Stochastic Porous Media Equation on General Measure Spaces with Increasing Lipschitz Nonlinearties*

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Abstract. We prove the existence and uniqueness of probabilistically strong solutions to stochastic porous media equations driven by time-dependent multiplicative noise on a general measure space $(E, \mathscr{B}(E), \mu)$, and the Laplacian replaced by a self-adjoint operator L. In the case of Lipschitz nonlinearities Ψ, we in particular generalize previous results for open $E \subset \mathbb{R}^d$ and L=Laplacian to fractional Laplacians. We also generalize known results on general measure spaces, where we succeeded in dropping the transience assumption on L, in extending the set of allowed initial data and in avoiding the restriction to superlinear behavior of Ψ at infinity for $L^2(\mu)$-initial data.

Keywords: Wiener process; Porous media equation; Sub-Markovian contractive semigroup.

1 Introduction

In this paper, we consider stochastic porous media equations (SPMEs) of the following type:

$$\begin{cases} dX(t) - L\Psi(X(t))dt = B(t, X(t))dW(t), \text{ in } [0,T] \times E, \\ X(0) = x \text{ on } E \text{ (with } x \in F^*_{1,2} \text{ or } L^2(\mu)), \end{cases} \tag{1.1}$$

where L is the self-adjoint generator of a sub-Markovian strongly continuous contraction semigroup $(P_t)_{t\geq 0}$ on $L^2(\mu) := L^2(E, \mathscr{B}(E), \mu)$, and $(E, \mathscr{B}(E), \mu)$ is a σ-finite measure space. $\Psi(\cdot) : \mathbb{R} \to \mathbb{R}$ is a monotonically nondecreasing Lipschitz continuous function, B is a progressively measurable process in the space of Hilbert-Schmidt operator from $L^2(\mu)$ to $F^*_{1,2}$, $W(t)$ is an $L^2(\mu)$-valued cylindrical $\mathscr{F}_t$-adapted Wiener process on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ with normal filtration $(\mathscr{F}_t)_{t\geq 0}$. For the definition of the Hilbert space $F^*_{1,2}$ and the precise conditions on B we refer to the next section.

In the special case when $E = \mathbb{R}^d$, L is equal to the Laplace operator Δ and B is time-independent linear multiplicative, equation (1.1) was recently analyzed in [3]. The aim of

*Research is supported by the DFG' through CRC 701, the National Natural Science Foundation of China (No.11271169) and the Project Funded by the Priority Academic Program Development of Jiangsu Higher Education Institutions.

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this paper is to prove analogous results as in [3] for the general case. The above framework is inspired by the work of Fukushima and Kaneko [5] (see also [7]).

The main motivation for this generality is that we would like to cover fractional powers of the Laplacian, i.e., $L = -(-\Delta)^\alpha$, $\alpha \in (0,1)$, generalized Schrödinger operators, i.e., $L = \Delta + 2\frac{\nabla\rho}{\rho} \cdot \nabla$, and Laplacians on fractals (see Section 4 below).

Recently, there has been much work on stochastic versions of the porous media equations. Based on the variational approach and monotonicity assumptions on the coefficients, [15] presents a generalization of Krylov-Rozovskii's result [10] on the existence and uniqueness of solutions to monotone stochastic differential equations, which applies to a large class of stochastic porous media equations. It should be said that in [15] (see also [16]), Ψ is assumed to be continuous such that $r\Psi(r) \to \infty$ as $r \to \infty$. In this paper we show that for Lipschitz continuous Ψ this condition can be dropped for initial data in $L^2(\mu)$, extending the corresponding result from [3] to general operators L as above. We would also like to emphasize that in contrast to [15, 16], in this paper, we do not assume that L is the generator of a transient Dirichlet form on $L^2(E, \mathscr{B}(E), \mu)$. In our case we can drop the transience assumption. In particular, in contrast to [15] (and [16]), we do not need any restriction on d when $E = \mathbb{R}^d$ and $L = -(-\Delta)^\alpha$, $\alpha \in (0,1]$. For more references on stochastic porous media equations we refer to [2]. In addition, we work in the state space $F^*_{1,2}$ which is larger than the state space $\mathscr{F}^*_e$ considered in [15], hence we can allow more general initial conditions (as done in [16] under assumptions much stronger than transience).

Section 4 of [3] deals with the case where Ψ is a maximal monotone multivalued function with at most polynomial growth. However, due to the multiplier problem, the existence is obtained for $d \geq 3$ only. We plan to extend also this result to our more general equation (1.1). This will be the subject of our future work.

The paper is organized as follows: in Section 2, we recall some notions concerning sub-Markovian semi-groups and introduce a suitable Gelfand triple. Section 3 is devoted to verify the existence and uniqueness of strong solutions to (1.1). Note that the Riesz isomorphism $1 - L$, through which we identify $H := F^*_{1,2}$ and $H^* := F_{1,2}$, plays an essential role in the proof. In Section 4, we will apply our results to a number of examples.

2 Preliminaries

First of all, let us recall some basic definitions and spaces which will be used throughout the paper (see [5, 6, 7]).

Let $(E, \mathscr{B}(E), \mu)$ be a σ-finite measure space. Let $\{P_t\}_{t\geq 0}$ be a strongly continuous sub-Markovian semigroup on $L^2(\mu)$ with self-adjoint generator $(L, D(L))$.

The gamma-transform $V_r (r > 0)$ of $\{P_t\}_{t\geq 0}$ is defined by

$$V_r = \Gamma(\frac{r}{2})^{-1} \int_0^\infty s^{\frac{r}{2}-1} e^{-s} P_s ds.$$

In this paper, we consider the Hilbert space $(F_{1,2}, \|\cdot\|_{F_{1,2}})$ defined by

$$F_{1,2} = V_1(L^2(\mu)), \text{ with norm } \|u\|_{F_{1,2}} = |f|_2 \text{ for } u = V_1 f, \quad f \in L^2(\mu),$$

where the norm $|\cdot|_2$ is defined as $|f|_2 = (\int_E |f|^2 d\mu)^{\frac{1}{2}}$. Clearly, $F_{1,2} \subset L^2(\mu)$ continuously and densely. In particular,

$$V_1 = (1-L)^{-\frac{1}{2}}, \text{ so that } \|u\|_{F_{1,2}} = |V_1^{-1} u|_2 = |(1-L)^{\frac{1}{2}} u|_2.$$

The dual space of $F_{1,2}$ is denoted by $F_{1,2}^*$.

In the following, we concentrate on finding a suitable Gelfand triple $V \subset H \equiv H^* \subset V^*$ with $H := F_{1,2}^*$. Let ${}_{F_{1,2}}\langle \cdot,\ \cdot \rangle_{F_{1,2}^*}$ denote the duality between $F_{1,2}$ and $F_{1,2}^*$, define $(1-L): F_{1,2} \to F_{1,2}^*$ as follows, given $u \in F_{1,2}$,

$$ {}_{F_{1,2}^*}\langle (1-L)u,\ v \rangle_{F_{1,2}} := \int_E (1-L)^{\frac{1}{2}} u \cdot (1-L)^{\frac{1}{2}} v \, d\mu \quad \text{for all } v \in F_{1,2}. \tag{2.1} $$

To show that $(1-L): F_{1,2} \to F_{1,2}^*$ is well-defined, we have to prove that the right-hand side of (2.1) defines a linear continuous function on $v \in F_{1,2}$ with respect to $\|\cdot\|_{F_{1,2}}$. But for $u \in F_{1,2}$, we have for all $v \in F_{1,2}$,

$$ \begin{aligned} \left| {}_{F_{1,2}^*}\langle (1-L)u,\ v \rangle_{F_{1,2}} \right| &= \left| \int_E (1-L)^{\frac{1}{2}} u \cdot (1-L)^{\frac{1}{2}} v \, d\mu \right| \\ &= \left| \left\langle (1-L)^{\frac{1}{2}} u,\ (1-L)^{\frac{1}{2}} v \right\rangle_2 \right| \\ &\leq |(1-L)^{\frac{1}{2}} u|_2 \cdot |(1-L)^{\frac{1}{2}} v\rangle|_2 \\ &= \|u\|_{F_{1,2}} \cdot \|v\|_{F_{1,2}}. \end{aligned} $$

This implies

$$ \|(1-L)u\|_{F_{1,2}^*} \leq \|u\|_{F_{1,2}}. $$

Now we would like to identify $F_{1,2}^*$ with its dual $F_{1,2}$ via the corresponding Riesz isomorphism $R: F_{1,2}^* \to F_{1,2}$ defined by $Rx = \langle x,\ \cdot \rangle_{F_{1,2}^*}$, $x \in F_{1,2}^*$.

Lemma 2.1 *The map $(1-L): F_{1,2} \to F_{1,2}^*$ is an isometric isomorphism. In particular,*

$$ \left\langle (1-L)u,\ (1-L)v \right\rangle_{F_{1,2}^*} = \langle u,\ v \rangle_{F_{1,2}} \quad \textit{for all} \ \ u, v \in F_{1,2}. \tag{2.2} $$

Furthermore, $(1-L)^{-1}: F_{1,2}^* \to F_{1,2}$ is the Riesz isomorphism for $F_{1,2}^*$, i.e., for every $u \in F_{1,2}^*$,

$$ \langle u,\ \cdot \rangle_{F_{1,2}^*} = {}_{F_{1,2}}\langle (1-L)^{-1} u,\ \cdot \rangle_{F_{1,2}^*}. \tag{2.3} $$

Proof For all $u, v \in F_{1,2}$, by (2.1) we know

$$ {}_{F_{1,2}^*}\langle (1-L)u,\ v \rangle_{F_{1,2}} = \langle (1-L)^{\frac{1}{2}} u,\ (1-L)^{\frac{1}{2}} v \rangle_2 = \langle u,\ v \rangle_{F_{1,2}}, $$

i.e., $(1-L): F_{1,2} \to F_{1,2}^*$ is the Riesz isomorphism for $F_{1,2}$.

In particular, for all $u, v \in F_{1,2}$, since the Riesz isomorphism is isometric,

$$ \langle (1-L)u,\ (1-L)v \rangle_{F_{1,2}^*} = \langle u,\ v \rangle_{F_{1,2}}. \tag{2.4} $$

Furthermore, for all $u, v \in F_{1,2}^*$,

$$ \langle u,\ v \rangle_{F_{1,2}^*} = \langle (1-L)^{-1} u,\ (1-L)^{-1} v \rangle_{F_{1,2}} = {}_{F_{1,2}}\langle (1-L)^{-1} u,\ v \rangle_{F_{1,2}^*}. $$

□

In this sense, we identify $F_{1,2}^*$ with $F_{1,2}$ via the Riesz map $(1-L)^{-1}: F_{1,2}^* \to F_{1,2}$, thus $F_{1,2}^* \equiv F_{1,2}$. Note that $L^2(\mu)$ can be considered as a subset of $F_{1,2}^*$, since for $u \in L^2(\mu)$, the map

$$ v \longmapsto \langle u, v \rangle_2, \quad v \in F_{1,2}, $$

belongs to $F_{1,2}^*$. Here $\langle\cdot,\cdot\rangle_2$ denotes the usual inner product on $L^2(\mu)$. Obviously, in this sense $L^2(\mu) \subset F_{1,2}^*$ continuously and densely. Consequently, we get a Gelfand triple with $V := L^2(\mu)$, $H := F_{1,2}^*$,

$$V = L^2(\mu) \subset F_{1,2}^* \subset (L^2(\mu))^*,$$

which satisfies

$$_{V^*}\langle u, v\rangle_V = \langle u, v\rangle_H, \text{ for all } u \in H, v \in V. \tag{2.5}$$

Lemma 2.2 *The map*

$$1 - L : F_{1,2} \to F_{1,2}^*$$

extends to a linear isometry

$$1 - L : L^2(\mu) \to (L^2(\mu))^*,$$

and for all $u, v \in L^2(\mu)$,

$$_{(L^2(\mu))^*}\langle (1-L)u, v\rangle_{L^2(\mu)} = \int_E u \cdot v \, d\mu. \tag{2.6}$$

Proof Let $u \in F_{1,2}$. Since $(1-L)u \in F_{1,2}^*$, from (2.3) and (2.5) we obtain that for all $v \in L^2(\mu)$,

$$_{(L^2(\mu))^*}\langle (1-L)u, v\rangle_{L^2(\mu)} = \langle (1-L)u, v\rangle_{F_{1,2}^*} =_{F_{1,2}} \langle u, v\rangle_{F_{1,2}^*} = \langle u, v\rangle_2, \tag{2.7}$$

the last equality holds since $F_{1,2} \subset L^2(\mu) \subset F_{1,2}^*$ densely and continuously. Therefore,

$$|(1-L)u|_{(L^2(\mu))^*} \le |u|_2.$$

In this sense, $1 - L$ extends to a continuous linear map

$$1 - L : L^2(\mu) \to (L^2(\mu))^*$$

such that (2.7) holds for all $u \in L^2(\mu)$, i.e., (2.6) is proved.

So, applying it to $u \in L^2(\mu)$ and

$$v := |u|_2^{-1} u \in L^2(\mu),$$

by (2.7) we obtain that

$$_{V^*}\langle (1-L)u, v\rangle_V = \langle u, v\rangle_2 = \langle u, |u|_2^{-1}u\rangle_2 = |u|_2,$$

and $|v|_2 = 1$, so $|(1-L)u|_{V^*} = |u|_V$ and the assertion is completely proved. □

Thoughout the paper, let $L^2([0,T]\times\Omega; L^2(\mu))$ denote the space of all $L^2(\mu)$-valued square-integrable functions on $[0,T] \times \Omega$, and $C([0,T]; F_{1,2}^*)$ the space of all continuous $F_{1,2}^*$-valued functions on $[0,T]$. For two Hilbert spaces H_1 and H_2, the space of Hilbert-Schmidt operators from H_1 to H_2 is denoted by $L_2(H_1, H_2)$. For simplicity, the positive constants c, C, C_1 and C_2 used in this paper may change from line to line. We would like to refer [2] for more background information and results on SPMEs.

3 The Main Result

Consider (1.1) under the following conditions:

(H1) $\Psi(\cdot) : \mathbb{R} \to \mathbb{R}$ is a monotonically nondecreasing Lipschitz function with $\Psi(0) = 0$.

(H2) For every $t > 0$, $B : [0,T] \times L^2(\mu) \times \Omega \to L_2(L^2(\mu), F^*_{1,2})$ is progressively measurable such that

(i) there exists $C_1 \in [0,\infty)$ satisfying

$$\|B(\cdot,u) - B(\cdot,v)\|^2_{L_2(L^2(\mu),F^*_{1,2})} \le C_1\|u-v\|^2_{F^*_{1,2}} \text{ for all } u,v \in L^2(\mu) \text{ on } [0,T]\times\Omega;$$

(ii) there exists $C_2 \in (0,\infty)$ satisfiying

$$\|B(\cdot,u)\|^2_{L_2(L^2(\mu),F^*_{1,2})} \le C_2\|u\|^2_{F^*_{1,2}} \text{ for all } u \in L^2(\mu) \text{ on } [0,T]\times\Omega.$$

Definition 3.1 *Let $x \in F^*_{1,2}$. A continuous $(\mathscr{F}_t)_{t\ge0}$-adapted process $X : [0,T] \to F^*_{1,2}$ is called strong solution to (1.1) if the following conditions are satisfied:*

$$X \in L^2([0,T]\times\Omega; L^2(\mu)) \cap L^2(\Omega; C([0,T]; F^*_{1,2})), \tag{3.1}$$

$$\int_0^\bullet \Psi(X(s))ds \in C([0,T]; F_{1,2}),\ \mathbb{P}\text{-a.s.}, \tag{3.2}$$

$$X(t) - L\int_0^t \Psi(X(s))ds = x + \int_0^t B(s,X(s))dW(s),\ \forall t \in [0,T],\ \mathbb{P}\text{-a.s.}. \tag{3.3}$$

Theorem 3.1 *Suppose **(H1)** and **(H2)** are satisfied. Then, for each $x \in L^2(\mu)$, there is a unique strong solution X to (1.1) and exists $C \in [0,\infty)$ satisfying*

$$\mathbb{E}\Big[\sup_{t\in[0,T]} |X(t)|_2^2\Big] \le 2|x|_2^2 e^{CT}.$$

Assume further that

$$\Psi(r)r \ge cr^2,\ \forall\ r \in \mathbb{R}, \tag{3.4}$$

where $c \in (0,\infty)$. Then, there is a unique strong solution X to (1.1) for all $x \in F^*_{1,2}$.

For the proof of the above theorem, we firstly consider the approximating equations for (1.1):

$$\begin{cases} dX^\nu(t) + (\nu - L)\Psi(X^\nu(t))dt = B(t,X^\nu(t))dW(t), \text{ in } (0,T)\times E, \\ X^\nu(0) = x \text{ on } E, \end{cases} \tag{3.5}$$

where $\nu \in (0,1)$. And we have the following results for (3.5).

Lemma 3.1 *Suppose **(H1)** and **(H2)** are satisfied. Then, for each $x \in L^2(\mu)$, there is a unique $(\mathscr{F}_t)_{t\ge0}$-adapted solution to (3.5), denoted by X^ν, i.e., in particular it has the following properties,*

$$X^\nu \in L^2\big([0,T]\times\Omega; L^2(\mu)\big) \cap L^2\big(\Omega; C([0,T]; F^*_{1,2})\big), \tag{3.6}$$

$$X^\nu(t) + (\nu - L)\int_0^t \Psi(X^\nu(s))ds = x + \int_0^t B(s,X^\nu(s))dW(s),\ \forall t\in[0,T],\ \ \mathbb{P}-a.s.. \tag{3.7}$$

Furthermore, there exists $C \in (0,\infty)$ such that for all $\nu \in (0,1)$,

$$\mathbb{E}\Big[\sup_{t\in[0,T]} |X^\nu(t)|_2^2\Big] \le 2|x|_2^2 e^{CT}. \tag{3.8}$$

In addition, if (3.4) is satisfied, there is a unique solution X^ν to (3.5) satisfying (3.6) and (3.7) for all $x \in F^*_{1,2}$.

Proof We proceed it in two steps.

Step 1: Assume $x \in F_{1,2}^*$ and that (3.4) is satisfied. Set $V := L^2(\mu)$, $H := F_{1,2}^*$, $Au := (L-\nu)\Psi(u)$ for $u \in V$. The space $F_{1,2}^*$ is equipped with the equivalent norm

$$\|\eta\|_{F_{1,2,\nu}^*} := \langle \eta, (\nu - L)^{-1}\eta\rangle^{\frac{1}{2}}, \quad \eta \in F_{1,2}^*.$$

Under the Gelfand triple $V \subset H \subset V^*$, we shall prove the existence and uniqueness of the solution to (3.5) by using [12, Theorem 4.2.4] (or [14, Section 4.2]).

In the following, we shall verify the four conditions of the existence and uniqueness theorem in [12, 14].

(i) (Hemicontinuity)
Let $u, v, w \in V = L^2(\mu)$. We have to show for $\lambda \in \mathbb{R}$, $|\lambda| \le 1$,

$$\lim_{\lambda\to 0} \ {}_{V^*}\langle A(u+\lambda v), w\rangle_V - {}_{V^*}\langle Au, w\rangle_V = 0.$$

By Lemma 2.2

$$\begin{aligned}
& {}_{V^*}\langle A(u+\lambda v), w\rangle_V \\
= & {}_{V^*}\langle (L-\nu)\Psi(u+\lambda v), w\rangle_V \\
= & -{}_{V^*}\langle (1-L)\Psi(u+\lambda v), w\rangle_V + (1-\nu)\,{}_{V^*}\big\langle (1-L)(1-L)^{-1}\Psi(u+\lambda v), w\big\rangle_V \\
= & -\langle \Psi(u+\lambda v), w\rangle_2 + (1-\nu)\langle (1-L)^{-1}\Psi(u+\lambda v), w\rangle_2 \\
= & -\int_E \Psi(u+\lambda v)\cdot w d\mu + (1-\nu)\int_E (1-L)^{-1}\Psi(u+\lambda v)\cdot w d\mu.
\end{aligned}$$

By the Lipschitz continuity of Ψ and denoting $k := Lip\Psi$, the first integrand in the right-hand side of the above equality is bounded by

$$|\Psi(u+\lambda v)|\cdot|w| \le k(|u|+|v|)\cdot|w|,$$

which by Hölder's inequality is in $L^1(\mu)$. Since $(1-L)^{-1}$ is a contraction, in order to prove the convergence of $(1-L)^{-1}\Psi(u+\lambda v)\cdot w$ in $L^1(\mu)$, it is sufficient to show the convergence of $\Psi(u+\lambda v)$ in $L^2(\mu)$, which is obvious because Ψ is Lipschitz and

$$|\Psi(u+\lambda v)| \le k(|u|+|v|).$$

(ii) (Weak Monotonicity)
Let $u, v \in V = L^2(\mu)$, then by Lemma 2.2 and (2.5)

$$\begin{aligned}
& 2\,{}_{V^*}\langle Au - Av, u-v\rangle_V + \|B(\cdot,u) - B(\cdot,v)\|^2_{L_2(L^2(\mu),F_{1,2}^*)} \\
= & 2\,{}_{V^*}\big\langle (L-\nu)(\Psi(u)-\Psi(v)), u-v\big\rangle_V + \|B(\cdot,u) - B(\cdot,v)\|^2_{L_2(L^2(\mu),F_{1,2}^*)} \\
= & -2\,{}_{V^*}\langle (1-L)(\Psi(u)-\Psi(v)), u-v\rangle_V \\
& + 2(1-\nu)\,{}_{V^*}\big\langle \Psi(u)-\Psi(v), u-v\big\rangle_V + \|B(\cdot,u) - B(\cdot,v)\|^2_{L_2(L^2(\mu),F_{1,2}^*)} \\
= & -2\langle (\Psi(u)-\Psi(v)), u-v\rangle_2 \\
& + 2(1-\nu)\big\langle \Psi(u)-\Psi(v), u-v\big\rangle_{F_{1,2}^*} + \|B(\cdot,u) - B(\cdot,v)\|^2_{L_2(L^2(\mu),F_{1,2}^*)}.
\end{aligned} \tag{3.9}$$

Set $\tilde\alpha := (Lip\Psi + 1)^{-1}$. By assumption **(H1)** on Ψ, we know that

$$\big(\Psi(r) - \Psi(r')\big)(r-r') \ge \tilde\alpha|\Psi(r)-\Psi(r')|^2, \quad \forall r,\ r' \in \mathbb{R}. \tag{3.10}$$

Since $L^2(\mu) \subset F_{1,2}^*$ continuously, by Young's inequality

$$\begin{aligned}
&\big\langle \Psi(u) - \Psi(v), u - v \big\rangle_{F_{1,2}^*} \\
&\le \|\Psi(u) - \Psi(v)\|_{F_{1,2}^*} \cdot \|u - v\|_{F_{1,2}^*} \\
&\le |\Psi(u) - \Psi(v)|_2 \cdot \|u - v\|_{F_{1,2}^*} \\
&\le \frac{\tilde{\alpha}}{1-\nu} |\Psi(u) - \Psi(v)|_2^2 + \frac{1-\nu}{\tilde{\alpha}} \|u - v\|_{F_{1,2}^*}^2.
\end{aligned} \tag{3.11}$$

By **(H2) (ii)**, and taking (3.10), (3.11) into account, (3.9) is dominated by

$$\begin{aligned}
&-2\tilde{\alpha}|\Psi(u) - \Psi(v)|_2^2 + 2\tilde{\alpha}|\Psi(u) - \Psi(v)|_2^2 + \frac{2(1-\nu)^2}{\tilde{\alpha}} \|u - v\|_{F_{1,2}^*}^2 + C_1 \|u - v\|_{F_{1,2}^*}^2 \\
&= \Big[\frac{2(1-\nu)^2}{\tilde{\alpha}} + C_1\Big] \cdot \|u - v\|_{F_{1,2}^*}^2.
\end{aligned}$$

Hence weak monotonicity holds.

(iii) (Coercivity)

Let $u \in L^2(\mu)$. By Lemma 2.2 and (2.5)

$$\begin{aligned}
&2_{V^*}\langle Au, u\rangle_V + \|B(\cdot, u)\|_{L_2(L^2(\mu), F_{1,2}^*)}^2 \\
&= -2_{V^*}\big\langle (1-L)\Psi(u), u\big\rangle_V + 2(1-\nu)_{V^*}\langle \Psi(u), u\rangle_V + \|B(\cdot, u)\|_{L_2(L^2(\mu), F_{1,2}^*)}^2 \\
&= -2\langle \Psi(u), u\rangle_2 + 2(1-\nu)\langle \Psi(u), u\rangle_{F_{1,2}^*} + \|B(\cdot, u)\|_{L_2(L^2(\mu), F_{1,2}^*)}^2.
\end{aligned} \tag{3.12}$$

By (3.4)

$$-2\langle \Psi(u), u\rangle_2 = -2\int_E \Psi(u) \cdot u d\mu \le -2c|u|_2^2. \tag{3.13}$$

Since $L^2(\mu) \subset F_{1,2}^*$ continuously, by Young's inequality for $\varepsilon \in (0,1)$

$$\begin{aligned}
\big\langle \Psi(u), u \big\rangle_{F_{1,2}^*} &\le \|\Psi(u)\|_{F_{1,2}^*} \cdot \|u\|_{F_{1,2}^*} \\
&\le |\Psi(u)|_2 \cdot \|u\|_{F_{1,2}^*} \\
&\le \varepsilon^2 k^2 |u|_2^2 + \frac{1}{\varepsilon^2} \|u\|_{F_{1,2}^*}^2.
\end{aligned} \tag{3.14}$$

By **(H2) (ii)**, and taking (3.13) and (3.14) into account, (3.12) is dominated by

$$\big[-2c + 2\varepsilon^2 k^2 (1-\nu)\big] \cdot |u|_2^2 + \Big[\frac{2(1-\nu)}{\varepsilon^2} + C_2\Big] \cdot \|u\|_{F_{1,2}^*}^2.$$

Choosing ε small enough, $-2c + 2\varepsilon^2 k^2 (1-\nu)$ becomes negative, which implies the coercivity.

(iv) (Boundedness)

Let $u \in L^2(\mu)$. Since

$$|Au|_{V^*} = |(L-\nu)\Psi(u)|_{V^*} = \sup_{|v|_2 = 1} {}_{V^*}\langle (L-\nu)\Psi(u), v\rangle_V,$$

by Lemma 2.2 and since $(1-L)^{-1}$ is a contraction, we deduce

$$\begin{aligned}
&{}_{V^*}\langle (L-\nu)\Psi(u), v\rangle_V \\
&= -_{V^*}\langle (1-L)\Psi(u), v\rangle_V + (1-\nu)_{V^*}\big\langle (1-L)(1-L)^{-1}\Psi(u), v\big\rangle_V \\
&= -\langle \Psi(u), v\rangle_2 + (1-\nu)\langle (1-L)^{-1}\Psi(u), v\rangle_2 \\
&\le |\Psi(u)|_2 \cdot |v|_2 + (1-\nu)|\Psi(u)|_2 \cdot |v|_2.
\end{aligned}$$

So

$$|Au|_{V^*} \leq 2|\Psi(u)|_2 \leq 2k|u|_2.$$

Hence the boundedness holds.

By [12, Theorem 4.2.4], there exists a unique solution to (3.5), denoted by X^ν, which takes values in $F_{1,2}^*$ and satisfies (3.6) and (3.7).

Step 2: If Ψ does not satisfy (3.4) and $x \in L^2(\mu)$, the above (i), (ii) and (iv) still hold, but (iii) not in general. In this case, we will approximate Ψ by $\Psi + \lambda I$, $\lambda \in (0,1)$.

Consider the approximating equation:

$$\begin{cases} X_\lambda^\nu(t) + (\nu - L)\big(\Psi(X_\lambda^\nu(t)) + \lambda X_\lambda^\nu(t)\big)dt = B(t, X_\lambda^\nu(t))dW(t), \text{ in } [0,T]\times E, \\ X_\lambda^\nu(0) = x \in F_{1,2}^* \text{ on } E. \end{cases} \tag{3.15}$$

By [12, Theorem 4.2.4], it is easy to prove that there is a solution X_λ^ν to (3.15) which satisfies $X_\lambda^\nu \in L^2\big([0,T]\times\Omega; L^2(\mu)\big) \cap L^2\big(\Omega; C([0,T]; F_{1,2}^*)\big)$,

$$X_\lambda^\nu(t) + (\nu - L)\int_0^t \Psi(X_\lambda^\nu(t)) + \lambda X_\lambda^\nu(t)ds = x + \int_0^t B(s, X_\lambda^\nu(s))dW(s), \quad \mathbb{P}-a.s.$$

and

$$\mathbb{E}\Big[\sup_{t\in[0,T]} \|X_\lambda^\nu(t)\|_{F_{1,2}^*}^2\Big] < \infty. \tag{3.16}$$

In the following, we want to prove that X_λ^ν converges to the solutions of (3.5) as $\lambda \to 0$. From now on, we assume the initial value $x \in L^2(\mu)$.

Claim 3.1

$$\mathbb{E}\Big[\sup_{t\in[0,T]} |X_\lambda^\nu(t)|_2^2\Big] + 4\lambda\nu\mathbb{E}\int_0^t \|X_\lambda^\nu(s)\|_{F_{1,2}}^2 ds \leq 2|x|_2^2 e^{CT}, \quad \textit{for all } \nu,\ \lambda \in (0,1),$$

and X_λ^ν has continuous sample path in $L^2(\mu)$, $\mathbb{P}$-a.s..

Proof Rewrite (3.15), for $t \in [0,T]$,

$$X_\lambda^\nu(t) = x + \int_0^t (L-\nu)\big(\Psi(X_\lambda^\nu(s)) + \lambda X_\lambda^\nu(s)\big)ds + \int_0^t B(s, X_\lambda^\nu(s))dW(s). \tag{3.17}$$

For $\alpha > \nu$, applying the operator $(\alpha - L)^{-\frac{1}{2}} : F_{1,2}^* \to L^2(\mu)$ to both sides of the above equation, we get

$$\begin{aligned} &(\alpha - L)^{-\frac{1}{2}} X_\lambda^\nu(t) \\ &= (\alpha - L)^{-\frac{1}{2}} x + \int_0^t (L-\nu)(\alpha - L)^{-\frac{1}{2}}\big(\Psi(X_\lambda^\nu(s)) + \lambda X_\lambda^\nu(s)\big)ds \\ &\quad + \int_0^t (\alpha - L)^{-\frac{1}{2}} B(s, X_\lambda^\nu(s))dW(s). \end{aligned}$$

Applying Itô's formula ([12, Theorem 4.2.5]) with $H = L^2(\mu)$, we obtain, for $t \in [0, T]$,

$$\begin{aligned}
&\left|(\alpha - L)^{-\frac{1}{2}} X_\lambda^\nu(t)\right|_2^2 \\
&= \left|(\alpha - L)^{-\frac{1}{2}} x\right|_2^2 + 2\int_0^t {}_{F_{1,2}^*}\left\langle (L-\nu)(\alpha - L)^{-\frac{1}{2}}\Psi(X_\lambda^\nu(s)), (\alpha - L)^{-\frac{1}{2}} X_\lambda^\nu(s)\right\rangle_{F_{1,2}} ds \\
&+2\lambda \int_0^t {}_{F_{1,2}^*}\left\langle (L-\nu)(\alpha - L)^{-\frac{1}{2}} X_\lambda^\nu(s), (\alpha - L)^{-\frac{1}{2}} X_\lambda^\nu(s)\right\rangle_{F_{1,2}} ds \\
&+ \int_0^t \left\|(\alpha - L)^{-\frac{1}{2}} B(s, X_\lambda^\nu(s))\right\|_{L_2(F_{1,2}^*, L^2(\mu))}^2 ds \\
&+2\int_0^t \left\langle (\alpha - L)^{-\frac{1}{2}} X_\lambda^\nu(s), (\alpha - L)^{-\frac{1}{2}} B(s, X_\lambda^\nu(s)) dW(s)\right\rangle_2.
\end{aligned} \tag{3.18}$$

Set $P := (\alpha - \nu)(\alpha - L)^{-1}$. For $f \in L^2(\mu)$, we have

$$\begin{aligned}
(P - I)f &= \left[(\alpha - L)^{-\frac{1}{2}}(\alpha - \nu)(\alpha - L)^{-\frac{1}{2}} - (\alpha - L)^{-\frac{1}{2}}(\alpha - L)(\alpha - L)^{-\frac{1}{2}}\right] f \\
&= \left[(\alpha - L)^{-\frac{1}{2}}(L - \nu)(\alpha - L)^{-\frac{1}{2}}\right] f.
\end{aligned}$$

Let g_α denote the Green function of $\alpha - L$. For $f \in L^2(\mu)$, we have

$$Pf = (\alpha - \nu)\int_E f(x) g_\alpha(\cdot, x) d\mu.$$

Applying [16, Lemma 5.1] with $f := X_\lambda^\nu(s)$ and $g := \Psi(X_\lambda^\nu(s))$, one obtains

$$\begin{aligned}
&2\int_0^t {}_{F_{1,2}^*}\left\langle (L-\nu)(\alpha - L)^{-\frac{1}{2}}\Psi(X_\lambda^\nu(s)), (\alpha - L)^{-\frac{1}{2}} X_\lambda^\nu(s)\right\rangle_{F_{1,2}} ds \\
&= 2\int_0^t \langle \Psi(X_\lambda^\nu(s)), (P - I) X_\lambda^\nu(s)\rangle_2 ds \\
&= -\frac{1}{2}\int_E\int_E \left[\Psi(f(\widetilde{\xi})) - \Psi(f(\xi))\right]\left[f(\widetilde{\xi}) - f(\xi)\right] g_\alpha(\xi, \widetilde{\xi}) d\widetilde{\xi} d\xi \\
&\quad - \int_E (1 - P1(\xi)) f(\xi)\cdot \Psi(f(\xi)) d\xi.
\end{aligned}$$

Since Ψ is monotone, $\Psi(0) = 0$ and $P1 \leq 1$, we have

$$2\int_0^t \langle \Psi(X_\lambda^\nu(s)), (P - I) X_\lambda^\nu(s)\rangle_2 ds \leq 0. \tag{3.19}$$

For the second integral on the right hand side of (3.18), since $(1 - L)^{-1}$ is a contraction, one has

$$
\begin{aligned}
& 2\lambda \int_0^t {}_{F_{1,2}^*}\big\langle (L-\nu)(\alpha-L)^{-\frac{1}{2}} X_\lambda^\nu(s), (\alpha-L)^{-\frac{1}{2}} X_\lambda^\nu(s)\big\rangle_{F_{1,2}} ds \\
= & -2\lambda \int_0^t {}_{F_{1,2}^*}\big\langle (1-L)(\alpha-L)^{-\frac{1}{2}} X_\lambda^\nu(s), (\alpha-L)^{-\frac{1}{2}} X_\lambda^\nu(s)\big\rangle_{F_{1,2}} ds \\
& +(1-\nu)2\lambda \int_0^t {}_{F_{1,2}^*}\big\langle (1-L)(1-L)^{-1}(\alpha-L)^{-\frac{1}{2}} X_\lambda^\nu(s), (\alpha-L)^{-\frac{1}{2}} X_\lambda^\nu(s)\big\rangle_{F_{1,2}} ds \\
= & -2\lambda \int_0^t \|(\alpha-L)^{-\frac{1}{2}} X_\lambda^\nu(s)\|_{F_{1,2}}^2 ds \\
& +(1-\nu)2\lambda \int_0^t \big\langle (1-L)^{-1}(\alpha-L)^{-\frac{1}{2}} X_\lambda^\nu(s), (\alpha-L)^{-\frac{1}{2}} X_\lambda^\nu(s)\big\rangle_{F_{1,2}} ds \\
\le & -2\lambda \int_0^t \|(\alpha-L)^{-\frac{1}{2}} X_\lambda^\nu(s)\|_{F_{1,2}}^2 ds + (1-\nu)2\lambda \int_0^t \|(\alpha-L)^{-\frac{1}{2}} X_\lambda^\nu(s)\|_{F_{1,2}}^2 ds \\
= & -2\lambda\nu \int_0^t \|(\alpha-L)^{-\frac{1}{2}} X_\lambda^\nu(s)\|_{F_{1,2}}^2 ds. \qquad (3.20)
\end{aligned}
$$

Multiplying both sides of (3.18) by α, (3.19) and (3.20) yield that, for all $t \in [0,T]$,

$$
\begin{aligned}
& \big|\sqrt{\alpha}(\alpha-L)^{-\frac{1}{2}} X_\lambda^\nu(s)\big|_2^2 + 2\lambda\nu \int_0^t \|\sqrt{\alpha}(\alpha-L)^{-\frac{1}{2}} X_\lambda^\nu(s)\|_{F_{1,2}}^2 ds \\
\le & \big|\sqrt{\alpha}(\alpha-L)^{-\frac{1}{2}} x\big|_2^2 + \int_0^t \big\|\sqrt{\alpha}(\alpha-L)^{-\frac{1}{2}} B(s, X_\lambda^\nu(s))\big\|_{L_2(F_{1,2}^*, L^2(\mu))}^2 ds \\
& +2 \int_0^t \big\langle \sqrt{\alpha}(\alpha-L)^{-\frac{1}{2}} X_\lambda^\nu(s), \sqrt{\alpha}(\alpha-L)^{-\frac{1}{2}} B(s, X_\lambda^\nu(s)) dW(s)\big\rangle_2. \qquad (3.21)
\end{aligned}
$$

Since $\sqrt{\alpha}(\alpha-L)^{-\frac{1}{2}}$ is a contraction operator on $L^2(\mu)$, **(H2)(ii)** implies

$$
\begin{aligned}
& \int_0^t \big\|\sqrt{\alpha}(\alpha-L)^{-\frac{1}{2}} B(s, X_\lambda^\nu(s))\big\|_{L_2(F_{1,2}^*, L^2(\mu))}^2 ds \\
\le & \int_0^t \|B(s, X_\lambda^\nu(s))\|_{L_2(L^2(\mu), F_{1,2}^*)}^2 ds \\
\le & C_2 \int_0^t \|X(s)\|_{F_{1,2}^*}^2 ds.
\end{aligned}
$$

Using the BDG inequality, we obtain

$$
\begin{aligned}
& \mathbb{E}\Big[\sup_{s\in[0,t]} |\sqrt{\alpha}(\alpha-L)^{-\frac{1}{2}} X_\lambda^\nu(s)|_2^2\Big] + 2\lambda\nu \mathbb{E} \int_0^t \|\sqrt{\alpha}(\alpha-L)^{-\frac{1}{2}} X_\lambda^\nu(s)\|_{F_{1,2}}^2 ds \\
\le & \big|\sqrt{\alpha}(\alpha-L)^{-\frac{1}{2}} x\big|_2^2 + C_2 \mathbb{E} \int_0^t \|X_\lambda^\nu(s)\|_{F_{1,2}^*}^2 ds \\
& +6\mathbb{E}\left[\int_0^t |\sqrt{\alpha}(\alpha-L)^{-\frac{1}{2}} X_\lambda^\nu(s)|_2^2 \cdot |\sqrt{\alpha}(\alpha-L)^{-\frac{1}{2}} B(s, X_\lambda^\nu(s))|_{L_2(F_{1,2}^*, L^2(\mu))}^2 ds\right]^{\frac{1}{2}}. \qquad (3.22)
\end{aligned}
$$

The last term of the right hand side of the above inequality can be estimated by

$$
\begin{aligned}
& 6\mathbb{E}\left[\sup_{s\in[0,t]} |\sqrt{\alpha}(\alpha-L)^{-\frac{1}{2}} X_\lambda^\nu(s)|_2^2 \cdot \int_0^t |\sqrt{\alpha}(\alpha-L)^{-\frac{1}{2}} B(s, X_\lambda^\nu(s))|_{L_2(F_{1,2}^*, L^2(\mu))}^2 ds\right]^{\frac{1}{2}} \\
\le & \frac{1}{2}\mathbb{E}\Big[\sup_{s\in[0,t]} |\sqrt{\alpha}(\alpha-L)^{-\frac{1}{2}} X_\lambda^\nu(s)|_2^2\Big] + C\mathbb{E} \int_0^t \|X_\lambda^\nu(s)\|_{F_{1,2}^*}^2 ds. \qquad (3.23)
\end{aligned}
$$

Since $L^2(\mu)$ is continuously embedded into $F_{1,2}^*$, by (3.21)-(3.23), we obtain that, for $t \in [0,T]$,

$$\begin{aligned}
&\mathbb{E}\left[\sup_{s\in[0,t]} |\sqrt{\alpha}(\alpha-L)^{-\frac{1}{2}}X_\lambda^\nu(s)|_2^2\right] + 2\lambda\nu\mathbb{E}\int_0^t \|\sqrt{\alpha}(\alpha-L)^{-\frac{1}{2}}X_\lambda^\nu(s)\|_{F_{1,2}}^2 ds \\
&\le \left|\sqrt{\alpha}(\alpha-L)^{-\frac{1}{2}}x\right|_2^2 + C_1\mathbb{E}\int_0^t |X_\lambda^\nu(s)|_2^2 ds \\
&\quad + \frac{1}{2}\mathbb{E}\left[\sup_{s\in[0,t]} |\sqrt{\alpha}(\alpha-L)^{-\frac{1}{2}}X_\lambda^\nu(t)|_2^2\right] + C_2\mathbb{E}\int_0^t |X_\lambda^\nu(s)|_2^2 ds. \qquad (3.24)
\end{aligned}$$

Note that the first summand of the left hand side of the above inequality is finite by (3.16), since $|\sqrt{\alpha}(\alpha-L)^{-\frac{1}{2}}\cdot|_2$ is equivalent to $\|\cdot\|_{F_{1,2}^*}$. (3.24) shows that

$$\begin{aligned}
&\mathbb{E}\left[\sup_{s\in[0,t]} |\sqrt{\alpha}(\alpha-L)^{-\frac{1}{2}}X_\lambda^\nu(s)|_2^2\right] + 4\lambda\nu\mathbb{E}\int_0^t \|\sqrt{\alpha}(\alpha-L)^{-\frac{1}{2}}X_\lambda^\nu(s)\|_{F_{1,2}}^2 ds \\
&\le 2|\sqrt{\alpha}(\alpha-L)^{-\frac{1}{2}}x|_2^2 + C\mathbb{E}\int_0^t |X_\lambda^\nu(s)|_2^2 ds. \qquad (3.25)
\end{aligned}$$

Note that the left hand side of (3.25) is an increasing function with respect to α and $\sqrt{\alpha}(\alpha-L)^{-\frac{1}{2}}$ is a contraction operator on $L^2(\mu)$. Letting $\alpha\to\infty$, the monotone convergence theorem implies

$$\mathbb{E}\left[\sup_{s\in[0,T]} |X_\lambda^\nu(s)|_2^2\right] + 4\lambda\nu\mathbb{E}\int_0^t \|X_\lambda^\nu(s)\|_{F_{1,2}}^2 ds \le 2|x|_2^2 + C\mathbb{E}\int_0^t |X_\lambda^\nu(s)|_2^2 ds.$$

Then Gronwall's inequality yields

$$\mathbb{E}\Big[\sup_{s\in[0,T]} |X_\lambda^\nu(s)|_2^2\Big] + 4\lambda\nu\mathbb{E}\int_0^t \|X_\lambda^\nu(s)\|_{F_{1,2}}^2 ds \le 2|x|_2^2 e^{CT}.$$

Furthermore, the continuity of X_λ^ν on $L^2(\mu)$ follows from [9, Theorem 2.1]. □

Claim 3.2 $\{X_\lambda^\nu\}_{\lambda\in(0,1)}$ *converges to an element* $X^\nu \in L^2([0,T]\times\Omega; L^2(\mu))$ *as* $\lambda\to 0$.

Proof By Itô's formula we get that, for $\lambda,\lambda'\in(0,1)$ and $t\in[0,T]$,

$$\begin{aligned}
&\|X_\lambda^\nu(t) - X_{\lambda'}^\nu(t)\|_{F_{1,2,\nu}^*}^2 \\
&+2\int_0^t \left\langle \Psi(X_\lambda^\nu(s) - \Psi(X_{\lambda'}^\nu(s) + \lambda X_\lambda^\nu(s) - \lambda' X_{\lambda'}^\nu(s), X_\lambda^\nu(s) - X_{\lambda'}^\nu(s))\right\rangle_2 ds \\
&= \int_0^t \left\|B(s,X_\lambda^\nu(s)) - B(s,X_{\lambda'}^\nu(s))\right\|_{L_2(L^2(\mu),F_{1,2,\nu}^*)}^2 ds \\
&+2\int_0^t \left\langle X_\lambda^\nu(s) - X_{\lambda'}^\nu(s), \big(B(s,X_\lambda^\nu(s)) - B(s,X_{\lambda'}^\nu(s))\big)dW(s)\right\rangle_{F_{1,2,\nu}^*}. \qquad (3.26)
\end{aligned}$$

(3.10) implies that for the second term on the left hand side in (3.26) we have

$$\begin{aligned}
&2\int_0^t \left\langle \Psi(X_\lambda^\nu(s)) - \Psi(X_{\lambda'}^\nu(s)) + \lambda X_\lambda^\nu(s) - \lambda' X_{\lambda'}^\nu(s), X_\lambda^\nu(s) - X_{\lambda'}^\nu(s)\right\rangle_2 ds \\
&\ge 2\tilde{\alpha}\int_0^t \left|\Psi(X_\lambda^\nu(s)) - \Psi(X_{\lambda'}^\nu(s))\right|_2^2 ds \\
&+2\int_0^t \left\langle \lambda X_\lambda^\nu(s) - \lambda' X_{\lambda'}^\nu(s), X_\lambda^\nu(s) - X_{\lambda'}^\nu(s)\right\rangle_2. \qquad (3.27)
\end{aligned}$$

The assumption **(H2)(i)** yields

$$\int_0^t \left\|B(s,X_\lambda^\nu(s))-B(s,X_{\lambda'}^\nu(s))\right\|^2_{L_2(L^2(\mu),F^*_{1,2,\nu})}ds \le C_1\int_0^t \|X_\lambda^\nu(s)-X_{\lambda'}^\nu(s)\|^2_{F^*_{1,2,\nu}}. \quad (3.28)$$

Using the BDG inequality and Young's inequality, for $t\in[0,T]$, (3.26)-(3.28) imply

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{s\in[0,t]}\left\|X_\lambda^\nu(s)-X_{\lambda'}^\nu(s)\right\|^2_{F^*_{1,2,\nu}}\Big]+2\tilde{\alpha}\mathbb{E}\int_0^t\left|\Psi(X_\lambda^\nu(s))-\Psi(X_{\lambda'}^\nu(s))\right|_2^2ds\\
\le{}& C_1\mathbb{E}\int_0^t\|X_\lambda^\nu(s)-X_{\lambda'}^\nu(s)\|^2_{F^*_{1,2,\nu}}ds\\
&-2\mathbb{E}\int_0^t\big\langle\lambda X_\lambda^\nu(s)-\lambda' X_{\lambda'}^\nu(s),X_\lambda^\nu(s)-X_{\lambda'}^\nu(s)\big\rangle_2ds\\
&+2\mathbb{E}\Big[\int_0^t\|X_\lambda^\nu(s)-X_{\lambda'}^\nu(s)\|^2_{F^*_{1,2,\nu}}\cdot\|B(s,X_\lambda^\nu(s))-B(s,X_{\lambda'}^\nu(s))\|^2_{F^*_{1,2,\nu}}ds\Big]^{\frac12}\\
\le{}&\frac12\mathbb{E}\Big[\sup_{s\in[0,t]}\|X_\lambda^\nu(s)-X_{\lambda'}^\nu(s)\|^2_{F^*_{1,2,\nu}}\Big]+C\mathbb{E}\int_0^t\|X_\lambda^\nu(s)-X_{\lambda'}^\nu(s)\|^2_{F^*_{1,2,\nu}}ds\\
&+4(\lambda+\lambda')\mathbb{E}\int_0^t\big(|X_\lambda^\nu(s)|_2^2+|X_{\lambda'}^\nu(s)|_2^2\big)ds. \qquad (3.29)
\end{aligned}$$

Since $x\in L^2(\mu)$, Gronwall's lemma and Claim 3.1 imply for some constant $C\in(0,\infty)$ independent of λ, λ' (and ν),

$$\mathbb{E}\Big[\sup_{s\in[0,T]}\|X_\lambda^\nu(s)-X_{\lambda'}^\nu(s)\|^2_{F^*_{1,2}}\Big]+\mathbb{E}\int_0^T\left|\Psi(X_\lambda^\nu(s))-\Psi(X_{\lambda'}^\nu(s))\right|_2^2ds\le C(\lambda+\lambda'). \quad (3.30)$$

(3.30) implies that there exists an $\mathscr{F}_t$-adapted continuous $F^*_{1,2}$-valued process $\{X^\nu(t)\}_{t\in[0,T]}$ such that $X^\nu\in L^2(\Omega;C([0,T],F^*_{1,2}))$. This together with Claim 3.1 implies that $X^\nu\in L^2([0,T]\times\Omega;L^2(\mu))$. ☐

Claim 3.3 *X^ν satisfies (3.7).*

Proof From Claim 3.2, we know that

$$X_\lambda^\nu\to X^\nu \ \text{ and } \int_0^\bullet B(s,X_\lambda^\nu(s))dW(s)\to\int_0^\bullet B(s,X^\nu(s))dW(s),\qquad \lambda\to0 \qquad (3.31)$$

in $L^2(\Omega;C([0,T],F^*_{1,2}))$. (3.17), (3.31) yield that

$$\int_0^\bullet\big(\Psi(X_\lambda^\nu(s)+\lambda X_\lambda^\nu(t))\big)ds,\ \lambda>0,$$

converge to some element in $L^2(\Omega;C([0,T],F_{1,2}))$ as $\lambda\to0$. In addition, by Claim 3.1, we have that, as $\lambda\to0$,

$$\int_0^\bullet(\Psi(X_\lambda^\nu(s))+\lambda X_\lambda^\nu(s))ds\to\int_0^\bullet\Psi(X^\nu(s))ds$$

in $L^2(\Omega;L^2([0,T];L^2(\mu)))$. This and (3.31) imply the claim. ☐

By lower semi-continuity, (3.8) follows immediately from Claim 3.1. Hence the proof of Lemma 3.1 is complete. ☐

Based on Lemma 3.1, we shall now give the proof of our main result Theorem 3.1. The idea is to prove that $\{X^\nu\}_{\nu\in(0,1)}$ converges to the solution of (1.1) as $\nu \to 0$. The method that we use here is similar to that in Lemma 3.1.

Proof of Theorem 3.1

First, we rewrite (3.5) as

$$dX^\nu(t) + (1-L)\Psi(X^\nu(t))dt = (1-\nu)\Psi(X^\nu(t))dt + B(t, X^\nu(t))dW(t).$$

For the function $\varphi(x) = \frac{1}{2}\|x\|^2_{F^*_{1,2}}$ with $x \in F^*_{1,2}$, Itô's formula yields

$$\begin{aligned}
&\frac{1}{2}\mathbb{E}\|X^\nu(t)\|^2_{F^*_{1,2}} + \int_0^t \langle \Psi(X^\nu(s)), X^\nu(s)\rangle_2 ds \\
= &\frac{1}{2}\|x\|^2_{F^*_{1,2}} + (1-\nu)\mathbb{E}\int_0^t \langle \Psi(X^\nu(s)), X^\nu(s)\rangle_{F^*_{1,2}} ds \\
&+\frac{1}{2}\mathbb{E}\int_0^t \|B(s, X^\nu(s))\|^2_{L_2(L^2(\mu),F^*_{1,2})} ds.
\end{aligned} \tag{3.32}$$

The condition **(H1)** implies

$$\Psi(r)r \geq \tilde{\alpha}\cdot|\Psi(r)|^2, \ r \in \mathbb{R}. \tag{3.33}$$

By (3.32) and (3.33), we have

$$\begin{aligned}
&\frac{1}{2}\mathbb{E}\|X^\nu(t)\|^2_{F^*_{1,2}} + \tilde{\alpha}\cdot\mathbb{E}\int_0^t |\Psi(X^\nu(s))|_2^2 ds \\
\leq &\frac{1}{2}\|x\|^2_{F^*_{1,2}} + \mathbb{E}\int_0^t \|\Psi(X^\nu(s))\|_{F^*_{1,2}}\cdot\|X^\nu(s)\|_{F^*_{1,2}} ds \\
&+\frac{1}{2}C_2\mathbb{E}\int_0^t \|X^\nu(s)\|^2_{F^*_{1,2}} ds.
\end{aligned}$$

Since $L^2(\mu)$ is continuously embedded into $F^*_{1,2}$, Young's inequality and the Gronwall's inequality yield that there exists a constant $C \in (0,\infty)$ such that, for $t \in [0,T]$ and $\nu \in (0,1)$,

$$\mathbb{E}\|X^\nu(t)\|^2_{F^*_{1,2}} \leq C\|x\|^2_{F^*_{1,2}}. \tag{3.34}$$

In the following, we will prove the convergence of $\{X^\nu\}_{\nu\in(0,1)}$. Applying Itô's formula to $\|X^\nu(t) - X^{\nu'}(t)\|^2_{F^*_{1,2}}$, we get that, for all $t \in [0,T]$,

$$\begin{aligned}
&\|X^\nu(t) - X^{\nu'}(t)\|^2_{F^*_{1,2}} + 2\int_0^t \big\langle (\Psi(X^\nu(s)) - \Psi(X^{\nu'}(s)), X^\nu(s) - X^{\nu'}(s)\big\rangle_2 ds \\
= &2\int_0^t \big\langle \Psi(X^\nu(s)) - \Psi(X^{\nu'}(s)), X^\nu(s) - X^{\nu'}(s)\big\rangle_{F^*_{1,2}} ds \\
&-2\int_0^t \big\langle \nu\Psi(X^\nu(s)) - \nu'\Psi(X^{\nu'}(s)), X^\nu(s) - X^{\nu'}(s)\big\rangle_{F^*_{1,2}} ds \\
&+2\int_0^t \|B(s, X^\nu(s)) - B(s, X^{\nu'}(s)\|^2_{L^2(L^2(\mu),F^*_{1,2})} \\
&+2\int_0^t \big\langle X^\nu(s) - X^{\nu'}(s), (B(s, X^\nu(s)) - B(s, X^{\nu'}(s)))dW(s)\big\rangle_{F^*_{1,2}} ds.
\end{aligned} \tag{3.35}$$

The second term on the right hand side of (3.35) can be dominated by

$$
\begin{aligned}
&-2\int_0^t \big\langle \nu\Psi(X^\nu(s)) - \nu'\Psi(X^{\nu'}(s)), X^\nu(s) - X^{\nu'}(s)\big\rangle_{F^*_{1,2}} ds \\
&\le 2C\int_0^t (\nu|\Psi(X^\nu(s))|_2 + \nu'|\Psi(X^{\nu'}(s))|_2)\cdot \|X^\nu(s) - X^{\nu'}(s)\|_{F^*_{1,2}} ds. \qquad (3.36)
\end{aligned}
$$

By assumption **(H1)** on Ψ and (3.33), we obtain

$$
\begin{aligned}
&2\int_0^t \big\langle (\Psi(X^\nu(s)) - \Psi(X^{\nu'}(s)), X^\nu(s) - X^{\nu'}(s)\big\rangle_2 ds \\
&= 2\int_0^t \int_E \big(\Psi(X^\nu(s)) - \Psi(X^{\nu'}(s))\big)\cdot (X^\nu(s) - X^{\nu'}(s)) d\mu ds \\
&\ge 2\int_0^t \int_E \tilde{\alpha}|\Psi(X^\nu(s)) - \Psi(X^{\nu'}(s))|^2 d\mu ds \\
&= 2\tilde{\alpha}\int_0^t |\Psi(X^\nu(s)) - \Psi(X^{\nu'}(s))|_2^2 ds. \qquad (3.37)
\end{aligned}
$$

(3.35)-(3.37) imply

$$
\begin{aligned}
&\|X^\nu(t) - X^{\nu'}(t)\|^2_{F^*_{1,2}} + 2\tilde{\alpha}\int_0^t |\Psi(X^\nu(s)) - \Psi(X^{\nu'}(s))|_2^2 ds \\
&\le C_1\int_0^t |\Psi(X^\nu(s)) - \Psi(X^{\nu'}(s))|_2 \cdot \|X^\nu(s) - X^{\nu'}(s)\|_{F^*_{1,2}} ds \\
&\quad + C_2\int_0^t (\nu|\Psi(X^\nu(s))|_2 + \nu'|\Psi(X^{\nu'}(s))|_2)\cdot \|X^\nu(s) - X^{\nu'}(s)\|_{F^*_{1,2}} ds \\
&\quad + C_3\int_0^t \|X^\nu(s) - X^{\nu'}(s)\|^2_{F^*_{1,2}} ds \\
&\quad + 2\int_0^t \big\langle X^\nu(s) - X^{\nu'}(s), \big(B(s,X^\nu(s)) - B(s,X^{\nu'}(s))\big)dW(s)\big\rangle_{F^*_{1,2}} ds.
\end{aligned}
$$

Taking expectation of both sides of the above inequality and using Young's and the BDG inequalities, we obtain, for all $t \in [0,T]$,

$$
\begin{aligned}
&\mathbb{E}\Big[\sup_{s\in[0,t]} \|X^\nu(s) - X^{\nu'}(s)\|^2_{F^*_{1,2}}\Big] + 2\tilde{\alpha}\mathbb{E}\int_0^t |\Psi(X^\nu(s)) - \Psi(X^{\nu'}(s))|_2^2 ds \\
&\le \frac{1}{2}\mathbb{E}\Big[\sup_{s\in[0,t]} \|X^\nu(s) - X^{\nu'}(s)\|^2_{F^*_{1,2}}\Big] + \tilde{\alpha}\mathbb{E}\int_0^t |\Psi(X^\nu(s)) - \Psi(X^{\nu'}(s))|_2^2 ds \\
&\quad + C_1\mathbb{E}\int_0^t \|X^\nu(s) - X^{\nu'}(s)\|_{F^*_{1,2}} ds + C_2\mathbb{E}\int_0^t \Big(\nu|\Psi(X^\nu(s))|_2^2 + \nu'|\Psi(X^{\nu'}(s))|_2^2\Big) ds.
\end{aligned}
$$

This yields

$$
\begin{aligned}
&\mathbb{E}\Big[\sup_{s\in[0,t]} \|X^\nu(s) - X^{\nu'}(s)\|^2_{F^*_{1,2}}\Big] + 2\tilde{\alpha}\mathbb{E}\int_0^t |\Psi(X^\nu(s)) - \Psi(X^{\nu'}(s))|_2^2 ds \\
&\le C_1\mathbb{E}\int_0^t \|X^\nu(s) - X^{\nu'}(s)\|_{F^*_{1,2}} ds \\
&\quad + C_2(\nu+\nu')\mathbb{E}\int_0^t (|\Psi(X^\nu(s))|_2^2) + |\Psi(X^{\nu'}(s))|_2^2\big) ds. \qquad (3.38)
\end{aligned}
$$

Note that if the initial value $x \in F_{1,2}^*$ and (3.4) is satisfied, we have (3.34). If $x \in L^2(\mu)$, we have (3.8). Hence, Gronwall's inequality and Young's inequality yield that there exists a positive constant $C \in (0, \infty)$ which is independent of ν, ν' such that

$$\mathbb{E}\Big[\sup_{s\in[0,T]} \|X^\nu(s) - X^{\nu'}(s)\|^2_{F_{1,2}^*}\Big] + \mathbb{E}\int_0^T \big|\Psi(X^\nu(s)) - \Psi(X^{\nu'}(s))\big|_2^2 ds$$
$$\leq C(\nu + \nu').$$

Hence, there exists an $\mathscr{F}_t$-adapted continuous $F_{1,2}^*$-valued process $X = (X_t)_{t\in[0,T]}$ such that $X \in L^2(\Omega; C([0,T], F_{1,2}^*)) \cap L^2([0,T] \times \Omega; L^2(\mu))$.

The remaining part of the proof is similar to that in Claim 3.3. Consequently, Theorem 3.1 is completely proved. □

4 Some Examples

4.1 Classical Dirichlet forms with densities

We apply Theorem 3.1 to the Friedrichs extension of the operator

$$Lu = \Delta u + 2\frac{\nabla \rho}{\rho} \cdot \nabla u, \quad u \in C_0^\infty(\mathbb{R}^d), \tag{4.1}$$

on $L^2(\rho^2 dx)$, where dx denotes Lebesgue measure and $\rho \in H^1(\mathbb{R}^d)$. Here H^1 is the usual Sobolev space and H^{-1} denotes its dual space.

In this case, equation (1.1) can be written as:

$$\begin{cases} dX(t) - (\Delta + 2\frac{\nabla \rho}{\rho} \cdot \nabla)\Psi(X(t))dt = B(t, X(t))dW(t), \text{ on } [0,T] \times \mathbb{R}^d, \\ X(0) = x \text{ on } \mathbb{R}^d, \end{cases} \tag{4.2}$$

i.e., here we choose E to be $\mathbb{R}^d$, $\mathscr{B}(E)$ to be $\mathscr{B}(\mathbb{R}^d)$, $\mu := \rho^2 dx$. Now let us determine $F_{1,2}$ and hence $F_{1,2}^*$. Clearly, $\Delta u \in L^2(\rho^2 dx)$, since $u \in C_0^\infty(\mathbb{R}^d)$. In addition, since $\rho \in H^1$,

$$2\frac{\nabla \rho}{\rho} \cdot \nabla u \in L^2(\rho^2 dx).$$

Hence L is a well-defined linear operator from $C_0^\infty(\mathbb{R}^d)$ to $L^2(\rho^2 dx)$. To apply Theorem 3.1, we need to find a strongly continuous contraction semigroup on $L^2(\rho^2 dx)$. The tool we use here is based on Dirichlet space theory, we refer to [13].

Since

$$\begin{aligned} \int Lu \cdot v\rho^2 dx &= \int (\Delta u + 2\frac{\nabla \rho}{\rho} \cdot \nabla u) \cdot v\rho^2 dx \\ &= \int \Delta u \cdot v\rho^2 dx + 2\int \frac{\nabla \rho}{\rho} \cdot \nabla u \cdot v\rho^2 dx \\ &= \int div\nabla u \cdot v\rho^2 dx = -\int \nabla u \cdot \nabla(v\rho^2) dx \\ &= -\int \nabla u \cdot \nabla v\rho^2 dx - \int \nabla u \cdot v \cdot 2\rho \cdot \nabla \rho dx \\ &= -\int \nabla u \cdot \nabla v\rho^2 dx = \int u \cdot Lv\rho^2 dx, \end{aligned}$$

which implies both that L is a symmetric operator and

$$\langle Lu, u\rangle \leq 0.$$

According to [13, Proposition 3.3], we hence know that there exists a Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(\rho^2 dx)$, which is in fact the closure of

$$\mathcal{E}(u,v) = \int \langle \nabla u, \nabla v\rangle_{\mathbb{R}^d} \rho^2 dx, \quad \text{for all} \quad u, v \in C_0^\infty(\mathbb{R}^d),$$

on $L^2(\rho^2 dx)$ such that its generator $(L, D(L))$ is an extension of the operator defined in (4.1). $(L, D(L))$ is thus the Friedrichs extension of $(L, C_0^\infty(\mathbb{R}^d))$ on $L^2(\mathbb{R}^d, \rho^2 dx)$.

As a result, we know $T_t = e^{tL}$ is the desired strongly continuous contraction sub-Markovian semigroup on $L^2(\mathbb{R}^d, \rho^2 dx)$ and $F_{1,2} = \mathscr{D}(\mathcal{E})$ with inner product

$$\langle u, v\rangle_{F_{1,2}} = \int \big(\langle \nabla u, \nabla v\rangle_{\mathbb{R}^d} + u \cdot v\big)\rho^2 dx, \; u, \; v \in \mathscr{D}(\mathcal{E}).$$

Now, we can use Theorem 3.1 to get the existence and uniqueness of the solutions to equation (4.2) for any B, Ψ satisfying **(H1)**, **(H2)** with $L^2(\rho^2 dx)$ and $F_{1,2}$ as above.

4.2 General regular symmetric case

The example in Section 4.1 is a special case of the example in [13, Chapter 2]: Let $E := U \subset \mathbb{R}^d$, U open, and m a positive Radon measure on U such that $\text{supp}[m] = U$. For $u, v \in C_0^\infty(U)$, define

$$\begin{aligned} \mathcal{E}(u,v) := &\sum_{i,j=1}^d \int \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} d\nu_{ij} \\ &+ \int_{U\times U\setminus\Delta} (u(x)-u(y))(v(x)-v(y))J(dx,dy) + \int uv \; dk. \end{aligned} \tag{4.3}$$

Here k is a positive Radon measure on U and J is a symmetric positive Radon measure on $U \times U \setminus \Delta$, where $\Delta := \{(x,x)|x \in U\}$, such that for all $u \in C_0^\infty(U)$

$$\int |u(x) - u(y)|^2 J(dxdy) < \infty. \tag{4.4}$$

For $1 \leq i, j \leq d$, ν_{ij} is a Radon measure on U such that for every $K \subset U$, K compact, $\nu_{ij}(K) = \nu_{ji}(K)$ and $\sum_{i,j=1}^d \xi_i \xi_j \nu_{ij}(K) \geq 0$ for all $\xi_i, \cdots, \xi_d \in \mathbb{R}^d$.

Then $(\mathcal{E}, C_0^\infty(U))$ is a densely defined symmetric positive definite bilinear form on $L^2(U; m)$.

Suppose that $(\mathscr{E}, C_0^\infty(U))$ is closable on $L^2(U; m)$ and let $(\mathscr{E}, D(\mathscr{E}))$ be its closure, then $(\mathscr{E}, D(\mathscr{E}))$ is a symmetric Dirichlet form. Hence by [13] we know there exists a self-adjoint negative definite linear operator $(L, D(L))$ on $L^2(U; m)$ defined by

$$D(L) := \{u \in D(\mathscr{E}) | \exists \; Lu \in L^2(m), \quad s.t. \;\; \mathscr{E}(u,v) = (-Lu, v), \forall v \in D(\mathscr{E})\}.$$

Hence $(L, D(L))$ is the generator of a sub-Markovian strongly continuous contraction semigroup $(T_t)_{t>0}$ on $L^2(U; m)$ given by

$$T_t := e^{tL}, \; t > 0.$$

Hence we can apply our Theorem 3.1 with the above generator $(L, D(L))$ to obtain a solution to SDE (1.1) for this L, and $F_{1,2} := D(\mathcal{E})$.

Remark:

(i) Our result thus in particular applies to the case where L is the fractional Laplace operator

$$L := -(-\Delta)^{\alpha}, \ \alpha \in (0, 1],$$

since it is just a special case of the above (see [13, Chapter 2]).

(ii) Similarly, using Dirichlet form theory on fractals, Theorem 3.1 applies when L is the Laplace operator on a fractal to solve (1.1) where the state space E is this fractal, (see, e.g., in [8, 11] for details).

Acknowledgements

The second and third authors acknowledge the financial support from NSF of China and the project funded by the Priority Academic Program Development of Jiangsu Higher Education Institutions. Support by the DFG' through CRC 701 is also gratefully acknowledged.

References

[1] S. Albeverio, M. Röckner, *Stochastic differential equations in infinite dimensions: solutions via Dirichlet form*, Probab. Th. Rel. Fields, 89(1991) 347-386.

[2] V. Barbu, G. Da Prato, M. Röckner, *Stochastic Porous Media Equations*. To appear as a Springer Lecture Notes.

[3] V. Barbu, M. Röckner, F. Russo, *Stochastic porous media equation in* $\mathbb{R}^d$, J. Math. Pures Appl. (9) 103(2015), no.4, 1024-1052.

[4] Y. Egorov, M. Shubin, *Foundations of the classical theory of partial differential equations*, Springer-Verlag Berlin Heidelberg, 1998.

[5] M. Fukushima, H. Kaneko, *(r,p)-capacities for general Markovian semigroups*, Infinite-dimensional analysis and stochastic processes (Bielefeld, 1983), 41-47, Res. Notes in Math., 124, Pitman, Boston, MA, 1985.

[6] N. Jacob, R. Schilling, *Towards an L^p potential theory for sub-Markovian semigroups: kernels and capacities*, Acta Math. Sin. (Engl. Ser.), 22(2006), no.4, 1227-1250.

[7] H. Kaneko, *On (r,p)-capacities for Markov process*, Osaka J. Math. 23(1986), no.2, 325-336.

[8] J. Kigami, *A harmonic calculus on the Sierpiński spaces*, Japan J. Appl. Math. 6(1989), no.2, 259-290.

[9] N.V. Krylov, *Itô's formula for the L_p-norm of stochastic W_p^1-valued processes*, Probab. Theory Related Fields, 147 (3-4)(2010) 583-605.

[10] N.V. Krylov, B.L. Rozovskii, *Stochastic evolution equation*, Plenum Publishing Corp., 1981; Translated from Itogi Naukii Tekhniki, Seriya Sovremennye Problemy Matematiki 14(1979), 71-146.

[11] S. Kusuoka, *Dirichlet forms on fractals and products of random matrices*, Publ. Res. Inst. Math. Sci. 25(1989), no.4, 659-680.

[12] W. Liu, M. Röckner, *Stochastic partial differential equations: an introduction*. Springer International Publishing Switzerland, 2015.

[13] Z.M. Ma, M. Röckner, *Introduction to the theory of (non-symmetric) Dirichlet forms*, Springer-Verlag Berlin Heidelberg, 1992.

[14] C. Prévôt, M. Röckner, *A concise course on stochastic partial differential equation*, Vol. 1905 of Lecture Notes in Mathematics, Springer, Berlin, 2007.

[15] J. Ren, M. Röckner, F.Y. Wang, *Stochastic generalized porous media and fast diffusion equations*, J. Differential Equations 238(2007), no.1, 118-152.

[16] M. Röckner, F.Y. Wang, *Non-monotone stochastic generalized porous media equations*, J. Differential Equations 245(2008), no.12, 3898-3935.

[17] E.M. Stein, *Singular integral and differentiability properties of functions*, Princeton Univ. Press, Princeton, 1970.