

# The stochastic logarithmic Schrödinger equation

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**Abstract.** In this paper we prove global existence and uniqueness of solutions to the stochastic logarithmic Schrödinger equation with linear multiplicative noise. Our approach is mainly based on the rescaling approach and the method of maximal monotone operators. In addition, uniform estimates of solutions in the energy space  $H^1(\mathbb{R}^d)$  and in an appropriate Orlicz space are also obtained here.

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# 1 Introduction and main result.

The logarithmic Schrödinger equation

$$i\frac{du}{dt} + \Delta u + u \log |u|^2 = 0, \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^d, \quad (1.1)$$

has wide applications in quantum mechanics, quantum optics, nuclear physics, open quantum systems, Bose-Einstein condensation and so on. It was first proposed in [9] as a model of nonlinear wave mechanics. As a matter of fact, as shown in [9], the logarithmic nonlinearity arising in (1.1) is the unique nonlinearity for which the separability hypothesis of noninteracting subsystems of the Schrödinger theory holds. It also possesses many other attractive features, including the additivity of the energy for noninteracting subsystems, the validity of the lower energy bound and Planck's relation for all stationary states. All these make this equation unique among nonlinear wave equations. See e.g. [9, 10, 16, 26]. We also refer to [19, 21] for the derivation of this equation from Nelson's stochastic quantum mechanics [22].

Motivated by the physical significance above, we are here mainly concerned with well-posedness of the logarithmic Schrödinger equation in the stochastic case, that is,

$$\begin{aligned} idX &= \Delta X dt + \lambda X \log |X|^2 dt - i\mu X dt + iX dW, \quad t \in (0, T), \\ X(0) &= x \in L^2, \end{aligned} \quad (1.2)$$

Here,  $\lambda \in \mathbb{R}$ ,  $W$  is the Wiener process

$$W(t, \xi) = \sum_{j=1}^n \mu_j e_j(\xi) \beta_j(t), \quad t \geq 0, \quad \xi \in \mathbb{R}^d, \quad (1.3)$$

where  $d \geq 1$ ,  $\{\mu_j\}_{j=1}^n$  are complex numbers,  $\{e_j\}_{j=1}^n$  are real-valued functions, and  $\{\beta_j\}_{j=1}^n$  is a family of independent real valued Brownian motions on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  with normal (in particular right-continuous) filtration  $(\mathcal{F}_t)_{t \geq 0}$ . For simplicity, we assume that  $n < \infty$ .

Moreover,

$$\mu(\xi) = \frac{1}{2} \sum_{j=1}^n |\mu_j|^2 e_j^2(\xi), \quad \xi \in \mathbb{R}^d. \quad (1.4)$$

The stochastic equation (1.2) can be derived from (1.1) with an additional potential  $VX$ , where the random potential  $V$  fluctuates rapidly and so can be

approximated by the Gaussian noise  $\dot{W}$ . Moreover, the linear multiplicative noise  $iXdW$  together with the term  $-i\mu Xdt$  also plays an important role in the theory of measurements continuous in time in open quantum systems. In this case, one main feature is that  $|X(t)|_2^2$  is a continuous martingale. This fact implies the mean norm square conservation of  $X(t)$  and allows to define a new probability law, the “physical” probability law, which has important applications to open quantum systems. For more physical interpretations, we refer to [6], [4, 5] and the references therein.

The stochastic nonlinear Schrödinger equation with the polynomial nonlinearity  $\lambda|X|^{\alpha-1}X$  was first studied in [7, 8], based on the mild formulation of the stochastic equation. The optimal exponents of the nonlinearity for the global well-posedness were recently achieved in [4, 5], based on the rescaling transformation (see (2.2) below) and the Strichartz estimates established in [20] for lower order perturbations of the Laplacian. However, the contraction mapping arguments used in the mentioned works are not applicable here, due to the fact that the function  $y \rightarrow y \log |y|^2$  is not locally Lipschitz.

One of the main features of the logarithmic nonlinearity is the quasi-monotonicity. Based on this, the global well-posedness of the deterministic equation (1.1) was first studied in [13] in the distribution sense for initial data in  $L^2$  or  $H^1$ . Later, the global well-posedness was also proved in [11] for initial data in  $H^1$  and in some convenient Orlicz space, which is closely related to the logarithmic nonlinearity. We also refer to [16] for the global well-posedness for initial data in  $H^1$  with finite momentum.

Furthermore, stochastic partial differential equations with monotone coefficients are also extensively studied in the literature. We refer to [18], [23], [24] and the references therein. Recently, based on the rescaling approach and operatorial reformulation, the approach of maximal monotone operators was developed in [3] in a general infinite dimensional setting, which has applications to new existence and uniqueness results of various stochastic models with linear multiplicative noise.

Inspired by the quasi-monotone feature of the logarithmic nonlinearity and the works mentioned above, we shall employ the rescaling transformation and the method of maximal monotone operators to study the global well-posedness of (1.2).

However, it should be mentioned that, the results in [3] are not applicable here, since the operator  $i\Delta$  in (1.2) is not coercive (see [3, (2.3)]).

Moreover, another difficulty arises from the passage to the limit in the approximating equation (see (3.1) below). Because even if a space  $\mathcal{X}$  is compactly imbedded into another one  $\mathcal{Y}$ , we generally do not have the compact imbedding

from  $L^p(\Omega; \mathcal{X})$  to  $L^p(\Omega; \mathcal{Y})$ ,  $1 \leq p \leq \infty$ , the classical deterministic method as in [11, 12, 13, 16] to pass to the limit in the nonlinear term can not directly be applied here.

In order to overcome these difficulties, inspired by [11, 12], we will consider the initial data in the energy space  $H^1(\mathbb{R}^d)$  and an appropriate Orlicz space  $V$  (see (1.6) below). These spaces allow to control the singularity of the logarithmic nonlinearity at infinity and at the origin respectively. More importantly, they are also suitable spaces for the maximal monotonicity of the logarithmic nonlinearity, which makes the passage to the limit in the approximating equation possible, thereby yielding the global well-posedness.

To state our results precisely, let us first introduce some necessary notations. Take  $H = L^2(\mathbb{R}^d; \mathbb{C}) =: L^2$  with the scalar product defined by  $\langle u, v \rangle = \int_{\mathbb{R}^d} u \bar{v} d\xi$ ,  $u, v \in H$ , and the norm  $|u|_2 = \langle u, u \rangle^{\frac{1}{2}}$ . Let  $H^1$  denote the classical Sobolev space, i.e.  $H^1 = \{u \in L^2 : \nabla u \in L^2\}$  with norm  $|u|_{H^1}^2 = |u|_2^2 + |\nabla u|_2^2$ . We also use the standard notation  $L^p = L^p(\mathbb{R}^d)$ ,  $1 \leq p \leq \infty$ , for the space of all  $p$ -integrable complex functions with the norm  $|\cdot|_{L^p}$ .

Moreover, as in [11], define the function

$$N(x) = \begin{cases} -x^2 \log x^2, & \text{if } 0 \leq x \leq e^{-3}; \\ 3x^2 + 4e^{-3}x - e^{-6}, & \text{if } e^{-3} \leq x. \end{cases} \quad (1.5)$$

$N$  is a positive convex and increasing function, and  $N \in C^1([0, \infty)) \cap C^2((0, \infty))$ . The Orlicz space  $V$  corresponding to  $N$  is defined by

$$V = \{u \in L^1_{loc} : N(|u|) \in L^1\}, \quad (1.6)$$

equipped with the Luxembourg norm

$$\|u\|_V = \inf\{k > 0 : \int N(k^{-1}|u(\xi)|)d\xi \leq 1\}. \quad (1.7)$$

Here as usual  $L^1_{loc}$  is the space of all locally Lebesgue integrable functions. It is proved in [11, Lemma 2.1] that  $N$  is a Young-function which is  $\Delta_2$ -regular and  $(V, \|\cdot\|_V)$  is a separable reflexive Banach space (see also [12] and [1]). We also have that (see [11, (2.2)]) for any  $u \in V$ ,

$$\min\{\|u\|_V, \|u\|_V^2\} \leq \int N(|u(\xi)|)d\xi \leq \max\{\|u\|_V, \|u\|_V^2\} \quad (1.8)$$

Now, set  $U := H^1 \cap V$ .  $U$  is a reflexive Banach space equipped with the norm  $\|u\|_U = |u|_{H^1} + \|u\|_V$ , for any  $u \in U$ , and its dual space is  $U' = H^{-1} + V'$  with the

norm  $\|u\|_{U'} = \inf\{\|u_1\|_{H^{-1}} + \|u_2\|_{V'} : u = u_1 + u_2, u_1 \in H^{-1}, u_2 \in V'\}$ . One advantage for introducing the space  $U$  is that the nonlinear operator  $u \mapsto u \log |u|^2$  is continuous from  $U$  to  $U'$  (see [11, Lemma 2.6]).

The precise definition of solutions to (1.2) is given below.

**Definition 1.1** *A continuous  $H$ -valued  $(\mathcal{F}_t)$ -adapted process  $X$  is said to be a solution to (1.2) if for any  $p \geq 3$ ,  $X \in L^p(\Omega \times (0, T); U)$ ,  $X \log |X|^2 \in L^{p'}(\Omega \times (0, T); U')$ , and it satisfies  $\mathbb{P}$ -a.s. for all  $t \in [0, T]$*

$$X(t) = x - \int_0^t (i\Delta X(s) ds + \mu X(s) + \lambda X(s) \log |X(s)|^2) ds + \int_0^t X(s) dW(s), \quad (1.9)$$

where the stochastic term is taken in Itô's sense.

We also assume that the spatial functions  $\{e_j\}_{j=1}^n$  in the noise  $W$  satisfy the hypothesis:

(H)  $e_j \in \mathcal{B}^\infty(\mathbb{R}^d)$  such that for each  $1 \leq k \leq d$ ,  $1 \leq m \leq n$ ,

$$|\partial_k e_m(\xi)| \leq \lambda(|\xi|), \quad \xi \in \mathbb{R}^d,$$

where  $\mathcal{B}^\infty = \{f \in C^\infty(\mathbb{R}^d), \partial_\alpha f \in L^\infty, \text{ for all } \alpha\}$ , and  $\lambda(\cdot)$  is a positive non-increasing function in  $C([0, \infty]) \cap L^1([0, \infty))$ .

The main result of this article is formulated as follows.

**Theorem 1.2** *Under Hypothesis (H), for any initial datum  $x \in U$  and  $0 < T < \infty$ , there exists a unique solution  $X$  to (1.2) in the sense of Definition 1.1.*

Moreover, for any  $p \geq 2$ ,

$$\mathbb{E} \|X(t)\|_{L^\infty(0, T; U)}^p < \infty, \quad (1.10)$$

$$\mathbb{E} \|X(t) \log |X(t)|^2\|_{L^\infty(0, T; U')}^p < \infty, \quad (1.11)$$

and

$$\mathbb{E} \|e^{W(t)} \frac{d}{dt} (e^{-W(t)} X(t))\|_{L^\infty(0, T; U')}^p < \infty. \quad (1.12)$$

The remainder of this paper is organized as follows. In Section 2, we first apply the rescaling transformation to reduce the original stochastic equation (1.2) to a random equation (see (2.3)), and then we introduce some appropriate spaces and prove the maximal monotonicity of the logarithmic nonlinearity. Section 3 is mainly concerned with the approximating equation. We first obtain the  $H^1$ -global well-posedness and derive the uniform estimate in the energy space in Subsection 3.1. Then in Subsection 3.2, in order to control the singularity of the logarithmic nonlinearity at the origin, we start with the analysis of the entropy function and then prove the uniform estimates in the Orlicz space. Section 4 is mainly devoted to the proof of the main result. As mentioned above, the maximal monotonicity will play an important role in the passage to the limit in the approximating equation. Some technical details are postponed to the Appendix.

Throughout this paper,  $C$  denotes various constants which may change from line to line.

## 2 Random equation

Taking into account the quasi-monotone feature of the logarithmic nonlinearity, we first use the change of variable  $X \rightarrow e^{-2|\lambda|t}X$  to reformulate the original equation (1.2) as

$$\begin{aligned} idX &= \Delta X dt + \lambda X \log |X|^2 dt + (4\lambda|\lambda|t - 2i|\lambda| - i\mu)X dt + iX dW, \\ X(0) &= x \in L^2. \end{aligned} \tag{2.1}$$

Then, applying the rescaling transformation

$$X = e^W y, \tag{2.2}$$

which can be seen as a Doss-Sussman transformation generalized to infinite dimensions, we can reduce the stochastic equation (2.1) to a random Schrödinger equation

$$\begin{aligned} \frac{dy}{dt}(t) &= -ie^{-W(t)} \Delta(e^{W(t)}y(t)) - (2|\lambda| + 4i\lambda|\lambda|t + \widehat{\mu})y(t) \\ &\quad - \lambda iy(t) \log |e^{W(t)}y(t)|^2, \quad a.e. t \in (0, T), \\ y(0) &= x, \end{aligned} \tag{2.3}$$

where  $\widehat{\mu} = \frac{1}{2} \sum_{j=1}^N (\mu_j^2 + |\mu_j|^2) e_j^2$ .

In order to formulate the definition of solutions to (2.3), proceeding as in [3], we consider the Hilbert space  $\mathcal{H}$  of all  $H$ -valued  $(\mathcal{F}_t)_{t \geq 0}$ -adapted processes  $y : [0, T] \rightarrow H$  with the scalar product

$$\langle y, z \rangle_{\mathcal{H}} = \mathbb{E} \int_0^T \langle e^{W(t)} y(t), e^{W(t)} z(t) \rangle dt, \quad (2.4)$$

and the norm

$$\|y\|_{\mathcal{H}} = \left( \mathbb{E} \int_0^T \|e^{W(t)} y(t)\|_H^2 dt \right)^{\frac{1}{2}}.$$

For any  $p \geq 3$ , consider the space  $\mathcal{U}$  of all  $(\mathcal{F}_t)_{t \geq 0}$ -adapted processes  $y := [0, T] \rightarrow U$  such that

$$\|y\|_{\mathcal{U}}^p = \mathbb{E} \int_0^T \|e^{W(t)} y(t)\|_U^p dt < \infty, \quad (2.5)$$

Let  $\mathcal{U}'$  denote the dual space of  $\mathcal{U}$ . In fact,  $\mathcal{U}'$  is the space of all  $(\mathcal{F}_t)_{t \geq 0}$ -adapted processes  $y : [0, T] \rightarrow \mathcal{U}'$  such that

$$\|y\|_{\mathcal{U}'}^{p'} = \mathbb{E} \int_0^T \|e^{W(t)} y(t)\|_{U'}^{p'} dt < \infty. \quad (2.6)$$

We have  $\mathcal{U} \subset \mathcal{H} \subset \mathcal{U}'$ , algebraically and topologically.

Set

$$(\mathcal{G}y)(t) := \lambda i y(t) \log |e^{W(t)} y(t)|^2 + 2|\lambda| |y(t), \quad y \in D(\mathcal{G}) = \mathcal{U}, \quad (2.7)$$

Analogously to Definition 1.1, the solutions to (2.3) is now defined below.

**Definition 2.1** *A solution to (2.3) is a continuous  $H$ -valued  $(\mathcal{F}_t)$ -adapted process  $y$ , such that  $y \in \mathcal{U}$ ,  $y \log |e^{W} y|^2 \in \mathcal{U}'$ , and it satisfies  $\mathbb{P}$ -a.s. for all  $t \in [0, T]$*

$$y(t) = x - \int_0^t (ie^{-W(s)} \Delta(e^{W(s)} y(s)) + (4i\lambda|\lambda|t + \widehat{\mu})y(s) + \mathcal{G}(y(s))) ds. \quad (2.8)$$

We refer to [3, Lemma 8.1] for a rigorous proof of the equivalence of solutions to (1.9) and (2.8). Therefore, the proof of Theorem 1.2 is now reduce to the theorem as follows.

**Theorem 2.2** *Under Hypothesis (H), for any initial datum  $x \in U$  and  $0 < T < \infty$ , there exists a unique solution  $y$  to (2.8) in the sense of Definition 2.1.*

*Moreover, for all  $p \geq 2$ ,*

$$\mathbb{E} \|e^{W(t)} y(t)\|_{L^\infty(0,T;U)}^p < \infty, \quad (2.9)$$

$$\mathbb{E} \|e^{W(t)} y(t) \log |e^{W(t)} y(t)|^2\|_{L^\infty(0,T;U')}^p < \infty, \quad (2.10)$$

and

$$\mathbb{E} \|e^{W(t)} \frac{d}{dt} y(t)\|_{L^\infty(0,T;U')}^p < \infty. \quad (2.11)$$

The remainder of this paper is devoted to the proof of Theorem 2.2. We will mainly consider the case  $d \geq 3$ . The simpler cases  $d = 1, 2$  can be proved similarly.

In the end of this section, let us show the maximal monotonicity of the operator  $\mathcal{G}$ . Recall that an operator  $A : \mathcal{X} \rightarrow \mathcal{X}'$  (possibly nonlinear) from a Banach space  $\mathcal{X}$  to its dual  $\mathcal{X}'$  is said to be monotone if

$$\operatorname{Re} {}_{\mathcal{X}'} \langle Ay_1 - Ay_2, y_1 - y_2 \rangle_{\mathcal{X}} \geq 0, \quad \forall y_1, y_2 \in D(A),$$

and maximal monotone if it has no nontrivial monotone extensions in  $\mathcal{X} \times \mathcal{X}'$ .

**Proposition 2.3** *For any  $p \geq 3$ , the operator  $\mathcal{G}$  is maximal monotone from  $\mathcal{U}$  to  $\mathcal{U}'$ .*

**Proof.** In view of [2, Theorem 2.4] the maximality, since the demicontinuity implies the hemicontinuity, it suffices to prove that  $\mathcal{G}$  is monotone and demicontinuous from  $\mathcal{U}$  to  $\mathcal{U}'$ , i.e., if  $y_n, y \in \mathcal{U}$  such that  $y_n \rightarrow y$  in  $\mathcal{U}$ , then

$${}_{\mathcal{U}'} \langle \mathcal{G}(y_n), z \rangle_{\mathcal{U}} \rightarrow {}_{\mathcal{U}'} \langle \mathcal{G}(y), z \rangle_{\mathcal{U}}, \quad z \in \mathcal{U}. \quad (2.12)$$

For this purpose, we first note that by the definition of  $\mathcal{G}$  in (2.7),

$$\begin{aligned} & \operatorname{Re} {}_{\mathcal{U}'} \langle \mathcal{G}(y_1) - \mathcal{G}(y_2), y_1 - y_2 \rangle_{\mathcal{U}} \\ &= 2|\lambda| \|y_1 - y_2\|_{\mathcal{H}}^2 - 2\lambda \operatorname{Im} {}_{\mathcal{U}'} \langle y_1 \log |e^W y_1| - y_2 \log |e^W y_2|, y_1 - y_2 \rangle_{\mathcal{U}} \geq 0. \end{aligned}$$

where in the last step we used (3.5) below with  $\varepsilon = 0$ , and so the monotonicity of  $\mathcal{G}$  follows.



In order to prove the demicontinuity (2.12), we will show that

$$\|e^W \mathcal{G}(y_n)\|_{L^{p'}(\Omega \times (0,T); U')} \leq C < \infty, \quad (2.13)$$

where  $C$  is independent of  $n$ . Then, for any subsequence of  $\{n\} \rightarrow \infty$ , there exists a further subsequence (still denoted by  $\{n\}$ ) such that  $e^W \mathcal{G}(y_n) \rightharpoonup \eta$ , in  $L^{p'}(\Omega \times (0,T); U')$ , where  $\rightharpoonup$  stands for weak convergence. But, since  $y_n \rightarrow y$  in  $\mathcal{U}$ , we have  $e^W \mathcal{G}(y_n) \rightarrow e^W \mathcal{G}(y)$  in measure  $\mathbb{P} \otimes dt \otimes d\xi$ . Hence, we conclude that  $\eta = e^W \mathcal{G}(y)$ , which implies (2.12), since the subsequence was arbitrary.

It remains to prove (2.13). Set  $X_n := e^W y_n$  and  $L(|X_n|^2) := \log |X_n|^2$ . By the definition of  $U'$  and  $\mathcal{G}$  we have

$$\begin{aligned} & \|e^W \mathcal{G}(y_n)\|_{L^{p'}(\Omega \times (0,T); U')} \\ & \leq 2|\lambda| \|X_n\|_{L^{p'}(\Omega \times (0,T); H^{-1})} + |\lambda| \|I_{\{|X_n| > e^{-3}\}} X_n L(|X_n|^2)\|_{L^{p'}(\Omega \times (0,T); H^{-1})} \\ & \quad + |\lambda| \|I_{\{|X_n| \leq e^{-3}\}} X_n L(|X_n|^2)\|_{L^{p'}(\Omega \times (0,T); V')}. \end{aligned} \quad (2.14)$$

Since for each  $\xi \in \{|X_n| > e^{-3}\}$ ,  $|X_n(\xi) L(|X_n(\xi)|^2)| \leq C_\delta (|X_n(\xi)| + |X_n(\xi)|^{1+\delta})$  with  $C_\delta$  independent of  $n$ . By Sobolev's imbedding theorem with  $\delta \leq \frac{2}{d-2}$ ,

$$\begin{aligned} & \|X_n\|_{L^{p'}(\Omega \times (0,T); H^{-1})} + \|I_{\{|X_n| > e^{-3}\}} X_n L(|X_n|^2)\|_{L^{p'}(\Omega \times (0,T); H^{-1})} \\ & \leq C (\|X_n\|_{L^{p'}(\Omega \times (0,T); L^2)} + \|X_n\|_{L^{(1+\delta)p'}(\Omega \times (0,T); L^{2(1+\delta)}}^{1+\delta}) \\ & \leq C (\|X_n\|_{L^{p'}(\Omega \times (0,T); L^2)} + \|X_n\|_{L^{(1+\delta)p'}(\Omega \times (0,T); H^1)}^{1+\delta}). \end{aligned}$$

Then, taking  $\delta$  such that  $0 < \delta < p-2$ , we have  $(1+\delta)p' < p$  and, via the Hölder inequality,

$$\begin{aligned} & \|X_n\|_{L^{p'}(\Omega \times (0,T); H^{-1})} + \|I_{\{|X_n| > e^{-3}\}} X_n L(|X_n|^2)\|_{L^{p'}(\Omega \times (0,T); H^{-1})} \\ & \leq C_T (\|X_n\|_{L^p(\Omega \times (0,T); L^2)} + \|X_n\|_{L^p(\Omega \times (0,T); H^1)}^{1+\delta}) \leq C_T < \infty, \end{aligned} \quad (2.15)$$

where  $C_T$  is independent of  $n$ .

On the other hand, for each  $\xi \in \{|X_n| \leq e^{-3}\}$ , as in the proof of [11, Lemma 2.5] we have

$$\tilde{N}(|X_n(\xi) L(|X_n(\xi)|^2)|) \leq 2N(|X_n(\xi)|), \quad (2.16)$$

where  $\tilde{N}$  is the convex conjugate of  $N$ . Then, since  $\tilde{N}(0) = 0$ , by (1.8),

$$\begin{aligned} \int \tilde{N}(I_{\{|X_n| \leq e^{-3}\}} |X_n L(|X_n|^2)|) d\xi & = \int I_{\{|X_n| \leq e^{-3}\}} \tilde{N}(|X_n L(|X_n|^2)|) d\xi \\ & \leq 2 \int I_{\{|X_n| \leq e^{-3}\}} N(|X_n|) d\xi \\ & \leq 2 \max\{\|X_n\|_V, \|X_n\|_V^2\}. \end{aligned} \quad (2.17)$$

Moreover, similarly to (1.8), there exist  $\kappa, C \in (2, \infty)$  such that

$$\min\{\|u\|_{V'}, \|u\|_{V'}^\kappa\} \leq C \int \tilde{N}(|u|) d\xi. \quad (2.18)$$

(See the Appendix for a proof.) Then, (2.17) and (2.18) imply that

$$\begin{aligned} \|I_{\{|X_n| \leq e^{-3}\}} X_n L(|X_n|^2)\|_{V'} &\leq C \max\{\|X_n\|_V, \|X_n\|_V^2, \|X_n\|_V^{1/\kappa}, \|X_n\|_V^{2/\kappa}\} \\ &\leq C(\|X_n\|_V^2 + 1), \end{aligned} \quad (2.19)$$

Hence, since  $p \geq 3$ ,  $2p' \leq p$ , Hölder's inequality yields

$$\begin{aligned} \|I_{\{|X_n| \leq e^{-3}\}} X_n L(|X_n|^2)\|_{L^{p'}(\Omega \times (0, T); V')} &\leq C_T (\|X_n\|_{L^{2p'}(\Omega \times (0, T); V)}^2 + 1) \\ &\leq C_T (\|X_n\|_{L^p(\Omega \times (0, T); V)}^2 + 1) \\ &\leq C_T < \infty, \end{aligned} \quad (2.20)$$

where  $C_T$  is independent of  $n$ .

Consequently, (2.14), (2.15) and (2.20) together yield (2.13), thereby completing the proof of Proposition 2.3.  $\square$

**Remark 2.4** *As in [3], we can also define the operators  $\mathcal{B}, \mathcal{A} : \mathcal{U} \rightarrow \mathcal{U}'$  by*

$$\begin{aligned} (\mathcal{A}y)(t) &= ie^{-W(t)} \Delta(e^{W(t)} y(t)) + 4i\lambda|\lambda|ty(t), \quad y \in D(\mathcal{A}) = \mathcal{U}, \\ (\mathcal{B}y)(t) &= \frac{dy(t)}{dt} + \hat{\mu}y(t), \quad a.e. \ t \in (0, T), \quad y \in D(\mathcal{B}), \end{aligned}$$

where  $D(\mathcal{B}) = \{y \in \mathcal{U} : y \in AC([0, T]; U') \cap C([0, T]; H), \mathbb{P} - a.s., \frac{dy}{dt} \in \mathcal{U}', y(0) = x\}$ . Then, (2.3) can be reformulated as an operatorial equation

$$\mathcal{B}y + \mathcal{A}y + \mathcal{G}y = 0.$$

*It is clear that  $\mathcal{A}$  is maximal monotone from  $\mathcal{U}$  to  $\mathcal{U}'$ . The same assertion holds also for  $\mathcal{B}$ , by similar arguments as in the proof of [3, Lemma 4.1, Lemma 4.2]. Then, since  $D(\mathcal{A}) = D(\mathcal{G}) = \mathcal{U}$ , we deduce from [2, Theorem 2.6] that  $\mathcal{A} + \mathcal{B} + \mathcal{G}$  is also maximal monotone. However, unlike in [3], we do not have the coercivity (see [3, (2.3)]) in the Schrödinger case, the proof of [3, Proposition 3.3] is not applicable here. In order to obtain existence of solutions to (2.8), we shall introduce and study an associated approximating equation in the next section.*

### 3 Approximating equation

Consider the approximating equation,

$$y(t) = x - \int_0^t (ie^{-W(s)}\Delta(e^{W(s)}y(s)) + (4i\lambda|\lambda|t + \widehat{\mu})y(s) + \mathcal{G}_\varepsilon(y(s))) ds, \quad (3.1)$$

$$y(0) = x,$$

Here,  $t \in (0, T)$ ,  $0 \leq \varepsilon \leq 1$ ,

$$\mathcal{G}_\varepsilon(y) := 2\lambda iyL_\varepsilon(e^W y) + 2|\lambda|y, \quad (3.2)$$

and

$$L_\varepsilon(u) = \log\left(\frac{|u| + \varepsilon}{1 + \varepsilon|u|}\right), \quad \forall u \in \mathbb{C}. \quad (3.3)$$

For  $\varepsilon = 0$ , set  $L(u) := L_0(u) = \log|u|$ ,  $u \in \mathbb{C}$ .

We collect some properties of  $L_\varepsilon$  in the following lemma, whose proof is included in the Appendix for completeness.

**Lemma 3.1** *Let  $0 < \varepsilon < 1$ . Then:*

- (i) *For all  $u > 0$ ,  $|L_\varepsilon(u)| \leq |\log \varepsilon|$ , and  $|uL_\varepsilon(u)| \leq |uL(u)|$ .*
- (ii) *For all  $u_1, u_2 \in \mathbb{C}$ ,*

$$|u_1L_\varepsilon(u_1) - u_2L_\varepsilon(u_2)| \leq (1 + \log(1/\varepsilon))|u_1 - u_2|. \quad (3.4)$$

- (iii) *For all  $u_1, u_2 \in \mathbb{C}$ ,*

$$|Im(\overline{u_1} - \overline{u_2})(u_1L_\varepsilon(u_1) - u_2L_\varepsilon(u_2))| \leq (1 - \varepsilon^2)|u_1 - u_2|^2. \quad (3.5)$$

The main result in this section is as follows.

**Proposition 3.2** *Assume (H) and let  $0 < \varepsilon < 1$  be fixed. For any initial datum  $x \in U$  and  $0 < T < \infty$ , there exists a unique  $U$ -valued  $(\mathcal{F}_t)$ -adapted process  $y_\varepsilon$ , such that  $y_\varepsilon \in C([0, T]; H^1)$ ,  $\mathbb{P}$ -a.s., and it satisfies (3.1) in the space  $U'$  on  $[0, T]$ ,  $\mathbb{P}$ -a.s.*

Moreover, for any  $p \geq 2$ ,

$$\mathbb{E} \sup_{0 \leq t \leq T} \|e^{W(t)}y_\varepsilon(t)\|_U^p \leq C(T, p) < \infty, \quad (3.6)$$

and

$$\mathbb{E} \sup_{0 \leq t \leq T} \|e^{W(t)}\mathcal{G}_\varepsilon(y_\varepsilon(t))\|_{U'}^p \leq C(T, p) < \infty, \quad (3.7)$$

where  $C(T, p)$  is independent of  $\varepsilon$ .

The proof will proceed in two steps. We first prove the global well-posedness of (3.1) in the state space  $H^1$  in Subsection 3.1, and then we prove the necessary uniform estimates in the Orlicz space in Subsection 3.2.

### 3.1 $H^1$ global well-posedness

**Proposition 3.3** *Assume (H) and let  $0 < \varepsilon < 1$  be fixed. For each  $x \in H^1$  and  $0 < T < \infty$ , there exists a unique  $H^1$ -valued  $(\mathcal{F}_t)$ -adapted process  $y_\varepsilon$ , such that  $y_\varepsilon \in C([0, T]; H^1)$ , and it solves (3.1) in the space  $H^{-1}$  on  $[0, T]$ ,  $\mathbb{P}$ -a.s.*

Moreover, for any  $p \geq 2$ ,

$$\mathbb{E} \sup_{t \in [0, T]} |e^{W(t)} y_\varepsilon(t)|_{H^1}^p \leq C(T, p) < \infty, \quad (3.8)$$

where  $C(T, p)$  is independent of  $\varepsilon$ .

The key observation for the proof lies in the fact that the operator  $y \rightarrow yL_\varepsilon(e^W y)$  is Lipschitz on  $L^2$  and bounded on  $H^1$ . This fact allows to apply a fixed point argument as in [5]. Below, the proof will rely on three lemmas. We first introduce the evolution operators in Lemma 3.4, and then we prove the local existence in Lemma 3.5. Finally, in Lemma 3.6 we derive the a priori estimate in  $H^1$ -norm, which in turn implies the global well-posedness.

**Lemma 3.4**  *$\mathbb{P}$  - a.e., the operator  $y \rightarrow -ie^{-W} \Delta(e^W y) - (2|\lambda| + 4i\lambda|\lambda|t + \widehat{\mu})y$  generates evolution operators  $U(t, s) = U(t, s, \omega)$  in the space  $H^1(\mathbb{R}^d)$ ,  $0 \leq s \leq t \leq T$ . For each  $x \in H^1(\mathbb{R}^d)$  and  $s \in [0, T]$ , the process  $[s, T] \ni t \rightarrow U(t, s)x$  is continuous and  $(\mathcal{F}_t)$ -adapted, hence progressively measurable with respect to the filtration  $(\mathcal{F}_t)_{t \geq s}$ .*

Moreover, for any  $f \in L^1(0, T; H^1)$ , then  $H^1$ -path

$$y(t) = U(t, 0)x + \int_0^t U(t, s)f(s)ds, \quad 0 \leq t \leq T, \quad (3.9)$$

satisfies the estimates

$$\|y\|_{C([0, T]; H)} \leq C_T(|x|_H + \|f\|_{L^1(0, T; H)}), \quad (3.10)$$

and

$$\|y\|_{C([0, T]; H^1)} \leq C_T(|x|_{H^1} + \|f\|_{L^1(0, T; H^1)}). \quad (3.11)$$

Here, the process  $C_t$ ,  $t \geq 0$ , can be taken to be  $(\mathcal{F}_t)$ -adapted progressively measurable, increasing and continuous.

(For the proof see the Appendix.)

**Lemma 3.5** *Assume (H) and let  $0 < \varepsilon < 1$  be fixed. For each  $x \in H^1$  and  $0 < T < \infty$ , there exists an  $H^1$ -valued  $(\mathcal{F}_t)$ -adapted process  $y_\varepsilon$  and a stopping time  $\tau_\varepsilon^*(x) \leq T$ , such that  $y_\varepsilon \in C([0, \tau_\varepsilon^*(x)]; H^1)$ , and  $y_\varepsilon$  solves the equation (3.1) in  $H^{-1}$  on  $[0, \tau_\varepsilon^*(x))$ ,  $\mathbb{P}$ -a.s.*

*Moreover,  $\tau_\varepsilon^*(x) = T$ ,  $\mathbb{P}$ -a.s., if*

$$\sup_{t \in [0, \tau_\varepsilon^*(x))} |y_\varepsilon(t)|_{H^1} < \infty, \quad \mathbb{P} - a.s. \quad (3.12)$$

*Proof.* Using the evolution operators introduced in Lemma 3.4, we reformulate the equation (3.1) in the mild form

$$y = U(t, 0)x - 2\lambda i \int_0^t U(t, s) (y(s)L_\varepsilon(e^{W(s)}y(s))) ds, \quad (3.13)$$

(Note that, since for  $y \in C([0, T]; H^1)$ ,  $yL_\varepsilon(e^W y) \in L^1(0, T; H^1)$ , the equivalence between (3.1) and (3.13) can be proved similarly as in [25, Theorem 2.2.2].)

Consider the integral operator  $F$  defined for any  $y \in C([0, T]; H^1)$  by

$$F(y)(t) := U(t, 0)x - 2\lambda i \int_0^t U(t, s)(y(s)L_\varepsilon(e^{W(s)}y(s)))ds, \quad t \in [0, T].$$

We first show that

$$F(C([0, T]; H^1)) \subset C([0, T]; H^1). \quad (3.14)$$

Indeed, by (3.11),

$$\|F(y)\|_{C([0, T]; H^1)} \leq C_T (|x|_{H^1} + 2|\lambda| \|yL_\varepsilon(e^W y)\|_{L^1(0, T; H^1)}).$$

By Lemma 3.1 (i) we have

$$|yL_\varepsilon(e^W y)|_{H^1} \leq \sqrt{2} |\log \varepsilon| |y|_{H^1} + |y \nabla(L_\varepsilon(e^W y))|_2.$$

Moreover, straightforward computations show that

$$\nabla(L_\varepsilon(e^W y)) = \frac{(1 - \varepsilon^2)|e^W y|^{-1} \operatorname{Re}(\overline{e^W y} \nabla(e^W y))}{(\varepsilon + |e^W y|)(1 + \varepsilon|e^W y|)}, \quad (3.15)$$

which implies that

$$|\nabla(L_\varepsilon(e^W y))| \leq |e^W y|^{-1} |\nabla(e^W y)|. \quad (3.16)$$

Then,

$$|y\nabla(L_\varepsilon(e^W y))|_2 \leq |e^{-W}\nabla(e^W y)|_2 \leq \sqrt{2}\exp(2|W|_{L^\infty})(1 + |\nabla W|_{L^\infty})|y|_{H^1}.$$

Hence,

$$|yL_\varepsilon(e^W y)|_{H^1} \leq \sqrt{2}(|\log \varepsilon| + \exp(2|W|_{L^\infty})(1 + |\nabla W|_{L^\infty}))|y|_{H^1}.$$

It follows that

$$\|F(y)\|_{C([0,T];H^1)} \leq C_T|x|_{H^1} + C_T D_1(T)T\|y\|_{C([0,T];H^1)} \quad (3.17)$$

with  $D_1(T) := 2\sqrt{2}|\lambda|(|\log \varepsilon| + \sup_{t \leq T} \exp(2|W(t)|_{L^\infty})(1 + \sup_{t \leq T} |\nabla W(t)|_{L^\infty}))$ , thereby yielding (3.14) as claimed.

Next, we will apply the iteration arguments as in [5] to construct the local solution to (3.1).

Fix  $\omega \in \Omega$ . Set  $\mathcal{Y}_{M_1}^{\tau_1} := \{y \in C([0, \tau_1]; H^1) : \|y\|_{C([0, \tau_1]; H^1)} \leq M_1\}$ , where  $\tau_1$  and  $M_1$  are random variables to be chosen later.

Similarly to (3.17), for any  $y \in \mathcal{Y}_{M_1}^{\tau_1}$ ,

$$\|F(y)\|_{C([0, \tau_1]; H^1)} \leq C_{\tau_1}|x|_{H^1} + C_{\tau_1} D_1(\tau_1)M_1\tau_1. \quad (3.18)$$

Moreover, for any  $y, \tilde{y} \in \mathcal{Y}_{M_1}^{\tau_1}$ , by (3.10),

$$\|F(y) - F(\tilde{y})\|_{C([0, \tau_1]; L^2)} \leq 2|\lambda|C_{\tau_1}\|yL_\varepsilon(e^W y) - \tilde{y}L_\varepsilon(e^W \tilde{y})\|_{L^1(0, \tau_1; L^2)},$$

which implies by (3.4) that

$$\|F(y) - F(\tilde{y})\|_{C([0, \tau_1]; L^2)} \leq C_{\tau_1} D_2(\tau_1)\tau_1\|y - \tilde{y}\|_{C([0, \tau_1]; L^2)}, \quad (3.19)$$

where  $D_2(t) = 2|\lambda|(1 + |\log \varepsilon|) \sup_{s \leq t} \exp(2|W(s)|_{L^\infty})$ .

Then, we define the real-valued continuous,  $(\mathcal{F}_t)$ -adapted process  $Z(t) := D_1(t) + D_2(t)$ , and denote the  $(\mathcal{F}_t)$ -stopping time  $\tau_1 := \inf\{t \in [0, T] : C_t Z(t)t \geq \frac{1}{2}\} \wedge T$  and  $M := 2C_{\tau_1}|x|_{H^1}$ . (3.18) and (3.19) imply that  $F(\mathcal{Y}_{M_1}^{\tau_1}) \subset \mathcal{Y}_{M_1}^{\tau_1}$  and  $F$  is a contraction in  $C([0, \tau_1]; L^2)$ . Hence, Banach's fixed point theorem yields a unique  $y \in \mathcal{Y}_{M_1}^{\tau_1}$ , such that  $y = F(y)$  on  $[0, \tau_1]$ . Setting  $y_1(t) := y(t \wedge \tau_1)$  and arguing as in the proof of [5, Proposition 2.5] we deduce that  $y_1|_{[0, \tau_1]} \in C([0, \tau_1]; H^1)$ ,  $y_1$  is  $(\mathcal{F}_t)$ -adapted and it solves (3.1) on  $[0, \tau_1]$ ,  $\mathbb{P}$ -a.s.

Applying similar arguments as in [5], we can extend the solution step by step and construct a sequence  $\{(y_m, \tau_m)\}_{m \geq 1}$ , such that for each  $m \geq 1$ ,  $\tau_m$  is an  $(\mathcal{F}_t)$ -stopping time,  $\tau_{m+1} \geq \tau_m$ ,  $y_m$  is an  $H^1$ -valued  $(\mathcal{F}_t)$ -adapted process, such

that  $y_m|_{[0, \tau_m]} \in C([0, \tau_m]; H^1)$ ,  $y_m(t) = y_m(t \wedge \tau_m)$ ,  $t \in [0, T]$ , and  $y_m$  solves (3.1) on  $[0, \tau_m]$ ,  $\mathbb{P}$ -a.s.

More precisely, given the pair  $(y_m, \tau_m)$  with such properties above at the  $m$ -th step, we set  $\mathcal{Y}_{M_{m+1}}^{\sigma_m} := \{z \in C([0, \sigma_m]; H^1) : \|z\|_{C([0, \sigma_m]; H^1)} \leq M_{m+1}\}$ , and define for  $z \in C([0, T]; H^1)$ ,

$$F_m(z)(t) := U(\tau_m + t, \tau_m)y_m(\tau_m) - 2\lambda i \int_0^t U(\tau_m + t, \tau_m + s)(z(s)L_\varepsilon(e^{W(\tau_m+s)}z(s)))ds.$$

Similarly to (3.18) and (3.19), for  $z \in \mathcal{Y}_{M_{m+1}}^{\sigma_m}$ ,

$$\|F_m(z)\|_{C([0, \sigma_m]; H^1)} \leq C_{\tau_m + \sigma_m} (\|y_m(\tau_m)\|_{H^1} + D_1(\tau_m + \sigma_m)M_{m+1}\sigma_m).$$

and for  $z, \tilde{z} \in \mathcal{Y}_{M_{m+1}}^{\sigma_m}$ ,

$$\|F_m(z) - F_m(\tilde{z})\|_{C([0, \sigma_m]; L^2)} \leq C_{\tau_m + \sigma_m} D_2(\tau_m + \sigma_m)\sigma_m \|z - \tilde{z}\|_{C([0, \sigma_m]; L^2)}.$$

Then, define  $Z_t^{(m)} := D_1(\tau_m + t) + D_2(\tau_m + t)$  and  $\sigma_m = \inf\{t \in [0, T - \tau_m] : C_{\tau_m + t} Z_t^{(m)} t > \frac{1}{2}\} \wedge (T - \tau_m)$ . It follows that  $F_m(\mathcal{Y}_{M_{m+1}}^{\sigma_m}) \subset \mathcal{Y}_{M_{m+1}}^{\sigma_m}$  and  $F_m$  is a contraction in  $C([0, \sigma_m]; L^2)$ . By Banach's fixed point theorem, we obtain a unique  $z_{m+1} \in \mathcal{Y}_{M_{m+1}}^{\sigma_m}$ , such that  $z_{m+1} = F_m(z_{m+1})$  on  $[0, \sigma_m]$ .

Therefore, set  $\tau_{m+1} := \tau_m + \sigma_m$  and

$$y_{m+1}(t) := \begin{cases} y_m(t), & t \in [0, \tau_m]; \\ z_{m+1}((t - \tau_m) \wedge \sigma_m), & t \in (\tau_m, T]. \end{cases}$$

Then, we construct a new pair  $(y_{m+1}, \tau_{m+1})$  with the properties mentioned above. In particular,  $y_{m+1}$  solves (3.1) on  $[0, \tau_{m+1}]$ ,  $\mathbb{P}$ -a.s. Iterating this procedure gives us the desired sequence  $\{(y_m, \tau_m)\}_{m \geq 1}$ .

Now, let  $\tau_\varepsilon^*(x) := \lim_{m \rightarrow \infty} \tau_m$  and  $y_\varepsilon := \lim_{m \rightarrow \infty} y_m I_{[0, \tau_\varepsilon^*(x))}$ . It follows that  $\tau_\varepsilon^*(x)$  is an  $(\mathcal{F}_t)$ -stopping time,  $y_\varepsilon$  is an  $H^1$ -valued  $(\mathcal{F}_t)$ -adapted process,  $y_\varepsilon \in C([0, \tau_\varepsilon^*(x)]; H^1)$ , and it solves the equation (3.1) on  $[0, \tau_\varepsilon^*(x))$ ,  $\mathbb{P}$ -a.s.

Finally, by the construction of  $\{(y_m, \tau_m)\}_{m \geq 1}$ , we use similar arguments as in [5] to obtain the blow-up alternative, i.e. for  $\mathbb{P}$ -a.e.  $\omega$ , if  $\tau_m(\omega) < \tau_\varepsilon^*(x)(\omega)$ ,  $\forall m \in \mathbb{N}$ , then  $\lim_{t \rightarrow \tau_\varepsilon^*(x)(\omega)} |y_\varepsilon(t)(\omega)|_{H^1} = \infty$ . By the construction of  $\sigma_m$  above, we consequently conclude that  $\tau_\varepsilon^*(x) = T$  if (3.12) holds.  $\square$

**Lemma 3.6** *Assume the conditions of Lemma 3.5 to hold, and let  $\tau_\varepsilon^*(x)$  and  $y_\varepsilon$  be as in Lemma 3.5. Then, for any  $p \geq 2$ ,*

$$\mathbb{E} \sup_{t \in [0, \tau_\varepsilon^*(x)]} |e^{W(t)} y_\varepsilon(t)|_{H^1}^p \leq C(T, p) < \infty, \quad (3.20)$$

where  $C(T, p)$  is independent of  $\varepsilon$ .

*Proof.* Let  $X_m := e^W y_\varepsilon$ ,  $\phi_j = \mu_j e_j$ ,  $1 \leq j \leq m$ , and  $\{\tau_m\}_{m \geq 1}$  be the sequence of stopping times constructed in the proof of Lemma 3.5. Since  $X_\varepsilon L_\varepsilon(X_\varepsilon) \in L^2 \subset H^{-1}$ , as in the proof of [5, Lemma 5.2], we derive that  $\mathbb{P}$ -a.s., for  $t \in [0, \tau_m]$ ,

$$\begin{aligned} & |X_\varepsilon(t)|_{H^1}^2 \\ &= |x|_{H^1}^2 - 4|\lambda| \int_0^t |X_\varepsilon|_{H^1}^2 ds - 2 \int_0^t \operatorname{Re} \int \nabla \overline{X_\varepsilon} \nabla (\mu X_\varepsilon) d\xi ds \\ &\quad + \sum_{j=1}^n \int_0^t |\nabla (X_\varepsilon \phi_j)|_2^2 ds + 4\lambda \int_0^t \operatorname{Im} \int \nabla \overline{X_\varepsilon} \nabla (X_\varepsilon L_\varepsilon(X_\varepsilon)) d\xi ds \\ &\quad + 2 \sum_{j=1}^n \int_0^t \int |X_\varepsilon|^2 \operatorname{Re} \phi_j d\xi d\beta_j(s) + 2 \sum_{j=1}^n \int_0^t \operatorname{Re} \int \nabla \overline{X_\varepsilon} \nabla (X_\varepsilon \phi_j) d\xi d\beta_j(s). \end{aligned} \quad (3.21)$$

Then, applying Itô's formula we obtain for any  $p \geq 2$ ,

$$\begin{aligned} & |X_\varepsilon(t)|_{H^1}^p \\ &= |x|_{H^1}^p - 2p|\lambda| \int_0^t |X_\varepsilon|_{H^1}^p ds \\ &\quad - p \int_0^t |X_\varepsilon|_{H^1}^{p-2} \operatorname{Re} \int \nabla \overline{X_\varepsilon} \nabla (\mu X_\varepsilon) d\xi ds + \frac{p}{2} \sum_{j=1}^n \int_0^t |X_\varepsilon|_{H^1}^{p-2} |\nabla (X_\varepsilon \phi_j)|_2^2 ds \\ &\quad + \frac{1}{2} p(p-2) \sum_{j=1}^n \int_0^t |X_\varepsilon|_{H^1}^{p-4} \left( \int |X_\varepsilon|^2 \operatorname{Re} \phi_j d\xi + \operatorname{Re} \int \nabla \overline{X_\varepsilon} \nabla (X_\varepsilon \phi_j) d\xi \right)^2 ds \\ &\quad + 2p\lambda \int_0^t |X_\varepsilon|_{H^1}^{p-2} \operatorname{Im} \int \nabla \overline{X_\varepsilon} \nabla (X_\varepsilon L_\varepsilon(X_\varepsilon)) d\xi ds \\ &\quad + p \sum_{j=1}^n \int_0^t |X_\varepsilon|_{H^1}^{p-2} \int |X_\varepsilon|^2 \operatorname{Re} \phi_j d\xi d\beta_j(s) \\ &\quad + p \sum_{j=1}^n \int_0^t |X_\varepsilon|_{H^1}^{p-2} \operatorname{Re} \int \nabla \overline{X_\varepsilon} \nabla (X_\varepsilon \phi_j) d\xi d\beta_j(s) \end{aligned}$$



$$=: |x|_{H^1}^p + \sum_{k=1}^7 J_k(t), \quad t \in [0, \tau_m], \quad \mathbb{P} - a.s. \quad (3.22)$$

Since  $e_j \in C_b^\infty$ ,  $1 \leq j \leq n$  and since by (3.16) we have

$$\left| \operatorname{Im} \int \nabla \overline{X_\varepsilon} \nabla (X_\varepsilon L_\varepsilon(X_\varepsilon)) d\xi \right| = \left| \operatorname{Im} \int \nabla \overline{X_\varepsilon} (X_\varepsilon \nabla L_\varepsilon(X_\varepsilon)) d\xi \right| \leq |\nabla X_\varepsilon|_2^2,$$

it follows that

$$\sum_{k=1}^5 \mathbb{E} \sup_{s \leq t} |J_k(s)| \leq C_p \int_0^t \mathbb{E} \sup_{r \leq s} |X_\varepsilon(r)|_{H^1}^p ds, \quad t \in [0, \tau_m], \quad (3.23)$$

where  $C_p$  is independent of  $\varepsilon$  and  $m$ .

As regards the remaining stochastic terms, it follows from the Burkholder-Davis-Gundy inequality that

$$\begin{aligned} \mathbb{E} \sup_{s \leq t \wedge \tau_m} |J_6(s)| &\leq C_p \mathbb{E} \left[ \int_0^{t \wedge \tau_m} \sum_{j=1}^n |X_\varepsilon|_{H^1}^{2p-4} \left( \int |X_\varepsilon|^2 \operatorname{Re} \phi_j d\xi \right)^2 ds \right]^{\frac{1}{2}} \\ &\leq C_p \mathbb{E} \left( \int_0^{t \wedge \tau_m} |X_\varepsilon|_{H^1}^{2p} ds \right)^{\frac{1}{2}} \\ &\leq C_p \delta \mathbb{E} \sup_{s \leq t \wedge \tau_m} |X_\varepsilon|_{H^1}^p + C(p, \delta) \int_0^t \mathbb{E} \sup_{r \leq s \wedge \tau_m} |X_\varepsilon|_{H^1}^p ds, \end{aligned} \quad (3.24)$$

where we used [5, Lemma 3.3] in the last step,  $\delta > 0$ , and  $C_p, C(p, \delta)$  are independent of  $\varepsilon$  and  $m$ .

Similarly,

$$\begin{aligned} \mathbb{E} \sup_{s \leq t \wedge \tau_m} |J_7(s)| &\leq C_p \mathbb{E} \left[ \int_0^{t \wedge \tau_m} \sum_{j=1}^n |X_\varepsilon|_{H^1}^{2p-4} \left( \operatorname{Re} \int \nabla \overline{X_\varepsilon} \nabla (X_\varepsilon \phi_j) d\xi \right)^2 ds \right]^{\frac{1}{2}} \\ &\leq C_p \delta \mathbb{E} \sup_{s \leq t \wedge \tau_m} |X_\varepsilon|_{H^1}^p + C(p, \delta) \int_0^t \mathbb{E} \sup_{r \leq s \wedge \tau_m} |X_\varepsilon|_{H^1}^p ds, \end{aligned} \quad (3.25)$$

where  $C(p, \delta)$  is independent of  $\varepsilon$  and  $m$ .

Therefore, combining (3.23)-(3.25), taking  $\delta$  sufficiently small, and applying the Gronwall inequality we obtain

$$\mathbb{E} \sup_{t \in [0, \tau_m]} |X_\varepsilon(t)|_{H^1}^p \leq C(T, p) < \infty,$$

where  $C(T, p)$  is independent of  $\varepsilon$  and  $m$ . Taking  $m \rightarrow \infty$  and using Fatou's lemma we consequently obtain (3.20).  $\square$

**Proof of Proposition 3.3.** It follows from (3.20) that  $\sup_{t \in [0, \tau_\varepsilon^*(x)]} |e^{W(t)} y_\varepsilon(t)|_{H^1} < \infty$ ,  $\mathbb{P}$ -a.s. Then, since  $e_j \in C_b^\infty$ ,  $1 \leq j \leq n$ ,  $\sup_{t \in [0, \tau_\varepsilon^*(x)]} |y_\varepsilon(t)|_{H^1} < \infty$ ,  $\mathbb{P}$ -a.s., which along with Lemma 3.5 implies the global existence of the solution to (3.1).

Uniqueness for (3.1) follows from monotonicity. Indeed, consider any two solutions  $y_1, y_2$  to (3.1) with the initial datum  $x$ , and set  $X_i = e^W y_i$ ,  $i = 1, 2$ . Then, similarly to (3.21), we derive that

$$\begin{aligned} \mathbb{E}|X_1(t) - X_2(t)|_2^2 &= -4|\lambda| \mathbb{E} \int_0^t |X_1 - X_2|_2^2 ds \\ &\quad + 4\lambda \mathbb{E} \int_0^t \text{Im} \int (\overline{X_1} - \overline{X_2})(X_1 L_\varepsilon(X_1) - X_2 L_\varepsilon(X_2)) d\xi ds. \end{aligned} \tag{3.26}$$

By (3.5),

$$\begin{aligned} &\left| \mathbb{E} \int_0^t \text{Im} \int (\overline{X_1} - \overline{X_2})(X_1 L_\varepsilon(X_1) - X_2 L_\varepsilon(X_2)) d\xi ds \right| \\ &\leq (1 - \varepsilon^2) \mathbb{E} \int_0^t |X_1 - X_2|_2^2 ds \leq \mathbb{E} \int_0^t |X_1 - X_2|_2^2 ds. \end{aligned} \tag{3.27}$$

Then, it follows that

$$\mathbb{E}|X_1(t) - X_2(t)|_2^2 \leq 0,$$

which implies that for each  $t \in [0, T]$ ,  $X_1(t) = X_2(t)$ ,  $\mathbb{P}$ -a.s. Thus, by the continuity of  $y_i$  in  $H^1$ ,  $i = 1, 2$ , we deduce that  $X_1(t) = X_2(t)$ ,  $\forall t \in [0, T]$ ,  $\mathbb{P}$ -a.s., thereby obtaining the uniqueness.  $\square$

In the next subsection, we shall derive some uniform estimates in the Orlicz space, which allows to apply the method of maximal monotone operators to take the limit in the approximating equations.

## 3.2 Uniform estimates

This subsection is mainly devoted to uniform estimates in the Orlicz space  $V$ . Taking into account the definition (1.6), let us begin with the estimate of the entropy function below.

**Lemma 3.7** Fix  $0 < \varepsilon \leq 1$ . Let  $x \in U$ ,  $0 < T < \infty$ ,  $y_\varepsilon$  be the approximating solution in Proposition 3.3 and  $X_\varepsilon = e^W y_\varepsilon$ . We have for any  $p \geq 2$ ,

$$\mathbb{E} \sup_{t \leq T} \left| \int |X_\varepsilon(t)|^2 \log |X_\varepsilon(t)|^2 d\xi \right|^p \leq C(T, p) < \infty, \quad (3.28)$$

where  $C(T, p)$  is independent of  $\varepsilon$ .

**Proof.** For  $u > 0$ , set  $F_m(u) := \int_0^u (L_{1/m}(\nu) + 1) d\nu$ , where  $L_{1/m}(\cdot)$  is as defined in (3.3). Using the techniques as in [17] and [5, Lemma 5.1] we can derive that  $\mathbb{P}$ -a.s., for  $t \in [0, T]$ ,

$$\begin{aligned} & \int F_m(|X_\varepsilon(t)|^2) d\xi \\ = & \int F_m(|x|^2) d\xi - 2 \int_0^t \int g_m(|X_\varepsilon|^2) \text{Im}(\overline{X_\varepsilon} \nabla X_\varepsilon) \text{Re}(\overline{X_\varepsilon} \nabla X_\varepsilon) d\xi ds \\ & - 4|\lambda| \int_0^t \int (L_{1/m}(|X_\varepsilon|^2) + 1) |X_\varepsilon|^2 d\xi ds + \int_0^t \int g_m(|X_\varepsilon|^2) (\text{Re} \phi_j)^2 |X_\varepsilon(s)|^4 d\xi ds \\ & + 2 \sum_{j=1}^n \int_0^t \int (L_{1/m}(|X_\varepsilon|^2) + 1) |X_\varepsilon|^2 \text{Re} \phi_j d\xi d\beta_j(s), \end{aligned} \quad (3.29)$$

where  $g_m(|X_\varepsilon|^2) := 2(1 - m^{-2})(m^{-1} + |X_\varepsilon|^2)^{-1}(1 + m^{-1}|X_\varepsilon|^2)^{-1}$ , and  $\phi_j = \mu_j e_j$ ,  $1 \leq j \leq n$ . (See the Appendix for the proof.)

Then, applying Itô's formula we derive that  $\mathbb{P}$ -a.s. for  $t \in [0, T]$ ,

$$\begin{aligned} & \left( \int F_m(|X_\varepsilon(t)|^2) d\xi \right)^p \\ = & \left( \int F_m(|x|^2) d\xi \right)^p \\ & - 2p \int_0^t \left( \int F_m(|X_\varepsilon|^2) d\xi \right)^{p-1} \left( \int g_m(|X_\varepsilon|^2) \text{Im}(\overline{X_\varepsilon} \nabla X_\varepsilon) \text{Re}(\overline{X_\varepsilon} \nabla X_\varepsilon) d\xi \right) ds \\ & - 4|\lambda| p \int_0^t \left( \int F_m(|X_\varepsilon|^2) d\xi \right)^{p-1} \left( \int (L_{1/m}(|X_\varepsilon|^2) + 1) |X_\varepsilon|^2 d\xi \right) ds \\ & + p \int_0^t \left( \int F_m(|X_\varepsilon|^2) d\xi \right)^{p-1} \left( \int g_m(|X_\varepsilon|^2) (\text{Re} \phi_j)^2 |X_\varepsilon|^4 d\xi \right) ds \\ & + 2p(p-1) \sum_{j=1}^N \int_0^t \left( \int F_m(|X_\varepsilon|^2) d\xi \right)^{p-2} \left( \int (L_{1/m}(|X_\varepsilon|^2) + 1) |X_\varepsilon|^2 \text{Re} \phi_j d\xi \right)^2 ds \end{aligned}$$

$$\begin{aligned}
& + 2p \sum_{j=1}^n \int_0^t \left( \int F_m(|X_\varepsilon|^2) d\xi \right)^{p-1} \left( \int |X_\varepsilon|^2 \text{Re} \phi_j d\xi \right) d\beta_j(s) \\
& + 2p \sum_{j=1}^n \int_0^t \left( \int F_m(|X_\varepsilon|^2) d\xi \right)^{p-1} \left( \int |X_\varepsilon|^2 L_{1/m}(|X_\varepsilon|^2) \text{Re} \phi_j d\xi \right) d\beta_j(s). \\
& = \left( \int F_m(|x|^2) d\xi \right)^p + \sum_{j=1}^6 K_j(t). \tag{3.30}
\end{aligned}$$

Since for  $u > 0$ ,

$$F_m(u) = u L_{1/m}(u) + u - (1 - m^{-2}) \int_0^u \nu(m^{-1} + \nu)^{-1} (1 + m^{-1}\nu)^{-1} d\nu, \tag{3.31}$$

it follows that

$$|F_m(|x|^2)| \leq ||x|^2 L_{1/m}(|x|^2)| + 2|x|^2 \leq ||x|^2 L(|x|^2)| + 2|x|^2 \in L^1(\mathbb{R}^d).$$

Then, since  $F_m(|x|^2) \rightarrow |x|^2 L(|x|^2)$ , as  $m \rightarrow \infty$ , the dominated convergence theorem yields

$$\int F_m(|x|^2) d\xi \rightarrow \int |x|^2 L(|x|^2) d\xi. \tag{3.32}$$

In particular,

$$\sup_{m \geq 1} \left| \int F_m(|x|^2) d\xi \right| \leq C < \infty. \tag{3.33}$$

For the other deterministic terms in (3.30), since  $e_j \in C_b^\infty$ ,  $1 \leq j \leq n$ , and  $g_m(|X_\varepsilon|^2) \leq 2|X_\varepsilon|^{-2}$ , using the Young inequality  $a^{p-1}b \leq \frac{p-1}{p}\delta a^p + \frac{1}{p}\delta^{-(p-1)}b^p$ ,  $a^{p-2}b \leq \frac{p-2}{p}\delta a^p + \frac{2}{p}\delta^{-\frac{p-2}{2}}b^{\frac{p}{2}}$ ,  $\delta > 0$ , and the boundedness of  $H^1$ -norm in (3.8), we derive that  $\mathbb{P}$ -a.s.

$$\begin{aligned}
& \sum_{j=1}^4 \mathbb{E} \sup_{s \leq t} |K_j(s)| \\
& \leq C_p T \delta \mathbb{E} \sup_{s \leq t} \left| \int F_m(|X_\varepsilon|^2) d\xi \right|^p \\
& \quad + C(p, \delta) \mathbb{E} \sup_{s \leq t} \left[ \int_0^s |X_\varepsilon|_{H^1}^{2p} ds + \int_0^s \left( \int |X_\varepsilon|^2 L_{1/m}(|X_\varepsilon|^2) d\xi \right)^p dr \right] \\
& \leq C(T, p, \delta) + C_p T \delta \mathbb{E} \sup_{s \leq t} \left| \int F_m(|X_\varepsilon|^2) d\xi \right|^p \\
& \quad + C(p, \delta) \mathbb{E} \sup_{s \leq t} \int_0^s \left( \int |X_\varepsilon|^2 L_{1/m}(|X_\varepsilon|^2) d\xi \right)^p dr, \quad t \in [0, T], \tag{3.34}
\end{aligned}$$

where  $C_p, C(p, \delta)$  and  $C(T, p, \delta)$  are independent of  $\varepsilon$  and  $m$ .

Moreover, it follows from the Burkholder-Davis-Gundy inequality, the Young inequality  $a^{2p-2}b^2 \leq \frac{p-1}{p}a^{2p} + \frac{1}{p}b^{2p}$ , (3.8) and Lemma 3.3 in [5] that

$$\begin{aligned}
& \mathbb{E} \sup_{s \leq t} |K_5(s)| \\
& \leq C_p \mathbb{E} \left[ \int_0^t \sum_{j=1}^n \left| \int F_m(|X_\varepsilon|^2) d\xi \right|^{2p-2} \left| \int |X_\varepsilon|^2 \operatorname{Re} \phi_j d\xi \right|^2 ds \right]^{\frac{1}{2}} \\
& \leq C_p \mathbb{E} \left[ \int_0^t \left| \int F_m(|X_\varepsilon|^2) d\xi \right|^{2p} \right]^{\frac{1}{2}} + C_p \mathbb{E} \left[ \int_0^t \left( \sum_{j=1}^n \left| \int |X_\varepsilon|^2 \operatorname{Re} \phi_j d\xi \right|^2 \right)^p ds \right]^{\frac{1}{2}} \\
& \leq C(T, p, \delta) + C_p \delta \mathbb{E} \sup_{s \leq t} \left| \int F_m(|X_\varepsilon|^2) d\xi \right|^p + C(p, \delta) \int_0^t \mathbb{E} \sup_{r \leq s} \left| \int F_m(|X_\varepsilon|^2) d\xi \right|^p ds.
\end{aligned} \tag{3.35}$$

Similarly,

$$\begin{aligned}
& \mathbb{E} \sup_{s \leq t} |K_6(s)| \\
& \leq C_p \mathbb{E} \left[ \int_0^t \left| \int F_m(|X_\varepsilon|^2) d\xi \right|^{2p} \right]^{\frac{1}{2}} \\
& \quad + C_p \mathbb{E} \left[ \int_0^t \left( \sum_{j=1}^n \left| \int |X_\varepsilon|^2 L_{1/m}(|X_\varepsilon|^2) \operatorname{Re} \phi_j d\xi \right|^2 \right)^p ds \right]^{\frac{1}{2}} \\
& \leq C_p \delta \mathbb{E} \sup_{s \leq t} \left[ \left| \int F_m(|X_\varepsilon|^2) d\xi \right|^p + \left( \int |X_\varepsilon|^2 L_{1/m}(|X_\varepsilon|^2) d\xi \right)^p \right] \\
& \quad + C(p, \delta) \int_0^t \mathbb{E} \sup_{r \leq s} \left[ \left| \int F_m(|X_\varepsilon|^2) d\xi \right|^p + \left( \int |X_\varepsilon|^2 L_{1/m}(|X_\varepsilon|^2) d\xi \right)^p \right] ds
\end{aligned} \tag{3.36}$$

Thus, it follows from (3.33)-(3.36) that

$$\begin{aligned}
& \mathbb{E} \sup_{s \leq t} \left| \int F_m(|X_\varepsilon(s)|^2) d\xi \right|^p \\
& \leq C(T, p, \delta) + C(T, p) \delta \mathbb{E} \sup_{s \leq t} \left[ \left| \int F_m(|X_\varepsilon(s)|^2) d\xi \right|^p + \left( \int |X_\varepsilon|^2 L_{1/m}(|X_\varepsilon|^2) d\xi \right)^p \right] \\
& \quad + C(T, p, \delta) \int_0^t \mathbb{E} \sup_{r \leq s} \left[ \left| \int F_m(|X_\varepsilon|^2) d\xi \right|^p + \left( \int |X_\varepsilon|^2 L_{1/m}(|X_\varepsilon|^2) d\xi \right)^p \right] ds.
\end{aligned} \tag{3.37}$$

Since by (3.31),

$$\left| \left| \int F_m(|X_\varepsilon|^2) d\xi \right| - \left| \int |X_\varepsilon|^2 L_{1/m}(|X_\varepsilon|^2) d\xi \right| \right| \leq 2|X_\varepsilon|_2^2, \quad (3.38)$$

using (3.8) we obtain

$$\begin{aligned} & \mathbb{E} \sup_{s \leq t} \left| \int |X_\varepsilon|^2 L_{1/m}(|X_\varepsilon|^2) d\xi \right|^p \\ & \leq C(T, p, \delta) + C(T, p) \delta \mathbb{E} \sup_{s \leq t} \left( \int |X_\varepsilon|^2 L_{1/m}(|X_\varepsilon|^2) d\xi \right)^p \\ & \quad + C(T, p, \delta) \int_0^t \mathbb{E} \sup_{r \leq s} \left( \int |X_\varepsilon|^2 L_{1/m}(|X_\varepsilon|^2) d\xi \right)^p ds. \end{aligned} \quad (3.39)$$

Note that  $u^2 L_{1/m}(u^2) \leq u^2 \log u^2 \leq C_\delta(u^2 + u^{2+\delta})$  for  $u > 1$ . The Sobolev imbedding theorem implies that for  $0 < \delta < \frac{4}{d-2}$ ,

$$\int I_{\{|X_\varepsilon| > 1\}} |X_\varepsilon|^2 L_{1/m}(|X_\varepsilon|^2) d\xi \leq C_\delta(|X_\varepsilon|_{H^1}^2 + |X_\varepsilon|_{H^1}^{2+\delta}). \quad (3.40)$$

Hence,

$$\begin{aligned} & \left| \int I_{\{|X_\varepsilon| \leq 1\}} |X_\varepsilon|^2 L_{1/m}(|X_\varepsilon|^2) d\xi \right|^p \\ & = \left| \int |X_\varepsilon|^2 L_{1/m}(|X_\varepsilon|^2) d\xi - \int I_{\{|X_\varepsilon| > 1\}} |X_\varepsilon|^2 L_{1/m}(|X_\varepsilon|^2) d\xi \right|^p \\ & \leq C_p \left| \int |X_\varepsilon|^2 L_{1/m}(|X_\varepsilon|^2) d\xi \right|^p + C(p, \delta)(|X_\varepsilon|_{H^1}^{2p} + |X_\varepsilon|_{H^1}^{(2+\delta)p}). \end{aligned}$$

and

$$\begin{aligned} & \left( \int |X_\varepsilon|^2 L_{1/m}(|X_\varepsilon|^2) d\xi \right)^p \\ & = \left( - \int I_{\{|X_\varepsilon| \leq 1\}} |X_\varepsilon|^2 L_{1/m}(|X_\varepsilon|^2) d\xi + \int I_{\{|X_\varepsilon| > 1\}} |X_\varepsilon|^2 L_{1/m}(|X_\varepsilon|^2) d\xi \right)^p \\ & \leq C_p \left| \int I_{\{|X_\varepsilon| \leq 1\}} |X_\varepsilon|^2 L_{1/m}(|X_\varepsilon|^2) d\xi \right|^p + C(p, \delta)(|X_\varepsilon|_{H^1}^{2p} + |X_\varepsilon|_{H^1}^{(2+\delta)p}). \end{aligned}$$

Therefore, inserting the two estimates above into (3.39) and then using (3.8) we get

$$\begin{aligned} & \mathbb{E} \sup_{s \leq t} \left| \int I_{\{|X_\varepsilon| \leq 1\}} |X_\varepsilon|^2 L_{1/m}(|X_\varepsilon|^2) d\xi \right|^p \\ & \leq C(T, p, \delta) + C(T, p) \delta \mathbb{E} \sup_{s \leq t} \left| \int I_{\{|X_\varepsilon| \leq 1\}} |X_\varepsilon|^2 L_{1/m}(|X_\varepsilon|^2) d\xi \right|^p \\ & \quad + C(T, p, \delta) \int_0^t \mathbb{E} \sup_{r \leq s} \left| \int I_{\{|X_\varepsilon| \leq 1\}} |X_\varepsilon|^2 L_{1/m}(|X_\varepsilon|^2) d\xi \right|^p ds. \end{aligned}$$

Then, taking  $\delta$  sufficiently small and applying Gronwall's inequality we have

$$\mathbb{E} \sup_{t \leq T} \left| \int I_{\{|X_\varepsilon| \leq 1\}} |X_\varepsilon(t)|^2 L_{1/m}(|X_\varepsilon(t)|^2) d\xi \right|^p \leq C(T, p),$$

where  $C(T, p)$  is independent of  $\varepsilon$  and  $m$ . Hence, by Fatou's lemma,

$$\mathbb{E} \sup_{t \leq T} \left| \int I_{\{|X_\varepsilon| \leq 1\}} |X_\varepsilon(t)|^2 L(|X_\varepsilon(t)|^2) d\xi \right|^p \leq C(T, p), \quad (3.41)$$

Consequently, (3.28) follows immediately from (3.41), (3.40) and (3.8). Hence, the proof is complete.  $\square$

**Proof of Proposition 3.2.** By Proposition 3.3, there exists a unique  $(\mathcal{F}_t)$ -adapted solution  $y_\varepsilon$  to (3.1), and  $y_\varepsilon \in C([0, T]; H^1)$ ,  $\mathbb{P}$ -a.s. Moreover, since  $u \mapsto uL_\varepsilon(e^W u)$  is Lipschitz on  $L^2$  and  $L^2 \subset U'$ , we obtain  $\mathcal{G}_\varepsilon(y_\varepsilon) \in C([0, T]; U')$ ,  $\mathbb{P}$ -a.s.

It remains to prove (3.6) and (3.7). For the proof of (3.6), in view of (3.8), we only need to prove that for any  $p \geq 2$ ,

$$\mathbb{E} \sup_{0 \leq t \leq T} \|X_\varepsilon(t)\|_V^p \leq C(T, p) < \infty, \quad (3.42)$$

where  $X_\varepsilon := e^W y_\varepsilon$ , and  $V$  is the Orlicz space defined in (1.6).

To this end, set  $B(u) := -u^2 \log u^2 - N(u)$ , where  $u > 0$ , and  $N$  is as defined in (1.5). It follows from (2.6) in [11] and the inequality  $ab \leq a^2 + b^2$  that

$$\left| \int B(|X_\varepsilon(s)|) d\xi \right| \leq C |X_\varepsilon|_2 |X_\varepsilon|_{H^1}^{\frac{d}{d-2}} \leq C(|X_\varepsilon|_2^2 + |X_\varepsilon|_{H^1}^{\frac{2d}{d-2}}).$$

Then

$$\int N(|X_\varepsilon|) d\xi \leq \left| \int |X_\varepsilon|^2 \log |X_\varepsilon|^2 d\xi \right| + C(|X_\varepsilon|_2^2 + |X_\varepsilon|_{H^1}^{\frac{2d}{d-2}}).$$

Hence by (3.8) and Lemma 3.7,

$$\mathbb{E} \sup_{0 \leq t \leq T} \left| \int N(|X_\varepsilon|) d\xi \right|^p \leq C(T, p) < \infty,$$

which along with (1.8) implies (3.42), thereby proving (3.6).

As regards (3.7), we note that

$$\begin{aligned} & \|e^W \mathcal{G}_\varepsilon(y_\varepsilon)\|_{L^p(\Omega; C([0, T]; U'))} \\ & \leq 2|\lambda| \|X_\varepsilon\|_{L^p(\Omega; C([0, T]; L^2))} + 2|\lambda| \|I_{\{|X_\varepsilon| > e^{-3}\}} X_\varepsilon L_\varepsilon(X_\varepsilon)\|_{L^p(\Omega; C([0, T]; L^2))} \\ & \quad + 2|\lambda| \|I_{\{|X_\varepsilon| \leq e^{-3}\}} X_\varepsilon L_\varepsilon(X_\varepsilon)\|_{L^p(\Omega; C([0, T]; V'))} \end{aligned}$$

Since for  $\xi \in \{|X_\varepsilon| > e^{-3}\}$ ,

$$|X_\varepsilon(\xi) L_\varepsilon(X_\varepsilon(\xi))| \leq |X_\varepsilon(\xi) L(X_\varepsilon(\xi))| \leq C_\delta (|X_\varepsilon(\xi)| + |X_\varepsilon(\xi)|^{1+\delta}), \quad (3.43)$$

by the Sobolev imbedding theorem and (3.6), it follows that for  $0 \leq \delta \leq \frac{2}{d-2}$ ,

$$\begin{aligned} & \|X_\varepsilon\|_{L^p(\Omega; C([0, T]; L^2))} + \|I_{\{|X_\varepsilon| > e^{-3}\}} X_\varepsilon L_\varepsilon(X_\varepsilon)\|_{L^p(\Omega; C([0, T]; L^2))} \\ & \leq C(\|X_\varepsilon\|_{L^p(\Omega; C([0, T]; L^2))} + \|X_\varepsilon\|_{L^{(1+\delta)p}(\Omega; C([0, T]; L^{2(1+\delta)}))}^{1+\delta}) \\ & \leq C(\|X_\varepsilon\|_{L^p(\Omega; C([0, T]; L^2))} + \|X_\varepsilon\|_{L^{(1+\delta)p}(\Omega; C([0, T]; H^1))}^{1+\delta}) \leq C(T, p) < \infty, \end{aligned}$$

where  $C(T, p)$  is independent of  $\varepsilon$ .

Moreover, since  $\tilde{N}$  is increasing, by Lemma 3.1 (i), similarly to (2.17) we have

$$\begin{aligned} \int \tilde{N}(-2I_{\{|X_\varepsilon| \leq e^{-3}\}} X_\varepsilon L_\varepsilon(X_\varepsilon)) d\xi & \leq \int I_{\{|X_\varepsilon| \leq e^{-3}\}} \tilde{N}(-X_\varepsilon L(|X_\varepsilon|^2)) d\xi \\ & \leq 2 \max\{\|X_\varepsilon\|_V, \|X_\varepsilon\|_V^2\}. \end{aligned}$$

Then, as in (2.20),

$$\begin{aligned} & \|I_{\{|X_\varepsilon| \leq e^{-3}\}} X_\varepsilon L_\varepsilon(X_\varepsilon)\|_{L^p(\Omega; C([0, T]; V'))} \\ & \leq C_T (\|X_\varepsilon\|_{L^{2p}(\Omega; C([0, T]; V))}^2 + 1) \leq C(T, p) < \infty, \end{aligned}$$

where the last step is due to (3.6), thereby proving (3.7). The proof of Proposition 3.2 is now complete.  $\square$

## 4 Proof of Theorem 2.2

Let us start with the lemma below.



**Lemma 4.1** *Let  $L_\varepsilon$  be defined as in (3.3) and  $p \geq 3$ . For any  $X \in L^p(\Omega \times (0, T); U)$ ,*

$$\|XL_\varepsilon(X) - XL(X)\|_{L^{p'}(\Omega \times (0, T); U')} \rightarrow 0, \quad \text{as } \varepsilon \rightarrow 0. \quad (4.1)$$

**Proof.** First note that  $XL_\varepsilon(X) \rightarrow XL(X)$  a.e., as  $\varepsilon \rightarrow 0$ , and

$$\begin{aligned} & \|XL_\varepsilon(X) - XL(X)\|_{L^{p'}(\Omega \times (0, T); U')} \\ & \leq \|I_{\{|X| \leq e^{-3}\}}(XL_\varepsilon(X) - XL(X))\|_{L^{p'}(\Omega \times (0, T); V')} \\ & \quad + \|I_{\{|X| > e^{-3}\}}(XL_\varepsilon(X) - XL(X))\|_{L^{p'}(\Omega \times (0, T); H^{-1})}. \end{aligned} \quad (4.2)$$

Since  $|XL_\varepsilon(X)| \leq |XL(X)|$ , and as in the proof of (2.15), the Sobolev imbedding theorem and the Hölder inequality imply that for  $0 < \delta \leq \min\{\frac{2}{d-2}, p-2\}$ ,

$$\begin{aligned} & \|I_{\{|X| > e^{-3}\}}XL(X)\|_{L^{p'}(\Omega \times (0, T); H^{-1})} \\ & \leq C(\|X\|_{L^{p'}(\Omega \times (0, T); L^2)} + \|X\|_{L^{(1+\delta)p'}(\Omega \times (0, T); L^{2(1+\delta)}})^{1+\delta} \\ & \leq C(T, p)(\|X\|_{L^p(\Omega \times (0, T); H^1)} + \|X\|_{L^p(\Omega \times (0, T); H^1)}^{1+\delta}) < \infty, \end{aligned} \quad (4.3)$$

the dominated convergence theorem implies that, as  $\varepsilon \rightarrow 0$ ,

$$\begin{aligned} & \|I_{\{|X| > e^{-3}\}}(XL_\varepsilon(X) - XL(X))\|_{L^{p'}(\Omega \times (0, T); H^{-1})} \\ & \leq \|I_{\{|X| > e^{-3}\}}(XL_\varepsilon(X) - XL(X))\|_{L^{p'}(\Omega \times (0, T); L^2)} \rightarrow 0. \end{aligned} \quad (4.4)$$

For the last term in the right hand side of (4.2), note that since  $\tilde{N}$  is increasing, by Lemma 3.1 (i) and (2.16)

$$\tilde{N}(-2|X(\xi)|L_\varepsilon(X(\xi))) \leq \tilde{N}(-|X(\xi)|L(|X(\xi)|^2)) \leq 2N(|X(\xi)|). \quad (4.5)$$

Moreover, by (1.8) and Hölder's inequality

$$\left\| \int N(|X(\xi)|)d\xi \right\|_{L^{p'}(\Omega \times (0, T))} \leq C_T(\|X\|_{L^p(\Omega \times (0, T); V)}^2 + 1) < \infty,$$

which implies that  $N(|X|) \in L^1(\mathbb{R}^d)$ ,  $\mathbb{P} \otimes dt$ -a.e. Then, it follows from the dominated convergence theorem that  $\mathbb{P} \otimes dt$ -a.e.

$$\int \tilde{N}(-2XL_\varepsilon(X)I_{\{|X| \leq e^{-3}\}})d\xi \rightarrow \int \tilde{N}(-2XL(X)I_{\{|X| \leq e^{-3}\}})d\xi,$$

which yields by [11, (2.8)] that

$$-XL_\varepsilon(X)I_{\{|X| \leq e^{-3}\}} \rightarrow -XL(X)I_{\{|X| \leq e^{-3}\}}, \quad \text{in } V'.$$

Since by Lemma 3.1 (i),  $\| -XL_\varepsilon(X)I_{\{|X|\leq e^{-3}\}} \|_{V'} \leq \| -XL(X)I_{\{|X|\leq e^{-3}\}} \|_{V'}$ , and as in (2.20), we have  $\| -XL(X)I_{\{|X|\leq e^{-3}\}} \|_{V'} \in L^{p'}(\Omega \times (0, T))$ . Again, we apply the dominated convergence theorem and get

$$\|I_{\{|X|\leq e^{-3}\}}(XL_\varepsilon(X) - XL(X))\|_{L^{p'}(\Omega \times (0, T); V')} \rightarrow 0, \quad \varepsilon \rightarrow 0. \quad (4.6)$$

Consequently, (4.1) follows from (4.2), (4.4) and (4.6). The proof of Lemma 4.1 is thus complete.  $\square$

**Proof of Theorem 2.2.** For any  $p \geq 2$ , by the uniform estimates (3.6) and (3.7), we have along a subsequence  $\{\varepsilon_n\} \rightarrow 0$ ,

$$e^W y_{\varepsilon_n} \xrightarrow{\omega^*} e^W \tilde{y}, \text{ in } L^p(\Omega; L^\infty(0, T; U)), \quad (4.7)$$

$$e^W \mathcal{G}_{\varepsilon_n}(y_{\varepsilon_n}) \xrightarrow{\omega^*} e^W \eta, \text{ in } L^p(\Omega; L^\infty(0, T; U')), \quad (4.8)$$

where  $\xrightarrow{\omega^*}$  stands for weak-star convergence.

In particular,  $e^W \tilde{y} \in L^\infty(0, T; U)$ , and  $e^W \eta \in L^\infty(0, T; U')$ ,  $\mathbb{P}$ -a.s. Since by Hypothesis (H), for any  $u \in U$ , we have  $\|e^{-W}u\|_U \leq c(t)\|u\|_U$ , where  $c(t) = \sqrt{2}(|e^{-W(t)}|_{L^\infty} + |\nabla e^{-W(t)}|_{L^\infty})$ . It follows that

$$\tilde{y} \in L^\infty(0, T; U), \quad \eta \in L^\infty(0, T; U'), \quad \mathbb{P} - a.s.$$

Moreover, for any  $p \geq 3$ , since  $L^p(\Omega; L^\infty(0, T; U)) \subset L^p(\Omega \times (0, T); U)$  and  $L^p(\Omega; L^\infty(0, T; U')) \subset L^{p'}(\Omega \times (0, T); U')$ , we have (selecting a further subsequence if necessary)

$$y_{\varepsilon_n} \xrightarrow{\omega} \tilde{y}, \text{ in } \mathcal{U}, \quad (4.9)$$

$$\mathcal{G}_{\varepsilon_n}(y_{\varepsilon_n}) \xrightarrow{\omega} \eta, \text{ in } \mathcal{U}'. \quad (4.10)$$

where  $\xrightarrow{\omega}$  means weak convergence.

We next take the limit in the approximating equation (3.1). Set

$$F_n(y_{\varepsilon_n})(t) := ie^{-W(t)}\Delta(e^{W(t)}y_{\varepsilon_n}(t)) + (4i\lambda|\lambda|t + \widehat{\mu})y_{\varepsilon_n}(t) + \mathcal{G}_{\varepsilon_n}(y_{\varepsilon_n}(t)),$$

and

$$F(\tilde{y})(t) := ie^{-W(t)}\Delta(e^{W(t)}\tilde{y}(t)) + (4i\lambda|\lambda|t + \widehat{\mu})\tilde{y}(t) + \eta(t),$$

where  $t \in [0, T]$ . Then, from (4.9) and (4.10) it follows that  $F_n(y_{\varepsilon_n}) \xrightarrow{\omega} F(\tilde{y})$ , in  $\mathcal{U}'$ . Thus, for any  $u \in U$ ,  $\varphi \in L^\infty([0, T] \times \Omega)$ , by (3.1)

$$\begin{aligned} & \mathbb{E} \int_0^T \int_U \langle e^{W(t)} \tilde{y}(t), \varphi(t)u \rangle dt \\ &= \lim_{n \rightarrow \infty} \mathbb{E} \int_0^T \int_U \langle e^{W(t)} \tilde{y}_{\varepsilon_n}(t), \varphi(t)u \rangle dt \\ &= \mathbb{E} \int_0^T \langle e^{W(t)} x, \varphi(t)u \rangle_H dt - \lim_{n \rightarrow \infty} \mathbb{E} \int_0^T \int_U \langle F_n(y_{\varepsilon_n})(s), \overline{e^{W(t)}} \varphi(t)u \rangle ds dt. \end{aligned}$$

Note that, the second term in the right hand side above is equal to

$$\begin{aligned} & \lim_{n \rightarrow \infty} \mathbb{E} \int_0^T \int_U \left\langle e^{W(s)} F_n(y_{\varepsilon_n})(s), \int_s^T \overline{e^{W(t)-W(s)}} \varphi(t)u dt \right\rangle ds \\ &= \mathbb{E} \int_0^T \int_U \left\langle e^{W(s)} F(\tilde{y})(s), \int_s^T \overline{e^{W(t)-W(s)}} \varphi(t)u dt \right\rangle ds \\ &= \mathbb{E} \int_0^T \int_U \left\langle e^{W(t)} \int_0^t F(\tilde{y})(s) ds, \varphi(t)u \right\rangle dt. \end{aligned}$$

It follows that for any  $u \in U$ ,  $\varphi \in L^\infty([0, T] \times \Omega)$ ,

$$\mathbb{E} \int_0^T \int_U \langle e^{W(t)} \tilde{y}(t), \varphi(t)u \rangle dt = \mathbb{E} \int_0^T \int_U \left\langle e^{W(t)} \left( x - \int_0^t F(\tilde{y})(s) ds \right), \varphi(t)u \right\rangle dt.$$

Thus,  $\tilde{y} = x - \int_0^\cdot F(\tilde{y})(s) ds$ , in  $\mathcal{U}'$ .

Set  $y(t) := x - \int_0^t F(\tilde{y})(s) ds$ ,  $t \in [0, T]$ . Then,  $y \in AC([0, T]; U')$ ,  $\mathbb{P}$ -a.s., and  $y = \tilde{y}$  in  $\mathcal{U}'$ , which implies that  $y = \tilde{y}$ ,  $\mathbb{P} \otimes dt$ -a.e. Since for each  $t \in [0, T]$ ,  $\int_0^t F(y)(s) ds = \int_0^t F(\tilde{y})(s) ds$ ,  $\mathbb{P}$ -a.s., by the continuity of  $t \mapsto \int_0^t F(y)(s) ds - \int_0^t F(\tilde{y})(s) ds$ , we thus have that  $\mathbb{P}$ -a.s. for all  $t \in [0, T]$ ,  $\int_0^t F(y)(s) ds = \int_0^t F(\tilde{y})(s) ds$ , which yields that  $y(t) = \tilde{y}(t)$ , for all  $t \in [0, T]$ ,  $\mathbb{P}$ -a.s.

Therefore,

$$\begin{aligned} y(t) &= x - \int_0^t (ie^{-W(s)} \Delta(e^{W(s)} y(s)) + (4i\lambda|\lambda|s + \hat{\mu})y(s) + \eta(s)) ds, \\ &\text{for all } t \in [0, T], \mathbb{P} - \text{a.s.} \end{aligned} \quad (4.11)$$

Moreover, taking into account (4.11), (4.9) and (4.10), we have  $y \in W^{1,p'}(0, T; U') \cap L^p([0, T]; U)$ ,  $\mathbb{P}$ -a.s., which implies that  $y \in C([0, T]; H)$ ,  $\mathbb{P}$ -a.s.

In order to prove that  $y$  is a solution to (2.8), we need to show that

$$\eta = \mathcal{G}(y). \quad (4.12)$$

For this purpose, it suffices to prove that

$$\limsup_{n \rightarrow \infty} \int_0^T \operatorname{Re} \mathcal{U}'_t \langle \mathcal{G}_{\varepsilon_n}(y_{\varepsilon_n}), y_{\varepsilon_n} \rangle_{\mathcal{U}_t} dt \leq \int_0^T \operatorname{Re} \mathcal{U}'_t \langle \eta, y \rangle_{\mathcal{U}_t} dt, \quad (4.13)$$

where  $\mathcal{U}_t$  and  $\mathcal{U}'_t$  are defined as in (2.5) and (2.6) respectively, but with  $T$  replaced by  $t$ . Indeed, by the monotonicity of  $\mathcal{G}_{\varepsilon_n}$ , for any positive function  $\varphi \in C([0, T])$ ,

$$\int_0^T \operatorname{Re} \mathcal{U}'_t \langle \mathcal{G}_{\varepsilon_n}(y_{\varepsilon_n}) - \mathcal{G}_{\varepsilon_n}(u), y_{\varepsilon_n} - u \rangle_{\mathcal{U}_t} \varphi(t) dt \geq 0, \quad u \in \mathcal{U}.$$

Then, it follows from (4.1), (4.9) and (4.10) that

$$\limsup_{n \rightarrow \infty} \operatorname{Re} \mathcal{U}'_t \langle \mathcal{G}_{\varepsilon_n}(y_{\varepsilon_n}) - \mathcal{G}_{\varepsilon_n}(u), y_{\varepsilon_n} - u \rangle_{\mathcal{U}_t} \leq \operatorname{Re} \mathcal{U}'_t \langle \eta - \mathcal{G}(u), y - u \rangle_{\mathcal{U}_t}.$$

Moreover,

$$\begin{aligned} & |\operatorname{Re} \mathcal{U}'_t \langle \mathcal{G}_{\varepsilon_n}(y_{\varepsilon_n}) - \mathcal{G}_{\varepsilon_n}(u), y_{\varepsilon_n} - u \rangle_{\mathcal{U}_t}| \\ & \leq \sup_{n \geq 1} (\|\mathcal{G}_{\varepsilon_n}(y_{\varepsilon_n})\|_{\mathcal{U}'} + \|\mathcal{G}_{\varepsilon_n}(y)\|_{\mathcal{U}'}) (\|y_{\varepsilon_n}\|_{\mathcal{U}'} + \|y\|_{\mathcal{U}'}) < \infty. \end{aligned}$$

Hence, by Fatou's lemma,

$$\begin{aligned} 0 & \leq \limsup_{n \rightarrow \infty} \int_0^T \operatorname{Re} \mathcal{U}'_t \langle \mathcal{G}_{\varepsilon_n}(y_{\varepsilon_n}) - \mathcal{G}_{\varepsilon_n}(u), y_{\varepsilon_n} - u \rangle_{\mathcal{U}_t} \varphi(t) dt \\ & \leq \int_0^T \limsup_{n \rightarrow \infty} \operatorname{Re} \mathcal{U}'_t \langle \mathcal{G}_{\varepsilon_n}(y_{\varepsilon_n}) - \mathcal{G}_{\varepsilon_n}(u), y_{\varepsilon_n} - u \rangle_{\mathcal{U}_t} \varphi(t) dt \\ & = \int_0^T \operatorname{Re} \mathcal{U}'_t \langle \eta - \mathcal{G}(u), y - u \rangle_{\mathcal{U}_t} \varphi(t) dt. \end{aligned}$$

As the integrand is continuous in  $t$ , and  $\varphi$  is an arbitrary positive continuous function, we deduce that,

$$\operatorname{Re} \mathcal{U}' \langle \eta - \mathcal{G}(u), y - u \rangle_{\mathcal{U}} \geq 0,$$

which implies (4.12) by the maximal monotonicity of  $\mathcal{G}$ .

For the proof of (4.13), we note that by (3.1) we have, via Itô's formula,

$$\begin{aligned} \int_0^T \mathbb{E}|e^{W(t)}y_{\varepsilon_n}(t)|_2^2 dt &= |x|_2^2 T - 4|\lambda| \int_0^T \mathbb{E} \int_0^t |y_{\varepsilon_n}(s)|_2^2 ds dt \\ &= |x|_2^2 T - 2 \int_0^T \operatorname{Re} u_t' \langle \mathcal{G}_{\varepsilon_n}(y_{\varepsilon_n}), y_{\varepsilon_n} \rangle_{\mathcal{U}_t} dt. \end{aligned} \quad (4.14)$$

Moreover, as in the proof of [3, Lemma 8.1], applying Itô's formula to (4.11) we derive

$$\int_0^T \mathbb{E}|e^{W(t)}y(t)|_2^2 dt = |x|_2^2 T - 2 \int_0^T \operatorname{Re} u_t' \langle \eta, y \rangle_{\mathcal{U}_t} dt. \quad (4.15)$$

Thus, by (4.15), (4.10) and (4.14) we derive that

$$\begin{aligned} \int_0^T \operatorname{Re} u_t' \langle \eta, y \rangle_{\mathcal{U}_t} dt &= -\frac{1}{2} \int_0^T \mathbb{E}|e^{W(t)}y(t)|_2^2 dt + \frac{1}{2}|x|_2^2 T \\ &\geq \limsup_{n \rightarrow \infty} \left( -\frac{1}{2} \int_0^T \mathbb{E}|e^{W(t)}y_{\varepsilon_n}(t)|_2^2 dt + \frac{1}{2}|x|_2^2 T \right) \\ &= \limsup_{n \rightarrow \infty} \int_0^T \operatorname{Re} u_t' \langle \mathcal{G}_{\varepsilon_n}(y_{\varepsilon_n}), y_{\varepsilon_n} \rangle_{\mathcal{U}_t} dt, \end{aligned}$$

which yields (4.13) as claimed, thereby proving (4.12).

Therefore,  $y$  is a solution to (2.8) in the sense of Definition 2.1. Moreover, the estimates (2.9)-(2.11) follow immediately from (4.7), (4.8) and (4.11).

It is left to prove the uniqueness, which follows from the monotonicity. In fact, given any two solutions  $y_1, y_2$  to (2.8), setting  $X_i = e^W y_i$ ,  $i = 1, 2$ , by the Ito formula, we obtain similar formula as in (3.26) but with  $\varepsilon = 0$ . Thus, it follows from (3.5) with  $\varepsilon = 0$  and similar arguments as those below (3.26) that  $X_1(t) = X_2(t)$ ,  $\forall t \in [0, T]$ ,  $\mathbb{P}$ -a.s. The proof of Theorem 2.2 is, therefore, complete.  $\square$

## 5 Appendix

**Proof of Lemma 3.1.** (i). First note that, for each  $0 < \varepsilon < 1$  fixed,

$$\frac{d}{du} L_\varepsilon(u) = \frac{1 - \varepsilon^2}{(\varepsilon + u)(1 + \varepsilon u)} \geq 0, \quad u > 0,$$

which implies that  $L_\varepsilon(u)$  is increasing with  $u$ , and so  $|L_\varepsilon(u)| \leq |\log \varepsilon|$ .

Similarly, for each  $u > 0$  fixed,

$$\frac{d}{d\varepsilon}L_\varepsilon(u) = \frac{1 - u^2}{(\varepsilon + u)(1 + \varepsilon u)}, \quad u > 0,$$

which yields that  $\varepsilon \mapsto L_\varepsilon(u)$  is increasing with  $\varepsilon$  if  $u \in [0, 1]$ , but decreasing if  $u \in [1, \infty)$ . Hence, for  $u \in [0, 1]$ , we have  $\log u \leq L_\varepsilon(u) \leq 0$ , and for  $u \in [1, \infty)$ ,  $0 \leq L_\varepsilon(u) \leq \log u$ . Therefore, we obtain for all  $u > 0$ ,  $|uL_\varepsilon(u)| \leq |uL(u)|$ .

(ii). We may assume  $0 < |u_2| \leq |u_1|$  without loss of generality. Note that

$$u_1L_\varepsilon(u_1) - u_2L_\varepsilon(u_2) = u_2(L_\varepsilon(u_1) - L_\varepsilon(u_2)) + (u_1 - u_2)L_\varepsilon(u_1).$$

Since  $|L_\varepsilon(u_1)| \leq |\log \varepsilon|$ , and

$$\begin{aligned} |L_\varepsilon(u_1) - L_\varepsilon(u_2)| &\leq \frac{1 + \varepsilon|u_2|}{|u_2| + \varepsilon} \left| \frac{|u_1| + \varepsilon}{1 + \varepsilon|u_1|} - \frac{|u_2| + \varepsilon}{1 + \varepsilon|u_2|} \right| \\ &= \left| \frac{(1 - \varepsilon^2)(|u_1| - |u_2|)}{(|u_2| + \varepsilon)(1 + \varepsilon|u_1|)} \right| \\ &\leq (1 - \varepsilon^2)|u_2|^{-1}|u_1 - u_2|, \end{aligned} \tag{5.1}$$

we obtain immediately (3.4).

(iii). We assume  $0 < |u_2| \leq |u_1|$  without loss of generality. Note that

$$\operatorname{Im}(\overline{u_1} - \overline{u_2})(u_1L_\varepsilon(u_1) - u_2L_\varepsilon(u_2)) = (\operatorname{Im}(\overline{u_1}u_2))(L_\varepsilon(u_1) - L_\varepsilon(u_2)),$$

and

$$|\operatorname{Im}(\overline{u_1}u_2)| = \left| \frac{u_2(\overline{u_1} - \overline{u_2}) + \overline{u_2}(u_2 - u_1)}{2i} \right| \leq |u_2||u_1 - u_2|.$$

Thus, taking into account (5.1) we obtain (3.5).  $\square$

**Proof of Lemma 3.4.** This lemma follows essentially from [14, 15]. Using the notations in [15], we reformulate (3.9) in form

$$(\partial_t + i\Delta + \sum_{j=1}^d b^j D_j + c)y = f,$$

meant in the weak sense, where  $D_j = -i\partial_j$ ,  $b^j = -2\partial_j W$ , and  $c = i \sum_{j=1}^d (\partial_j W)^2 + i\Delta W + 2|\lambda| + 4i\lambda|\lambda|t + \widehat{\mu}$ .

Since for each  $1 \leq m \leq n$ ,  $e_m \in C_b^\infty$  and  $\beta(\cdot)$  is continuous,  $\mathbb{P}$ -a.s., we have  $b^j, c \in C_\omega([0, T]; \mathcal{B}^\infty)$ , where  $\mathcal{B}^\infty$  is as defined in Hypothesis (H), and  $C_\omega([0, T]; \mathcal{B}^\infty) = \{g \in C([0, T]; C^\infty), \{g(t, \cdot)\}_{0 \leq t \leq T} \text{ is uniformly bounded in } \mathcal{B}^\infty.\}$

Moreover, under Hypothesis (H),

$$|\operatorname{Re} b(t, \xi)| \leq \left( 2 \sum_{m=1}^n |\mu_m| \sup_{t \leq T} |\beta_m(t)| \right) \lambda(|\xi|).$$

Hence, the conditions in [15, Theorem 1.1] are verified, and we obtain the existence and uniqueness of the evolution operators  $U(t, s)$ .

Furthermore, as remarked by the author in [15], the results in [14] holds also for the time-dependent coefficients. Thus, similarly to [14, (1.6)], we have the estimates (3.10) and (3.11).

Finally, the measurabilities of the processes  $U(\cdot, s)x$  and  $C_t$ ,  $t \geq 0$ , can be proved similarly as in the proof of Lemma 3 and Lemma 4 in [4] (see also [25, Lemma 1.2.1, Lemma 1.2.3]). The proof is now complete.  $\square$

**Proof of (3.29).** Since the nonlinearity  $X_\varepsilon L_\varepsilon(X_\varepsilon) \in L^2 \subset H^{-1}$ , we can use similar arguments as in the proof of [5, Lemma 2.4, Proposition 6.1] to derive that  $X_\varepsilon := e^W y_\varepsilon$  satisfies  $\mathbb{P}$ -a.s. for all  $t \in [0, T]$ ,

$$\begin{aligned} X_\varepsilon(t) = & x - i \int_0^t \Delta X_\varepsilon ds - 2\lambda i \int_0^t X_\varepsilon L_\varepsilon(X_\varepsilon) ds \\ & - \int_0^t (4i\lambda|\lambda|s + 2|\lambda| + \mu) X_\varepsilon ds + \sum_{j=1}^n \int_0^t X_\varepsilon \phi_j d\beta_j(s), \end{aligned} \quad (5.2)$$

where the equation is taken in  $H^{-1}$ .

Proceeding as in [17] and [5], we set  $h_\delta = h * \psi_\delta$  for any locally integrable function  $h$  mollified by  $\psi_\delta$ , where  $\psi_\delta = \delta^{-d} \psi(\frac{x}{\delta})$  and  $\psi$  is a real-valued, nonnegative, compactly supported smooth function with unit integral.

Taking convolution of both sides of (5.2) with the mollifiers  $\psi_\delta$ , we have for each  $\xi \in \mathbb{R}^d$  that

$$\begin{aligned} (X_\varepsilon(t))_\delta(\xi) = & -i \int_0^t \Delta X_{\varepsilon, \delta}(\xi) ds - 2\lambda i \int_0^t (X_\varepsilon L_\varepsilon(X_\varepsilon))_\delta(\xi) ds \\ & - \int_0^t (4i\lambda|\lambda|s + 2|\lambda|) X_{\varepsilon, \delta}(\xi) ds - \int_0^t (\mu X_\varepsilon)_\delta(\xi) ds \\ & + \sum_{j=1}^n \int_0^t (X_\varepsilon \phi_j)_\delta(\xi) d\beta_j(s), \quad t \in [0, T], \end{aligned} \quad (5.3)$$

where  $X_{\varepsilon,\delta} = (X_\varepsilon)_\delta$ , and (5.3) holds on a set  $\Omega_\xi \in \mathcal{F}$  with  $\mathbb{P}(\Omega_\xi) = 1$ .

Since for any locally integrable function  $h$ ,  $h_\delta(\xi)$  is continuous in  $\xi$ , using the boundedness of the  $H^1$ -norm in (3.8) and similar arguments as in the proof of [5, Lemma 5.1] and [25, Lemma 2.3.11], we can prove the continuity in  $\xi$  of all terms in (5.3). Thus, (5.3) holds on a full probability set  $\tilde{\Omega} \in \mathcal{F}$ , which is independent of  $\xi \in \mathbb{R}^d$ . For simplicity, below we omit the argument  $\xi$  in (5.3).

Now, applying Itô's formula to the real valued function  $F_m(|X_{\varepsilon,\delta}|^2)$ , then integrating over  $\mathbb{R}^d$ , interchanging the integrals and integrating by parts, we obtain

$$\begin{aligned}
& \int F_m(X_{\varepsilon,\delta}(t)) d\xi \\
&= \int F_m(x) d\xi - 2 \int_0^t \int g_m(|X_{\varepsilon,\delta}|^2) \operatorname{Re}(\overline{X_{\varepsilon,\delta}} \nabla X_{\varepsilon,\delta}) \operatorname{Im}(\overline{X_{\varepsilon,\delta}} \nabla X_{\varepsilon,\delta}) d\xi ds \\
&\quad + 4\lambda \operatorname{Im} \int_0^t \int (L_{1/m}(|X_{\varepsilon,\delta}|^2) + 1) \overline{X_{\varepsilon,\delta}} (X_\varepsilon L_\varepsilon(X_\varepsilon))_\delta d\xi ds \\
&\quad - 4|\lambda| \int_0^t \int (L_{1/m}(|X_{\varepsilon,\delta}|^2) + 1) |X_{\varepsilon,\delta}|^2 d\xi ds \\
&\quad - 2 \int_0^t \int (L_{1/m}(|X_{\varepsilon,\delta}|^2) + 1) \operatorname{Re}(\overline{X_{\varepsilon,\delta}} (\mu X_\varepsilon)_\delta) d\xi ds \\
&\quad + \sum_{j=1}^n \int_0^t \int (L_{1/m}(|X_{\varepsilon,\delta}|^2) + 1) |(X_\varepsilon \phi_j)_\delta|^2 d\xi ds \\
&\quad + \sum_{j=1}^n \int_0^t \int g_m(|X_{\varepsilon,\delta}|^2) (\operatorname{Re}(\overline{X_{\varepsilon,\delta}} (X_\varepsilon \phi_j)_\delta))^2 d\xi ds \\
&\quad + 2 \sum_{j=1}^n \int_0^t \int (L_{1/m}(|X_{\varepsilon,\delta}|^2) + 1) \operatorname{Re}(\overline{X_{\varepsilon,\delta}} (X_\varepsilon \phi_j)_\delta) d\xi d\beta_j(s),
\end{aligned}$$

where  $g_m(|X_{\varepsilon,\delta}|^2) := 2(1 - \frac{1}{m^2})(\frac{1}{m} + |X_{\varepsilon,\delta}|^2)^{-1}(1 + \frac{1}{m}|X_{\varepsilon,\delta}|^2)^{-1}$ . (Note that, since  $|L_{1/m}(|X_{\varepsilon,\delta}|^2)| \leq \log m$ , we can use the (stochastic) Fubini theorem to interchange the integrals.)

Therefore, since  $|L_{1/m}(|X_{\varepsilon,\delta}|^2)| \leq \log m$ ,  $|g_m(|X_{\varepsilon,\delta}|^2)| \leq 2|X_{\varepsilon,\delta}|^{-2}$ , and  $h_\delta \rightarrow h$  in  $L^q$ , for any  $h \in L^q$ ,  $q > 1$ , using the boundedness of the  $H^1$ -norm in (3.8) and the generalized dominated convergence theorem, we can take the limit  $\delta \rightarrow 0$  above and consequently obtain (3.29).  $\square$



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