^{-1}(\mathbb{R})$-multiplier.*
Indeed, by duality arguments, to show this, it is enough to show the existence of a constant $\mathcal{C}(e)$ such that

$$\|eg\|_{H^1} \leqslant \mathcal{C}(e)\,\|g\|_{H^1}\,,\ \forall\, g \in \mathcal{S}(\mathbb{R}). \tag{2.1}$$

Now (2.1) follows easily by the derivation product rules with for instance

$$\mathcal{C}(e) = \sqrt{2}\left(\|e\|_\infty^2 + \|e'\|_\infty^2\right)^{\frac{1}{2}}.$$

Here we fix some conventions concerning measurability. Any topological space E is naturally equipped with its Borel σ-algebra $\mathcal{B}(E)$. For instance $\mathcal{B}(\mathbb{R})$ (resp. $\mathcal{B}([0,T])$) denotes the Borel σ-algebra of $\mathbb{R}$ (resp. $[0,T]$).

In the whole paper, the following assumption on μ will be in force.

Assumption 2.3.

1. *Each $e^i, i \ge 0$, belongs to the Sobolev space $W^{1,\infty}$.*
2. *$\sum_{i=1}^\infty \left(\|(e^i)'\|_\infty^2 + \|e^i\|_\infty^2\right) < \infty$.*

With respect to the random field μ, we introduce a notation for the Itô type stochastic integral below.

Let $Z = (Z(s,\xi), s \in [0,T], \xi \in \mathbb{R})$ be a random field on $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ such that $\int_0^T \left(\int_\mathbb{R} |Z(s,\xi)| \mathrm{d}\xi\right)^2 \mathrm{d}s < \infty$ a.s. and it is an $L^1(\mathbb{R})$-valued, $(\mathcal{F}_s)$-progressively measurable process. Then, provided, Assumption 2.3 holds, the stochastic integral

$$\int_{[0,t]\times\mathbb{R}} Z(s,\xi)\mu(\mathrm{d}s,\xi) := \sum_{i=0}^\infty \int_0^t \left(\int_\mathbb{R} Z(s,\xi)e^i(\xi)\right) \mathrm{d}W_s^i, t \ge 0,$$

is well-defined.
More generally, if $s \mapsto Z(s,\cdot)$ is a measurable map $[0,T] \times \Omega \mapsto \mathcal{M}(\mathbb{R})$, such that $\int_0^T \|Z(s,\cdot)\|_{\mathrm{var}}^2 \mathrm{d}s < \infty$, then the stochastic integral

$$\int_{[0,t]\times\mathbb{R}} Z(s,\mathrm{d}\xi)\mu(\mathrm{d}s,\xi) := \sum_{i=0}^\infty \int_0^t \left(\int_\mathbb{R} Z(s,\mathrm{d}\xi)e^i(\xi)\right) \mathrm{d}W_s^i, t \ge 0,$$

is well-defined.

3 On the uniqueness of a Fokker-Planck type SPDE

The theorem below plays the analogous role as Theorem 3.8 in [5] or Theorem 3.1 in [4]. We recall that our Fokker-Planck SPDE has possibly degenerate measurable coefficients.

Theorem 3.1. *We suppose that Assumptions 1.1 and 2.3 hold. Let z^1, z^2 be two measurable random fields belonging ω a.s. to $C([0,T], \mathcal{S}'(\mathbb{R}))$ such that $z^1, z^2 :]0,T] \times \Omega \to \mathcal{M}(\mathbb{R})$. We moreover suppose the following.*

i) $z^1 - z^2 \in L^2([0,T] \times \mathbb{R})$ *a.s.*

ii) $t \mapsto (z^1 - z^2)(t, \cdot)$ *is an* $(\mathcal{F}_t)$*-progressively measurable* $\mathcal{S}'(\mathbb{R})$*-valued process.*

iii) z^1, z^2 *are solutions to* (1.1)*. such that* $\int_0^T \|z^i(s,\cdot)\|^2_{\text{var}} \mathrm{d}s < \infty$ *a.s.*

Then $z^1 \equiv z^2$.

Remark 3.2. *a) By a solution* z *of equation* (1.1) *we mean the following: for every* $\varphi \in \mathcal{S}(\mathbb{R}), \forall t \in [0,T]$,

$$\int_{\mathbb{R}} \varphi(\xi) z(t, \mathrm{d}\xi) = \langle x_0, \varphi \rangle + \int_0^t \mathrm{d}s \int_{\mathbb{R}} a(s,\xi)\varphi''(\xi) z(s, \mathrm{d}\xi) + \int_{[0,t]\times\mathbb{R}} \varphi(\xi) z(s, \mathrm{d}\xi)\mu(\mathrm{d}s, \xi) \quad a.s.$$

b) Let $z = z^1 - z^2$*. Since* z *is* ω *a.s. in* $L^2([0,T]; L^2(\mathbb{R}) \cap \mathcal{M}(\mathbb{R})) \subset L^2([0,T]; H^{-1}(\mathbb{R}))$, $\int_0^t z(s,\cdot)\mu(\mathrm{d}s, \cdot)$ *belongs* ω *a.s. to* $C([0,T]; H^{-1}(\mathbb{R}))$ *and so also to* $C([0,T]; H^{-2}(\mathbb{R}))$ ω *a.s. On the other hand* $\int_0^t (az)''(s,\cdot)\mathrm{d}s$ *can be seen as a Bochner integral in* $H^{-2}(\mathbb{R})$*. In particular any solutions* z^1, z^2 *to* (1.1) *are such that* $z = z^1 - z^2$ *admits a modification whose paths belong (a.s.) to* $C([0,T]; H^{-2}(\mathbb{R})) \cap L^2([0,T]; L^2(\mathbb{R}))$*. Since* $z^i, i = 1, 2$*, are continuous with values in* $\mathcal{S}'(\mathbb{R})$*, their difference is indistinguishable with the mentioned modification.*
Consequently for ω *a.s.* $z(t,\cdot) \in C([0,T]; H^{-2}(\mathbb{R}))$ *and outside a* P*-null set* N_0*, we have (in* $\mathcal{S}'(\mathbb{R})$ *and* $H^{-2}(\mathbb{R})$ *)*

$$z(t,\cdot) = \int_0^t (az)''(s,\cdot)\mathrm{d}s + \int_0^t z(s,\cdot)\mu(\mathrm{d}s,\cdot). \tag{3.1}$$

c) By assumption i), possibly enlarging the P*-null set* N_0 *we get the following. For* $\omega \notin N_0$*, for almost all* $t \in]0,T]$, $\left(\int_0^t (az)(s,\cdot)\mathrm{d}s\right)'' \in H^{-1}(\mathbb{R})$ *and so* $\int_0^t (az)(s, \cdot)\mathrm{d}s \in H^1$ $\mathrm{d}t$ *a.e.*

Proof of Theorem 3.1. Let $z = z^1 - z^2$.

We fix the null set N_0 and so ω will always lie outside N_0 introduced in Remark 3.2 c). Let ϕ be a mollifier with compact support and $\phi_\varepsilon = \frac{1}{\varepsilon}\phi(\frac{\cdot}{\varepsilon})$ be a generalized sequence of mollifiers converging to the Dirac delta function. We set

$$g_\varepsilon(t) = \|z_\varepsilon(t)\|^2_{H^{-1}} = \int_{\mathbb{R}} z_\varepsilon(t,\xi)((I-\Delta)^{-1}z_\varepsilon)(t,\xi)\mathrm{d}\xi,$$

where $z_\varepsilon(t,\xi) = \int_{\mathbb{R}} \phi_\varepsilon(\xi - y)z(t,\mathrm{d}y)$. Since $t \mapsto z(t,\ \cdot\ )$ is continuous in $H^{-2}(\mathbb{R})$, $t \mapsto z_\varepsilon(t,\ \cdot\ )$ is continuous in $L^2(\mathbb{R})$ and so also in $H^{-1}(\mathbb{R})$. We look at the equation fulfilled by z_ε. The identity (3.1) produces the following equality in $L^2(\mathbb{R})$ and so in $H^{-1}(\mathbb{R})$:

$$\begin{aligned} z_\varepsilon(t,\ \cdot\ ) &= \int_0^t \left\{[(a(s,\ \cdot\ )z(s,\ \cdot\ ))\star\phi_\varepsilon]'' - (a(s,\ \cdot\ )z(s,\ \cdot\ ))\star\phi_\varepsilon\right\}\mathrm{d}s \qquad (3.2)\\ &+ \int_0^t (a(s,\ \cdot\ )z(s,\ \cdot\ ))\star\phi_\varepsilon \mathrm{d}s + \sum_{i=1}^\infty \int_0^t (e^i z)(s,\ \cdot\ )\star\phi_\varepsilon \mathrm{d}W_s^i. \end{aligned}$$

We apply $(I-\Delta)^{-1}$ and we get

$$\begin{aligned} (I-\Delta)^{-1}z_\varepsilon(t,\ \cdot\ ) &= -\int_0^t (a(s,\ \cdot\ )z(s,\ \cdot\ ))\star\phi_\varepsilon \mathrm{d}s \qquad (3.3)\\ &+ \int_0^t (I-\Delta)^{-1}\left[(a(s,\ \cdot\ )z(s,\ \cdot\ ))\star\phi_\varepsilon\right]\mathrm{d}s\\ &+ \sum_{i=1}^\infty \int_0^t (I-\Delta)^{-1}(e^i z)(s,\ \cdot\ )\star\phi_\varepsilon \mathrm{d}W_s^i. \end{aligned}$$

We apply Itô's formula to g_ε. For a general introduction to infinite dimensional Hilbert space valued stochastic calculus, see [8], [10] or [9]. Taking into account, (3.2), (3.3) and that $\langle f, g\rangle_{H^{-1}} = \left\langle f, (I-\Delta)^{-1}g\right\rangle_{L^2}$, we now obtain

$$\begin{aligned} g_\varepsilon(t) &= 2\int_0^t \langle z_\varepsilon(s,\ \cdot\ ),\ \mathrm{d}z_\varepsilon(s,\ \cdot\ )\rangle_{H^{-1}} \qquad (3.4)\\ &+ \sum_{i=1}^\infty \int_0^t \mathrm{d}s \left\langle (z(s,\ \cdot\ )e^i)\star\phi_\varepsilon,\ (z(s,\ \cdot\ )e^i)\star\phi_\varepsilon\right\rangle_{H^{-1}}\\ &= -2\int_0^t \langle z_\varepsilon(s,\ \cdot\ ), (a(s,\ \cdot\ )z(s,\ \cdot\ ))\star\phi_\varepsilon\rangle_{L^2}\mathrm{d}s\\ &+ 2\int_0^t \left\langle z_\varepsilon(s,\ \cdot\ ),\ (I-\Delta)^{-1}\left((a(s,\ \cdot\ )z(s,\ \cdot\ ))\star\phi_\varepsilon\right)\right\rangle_{L^2}\mathrm{d}s\\ &+ \sum_{i=1}^\infty \int_0^t \left\langle (z(s,\ \cdot\ )e^i)\star\phi_\varepsilon,\ (z(s,\ \cdot\ )e^i)\star\phi_\varepsilon\right\rangle_{H^{-1}}\mathrm{d}s\\ &+ 2\int_0^t \left\langle z_\varepsilon(s,\ \cdot\ ), (ze^0)(s,\cdot)\star\phi_\varepsilon\right\rangle_{H^{-1}}\mathrm{d}s + M_t^\varepsilon \end{aligned}$$

where

$$M_t^\varepsilon = 2\sum_{i=1}^\infty \int_0^t \left\langle z_\varepsilon(s,\ \cdot\ ),\ (e^i z)(s,\ \cdot\ ) \star \phi_\varepsilon \right\rangle_{H^{-1}} \mathrm{d}W_s^i. \tag{3.5}$$

Below we will justify that (3.5) is well-defined. We summarize (3.4) into

$$g_\varepsilon(t) = \widetilde{g}_\varepsilon(t) + M_t^\varepsilon, t \in [0,T].$$

We remark that

$$\sum_{i=1}^\infty \int_0^T \left(\left\langle z(s,\ \cdot\ ),\ e^i z(s,\ \cdot\ )\right\rangle_{H^{-1}}\right)^2 \mathrm{d}s = \sum_{i=1}^\infty \int_0^T \left(\left\langle e^i z(s,\ \cdot\ ),\ (I-\Delta)^{-1} z(s,\ \cdot\ )\right\rangle_{L^2}\right)^2 \mathrm{d}s \tag{3.6}$$

$$\leq \sum_{i=1}^\infty \int_0^T \|e^i z(s,\ \cdot\ )\|_{L^2}^2 \|z(s,\cdot)\|_{H^{-2}}^2 \mathrm{d}s$$

$$\leq \sum_{i=1}^\infty \|e^i\|_\infty^2 \sup_{s\in[0,T]} \|z(s,\cdot)\|_{H^{-2}}^2 \int_0^T \|z(s,\ \cdot\ )\|_{L^2}^2 \mathrm{d}s,$$

because $z : [0,T] \to H^{-2}$ is a.s. continuous by Remark 3.2 b).

Consequently,

$$M_t = \sum_{i=1}^\infty \int_0^t \left\langle z(s,\ \cdot\ ),\ e^i z(s,\ \cdot\ )\right\rangle_{H^{-1}} \mathrm{d}W_s^i,$$

is a well-defined local martingale.

It is also not difficult to show that for $\varepsilon > 0$,

$$\sum_{i=1}^\infty \int_0^T \left\langle z_\varepsilon(s,\ \cdot\ ),\ (e^i z(s,\ \cdot\ )) \star \phi_\varepsilon \right\rangle_{H^{-1}}^2 \mathrm{d}s < \infty, \text{ a.s.}$$

and so M^ε defined in (3.5) is a well-defined local martingale.

By assumption we have of course ($\omega \notin N_0$)

$$\int_{[0,T]\times\mathbb{R}} (z_\varepsilon(s,\xi) - z(s,\xi))^2 \mathrm{d}s\mathrm{d}\xi \underset{\varepsilon\to 0}{\longrightarrow} 0, \tag{3.7}$$

$$\int_{[0,T]\times\mathbb{R}} ((az)\star\phi_\varepsilon - az)^2(s,\xi) \mathrm{d}s\mathrm{d}\xi \underset{\varepsilon\to 0}{\longrightarrow} 0, \tag{3.8}$$

$$\int_{[0,T]\times\mathbb{R}} ((z(s,\cdot)e^i)\star\phi_\varepsilon - z(s,\cdot)e^i)^2(\xi) \mathrm{d}s\mathrm{d}\xi \underset{\varepsilon\to 0}{\longrightarrow} 0, \tag{3.9}$$

for every $i \geq 0$, because $z, az, e^i z \in L^2([0,T]\times\mathbb{R}), i \geq 0$. By usual estimates on convolutions, there is a universal constant C such that

$$\int_{[0,T]\times\mathbb{R}} ((z(s,\cdot)e^i)\star\phi_\varepsilon)^2(\xi)\mathrm{d}s\mathrm{d}\xi \leq \int_{[0,T]\times\mathbb{R}} z^2(s,\xi)(e^i)^2(\xi))\mathrm{d}s\mathrm{d}\xi \leq C\|e^i\|_\infty^2 \|z\|_{L^2([0,T]\times\mathbb{R})}^2. \tag{3.10}$$

By Lebesgue dominated convergence theorem, using (3.9), it follows that (for $\omega \notin N_0$),

$$\sum_{i=0}^{\infty} \int_0^T \|(z(s,\ \cdot)e^i) \star \phi_\varepsilon - z(s,\ \cdot\ )e^i\|_{L^2}^2 \mathrm{d}s \to_{\varepsilon \to 0} 0. \tag{3.11}$$

Using (3.7) and (3.11), it is not difficult to show that (for $\omega \notin N_0$)

$$\sum_{i=0}^{\infty} \int_0^T \left(\left\langle z_\varepsilon(s,\ \cdot\ ), (e^i z(s,\cdot)) \star \phi_\varepsilon \right\rangle_{H^{-1}} - \left\langle z(s,\ \cdot\ ), e^i z(s,\ \cdot\ ) \right\rangle_{H^{-1}} \right)^2 \mathrm{d}s \tag{3.12}$$

converges to zero. Now (for $\omega \notin N_0$),

$$\sum_{i=0}^{\infty} \int_0^T \left| \left\|(z(s,\ \cdot\ )e^i) \star \phi_\varepsilon \right\|_{H^{-1}}^2 - \left\|z(s,\ \cdot\ )e^i\right\|_{H^{-1}}^2 \right| \mathrm{d}s \tag{3.13}$$

$$\leq \sqrt{2} \sum_{i=0}^{\infty} \sqrt{\int_0^T \left(\|(z(s,\ \cdot\ )e^i) \star \phi_\varepsilon - z(s,\ \cdot\ )e^i\|_{H^{-1}}^2 \right) \mathrm{d}s \int_0^T \left(\|z(s,\ \cdot\ )e^i) \star \phi_\varepsilon\|^2 + \|z(s,\ \cdot\ )e^i\|_{H^{-1}}^2 \right) \mathrm{d}s}$$

$$\leq \sqrt{2} \sqrt{\sum_{i=0}^{\infty} \int_0^T \left(\|(z(s,\ \cdot\ )e^i) \star \phi_\varepsilon - z(s,\ \cdot\ )e^i\|_{H^{-1}}^2 \right) \mathrm{d}s} \sqrt{\sum_{i=0}^{\infty} \int_0^T \left(\|z(s,\ \cdot\ )e^i) \star \phi_\varepsilon\|^2 + \|z(s,\ \cdot\ )e^i\|_{H^{-1}}^2 \right) \mathrm{d}s}$$

converges to zero, because of (3.10) and (3.11).

Taking into account (3.7), (3.8), (3.12) and (3.13) we obtain (for $\omega \notin N_0$), that $\lim_{\varepsilon \to 0} \widetilde{g}_\varepsilon(t) = \widetilde{g}(t),\ t \in [0, T]$, where

$$\begin{aligned} \widetilde{g}(t) = &- 2 \int_0^t \langle z(s,\ \cdot\ ), a(s,\ \cdot\ ) z(s,\ \cdot\ ) \rangle_{L^2} \mathrm{d}s \\ &+ 2 \int_0^t \left\langle z(s,\ \cdot\ ), (I - \Delta)^{-1} (a(s,\ \cdot\ ) z(s,\ \cdot\ )) \right\rangle_{L^2} \mathrm{d}s \\ &+ 2 \int_0^t \left\langle z(s,\ \cdot\ ), z(s,\ \cdot\ ) e^0 \right\rangle_{H^{-1}} \mathrm{d}s \\ &+ \sum_{i=1}^{\infty} \int_0^t \left\langle z(s,\ \cdot\ ) e^i, z(s,\ \cdot\ ) e^i \right\rangle_{H^{-1}} \mathrm{d}s. \end{aligned} \tag{3.14}$$

The convergence of the second term of the right-hand side of (3.4) to the second term of the right-hand side of (3.14) holds again due to (3.7) and (3.8), cutting the difference in two pieces and using Cauchy-Schwarz's inequality. On the other hand the convergence of (3.12) to zero implies that $M^\varepsilon \to M$ ucp, so that the ucp limit of $\widetilde{g}_\varepsilon(t) + M_t^\varepsilon$ is equal to $\widetilde{g}(t) + M_t$. So, after a possible modification of the P-null set N_0, setting $g(t) := \|z(t,\,\cdot\,)\|^2_{H^{-1}}$, for $\omega \notin N_0$, we have

$$
\begin{aligned}
g(t) &+ 2\int_0^t \langle z(s,\,\cdot\,), a(s,\,\cdot\,)z(s,\,\cdot\,)\rangle_{L^2}\,\mathrm{d}s \\
&= 2\int_0^t \left\langle (I-\Delta)^{-1}z(s,\,\cdot\,), a(s,\,\cdot\,)z(s,\,\cdot\,)\right\rangle_{L^2}\mathrm{d}s \\
&+ 2\int_0^t \mathrm{d}s \left\langle z(s,\,\cdot\,), z(s,\,\cdot\,)e^0\right\rangle_{H^{-1}}\mathrm{d}s \\
&+ \sum_{i=1}^\infty \int_0^t \left\langle z(s,\,\cdot\,)e^i, z(s,\,\cdot\,)e^i\right\rangle_{H^{-1}}\mathrm{d}s + M_t.
\end{aligned} \tag{3.15}
$$

By the inequality

$$
2bc \leqslant b^2\,\|a\|_\infty + \frac{c^2}{\|a\|_\infty},
$$

$b, c \in \mathbb{R}$, it follows that

$$
\begin{aligned}
2&\int_0^t < (I-\Delta)^{-1}z(s,\,\cdot\,), (az)(s,\,\cdot\,) >_{L^2}\,\mathrm{d}s \\
&\leqslant \|a\|_\infty \int_0^t \left\|(I-\Delta)^{-1}z(s,\,\cdot\,)\right\|^2_{L^2}\mathrm{d}s \\
&+ \frac{1}{\|a\|_\infty}\int_0^t < (az)(s,\,\cdot\,), (az)(s,\,\cdot\,) >_{L^2}\,\mathrm{d}s \\
&\leqslant \|a\|_\infty \int_0^t \|z(s,\,\cdot\,)\|^2_{H^{-2}}\,\mathrm{d}s \\
&+ \frac{1}{\|a\|_\infty}\,\|a\|_\infty \int_0^t \langle z(s,\,\cdot\,), az(s,\,\cdot\,)\rangle_{L^2}\,\mathrm{d}s.
\end{aligned}
$$

Since $\|\cdot\|_{H^{-2}} \leqslant \|\cdot\|_{H^{-1}}$, (3.15) gives now (for $\omega \notin N_0$),

$$
\begin{aligned}
g(t) &+ \int_0^t \langle z(s,\,\cdot\,), (az)(s,\,\cdot\,)\rangle_{L^2}\,\mathrm{d}s \\
\leqslant M_t &+ \sum_{i=1}^\infty \int_0^t \left\langle z(s,\,\cdot\,)e^i, z(s,\,\cdot\,)e^i\right\rangle_{H^{-1}}\mathrm{d}s \\
&+ 2\int_0^t \left\langle z(s,\,\cdot\,), z(s,\,\cdot\,)e^0\right\rangle_{H^{-1}}\mathrm{d}s + \|a\|_\infty \int_0^t \|z(s,\,\cdot\,)\|^2_{H^{-1}}\,\mathrm{d}s.
\end{aligned}
$$

Since $e^i,\ i \geq 0$ are H^{-1}-multipliers with norm $\mathcal{C}(e^i)$, (for $\omega \notin N_0$)

$$
\begin{aligned}
g(t) &+ \int_0^t \langle z(s,\,\cdot\,), (az)(s,\,\cdot\,)\rangle_{L^2}\,\mathrm{d}s \\
&\leqslant M_t + \mathcal{C}\int_0^t \|z(s,\,\cdot\,)\|^2_{H^{-1}}\,\mathrm{d}s = M_t + \mathcal{C}\int_0^t g(s)\mathrm{d}s,\ \forall\, t \in [0,T],
\end{aligned} \tag{3.16}
$$

where

$$\mathcal{C} = \sum_{i=1}^{\infty} \mathcal{C}(e^i)^2 + 2\mathcal{C}(e^0) + \|a\|_\infty.$$

We proceed now via localization which is possible because $t \mapsto \int_0^t \|z(s,\ \cdot\ )\|^2 \, \mathrm{d}s$ and $t \mapsto \|z(t,\cdot)\|_{H^{-2}}$ are continuous P a.s. Let (ς^ℓ) be the sequence of stopping times

$$\varsigma^\ell := \inf\{t \in [0,T] | \int_0^t \mathrm{d}s \, \|z(s,\ \cdot\ )\|_{L^2}^2 \geq \ell, \|z(t,\cdot)\|_{H^{-2}}^2 \geqslant \ell\}. \tag{3.17}$$

If $\{\ \} = \emptyset$ we convene that $\varsigma^\ell = +\infty$. Clearly, the stopped processes M^{ς^ℓ} are (square integrable) martingales starting at zero. We evaluate (3.16) at $t \wedge \varsigma^\ell$. Taking expectation we get

$$E(g(t\wedge\varsigma^\ell)) \leqslant \underbrace{E(M_{\varsigma^\ell\wedge t})}_{=0} + \mathcal{C}E\left(\int_0^{t\wedge\varsigma^\ell} g(s)\mathrm{d}s\right) \leqslant \mathcal{C}\int_0^t \mathrm{d}s E(g(s\wedge\varsigma^\ell)).$$

By Gronwall's lemma it follows that $E(g(t\wedge\varsigma^\ell)) = 0 \quad \forall\ \ell \in \mathbb{N}^\star$. Since g is a.s. continuous and $\lim_{\ell\to\infty} t\wedge\varsigma^\ell = T$ a.s., for every $t \in [0,T]$, by Fatou's lemma we get

$$E(g(t)) = E\left(\liminf_{\ell\to\infty} g(t\wedge\varsigma^\ell)\right) \leqslant \liminf_{\ell\to\infty} E\left(g(t\wedge\varsigma^\ell)\right) = 0,$$

and the result follows. □

4 Uniqueness for the porous media equation with noise

We first discuss first in which sense the SPDE (1.2) has to be understood.

Definition 4.1. *A random field $X = (X(t,\xi,\omega), t \in [0,T], \xi \in \mathbb{R}, \omega \in \Omega)$ is said to be a solution to* (1.2) *if P a.s. we have the following.*

- $X \in C([0,T]; \mathcal{S}'(\mathbb{R})) \cap L^2([0,T]; L^1_{\mathrm{loc}}(\mathbb{R}))$.
- *X is an $\mathcal{S}'(\mathbb{R})$ -valued $(\mathcal{F}_t)$-progressively measurable process.*
- *For any test function $\varphi \in \mathcal{S}(\mathbb{R})$ with compact support, $t \in]0,T]$, we have*

$$\begin{aligned}\int_{\mathbb{R}} X(t,\xi)\varphi(\xi)\mathrm{d}\xi \quad &= \quad \int_{\mathbb{R}} \varphi(\xi)x_0(\mathrm{d}\xi) + \frac{1}{2}\int_0^t \mathrm{d}s \int_{\mathbb{R}} \psi(X(s,\xi,\cdot))\varphi''(\xi)d\xi \\ &+ \quad \int_{[0,t]\times\mathbb{R}} X(s,\xi)\varphi(\xi)\mu(\mathrm{d}s,\xi)\mathrm{d}\xi \text{ a.s.}\end{aligned}$$

We can state now the uniqueness theorem for the stochastic porous media equation.

Theorem 4.2. *Suppose that Assumptions 1.2 and 2.3 hold. Then equation* (1.2) *admits at most one solution among the random fields* $X :]0,T] \times \mathbb{R} \times \Omega \to \mathbb{R}$ *such that*

$$\int_{[0,T]\times\mathbb{R}} X^2(s,\xi)\mathrm{d}s\mathrm{d}\xi < \infty \quad a.s. \tag{4.2}$$

Remark 4.3. *Let* X *be a solution of* (1.2) *verifying* (4.2).

i) There is a P*-null set* N_0*, so that for* $\omega \notin N_0$, $X(t,\ \cdot\ ) \in L^2(\mathbb{R})$ *for almost all* $t \in [0,T]$.

ii) Condition (4.2) *also implies that* $\int_0^T \|X(s,\ \cdot\ )\|^2_{H^{-1}}\,\mathrm{d}s < \infty \quad a.s.$

iii) Since ψ *is Lipschitz and* $\psi(0) = 0$, (4.2) *implies that* $\int_0^T \|\psi(X(r,\cdot))\|^2_{L^2}\mathrm{d}r < \infty$ *a.s. So,* $\int_0^t \mathrm{d}s\psi(X(s,\cdot))$ *is a Bochner integral with values in* $L^2(\mathbb{R})$.

iv) Consequently, $t \mapsto \Delta\left(\int_0^t \psi(X(s,\ \cdot\ ))\mathrm{d}s\right)$ *is continuous from* $[0,T]$ *to* H^{-2} *and so also in* $\mathcal{S}'(\mathbb{R})$*; since* $e^i, i \geq 0$, *are* H^{-1}*-multipliers verifying Assumption 2.3, by Kolmogorov's lemma* $t \mapsto \int_0^t X(s,\cdot)\mu(\mathrm{d}s,\cdot)$ *admits a version which belongs to* $C\left([0,T]; H^{-1}(\mathbb{R})\right)$. *Since* $x_0 \in \mathcal{S}'(\mathbb{R})$ *and* $X \in C([0,T];\mathcal{S}'(\mathbb{R}))$ *a.s., it follows that for* ω *not belonging to a null set, we have*

$$X(t,\ \cdot\ ) = x_0 + \Delta\left(\int_0^t \psi(X(s,\ \cdot\ ))\mathrm{d}s\right) + \int_0^t X(s,\cdot)\mu(\mathrm{d}s,\cdot), \quad t \in [0,T],$$

as an identity in $\mathcal{S}'(\mathbb{R})$.

v) If $x_0 \in H^{-1}$*, then* $X \in C\left([0,T]; H^{-2}\right)$*, for* $\omega \notin N_0$, N_0 *a* P*-null set.*

vi) If $x_0 \in H^{-s}$ *for some* $s \geq 2$*, then* $X \in C\left([0,T]; H^{-s}\right)$*, for* $\omega \notin N_0$, N_0 *a* P*-null set.*

vii) We consider a sequence of mollifiers (ϕ_ε) *converging to the Dirac measure. Then* $X^\varepsilon(t,\ \cdot\ ) = X(t,\cdot) \star \phi_\varepsilon$ *belongs a.s. to* $C\left([0,T]; L^2(\mathbb{R})\right)$.

Remark 4.4. *Since* ψ *is Lipschitz, there is* $\alpha > 0$ *such that*

$$\left(\psi(r) - \psi(\bar{r})\right)(r - \bar{r}) \geqslant \alpha\left(\psi(r) - \psi(\bar{r})\right)^2.$$

Remark 4.5. *1. We note that condition 2. in Assumption 2.3 is more general than (3.1) of [3], which can be reformulated here as follows.*

Assumption 4.6. *(a)* $e^i \in W^{1,\infty}$ *for every* $i \geq 0$.

(b) $e^i, i \geq 0,$ belong to H^1.

(c) (e^i) *is an orthonormal system of* H^{-1} *and* $\sum_{i=1}^{\infty} \left(\|(e^i)'\|_\infty^2 + \|e^i\|_\infty^2 + \|e^i\|_{H^{-1}}^2 \right) < \infty.$

2. *An easy adaptation of Theorem 3.4 of [3], in order to take into account* e^0*, constitutes an existence result for* (1.2)*: it says the following.*
Besides Assumptions 4.6 and 1.2, let us suppose moreover that $x_0 \in L^2$ *or* ψ *is non-degenerate (i.e.* $\frac{\psi(x)}{x} \geq 0, \ \forall x \neq 0$*). Then, there is a random field* X *such that*
$$E \left(\int_{[0,T]\times\mathbb{R}} X^2(s,\xi) \mathrm{d}s \mathrm{d}\xi \right) < \infty,$$
with $t \mapsto \int_0^t \psi(X(s,\cdot)) \mathrm{d}s \in C([0,T]; H^1(\mathbb{R}))$ *a.s.*

3. *So, under the assumptions of item 2. above, the solution* X *is unique among those fulfilling* (4.2)*.*

Proof. Let $(\phi_\varepsilon, \varepsilon > 0)$ be a sequence of mollifiers as in Remark 4.3 vii). Let X^1, X^2 be two solutions of (1.2). For $i = 1, 2$, we set $(X^i)^\varepsilon(t,\cdot) = X^i(t,\cdot) \star \phi_\varepsilon$. We set $X = X^1 - X^2$ and $X^\varepsilon = (X^1)^\varepsilon - (X^2)^\varepsilon$ which a.s. belongs to $C\left([0,T];\ L^2(\mathbb{R})\right) \subset C\left([0,T];\ H^{-1}\right)$. We set

$$g_\varepsilon(t) := \|X^\varepsilon(t,\ \cdot\ )\|_{H^{-1}}^2 = \int_{\mathbb{R}} \left((I-\Delta)^{-1} X^\varepsilon(t,\ \cdot\ )\right)(\xi) X^\varepsilon(t,\xi) \mathrm{d}\xi.$$

Itô's formula gives

$$g_\varepsilon(t) = 2\int_0^t < X^\varepsilon(s,\ \cdot\ ), X^\varepsilon(\mathrm{d}s,\ \cdot\ ) >_{H^{-1}} + \sum_{i=1}^{\infty} \int_0^t \left\|(e^i X)(s,\cdot) \star \phi_\varepsilon\right\|_{H^{-1}}^2 \mathrm{d}s. \tag{4.3}$$

On the other hand we have

$$X^\varepsilon(t,\ \cdot\ ) = \int_0^t \Delta\left[\left\{\psi(X^1(s,\ \cdot\ )) - \psi(X^2(s,\ \cdot\ ))\right\} \star \phi_\varepsilon\right] \mathrm{d}s + \int_0^t \phi_\varepsilon \star (X\mu(\mathrm{d}s,\ \cdot\ )), \tag{4.4}$$

where the notation of the latter integral is self-explanatory. So

$$\begin{aligned}(I-\Delta)^{-1} X^\varepsilon(t,\ \cdot\ ) = &- \int_0^t \left(\psi(X^1(s,\ \cdot\ )) - \psi(X^2(s,\ \cdot\ ))\right) \star \phi_\varepsilon \mathrm{d}s \qquad (4.5)\\ &+ \int_0^t (I-\Delta)^{-1}\left(\psi(X^1(s,\ \cdot\ )) - \psi(X^2(s,\ \cdot\ ))\right) \star \phi_\varepsilon \mathrm{d}s \\ &+ \sum_{i=0}^{\infty} \int_0^t \left[(I-\Delta)^{-1}(e^i X(s,\ \cdot\ ))\right] \star \phi_\varepsilon \mathrm{d}W_s^i.\end{aligned}$$

We define

$$M_t = \sum_{i=1}^{\infty} \int_0^t < (I-\Delta)^{-1} X(s,\ \cdot\ ), e^i X(s,\ \cdot\ ) >_{L^2} \mathrm{d}W_s^i.$$

We observe that M is well-defined and it is a local martingale. Indeed, by Remark 4.3 v), $X \in C([0,T]; H^{-2})$. So by similar arguments as in (3.6),

$$\sum_{i=1}^{\infty} \int_0^t < (I-\Delta)^{-1} X(s,\ \cdot\ ), e^i X(s,\ \cdot\ ) >_{L^2}^2 \mathrm{d}s \le \sup_{s\in[0,T]} \|X(s,\cdot)\|_{H^{-2}}^2 \sum_{i=1}^{\infty} \|e^i\|_\infty^2$$
$$\int_0^T \|X(s,\ \cdot\ )\|_{L^2}^2 \mathrm{d}s < \infty. \tag{4.6}$$

Using (4.3), (4.4) and (4.5) we get

$$\begin{aligned} g_\varepsilon(t) =& \sum_{i=1}^{\infty} \int_0^t \left\| (e^i X)(s,\cdot) \star \phi_\varepsilon \right\|_{H^{-1}}^2 \mathrm{d}s \qquad (4.7)\\ &- 2\int_0^t < X^\varepsilon(s,\ \cdot\ ), \left[\psi(X^1(s,\ \cdot\ )) - \psi(X^2(s,\ \cdot\ ))\right] \star \phi_\varepsilon >_{L^2} \mathrm{d}s \\ &+ 2\int_0^t < X^\varepsilon(s,\ \cdot\ ), (I-\Delta)^{-1} \left[\psi(X^1(s,\ \cdot\ )) - \psi(X^2(s,\ \cdot\ ))\right] \star \phi_\varepsilon >_{L^2} \mathrm{d}s \\ &+ 2\int_0^t < X^\varepsilon(s,\ \cdot\ ), (I-\Delta)^{-1} \left[e^0 X(s,\ \cdot\ )\right] \star \phi_\varepsilon >_{L^2} \mathrm{d}s + M_t^\varepsilon, \end{aligned}$$

where M^ε is the local martingale defined by

$$M_t^\varepsilon = \sum_{i=1}^{\infty} \int_0^t < X^\varepsilon(s,\ \cdot\ ), (I-\Delta)^{-1} \left(e^i X(s,\ \cdot\ )\right) \star \phi_\varepsilon >_{L^2} \mathrm{d}W_s^i,$$

which is again well-defined by similar arguments as in the proof of (4.6). Taking into account (4.2) and the Lipschitz property for ψ, we can take the limit when $\varepsilon \to 0$ in (4.7) and for $g(t) := \|X(t,\ \cdot\ )\|_{H^{-1}}^2$, to obtain

$$\begin{aligned} & g(t) + 2\int_0^t \left\langle X(s,\ \cdot\ ),\ \psi\left(X^1(s,\ \cdot\ )\right) - \psi\left(X^2(s,\ \cdot\ )\right)\right\rangle_{L^2} \mathrm{d}s \qquad (4.8)\\ &= \sum_{i=1}^{\infty} \int_0^t \left\| e^i X(s,\ \cdot\ ) \right\|_{H^{-1}}^2 \mathrm{d}s \\ &+ 2\int_0^t \left\langle (I-\Delta)^{-1} X(s,\ \cdot\ ),\ \psi\left(X^1(s,\ \cdot\ )\right) - \psi\left(X^2(s,\ \cdot\ )\right)\right\rangle_{L^2} \mathrm{d}s \\ &+ 2\int_0^t \left\langle X(s,\ \cdot\ ), e^0 X(s,\ \cdot\ )\right\rangle_{H^{-1}} \mathrm{d}s + M_t. \end{aligned}$$

The convergence $M^\varepsilon \to M$ when $\varepsilon \to 0$ is ucp, since

$$\sum_{i=1}^{\infty} \int_0^t \left| \left\langle X^\varepsilon(s,\ \cdot\ ),\ [e^i X(s,\ \cdot\ )] \star \phi_\varepsilon \right\rangle_{H^{-1}} - \left\langle X(s,\ \cdot\ ),\ e^i X(s,\ \cdot\ )\right\rangle_{H^{-1}} \right|^2 \mathrm{d}s \underset{\varepsilon\to 0}{\longrightarrow} 0,$$

which follows by similar arguments as in the proof of (3.12), using (4.2).

Taking into account the inequality

$$2ab \leqslant \frac{a^2}{\alpha^2} + b^2\alpha^2,$$

for $a, b \in \mathbb{R}$, α being the constant appearing at Remark 4.4, the second term of the right-hand side of equality (4.8) is bounded by

$$\alpha^2 \int_0^t \left\|(I-\Delta)^{-1} X(s,\,\cdot\,)\right\|_{L^2}^2 \mathrm{d}s + \frac{1}{\alpha^2}\int_0^t \left(\psi\left(X^1(s,\,\cdot\,)\right) - \psi\left(X^2(s,\,\cdot\,)\right)\right)_{L^2}^2 \mathrm{d}s$$

$$\leqslant \alpha^2 \int_0^t \|X(s,\,\cdot\,)\|_{H^{-1}}^2 \mathrm{d}s + \int_0^t \left\langle \psi\left(X^1(s,\,\cdot\,)\right) - \psi\left(X^2(s,\,\cdot\,)\right),\ X(s,\,\cdot\,)\right\rangle_{L^2} \mathrm{d}s.$$

This together with (4.8) gives

$$\begin{aligned} g(t) &+ \int_0^t \left\langle X(s,\,\cdot\,),\ \psi\left(X^1(s,\,\cdot\,)\right) - \psi\left(X^2(s,\,\cdot\,)\right)\right\rangle_{L^2} \mathrm{d}s \\ \leqslant & 2\int_0^t \left\langle X(s,\,\cdot\,), e^0 X(s,\,\cdot\,)\right\rangle_{H^{-1}} \mathrm{d}s + \alpha^2 \int_0^t \|X(s,\,\cdot\,)\|_{H^{-1}}^2 \mathrm{d}s \\ &+ \sum_{i=1}^{\infty} \int_0^t \left\|(e^i X)(s,\cdot)\right\|_{H^{-1}}^2 \mathrm{d}s + M_t,\ t \in [0,T]\ \mathrm{d}P\text{-a.s.}. \end{aligned}$$

Since $e^i,\ i \in \mathbb{N}$, are H^{-1}-multipliers and taking into account Hypothesis ii), we get

$$g(t) \leqslant M_t + (2\mathcal{C}_0 + \sum_{i=1}^{\infty} \mathcal{C}(e^i)^2 + \alpha^2) \int_0^t g(s)\mathrm{d}s.$$

The proof is then completed by localization as in (3.17) at the end of Section 3. □

ACKNOWLEDGMENTS

Financial support through the SFB 701 at Bielefeld University and NSF-Grant 0606615 is gratefully acknowledged. The second named author benefited partially from the support of the "FMJH Program Gaspard Monge in optimization and operation research" (Project 2014-1607H). The authors are grateful to Viorel Barbu for stimulating discussions. The authors are grateful to the Referee whose comments have stimulated them to drastically improve the first version of the paper.

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