Optimal bilinear control of nonlinear stochastic Schrödinger equations driven by linear multiplicative noise [1]

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Abstract. We analyze the bilinear optimal control problem of quantum mechanical systems with final observation governed by a stochastic nonlinear Schrödinger equation perturbed by a linear multiplicative Wiener process. The existence of an open loop optimal control and first order Lagrange optimality conditions are derived, via Skorohod's representation theorem, Ekeland's variational principle and the existence for the linearized dual backward stochastic equation. Moreover, our approach in particular applies to the deterministic case.

Keywords: Backward stochastic equation, nonlinear Schrödinger equation, optimal control, Wiener process.

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1 Introduction

We consider the controlled stochastic system governed by the nonlinear Schrö-dinger equation

$$
\begin{aligned}
idX(t,\xi) \;=\; & \Delta X(t,\xi)dt + \lambda|X(t,\xi)|^{\alpha-1}X(t,\xi)dt - i\mu(\xi)X(t,\xi)dt \\
& +V_0(\xi)X(t,\xi)dt + \sum_{j=1}^{m} u_j(t)V_j(\xi)X(t,\xi)dt \\
& +iX(t,\xi)dW(t,\xi), \quad t \in (0,T),\ \xi \in \mathbb{R}^d,
\end{aligned}
$$
$$
X(0) \;=\; x \ \text{ in } \mathbb{R}^d. \tag{1.1}
$$

Here $\lambda = \pm 1,\ \alpha > 1,\ V_j \in W^{1,\infty}(\mathbb{R}^d),\ 0 \le j \le m$, are real valued functions, W is the Wiener process,

$$
W(t,\xi) = \sum_{j=1}^{N} \mu_j e_j(\xi)\beta_j(t),\ t \ge 0,\ \xi \in \mathbb{R}^d, \tag{1.2}
$$

and

$$
\begin{aligned}
u(t) \;&=\; (u_1(t),...,u_m(t)) \in \mathbb{R}^m,\ t \in (0,T), \\
\mu(\xi) \;&=\; \frac{1}{2}\sum_{j=1}^{N} |\mu_j|^2 e_j^2(\xi), \xi \in \mathbb{R}^d,\ d \ge 1,
\end{aligned}
$$

with μ_j purely imaginary numbers (i.e. $Re\mu_j = 0$), $e_j(\xi)$ real-valued functions and β_j independent real Brownian motions on a probability space $(\Omega,\mathcal{F},\mathbb{P})$ with natural filtration $(\mathcal{F}_t)_{t\ge0}$, $1 \le j \le N$. For simplicity, we assume $N < \infty$, but the arguments in this paper easily extend to the case where $N = \infty$.

The physical significance of (1.1) is well known. $X = X(t,\xi,\omega),\ \xi \in \mathbb{R}^d$, $t \ge 0,\ \omega \in \Omega$, represents the quantum state at time t, while the stochastic perturbation $iX\,dW$ represents a stochastic continuous measurement via the pointwise quantum observables $R_j(X) = \mu_j e_j X$. The functin V_0 describes an external potential.

In the conservative case considered in this paper (i.e. $Re\mu_j = 0,\ 1 \le j \le N$), $-i\mu X dt + iX dW$ is indeed the Stratonovitch differential. It follows by Itô's formula that $|X(t)|_2^2 = |x|_2^2,\ \forall t \ge 0$. Hence, normalizing the initial state $|x|_2 = 1$, we have $|X(t)|_2 = 1,\ \forall t \in [0,T]$, and so, the quantum system evolves on the unit ball of L^2 and verifies the conservation of probability. See e.g. [8, 9].

We also mention that, for the general case when μ_j are complex numbers, one of the main feature is that the mean norm square $|X(t)|_2^2$, $t \in [0, T]$, is a continuous martingale. This fact enables one to define the "physical" probability law and implies the conservation of $\mathbb{E}|X(t)|_2^2$, $t \in [0, T]$, which plays an important role in the application to open quantum systems. See, e.g., [4] for more details. See also [1, 2] for global well-posedness with exponents of the nonlinearity in the optimal subcritical range.

As regards the real valued input control u, in most situations it represents an external applied force due to the interaction of the quantum system with an electric field or a laser pulse applied to a quantum system.

Here, motivated by the optimal control of quantum mechanical systems (cf. e.g. [24], [29]), we shall study an optimal control problem associated with the control system (1.1) which, in a few words, can be described as follows (see Problem (P) below): find an input control u that steers in time T the state X as close as possible of a target state $\mathbb{X}_T$ and a given trajectory $\mathbb{X}_1$, and with a reasonable minimum energy. Roughly speaking, this means to find the quantum mechanical potential u from observation of the quantum state $X(T)$ at the end of time interval $[0, T]$.

It should be mentioned that, there is an extensive literature on the deterministic bilinear control equation (1.1) mainly concerned with exact controllability in time T of Schrödinger equations or with the optimal control problem (see, for instance, [5], [6], [7], [16], [18], [19], [23], [24], [29]). However, there are very few results on optimal control problems governed by nonlinear Schrödinger equations (i.e. $\alpha > 1$) and, to the best of our knowledge, none for stochastic control systems (1.1) with linear multiplicative noise. In the latter case, the existence of an optimal control is largely an open problem, since the cost functional is not simultaneously lower semicontinuous and coercive in the basic control space. It is also very interesting to consider the control in the diffusion coefficient (see e.g. [30]). In this paper, motivated by the quantum control problem mentioned above, we will focus on the case where the control acts in the potential of the quantum mechanical system.

The approach we used here is based on Skorohod's representation theorem and Ekeland's variational principle, and this is one of the main novelties of this work. The approach is also based on an existence result of the linearized backward dual stochastic equation, which is also new in the literature and uses sharp stochastic estimates for linear Schrödinger equations with time dependent coefficients (see [1, 2]). As a matter of fact, a great effort of this work is dedicated to this issue.

2 Formulation of problem and the main results

To begin with, we recall the definition of a strong solution to equation (1.1) (see [1], [2]).

Definition 2.1. Let $x \in L^2$ (resp. H^1), $0 < T < \infty$. Let α satisfy $1 < \alpha < 1 + \frac{4}{d}$ (resp. $1 < \alpha < 1 + \frac{4}{(d-2)_+}$), $d \geq 1$. A strong L^2-(resp. H^1-)solution to (1.1) on $[0, T]$ is an L^2-(resp. H^1-)valued continuous $(\mathcal{F}_t)_{t \geq 0}$-adapted process $X = X(t)$ such that $|X|^{\alpha-1}X \in L^1(0, T; H^{-1})$, and $\mathbb{P}$-a.s.,

$$
X(t) = x - \int_0^t \left(i\Delta X(s) + \mu X(s) + \lambda i |X(s)|^{\alpha-1} X(s, \xi) + iV_0(\xi)X(s) \right.
$$
$$
\left. + i \sum_{j=1}^m u_j(s)V_j(\xi)X(s) \right) ds + \int_0^t X(s)dW(s), \ t \in [0, T],
$$

$$(2.1)$$

as an Itô equation in H^{-2} (resp. H^{-1}). Here, the controller $u = (u_1, \cdots, u_m)$ is in the admissible set $\mathcal{U}_{ad}$, defined by

$$
\mathcal{U}_{ad} = \left\{ u \in L^2_{ad}(0, T; \mathbb{R}^m); \ u \in U, \ \ a.e. \ on \ (0, T) \times \Omega. \right\}, \qquad (2.2)
$$

where $L^2_{ad}(0, T; \mathbb{R}^m)$ is the space of all $(\mathcal{F}_t)_{t \geq 0}$-adapted $\mathbb{R}^m$-valued processes $u : [0, T] \to \mathbb{R}^m$ such that $u \in L^2((0, T) \times \Omega; \mathbb{R}^m)$, and U is a compact convex subset of $\mathbb{R}^m$.

It is easy to check that $\int_0^t X(s)dW(s)$ in Definition 2.1 is an L^2-(resp. H^1-)valued continuous stochastic integral. (We refer, e.g., to [10, 21] for the general theory of infinite dimensional stochastic integrals.) Moreover, $\sup_{u \in \mathcal{U}_{ad}} \|u\|_{L^\infty(0,T;\mathbb{R}^m)} \leq D_U < \infty$, where D_U is the diameter of U.

Following [1, 2], we introduce the hypotheses below.

(H0) $1 < \alpha < 1 + \frac{4}{d}$. For each $1 \leq j \leq N$, $e_j \in C_b^\infty(\mathbb{R}^d)$ satisfies

$$
\lim_{|\xi| \to \infty} \zeta(\xi) |\partial^\gamma e_j(\xi)| = 0, \qquad (2.3)
$$

where γ is a multi-index such that $1 \leq |\gamma| \leq 2$, and

$$
\zeta(\xi) = \begin{cases} 1 + |\xi|^2, & \text{if } d \neq 2, \\ (1 + |\xi|^2(\ln(3 + |\xi|^2))^2, & \text{if } d = 2. \end{cases}
$$

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(H1) In the defocusing case $\lambda = -1$, $1 < \alpha < 1 + \frac{4}{(d-2)_+}$, and in the focusing case $\lambda = 1$, $1 < \alpha < 1 + \frac{4}{d}$. For each $1 \le j \le N$, $e_j \in C_b^\infty(\mathbb{R}^d)$ satisfies (2.3) for any multi-index $1 \le |\gamma| \le 3$.

The global existence, uniqueness and uniform estimates of the solution to (1.1) used in this paper are summarized in Proposition 2.2 below.

Proposition 2.2. *Assume $(H0)$ (resp. $(H1)$). For each $x \in L^2$ (resp. H^1), $u \in \mathcal{U}_{ad}$ and $0 < T < \infty$, there exists a unique strong L^2-(resp. H^1-)solution X^u to (1.1), satisfying $|X(t)|_2 = |x|_2$, $t \in [0, T]$ (resp. for any $\rho \ge 1$,*

$$\sup_{u \in \mathcal{U}_{ad}} \mathbb{E}\|X^u\|_{C([0,T];H^1)}^\rho < \infty). \tag{2.4}$$

Moreover, assuming that the exponent α is in the range specified in $(H1)$ and that e_k are constants, $1 \le k \le N$, we have for any $\rho \ge 1$,

$$\sup_{u \in \mathcal{U}_{ad}} \left(\|X^u\|_{L^\infty(\Omega;L^q(0,T;L^p))} + \|X^u\|_{L^\rho(\Omega;L^q(0,T;W^{1,p}))} \right) < \infty, \tag{2.5}$$

where (p, q) is any Strichartz pair, i.e., $\frac{2}{q} = \frac{d}{2} - \frac{d}{p}$, $(p, q) \in [2, \infty] \times [2, \infty]$, and $(p, q, d) \ne (\infty, 2, 2)$.

The global existence and uniqueness can be proved similarly as in [1, 2] by the rescaling approach and the Strichartz estimates for lower order perturbations of the Laplacian. We refer to [1, Lemma 4.1] and [2, Lemma 2.7] for explicit formulations of Strichartz estimates in the L^p and Sobolev spaces respectively. The technical proof of the estimates (2.4) and (2.5) is postponed to the Appendix for simplicity of the exposition.

We mention that, for controlled stochastic Schrödinger equations with control also in the diffusion coefficient, i.e., $iXdW$ and μ in (1.1) are replaced by $iX \sum_{j=1}^N \sigma_j(u)\mu_j e_j d\beta_j(t)$ and $1/2 \sum_{j=1}^N (\sigma_j(u))^2 |\mu_j|^2 e_j^2$ respectively, one also has global well-posedness by using the rescaling transformation $y = e^{-\Phi(W)}X$, where $\Phi(W) = \sum_{j=1}^N \int_0^t \sigma(u(s))\mu_j e_j d\beta_j(s)$. However, the optimal control problem is technically more difficult. See Remark 2.9 below.

In the following, $L_{ad}^2(0, T; L^2(\Omega; L^2))$ is the space of L^2-valued $(\mathcal{F}_t)_{t \ge 0}$-adapted processes u such that $\mathbb{E}\int_0^T |u(t)|_2^2 dt < \infty$.

The optimal control problem we study in the following is

(P) *Minimize*

$$\mathbb{E}\left(|X(T) - \mathbb{X}_T|_2^2 + \gamma_1 \int_0^T |X(t) - \mathbb{X}_1(t)|_2^2 dt + \int_0^T (\gamma_2 |u(t)|_m^2 + \gamma_3 |u'(t)|_m^2) dt \right)$$

on all $(X, u) \in L^2_{ad}(0, T; L^2(\Omega; L^2)) \times \mathcal{U}_{ad}$ subject to (1.1).

Here, u' is the time derivative in the sense of distributions ($\Phi(u) = \infty$ if there is no such derivative), $\gamma_j \geq 0$, $1 \leq j \leq 3$, $\mathbb{X}_T \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; L^2)$ and $\mathbb{X}_1 \in L^2_{ad}(0, T; L^2(\Omega; L^2))$ are given. In most situations, $\mathbb{X}_1$ is a given trajectory of the uncontrolled system or, in particular, a steady state solution.

As seen earlier, due to the conservation of $|X(t)|^2_2$, by normalizing the initial state we have $|X(t)|_2 = 1$, and so Problem (P) reduces to

$$\underset{(u,X)}{\text{Min}} \left\{ -2\mathbb{E}Re \left\langle X(T), \mathbb{X}_T \right\rangle_2 - 2\gamma_1 \int_0^T Re \left\langle X(t), \mathbb{X}_1(t) \right\rangle_2 dt \right.$$
$$\left. + \int_0^T (\gamma_2 |u(t)|^2_m + \gamma_3 |u'(t)|^2_m) dt. \right\}$$

It should be said that in the quantum model V is a given potential which describes the spatial profile of an external field, while the control input $u = \{u_j\}^m_{j=1}$ is its intensity. The objective of the control process is to steer the quantum system from an initial state x to a target state $\mathbb{X}_T$ and also into the neighborhood of a given trajectory $\mathbb{X}_1$. The last term in the cost functional is the energy cost to obtain the desired objective.

Taking into account that in quantum mechanics the wave function X is not a physical observable, a more realistic situation is where in the cost functional $|X(T) - \mathbb{X}_T|^2_2$ is replaced by $\langle Q(X(T)) - \mathbb{X}_T, X(T) - \mathbb{X}_T \rangle_2$, where Q is a self-adjoint operator in L^2. However, its treatment is essentially the same.

By $\Phi : L^2_{ad}(0, T; \mathbb{R}^m) \to \mathbb{R}$ we denote the objective functional

$$\Phi(u) = \mathbb{E}|X^u(T) - \mathbb{X}_T|^2_2 + \gamma_1 \mathbb{E} \int_0^T |X^u(t) - \mathbb{X}_1(t)|^2_2 dt + \gamma_2 \mathbb{E} \int_0^T |u(t)|^2_m dt$$
$$+ \gamma_3 \mathbb{E} \int_0^T |u'(t)|^2_m dt, \tag{2.6}$$

we may reformulate Problem (P) as

$$\text{(P)} \quad \text{Min}\{\Phi(u); \ u \in \mathcal{U}_{ad}, \ X^u \ satisfies \ (1.1)\}. \tag{2.7}$$

It should be said that, since Problem (P) is a nonconvex minimization problem, in general it is not well posed. However, if $\gamma_2, \gamma_3 = 0$, we have the following generic existence result.

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Proposition 2.3. *Assume Hypothesis* $(H0)$, *and* $\gamma_2 = \gamma_3 = 0$. *Then, there is a residual set*

$$\mathcal{G} \subset L^2(\Omega, \mathcal{F}_T, \mathbb{P}, L^2) \times L^2_{ad}(0, T; L^2(\Omega; L^2))$$

such that, for every $(\mathbb{X}_T, \mathbb{X}_1) \in \mathcal{G}$, *Problem* (P) *has at least one solution* $u \in \mathcal{U}_{ad}$.

This is an immediate consequence of a well-known result of Edelstein [11] on existence of nearest points of closed sets in uniformly convex Banach spaces. Indeed, if we set $\mathcal{Y} = \{Y = (X^u(T), X^u); \ u \in \mathcal{U}_{ad}\}$, it follows that $\mathcal{Y}$ is a closed subset of $L^2(\Omega; \mathcal{F}_T, \mathbb{P}, L^2) \times L^2_{ad}(0, T; L^2(\Omega; L^2))$ (see e.g. the proof of Lemma 3.2 and 3.3) and so, rewriting Problem (P) as

$$\text{Min}\{\|(\mathbb{X}_T, \mathbb{X}_1) - Y\|^2_*; \ Y \in \mathcal{Y}\},$$

where $\| \cdot \|_*$ is the norm of $L^2(\Omega; \mathcal{F}_T, \mathbb{P}, L^2) \times L^2_{ad}(0, T; L^2(\Omega; L^2))$, we arrive at the desired conclusion.

However, for the general cases $\gamma_2, \gamma_3 \neq 0$, the existence of a solution in Problem (P) does not follow by standard compactness techniques used in deterministic optimization problems. The main reason is that, even if a space $\mathcal{Y}$ is compactly imbedded into another space $\mathcal{Z}$, one generally does not have the compact imbedding from $L^p(\Omega; \mathcal{Y})$ into $L^p(\Omega; \mathcal{Z})$, $1 \leq p \leq \infty$. Here, we consider the existence for relaxed versions of Problem (P) to be defined below.

Definition 2.4. Let $\mathcal{Y} := L^2(\mathbb{R}^d) \times L^2((0, T) \times \mathbb{R}^d) \times C([0, T]; \mathbb{R}^N) \times L^2(0, T; \mathbb{R}^m) \times L^2((0, T) \times \mathbb{R}^d)$ and $(\Omega^*, \mathcal{F}^*, (\mathcal{F}^*_t)_{t \geq 0})$ be a new filtered probability space, carrying $(\mathbb{X}^*_T, \mathbb{X}^*_1, \beta^*, u^*, X^*)$ in $\mathcal{Y}$. Define $L^2_{ad*}(0, T; L^2(\Omega; L^2))$, $\mathcal{U}_{ad*}$ and $\Phi^*(u^*)$ similarly as above on this new filtered probability space.

The system $(\Omega^*, \mathcal{F}^*, \mathbb{P}^*, (\mathcal{F}^*_t)_{t \geq 0}, \beta^*, u^*, X^*)$ is said to be *admissible*, if $\mathbb{X}^*_T \in L^2(\Omega, \mathcal{F}^*_T, \mathbb{P}^*; L^2)$, $\mathbb{X}^*_1 \in L^2_{ad*}(0, T; L^2(\Omega; L^2))$, $\beta^* = (\beta^*_1, \ldots, \beta^*_N)$ is an $(\mathcal{F}^*_t)_{t \geq 0}$-adapted $\mathbb{R}^N$-valued Wiener process, the joint distributions of $(\mathbb{X}^*_T, \mathbb{X}^*_1, \beta^*)$ and $(\mathbb{X}_T, \mathbb{X}_1, \beta)$ coincide, $u^* \in \mathcal{U}_{ad*}$, and X^* is an $(\mathcal{F}^*_t)_{t \geq 0}$-adapted L^2-valued process that satisfies equation (1.1) corresponding to (β^*, u^*).

The admissible system $(\Omega^*, \mathcal{F}^*, \mathbb{P}^*, (\mathcal{F}^*_t)_{t \geq 0}, \beta^*, u^*, X^*)$ is said to be a *relaxed solution* to the optimal control problem (P), if

$$\Phi^*(u^*) \leq \inf\{\Phi(u); \ u \in \mathcal{U}_{ad}, \ X^u \ satisfies \ (1.1)\}. \tag{2.8}$$

Similarly to martingale and strong solutions, one can have a non-relaxed optimal control by using e.g. the Gyöngy-Krylov argument as in [15], as long as the uniqueness of the optimal control is obtained, which reduces to the uniqueness of solutions to a forward-backward equation (see (1.1) and (2.9) below with u replaced by (2.12)) and which is generally very difficult due to the non-convexity of the optimal control problem (P).

We first prove that, under the regular condition of controls (i.e., $\gamma_3 > 0$), there exists a relaxed solutions for the exponents of the nonlinearity in exactly the mass-subcritical range. A similar problem was studied in [16] (see also [6], [19]) in the deterministic case. We have

Theorem 2.5. *Consider Φ with $\gamma_3 > 0$. Assume $(H0)$. Then, for each $x \in L^2$, $0 < T < \infty$, there exists at least one relaxed solution in the sense of Definition 2.4 to the optimal problem (P).*

The proof is mainly based on the Skorohod representation theorem and pathwise analysis of solutions by the rescaling approach devoloped in [1]. We would also like to mention that the rescaling approach allows to obtain pathwise continuous dependence of solutions on controls.

In order to construct a relaxed solution with equality in (2.8) in the more difficult irregular case (i.e., $\gamma_3 = 0$), we will employ the Ekeland principle and work with the dual backward stochastic equation below

$$dY = -i\Delta Y\, dt - \lambda i h_1(X^u)Y\, dt + \lambda i h_2(X^u)\overline{Y}\, dt + \mu Y\, dt - iV_0 Y\, dt - iu \cdot \nabla Y\, dt$$

$$+ \gamma_1(X^u - \mathbb{X}_1)dt - \sum_{k=1}^{N} \overline{\mu_k} e_k Z_k dt + \sum_{k=1}^{N} Z_k d\beta_k(t), \tag{2.9}$$

$$Y(T) = -(X^u(T) - \mathbb{X}_T),$$

where

$$h_1(X^u) := \frac{\alpha+1}{2}|X^u|^{\alpha-1}, \quad h_2(X^u) := \frac{\alpha-1}{2}|X^u|^{\alpha-3}(X^u)^2. \tag{2.10}$$

The functions h_j, $j = 1, 2$, are the complex derivatives of the complex function $z \to |z|^{\alpha-1}z$, i.e. $h_1(z) = \partial_z(|z|^{\alpha-1}z)$ and $h_2(z) = \partial_{\bar{z}}(|z|^{\alpha-1}z)$, $z \in \mathbb{C}$.

However, the singular coefficient $h_2(X^u)$ in (2.9) and the weak regularity effect of the Schrödinger group make it quite difficult to obtain the existence

and integrability of the backward solution. The standard method to derive a global estimate for $\mathbb{E}\|Y\|^2_{C([0,T];L^2)}$ from the Itô formula applied to $|Y(t)|^2_2$ are not applicable in the nonlinear case.

The idea here is to apply duality analysis to reduce the analysis of the backward stochastic equation to that of the dual equation (4.26) below (see also the equation of variation (4.3) below). By virtue of the forward character of the dual equation, we will apply the rescaling approach and the Strichartz estimates, instead of the Itô formula for $|Y(t)|^2_2$, to control the singular coefficient $h_2(X^u)$ and to obtain pathwise estimates of solutions on small intervals, which then by iteration yield the global pathwise estimates (4.10), (4.27) below. To this aim, we consider in this case the following basic hypothesis.

(H2) $2 \leq \alpha < 1 + \frac{4}{d}$, $1 \leq d \leq 3$, and e_k are constants, $1 \leq k \leq N$.

(In the case where e_k are not constant, which is ruled out here, there arise some delicate problems related to the nonintegrability of $(B_j)^{c(B_j)^{2/\theta}}$, where $B_k := \sup_{t\in[0,T]} |\beta_k(t)|$, $\theta \in (0,1)$ and $c > 0$, $1 \leq k \leq N$.)

It is easily seen that $(H2)$ implies $(H0)$ and $(H1)$ and also that $(H2)$ is fulfilled in some important physical models, for instance the Gross-Pitaevskii model when $d = 1$ ([16]). As a matter of fact, under Hypothesis $(H2)$, one has not only (2.8) with equality, but also that the optimal pair (X, u) satisfies the stochastic maximum principle. The main result is formulated below.

Theorem 2.6. *Consider* Φ *with* $\gamma_3 = 0$. *Assume Hypothesis* $(H2)$, *and* $\mathbb{X}_T \in L^{2+\nu}(\Omega; H^1)$, $\mathbb{X}_1 \in L^{2+\nu}(\Omega; L^2(0,T; H^1))$ *for some small* $\nu \in (0,1)$.

Then, for each $x \in H^1$, $0 < T < \infty$, *there exists a relaxed solution* $(\Omega^*, \mathscr{F}^*, \mathbb{P}^*, (\mathscr{F}_t^*)_{t\geq 0}, \beta^*, u^*, X^*)$ *in the sense of Definition 2.6 to Problem* (P), *such that*

$$\Phi^*(u^*) = \inf\{\Phi(u);\ u \in \mathcal{U}_{ad},\ X^u\ satisfies\ (1.1)\}. \qquad (2.11)$$

Moreover, we have (the stochastic maximum principle)

$$u^*(t) = P_U\left(\frac{1}{\gamma_2} \operatorname{Im} \int_{\mathbb{R}^d} V(\xi) X^*(t,\xi)\overline{Y^*}(t,\xi)d\xi\right),\ \forall t \in [0,T],\ \mathbb{P}^* - a.s. \tag{2.12}$$

where P_U *is the projection on* U, *and* (Y^*, Z^*) *is the solution to the dual backward stochastic equation* (2.9) *with* $\mathbb{X}_T, \mathbb{X}_1, \beta, u, X^u$ *replaced by* $\mathbb{X}_T^*, \mathbb{X}_1^*, \beta^*, u^*$, X^* *respectively.*

In the case where $\mu_k = 0$, $1 \le k \le N$, the noise W in (1.1) disappears and so, (1.1) is the deterministic controlled nonlinear Schrödinger equation. In this case, for the initial datum $x \in H^1$, the optimal control indeed exists for the exponent $\alpha \ge 2$ in the energy-subcritical case $(H1)$, which is also new in the literature.

Theorem 2.7. *In the deterministic case (i.e. $\mu_k = 0$, $1 \le k \le N$), consider Φ in (2.6) with $\gamma_3 = 0$ and the exponent $\alpha \ge 2$ in the range specified in Hypothesis $(H1)$. Assume that $\mathbb{X}_T \in H^1$ and $\mathbb{X}_1 \in L^2(0,T;H^1)$.*

Then, for each $x \in H^1$, $0 < T < \infty$, there exists an optimal control u to Problem (P) such that

$$\Phi(u) = \inf\{\Phi(v); \ v \in \mathcal{U}_{ad}, \ X^v \ satisfies \ (2.1)\}. \tag{2.13}$$

Moreover,

$$u(t) = P_U\left(\frac{1}{\gamma_2} \operatorname{Im} \int_{\mathbb{R}^d} V(\xi)X(t,\xi)\overline{Y}(t,\xi)d\xi\right), \ \forall t \in [0,T], \tag{2.14}$$

where P_U is the projection on U, and Y is the solution to the backward equation (2.9) with $Z = 0$.

Remark 2.8. Optimal bilinear control is also studied in [18] and [16] for linear and nonlinear deterministic Schrödinger equations respectively. In both papers, some compactness conditions of initial data or controls are needed for the existence of the optimal control. More precisely, in [18] the initial data belong to a compact subspace of L^2, while in [16] the minimizing controls are bounded in $H^1[0,T]$, hence compact in $L^2[0,T]$. In contrast to this, in Theorem 2.7 the existence of the optimal control is obtained without these conditions, and, unlike in [16], less regularity of the initial data is required for the maximum principle (2.14). The proof is quite different, it applies to more general initial data and as well to the stochastic case. Moreover, taking into account the importance of the energy state space H^1, we also expect the results and methods developed here to be applied to the study of blow-up and scattering phenomena (see e.g. [20], [3]).

Remark 2.9. For the control in the diffusion coefficient, it is technically more complicated to treat the equation of variation and the corresponding stochastic backward equation. It also involves the analysis of a second-order adjoint equation in the stochastic maximum principle (see e.g. Chapter 3 in [30]). Including control in the diffusion coefficient will be one subject that we plan to analyze in future.

The proof of Theorem 2.7 follows the lines of that of Theorem 2.6, the main part of which are the analysis of equation of variation (4.3) as well as of the backward stochastic equation (2.9), and the tightness of controls.

The key idea to obtain the tightness of controls in this irregular case is to employ the Ekeland principle, as well as the directional derivative of Φ, to obtain the representation formula of the minimizing controls (see (5.10) below). Then, by virtue of the integrability of the forward and backward solutions to (1.1) and (2.9) respectively, one is able to obtain the tightness of controls in the space $L^1(0, T; \mathbb{R}^d)$, which consequently yields equality in (2.8) by analogous arguments as in the proof of Theorem 2.5.

As mentioned above, the proof of integrability of the stochastic backward solution relies on duality analysis, which is also of independent interest.

The remaining part of this paper is organized as follows. Section 3 includes the proof of Theorem 2.5. Section 4 and Section 5 are mainly devoted to the proof of Theorem 2.6. Section 4 is concerned with the directional derivative of Φ, which requires the analysis of the equation of variation (4.3) and of the backward stochastic equation (2.9). Section 5 mainly contains the proof for the tightness of controls. The proof of Theorem 2.7 is also included there. For simplicity of the exposition, some auxiliary lemmas and technical proofs are postponed to the Appendix, i.e. Section 6.

Notations. For $1 \leq p \leq \infty$, we denote by $L^p(\mathbb{R}^d) = L^p$ the space of all Lebesgue p-integrable (complex-valued) functions on the real Euclidean space $\mathbb{R}^d$. The norm of L^p is denoted by $|\cdot|_{L^p}$, and $p' \in [1, \infty]$ denotes the unique number such that $\frac{1}{p} + \frac{1}{p'} = 1$. In particular, the Hilbert space $L^2(\mathbb{R}^d)$ is endowed with the scalar product

$$\langle y, z \rangle_2 = \int_{\mathbb{R}^d} y(\xi) \bar{z}(\xi) d\xi; \quad y, z \in L^2(\mathbb{R}^d),$$

where $\bar{z}$ is the complex conjugate of $z \in \mathbb{C}$. We also use $|\cdot|_2 = |\cdot|_{L^2}$. $W^{1,p} = W^{1,p}(\mathbb{R}^d)$ is the classical Sobolev space $\{v \in L^p; \nabla v \in L^p\}$ with the norm $\|v\|_{W^{1,p}} = |v|_{L^p} + |\nabla v|_{L^p}$, $H^1 = W^{1,2}$ and H^{-1} is the dual space of H^1.

By $L^q(0, T; L^p)$ we denote the space of all integrable L^p-valued functions $u : (0, T) \to L^p$ with the norm

$$\|u\|_{L^q(0,T;L^p)} = \left(\int_0^T \left(\int_{\mathbb{R}^d} |u(t, \xi)|^p d\xi \right)^{\frac{q}{p}} dt \right)^{\frac{1}{q}}.$$

By $C([0, T]; L^p)$ we denote the standard space of all L^p-valued continuous functions on $[0, T]$ with the sup norm in t. $L^q(0, T; W^{1,p})$ and $C([0, T]; H^1)$ are defined similarly. $\mathcal{D}(0, T; \mathbb{R}^m)$ is the set of all $\mathbb{R}^m$-valued smooth and compactly supported functions, and $\mathcal{D}'(0, T; \mathbb{R}^m)$ is its dual space.

Let $|\cdot|_m$ denote the Euclidean norm in $\mathbb{R}^m$ and $u \cdot v$ the scalar product of vectors $u, v \in \mathbb{R}^m$. We shall use standard notations to represent spaces of infinite dimensional stochastic processes (see, e.g., [10, 21]).

Throughout this paper, we use C for various constants that may change from line to line.

3 Proof of Theorem 2.5.

We set $I := \inf\{\Phi(u); u \in \mathcal{U}_{ad},\ X^u\ satisfies\ (1.1)\} > 0$ and consider a sequence $\{u_n\} \subset \mathcal{U}_{ad}$ such that

$$I \leq \Phi(u_n) \leq I + n^{-1}, \quad \forall n \in \mathbb{N}. \tag{3.1}$$

Since $\gamma_3 > 0$, this yields

$$\sup_{n \geq 1} \mathbb{E} \int_0^T (|u_n(t)|_m^2 + |u_n'(t)|_m^2) dt < \infty. \tag{3.2}$$

Lemma 3.1. Let $\mathbb{P} \circ u_n^{-1}$ be the laws of u_n, $n \geq 1$. Then, $\{\mathbb{P} \circ u_n^{-1}\}$ is tight on $C([0, T]; \mathbb{R}^m)$.

Proof. By the Arzelà theorem, it suffices to show that

$$\lim_{R \to \infty} \sup_{n \geq 1} \mathbb{P} \circ u_n^{-1} \left\{ v \in C([0, T]; \mathbb{R}^m) : \sup_{t \in [0, T]} |v(t)|_m > R \right\} = 0, \tag{3.3}$$

and for any $\varepsilon > 0$,

$$\lim_{\delta \to 0} \sup_{n \geq 1} \mathbb{P} \circ u_n^{-1} \left\{ v \in C([0, T]; \mathbb{R}^m) : \sup_{|t-s| \leq \delta} |v(t) - v(s)|_m > \varepsilon \right\} = 0. \tag{3.4}$$

In fact, (3.3) follows immediately form the uniform boundedness of $\{u_n\}$,

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while (3.4) follows by (3.2) and

$$\sup_{n\geq 1} \mathbb{P} \circ u_n^{-1} \left\{ v \in C([0,T];\mathbb{R}^m) : \sup_{|t-s|\leq \delta} |v(t) - v(s)|_m > \varepsilon \right\}$$

$$\leq \frac{1}{\varepsilon} \sup_{n\geq 1} \mathbb{E} \sup_{|t-s|\leq \delta} |u_n(t) - u_n(s)|_m$$

$$\leq \frac{\delta^{\frac{1}{2}}}{\varepsilon} \sup_{n\geq 1} \mathbb{E}\|u_n'\|_{L^2(0,T;\mathbb{R}^m)} \to 0, \quad as \ \delta \to 0. \tag{3.5}$$

$\square$

Now, consider the sequence $\mathcal{X}_n := (\mathbb{X}_T, \mathbb{X}_1, \beta, u_n)$ with $\beta = (\beta_1, \cdots, \beta_N)$ in the space $\mathcal{Y} := L^2(\mathbb{R}^d) \times L^2((0,T) \times \mathbb{R}^d) \times C([0,T];\mathbb{R}^N) \times C([0,T];\mathbb{R}^m)$. Lemma 3.1 implies that the laws of $\mathcal{X}_n$, $n \in \mathbb{N}$, are tight in the space $\mathcal{Y}$. Then, by Prohorov's theorem, they are weakly compact and so, by the Skorohod representation theorem, there exist a probability space $(\Omega^*, \mathcal{F}^*, \mathbb{P}^*)$ and $\mathcal{X}_n^* := ((\mathbb{X}_T^*)_n, \mathbb{X}_{1,n}^*, \beta_n^*, u_n^*)$, $\mathcal{X}^* := (\mathbb{X}_T^*, \mathbb{X}_1^*, \beta^*, u^*)$ in $\mathcal{Y}$, $n \in \mathbb{N}$, such that the joint distribution of $\mathcal{X}_n^*$ and $\mathcal{X}_n$ coincide, and $\mathbb{P}^*$-a.s., as $n \to \infty$,

$$\beta_n^* \to \beta^* \quad in \ C([0,T];\mathbb{R}^N), \tag{3.6}$$

$$(\mathbb{X}_T^*)_n \to \mathbb{X}_T^*, \quad in \ L^2(\mathbb{R}^d), \quad \mathbb{X}_{1,n}^* \to \mathbb{X}_1^*, \quad in \ L^2((0,T) \times \mathbb{R}^d), \tag{3.7}$$

and

$$u_n^* \to u^* \quad in \ C([0,T];\mathbb{R}^m). \tag{3.8}$$

Note that, $((\mathbb{X}_T^*)_n, \mathbb{X}_{1,n}^*, \beta_n^*)$ has the same distribution as $(\mathbb{X}_T, \mathbb{X}_1, \beta)$, and so does the limit $(\mathbb{X}_T^*, \mathbb{X}_1^*, \beta^*)$.

For each $n \geq 1$, define $\mathcal{F}_{t,n}^* := \sigma(\mathcal{X}_n^*(s), s \leq t)$. Then, $\mathbb{X}_n^*(T) \in L^2(\Omega, \mathcal{F}_T^*, \mathbb{P}^*; L^2)$, $\mathbb{X}_{1,n}^* \in L^2_{ad^*}(0,T; L^2(\Omega; L^2))$, $u_n^* \in \mathcal{U}_{ad^*}$, and $(\beta_n^*(t), \mathcal{F}_{t,n}^*)$, $t \in [0,T]$, is a Wiener process. It follows from Proposition 2.2 that, under the hypothesis $(H0)$, for each (β_n^*, u_n^*) there exists a unique strong L^2-solution X_n^* to (1.1). Hence, $(\Omega^*, \mathcal{F}^*, \mathbb{P}^*, (\mathcal{F}_t^*)_{t\geq 0}, \beta_n^*, u_n^*, X_n^*)$ is an admissible system.

Moreover, since the solution to (1.1) is a measurable map of Brownian motions and controls, we also have that the distribution of $((\mathbb{X}_T^*)_n, \mathbb{X}_{1,n}^*, \beta_n^*, u_n^*, X_n^*)$ is the same as that of $(\mathbb{X}_T, \mathbb{X}_1, \beta, u_n, X_n)$, where X_n is the solution to (1.1) corresponding to (β, u_n). In particular, $\Phi^*(u_n^*) = \Phi(u_n)$.

Similarly, set $\mathcal{F}_t^* := \sigma(\mathcal{X}^*(s), s \leq t)$ and let X^* be the unique strong L^2-solution to (1.1) corresponding to (β^*, u^*). Then, $(\Omega^*, \mathcal{F}^*, \mathbb{P}^*, (\mathcal{F}_t^*)_{t\geq 0}, \beta^*, u^*, X^*)$ is an admissible system.

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Below, we consider the derivatives of u_n^* and u^*. For each $n \geq 1$, define $(u_n^*)' \in \mathcal{D}'(0,T;\mathbb{R}^m)$ in the distribution sense, i.e., $((u_n^*)', v) = -(u_n^*, v'), \forall v \in \mathcal{D}(0,T;\mathbb{R}^m)$, where $(\ ,\ )$ denotes the pairing between $\mathcal{D}'(0,T;\mathbb{R}^m)$ and $\mathcal{D}(0,T; \mathbb{R}^m)$. We claim that $(u_n^*)'$ has the same distribution as u_n' in $\mathcal{D}'(0,T;\mathbb{R}^m)$, and $\mathbb{E}^* \|(u_n^*)'\|_{L^2(0,T;\mathbb{R}^m)}^2 = \mathbb{E}\|u_n'\|_{L^2(0,T;\mathbb{R}^m)}^2$. It follows from (3.2) that there exists $v^* \in L^2(\Omega^*; L^2(0,T;\mathbb{R}^m))$, such that

$$(u_n^*)' \to v^*, \quad weakly\ in\ L^2(\Omega^*; L^2(0,T;\mathbb{R}^m)), \ n \to \infty. \qquad (3.9)$$

Indeed, for any $l \geq 1$, $v_j \in \mathcal{D}(0,T;\mathbb{R}^m)$, $1 \leq j \leq l$, and $c \in \mathbb{R}^l$,

$$\mathbb{E}^* exp(i\sum_{j=1}^{l} c_j((u_n^*)', v_j)) = \mathbb{E}^* exp(-i(u_n^*, \sum_{j=1}^{l} c_j v_j'))$$

$$= \mathbb{E} exp(-i(u_n, \sum_{j=1}^{l} c_j v_j')) = \mathbb{E} exp(i\sum_{j=1}^{l} c_j(u_n', v_j)),$$

which implies that the distributions of $(u_n^*)'$ and u_n' coincide. Moreover, if $\mathscr{D} := \{v_n\}$ is a dense subset in $\mathcal{D}(0,T;\mathbb{R}^m)$ (hence also dense in $L^2(0,T;\mathbb{R}^m)$), we have

$$\mathbb{E}^* \sup_{n \geq 1} \frac{|((u_n^*)', v_n)|^2}{\|v_n\|_{L^2(0,T;\mathbb{R}^m)}^2} = \mathbb{E} \sup_{n \geq 1} \frac{|(u_n', v_n)|^2}{\|v_n\|_{L^2(0,T;\mathbb{R}^m)}^2} = \mathbb{E}\|u_n'\|_{L^2(0,T;\mathbb{R}^m)}^2 < \infty,$$

which implies that $\mathbb{E}^* \|(u_n^*)'\|_{L^2(0,T;\mathbb{R}^m)}^2 = \mathbb{E}\|u_n'\|_{L^2(0,T;\mathbb{R}^m)}^2$, as claimed.

Similarly, define $(u^*)' \in \mathcal{D}'(0,T;\mathbb{R}^m)$ in the distribution sense. We have

$$(u^*)' = v^*, \quad in\ L^2(0,T;\mathbb{R}^m), \ \mathbb{P}^* - a.s. \qquad (3.10)$$

Indeed, for any $v \in \mathscr{D}$ and $\psi \in L^\infty(\Omega^*)$, by (3.8), the dominated convergence theorem and the weak convergence (3.9), it follows that

$$\mathbb{E}^*((u^*)', v)\psi = -\mathbb{E}^*\psi(\int_0^T u^*(t) \cdot v'(t)dt) = -\lim_{n\to\infty} \mathbb{E}^*\psi(\int_0^T u_n^*(t) \cdot v'(t)dt)$$

$$= \lim_{n\to\infty} \mathbb{E}^* \int_0^T (u_n^*)'(t) \cdot v(t)\psi dt = \mathbb{E}^* \int_0^T v^*(t) \cdot v(t)\psi dt = \mathbb{E}^*(v^*, v)\psi.$$

Hence, $((u^*)', v) = (v^*, v)$, $\mathbb{P}^*$-a.s., $v \in \mathscr{D}$. Since $\mathscr{D}$ is countable and dense in $L^2(0,T;\mathbb{R}^m)$, (3.10) follows.

Next, we show that the solutions to (1.1) depend pathwisely continuous on the controllers, by using the rescaling approach developed in [1, 2].

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Lemma 3.2. *Let X_n^* (resp. X^*) be the solution to (1.1) corresponding to (β_n^*, u_n^*) (resp. (β^*, u^*)) as above, $n \geq 1$. Assume the conditions in Theorem 2.5 to hold. Then, for each $x \in L^2$, $0 < T < \infty$ and each Strichartz pair (p,q), we have $\mathbb{P}^*$-a.s., as $n \to \infty$,*

$$\|X_n^* - X^*\|_{C([0,T];L^2)} + \|X_n^* - X^*\|_{L^q(0,T;L^p)} \to 0. \tag{3.11}$$

Proof. Set $W_n^*(t,\xi) = \sum_{j=1}^N \mu_j e_j(\xi)\beta_{j,n}^*(t)$, $W^*(t,\xi) = \sum_{j=1}^N \mu_j e_j(\xi)\beta_j^*(t)$, $t \geq 0$, $\xi \in \mathbb{R}^d$. We may assume $T \geq 1$ without loss of generality.

Note that, in the conservative case,

$$|X_n^*(t)|_2 = |x|_2 < \infty, \quad t \in [0,T], \ n \geq 1, \ \mathbb{P}^* - a.s. \tag{3.12}$$

Using the rescaling transformation,

$$y_n^* = e^{-W_n^*}X_n^*, \tag{3.13}$$

we deduce from (1.1) with X, u, β_j replaced by X_n^*, u_n^* and $\beta_{j,n}^*$ respectively that

$$
\begin{aligned}
dy_n^* &= A_n^*(t)y_n^* dt - \lambda i|y_n^*|^{\alpha-1}y_n^* dt + f(u_n^*)y_n^* dt \\
y_n^*(0) &= x,
\end{aligned}
\tag{3.14}
$$

where $A_n^*(t) = -i(\Delta + b_n^*(t)\cdot\nabla + c_n^*(t))$, $b_n^*(t) = 2\nabla W_n^*(t)$, $c_n^*(t) = \sum_{j=1}^d (\partial_j W_n^*(t))^2 + \Delta W_n^*(t)$, and $f(u_n^*) = -i(V_0 + u_n^* \cdot V)$.

It suffices to prove that for each Strichartz pair (p,q), $\mathbb{P}^*$-a.s.,

$$\|y_n^* - y^*\|_{L^\infty(0,T;L^2)} + \|y_n^* - y^*\|_{L^q(0,T;L^p)} \to 0, \quad as \ n \to \infty. \tag{3.15}$$

Now, we will prove (3.15) for the Strichartz pair $(p,q) = (\alpha+1, \frac{4(\alpha+1)}{d(\alpha-1)})$. The general case will follow immediately from the Strichartz estimates (see, e.g., [1, Lemma 4.1] or [2, (A.1)]).

To this end, we prove first that

$$\sup_{n\geq 1} \|y_n^*\|_{L^q(0,T;L^p)} < \infty, \quad \mathbb{P}^* - a.s. \tag{3.16}$$

Applying the Strichartz estimates to (3.14) yields

$$\|y_n^*\|_{L^q(0,t;L^p)} \leq C_T\left[|x|_2 + \||\lambda i|y_n^*|^{\alpha-1}y_n^*\|_{L^{q'}(0,t;L^{p'})} + \|f(u_n^*)y_n^*\|_{L^1(0,t;L^2)}\right].$$

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(Note that, the Strichartz coefficient C_T is independent of n, since by (3.6) $\sup_{n \geq 1} \|W_n^*\|_{C([0,T];L^\infty)} < \infty$, $\mathbb{P}^*$-a.s.)

By Hölder's inequality,

$$\||\lambda i|y_n^*|^{\alpha-1}y_n^*\|_{L^{q'}(0,t;L^{p'})} \leq t^\theta \|y_n^*\|_{L^q(0,T;L^p)}^\alpha, \tag{3.17}$$

where $\theta = 1 - \frac{d(\alpha-1)}{4} > 0$, and by (3.12),

$$\|f(u_n^*)y_n^*\|_{L^1(0,t;L^2)} \leq T(|V_0|_{L^\infty} + D_U\|V\|_{L^\infty(\mathbb{R}^d;\mathbb{R}^m)})|x|_2.$$

Thus,

$$\|y_n^*\|_{L^q(0,t;L^p)} \leq C_T(D(T)|x|_2 + t^\theta \|y_n^*\|_{L^q(0,T;L^p)}^\alpha), \tag{3.18}$$

where $D(T) := 1 + T(|V_0|_\infty + D_U\|V\|_{L^\infty(0,T;\mathbb{R}^m)})$.

Choose $t \in [0,T]$ such that $C_T D(T)(|x|_2 + 1) = (1 - \frac{1}{\alpha})(\alpha C_T t^\theta)^{-\frac{1}{\alpha-1}}$, i.e.,

$$t = \alpha^{-\frac{\alpha}{\theta}}(\alpha-1)^{\frac{\alpha-1}{\theta}}(|x|_2+1)^{-\frac{\alpha-1}{\theta}}C_T^{-\frac{\alpha}{\theta}}D(T)^{-\frac{\alpha-1}{\theta}}(\leq T).$$

Then, by Lemma 6.1, we get

$$\|y_n^*\|_{L^q(0,t;L^p)} \leq \frac{\alpha}{\alpha-1}C_T D(T)|x|_2.$$

Iterating similar estimates on $[jt,(j+1)t \wedge T]$, $1 \leq j \leq [\frac{T}{t}]$, we obtain

$$\|y_n^*\|_{L^q(0,T;L^p)} \leq \left([\frac{T}{t}]+1\right)^{\frac{1}{q}}\frac{\alpha}{\alpha-1}C_T D(T)|x|_2, \tag{3.19}$$

which yields (3.16).

It remains to prove (3.15). Applying the Strichartz estimates to the equations of y_n^* and y^*, we have for any $t \in (0,T)$,

$$\|y_n^* - y^*\|_{L^\infty(0,t;L^2)} + \|y_n^* - y^*\|_{L^q(0,t;L^p)}$$
$$\leq C_T\left[\|f(u_n^*)y_n^* - f(u^*)y^*\|_{L^1(0,t;L^2)} + \||y_n^*|^{\alpha-1}y_n^* - |y^*|^{\alpha-1}y^*\|_{L^{q'}(0,t;L^{p'})}\right] + w_n(y^*),$$

where $w_n(y^*) = C_T(\|(b_n^* - b^*) \cdot \nabla y^*\|_{\widetilde{X}'_{[0,T]}} + \|(c_n^* - c^*)y^*\|_{L^1(0,T;L^2)})$, and $\widetilde{X}'_{[0,T]}$ is the dual space of the local smoothing space $\widetilde{X}_{[0,T]}$ introduced in [22] up

to time T (see also the proof of Lemma 2.7 in [2]). Since $y^* \in \widetilde{X}_{[0,T]}$, by Proposition 2.3(a) in [22] and (3.6), we note that $\mathbb{P}^*$-a.s. as $n \to \infty$,

$$w_n(y^*) \leq C_T \|\beta_n^* - \beta\|_{C([0,T];\mathbb{R}^N)} (\|y^*\|_{\widetilde{X}_{[0,T]}} + \|y^*\|_{L^1(0,T;L^2)}) \to 0.$$

Proceeding as in [1, (4.12)], we have

$$\||y_n^*|^{\alpha-1} y_n^* - |y^*|^{\alpha-1} y^*\|_{L^{q'}(0,t;L^{p'})}$$
$$\leq \alpha t^\theta (\|y_n^*\|_{L^q(0,T;L^p)}^{\alpha-1} + \|y^*\|_{L^q(0,T;L^p)}^{\alpha-1}) \|y_n^* - y^*\|_{L^q(0,t;L^p)}.$$

Moreover, for $t \leq 1$,

$$\|f(u_n^*)y_n^* - f(u^*)y^*\|_{L^1(0,t;L^2)} \leq t^{\frac{1}{2}} D \left(\|u_n^* - u^*\|_{L^2(0,T;\mathbb{R}^m)} + \|y_n^* - y^*\|_{L^\infty(0,t;L^2)} \right),$$

where $D = 2(|V_0|_{L^\infty} + \|V\|_{L^\infty(\mathbb{R}^d;\mathbb{R}^m)}(|x|_2 + D_U))$.

Hence,

$$\|y_n^* - y^*\|_{L^\infty(0,t;L^2)} + \|y_n^* - y^*\|_{L^q(0,t;L^p)}$$
$$\leq C_T \left[\alpha t^\theta (\|y_n^*\|_{L^q(0,T;L^p)}^{\alpha-1} + \|y^*\|_{L^q(0,T;L^p)}^{\alpha-1}) \|y_n^* - y^*\|_{L^q(0,t;L^p)} \right.$$
$$\left. + t^{\frac{1}{2}} D \|u_n^* - u^*\|_{L^2(0,T;\mathbb{R}^m)} + t^{\frac{1}{2}} D \|y_n^* - y^*\|_{L^\infty(0,t;L^2)} \right] + w_n(y^*)$$
$$\leq (t^\theta + t^{\frac{1}{2}}) C_T \widetilde{D}(T) \left(\|y_n^* - y^*\|_{L^\infty(0,t;L^2)} + \|y_n^* - y^*\|_{L^q(0,t;L^p)} \right.$$
$$\left. + \|u_n^* - u^*\|_{L^2(0,T;\mathbb{R}^m)} \right) + w_n(y^*), \tag{3.20}$$

where $\widetilde{D}(T) := \alpha(\sup_{n \geq 1} \|y_n^*\|_{L^q(0,T;L^p)}^{\alpha-1} + \|y^*\|_{L^q(0,T;L^p)}^{\alpha-1}) + D < \infty$, $\mathbb{P}^*$-a.s. Choosing t small enough and independent of n, such that $t^\theta + t^{\frac{1}{2}} \leq (2\widetilde{D}(T)C_T)^{-1}$, we get that $\mathbb{P}^*$-a.s. as $n \to \infty$,

$$\|y_n^* - y^*\|_{L^\infty(0,t;L^2)} + \|y_n^* - y^*\|_{L^q(0,t;L^p)} \leq \|u_n^* - u^*\|_{L^2(0,T;\mathbb{R}^m)} + 2w_n(y^*) \to 0. \tag{3.21}$$

Since t is independent of the initial data, iterating this procedure finite times we obtain (3.15), thereby completing the proof. $\qquad\square$

Lemma 3.3. *Let* X_n^*, $(\mathbb{X}_T^*)_n$, $\mathbb{X}_{1,n}^*$, X^*, $\mathbb{X}_T^*$ *and* $\mathbb{X}_1^*$ *be as above,* $n \geq 1$. *We have* $\mathbb{P}^*$-*a.s., as* $n \to \infty$,

$$\mathbb{E}^* Re \langle X_n^*(T), (\mathbb{X}_T^*)_n \rangle_2 \to \mathbb{E}^* Re \langle X^*(T), \mathbb{X}_T^* \rangle_2, \tag{3.22}$$

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and

$$\mathbb{E}^* \int_0^T Re \left\langle X_n^*(t), \mathbb{X}_{1,n}^*(t) \right\rangle_2 dt \to \mathbb{E}^* \int_0^T Re \left\langle X^*(t), \mathbb{X}_1^*(t) \right\rangle_2 dt. \qquad (3.23)$$

Proof. By (3.11) and (3.7), $\mathbb{P}^*$-a.s., as $n \to \infty$,

$$Re \left\langle X_n^*(T), (\mathbb{X}_T^*)_n \right\rangle_2 \to Re \left\langle X^*(T), \mathbb{X}_T^* \right\rangle_2, \qquad (3.24)$$

$$\int_0^T Re \left\langle X_n^*(t), \mathbb{X}_{1,n}^*(t) \right\rangle_2 dt \to \int_0^T Re \left\langle X^*(t), \mathbb{X}_1^*(t) \right\rangle_2 dt. \qquad (3.25)$$

Then, for any $\varepsilon \in (0,1)$ fixed, by the Young inequality $ab \leq \frac{1-\varepsilon}{2} a^{\frac{2}{1-\varepsilon}} + \frac{1+\varepsilon}{2} b^{\frac{2}{1+\varepsilon}}$, we get

$$\sup_{n \geq 1} \mathbb{E}^* | \left\langle X_n^*(T), (\mathbb{X}_T^*)_n \right\rangle_2 |^{1+\varepsilon} \leq \sup_{n \geq 1} \mathbb{E}^* |X_n^*(T)|_2^{1+\varepsilon} |(\mathbb{X}_T^*)_n|_2^{1+\varepsilon}$$

$$\leq \frac{1-\varepsilon}{2} \sup_{n \geq 1} \mathbb{E}^* |X_n^*(T)|_2^{\frac{2(1+\varepsilon)}{1-\varepsilon}} + \frac{1+\varepsilon}{2} \sup_{n \geq 1} \mathbb{E}^* |(\mathbb{X}_T^*)_n|_2^2$$

$$= \frac{1-\varepsilon}{2} |x|_2^{\frac{2(1+\varepsilon)}{1-\varepsilon}} + \frac{1+\varepsilon}{2} \mathbb{E} |\mathbb{X}_T|_2^2,$$

which implies the uniform integrability of $\{Re \left\langle X_n^*(T), (\mathbb{X}_T^*)_n \right\rangle_2\}_{n \geq 1}$, thereby yielding (3.22) by (3.24).

Similarly, for $\varepsilon \in (0,1)$ fixed, we have

$$\sup_{n \geq 1} \mathbb{E}^* \left| \int_0^T Re \left\langle X_n^*(t), \mathbb{X}_{1,n}^*(t) \right\rangle_2 dt \right|^{1+\varepsilon}$$

$$\leq \frac{1-\varepsilon}{2} \sup_{n \geq 1} \mathbb{E}^* \|X_n^*\|_{L^2(0,T;L^2)}^{\frac{2(1+\varepsilon)}{1-\varepsilon}} + \frac{1+\varepsilon}{2} \sup_{n \geq 1} \mathbb{E}^* \|\mathbb{X}_{1,n}^*\|_{L^2(0,T;L^2)}^2$$

$$= \frac{1-\varepsilon}{2} T^{\frac{1+\varepsilon}{1-\varepsilon}} |x|_2^{\frac{2(1+\varepsilon)}{1-\varepsilon}} + \frac{1+\varepsilon}{2} \mathbb{E} \int_0^T |\mathbb{X}_1(t)|_2^2 dt < \infty,$$

which in view of (3.25) implies (3.23), as claimed. $\qquad \square$

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Proof of Theorem 2.5. By (3.12) we have

$$
\Phi^*(u_n^*) = (1 + \gamma_1)|x|_2^2 + \mathbb{E}^*|(\mathbb{X}_T^*)_n|_2^2 + \gamma_1 \mathbb{E}^* \int_0^T |\mathbb{X}_{1,n}^*(t)|_2^2 dt
$$

$$
- 2\mathbb{E}^* Re \left\langle X_n^*(T), (\mathbb{X}_T^*)_n \right\rangle_2 - 2\gamma_1 \mathbb{E}^* \int_0^T Re \left\langle X_n^*(t), \mathbb{X}_{1,n}^*(t) \right\rangle_2 dt
$$

$$
+ \gamma_2 \mathbb{E}^* \int_0^T |u_n^*(t)|_m^2 dt + \gamma_3 \mathbb{E}^* \int_0^T |(u_n^*)'(t)|_m^2 dt,
$$

Note that, since the distributions of $(\mathbb{X}_T^*, \mathbb{X}_1^*)$ and $((\mathbb{X}_T^*)_n, \mathbb{X}_{1,n}^*)$ coincide for $n \geq 1$, we have

$$
\mathbb{E}^*|\mathbb{X}_T^*|_2^2 + \gamma_1 \mathbb{E}^* \int_0^T |\mathbb{X}_1^*(t)|_2^2 dt = \lim_{n \to \infty} \left(\mathbb{E}^*|(\mathbb{X}_T^*)_n|_2^2 + \gamma_1 \mathbb{E}^* \int_0^T |\mathbb{X}_{1,n}^*(t)|_2^2 dt \right).
$$

$$(3.26)$$

Moreover, by (3.8) and the bounded dominated convergence theorem, it follows that

$$
\gamma_2 \mathbb{E}^* \int_0^T |u_n^*(t)|_m^2 dt \to \gamma_2 \mathbb{E}^* \int_0^T |u^*(t)|_m^2 dt, \ as \ n \to \infty,
$$

and by (3.9) and (3.10),

$$
\mathbb{E}^* \int_0^T |(u^*)'(t)|_m^2 dt \leq \liminf_{n \to \infty} \mathbb{E}^* \int_0^T |(u_n^*)'(t)|_m^2 dt.
$$

Thus, taking into account Lemma 3.3, we obtain

$$
\Phi^*(u^*) \leq \liminf_{n \to \infty} \Phi^*(u_n^*) = \liminf_{n \to \infty} \Phi(u_n) = I.
$$

which completes the proof. $\qquad\square$

Remark 3.4. The proofs above also show that, in the case $\gamma_3 = 0$, the objective functional Φ depends continuously on controls.

4　The directional derivative of function Φ

This section is devoted to the calculation of the directional derivative of the function Φ on the convex set $\mathcal{U}_{ad}$. Namely, one has

Proposition 4.1. *Assume that $\gamma_3 = 0$ and that the conditions of Theorem 2.6 to hold. Then, for each $x \in L^2$ and all $u, v \in \mathcal{U}_{ad}$, we have*

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left(\Phi(u + \varepsilon\widetilde{u}) - \Phi(u) \right) = \mathbb{E} \int_0^T \eta(u)(t) \cdot \widetilde{u}(t) dt, \qquad (4.1)$$

where $\widetilde{u} = v - u$, and

$$\eta(u) = 2 \left(\gamma_2 u - \mathrm{Im} \int_{\mathbb{R}^d} V(\xi) X^u(\xi) \overline{Y^u}(\xi) d\xi \right). \qquad (4.2)$$

Here (Y^u, Z^u) is the solution to the dual backward stochastic equation (2.9).

To prove Proposition 4.1, we first study the *equation of variation* associated with Problem (P), namely,

$$\begin{aligned} id\varphi &= \Delta\varphi dt + \lambda h_1(X^u)\varphi dt + \lambda h_2(X^u)\overline{\varphi} dt - i\mu\varphi dt \\ &\quad + V_0\varphi dt + u \cdot V\varphi dt + \widetilde{u} \cdot VX^u dt + i\varphi dW(t), \\ \varphi(0) &= 0, \end{aligned} \qquad (4.3)$$

where $\widetilde{u} = v - u$, $u, v \in \mathcal{U}_{ad}$, X^u is the solution to (1.1), and $h_j(X^u)$, $j = 1, 2$, are defined as in (2.10). The strong H^1-(and L^2-)solution to (4.3) can be defined similarly as in Definition 2.1.

Lemma 4.2. (*i*) *Under Hypothesis (H0), for $u, v \in \mathcal{U}_{ad}$, $\widetilde{u} := v - u$, there exists a unique strong L^2-solution $\varphi^{u,\widetilde{u}}$ to (4.3) on $[0, T]$.*

(*ii*) *Under Hypothesis (H2), for any Strichartz pair (p, q),*

$$\sup_{u,v \in \mathcal{U}_{ad}} \left(\|\varphi^{u,\widetilde{u}}\|_{L^\infty(\Omega; C([0,T]; L^2))} + \|\varphi^{u,\widetilde{u}}\|_{L^\infty(\Omega; L^q(0,T; L^p))} \right) < \infty. \qquad (4.4)$$

Moreover, set $u_\varepsilon := u + \varepsilon\widetilde{u}$ and let X^u and X^{u_ε} be the corresponding solutions to (1.1) with the initial datum $x \in H^1$. Then,

$$\lim_{\varepsilon \to 0} \mathbb{E} \sup_{t \in [0,T]} |\varepsilon^{-1}(X^{u_\varepsilon}(t) - X^u(t)) - \varphi^{u,\widetilde{u}}(t)|_2^2 = 0. \qquad (4.5)$$

Remark 4.3. In comparison with (*ii*), the weaker Hypothesis (H0) is sufficient for the pathwise existence and uniqueness of the solution to (4.3), thanks to the linear structure of (4.3). However, as mentioned in Section 2, Hypothesis (H2) is needed in order that the estimate (4.4) holds. The arguments presented below, particularly in the proof of estimate (4.4), will also be used in the analysis of the dual backward stochastic equation in the proof of Proposition 4.4 below.

Proof of Lemma 4.2. (i) We set $z^{u,\widetilde{u}} := e^{-W}\varphi^{u,\widetilde{u}}$, $\widetilde{u} := v - u$, and for simplicity, we omit the dependence of $u, \widetilde{u}$ in $z^{u,\widetilde{u}}$ below. It follows from (4.3) that

$$dz = A(t)zdt - \lambda ih(X^u, z)dt + f(u)zdt - i\widetilde{u}\cdot Ve^{-W}X^u dt, \ t \in (0,T), \ (4.6)$$
$$z(0) = 0,$$

where $A(t)$ is similar to $A_n^*(t)$ in (3.14), i.e., $A(t) = -i(\Delta + b(t)\cdot\nabla + c(t))$, $b(t) = 2\nabla W(t)$, $c(t) = \sum_{j=1}^d (\partial_j W(t))^2 + \Delta W(t)$, $h(X^u, z) := h_1(X^u)z + h_2(X^u)e^{-2iImW}\overline{z}$, and $f(u) := -i(V_0 + u\cdot V)$.

It is equivalent to prove the existence and uniqueness of the solution to (4.6) (see the proof of [1, Lemma 6.1]).

To this purpose, we reformulate (4.6) in the mild form as

$$z(t) = \int_0^t U(t,s)\Big[-\lambda ih(X^u, z)(s) + f(u(s))z(s) - i\widetilde{u}(s)\cdot Ve^{-W(s)}X^u(s)\Big]ds,$$
$$(4.7)$$

where $0 \le t \le T$, and $U(t,s)$, $0 \le s, t \le T$, are the evolution operators corresponding to the operator $A(t)$ (see [1, Lemma 3..3]). Choose the Strichartz pair $(p,q) = (\alpha+1, \frac{4(\alpha+1)}{d(\alpha-1)})$. Define the operator F on $C([0,T]; L^2)\cap L^q(0,T; L^p)$ by

$$F(\phi)(t) := \int_0^t U(t,s)\Big[-\lambda ih(X^u, \phi)(s) + f(u(s))\phi(s) - i\widetilde{u}\cdot Ve^{-W(s)}X^u(s)\Big]ds,$$

where $0 \le t \le T$, $\phi \in C([0,T]; L^2) \cap L^q(0,T; L^p)$. Set $\mathscr{Z}_{M_1}^{\tau_1} := \{\phi \in C([0,\tau_1]; L^2) \cap L^q(0,\tau_1; L^p) : \|\phi\|_{C([0,\tau_1];L^2)} + \|\phi\|_{L^q(0,\tau_1;L^p)} \le M_1\}$, where τ_1 and M_1 are two random variables to be determined later.

Note that, by Hölder's inequality, for any $\phi_j \in C([0,T]; L^2)\cap L^q(0,T; L^p)$, $j = 1, 2$,

$$\|h(X^u, \phi_1) - h(X^u, \phi_2)\|_{L^{q'}(0,t;L^{p'})} \le \alpha t^\theta \|X^u\|_{L^q(0,t;L^p)}^{\alpha-1}\|\phi_1 - \phi_2\|_{L^q(0,t;L^p)},$$

where $\theta = 1 - d(\alpha - 1)/4 \in (0,1)$, and

$$\|f(u)(\phi_1 - \phi_2)\|_{L^1(0,t;L^2)} \le t\|f(u)\|_{L^\infty(0,t;L^\infty)}\|\phi_1 - \phi_2\|_{C([0,t];L^2)}.$$

Then, let $R_1(t) = \alpha t^\theta \|X^u\|_{L^q(0,t;L^p)}^{\alpha-1} + t\|f(u)\|_{L^\infty(0,t;L^\infty)}$, $t \in [0,T]$. By the Strichartz estimates and the above estimates we get for any $t \in [0,T]$,

$$\|F(\phi_1) - F(\phi_2)\|_{C([0,t];L^2)\cap L^q(0,t;L^p)} \le C_t R_1(t)\|\phi_1 - \phi_2\|_{C([0,t];L^2)\cap L^q(0,t;L^p)}.$$
$$(4.8)$$

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Similarly, for $\phi \in C([0,T]; L^2) \cap L^q(0,T; L^p)$, $t \in [0,T]$,

$$\|F(\phi)\|_{C([0,t];L^2)\cap L^q(0,t;L^p)} \leq C_t R_1(t)\|\phi\|_{C([0,t];L^2)\cap L^q(0,t;L^p)} + C_t\|\widetilde{u} \cdot VX^u\|_{L^1(0,t;L^2)}.$$

Setting $\tau_1 = \inf\{t \in [0,T] : C_t R_1(t) \geq \frac{1}{2}\} \wedge T$, $M_1 = 2C_{\tau_1}\|\widetilde{u}\cdot VX^u\|_{L^1(0,\tau_1;L^2)}$, it follows by (4.8) that F is a contraction map in $\mathcal{Z}_{M_1}^{\tau_1}$, implying that there exists $\widetilde{z}_1 \in \mathcal{Z}_{M_1}^{\tau_1}$ such that $F(\widetilde{z}_1) = \widetilde{z}_1$. Setting $z_1(\cdot) := \widetilde{z}_1(\cdot \wedge \tau_1)$ and using similar arguments as in [1], we deduce that z_1 is $(\mathcal{F}_t)$-adapted, continuous in L^2, and solves (4.6) on $[0, \tau_1]$, and $\|z_1\|_{C([0,\tau_1];L^2)\cap L^q(0,\tau_1;L^p)} \leq M_1$. We also note that $\tau_1 \geq \sigma_*$, where

$$\sigma_* := \inf\{t \in [0,T]; Z(t) \geq \frac{1}{2}\} \wedge T \tag{4.9}$$

with $Z(t) := t^\theta \alpha C_T \|X^u\|_{L^q(0,T;L^p)}^{\alpha-1} + tC_T\|f(u)\|_{L^\infty(0,T;L^\infty)}$, $t \in [0,T]$.

Suppose that at the nth-step ($n \geq 1$) we have an increasing sequence of stopping times $\{\tau_j\}_{j=0}^n$ and an L^2-valued continuous $(\mathcal{F}_t)$-adapted process z_n such that $\tau_0 = 0$, $\tau_j - \tau_{j-1} \geq \sigma_*$, $1 \leq j \leq n$, z_n solves (4.6) on $[0, \tau_n]$, $z_n(\cdot) = z_n(\cdot \wedge \tau_n)$, and

$$\|z_n\|_{C([0,\tau_n];L^2)\cap L^q(0,\tau_n;L^p)} \leq \sum_{j=1}^{n}(2C_{\tau_n})^{n+1-j}\|\widetilde{u} \cdot VX^u\|_{L^1(\tau_{j-1},\tau_j;L^2)}.$$

Set $\mathcal{Z}_{M_{n+1}}^{\sigma_n} = \{\phi \in C([0,\sigma_n]; L^2)\cap L^q(0,\sigma_n; L^p) : \|\phi\|_{C([0,\sigma_n];L^2)} + \|\phi\|_{L^q(0,\sigma_n;L^p)} \leq M_{n+1}\}$, where σ_n and M_{n+1} are random variables to be determined later. Define the operator F_n on $C([0,T]; L^2) \cap L^q(0,T; L^p)$ by

$$F_n(\phi)(t) := U(\tau_n + t, \tau_n)z_n(\tau_n) + \int_0^t U(\tau_n + t, \tau_n + s)\Big[-\lambda ih(X^u(\tau_n + s), \phi(s))$$
$$+ f(u(\tau_n + s))\phi(s) - i\widetilde{u}(\tau_n + s) \cdot Ve^{-W(\tau_n+s)}X^u(\tau_n + s)\Big]ds,$$

where $0 \leq t \leq T$, $\phi \in C([0,T]; L^2) \cap L^q(0,T; L^p)$.

Similarly, for any $\phi_j \in \mathcal{Z}_{M_{n+1}}^{\sigma_n}$, $j = 1, 2$,

$$\|F_n(\phi_1) - F_n(\phi_2)\|_{C([0,\sigma_n];L^2)\cap L^q(0,\sigma_n;L^p)}$$
$$\leq C_{\tau_n+\sigma_n}R_{n+1}(\sigma_n)\|\phi_1 - \phi_2\|_{C([0,\sigma_n];L^2)\cap L^q(0,\sigma_n;L^p)},$$

where $R_{n+1}(t) = \alpha t^\theta \|X^u\|_{L^q(\tau_n,\tau_n+t;L^p)}^{\alpha-1} + t\|f(u)\|_{L^\infty(\tau_n,\tau_n+t;L^\infty)}$, $t \in [0, T-\tau_n]$, while for $\phi \in \mathcal{Z}_{M_{n+1}}^{\sigma_n}$, we have

$$
\begin{aligned}
&\|F_n(\phi)\|_{C([0,\sigma_n];L^2)\cap L^q(0,\sigma_n;L^p)} \\
\leq & C_{\tau_n+\sigma_n}|z_n(\tau_n)|_2 + C_{\tau_n+\sigma_n}\|\widetilde{u}\cdot VX^u\|_{L^1(\tau_n,\tau_n+\sigma_n;L^2)} \\
& + C_{\tau_n+\sigma_n}R_{n+1}(\sigma_n)\|\phi\|_{C([0,\sigma_n];L^2)\cap L^q(0,\sigma_n;L^p)} \\
\leq & \frac{1}{2}\sum_{j=1}^{n+1}(2C_{\tau_n+\sigma_n})^{n+2-j}\|\widetilde{u}\cdot VX^u\|_{L^1(\tau_{j-1},\tau_j;L^2)} \\
& + C_{\tau_n+\sigma_n}R_{n+1}(\sigma_n)\|\phi\|_{C([0,\sigma_n];L^2)\cap L^q(0,\sigma_n;L^p)}.
\end{aligned}
$$

Then, let $\sigma_n(t) := \inf\{t \in [0, T-\tau_n] : C_{\tau_n+t}R_{n+1}(t) \geq \frac{1}{2}\} \wedge (T-\tau_n)$, $\tau_{n+1} := \tau_n + \sigma_n$, and $M_{n+1} := \sum_{j=1}^{n+1}(2C_{\tau_{n+1}})^{n+2-j}\|\widetilde{u}\cdot VX^u\|_{L^1(\tau_{j-1},\tau_j;L^2)}$. It follows that $\tau_{n+1} - \tau_n = \sigma_n \geq \sigma_*$, F_n is a contraction map in $\mathcal{Z}_{M_{n+1}}^{\sigma_n}$, and so there exists $\widetilde{z}_{n+1} \in \mathcal{Z}_{M_{n+1}}^{\sigma_n}$ satisfying $F_n(\widetilde{z}_{n+1}) = \widetilde{z}_{n+1}$. As in [1], letting

$$
z_{n+1}(t) = \begin{cases} z_n(t), & t \in [0,\tau_n]; \\ \widetilde{z}_{n+1}((t-\tau_n)\wedge\sigma_n), & t \in (\tau_n, T], \end{cases}
$$

it follows that z_{n+1} is continuous $(\mathcal{F}_t)$-adapted, satisfies (4.6) on $[0,\tau_{n+1}]$, $z_n(\cdot) = z_n(\cdot \wedge \tau_{n+1})$, and

$$
\|z_n\|_{C([0,\tau_{n+1}];L^2)\cap L^q(0,\tau_{n+1};L^p)} \leq \sum_{j=1}^{n+1}(2C_{\tau_{n+1}})^{n+2-j}\|\widetilde{u}\cdot VX^u\|_{L^1(\tau_{j-1},\tau_j;L^2)}.
$$

Iterating this procedure, since $\sigma_n \geq \sigma_*$, we see that after at most $[T/\sigma_*]+1$ steps the stopping time τ_n reaches T. Hence, $\mathbb{P}$-a.s. there exists a global solution (denoted by z) on $[0,T]$ which satisfies

$$
\|z\|_{C([0,T];L^2)\cap L^q(0,T;L^p)} \leq \sum_{j=1}^{[T/\sigma_*]+1}(2C_T)^{[T/\sigma_*]+2-j}\|\widetilde{u}\cdot VX^u\|_{L^1(\tau_{j-1},\tau_j;L^2)}.
$$

$$(4.10)$$

As regards the uniqueness, given any two solutions φ_j, we set $z_j = e^{-W}\varphi_j$, $j = 1, 2$. Then, similarly to (4.8), for any $s, t \in (0, T)$, $s + t \leq T$, we have

$$
\begin{aligned}
&\|z_1 - z_2\|_{C([s,s+t];L^2)\cap L^q(s,s+t;L^p)} \\
\leq & C_T(\alpha t^\theta\|X^u\|_{L^q(0,T;L^p)}^{\alpha-1} + t\|f(u)\|_{L^\infty(0,T;L^\infty)})\|z_1 - z_2\|_{C([s,s+t];L^2)\cap L^q(s,s+t;L^p)},
\end{aligned}
$$

which implies that $z_1 = z_2$ on $[s, s+t]$, $\mathbb{P}$-a.s., for t sufficiently small and independent of s, thereby yielding the uniqueness by the arbitrariness of s.

(ii) Under Hypothesis $(H2)$, the Strichartz coefficient C_T is now a deterministic constant. Moreover, (2.5) and (4.9) imply that σ_* has a deterministic lower bound, namely,

$$\sigma_* \geq t_* := \inf\{t \in [0, T] : Z^*(t) \geq T\} \wedge T, \quad \mathbb{P} - a.s.,$$

where $Z^*(t) := \alpha C(t^\theta + t) \sup_{u \in \mathcal{U}_{ad}}(\|X^u\|_{L^\infty(\Omega; L^q(0,T;L^p))}^{\alpha-1} + \|f(u)\|_{L^\infty(\Omega; L^\infty(0,T;L^\infty))})$. Thus, taking into account (4.10) and the uniform boundedness of $u, v \in \mathcal{U}_{ad}$, we obtain (4.4).

Now, set $\widetilde{X}_\varepsilon^u := \varepsilon^{-1}(X^{u_\varepsilon} - X^u) - \varphi$ and $\widetilde{y}_\varepsilon^u := e^{-W}\widetilde{X}_\varepsilon^u$. We need to prove that

$$\lim_{\varepsilon \to 0} \mathbb{E}\|\widetilde{y}_\varepsilon^u\|_{C([0,T];L^2)}^2 = 0. \tag{4.11}$$

To this purpose, note that

$$\varepsilon^{-1}(u_\varepsilon \cdot VX^{u_\varepsilon} - u \cdot VX^u) = \widetilde{u} \cdot VX^u + u_\varepsilon \cdot V(\widetilde{X}_\varepsilon^u + \varphi),$$

and

$$\varepsilon^{-1}(|X^{u_\varepsilon}|^{\alpha-1}X^{u_\varepsilon} - |X^u|^{\alpha-1}X^u) - (h_1(X^u)\varphi + h_2(X^u)\overline{\varphi})$$
$$= \left(\int_0^1 h_1(X_{u,r,\varepsilon})dr\right)\widetilde{X}_\varepsilon^u + \left(\int_0^1 h_2(X_{u,r,\varepsilon})dr\right)\overline{\widetilde{X}_\varepsilon^u}$$
$$+ \varphi \int_0^1 (h_1(X_{u,r,\varepsilon})dr - h_1(X^u))dr + \overline{\varphi}\int_0^1 (h_2(X_{u,r,\varepsilon})dr - h_2(X^u))dr.$$

where $X_{u,r,\varepsilon} = X^u + r(X^{u_\varepsilon} - X^u)$, $r \in [0,1]$. For simplicity, set $R_j(\varepsilon) := \int_0^1(h_j(X_{u,r,\varepsilon}) - h_j(X^u))dr$, $j = 1, 2$, and $R(\varepsilon, \varphi) := -i(\lambda R_1(\varepsilon)\varphi + \lambda R_2(\varepsilon)\overline{\varphi} + \varepsilon\widetilde{u} \cdot V\varphi)$.

Then, by (1.1) and (4.3), $\widetilde{X}_\varepsilon^u$ satisfies the equation

$$d\widetilde{X}_\varepsilon^u = -i\Delta\widetilde{X}_\varepsilon^u dt - \lambda i \int_0^1 h_1(X_{u,r,\varepsilon})dr\widetilde{X}_\varepsilon^u dt - \lambda i \int_0^1 h_2(X_{u,r,\varepsilon})dr\overline{\widetilde{X}_\varepsilon^u}dt$$
$$- (\mu + iV_0 + iu_\varepsilon \cdot V)\widetilde{X}_\varepsilon^u dt + R(\varepsilon, \varphi)dt + \widetilde{X}_\varepsilon^u dW(t). \tag{4.12}$$

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This yields

$$d\widetilde{y}_\varepsilon^u = -i\Delta\widetilde{y}_\varepsilon^u dt - \lambda i \int_0^1 h(X_{u,r,\varepsilon}, \widetilde{y}_\varepsilon^u) dr dt + f(u_\varepsilon)\widetilde{y}_\varepsilon^u dt + e^{-W} R(\varepsilon, \varphi) dt,$$

(4.13)

where $h(X_{u,r,\varepsilon}, \widetilde{y}_\varepsilon^u)$ and $f(u_\varepsilon)$ are similar as in (4.6), with X^u, z, u replaced by $X_{u,r,\varepsilon}$, $\widetilde{y}_\varepsilon^u$ and u_ε respectively.

Choose the Strichartz pair $(p,q) = (\alpha+1, \frac{4(\alpha+1)}{d(\alpha-1)})$. Then, by Strichartz's estimates, Hölder's inequality and Minkowski's inequality, we have

$$\|\widetilde{y}_\varepsilon^u\|_{C([0,t];L^2)} + \|\widetilde{y}_\varepsilon^u\|_{L^q(0,t;L^p)}$$

$$\leq C \int_0^1 \|h(X_{u,r,\varepsilon}, \widetilde{y}_\varepsilon^u)\|_{L^{q'}(0,t,L^{p'})} dr + Ct\|f(u_\varepsilon)\|_{L^\infty(0,T;L^\infty)}\|\widetilde{y}_\varepsilon^u\|_{C([0,t];L^2)}$$

$$+ C\|R(\varepsilon,\varphi)\|_{L^1(0,t;L^2)+L^{q'}(0,t;L^{p'})}.$$

Note that,

$$\|h(X_{u,r,\varepsilon}, \widetilde{y}_\varepsilon^u)\|_{L^{q'}(0,t,L^{p'})}$$

$$\leq \alpha t^\theta \|\widetilde{y}_\varepsilon^u\|_{L^q(0,t;L^p)}\|X_{u,r,\varepsilon}\|_{L^q(0,t;L^p)}^{\alpha-1}$$

$$\leq \alpha(1 \vee 2^{\alpha-1}) t^\theta \|\widetilde{y}_\varepsilon^u\|_{L^q(0,t;L^p)}(\|X^{u_\varepsilon}\|_{L^q(0,T;L^p)}^{\alpha-1} + \|X^u\|_{L^q(0,T;L^p)}^{\alpha-1}), \qquad (4.14)$$

where $\theta = 1 - d(\alpha-1)/4 > 0$. Then,

$$\|\widetilde{y}_\varepsilon^u\|_{C([0,t];L^2)} + \|\widetilde{y}_\varepsilon^u\|_{L^q(0,t;L^p)} \leq CD_3(T)(t^\theta + t)(\|\widetilde{y}_\varepsilon^u\|_{C([0,t];L^2)} + \|\widetilde{y}_\varepsilon^u\|_{L^q(0,t;L^p)})$$

$$+ C\|R(\varepsilon,\varphi)\|_{L^1(0,T;L^2)+L^{q'}(0,T;L^{p'})}.$$

where $D_3(T) = \alpha 2^{\alpha+1} \sup_{\varepsilon \in [0,1]}(\|X^{u_\varepsilon}\|_{L^\infty(\Omega;L^q(0,T;L^p))}^{\alpha-1} + \|f(u_\varepsilon)\|_{L^\infty(\Omega;L^\infty(0,T;L^\infty))})$. Using similar iterating arguments as in the proof of (4.4), we obtain

$$\sup_{u \in \mathcal{U}_{ad}} (\|\widetilde{y}_\varepsilon^u\|_{C([0,T];L^2)} + \|\widetilde{y}_\varepsilon^u\|_{L^q(0,T;L^p)}) \leq C(T)\|R(\varepsilon,\varphi)\|_{L^1(0,T;L^2)+L^{q'}(0,T;L^{p'})}$$

with $C(T) \in L^\infty(\Omega)$.

Thus, by Hölder's inequality,

$$\mathbb{E}\|\widetilde{y}_\varepsilon^u\|_{C([0,T];L^2)}^2 + \mathbb{E}\|\widetilde{y}_\varepsilon^u\|_{L^q(0,T;L^p)}^2$$

$$\leq C(T)\mathbb{E}\|R(\varepsilon,\varphi)\|_{L^1(0,T;L^2)+L^{q'}(0,T;L^{p'})}^2$$

$$\leq C(T)\big(\varepsilon^2 D_U^2 T^2 \|V\|_{L^\infty(0,T;\mathbb{R}^m)}^2 \mathbb{E}\|\varphi\|_{C([0,T];L^2)}^2 + \sum_{j=1}^2 \mathbb{E}\|R_j(\varepsilon)\varphi\|_{L^{q'}(0,T;L^{p'})}^2\big).$$

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Therefore, in order to prove (4.11), we only need to show that

$$\mathbb{E}\|R_j(\varepsilon)\varphi\|^2_{L^{q'}(0,T;L^{p'})} \to 0, \quad as \ \varepsilon \to 0, \ j = 1, 2. \tag{4.15}$$

Below, we prove (4.15) only for $R_1(\varepsilon)$, but the argument applies as well to $R_2(\varepsilon)$. As in the proof of (3.15) we get

$$\|X^{u_\varepsilon} - X^u\|_{C([0,T];L^2)} + \|X^{u_\varepsilon} - X^u\|_{L^q(0,T;L^p)} \to 0, \quad as \ \varepsilon \to 0, \ \mathbb{P} - a.s. \tag{4.16}$$

Note that

$$h_1(X_{u,r,\varepsilon}) - h_1(X^u) = \int_0^1 \partial_z h_1(X^u + r'(X_{u,r,\varepsilon} - X^u))dr'(X_{u,r,\varepsilon} - X^u)$$
$$+ \int_0^1 \partial_{\bar{z}} h_1(X^u + r'(X_{u,r,\varepsilon} - X^u))dr'(\overline{X_{u,r,\varepsilon}} - \overline{X^u}). \tag{4.17}$$

Since $|\partial_z h_1(z)| + |\partial_{\bar{z}} h_1(z)| \le C|z|^{\alpha-2}$ for $z \in \mathbb{C}$, using the Minkowski inequality and the Hölder inequality we get that $\mathbb{P}$-a.s. for each $r \in [0,1]$,

$$\|h_1(X_{u,r,\varepsilon}) - h_1(X^u)\|_{L^{\frac{q}{\alpha-1}}(0,T;L^{\frac{p}{\alpha-1}})}$$
$$\le C \int_0^1 \|X^u + r'(X_{u,r,\varepsilon} - X^u)\|^{\alpha-2}_{L^q(0,T;L^p)} dr' \|X_{u,r,\varepsilon} - X^u\|_{L^q(0,T;L^p)}$$
$$\le C \sup_{\varepsilon \in [0,1]} \|X^{u_\varepsilon}\|^{\alpha-2}_{L^q(0,T;L^p)} \|X^{u_\varepsilon} - X^u\|_{L^q(0,T;L^p)} \to 0, \quad as \ \varepsilon \to 0, \tag{4.18}$$

where we also used $\alpha \ge 2$ and the last step is due to (4.16).

Thus, using the Hölder inequality combined with the Minkowski inequality and the bounded dominated convergence theorem we obtain that

$$\|R_1(\varepsilon)\varphi\|_{L^{q'}(0,T;L^{p'})} \le T^\theta \|\varphi\|_{L^q(0,T;L^p)} \int_0^1 \|h_1(X_{u,r,\varepsilon}) - h_1(X^u)\|_{L^{\frac{q}{\alpha-1}}(0,T;L^{\frac{p}{\alpha-1}})} dr$$
$$\to 0, \quad as \ \varepsilon \to 0, \ \mathbb{P} - a.s. \tag{4.19}$$

Moreover, taking into account (4.18), (2.5) and (4.4), we have

$$\|R_1(\varepsilon)\varphi\|_{L^{q'}(0,T;L^{p'})} \le CT^\theta \sup_{\varepsilon \in [0,1]} \|X^{u_\varepsilon}\|^{\alpha-1}_{L^q(0,T;L^p)} \|\varphi\|_{L^q(0,T;L^p)} \in L^\infty(\Omega),$$

which along with (4.19) and the bounded dominated convergence theorem yields (4.15) for $j = 1$. Therefore, the proof is complete. $\qquad\square$

We shall prove now the following result.

Proposition 4.4. *(i) Assume Hypothesis (H2) and that $\mathbb{X}_T \in L^{2+\nu}(\Omega; L^2)$, $\mathbb{X}_1 \in L^{2+\nu}(\Omega; L^2(0,T;L^2))$ for some small $\nu \in (0,1)$.*

Then, there exists a unique $(\mathcal{F}_t)$-adapted solution (Y^u, Z^u) to (2.9) corresponding to $u \in \mathcal{U}_{ad}$, satisfying for any Stirchartz pair (p,q),

$$\sup_{u \in \mathcal{U}_{ad}} \left(\|Y^u\|_{L^{2+\nu}(\Omega; C([0,T];L^2))} + \|Y^u\|_{L^{2+\nu}(\Omega; L^q(0,T;L^p))} \right) < \infty, \tag{4.20}$$

and

$$\sup_{u \in \mathcal{U}_{ad}} \|Z^u_k\|_{L^{2+\nu}(\Omega; L^2(0,T;L^2))} < \infty, \quad 1 \le k \le N. \tag{4.21}$$

(ii) Assume in addition that $\mathbb{X}_T \in L^{2+\nu}(\Omega; H^1)$, $\mathbb{X}_1 \in L^{2+\nu}(\Omega; L^2(0,T;H^1))$ for some small $\nu \in (0,1)$. Then, for any $\rho \in [2, 2+\nu)$ and any Strichartz pair (p,q), we have

$$\sup_{u \in \mathcal{U}_{ad}} \left(\|Y^u\|_{L^\rho(\Omega; C([0,T];H^1))} + \|Y^u\|_{L^\rho(\Omega; L^q(0,T;W^{1,p}))} \right) < \infty, \tag{4.22}$$

and

$$\sup_{u \in \mathcal{U}_{ad}} \|Z^u_k\|_{L^\rho(\Omega; L^2(0,T;H^1))} < \infty, \quad 1 \le k \le N. \tag{4.23}$$

As mentioned in Section 2, the main difficulty in the analysis of the backward stochastic equation comes from the singular term $\lambda i h_2(X^u)\overline{Y}$. The proof of Proposition 4.4 follows the following steps.

First, we will consider the truncated approximating equation (4.24) and introduce the dual equation (4.26) below, which are related by the formula (4.29). Then, the uniform estimate (4.28) for the dual solutions imply, via duality arguments, those for the approximating solutions $\{Y_n\}$ (see (4.31) below), which in turn imply the uniform estimates for $\{Z_n\}$ (see (4.35) below). Consequently, in view of the linear structure of (4.24), one can pass to the limit and obtain the well-posedness of problem (2.9) as well as the estimates (4.20) and (4.21).

Analogous arguments are applicable to prove estimates (4.22) and (4.23) in Sobolev spaces, which requires the condition $\alpha \ge 2$ and the integrability conditions on $\mathbb{X}_T$ and $\mathbb{X}_1$ in Sobolev spaces.

Proof. *(i).* Let g be a radial smooth cut-off function such that $g = 1$ on $B_1(\mathbb{R})$, and $g = 0$ on $B_2^c(\mathbb{R})$. For $j = 1,2$, set $h_{j,n}(X^u) := g(\frac{|X^u|}{n})h_j(X^u)$. Note that $|h_{1,n}(X^u)| + |h_{2,n}(X^u)| \le \alpha 2^{\alpha-1}|g|_{L^\infty} n^{\alpha-1}$.

Consider the approximating backward stochastic equation

$$dY_n = -i\Delta Y_n\, dt - \lambda i h_{1,n}(X^u)Y_n dt + \lambda i h_{2,n}(X^u)\overline{Y_n}dt + \mu Y_n dt - iV_0 Y_n dt$$

$$- iu \cdot VY_n dt + \gamma_1(X^u - \mathbb{X}_1)dt - \sum_{k=1}^{N}\overline{\mu_k}e_k Z_{k,n}dt + \sum_{k=1}^{N} Z_{k,n}d\beta_k(t),$$

$$Y_n(T) = -(X^u(T) - \mathbb{X}_T). \tag{4.24}$$

By standard theory for stochastic backward infinite dimensional equations (see, e.g., [14], [17], [28]), it follows that there exists a unique $(\mathcal{F}_t)$-adapted solution $(Y_n, Z_n) \in L^2(\Omega; C([0,T]; L^2)) \times (L^2_{ad}(0,T; L^2(\Omega; L^2)))^N$ to (4.24).

In order to pass to the limit $n \to \infty$, we are going to obtain uniform estimates of Y_n in the space $\mathcal{Y}_{\rho_\nu} := L^{\rho_\nu}(\Omega; L^\infty(0,T; L^2)) \cap L^{\rho_\nu}(\Omega; L^q(0,T; L^p))$ with $\rho_\nu := 2 + \nu$ and $(p,q) = (\alpha + 1, \frac{4(\alpha+1)}{d(\alpha-1)})$.

To this purpose, for each $n \geq 1$, define the functional Λ_n on the space $L^\infty(\Omega \times (0,T) \times \mathbb{R}^d)$,

$$\Lambda_n(\Psi) := \mathbb{E}Re\,\langle X^u(T) - \mathbb{X}_T, \psi_n(T)\rangle_2 + \gamma_1\mathbb{E}\int_0^T Re\,\langle X^u(t) - \mathbb{X}_1(t), \psi_n(t)\rangle_2\, dt,$$
$$\tag{4.25}$$

where $\Psi \in L^\infty(\Omega \times (0,T) \times \mathbb{R}^d)$, X^u is the solution to (1.1), and ψ_n satisfies

$$d\psi_n = -i\Delta\psi_n dt - \lambda i h_{1,n}(X^u)\psi_n dt - \lambda i h_{2,n}(X^u)\overline{\psi_n}dt - \mu\psi_n dt$$
$$- iV_0\psi_n dt - iu \cdot V\psi_n dt - \Psi dt + \psi_n dW(t),$$
$$\psi_n(0) = 0. \tag{4.26}$$

(Note that, (4.26) is similar to (4.3) but with Ψ in place of $i\widetilde{u} \cdot VX^u$.)

Since $|h_{j,n}(X^u)| \leq \alpha|g|_{L^\infty}|X^u|^{\alpha-1}$, $j = 1,2$, arguing as in the proof of Lemma 4.2 we infer that there exists a unique strong L^2-solution ψ_n to (4.26) on $[0,T]$, satisfying

$$\sup_n(\|\psi_n\|_{C([0,T];L^2)} + \|\psi_n\|_{L^q(0,T;L^p)}) \leq C(T)\|\Psi\|_{L^1(0,T;L^2)+L^{q'}(0,T;L^{p'})}, \tag{4.27}$$

where $C(T) \in L^\infty(\Omega)$ is independent of n and Ψ. It follows also that for $\rho > 1$,

$$\sup_{n\geq 1}\|\psi_n\|_{\mathcal{Y}_{\rho'}} \leq C(\rho, T)\|\Psi\|_{\mathcal{Y}'_{\rho'}}, \tag{4.28}$$

where $C(\rho, T)$ is independent of n and Ψ and $\mathcal{Y}'_{\rho'} := L^{\rho'}(\Omega; L^1(0, T; L^2)) + L^{\rho'}(\Omega; L^{q'}(0, T; L^{p'}))$.

Moreover, by Itô's formula, for every $n \geq 1$ and $\Psi \in L^\infty(\Omega \times (0, T) \times \mathbb{R}^d)$, we have

$$\Lambda_n(\Psi) = \mathbb{E} \int_0^T Re \langle \Psi, Y_n \rangle_2 \, dt. \tag{4.29}$$

Thus, since $|X^u(t)|_2$ is $\mathbb{P}$-a.s. conserved and by estimates (4.25), (4.28), we have

$$
\begin{aligned}
|\Lambda_n(\Psi)| \leq & \gamma_1 \|X^u - \mathbb{X}_1\|_{L^{\rho_\nu}(\Omega; L^2(0,T;L^2))} \|\psi_n\|_{L^{\rho'_\nu}(\Omega; L^2(0,T;L^2))} \\
& + \|X^u(T) - \mathbb{X}_T\|_{L^{\rho_\nu}(\Omega; L^2)} \|\psi_n(T)\|_{L^{\rho'_\nu}(\Omega; L^2)} \\
\leq & C\|\Psi\|_{\mathcal{Y}'_{\rho'_\nu}},
\end{aligned} \tag{4.30}
$$

where C is independent of n. Since $L^\infty(\Omega \times (0, T) \times \mathbb{R}^d)$ is dense in $\mathcal{Y}'_{\rho'_\nu}$ and $\mathcal{Y}_{\rho_\nu}$ is the dual space of $\mathcal{Y}'_{\rho'_\nu}$, it follows by (4.29), (4.30) that

$$\sup_{n \geq 1} \|Y_n\|_{\mathcal{Y}_{\rho_\nu}} < \infty. \tag{4.31}$$

Hence, there exists $\widetilde{Y} \in \mathcal{Y}_{\rho_\nu}$, such that along a subsequence of $\{n\} \to \infty$ (still denoted by $\{n\}$),

$$Y_n \to \widetilde{Y}, \quad weak^* \ in \ \mathcal{Y}_{\rho_\nu}. \tag{4.32}$$

Note that, for each $j = 1, 2$, $h_{j,n}(X^u) \to h_j(X^u)$, $d\mathbb{P} \otimes dt \otimes d\xi - a.e.$, and $\sup_{n \geq 1} |h_{j,n}(X^u)| \leq \alpha |g|_{L^\infty} |X^u|^{\alpha-1} \in L^{2\rho'_\nu}(\Omega; L^{\frac{q}{\alpha-1}}(0, T; L^{\frac{p}{\alpha-1}}))$. Hence, by the dominated convergence theorem,

$$h_{j,n}(X^u) \to h_j(X^u), \ in \ L^{2\rho'_\nu}(\Omega; L^{\frac{q}{\alpha-1}}(0, T; L^{\frac{p}{\alpha-1}})), \quad as \ n \to \infty. \tag{4.33}$$

Since $\frac{1}{(2\rho'_\nu)'} = \frac{1}{\rho_\nu} + \frac{1}{2\rho'_\nu}$, from Hölder's inequality, (4.32) and (4.33) it follows that

$$h_{1,n}(X^u)Y_n \to h_1(X^u)\widetilde{Y}, \ h_{2,n}(X^u)\overline{Y_n} \to h_2(X^u)\overline{\widetilde{Y}}, \tag{4.34}$$

weakly in $L^{(2\rho'_\nu)'}(\Omega; L^{\frac{q}{\alpha}}(0, T; L^{p'}))$.

Moreover, we claim that for $1 \le k \le N$,

$$\sup_{n \ge 1} \|Z_{k,n}\|_{L^{\rho_\nu}(\Omega; L^2(0,T;L^2))} \le C < \infty. \tag{4.35}$$

In particular, for each $1 \le k \le N$, there exists $Z_k^u \in L^{\rho_\nu}(\Omega; L^2(0,T;L^2))$ such that (selecting a further subsequence if necessary)

$$Z_{k,n} \to Z_k^u, \quad weakly \ in \ L^2(\Omega; L^2(0,T;L^2)). \tag{4.36}$$

Since $v \mapsto \int_{\cdot}^{T} v d\beta_k(s)$ is a bounded linear operator in $L^2(\Omega; L^2(0,T;L^2))$, it follows that for each $1 \le k \le N$,

$$\int_{\cdot}^{T} Z_{k,n} d\beta_k(s) \to \int_{\cdot}^{T} Z_k^u d\beta_k(s), \quad weakly \ in \ L^2(\Omega; L^2(0,T;L^2)). \tag{4.37}$$

To prove (4.35), we apply Itô's formula to (4.24) to get that for $\eta > 0$,

$$e^{\eta t}|Y_n(t)|_2^2$$

$$= e^{\eta T}|X^u(T) - \mathbb{X}_T|_2^2 - \eta \int_t^T e^{\eta s}|Y_n|_2^2 ds$$

$$+ 2\lambda \int_t^T Im \int e^{\eta s} h_{2,n}(X^u)\overline{Y_n^2} d\xi ds - 2\int_t^T \int e^{\eta s}\mu|Y_n|^2 d\xi ds$$

$$- 2\gamma_1 \int_t^T Re \int e^{\eta s}Y_n(\overline{X^u} - \overline{\mathbb{X}_1})d\xi ds + 2\sum_{k=1}^N \int_t^T Re \int e^{\eta s}\mu_k e_k Y_n \overline{Z_{k,n}} d\xi ds$$

$$- \sum_{k=1}^N \int_t^T e^{\eta s}|Z_{k,n}|_2^2 ds - 2\sum_{k=1}^N \int_t^T Re \int e^{\eta s}Y_n \overline{Z_{k,n}} d\xi d\beta_k(s)$$

$$=: e^{\eta T}|X^u(T) - \mathbb{X}_T|_2^2 - \eta \int_t^T e^{\eta s}|Y_n|_2^2 ds + \sum_{j=1}^6 K_j(t). \tag{4.38}$$

Note that, since $|h_{2,n}(X^u)| \le \alpha|g|_{L^\infty}|X^u|^{\alpha-1}$, by Hölder's inequality,

$$|K_1(t)| \le \alpha|g|_{L^\infty} e^{\eta T} T^\theta \|X^u\|_{L^q(0,T;L^p)}^{\alpha-1} \|Y_n\|_{L^q(0,T;L^p)}^2, \tag{4.39}$$

where $\theta = 1 - d(\alpha - 1)/4 \in (0,1)$.

Moreover, using $ab \leq ca^2 + c^{-1}b^2$, $c > 0$, we get

$$\sum_{j=2}^{4} |K_j(t)| \leq (2|\mu|_\infty + 2\gamma_1 + 8\sum_{k=1}^{N} |\mu_k e_k|_{L^\infty}^2) \int_t^T e^{\eta s} |Y_n|_2^2 ds$$

$$+ 2\gamma_1 \int_t^T e^{\eta s} |X^u - \mathbb{X}_1|_2^2 ds + \frac{1}{2} \int_t^T e^{\eta s} |Z_{k,n}|_2^2 ds. \qquad (4.40)$$

Thus, choosing $\eta > 2|\mu|_\infty + 2\gamma_1 + 8\sum_{k=1}^{N} |\mu_k e_k|_{L^\infty}^2$, it follows that for any $t \in [0, T]$,

$$e^{\eta t} |Y_n(t)|_2^2 + \frac{1}{2} \sum_{k=1}^{N} \int_t^T e^{\eta s} |Z_{k,n}|_2^2 ds \leq V_{T,n} - 2\sum_{k=1}^{N} \int_t^T Re \int e^{\eta s} Y_n \overline{Z_{k,n}} d\xi d\beta_k(s),$$

$$\qquad (4.41)$$

where

$$V_{T,n} := e^{\eta T} |X^u(T) - \mathbb{X}_T|_2^2 + 2\gamma_1 \int_0^T e^{\eta s} |X^u - \mathbb{X}_1|_2^2 ds$$

$$+ \alpha |g|_{L^\infty} e^{\eta T} T^\theta \|X^u\|_{L^q(0,T;L^p)}^{\alpha-1} \|Y_n\|_{L^q(0,T;L^p)}^2. \qquad (4.42)$$

This yields

$$\sum_{k=1}^{N} \mathbb{E} \left(\int_0^T e^{\eta s} |Z_{k,n}|_2^2 ds \right)^{\frac{\rho_\nu}{2}}$$

$$\leq C(\rho_\nu) \left(\mathbb{E} V_{T,n}^{\frac{\rho_\nu}{2}} + \mathbb{E} \sup_{t \in [0,T]} \left| \sum_{k=1}^{N} \int_0^t Re \int e^{\eta s} Y_n \overline{Z_{k,n}} d\xi d\beta_k(s) \right|^{\frac{\rho_\nu}{2}} \right). \qquad (4.43)$$

Note that, $\mathbb{E} V_{T,n}^{\rho_\nu/2} < \infty$, due to (2.5), (4.31) and to the integrability conditions on $\mathbb{X}_T$ and $\mathbb{X}_1$. Moreover, by the Burkholder-Davis-Gundy inequality, the second term in the right hand side of (4.43) is bounded by

$$C(\rho_\nu) \mathbb{E} \left(\int_0^T \sum_{k=1}^{N} \left| \int e^{\eta s} Y_n \overline{Z_{k,n}} d\xi \right|^2 ds \right)^{\frac{\rho_\nu}{4}}$$

$$\leq C(\rho_\nu) \sum_{k=1}^{N} \mathbb{E} \sup_{t \in [0,T]} e^{\frac{1}{4}\rho_\nu \eta t} |Y_n(t)|_2^{\frac{\rho_\nu}{2}} \left(\int_0^T e^{\eta s} |Z_{k,n}|_2^2 ds \right)^{\frac{\rho_\nu}{4}}$$

$$\leq C(\rho_\nu, N) e^{\frac{1}{2}\rho_\nu \eta T} \mathbb{E} |Y_n|_{C([0,T];L^2)}^{\rho_\nu} + \frac{1}{2} \sum_{k=1}^{N} \mathbb{E} \left(\int_0^T e^{\eta s} |Z_{k,n}|_2^2 ds \right)^{\frac{\rho_\nu}{2}}.$$

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Plugging this into (4.43) we get

$$\sup_{n \geq 1} \frac{1}{2} \sum_{k=1}^{N} \mathbb{E} \left(\int_0^T e^{\eta s} |Z_{k,n}|_2^2 ds \right)^{\frac{\rho_\nu}{2}}$$

$$\leq C(\rho_\nu) \sup_{n \geq 1} \mathbb{E} V_{T,n}^{\frac{\rho_\nu}{2}} + C(\rho_\nu, N) e^{\frac{1}{2}\rho_\nu \eta T} \sup_{n \geq 1} \mathbb{E} |Y_n|_{C([0,T];L^2)}^{\rho_\nu}, \qquad (4.44)$$

which by (4.31) implies (4.35), as claimed.

Now, set

$$Y^u(\cdot) := -(X^u(T) - \mathbb{X}_T) - \int_{\cdot}^{T} \left(-i\Delta \widetilde{Y} - \lambda i h_1(X^u)\widetilde{Y} + \lambda i h_2(X^u)\overline{\widetilde{Y}} \right.$$

$$\left. + \mu \widetilde{Y} - iV_0 \widetilde{Y} - iu \cdot V \widetilde{Y} + \gamma_1(X^u - \mathbb{X}_1) - \sum_{k=1}^{N} \overline{\mu_k} e_k Z_k^u \right) ds$$

$$- \sum_{k=1}^{N} \int_{\cdot}^{T} Z_k^u d\beta_k(s).$$

By virtue of (4.32), (4.34), (4.36) and (4.37), we may pass to the limit in (4.24) and obtain that for any $v \in H^2$, $f \in L^\infty(\Omega \times (0,T))$,

$$\mathbb{E} \int_0^T {}_{H^{-2}}\left\langle \widetilde{Y}(t), f(t)v \right\rangle_{H^2} dt = \mathbb{E} \int_0^T {}_{H^{-2}}\left\langle Y^u(t), f(t)v \right\rangle_{H^2} dt, \qquad (4.45)$$

which implies that $Y^u = \widetilde{Y}$, in H^{-2}, $d\mathbb{P} \otimes dt$-a.e. Since Y^u is continuous in H^{-2}, $\mathbb{P}$-a.s., we can find a null set N', such that for any $\omega \notin N'$, $(Y^u(\omega), Z^u(\omega))$ solves (2.9) in H^{-2} for all $t \in [0,T]$, which proves the existence of solution to (2.9). Estimates (4.20) and (4.21) follow immediately by (4.31) and (4.35) respectively. Moreover, as in the proof of [1, Lemma 4.3], we have for $|Y^u(t) - Y^u(s)|_2^2$ an Itô formula similar to (4.38), which implies $t \to Y^u(t)$ is L^2 continuous.

The uniqueness can also be proved by duality arguments. Indeed, let (Y_j^u, Z_j^u), $j = 1, 2$, be any two solutions to (2.9). For any $\Psi \in L^\infty(\Omega \times (0,T) \times \mathbb{R}^d)$, let ψ be the unique solution to (4.26) but with $h_{j,n}(X^u)$ replaced by $h_j(X^u)$, $j = 1, 2$. Define $\Lambda(\Psi)$ similarly as in (4.25) with ψ_n replaced by ψ. Then, similarly to (4.29), $\Lambda(\Psi) = \mathbb{E} \int_0^T Re \left\langle \Psi, Y_j^u \right\rangle_2 dt$, $j = 1, 2$. It follows that $Y_1^u = Y_2^u$ by the arbitrariness of Ψ, and so $Z_1^u = Z_2^u$ by an estimate

similar to (4.44).

(ii). Fix $1 \leq j \leq N$. Consider the approximating equation of $(\partial_j Y^u, \partial_j Z^u)$ below ($\partial_j := \frac{\partial}{\partial \xi_j}$),

$$
dY_n' = -i\Delta Y_n' dt + G_n(Y_n') dt - \sum_{k=1}^{N} \overline{\mu}_k e_k Z_{k,n}' dt + \gamma_1 \partial_j (X^u - \mathbb{X}_1) dt
$$
$$
+ F_n(X^u, Y^u, Z^u) dt + \sum_{k=1}^{N} Z_{k,n}' d\beta_k(t),
$$
$$
Y_n' = -(\partial_j X^u(T) - \partial_j \mathbb{X}_T), \tag{4.46}
$$

where X^u and (Y^u, Z^u) are the solutions to (1.1) and (2.9) respectively,

$$
G_n(Y_n') := -\lambda i h_{1,n}(X^u) Y_n' + \lambda i h_{2,n}(X^u) \overline{Y_n'} + (\mu - iV_0 - iu \cdot V) Y_n',
$$

$$
F_n(X^u, Y^u, Z^u) := -\lambda i h_{1,n}'(X^u) Y^u + \lambda i h_{2,n}'(X^u) \overline{Y^u} + \partial_j \mu Y^u - i\partial_j V_0 Y^u
$$
$$
- iu \cdot \partial_j V Y^u - \sum_{k=1}^{N} \partial_j (\overline{\mu_k} e_k) Z_k^u,
$$

$h_{k,n}(X^u)$ is as in (4.24) and $h_{k,n}'(X^u) = g(\frac{|X^u| + |\nabla X^u|}{n}) \partial_j (h_k(X^u))$, $k = 1, 2$. By truncation, $|h_{k,n}(X^u)| + |h_{k,n}'(X^u)| \leq Cn^{\alpha-1}$, $k = 1, 2$, it follows that there exists a unique $(\mathcal{F}_t)$-adapted solution $(Y_n', Z_n') \in L^2(\Omega; C([0,T]; L^2)) \times (L^2_{ad}(0,T; L^2(\Omega; L^2)))^N$ to (4.46).

For each $\Psi \in L^\infty(\Omega \times (0,T) \times \mathbb{R}^d)$, let ψ_n be the solution to (4.26). Similarly to (4.25), set

$$
\widetilde{\Lambda}_{j,n}(\Psi) := \mathbb{E} Re \langle \partial_j X^u(T) - \partial_j \mathbb{X}_T, \psi_n(T) \rangle_2
$$
$$
+ \gamma_1 \mathbb{E} \int_0^T Re \langle \partial_j X^u(t) - \partial_j \mathbb{X}_1(t), \psi_n(t) \rangle_2 dt.
$$

By Itô's formula, we have

$$
\widetilde{\Lambda}_{j,n}(\Psi) = \mathbb{E} \int_0^T Re \langle \Psi, Y_n' \rangle_2 dt - \mathbb{E} \int_0^T Re \int F_n(X^u, Y^u, Z^u) \overline{\psi_n} d\xi dt. \tag{4.47}
$$

Note that, since $|h'_{1,n}(X^u)| \leq C|X^u|^{\alpha-2}|\partial_j X^u|$ and $2 \leq \alpha < 1 + 4/d$, by Hölder's inequality, we have for $(p,q) = (\alpha + 1, \frac{4(\alpha+1)}{d(\alpha-1)})$,

$$\left| \int_0^T Re \int (-\lambda i) h'_{1,n}(X^u) Y^u \overline{\psi_n} d\xi dt \right|$$
$$\leq \|h'_{1,n}(X^u) Y^u\|_{L^{q'}(0,T;L^{p'})} \|\psi_n\|_{L^q(0,T;L^p)}$$
$$\leq C_\alpha T^\theta \|X^u\|_{L^q(0,T;L^p)}^{\alpha-2} \|\partial_j X^u\|_{L^q(0,T;L^p)} \|Y^u\|_{L^q(0,T;L^p)} \|\psi_n\|_{L^q(0,T;L^p)}$$
$$\leq C_\alpha T^\theta \|X^u\|_{L^q(0,T;W^{1,p})}^{\alpha-1} \|Y^u\|_{L^q(0,T;L^p)} \|\psi_n\|_{L^q(0,T;L^p)}, \tag{4.48}$$

where $\theta = 1 - d(\alpha-1)/4 \in (0,1)$. Hence, for any $\rho \in [2, \rho_\nu)$,

$$\left| \mathbb{E} \int_0^T Re \int (-\lambda i) h'_{1,n}(X^u) Y^u \overline{\psi_n} d\xi dt \right|$$
$$\leq C(T) \mathbb{E}(\|X^u\|_{L^q(0,T;W^{1,p})}^{\alpha-1} \|Y^u\|_{L^q(0,T;L^p)} \|\psi_n\|_{L^q(0,T;L^p)})$$
$$\leq C(T) \|X^u\|_{L^\eta(\Omega;L^q(0,T;W^{1,p}))}^{\alpha-1} \|Y^u\|_{L^{\rho_\nu}(\Omega;L^q(0,T;L^p))} \|\psi_n\|_{L^{\rho'}(\Omega;L^q(0,T;L^p))}),$$

where η satisfies $\frac{1}{(\alpha-1)\eta} = \frac{1}{\rho'_\nu} - \frac{1}{\rho'} > 0$. Similar arguments apply to the term involving $\lambda i h'_{2,n}(X^u) \overline{Y^u}$. Moreover, the other terms in the integration $\mathbb{E} \int_0^T Re \int F_n(X^u, Y^u, Z^u) \overline{\psi_n} d\xi dt$ are bounded by

$$C \left(\|Y^u\|_{L^\rho(\Omega;L^\infty(0,T;L^2))} + \sum_{k=1}^N \|Z_k^u\|_{L^\rho(\Omega;L^2(0,T;L^2))} \right) \|\psi_n\|_{L^{\rho'}(\Omega;L^\infty(0,T;L^2))}.$$

Plugging the estimates above into (4.47) and using (4.28), (2.5), (4.20) and (4.21) we obtain for any $\rho \in [2, \rho_\nu)$,

$$\left| \mathbb{E} \int_0^T Re \langle \Psi, Y'_n \rangle_2 dt \right| \leq |\widetilde{\Lambda}_{j,n}(\Psi)| + \left| \mathbb{E} \int_0^T Re \int F_n(X^u, Y^u, Z^u) \overline{\psi_n} d\xi dt \right|$$
$$\leq C \|\Psi\|_{\mathcal{Y}'_{\rho'}}$$

with C independent of n and Ψ, which implies that for any $\rho \in [2, \rho_\nu)$,

$$\sup_{n \geq 1} \|Y'_n\|_{\mathcal{Y}_\rho} \leq C. \tag{4.49}$$

Once we have obtained (4.49), using similar arguments as those below (4.31), we can prove the assertion (ii). The details are omitted. $\square$

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Proof of Proposition 4.1. Using (4.5) in Lemma 4.2 we have

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left(\Phi(u + \varepsilon \widetilde{u}) - \Phi(u) \right)$$

$$= 2\mathbb{E} \bigg(\mathrm{Re} \left\langle X^u(T) - \mathbb{X}_T, \varphi^{u,\widetilde{u}}(T) \right\rangle_2$$

$$+ \gamma_1 \mathbb{E} \int_0^T \mathrm{Re} \left\langle X^u(t) - \mathbb{X}_1(t), \varphi^{u,\widetilde{u}}(t) \right\rangle_2 dt + \gamma_2 \int_0^T u \cdot \widetilde{u} \, dt \bigg). \quad (4.50)$$

Then, similarly to (4.29), by (2.9) and (4.3), we obtain via Itô's formula,

$$\mathbb{E} \, \mathrm{Re} \left\langle X^u(T) - \mathbb{X}_T, \varphi^{u,\widetilde{u}}(T) \right\rangle_2 + \gamma_1 \mathbb{E} \int_0^T \mathrm{Re} \left\langle X^u(t) - \mathbb{X}_1(t), \varphi^{u,\widetilde{u}}(t) \right\rangle_2 dt$$

$$= -\mathbb{E} \, \mathrm{Im} \int_0^T \int_{\mathbb{R}^d} \widetilde{u} \cdot V \, X^u \overline{Y^u} d\xi dt.$$

Combining these formulas we get (4.1) as claimed. □

5 Proof of Theorem 2.6.

As in the proof of Lemma 3.3, we note that Φ is continuous on the metric space $\mathcal{U}_{ad}$ endowed with the distance $d(u,v) = \|u - v\| = (\mathbb{E} \int_0^T |u(t) - v(t)|_m^2 dt)^{1/2}$. Applying Ekeland's variational principle in $\mathcal{U}_{ad}$ (see [13, Theorem 1], or [12]), for every $n \in \mathbb{N}$ we get $u_n \in \mathcal{U}_{ad}$ such that

$$\Phi(u_n) \leq \Phi(u) + \frac{1}{n} d(u_n, u), \quad \forall u \in \mathcal{U}_{ad}. \quad (5.1)$$

In particular, it follows that

$$u_n = \arg\min \left\{ \Phi(u) + \frac{1}{n} \|u_n - u\|; \ u \in \mathcal{U}_{ad} \right\}. \quad (5.2)$$

We define the function $\widetilde{\Phi} : L^2_{ad}(0, T; \mathbb{R}^m) \to \overline{\mathbb{R}} = (-\infty, +\infty]$ by

$$\widetilde{\Phi} = \Phi(u) + I_{\mathcal{U}_{ad}}(u), \quad \forall u \in L^2_{ad}(0, T; \mathbb{R}^m),$$

where

$$I_{\mathcal{U}_{ad}}(u) = \begin{cases} 0, & \text{if } u \in \mathcal{U}_{ad}; \\ +\infty, & \text{otherwise.} \end{cases}$$

The subdifferential $\partial\widetilde{\Phi}(u) \subset L^2_{ad}(0,T;\mathbb{R}^m)$ of $\widetilde{\Phi}$ at u in the sense of R.T. Rockafellar [25] is defined as the set of all $z \in L^2_{ad}(0,T;\mathbb{R}^m)$ such that the function $v \to \widetilde{\Phi}(v) - \mathbb{E}\int_0^T v(t)z(t)dt$ has u as a substationary point in the sense of [25].

We have

$$\partial\widetilde{\Phi}(u) \subset \eta(u) + \mathcal{N}_{\mathcal{U}_{ad}}(u), \quad \forall u \in \mathcal{U}_{ad}, \tag{5.3}$$

where $\eta(u)$ is defined by (4.2), $\mathcal{N}_{\mathcal{U}_{ad}}(u)$ is the normal cone to $\mathcal{U}_{ad}$ at u_n, i.e.,

$$\mathcal{N}_{\mathcal{U}_{ad}}(u_n) = \{v \in L^2_{ad}(0,T;\mathbb{R}^m); \ \langle v, u_n - \widetilde{v}\rangle \geq 0, \quad \forall \widetilde{v} \in \mathcal{U}_{ad}\},$$

and $\langle \ , \ \rangle$ denotes the inner product of $L^2_{ad}(0,T;\mathbb{R}^m)$.

To prove (5.3), as mentioned in [25, (2.4)], for each $u \in \mathcal{U}_{ad}$, one has

$$\partial\widetilde{\Phi}(u) = \{z \in L^2_{ad}(0,T;\mathbb{R}^m) : \widetilde{\Phi}^\uparrow(u,y) \geq \langle y, z\rangle, \forall y \in L^2_{ad}(0,T;\mathbb{R}^m)\},$$

where $\widetilde{\Phi}^\uparrow(u,y)$ is the subderivative at u with respect to y

$$\widetilde{\Phi}^\uparrow(u,y) = \sup_{V \subset \mathcal{N}(y)} \left[\limsup_{\substack{u' \to u, \alpha' \to \widetilde{\Phi}(u) \\ \alpha' \geq \widetilde{\Phi}(u'), t \to 0}} \inf_{y' \in V} \left(\frac{\Phi(u' + ty') - \alpha'}{t} + \frac{I_{\mathcal{U}_{ad}}(u' + ty')}{t} \right) \right],$$

and $\mathcal{N}(y)$ is the set of all neighborhoods of y. This yields

$$\widetilde{\Phi}^\uparrow(u,y) = \lim_{t \to 0} \frac{\Phi(u + ty) - \Phi(u)}{t} + I'_{\mathcal{U}_{ad}}(u,y),$$

where $I'_{\mathcal{U}_{ad}}(u,y) = 0$ if $y \in T_{\mathcal{U}_{ad}}(u)$, and $I'_{\mathcal{U}_{ad}}(u,y) = \infty$ if $y \notin T_{\mathcal{U}_{ad}}(u)$, $T_{\mathcal{U}_{ad}}(u)$ is the (Clarke) tangent cone to $\mathcal{U}_{ad}$ at u defined in [25]. Then, by Proposition 4.1, for any $z \in \partial\widetilde{\Phi}(u)$, we have that $\langle \eta(u), y\rangle \geq \langle y, z\rangle$, $\forall y = v - u$, $v \in \mathcal{U}_{ad}$. Thus, $z \in \eta(u) + \mathcal{N}_{\mathcal{U}_{ad}}(u)$. This implies (5.3).

On the other hand, by Theorem 2 in [25] we have

$$\partial(\widetilde{\Phi}(u) + \frac{1}{n}\|u_n - u\|) \subset \partial\widetilde{\Phi}(u) + \frac{1}{n}\partial\|u_n - u\|. \tag{5.4}$$

Thus, by (5.2)-(5.4) we get

$$0 \in \partial(\widetilde{\Phi}(u) + \frac{1}{n}\|u_n - u\|)(u = u_n)$$

$$\subset \eta(u_n) + \frac{1}{n}(\partial\|u_n - u\|)(u = u_n) + \mathcal{N}_{\mathcal{U}_{ad}}(u_n),$$

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which implies that there exist $\zeta_n \in \mathcal{N}_{\mathcal{U}_{ad}}(u_n)$ and $\eta_n \in (\partial\|u_n - u\|)(u = u_n)$, such that

$$\eta(u_n) + \zeta_n + \frac{1}{n}\eta_n = 0. \qquad (5.5)$$

We claim that,

$$\mathcal{N}_{\mathcal{U}_{ad}}(u_n) = \{v \in L^2_{ad}(0, T; \mathbb{R}^m) : v \in N_U(u_n), \quad a.e. \ on \ (0, T) \times \Omega.\}, \qquad (5.6)$$

where $N_U(u_n)$ is the normal cone to $U \subset \mathbb{R}^m$ at $u_n \in U$, that is, $N_U(u_n) = \{v \in \mathbb{R}^m; \ v \cdot (u_n - \widetilde{v}) \geq 0, \quad \forall \widetilde{v} \in U\}$.

Indeed, for any $\eta \in \mathcal{N}_{\mathcal{U}_{ad}}(u_n)$, we have

$$\mathbb{E}\int_0^T \eta \cdot (u_n - v)dt \geq 0, \quad \forall \ v \in \mathcal{U}_{ad}. \qquad (5.7)$$

Since for each closed convex set U, $\forall \nu > 0$, $(I + \nu N_U)^{-1} = P_U$, where P_U is the projection on U, there exists a unique $v \in \mathcal{U}_{ad}$, such that

$$v + N_U(v) \ni u_n + \eta, \quad a.e. \ on \ (0, T) \times \Omega, \qquad (5.8)$$

i.e. $v = P_U(u_n + \eta)$, a.e. on $(0, T) \times \Omega$. Hence, there exists $\zeta_v \in N_U(v)$, such that $v + \zeta_v = u_n + \eta$, $d\mathbb{P} \otimes dt$-a.e. Then, by (5.7),

$$0 \leq \mathbb{E}\int_0^T (v - u_n + \zeta_v) \cdot (u_n - v)dt = -\|u_n - v\|^2 + \mathbb{E}\int_0^T \zeta_v(u_n - v)dt.$$

Since $d\mathbb{P} \otimes dt$-a.e., $\zeta_v \in N_U(v)$, $\zeta_v \cdot (v - u_n) \geq 0$, we get

$$\|u_n - v\|^2 \leq \mathbb{E}\int_0^T \zeta_v \cdot (u_n - v)dt \leq 0.$$

It follows that $u_n = v$, $d\mathbb{P} \otimes dt$-a.e., which yields by (5.8) that $\eta \in N_U(u_n)$, $d\mathbb{P} \otimes dt$-a.e., thereby implying

$$\mathcal{N}_{\mathcal{U}_{ad}}(u_n) \subset \{v \in L^2_{ad}(0, T; \mathbb{R}^m) : v \in N_U(u_n), \quad a.e. \ on \ (0, T) \times \Omega.\}.$$

The inverse inclusion is obvious. Thus we obtain (5.6), as claimed.

Now, by virtue of (4.2), we may rewrite (5.5) as

$$u_n(t) + \frac{1}{2\gamma_2} \zeta_n^0(t) = \frac{1}{\gamma_2} \operatorname{Im} \int_{\mathbb{R}^d} V(\xi) X_n(t,\xi) \overline{Y_n}(t,\xi) d\xi - \frac{1}{2\gamma_2 n} \eta_n(t) \tag{5.9}$$
$$\text{a.e. on } (0,T) \times \Omega,$$

where $\zeta_n^0(t) \in N_U(u_n(t))$, $X_n := X^{u_n}$, and (Y_n, Z_n) is the solution to (2.9) corresponding to u_n. We get by (5.9) that

$$u_n(t) = P_U \left(\frac{1}{\gamma_2} \operatorname{Im} \int V(\xi) X_n(t,\xi) \overline{Y_n}(t,\xi) d\xi - \frac{1}{2\gamma_2 n} \eta_n(t) \right), \tag{5.10}$$

with

$$\mathbb{E} \int_0^T |\eta_n(t)|_m^2 dt = 1. \tag{5.11}$$

We claim that there exists a probability space $(\Omega^*, \mathcal{F}^*, \mathbb{P}^*)$, $u_n^*, u^* \in \mathcal{U}_{ad^*}$, $n \geq 1$, such that the distributions of u_n^* and u_n coincide on $L^1(0,T;\mathbb{R}^m)$, and as $n \to \infty$,

$$u_n^* \to u^*, \quad in \ L^1(0,T;\mathbb{R}^m), \quad \mathbb{P}^* - a.s. \tag{5.12}$$

Then, it follows from the boundedness of $\{u_n^*\}$ that

$$u_n^* \to u^*, \quad in \ L^2(0,T;\mathbb{R}^m), \quad \mathbb{P}^* - a.s.$$

Hence, similar arguments as in the proof of Theorem 2.5 imply that

$$\Phi^*(u^*) = \lim_{n \to \infty} \Phi^*(u_n^*) = \lim_{n \to \infty} \Phi(u_n) = I,$$

thereby yielding the equality in (2.11).

It remains to prove (5.12). By virtue of Skorohod's representation theorem, we only need to show the tightness of the distributions of u_n on $L^1(0,T;\mathbb{R}^m)$, $n \geq 1$. For this purpose, in view of Lemma 6.2, it suffices to prove that $\mu_n := \mathbb{P} \circ u_n^{-1}$, $n \geq 1$, satisfy (6.14) and (6.15).

Indeed, (6.14) follows immediately from the uniform boundedness of $\{u_n\}$. Regarding (6.15), by Markov's inequality, it suffices to show that there exists a positive exponent $b > 0$ such that for any $\delta \in (0,1)$,

$$\limsup_{n \to \infty} \mathbb{E} \sup_{0 < h \leq \delta} \int_0^{T-h} |u_n(t+h) - u_n(t)|_m dt \leq C\delta^b. \tag{5.13}$$

To this end, since P_U is Lipschitz, using (5.10), the Cauchy inequality, (2.4) and (4.22) we get

$$\mathbb{E} \sup_{0<h\leq\delta} \int_0^{T-h} |u_n(t+h) - u_n(t)|_m dt$$

$$\leq \frac{1}{\gamma_2 n} T^{\frac{1}{2}} + \frac{1}{\gamma_2} |V|_{L^\infty} \mathbb{E} \int_0^T \sup_{0<h\leq\delta} \Big(|X_n(t+h) - X_n(t)|_{H^{-1}} |Y_n(t+h)|_{H^1}$$

$$+ |Y_n(t+h) - Y_n(t)|_{H^{-1}} |X_n(t)|_{H^1} \Big) dt$$

$$\leq \frac{1}{\gamma_2 n} T^{\frac{1}{2}} + C \left(\mathbb{E} \int_0^T \sup_{0<h\leq\delta} |X_n(t+h) - X_n(t)|_{H^{-1}}^2 dt \right)^{\frac{1}{2}}$$

$$+ C \left(\mathbb{E} \int_0^T \sup_{0<h\leq\delta} |Y_n(t+h) - Y_n(t)|_{H^{-1}}^2 dt \right)^{\frac{1}{2}}. \tag{5.14}$$

Let us estimate $\mathbb{E} \int_0^T \sup_{0<h\leq\delta} |Y_n(t+h) - Y_n(t)|_{H^{-1}}^2 dt$ in the right hand side of (5.14). Similar arguments apply to $\mathbb{E} \int_0^T \sup_{0<h\leq\delta} |X_n(t+h) - X_n(t)|_{H^{-1}}^2 dt$.
By the backward equation (2.9),

$$\mathbb{E} \int_0^T \sup_{0<h\leq\delta} |Y_n(t+h) - Y_n(t)|_{H^{-1}}^2 dt$$

$$\leq \mathbb{E} \int_0^T \sup_{0<h\leq\delta} \left| \int_t^{t+h} i\Delta Y(s) ds \right|_{H^{-1}}^2 dt$$

$$+ \mathbb{E} \int_0^T \sup_{0<h\leq\delta} \left| \int_t^{t+h} (-\lambda i h_1(X_n(s)) Y_n(s) + \lambda i h_2(X_n(s)) \overline{Y_n}(s)) ds \right|_{H^{-1}}^2 dt$$

$$+ \mathbb{E} \int_0^T \sup_{0<h\leq\delta} \left| \int_t^{t+h} (\mu - iV_0 - iu_n(s) \cdot V) Y_n(s) ds \right|_{H^{-1}}^2 dt$$

$$+ \mathbb{E} \int_0^T \sup_{0<h\leq\delta} \left| \int_t^{t+h} (\gamma_1(X_n(s) - \mathbb{X}_1(s)) - \sum_{k=1}^N \overline{\mu_k} e_k Z_{k,n}(s)) ds \right|_{H^{-1}}^2 dt$$

$$+ \mathbb{E} \int_0^T \sup_{0<h\leq\delta} \left| \int_t^{t+h} \sum_{k=1}^N Z_{k,n}(s) d\beta_k(s) \right|_{H^{-1}}^2 dt =: \sum_{j=1}^5 K_j. \tag{5.15}$$

For K_1, by (4.22),

$$K_1 \leq \mathbb{E} \int_0^T \sup_{0 < h \leq \delta} \left(\int_t^{t+h} |Y_n(s)|_{H^1} ds \right)^2 dt \leq \delta^2 T \, \mathbb{E} \sup_{t \in [0,T+1]} |Y_n(t)|_{H^1}^2 \leq C\delta^2,$$

(5.16)

where C is independent of n.

Similarly, by Cauchy's inequality and by (4.20),

$$K_3 + K_4 \leq \delta^2 T (|\mu|_\infty + |V_0|_\infty + D_U \|V\|_{L^\infty(0,T+1;L^\infty)}) \mathbb{E} \sup_{t \in [0,T+1]} |Y_n(t)|_2^2$$

$$+ \delta \gamma_1 \mathbb{E} \int_0^T \int_t^{t+\delta} |X_n(s) - \mathbb{X}_1(s)|_2^2 ds dt$$

$$+ \delta \sum_{k=1}^N |\mu_k| |e_k|_\infty \mathbb{E} \int_0^T \int_t^{t+\delta} |Z_{k,n}(s)|_2^2 ds dt$$

$$\leq C(\delta + \delta^2),$$

(5.17)

where C is independent of n.

Regarding K_2, choose the Strichartz pair $(p, q) = (\alpha + 1, \frac{4(\alpha+1)}{d(\alpha-1)})$. Since $p \in (2, \frac{2d}{d-2})$, $L^{p'}(\mathbb{R}^d) \hookrightarrow H^{-1}(\mathbb{R}^d)$, we have

$$K_2 \leq \mathbb{E} \int_0^T \sup_{0 < h \leq \delta} \left(\int_t^{t+h} \left| -\lambda i h_1(X_n(s)) Y_n(s) + \lambda i h_2(X_n(s)) \overline{Y}_n(s) \right|_{L^{p'}} ds \right)^2 dt$$

$$\leq \alpha^2 \mathbb{E} \int_0^T \sup_{0 < h \leq \delta} \left(\int_t^{t+h} \left| X_n^{\alpha-1}(s) Y_n(s) \right|_{L^{p'}} ds \right)^2 dt$$

$$\leq \delta^{2/q} \alpha^2 \mathbb{E} \int_0^T \|X_n^{\alpha-1} Y_n\|_{L^{q'}(t,t+\delta;L^{p'})}^2 dt$$

$$\leq \delta^{2/q} \alpha^2 T \, \mathbb{E} \|X_n^{\alpha-1} Y_n\|_{L^{q'}(0,T+1;L^{p'})}^2.$$

Note that, by Hölder's inequality,

$$\|X_n^{\alpha-1} Y_n\|_{L^{q'}(0,T;L^{p'})} \leq T^\theta \|X_n\|_{L^q(0,T;L^p)}^{\alpha-1} \|Y_n\|_{L^q(0,T;L^p)},$$

where $\theta = 1 - d(\alpha - 1)/4 \in (0, 1)$. Hence

$$K_2 \leq \delta^{2/q} \alpha^2 T^{2\theta+1} \, \mathbb{E} \|X_n\|_{L^q(0,T;L^p)}^{2(\alpha-1)} \|Y_n\|_{L^q(0,T+1;L^p)}^2$$

$$\leq \delta^{2/q} \alpha^2 T^{2\theta+1} \|X_n\|_{L^\infty(\Omega;L^q(0,T;L^p))}^{2(\alpha-1)} \|Y_n\|_{L^2(\Omega;L^q(0,T+1;L^p))}^2$$

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Then, by (2.5) and (4.20) we obtain

$$K_4 \leq C\delta^{2/q}, \tag{5.18}$$

where C is independent of n.

For K_5, using the Burkholder-Davis-Gundy inequality we get

$$K_5 \leq C \int_0^T \mathbb{E} \int_t^{t+\delta} \sum_{k=1}^N |Z_{k,n}(s)|_2^2 ds dt.$$

Then, using Fubini's theorem to interchange the sum and integrals, by (4.21) we have,

$$K_5 \leq C \sum_{k=1}^N \mathbb{E} \left(\int_0^\delta \int_0^s + \int_\delta^T \int_{s-\delta}^s + \int_T^{T+\delta} \int_{s-\delta}^T \right) |Z_{k,n}(s)|_2^2 \, dt ds$$

$$\leq 3\delta C \sum_{k=1}^N \mathbb{E} \int_0^{T+1} |Z_{k,n}(s)|_2^2 ds \leq C\delta, \tag{5.19}$$

where C is independent of n.

Plugging (5.16)-(5.19) into (5.15), since $2/q < 1$ and $\delta < 1$, we obtain

$$\mathbb{E} \int_0^T \sup_{0 < h \leq \delta} |Y_n(t+h) - Y_n(t)|_{H^{-1}}^2 dt \leq C(\delta + \delta^2 + \delta^{\frac{2}{q}}) \leq C\delta^{\frac{2}{q}},$$

where C is independent of n.

The term $\mathbb{E} \int_0^T \sup_{0 < h \leq \delta} |X_n(t+h) - X_n(t)|_{H^{-1}}^2 dt$ in the right hand side of (5.14) can be estimated similarly.

Therefore, in view of (5.14) we obtain (5.13) with $b = 1/q$, thereby proving the tightness of $\{\mu_n\}$ and yielding the equality in (2.11).

Finally, the stochastic maximum principle (2.12) follows from Proposition 4.1, taking into account that (see (5.5)) for the optimal u^*, $\eta(u^*) + \zeta^* = 0$, where $\zeta^* \in \mathcal{N}_{\mathcal{U}_{ad^*}}(u^*)$. Now, the proof is complete. $\square$

An example. We consider the case $m = 1$ and $U = [0, \ell]$, where $\ell > 0$. Then, equation (2.12) reduces to

$$u^*(t) = \begin{cases} 0 & \text{if } \operatorname{Im} \int_{\mathbb{R}} V(\xi) X^*(t, \xi) \overline{Y^*}(t, \xi) d\xi \leq 0, \\ \frac{\ell}{\gamma_2} & \text{if } \operatorname{Im} \int_{\mathbb{R}} V(\xi) X^*(t, \xi) \overline{Y^*}(t, \xi) d\xi \geq \ell, \\ \frac{1}{\gamma_2} \operatorname{Im} \int_{\mathbb{R}} V(\xi) X^*(t, \xi) \overline{Y^*}(t, \xi) d\xi & \text{otherwise.} \end{cases}$$

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For the numerical computation of the optimal controller u^*, one can use the standard gradient descent algorithm suggested by (2.12). Namely,

$$u_{n+1} = P_U\big(\frac{1}{1+2\gamma_2\rho_n}u_n + \frac{2\rho_n}{1+2\gamma_2\rho_n}\mathrm{Im}\int_{\mathbb{R}^d} V(\xi)X_n(t,\xi)\overline{Y}_n(t,\xi)d\xi\big), \quad (5.20)$$

where $\rho_n > 0$ are suitably chosen and X_n, Y_n are solutions to the forward–backward system (1.1), (2.9) with $u = u_n$.

Proof of Theorem 2.7. The proof is similar to that of Theorem 2.6. As a matter of fact, in the deterministic case, the analysis of the equation of variation and of the backward equation is much easier.

Similarly to Lemma 4.2, we have

$$\sup_{u,v\in\mathcal{U}_{ad}} \big(\|\varphi^{u,\widetilde{u}}\|_{C([0,T];H^1)} + \|\varphi^{u,\widetilde{u}}\|_{L^q(0,T;W^{1,p})}\big) < \infty,$$

where $\widetilde{u} = v - u$, $u,v \in \mathcal{U}_{ad}$, and $\varphi^{u,\widetilde{u}}$ is the solution to the deterministic equation of variation (i.e. (4.3) without W). Moreover,

$$\lim_{\varepsilon\to 0}\sup_{t\in[0,T]} |\varepsilon^{-1}(X^{u_\varepsilon}(t) - X^u(t)) - \varphi^{u,\widetilde{u}}(t)|_2^2 \to 0, \quad (5.21)$$

where X^{u_ε} and X^u are the solutions to (1.1) corresponding to $u_\varepsilon := u + \varepsilon\widetilde{u}$ and u respectively.

Regarding the backward deterministic equation, we can now use time reversal arguments and the Strichartz estimates to obtain directly the estimate below,

$$\sup_{u\in\mathcal{U}_{ad}} \big(\|Y^u\|_{L^\infty(0,T;H^1)} + \|Y^u\|_{L^q(0,T;W^{1,p})}\big) < \infty.$$

Based on these, one also has the directional derivative of Φ as in Proposition 4.1, and similarly to (5.13), the estimate below for the minimizing sequence of controls $\{u_n\}$ from Ekeland's principle,

$$\limsup_{n\to\infty}\sup_{0<h\leq\delta}\int_0^{T-h} |u_n(t+h) - u_n(t)|_m dt \leq C\delta^{1/q},$$

which by the Riesz-Kolmogorov theorem implies that $\{u_n\}$ is relatively compact in $L^1(0,T;\mathbb{R}^m)$, thereby yielding the result. $\qquad\square$

6 Appendix

Lemma 6.1. *([26, Lemma 2.17]) Let $T > 0$ and $f \in C([0,T]; \mathbb{R}_+)$, such that*

$$f \leq a + bf^\alpha, \quad on \ [0,T],$$

where $a, b > 0$, $\alpha > 1$, $a < (1 - \frac{1}{\alpha})(\alpha b)^{-\frac{1}{\alpha-1}}$, and $f(0) \leq (\alpha b)^{-\frac{1}{\alpha-1}}$. Then,

$$f \leq \frac{\alpha}{\alpha - 1} a, \quad on \ [0,T].$$

Proof of (2.4) **and** (2.5)**.** For simplicity, we omit the dependence of u in X^u. We may assume $T \geq 1$ without loss of generality. Set

$$H(X(t)) := \frac{1}{2}|\nabla X(t)|_2^2 - \frac{\lambda}{\alpha+1}|X(t)|_{L^{\alpha+1}}^{\alpha+1} \tag{6.1}$$

As in the proof of [2, Theorem 3.1] we have for $t \in [0,T]$,

$$
\begin{aligned}
&H(X(t)) - H(x) \\
&= - \int_0^t \left(Im \left\langle (\nabla V_0 + u(s) \cdot \nabla V) X(s), \nabla X(s) \right\rangle \right) ds \\
&\quad + \int_0^t \left(Re \left\langle -\nabla(\mu X(s)), \nabla X(s) \right\rangle_2 ds + \frac{1}{2} \sum_{j=1}^N |\nabla(X(s)\phi_j)|_2^2 \right) ds \\
&\quad - \frac{1}{2}\lambda(\alpha - 1) \sum_{j=1}^N \int_0^t \int (Re\phi_j)^2 |X(s)|^{\alpha+1} d\xi ds + M(t), \tag{6.2}
\end{aligned}
$$

where $M(t) := \sum_{j=1}^N \int_0^t (Re \langle \nabla(\phi_j X(s)), \nabla X(s) \rangle_2 - \lambda \int Re\phi_j |X(s)|^{\alpha+1} d\xi) d\beta_j(s)$.

Below we shall treat the focusing and defocusing cases respectively.

(i) (The focusing case $\lambda = 1$.) Note that, by [2, Lemma 3.5],

$$|X(t)|_{L^{\alpha+1}}^{\alpha+1} \leq C_\varepsilon |X(t)|_2^p + \varepsilon |\nabla X(t)|_2^2, \tag{6.3}$$

where $p = 2\frac{2(\alpha+1)-d(\alpha-1)}{4-d(\alpha-1)} > 2$. As in the proof of [2, Theorem 3.7], the first three terms on the right hand side of (6.2) are bounded by $C \int_0^t (|X(s)|_2^p + |X(s)|_2^2 + |\nabla X(s)|_2^2) ds$, where C is independent of u. Thus, taking $\varepsilon < \frac{\alpha+1}{4}$ yields

$$|\nabla X(t)|_2^2 \leq 4H(x) + C(|X(t)|_2^p + D(t)) + 4M(t), \tag{6.4}$$

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where $D(t) := \int_0^t (|X(s)|_2^p + |X(s)|_2^2 + |\nabla X(s)|_2^2)ds$. It follows that for any $\rho \geq 4$,

$$|\nabla X(t)|_2^{2\rho} \leq C + C(|X(t)|_2^{\rho p} + D^\rho(t) + |M(t)|^\rho) \tag{6.5}$$

with C independent of u.

Note that, by Jensen's inequality and

$$|X(t)|_2 = |x|_2, \quad t \in [0, T], \tag{6.6}$$

we have

$$\mathbb{E} \sup_{s \in [0,t]} D^\rho(s) \leq \mathbb{E} \sup_{s \in [0,t]} s^{\rho-1} \int_0^s (|X(r)|_2^{p\rho} + |X(r)|_2^{2\rho} + |\nabla X(r)|_2^{2\rho})dr$$

$$\leq C(\rho, T) \left(1 + \int_0^t \mathbb{E} \sup_{r \in [0,s]} |\nabla X(r)|_2^{2\rho} ds \right). \tag{6.7}$$

Moreover, by the BDG inequality we get

$$\mathbb{E} \sup_{s \in [0,t]} |M(s)|^\rho$$

$$\leq C(\rho)\mathbb{E} \left(\int_0^t \sum_{j=1}^N \left(|Re \langle \nabla(\phi_j X(s)), \nabla X(s) \rangle_2|^2 + \left| \int Re\phi_j |X(s)|^{\alpha+1} d\xi \right|^2 \right) ds \right)^{\frac{\rho}{2}}$$

$$\leq C(\rho)\mathbb{E} \left(\int_0^t |\nabla X(s)|_2^4 + |X(s)|_2^4 + |X(s)|_{L^{\alpha+1}}^{2(\alpha+1)} ds \right)^{\frac{\rho}{2}}$$

$$\leq C(\rho, T)\mathbb{E} \int_0^t |\nabla X(s)|_2^{2\rho} + |X(s)|_2^{2\rho} + |X(s)|_{L^{\alpha+1}}^{(\alpha+1)\rho} ds, \tag{6.8}$$

Then, using (6.6) and (6.3) one obtains the estimate

$$\mathbb{E} \sup_{s \in [0,t]} |M(s)|^\rho \leq C(\rho, T) \left(1 + \int_0^t \mathbb{E} \sup_{r \in [0,s]} |\nabla X(r)|_2^{2\rho} ds \right). \tag{6.9}$$

Thus, plugging (6.7) and (6.9) into (6.5) and using (6.6) yield

$$\mathbb{E} \sup_{s \in [0,t]} |\nabla X(s)|_2^{2\rho} \leq C + C \int_0^t \sup_{r \in [0,s]} |\nabla X(r)|_2^{2\rho} ds,$$

which implies (2.4) by Gronwall's inequality.

(ii) (The defocusing case $\lambda = -1$.) Similarly to (6.4), we have by (6.2),

$$\frac{1}{2}|\nabla X(t)|_2^2 + \frac{1}{\alpha+1}|X(t)|_{L^{\alpha+1}}^{\alpha+1}$$

$$\leq H(x) + C\int_0^t (|X(s)|_2^2 + |\nabla X(s)|_2^2 + |X(s)|_{L^{\alpha+1}}^{\alpha+1})ds + M(t).$$

Using (6.6) and (6.8) we get for $\rho \geq 4$,

$$\mathbb{E} \sup_{s\in[0,t]} (|\nabla X(t)|_2^{2\rho} + |X(t)|_{L^{\alpha+1}}^{(\alpha+1)\rho})$$

$$\leq C + C\int_0^t \mathbb{E} \sup_{r\in[0,s]} (|\nabla X(r)|_2^{2\rho} + |X(r)|_{L^{\alpha+1}}^{(\alpha+1)\rho})ds,$$

and so (2.4) follows.

It remains to prove (2.5). Indeed, in the case where e_k are constants, $1 \leq k \leq N$, by the rescaling transformation $y = e^{-W}X$, we have

$$\partial_t y = -i\Delta y - \lambda i|y|^{\alpha-1}y + f(u)y, \tag{6.10}$$

where $f(u) := -i(V_0 + u \cdot V)$. Note that, the Strichartz coefficient C_T is now a deterministic constant. Then, arguing as in (3.19) we obtain that $\sup_{u\in\mathcal{U}_{ad}} \|y^u\|_{L^\infty(\Omega;L^q(0,T;L^p))} < \infty$ for any Strichartz pair (p,q).

As regards the estimate for $\|X^u\|_{L^\rho(\Omega;L^q(0,T;W^{1,p}))}$, it suffices to prove that for any $\rho \geq 1$,

$$\sup_{u\in\mathcal{U}_{ad}} \mathbb{E}\|y^u\|_{L^q(0,T;W^{1,p})}^\rho < \infty, \tag{6.11}$$

where $(p,q) = (\alpha+1, \frac{4(\alpha+1)}{d(\alpha-1)})$.

Since $|\nabla(|y|^{\alpha-1}y)| \leq \alpha|y|^{\alpha-1}|\nabla y|$, $|y|_{L^p} \leq D|y|_{H^1}$, the Hölder inequality implies that (see e.g. [2, (2.25)])

$$\||y|^{\alpha-1}y\|_{L^{q'}(0,t;W^{1,p'})} \leq 2\alpha D^{\alpha-1}t^\theta\|y\|_{C([0,T];H^1)}^{\alpha-1}\|y\|_{L^q(0,t;W^{1,p})}, \tag{6.12}$$

where $\theta = 1 - 2/q > 0$. Moreover,

$$\|f(u)y\|_{L^1(0,t;H^1)} \leq T\|f(u)\|_{L^\infty(0,T;W^{1,\infty})}\|y\|_{C([0,t];H^1)},$$

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Thus, applying Strichartz estimates to (6.10) and using the estimates above, we get

$$
\begin{aligned}
&\|y\|_{L^q(0,t;W^{1,p})}\\
&\leq C\big(|x|_{H^1} + 2\alpha D^{\alpha-1}t^\theta \|y\|_{C([0,T];H^1)}^{\alpha-1}\|y\|_{L^q(0,t;W^{1,p})}\\
&\quad + T\|f(u)\|_{L^\infty(0,T;W^{1,\infty})}\|y\|_{C([0,t];H^1)}\big)\\
&\leq D(T)(\|y\|_{C([0,T];H^1)} + t^\theta \|y\|_{L^q(0,t;W^{1,p})}),
\end{aligned}
\tag{6.13}
$$

where $D(T) = 1 + 2\alpha D^{\alpha-1}\|y\|_{C([0,T];H^1)}^{\alpha-1} + T\sup_{u\in\mathcal{U}_{ad}}\|f(u)\|_{L^\infty(\Omega;L^\infty(0,T;W^{1,\infty}))}$.

Then, similarly to (3.19) we have

$$
\|y^u\|_{L^q(0,T;W^{1,p})} \leq 2\left([\tfrac{T}{t}]+1\right)^{\frac{1}{q}} D(T)\|y^u\|_{C([0,T];H^1)},
$$

where $t = 2^{-1/\theta}D(T)^{-1/\theta} \wedge T$.

Therefore, taking into account (2.4) we obtain (6.11), thereby completing the proof. $\qquad\square$

Lemma 6.2. *Let μ_n, $n \geq 1$, be a family of probability measures on $L^1(0,T;\mathbb{R}^m)$. Assume that*

$$
\lim_{R\to\infty}\limsup_{n\to\infty}\mu_n\left\{v \in L^1(0,T;\mathbb{R}^m) : \int_0^T |v(t)|_m dt > R\right\} = 0,
\tag{6.14}
$$

and for any $\varepsilon > 0$,

$$
\lim_{\delta\to 0}\limsup_{n\to\infty}\mu_n\left\{v \in L^1(0,T;\mathbb{R}^m) : \sup_{0<h\leq\delta}\int_0^{T-h} |v(t+h) - v(t)|_m dt > \varepsilon\right\} = 0.
\tag{6.15}
$$

Then, $\{\mu_n\}_{n\geq 1}$ is tight on $L^1(0,T;\mathbb{R}^m)$.

Proof. Set $K_1(R) = \{v \in L^1(0,T;\mathbb{R}^m) : \int_0^T |v(t)|_m dt \leq R\}$ and $K_2(\delta,\varepsilon) = \{v \in L^1(0,T;\mathbb{R}^m) : \sup_{0<h\leq\delta}\int_0^{T-h} |v(t+h)-v(t)|_m dt \leq \varepsilon\}$, where $R,\delta,\varepsilon > 0$.

Fix $\varepsilon > 0$. By (6.14) there exists $N(= N(\varepsilon))$, $R_1(= R_1(\varepsilon)) \geq 1$, such that $\sup_{n\geq N}\mu_n(K_1^c(R_1)) \leq \frac{\varepsilon}{2}$. Since for each $n \geq 1$, $\lim_{R\to\infty}\mu_n(K_1^c(R)) = 0$, we can choose $R_2(= R_2(\varepsilon))$ sufficiently large, such that $\sup_{1\leq n\leq N}\mu_n(K_1^c(R_2)) \leq \frac{\varepsilon}{2N}$. Thus, letting $R = R_1 \vee R_2$ we get $\sup_{n\geq 1}\mu_n(K_1^c(R)) \leq \varepsilon$.

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Similarly, since for each $k, n \geq 1$, $\lim_{\delta \to 0} \mu_n(K_2^c(\delta, \frac{1}{k})) = 0$, by (6.15) and similar arguments as above, we can choose $\delta_k > 0$ sufficiently small such that $\sup_{n \geq 1} \mu_n(K_2^c(\delta_k, \frac{1}{k})) \leq \frac{\varepsilon}{2^k}$.

Then, set $K := K_1(R) \cap (\bigcap_{k \geq 1} K_2(\delta_k, \frac{1}{k}))$. It follows from [27, Theorem 1] that K is relatively compact in $L^1(0, T; \mathbb{R}^m)$, and by the estimates above we have $\sup_{n \geq 1} \mu_n(K^c) \leq 2\varepsilon$, which implies the tightness of $\{\mu_n\}_{n \geq 1}$ on $L^1(0, T; \mathbb{R}^m)$. $\qquad\square$

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