Global solutions to random 3D vorticity equations for small initial data

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Abstract

One proves the existence and uniqueness in $(L^p(\mathbb{R}^3))^3$, $\frac{3}{2} < p < 2$, of a global mild solution to random vorticity equations associated to stochastic $3D$ Navier-Stokes equations with linear multiplicative Gaussian noise of convolution type, for sufficiently small initial vorticity. This resembles some earlier deterministic results of T. Kato [15] and are obtained by treating the equation in vorticity form and reducing the latter to a random nonlinear parabolic equation. The solution has maximal regularity in the spatial variables and is weakly continuous in $(L^3 \cap L^{\frac{3p}{4p-6}})^3$ with respect to the time variable. Furthermore, we obtain the pathwise continuous dependence of solutions with respect to the initial data.

Keywords: stochastic Navier-Stokes equation, vorticity, Biot-Savart operator.
MSC: 60H15, 35Q30.

1 Introduction

Consider the stochastic $3D$ Navier–Stokes equation

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$$
\begin{aligned}
&dX - \Delta X\,dt + (X\cdot\nabla)X\,dt = \sum_{i=1}^{N}(B_i(X)+\lambda_i X)d\beta_i(t) + \nabla\pi\,dt \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{on } (0,\infty)\times\mathbb{R}^3, \\
&\nabla\cdot X = 0 \qquad\qquad\qquad\qquad\qquad\qquad\quad \text{on } (0,\infty)\times\mathbb{R}^3, \\
&X(0) = x \qquad\qquad\qquad\qquad\qquad\qquad\quad\ \text{in } (L^p(\mathbb{R}^3))^3,
\end{aligned}
\tag{1.1}
$$

where $\lambda_i \in \mathbb{R}$, $x : \Omega \to \mathbb{R}^3$ is a random variable. Here π denotes the pressure and $\{\beta_i\}_{i=1}^N$ is a system of independent Brownian motions on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with normal filtration $(\mathcal{F}_t)_{t\geq 0}$, x is $\mathcal{F}_0$–measurable, and B_i are the convolution operators

$$
B_i(X)(\xi) = \int_{\mathbb{R}^3} h_i(\xi-\bar{\xi})X(\bar{\xi})d\bar{\xi} = (h_i * X)(\xi),\ \xi\in\mathbb{R}^3, \tag{1.2}
$$

where $h_i \in L^1(\mathbb{R}^3),\ i = 1, 2, ..., N$, and Δ is the Laplacian on $(L^2(\mathbb{R}^3))^3$.

It is not known whether (1.1) has a probabilistically strong solution in the mild sense for all time. Therefore, we shall rewrite (1.1) in vorticity form and then transform it into a random PDE, which we shall prove, has a global in time solution for $\mathbb{P}$-a.e. fixed $\omega \in \Omega$, provided the initial condition is small enough.

Consider the vorticity field

$$
U = \nabla \times X = \text{curl } X \tag{1.3}
$$

and apply the curl operator to equation (1.1). We obtain (see e.g. [4], [8]) the transport vorticity equation

$$
\begin{aligned}
&dU - \Delta U\,dt + ((X\cdot\nabla)U - (U\cdot\nabla)X)dt = \sum_{i=1}^{N}(h_i * U + \lambda_i U)d\beta_i \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in } (0,\infty)\times\mathbb{R}^3, \\
&U(0,\xi) = U_0(\xi) = (\text{curl}\, x)(\xi),\ \xi\in\mathbb{R}^3.
\end{aligned}
\tag{1.4}
$$

The vorticity U is related to the velocity X by the equation

$$
X(t,\xi) = K(U(t))(\xi),\ t\in(0,\infty),\ \xi\in\mathbb{R}^3, \tag{1.5}
$$

where K is the Biot–Savart integral operator

$$
K(u)(\xi) = -\frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{\xi-\bar{\xi}}{|\xi-\bar{\xi}|^3}\times u(\bar{\xi})d\tilde{\xi},\ \xi\in\mathbb{R}^3. \tag{1.6}
$$

Then one can rewrite the vorticity equation (1.4) as

$$\begin{aligned} &dU - \Delta U\, dt + ((K(U) \cdot \nabla)U - (U \cdot \nabla)K(U))dt \\ &\qquad = \sum_{i=1}^{N}(h_i * U + \lambda_i U)d\beta_i \quad \text{in } (0,\infty) \times \mathbb{R}^3, \\ &U(0,\xi) = U_0(\xi),\ \xi \in \mathbb{R}^3. \end{aligned} \tag{1.7}$$

Equivalently,

$$\begin{aligned} U(t) = e^{t\Delta}U_0 - &\int_0^t e^{(t-s)\Delta}((K(U(s))\cdot\nabla)U(s) - (U(s)\cdot\nabla)K(U(s)))ds \\ &+ \int_0^t \sum_{i=1}^{N} e^{-(t-s)\Delta}(h_i * U(s)) + \lambda_i(U(s))d\beta_i(s),\ t \geq 0. \end{aligned} \tag{1.8}$$

Now, we consider the transformation

$$U(t) = \Gamma(t)y(t),\ t \in [0,\infty), \tag{1.9}$$

where $\Gamma(t) : (L^2(\mathbb{R}^3))^3 \to (L^2(\mathbb{R}^3))^3$ is the linear continuous operator defined by the equations

$$d\Gamma(t) = \sum_{i=1}^{N}(B_i + \lambda_i I)\Gamma(t)d\beta_i(t),\ t \geq 0, \qquad \Gamma(0) = I, \tag{1.10}$$

where (see (1.2))

$$B_i u = h_i * u,\ \forall u \in (L^p(\mathbb{R}^3))^3,\ i = 1, ..., N,\ p \in (1,\infty). \tag{1.11}$$

We also set

$$\widetilde{B}_i = B_i + \lambda_i I,\ i = 1, ..., N, \tag{1.12}$$

where I is the identity operator.

Since $B_iB_j = B_jB_i$, equation (1.10) has a solution Γ and can be equivalently expressed as (see [9], Section 7.4)

$$\Gamma(t) = \prod_{i=1}^{N} \exp\left(\beta_i(t)\widetilde{B}_i - \frac{t}{2}\ \widetilde{B}_i^2\right),\ t \geq 0. \tag{1.13}$$

Here (1.10) is meant in the sense that, for every $z_0 \in (L^2(\mathbb{R}^3))^3$, the continuous $(\mathcal{F}_t)$–adapted $(L^2(\mathbb{R}^3))^3$–valued process $z(t) := \Gamma(t)z_0$, $t \geq 0$, solves the following SDE on $H := (L^2(\mathbb{R}^3))^3$,

$$dz(t) = \sum_{i=1}^{N} \widetilde{B}_i z(t) d\beta_i(t), \quad z(0) = z_0,$$

where H is equipped with the usual scalar product $\langle, \cdot, \rangle$.

Applying the Itô formula in (1.7) (the justification for this is as in [2]), we obtain for y the random differential equation

$$\begin{aligned}
&\frac{dy}{dt}(t) - \Gamma^{-1}(t)\Delta(\Gamma(t)y(t)) + \Gamma^{-1}(t)(K(\Gamma(t)y(t)) \cdot \nabla)(\Gamma(t)y(t)) \\
&\qquad -(\Gamma(t)y(t) \cdot \nabla)(K(\Gamma(t)y(t))) = 0, \quad t \in [0, \infty), \\
&y(0) = U_0.
\end{aligned} \tag{1.14}$$

Taking into account that, for all i, $B_i\Delta = \Delta B_i$ on $H^2(\mathbb{R}^3)$, it follows by (1.10), (1.13) that $\Delta\Gamma(t) = \Gamma(t)\Delta$ on $H^2(\mathbb{R}^3)$, $\forall t \geq 0$.

In what follows, equation (1.14) will be taken in the following mild sense

$$y(t) = e^{t\Delta}U_0 + \int_0^t e^{(t-s)\Delta}\Gamma^{-1}(s)M(\Gamma(s)y(s))ds, \quad t \in [0, \infty), \tag{1.15}$$

where

$$(e^{t\Delta}u)(\xi) = \frac{1}{(4\pi t)^{\frac{3}{2}}} \int_{\mathbb{R}^3} \exp\left(-\frac{|\xi - \bar{\xi}|^2}{4t}\right) u(\bar{\xi})d\bar{\xi}, \ t \in [0, \infty), \ \xi \in \mathbb{R}^3, \tag{1.16}$$

and M is defined by

$$M(u) = -[(K(u) \cdot \nabla)(u) - (u \cdot \nabla)(K(u))], \ t \in [0, \infty). \tag{1.17}$$

We note that $U(t) = \Gamma(t)y(t)$ is the solution to the equation

$$U(t) = e^{t\Delta}\Gamma(t)U_0 + \int_0^t e^{(t-s)\Delta}\Gamma(t)\Gamma^{-1}(s)M(U(s))ds, \tag{1.18}$$

which may be viewed as the random version of the stochastic vorticity equation (1.8).

Our aim here and the principal contribution of this work is to show that, for every $\varepsilon \in (0,1)$, there exists $\Omega_\varepsilon \in \mathcal{F}$ such that $\mathbb{P}(\Omega_\varepsilon) \geq 1-\varepsilon$ and, for all $\omega \in \Omega_\varepsilon$, we have the existence and uniqueness of a solution (in the mild sense) for (1.15) if the vorticity of x, i.e., $U_0 = \operatorname{curl} x$, is $\mathbb{P}$-a.s. sufficiently small in a sense to be made precise in Theorem 1.1 below. We recall that, for a deterministic Navier–Stokes equation, such a result was first established by T. Kato [15] (see also T. Kato and H. Fujita [16]) and extended later to more general initial data by Y. Giga and T. Miyakawa [14], M. Taylor [21], H. Koch and D. Tataru [17]. However, the standard approach [15], [16] cannot be applied in the present case for one reason: the nonlinear inertial term $(X\cdot\nabla)X$ cannot be conveniently estimated in the space $C_b([0,\infty);L^p(\Omega\times\mathbb{R}^d))$ and similarly for the nonlinearity arising in (1.7). As regards the stochastic $3D$ Navier-Stokes equations, to best of our knowledge all global existence results were limited to martingale solutions. Since the fundamental work [11], the literature on (global) martingale solutions for stochastic $3D$-Navier-Stokes equations has grown enormously. We refer, e.g., to [6], [10], [12], [13], [18], and the references therein.

In the following, we denote by L^p, $1 \leq p \leq \infty$, the space $(L^p(\mathbb{R}^3))^3$ with the norm $|\cdot|_p$, by $W^{1,p}$ the corresponding Sobolev space and by $C_b([0,\infty);L^p)$ the space of all bounded and continuous functions $u:[0,\infty)\to L^p$ with the sup norm. We also set $D_i = \frac{\partial}{\partial\xi_i}$, $i=1,2,3$, and denote by $\nabla\cdot u$ the divergence of u, while

$$((u\cdot\nabla)v)_j = u_iD_iv_j,\ j=1,2,3,\ u=\{u_i\}_{i=1}^3,\ v=\{v_j\}_{j=1}^3.$$

As usual

$$q' = \frac{q}{q-1} \quad \text{for } q \in (1,\infty).$$

We set for $p \in \left(\frac{3}{2},3\right)$

$$\eta(t) = \|\Gamma(t)\|_{L(L^p,L^p)}\|\Gamma(t)\|_{L(L^{\frac{3p}{3-p}},L^{\frac{3p}{3-p}})}\|\Gamma^{-1}(t)\|_{L^q,L^q)},\ t \geq 0, \tag{1.19}$$

where for $q \in (1,\infty)$, $\|\cdot\|_{L(L^q,L^q)}$ is the norm of the space $L(L^q,L^q)$ of linear continuous operators on L^q.

For $p \in [1,\infty)$, we denote by $\mathcal{Z}_p$ the space of all functions $y:(0,\infty)\times\mathbb{R}^3\to\mathbb{R}^3$ such that

$$\begin{array}{rcl} t^{1-\frac{3}{2p}}y & \in & C_b([0,\infty);L^p), \\ t^{\frac{3}{2}\left(1-\frac{1}{p}\right)}D_iy & \in & C_b([0,\infty);L^p),\ i=1,2,3. \end{array} \tag{1.20}$$

The space $\mathcal{Z}_p$ is endowed with the norm

$$\|y\|_{p,\infty} = \sup \left\{ t^{1-\frac{3}{2p}} |y(t)|_p + t^{\frac{3}{2}\left(1-\frac{1}{p}\right)} |D_i y(t)|_p ;\ t \in (0,\infty),\ i = 1,2,3 \right\}. \quad (1.21)$$

In the following, we take $\lambda_i \in \mathbb{R}$ such that

$$|\lambda_i| > (\sqrt{12}+3)|h_i|_1, \quad \forall i = 1,2,...,N. \quad (1.22)$$

We note that

$$\|B_i\|_{L(L^q,L^q)} \leq |h_i|_{L^1},\ \forall i = 1,...,N.$$

Theorem 1.1 is the main result.

Theorem 1.1. *Let $p,q \in (1,\infty)$ such that*

$$\frac{3}{2} < p < 2, \quad \frac{1}{q} = \frac{2}{p} - \frac{1}{3} \cdot \quad (1.23)$$

Let $\Omega_0 = \left\{\sup_{t\geq 0} \eta(t) < \infty\right\}$ and consider (1.15) for fixed $\omega \in \Omega_0$. Set $\Gamma(t) := \Gamma(t)(\omega)$, $\eta(t) := \eta(t,\omega)$. Then $\mathbb{P}(\Omega_0) = 1$ and there is a positive constant C^* independent of $\omega \in \Omega_0$ such that, if $U_0 \in L^{\frac{3}{2}}$ is such that

$$\sup_{t\geq 0} \eta(t)|U_0|_{\frac{3}{2}} \leq C^*, \quad (1.24)$$

then the random equation (1.15) has a unique solution $y \in \mathcal{Z}_p$ which satisfies

$$M(\Gamma(t)y) \in L^1(0,T;L^q). \quad (1.25)$$

Moreover, for each $\varphi \in L^3 \cap L^{q'}$, the function $t \to \int_{\mathbb{R}^3} y(t,\xi)\varphi(\xi)d\xi$ is continuous on $[0,\infty)$. The map $U_0 \to y$ is Lipschitz form $L^{\frac{3}{2}}$ to $\mathcal{Z}_p$.

In particular, the random vorticity equation (1.18) has a unique solution U such that $\Gamma^{-1}U \in \mathcal{Z}_p$.

Remark 1.2. Concerning condition (1.24), we note that an elementary calculation shows that

$$\eta(t) \leq \prod_{i=1}^{N} \exp(3|\beta_i(t)|(|h_i|_1| + |\lambda_i|) - t\alpha_i),\ t \in [0,\infty),$$

where $\alpha_i := \frac{1}{2}\,\lambda_i^2 - \frac{3}{2}\,(|h_i|_1^2 + 2|\lambda_i|\,|h_i|_1)$, which is strictly positive by (1.22).

By the law of the iterated logarithm, it follows that

$$\sup_{t\geq 0} \eta(t) < \infty, \quad \mathbb{P}\text{-a.e.},$$

hence for $\Omega_r := \{\sup_{t\geq 0} \eta(t) \leq r\}$ we have $\mathbb{P}(\Omega_r^c) \to 0$ as $r \to \infty$.

But, taking into account that, for each $r > 0$ and all $\nu > 0$, $i = 1, ..., N$, we have (see Lemma 3.4 in [1])

$$\mathbb{P}\left[\sup_{t\geq 0} \{\exp(\beta_i(t) - \nu t)\} \geq r\right] = r^{-2\nu},$$

and, more explicitly, we get that

$$\mathbb{P}(\Omega_r^c) \leq 2N r^{-\frac{N\alpha}{\gamma^2}}, \quad \forall r > 0,$$

where $\alpha = \min_{1\leq i\leq N} \alpha_i$, $\gamma = 3\max\{(|h_i|_1 + |\lambda_i|);\ i \leq N\}$. Therefore, if $\omega \in \Omega_r$ and $U_0 = U_0(\omega)$ satisfies

$$|U_0|_{\frac{3}{2}} \leq \frac{C^*}{r}, \tag{1.26}$$

then condition (1.24) holds. It is trivial to define such an $\mathcal{F}$-measurable function $U_0 : \Omega \to L^{\frac{3}{2}}$, for instance,

$$U_0 := \sum_{n=1}^{\infty} \frac{C^*}{n}\ 1_{\{n-1\leq \sup_{t\geq 0} \eta(t) < n\}} u_0,$$

for some $u_0 \in L^{\frac{3}{2}}$. But, of course, U_0 is not $\mathcal{F}_0$-measurable and so the process $U(t)$, $t \geq 0$, given by Theorem 1.1, is not $(\mathcal{F}_t)_{t\geq 0}$-adapted. Therefore, $U = \Gamma(t)y$ is not a solution to the stochastic vorticity equation (1.8). However, it can be viewed as a generalized solution to (1.8).

It should also be mentioned that assumption (1.22) is not necessary for existence of a solution to equation (1.15), but only to make sure that condition (1.24) is not void.

2 Proof of Theorem 1.1

To begin with, we note below in Lemma 2.1 a few immediate properties of the operator Γ defined in (1.10)–(1.13).

Lemma 2.1. *We have*

$$|\Gamma(t)z|_q + |\Gamma^{-1}(t)z|_q \le C_t|z|_q,\ t \in [0,\infty),\ \forall z \in L^q,\ \forall q \in [1,\infty), \tag{2.1}$$

and

$$|\nabla(\Gamma(t)z)|_q \le \|\Gamma(t)\|_{L(L^q,L^q)}|\nabla z|_q,\ \textit{for all } z \in W^{1,q}(\mathbb{R}^3). \tag{2.2}$$

Proof. By (1.2), (1.11) and by the Young inequality, we see that

$$|B_i(u)|_q \le (|h_i|_1 + |\lambda_i|)|u|_q,\ \forall u \in L^q,\ i = 1,...,N. \tag{2.3}$$

Recalling (1.13), we see by (2.3) that (2.1), (2.2) hold, as claimed. ∎

Lemma 2.2. *Let* $\frac{1}{q} = \frac{1}{r_1} + \frac{1}{r_2}$, $\frac{3}{2} < r_1 < \infty$, $r_1^* = \frac{3r_1}{3+r_1}$, $1 < q < \infty$. *Then, for some* $C > 0$ *independent of* ω,

$$|M(\Gamma(t)z)|_q \le C\|\Gamma(t)\|_{L(L^{r_1},L^{r_1})}\|\Gamma(t)\|_{L(L^{r_2},L^{r_2})}(|z|_{r_1}|z|_{r_2} + |z|_{r_1^*}|\nabla z|_{r_2}),\quad t \in [0,\infty), \tag{2.4}$$

for all $z \in L^{r_1} \cap L^{r_2} \cap L^{r_1^*}$ *with* $\nabla z \in L^{r_2}$.

Proof. We have by (1.17) and (2.1)

$$|M(\Gamma(t)z)|_q \le |(K(\Gamma(t)z)\cdot\nabla)(\Gamma(t)z)|_q + |(\Gamma(t)z\cdot\nabla)K(\Gamma(t)z)|_q. \tag{2.5}$$

On the other hand, by (2.2) and the Hölder inequality we have

$$\begin{aligned}|(K(\Gamma(t)z)\cdot\nabla)(\Gamma(t)z)|_q &\le |K(\Gamma(t)z)|_{r_1}|\nabla(\Gamma(t)z)|_{r_2}\\ &\le \|\Gamma(t)\|_{L(L^{r_1},L^{r_1})}\|\Gamma(t)\|_{L(L^{r_2},L^{r_2})}|K(z)|_{r_1}|\nabla z|_{r_2}.\end{aligned} \tag{2.6}$$

Now, we recall the classical estimate for Riesz potentials (see [20], p. 119)

$$\left|\int_{\mathbb{R}^3} \frac{f(\bar\xi)}{|\xi-\bar\xi|^2}\,d\xi\right|_\beta \le C|f|_\alpha,\ \forall f \in L^\alpha,$$

where $\frac{1}{\beta} = \frac{1}{\alpha} - \frac{1}{3}$, $\alpha \in (1,3)$. By virtue of (1.6), this yields

$$|K(u)|_\beta \le C|u|_\alpha;\ \forall u \in L^\alpha,\ \frac{1}{\beta} = \frac{1}{\alpha} - \frac{1}{3}, \tag{2.7}$$

and so, for $\beta = r_1$, $\alpha = \frac{3r_1}{3+r_1} = r_1^*$, we get by (2.2) and (2.6) the estimate

$$|(K(\Gamma(t)z)\cdot\nabla)(\Gamma(t)z)|_q \le C\|\Gamma(t)\|_{L(L^{r_1},L^{r_1})}\|\Gamma(t)\|_{L(L^{r_2},L^{r_2})}|z|_{r_1^*}|\nabla z|_{r_2}. \tag{2.8}$$

(Here and everywhere in the following, $|\nabla z|_p$ means $\sup\{|D_i z|_p\,;\ i=1,2,3\}$.) Taking into account that, by the Calderon–Zygmund inequality (see [7], Theorem 1),

$$|\nabla K(z)|_{\tilde{p}} \le C|z|_{\tilde{p}},\ \forall z \in L^{\tilde{p}},\ 1 \le \tilde{p} < \infty, \tag{2.9}$$

one obtains that

$$|(\Gamma(t)z\cdot\nabla)(K(t)\Gamma(t)z)|_q \le C\|\Gamma(t)\|_{L(L^{r_1},L^{r_1})}\|\Gamma(t)\|_{L(L^{r_2},L^{r_2})}|z|_{r_1}|z|_{r_2}. \tag{2.10}$$

Substituting (2.8), (2.10) in (2.5), one obtains estimate (2.4), as claimed. ■

Lemma 2.3. *Let* $r_1 = \frac{3r_2}{3-r_2}$, $\frac{3}{2} < r_2 < 3$, $q = \frac{3r_1}{r_1+6}$. *Then, we have, for some* $C > 0$ *independent of* ω,

$$|M(\Gamma(t)z)|_q \le C\|\Gamma(t)\|_{L(L^{r_1},L^{r_1})}\|\Gamma(t)\|_{L(L^{r_2},L^{r_2})}|z|_{r_2}|\nabla z|_{r_2},\ \forall z \in W^{1,r_2}. \tag{2.11}$$

Proof. We have by the Sobolev–Gagliardo-Nirenberg inequality (see, e.g., [5], p. 278)

$$|z|_{r_1} \le C|\nabla z|_{r_2},\ \forall z \in W^{1,r_2}(\mathbb{R}^3).$$

Substituting in (2.4) and taking into account that $r_1^* = r_2$, we obtain (2.11), as claimed. ■

In the following, we fix $p = r_2$, r_1 and q as in Lemma 2.3, (2.11), that is,

$$\frac{3}{2} < p < 2,\ r_1 = \frac{3p}{3-p},\ \frac{1}{q} = \frac{2}{p} - \frac{1}{3}. \tag{2.12}$$

We write equation (1.15) as

$$y(t) = G(y)(t) = e^{t\Delta}U_0 + F(y)(t),\ t \in [0,\infty), \tag{2.13}$$

where

$$F(z)(t) = \int_0^t e^{(t-s)\Delta}\Gamma^{-1}(s)M(\Gamma(s)z(s))ds,\ t \in [0,\infty). \tag{2.14}$$

By (1.16), we have for $1 < \tilde{q} \le \tilde{p} < \infty$ the estimates

$$|e^{t\Delta}u|_{\tilde{p}} \ \le\ Ct^{-\frac{3}{2}\left(\frac{1}{\tilde{q}}-\frac{1}{\tilde{p}}\right)}|u|_{\tilde{q}},\quad u \in L^{\tilde{q}}, \tag{2.15}$$

$$|D_j e^{t\Delta}u|_{\tilde{p}} \ \le\ Ct^{-\frac{3}{2}\left(\frac{1}{\tilde{q}}-\frac{1}{\tilde{p}}\right)-\frac{1}{2}}|u|_{\tilde{q}},\ j = 1,2,3. \tag{2.16}$$

(Everywhere in the following, we shall denote by C several positive constants independent of ω and $t \geq 0$.)

We apply (2.15), with $\tilde{q} = q,\ \tilde{p} = p$. By (2.11)–(2.14), we obtain that

$$
\begin{aligned}
|F(z(t))|_p &\leq C\int_0^t (t-s)^{-\frac{1}{2}\left(\frac{3}{p}-1\right)} |\Gamma^{-1}(s)M(\Gamma(s)z(s))|_q ds \\
&\leq C\int_0^t (t-s)^{-\frac{1}{2}\left(\frac{3}{p}-1\right)} \|\Gamma(s)\|_{L(L^{\frac{3p}{3-p}},L^{\frac{3p}{3-p}})} \\
&\qquad \|\Gamma(s)\|_{L(L^p,L^p)} \|\Gamma^{-1}(s)\|_{L(L^q,L^q)} |z(s)|_p |\nabla z(s)|_p \, ds.
\end{aligned} \tag{2.17}
$$

Similarly, we obtain by (2.16) that

$$
\begin{aligned}
|D_jF(z(t))|_p \leq C\int_0^t (t-s)^{-\frac{3}{2p}} \|\Gamma(s)\|_{L(L^{\frac{3p}{3-p}},L^{\frac{3p}{3-p}})} \\
\|\Gamma(s)\|_{L(L^p,L^p)} \|\Gamma^{-1}(s)\|_{L(L^q,L^q)} |z(s)|_p |\nabla z(s)|_p \, ds,\ j = 1,2,3.
\end{aligned} \tag{2.18}
$$

We consider the Banach space $\mathcal{Z}_p$ defined by (1.20), that is,

$$
\begin{aligned}
\mathcal{Z}_p = \big\{y;\ t^{1-\frac{3}{2p}} y \in C_b([0,\infty);L^p),\ t^{\frac{3}{2}\left(1-\frac{1}{p}\right)} D_j y \in C_b([0,\infty);L^p), \\
j = 1,2,3\big\},
\end{aligned} \tag{2.19}
$$

with the norm

$$
\|z\|_{p,\infty} = \|z\| = \sup_{t>0} \left\{ \left(t^{1-\frac{3}{2p}} |z(t)|_p + t^{\frac{3}{2}\left(1-\frac{1}{p}\right)} |D_i z(t)|_p \right),\ i = 1,2,3 \right\}. \tag{2.20}
$$

We note that

$$
|z(t)|_p |\nabla z(t)|_p \leq Ct^{-\frac{5}{2}+\frac{3}{p}} \|z\|^2,\ \forall z \in \mathcal{Z}_p,\ t \in (0,\infty). \tag{2.21}
$$

By (2.17) and (2.20) we see that, for $z \in \mathcal{Z}_p$, we have

$$
\begin{aligned}
|F(z(t))|_p &\leq \int_0^t (t-s)^{-\frac{1}{2}\left(\frac{3}{p}-1\right)} \|\Gamma(s)\|_{L(L^{\frac{3p}{3-p}},L^{\frac{3p}{3-p}})} \|\Gamma(s)\|_{L(L^p,L^p)} \\
&\qquad \|\Gamma^{-1}(s)\|_{L(L^q,L^q)} |z(s)|_p |\nabla z(s)|_p \, ds \\
&\leq C\int_0^t (t-s)^{-\frac{1}{2}\left(\frac{3}{p}-1\right)} |s|^{-\frac{5}{2}+\frac{3}{p}} \|\Gamma(s)\|_{L(L^{\frac{3p}{3-p}},L^{\frac{3p}{3-p}})} \\
&\qquad \|\Gamma(s)\|_{L(L^p,L^p)} \|\Gamma^{-1}(s)\|_{L(L^q,L^q)} \, ds \|z\|^2 \\
&\leq Ct^{\frac{3}{2p}-1} \sup_{0\leq s\leq t} \eta(s) \int_0^1 (1-s)^{-\frac{1}{2}\left(\frac{3}{p}-1\right)} s^{-\frac{5}{2}+\frac{3}{p}} ds\, \|z\|^2,\ \forall t > 0,
\end{aligned} \tag{2.22}
$$

where η is given by (1.19). This yields

$$t^{1-\frac{3}{2p}}|F(z(t))|_p \le C \sup_{0\le s\le t}\{\eta(s)\}B\left(\frac{3}{2}\Big(\frac{2}{p}-1\Big),\frac{3}{2}\left(1-\frac{1}{p}\right)\right)\|z\|^2,\ \forall t>0, \tag{2.23}$$

where B is the classical beta function (which is finite by virtue of (1.23)).

Similarly, by (2.16) and (2.21), we have, for $j=1,2,3$,

$$\begin{aligned}|D_jF(z)(t)|_p &\le C\int_0^t (t-s)^{-\frac{3}{2p}}s^{-\frac{5}{2}+\frac{3}{p}}\|\Gamma(s)\|_{L(L^{\frac{3p}{3-p}},L^{\frac{3p}{3-p}})}\\ &\qquad \|\Gamma(s)\|_{L(L^p,L^p)}\|\Gamma^{-1}(s)\|_{L(L^q,L^q)}ds\|z\|^2 \qquad (2.24)\\ &\le C\sup_{0\le s\le t}\{\eta(s)\}t^{-\frac{3}{2}\left(1-\frac{1}{p}\right)}B\left(3\left(\frac{1}{p}-\frac{1}{2}\right),1-\frac{3}{2p}\right)\|z\|^2,\ t>0.\end{aligned}$$

Hence,

$$t^{\frac{3}{2}\left(1-\frac{1}{p}\right)}|D_jF(z(t))|_p \le C\sup_{0\le s\le t}\eta(s)\|z\|^2,\ \forall z\in\mathcal{Z}_p,\ t>0,\ j=1,2,3. \tag{2.25}$$

By (2.15)–(2.16), we have

$$\begin{aligned}|e^{t\Delta}U_0|_p &\le Ct^{\frac{3}{2p}-1}|U_0|_{\frac{3}{2}}, \quad t>0,\\ |D_je^{t\Delta}U_0|_p &\le Ct^{\frac{3}{2p}-\frac{3}{2}}|U_0|_{\frac{3}{2}}, \quad t>0,\ j=1,2,3.\end{aligned}$$

Therefore, by (2.20) we get

$$\|e^{t\Delta}U_0\| \le C|U_0|_{\frac{3}{2}}. \tag{2.26}$$

By (2.20), (2.23), (2.25), (2.26), we get,

$$\|G(z)\| \le C_1\left(|U_0|_{\frac{3}{2}}+\sup_{t\ge 0}\eta(t)\|z\|^2\right),\ \forall z\in\mathcal{Z}_p, \tag{2.27}$$

where $C_1>0$ is independent of ω.

We set

$$\eta_\infty = \sup_{t\ge 0}\eta(t), \tag{2.28}$$

and so (2.27) yields

$$\|G(z)\| \le C_1(|U_0|_{\frac{3}{2}}+\eta_\infty\|z\|^2),\ \forall z\in\mathcal{Z}_p. \tag{2.29}$$

We set

$$\Sigma = \{z \in \mathcal{Z}_p;\ \|z\| \le R^*\}$$

and note that, by (2.29), it follows that $G(\Sigma) \subset \Sigma$ if

$$|U_0|_{\frac{3}{2}}\eta_\infty \le (4C_1^2)^{-1}, \tag{2.30}$$

(so U_0 must depend on ω) and $R^* = R^*(\omega)$ is given by

$$R^* = 2C_1|U_0|_{\frac{3}{2}}. \tag{2.31}$$

(We recall that C_1 is independent of ω and U_0.) Moreover, by (1.17) and (2.14), we have, for all $z, \bar{z} \in \mathcal{Z}_p$,

$$\begin{aligned} G(z)(t) - \ G(\bar{z})(t) = -\int_0^t e^{(t-s)\Delta}\Gamma^{-1}(s)[(K\Gamma(s)(z(s) - \bar{z}(s)) \cdot \nabla)\Gamma(s)z(s) \\ + \ (K(\Gamma(s)\bar{z}(s)) \cdot \nabla)\Gamma(s)(z(s) - \bar{z}(s)) - \Gamma(s)(z(s) - \bar{z}(s)) \cdot \nabla K(\Gamma(s)z(s)) \\ - \ (\Gamma(s)\bar{z}(s) \cdot \nabla)K(\Gamma(s)(z(s) - \bar{z}(s)))]ds. \end{aligned}$$

Proceeding as above, we get, as in (2.17), (2.22), (2.23), that

$$\begin{aligned} &|G(z)(t) - G(\bar{z})(t)|_p \\ &\quad \le C\int_0^t (t-s)^{-\frac{1}{2}\left(\frac{3}{p}-1\right)} \|\Gamma(s)\|_{L(L^{\frac{3p}{p-3}}, L^{\frac{3p}{p-3}})} \|\Gamma(s)\|_{L(L^p,L^p)} \\ &\qquad\qquad \|\Gamma^{-1}(s)\|_{L(L^q,L^q)}(|z(s) - \bar{z}(s)|_p(|\nabla z(s)|_p + |\nabla \bar{z}(s)|_p) \\ &\qquad\qquad + |\nabla z(s) - \nabla \bar{z}(s)|_p(|z(s)|_p + |\bar{z}(s)|_p))ds \\ &\quad \le Ct^{-\left(1-\frac{3}{2p}\right)} \sup_{0\le s\le t} \eta(s)\|z - \bar{z}\|(\|z\| + \|\bar{z}\|),\ \forall t > 0, \end{aligned} \tag{2.32}$$

and also (see (2.18), (2.24), (2.25))

$$|D_jG(z)(t) - D_jG(\bar{z}(t))|_p \le Ct^{-\frac{3}{2}\left(1-\frac{1}{p}\right)} \sup_{0\le s\le t} \eta(s)\|z - \bar{z}\|(\|z\| + \|\bar{z}\|),\ \forall t > 0,$$

for $j = 1, 2, 3$. Hence, by (2.20) and (2.28), we obtain that

$$\|G(z) - G(\bar{z})\| \le C_2\eta_\infty R^*\|z - \bar{z}\|,\ \forall z, \bar{z} \in \Sigma, \tag{2.33}$$

where C_2 is independent of ω.

Then, by (2.31), (2.33), it follows that, if (2.30) and

$$2C_1C_2\eta_\infty|U_0|_{\frac{3}{2}} < 1, \tag{2.34}$$

hold, then the operator G is a contraction on Σ and so there is a unique solution $U \in \Sigma$ to (1.15) provided (1.24) holds with $C^* < (2C_1C_2)^{-1}$.

Now, as seen earlier, by (2.11), (1.15) and (2.21) we have

$$\begin{aligned} |M(\Gamma(t)y(t))|_q &\le C\|\Gamma(t)\|_{L(L^{\frac{3p}{3-p}},L^{\frac{3p}{3-p}})}\|\Gamma(t)\|_{L(L^p,L^p)}|y(t)|_p|\nabla y(t)|_p \\ &\le C\|\Gamma(t)\|_{L(L^{\frac{3p}{3-p}},L^{\frac{3p}{3-p}})}\|\Gamma(t)\|_{L(L^p,L^p)}t^{-\frac{5}{2}+\frac{3}{p}}\|y\|^2,\ \forall t > 0. \end{aligned} \tag{2.35}$$

On the other hand, we have for all $\varphi \in C_0^\infty(\mathbb{R}^3)$,

$$\begin{aligned} \int_{\mathbb{R}^3} y(t,\xi)\cdot\varphi(\xi)d\xi &= \int_{\mathbb{R}^3}(e^{t\Delta})U_0(\xi)\cdot\varphi(\xi)d\xi \\ &+ \int_0^t\int_{\mathbb{R}^3}\Gamma^{-1}(s)M(\Gamma(s)y(s))\cdot e^{(t-s)\Delta}\varphi(\xi)d\xi\,ds. \end{aligned} \tag{2.36}$$

Recalling that, for all $1 \le \tilde{p} < \infty$, $|e^{t\Delta}\varphi|_{\tilde{p}} \le |\varphi|_{\tilde{p}}$, it follows by (2.35) that

$$\begin{aligned} &\left|\int_0^t\int_{\mathbb{R}^3}\Gamma^{-1}(s)M(\Gamma(s)y(s))\cdot e^{(t-s)\Delta}\varphi(\xi)d\xi\,ds\right| \\ &\qquad \le C\sup_{0\le s\le t}\eta(s)\int_0^t s^{-\frac{5}{2}+\frac{3}{p}}ds\|y\|^2\,|\varphi|_{q'} \\ &\qquad \le C\sup_{0\le s\le t}\eta(s)\ t^{\frac{3}{p}-\frac{3}{2}}\|y\|^2\,|\varphi|_{q'},\ \forall t\in(0,\infty). \end{aligned} \tag{2.37}$$

We also have by (2.26)

$$\left|\int_{\mathbb{R}^3} e^{t\Delta}U_0(\xi)\varphi(\xi)d\xi\right| \le C|U_0|_{\frac{3}{2}}|\varphi|_3,\ \forall t\in[0,\infty).$$

Combining the latter with (2.36), (2.37), we obtain that, for $T > 0$,

$$\left|\int_{\mathbb{R}^3} y(t,\xi)\cdot\varphi(\xi)d\xi\right| \le CT^{\frac{3}{p}-\frac{3}{2}}(|\varphi|_{q'}+|\varphi|_3),\ \forall\varphi\in L^{q'}\cap L^3,\ t\in[0,T].$$

Hence, by (2.36) and since $t \to e^{t\Delta}U_0$ is continuous on $L^{\frac{3}{2}}$, the function $t \to y(t)$ is $L^3\cap L^{q'}$ weakly continuous on $[0,\infty)$, where $q' = \frac{3p}{4p-6}$.

If $U = y(t, U_0) \in \mathcal{Z}_p$ is the solution to (1.15), equivalently (2.13), we have for all $U_0, \overline{U}_0$ satisfying (1.24) (see (2.26) and (2.33))

$$\begin{aligned} \|y(\cdot, U_0) - y(\cdot, \overline{U}_0)\| &\le \|e^{t\Delta}(U_0 - \overline{U}_0)\| + \|F(y(\cdot, U_0)) - F(y(\cdot, \overline{U}_0))\| \\ &\le C|U_0 - \overline{U}_0|_{\frac{3}{2}} + \eta_\infty R^* C_2 \|y(\cdot, U_0) - y(\cdot, \overline{U}_0)\|. \end{aligned}$$

Recalling that by (2.31) and (2.34) we have $R^* C_2 \eta_\infty < 1$, this yields

$$\|y(\cdot, U_0) - y(\cdot, \overline{U}_0)\| \le \frac{C}{1 - R^* C_2 \eta_\infty} \, |U_0 - \overline{U}_0|_{\frac{3}{2}} \le C(\omega)|U_0 - \overline{U}_0|_{\frac{3}{2}},$$

and so, the map $y \to U(\cdot, U_0)$ is Lipschitz from $L^{\frac{3}{2}}$ to $\mathcal{Z}_p$. This completes the proof of Theorem 1.1. ∎

It should be noted that, by (2.30) and (2.31), we have by the Fernique theorem

$$|U_0|_{\frac{3}{2}}, R^* \in \bigcap_{r \ge 1} L^r(\Omega),$$

and so, taking into account that $y \in \Sigma$, we see by (2.19), (2.20) that

$$\sup_{t \ge 0} \left\{ t^{1 - \frac{3}{2p}} |y(t)|_p + t^{\frac{3}{2}\left(1 - \frac{1}{p}\right)} |D_i y(t)|_p \right\} \in \bigcap_{r \ge 1} L^r(\Omega), \ i = 1, 2, 3. \tag{2.38}$$

We have, therefore, the following completion of Theorem 1.1.

Corollary 2.4. *Under the assumptions of Theorem* 1.1*, the solution* $y = y(t, \omega)$ *to the equation* (1.15) *satisfies* (2.38)*. The same result holds for the solution* $U(t) = \Gamma(t) y(t)$ *of the random vorticity equation* (1.18)*.*

3 The random version of the $3D$ Navier-Stokes equation

We fix in (1.1) the initial random variable x by the formula

$$x = K(U_0), \tag{3.1}$$

where $U_0 = \operatorname{curl} x$, $U_0 = U_0(\omega)$ satisfies condition (1.24) for all $\omega \in \Omega_0$ (see Remark 1.2). If y is the corresponding solution to equation (1.15) given by Theorem 1.1, we define the process X by formula (1.5), that is,

$$X(t) = K(U(t)) = K(\Gamma(t) y(t)), \quad \forall t \in [0, \infty). \tag{3.2}$$

By (2.7), where U is the solution to the vorticity equation (1.1), we have

$$|X(t)|_{\frac{3p}{3-p}} \leq C|U(t)|_p, \quad \forall t \in [0,\infty). \tag{3.3}$$

(Everywhere in the following, C are positive constants independent of $\omega \in \Omega$.)

On the other hand, by the Carlderon–Zygmund inequality (2.9), we have

$$|D_i X(t)|_p \leq C|U(t)|_p, \; i = 1,2,3. \tag{3.4}$$

By (3.3) and by Theorem 1.1, it follows that

$$t^{1-\frac{3}{2p}} X \in C_b([0,\infty); L^{\frac{3p}{3-p}}), \tag{3.5}$$

while, by (3.4), we have for $i = 1,2,3$

$$t^{\frac{3}{2}\left(1-\frac{1}{p}\right)} D_i X \in C_b([0,\infty); L^p). \tag{3.6}$$

Now, if in (2.9) we take $z = D_j X$, we get that, for all $i,j = 1,2,3$,

$$t^{\frac{3}{2}\left(1-\frac{1}{p}\right)} |D_i D_j X|_p \leq C t^{\frac{3}{2}\left(1-\frac{1}{p}\right)} |D_i U|_p \leq C, \; \forall t \in [0,\infty).$$

This yields

$$t^{\frac{3}{2}\left(1-\frac{1}{p}\right)} D^2_{ij} X \in C_b([0,\infty); L^p), \; i,j = 1,2,3. \tag{3.7}$$

Moreover, by Corollary 2.4, we also have

$$t^{1-\frac{3}{2p}} X \in C_b([0,\infty); L^r(\Omega; L^{\frac{3p}{3-p}})), \; \forall r \geq 1, \tag{3.8}$$

$$t^{\frac{3}{2}\left(1-\frac{1}{p}\right)} D_i X \in C_b([0,\infty); L^r(\Omega; L^p)), \; i = 1,2,3, \tag{3.9}$$

$$t^{\frac{3}{2}\left(1-\frac{1}{p}\right)} D_{ij} X \in C_b([0,\infty); L^r(\Omega; L^p)), \; i,j = 1,2,3. \tag{3.10}$$

Now, if in equation (1.18) one applies the Biot–Savart operator K, we obtain for X the equation

$$X(t) = K(e^{t\Delta}\Gamma(t)\operatorname{curl} x) + \int_0^t K(e^{(t-s)\Delta}\Gamma(t)\Gamma^{-1}(s)M(\operatorname{curl} X(s)))ds, \quad t \geq 0, \tag{3.11}$$

where M is given by (1.17). It should be noted that, by virtue of (3.7)-(3.10), the right hand side of (3.11) is well defined.

Equation (3.11) can be viewed as the random version of the Navier-Stokes equation (1.1). However, since, as seen earlier, U_0 is not $\mathcal{F}_0$-measurable, the processes $t \to y(t)$, $t \to U(t)$ are not $(\mathcal{F}_t)_{t\geq 0}$-adapted, and so X is not $(\mathcal{F}_t)_{t\geq 0}$-adapted, too. Therefore, (3.11) cannot be transformed back into (1.1). By Theorem 1.1, it follows that

Theorem 3.1. *Under assumptions* (1.24)*, the random Navier-Stokes equation* (3.11) *has a unique solution X satisfying* (3.7)-(3.10)*.*

Remark 3.2. As easily seen from the proofs, Theorem 1.1 extends mutatis-mutandis to the noises $\sum\limits_{i=1}^{N} \sigma_i(t,X)\dot{\beta}_i(t)$, where

$$\sigma_i(t,x)(\xi) = \int_{\mathbb{R}^3} h_i(t,\xi-\bar{\xi})x(\bar{\xi})d\bar{\xi},\ \xi \in \mathbb{R}^3,\ i=1,...,N,$$

where $t \to h_i(t,\xi)$ is continuous and

$$|h_i(t)|_1 \leq C, \quad \forall t \geq 0,\ i=1,...,N.$$

Remark 3.3. The linear multiplicative case $B_i(X) := \alpha_i X$, $i=1,...,N$, that is $h_i := \delta$, where δ is the Dirac measure, can be approximated by taking $h_i(\xi) = \frac{1}{\varepsilon^d}\,\rho\left(\frac{\xi}{\varepsilon}\right)$, where $\rho \in C_0^\infty(\mathbb{R}^d)$, support $\rho \subset \{\xi;\ |\xi|_d \leq 1\}$, $\int \rho(\xi)d\xi = 1$.

Acknowledgements. Financial support through SFB701 at Bielefeld University is gratefully acknowledged. V. Barbu was also partially supported by a CNCS UEFISCDI (Romania) grant, project PN-II-ID-PCE-2012-4-0156.

References

[1] V. Barbu, G. Da Prato, Internal stabilization by noise of the Navier-Stokes equation, *SIAM J. Control Optim.*, 49 (1) (2012), 1-20.

[2] V. Barbu, M. Röckner, An operatorial approach to stochastic partial differential equations driven by linear multiplicative noise, *J. Eur. Math. Soc.*, 17 (2015), 1789-1815.

[3] V. Barbu, M. Röckner, D. Zhang, Stochastic nonlinear Schrödinger equations with linear multiplicative noise: rescaling approach, *J. Nonlinear Sci.*, vol. 34 (2014), 383-409.

[4] G.K. Batchelor, *Fluid Dynamics*, Cambridge University Press, New York, 1988.

[5] H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer, New York, Dordrecht, Heidelberg, London, 2011.

[6] Z. Brzezniak, E. Motyl, Existence and martingale solution of the stochastic Navier-Stokes equations in unbounded $2D$ and $3D$ domains, *J. Differential Equations*, 254 (2013), 1627-1642.

[7] A.P. Calderon, A. Zygmund, On the existence of certain singular integrals, *Acta Mathematica*, 88 (1) (1952), 85-139.

[8] A.J. Chorin, J.E. Mardsen, *A Mathematical Introduction to Fluid Mechanics*, Springer–Verlag, 1975.

[9] G. Da Prato, J.Zabczyk, *Stochastic Equations in Infinite Dimensions*, Second Edition, Cambridge University Press, Cambridge, 2014.

[10] A. Debussche, C. Odasso, Markov solutions for $3D$ stochastic Navier-Stokes equations, with state dependent noise, *J. Evol. Equ.*, 6 (2006), 305-324.

[11] F. Flandoli, D. Gatarek, Martingale and stationary solutions for stochastic Navier-Stokes equations, *Probab. Theory Related Fields*, 102 (1995), 367-391.

[12] F. Flandoli, M. Romito, Markov solutions for the $3D$ stochastic Navier-Stokes equations, *Probab. Theory Related Fields*, 140 (2008), 407-458.

[13] B. Goldys, M. Röckner, X. Zhang, martingale solutions and Markov selections for stochastic partial differential equations, *Stochastic Process. Appl.*, 119 (2009), 1725-1764.

[14] Y. Giga, T. Miyakawa, Navier–Stokes flow in $\mathbb{R}^3$ with measures as initial vorticity and Morrey spaces, *Comm. Partial Differential Equations*, 14 (1988), 577–618.

[15] T. Kato, Strong L^p-solutions of the Navier–Stokes equation in $\mathbb{R}^m$ with applications to weak solutions, *Math. Z.*, 187 (1984), 471–480.

[16] T. Kato, H. Fujita, On the nonstationary Navier–Stokes system, *Rend. Sem. mat. Univ. Padova*, 32 (1962), 243-260.

[17] H. Koch, D. Tataru, Well-posedness for the Navier–Stokes equations, *Advances in Mathematics*, 157 (1) (2001), 22–35.

[18] R. Mikulevicius, B. Rozovskii, Global L_2-solutions of stochastic Navier-Stokes equations, *Ann. Probab.*, 33 (1) (2005), 137-176.

[19] R. Mikulevicius, B. Rozovskii, Stochastic Navier-Stokes equations for turbulent flows, *SIAM J. Math. Anal.*, 35 (2004), 1250-1320.

[20] E.M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, 1970.

[21] M. Taylor, Analysis on Morrey spaces and applications to Navier–Stokes equation, *Comm. PDE*, 17 (1992), 1407–1456.