A splitting algorithm for stochastic partial differential equations driven by linear multiplicative noise

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Abstract

We study the convergence of a Douglas-Rachford type splitting algorithm for the infinite dimensional stochastic differential equation

$$dX + A(t)(X)dt = X\, dW \text{ in } (0,T);\ X(0) = x,$$

where $A(t) : V \to V'$ is a nonlinear, monotone, coercive and demicontinuous operator with sublinear growth and V is a real Hilbert space with the dual V'. V is densely and continuously embedded in the Hilbert space H and W is an H-valued Wiener process. The general case of a maximal monotone operators $A(t) : H \to H$ is also investigated.

Keywords: Maximal monotone operator, stochastic process, parabolic stochastic equation.

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1 Introduction

We consider here the stochastic differential equation

$$\begin{aligned} &dX(t) + A(t)X(t)dt = X(t)dW(t),\ t \in (0,T), \\ &X(0) = x, \end{aligned} \tag{1.1}$$

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in a real separable Hilbert space H, whose elements are functions or distributions on a bounded and open set $\mathcal{O} \subset \mathbb{R}^d$ with smooth boundary $\partial\mathcal{O}$. In particular, H can be any of the spaces $L^2(\mathcal{O})$, $H_0^1(\mathcal{O})$, $H^{-1}(\mathcal{O})$, $H^1(\mathcal{O})$, $k = 1, 2, \ldots$, with the corresponding Hilbertian structure. Here $H_0^1(\mathcal{O})$, $H^k(\mathcal{O})$ are the standard L^2-Sobolev spaces on $\mathcal{O}$, and W is a Wiener process of the form

$$W(t,\xi) = \sum_{j=1}^{\infty} \mu_j e_j(\xi)\beta_j(t),\ \xi \in \mathcal{O},\ t \geq 0, \tag{1.2}$$

where $\{\beta_j\}_{j=1}^{\infty}$ is an independent system of real-valued Brownian motions on a probability space $\{\Omega, \mathcal{F}, \mathbb{P}\}$ with natural filtration $(\mathcal{F}_t)_{t\geq 0}$. Here, $e_j \in C^2(\overline{\mathcal{O}}) \cap H$, $j \in \mathbb{N}$, is an orthonormal basis in H, and $\mu_j \in \mathbb{R}$, $j = 1, 2, \ldots$.

The following hypotheses will be in effect throughout this work.

(i) There is a Hilbert space V with dual V' such that $V \subset H$, continuously and densely. Hence $V \subset H\ (\equiv H') \subset V'$ continuously and densely.

(ii) $A : [0,T] \times V \times \Omega \to V'$ is *progressively measurable*, i.e., for every $t \in [0,T]$, this operator restricted to $[0,t]\times V\times\Omega$ is $\mathcal{B}([0,t])\otimes\mathcal{B}(V)\otimes\mathcal{F}_t$-measurable.

(iii) There is $\delta \geq 0$ such that, for each $t \in [0,T]$, $\omega \in \Omega$, the operator $u \mapsto \delta u + A(t,\omega)u$ is monotone and *demicontinuous* (that is, strongly-weakly continuous) from V to V'.

Moreover, there are $\alpha_i, \gamma_i \in \mathbb{R}$, $i = 1, 2, 3$, $\alpha_1 > 0$, such that, $\mathbb{P}$-a.s.,

$$\langle A(t,\omega)u, u\rangle \geq \alpha_1|u|_V^2 + \alpha_2|u|_H^2 + \alpha_3,\ \forall u \in V,\ t \in [0,T], \tag{1.3}$$

$$|A(t,\omega)u|_{V'} \leq \gamma_1|u|_V + \gamma_2,\qquad \forall u \in V,\ t \in [0,T]. \tag{1.4}$$

(iv) $e^{\pm W(t)}$ is, for each t, a multiplier in V and a multiplier in H such that there exists an $(\mathcal{F}_t)$-adapted, $\mathbb{R}_+$-valued process $Z(t)$, $t \in [0,T]$, with $\mathbb{E}\left[\sup\limits_{t\in[0,T]} |Z(t)|\right] < \infty$ for all $r \in [1,\infty)$ and such that, $\mathbb{P}$-a.s.,

$$\begin{aligned} |e^{\pm W(t)}y|_V &\leq Z(t)|y|_V,\quad \forall t \in [0,T],\ \forall y \in V, \\ |e^{\pm W(t)}y|_H &\leq Z(t)|y|_H,\quad \forall t \in [0,T],\ \forall y \in H. \end{aligned} \tag{1.5}$$

One assumes also that, for each $\omega \in \Omega$, the function $t \to e^{\pm W(t)}$ is H-valued continuous on $[0, T]$.

Throughout in the following, $|\cdot|_V$ and $|\cdot|_{V'}$ denote the norms of V and V', respectively, and by $\langle \cdot, \cdot \rangle$ we denote the duality pairing between V and V' with H as pivot space; on $H \times H$, $\langle \cdot, \cdot \rangle$ is just the scalar product of H. The norms of V and V' are denoted by $|\cdot|_H$ and $|\cdot|_V$, $|\cdot|_{V'}$, respectively, $\mathcal{B}(H)$, $\mathcal{B}(V)$ etc. are the classes of Borel sets in the corresponding spaces.

As regards the orthonormal basis $\{e_j\}_{j=1}^{\infty}$ in (1.2), we assume that there exist $\widetilde{\gamma}_j \in [1, \infty)$ such that

$$|ye_j|_H \leq \widetilde{\gamma}_j |y|_H, \ \forall y \in H, \ j = 1, 2, ..., \ \nu := \sum_{j=1}^{\infty} \mu_j^2 \widetilde{\gamma}_j^2 |e_j|_{\infty}^2 < \infty. \tag{1.6}$$

and we assume also that

$$\mu := \frac{1}{2} \sum_{j=1}^{\infty} \mu_j^2 e_j^2 \tag{1.7}$$

is a multiplier in V, V' and H.

It should be noted that $X\,dW = \sigma(X) d\widetilde{W}$ where $\sigma : H \to L_2(H)$ (the space of Hilbert-Schmidt operators on H) is defined by

$$\sigma(u)v = \sum_{j=1}^{\infty} \mu_j u \langle v, e_j \rangle e_j, \ \forall v \in H,$$

and so, $\widetilde{W} = \sum\limits_{j=1}^{\infty} e_j \beta_j$ is a cylindrical Wiener process on H (see [5]).

Definition 1.1. By a *solution* to (1.1) for $x \in H$, we mean an $(\mathcal{F}_t)_{t \geq 0}$-adapted process $X : [0, T] \to H$ with continuous sample paths which satisfies

$$X(t) + \int_0^t A(s) X(s) ds = x + \int_0^t X(s) dW(s), \ t \in [0, T], \tag{1.8}$$

$$X \in L^2((0, T) \times \Omega; V). \tag{1.9}$$

The stochastic integral arising in (1.8) is considered in Itô's sense.

In [3], the authors developed an operatorial approach to (1.1) under the more general hypotheses than (i)–(iv) above. As a special case (see Theorem 3.1 in [3]), we have

Theorem 1.2. *Under Hypotheses* (i)–(iv)*, for each $x \in H$, equation* (1.1) *has a unique solution X (in the sense of Definition* 1.1*). Moreover, the function $t \mapsto e^{-W(t)}X(t)$ is V'-absolutely continuous on $[0,T]$ and*

$$\mathbb{E}\int_0^T \left| e^{W(t)}\,\frac{d}{dt}\,\left(e^{-W(t)}X(t)\right)\right|_{V'}^2 dt < \infty. \tag{1.10}$$

In a few words, the method developed in [3] is the following. By the transformation

$$X(t) = e^{W(t)}y(t),\ \ t \geq 0, \tag{1.11}$$

one reduces equation (1.1) to the random differential equation

$$\begin{aligned}&\frac{dy}{dt}\,(t) + e^{-W(t)}A(t)\left(e^{W(t)}y(t)\right) + \mu y(t) = 0,\ \text{ a.e. } t \in (0,T),\\ &y(0) = x,\end{aligned} \tag{1.12}$$

and treat (1.12) as an operatorial equation of the form

$$\mathcal{B}y + \mathcal{A}y = 0 \tag{1.13}$$

in a suitable Hilbert space $\mathcal{H}$ of stochastic processes on $[0,T]$. Here, $\mathcal{A}$ and $\mathcal{B}$ are maximal monotone operators suitable defined from $\mathcal{V}$ to $\mathcal{V}'$, where $(\mathcal{V},\mathcal{V}')$ is a dual pair of spaces such that $\mathcal{V} \subset \mathcal{H} \subset \mathcal{V}'$ with dense and continuous embeddings.

The operatorial form (1.13) of equation (1.12) suggests to approximate the solution y by the Douglas–Rachford splitting algorithm ([6]–[8]).

The exact form and convergence of the corresponding splitting algorithm for equation (1.13) will be given below in Section 2. As seen later on in Theorem 2.1, it leads to a convergent splitting algorithm for the stochastic differential equation (1.1).

In this way, the operator theoretic approach to equation (1.1) written in the form (1.13) allows to design a convergent splitting scheme for equation (1.1) inspired by the Rockafellar [9] proximal point algorithm for nonlinear operatorial equations (on these lines see also [4]). By our knowledge, the splitting algorithm obtained here for the stochastic equation is new and might have implications in numerical approximation of stochastic PDEs.

Notations. If U is a Banach space, we denote by $L^p(0,T;U)$, $1 \leq p \leq \infty$, the space of all L^p-integrable U-valued functions on $(0,T)$. The space

$L^p((0,T)\times\Omega;U)$ is defined similarly. We refer to [2] for notation and standard results of the theory of maximal monotone operators in Banach spaces. If $\mathcal{O}$ is an open domain of $\mathbb{R}^d$, we denote by $W^{1,p}(\mathcal{O})$, $1\le p\le\infty$ and $H^1(\mathcal{O}), H^{-1}(\mathcal{O})$ the standard Sobolev spaces on $\mathcal{O}$.

2 Main results

Without loss of generality, we may assume that, besides assumptions (i)–(iii), $A(t)$ satisfies also the strong monotonicity condition

$$\langle A(t)u-A(t)v,u-v\rangle \ge \nu|u-v|_H^2,\ \forall u,v\in V, \tag{2.1}$$

where $\nu>0$ is given by (1.6). (In fact, as easily seen, by the substitution $X\to\exp(-(\nu+\delta)t)X$ with a suitable δ, equation (1.1) can be rewritten as

$$dX+\widetilde{A}(t)X\,dt = X\,dW,$$

where the operator $X\to\widetilde{A}(t)X=e^{-(\nu+\delta)t}A(t)(e^{(\nu+\delta)t}X)+(\nu+\delta)X$ satisfies conditions (i)–(iii) and (2.1).)

We associate with equation (1.1) the following splitting algorithm

$$\begin{aligned} &\lambda dZ_{n+1}+J(Z_{n+1})dt+\lambda\nu Z_{n+1}dt=\lambda Z_{n+1}dW-\lambda A(t)X_ndt\\ &\qquad\qquad\qquad\qquad\qquad +\lambda\nu X_n\,dt+J(X_n)dt,\ t\in(0,T), \\ &Z_{n+1}(0)=x,\ n=0,1,... \end{aligned} \tag{2.2}$$

$$\begin{aligned} &\lambda A(t)X_{n+1}(t)+J(X_{n+1}(t))-\lambda\nu X_{n+1}(t)\\ &\qquad = J(Z_{n+1}(t))+\lambda A(t)X_n(t)-\lambda\nu X_n(t), \end{aligned} \tag{2.3}$$

where $X_0\in L^2((0,T)\times\Omega;V)$ is $(\mathcal{F}_t)_{t\ge0}$-adapted and arbitrary. Here, the parameter $\lambda>0$ is arbitrary but fixed and $J:V\to V'$ is the canonical isomorphism of the space V onto its dual V'.

Taking into account assumptions (i)–(iii) and (2.1), which, in particular, implies that the operator $\Gamma_0: L^2(0,T;V)\to L^2(0,T;V')$, $\Gamma_0u=\lambda A(t)u+J(u)-\lambda\nu u$, $u\in L^2(0,T;V)$, is demicontinuous, locally bounded, and with inverse continuous, we see that the sequence (Z_n,X_n) is well defined by (2.2), (2.3) and we have also

$$X_n,Z_n\in L^2((0,T)\times\Omega;V) \text{ and } Z_n\in L^2(\Omega;C([0,T];H)),\ n=1,2,... \tag{2.4}$$

Moreover, the processes X_n,Z_n are $(\mathcal{F}_t)_{t\ge0}$-adapted on $[0,T]$.

Theorem 2.1 is the main result.

Theorem 2.1. *Under Hypotheses* (i)–(iv) *and* (2.1)*, assume that $x \in V$ and $\lambda > 0$. If (X_n, Z_n) is the sequence defined by* (2.2)*,* (2.3)*, we have for $n \to \infty$*

$$X_n \to X \text{ weakly in } L^2((0,T) \times \Omega; V), \tag{2.5}$$

where X is the solution to equation (1.1) *given by Theorem* 1.2*. Assume further that the operator $u \to A(t)u$ is odd, that is, $A(t)(-u) = -A(t)u$, $\forall u \in V$. Then, for $n \to \infty$,*

$$X_n \to X \text{ strongly in } L^2((0,T) \times \Omega; V). \tag{2.6}$$

The splitting scheme (2.2)–(2.3) reduces the approximation of problem (1.1) to a sequence of simpler linear equations. In fact, at each step n, one should solve a linear stochastic differential equation of the form

$$\begin{aligned} &dZ_{n+1} + \frac{1}{\lambda}\, J(Z_{n+1})dt + \nu Z_{n+1}dt = Z_{n+1}dW + F_n dt,\ t \in (0,T), \\ &Z_{n+1}(0) = x, \end{aligned} \tag{2.7}$$

and the stationary random equation (2.3), where

$$F_n = -\lambda A(t)X_n + \lambda\nu X_n + J(X_n).$$

By Itô's formula (see, e.g., [3]), equation (2.7) has, for each n, the solution

$$Z_{n+1} = e^W z_{n+1},$$

where z_{n+1} is the solution to the random differential equation

$$\begin{aligned} &\frac{d}{dt}\, z_{n+1} + \frac{1}{\lambda}\, e^{-W} J(e^W z_{n+1}) + (\mu + \nu) z_{n+1} = e^{-W} F_n, \\ &z_{n+1}(0) = x. \end{aligned} \tag{2.8}$$

If $F : L^2((0,T) \times \Omega; V') \to L^2((0,T) \times \Omega; V)$ is the linear continuous operator defined by

$$F(f) = Y,$$

where Y is the solution to the stochastic equation

$$dY + \frac{1}{\lambda}\, J(Y)dt + \nu Y\, dt = Y\, dW + f\, dt;\ Y(0) = x,$$

then we may rewrite (2.2)–(2.3) as

$$X_{n+1} = (\lambda(A-\nu I)+J)^{-1}[JF((\lambda(\nu I-A)+J)(X_n))+\lambda(A-\nu I)X_n],\ n=0,1,...$$

Equivalently,

$$X_{n+1} = \Gamma^n X_0,\ \ \forall n \in \mathbb{N}, \tag{2.9}$$

where $\Gamma : L^2((0,T)\times\Omega;V) \to L^2((0,T)\times\Omega;V)$ is the Lipschitzian and given by

$$\Gamma = (\lambda(A-\nu I)+J)^{-1}[JF(\lambda(\nu I-A)+J)+\lambda(A-\nu I)]. \tag{2.10}$$

Then, by Theorem 2.1, we get

Corollary 2.2. *Under assumptions* (i)-(iv)*,* (2.1)*, for each* $\lambda > 0$ *the solution* X *to* (1.1) *is expressed as*

$$X = w - \lim_{n\to\infty} \Gamma^n X_0 \ \ \text{in } L^2((0,T)\times\Omega;V), \tag{2.11}$$

where $X_0 \in L^2((0,T)\times\Omega;V)$ *is an arbitrary* $(\mathcal{F}_t)_{t\geq 0}$*-adapted process.*

Here $w - \lim$ indicates the weak limit.

3 Proof of Theorem 2.1

Proceeding as in [3], we consider the spaces $\mathcal{H}, \mathcal{V}$ and $\mathcal{V}'$, defined as follows. $\mathcal{H}$ is the Hilbert space of all $(\mathcal{F}_t)_{t\geq 0}$-adapted processes $y : [0,T] \to H$ such that

$$|y|_{\mathcal{H}} = \left(\mathbb{E}\int_0^Y |e^{W(t)}y(t)|_H^2 dt\right)^{\frac{1}{2}} < \infty,$$

where $\mathbb{E}$ denotes the expectation in the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The space $\mathcal{H}$ is endowed with the norm $|\cdot|_{\mathcal{H}}$ generated by the scalar product

$$\langle y, z\rangle_H = \mathbb{E}\int_0^T \left\langle e^{W(t)}y(t), e^{E(t)}y(t)\right\rangle dt.$$

$\mathcal{V}$ is the space of all $(\mathcal{F}_t)_{t\geq 0}$-adapted processes $y : [0,T] \to V$ such that

$$|y|_{\mathcal{V}} = \left(\mathbb{E}\int_0^T |e^{W(t)}y(t)|_V^2 dt\right)^{\frac{1}{2}} < \infty.$$

$\mathcal{V}'$ (the dual of $\mathcal{V}$) is the space of all $(\mathcal{F}_t)_{t\geq 0}$-adapted processes $y:[0,T]\to V'$ such that

$$|y|_{\mathcal{V}'}=\left(\mathbb{E}\int_0^T |e^{W(t)}y(t)|^2_{V'}dt\right)^{\frac{1}{2}}<\infty.$$

We have $\mathcal{V}\subset\mathcal{H}\subset\mathcal{V}'$ with continuous and dense embeddings. Moreover,

$${}_{\mathcal{V}'}\left\langle u,v\right\rangle_{\mathcal{V}}=\mathbb{E}\int_0^T\left\langle e^{W(t)}u(t),e^{W(t)}v(t)\right\rangle dt,\ v\in\mathcal{V},\ u\in\mathcal{V}',$$

is the duality pairing between $\mathcal{V}$ and $\mathcal{V}'$, with the pivot space $\mathcal{H}$, that is,

$${}_{\mathcal{V}'}\left\langle u,v\right\rangle_{\mathcal{V}}=\left\langle u,v\right\rangle_{\mathcal{H}},\ \forall u\in\mathcal{H},\ v\in\mathcal{V}.$$

Now, for $x\in H$, define the operators $\mathcal{A}:\mathcal{V}\to\mathcal{V}'$ and $\mathcal{B}:D(\mathcal{B})\subset\mathcal{V}\to\mathcal{V}'$ as follows:

$$\begin{array}{rcl}(\mathcal{A}y)(t)&=&e^{-W(t)}A(t)(e^{W(t)}y(t))-\nu y(t),\quad\text{a.e. } t\in(0,T),\ y\in\mathcal{V},\\ (\mathcal{B}y)(t)&=&\dfrac{dy}{dt}(t)+(\mu+\nu)y(t),\ \text{a.e. } t\in(0,T),\ y\in D(\mathcal{B}),\end{array}\tag{3.1}$$

$$\begin{aligned}D(\mathcal{B})=\Big\{y\in\mathcal{V}:y\in AC([0,T];V')\cap C([0,T];H),\ \mathbb{P}\text{-a.s.},\\ \frac{dy}{dt}\in\mathcal{V}',\ y(0)=x\Big\}.\end{aligned}\tag{3.2}$$

Here, $AC([0,T];V')$ is the space of all absolutely continuous V'-valued functions on $[0,T]$. If $y\in D(\mathcal{B})$, then $y\in C([0,T];H)$ and $\frac{dy}{dt}$ is the derivative of y in the sense of V'-valued distributions on $(0,T)$. Then, equation (1.12) can be expressed as

$$\mathcal{B}y+\mathcal{A}y=0.\tag{3.3}$$

Then, the map $\Lambda:\mathcal{V}\to\mathcal{V}'$ defined by

$$\Lambda v=e^{-W}J(e^Wv),\ v\in V,\tag{3.4}$$

is the canonical isomorphism of $\mathcal{V}$ onto $\mathcal{V}'$ and the scalar product ${}_{\mathcal{V}}\left\langle\cdot,\cdot\right\rangle_{\mathcal{V}}$ of the space $\mathcal{V}$ can be expressed as

$${}_{\mathcal{V}}\left\langle v,\bar{v}\right\rangle_{\mathcal{V}}={}_{\mathcal{V}}\left\langle v,\Lambda\bar{v}\right\rangle_{\mathcal{V}'},\ \forall v,\bar{v}\in\mathcal{V}.\tag{3.5}$$

We set

$$(\mathcal{A}^*u)(t) \;=\; \Lambda^{-1}\mathcal{A}u(t) = e^{-W}J^{-1}(A(t)(e^W u) - \nu e^W u),\ \forall u \in \mathcal{V}, \tag{3.6}$$

$$(\mathcal{B}^*u)(t) \;=\; \Lambda^{-1}\mathcal{B}u(t) = e^{-W}J^{-1}\left(e^W\left(\frac{du}{dt} + (\mu+\nu)u\right)\right), \quad \forall u \in D(\mathcal{B}^*) = D(\mathcal{B}). \tag{3.7}$$

Since the operators $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{A}+\mathcal{B}$ are maximal monotone in $\mathcal{V}\times\mathcal{V}'$ ([3], Lemma 4.1, Lemma 4.2), it is easily seen by (3.6)-(3.7) that $\mathcal{A}^*$, $\mathcal{B}^*$ and $\mathcal{A}^*+\mathcal{B}^*$ are maximal monotone in $\mathcal{V}\times\mathcal{V}$.

On the other hand, by (3.3) we can rewrite equation (3.3) as

$$\mathcal{B}^*y + \mathcal{A}^*y = 0. \tag{3.8}$$

Let $y \in D(\mathcal{B})$ be the unique solution to equation (3.3) (see [3], Proposition 3.3). Then, y is also the solution to (3.8) and so, by Theorem 1 in [8] (see, also, Corollary 6.1 in [7]), we have that

$$y = \lim_{n\to\infty}(I+\lambda\mathcal{A}^*)^{-1}v_n \text{ weakly in } \mathcal{V} \text{ as } n\to\infty, \tag{3.9}$$

where $\{v_n\} \subset \mathcal{V}$ is, for $n \geq 0$, defined by

$$v_{n+1} = (I+\lambda\mathcal{B}^*)^{-1}(2(I+\lambda\mathcal{A}^*)^{-1}v_n - v_n) + (I-(I+\lambda\mathcal{A}^*)^{-1})v_n, \tag{3.10}$$

and v_0 is arbitrary in $\mathcal{V}$. Here, I is the identity operator in $\mathcal{V}$.

The splitting algorithm (3.9)–(3.10) is just the Douglas–Rachford algorithm ([6]) for equation (3.8) and it can be equivalently expressed as

$$y = \lim_{n\to\infty} y_n \text{ weakly in } \mathcal{V}, \tag{3.11}$$

$$y_n = (I+\lambda\mathcal{A}^*)^{-1}v_n,\ n = 0,1,..., \tag{3.12}$$

$$y_{n+1} + \lambda\mathcal{A}^*y_{n+1} = z_{n+1} + v_n - y_n, \tag{3.13}$$

$$z_{n+1} + \lambda\mathcal{B}^*z_{n+1} = 2y_n - v_n, \tag{3.14}$$

where $v_0 \in \mathcal{V}$. (To get (3.12)-(3.14) from (3.10), we have used the identity $(I+\lambda\mathcal{B}^*)^{-1}(v+\lambda\mathcal{B}^*v) = v,\ \forall v \in D(\mathcal{B}^*)$ and the linearity of $\mathcal{B}^*$.)

In fact, the weak convergence of $\{v_n\}$ in the space $\mathcal{V}$ is also a consequence of the convergence of the Rockafellar proximal point algorithm [9] for the maximal monotone operator $v \to G^{-1}(v) - v$, where

$$G(z) = (I+\lambda\mathcal{B}^*)^{-1}(2(I+\lambda\mathcal{A}^*)^{-1}z - z) + z - (I+\lambda\mathcal{A}^*)^{-1}z,\ \forall z \in \mathcal{V}. \tag{3.15}$$

(See [7], Theorem 4.) Taking into account (3.6), (3.7), (3.12) we rewrite (3.14) as

$$\begin{aligned} & e^{-W}J(e^W z_{n+1}) + \lambda\left(\frac{dz_{n+1}}{dt} + (\mu+\nu)z_{n+1}\right) \\ & \quad = e^{-W}J(e^W(2y_n - v_n)) = e^{-W}J(e^W(-\lambda\mathcal{A}^* y_n + y_n)) \\ & \quad = -\lambda e^{-W}A(t)(e^W y_n) + \lambda\nu y_n + e^{-W}J(e^W y_n) \end{aligned} \tag{3.16}$$

and (3.13) as

$$\begin{aligned} J(e^W y_{n+1}) + \lambda A(t)(e^W y_{n+1}) - \lambda\nu e^W y_{n+1} \\ = J(e^W(z_{n+1} + v_n - y_n)). \end{aligned} \tag{3.17}$$

We set

$$X_n = e^W y_n,\ Z_n = e^W z_n.$$

Then, by (3.16), we get via Itô's formula (see [3] and (2.8), (2.7))

$$\begin{aligned} & \lambda dZ_{n+1} + J(Z_{n+1})dt + \lambda\nu Z_{n+1}dt = \lambda Z_{n+1}dW - \lambda A(t)X_n dt \\ & \qquad\qquad\qquad\qquad\qquad\qquad + \lambda\nu X_n dt + J(X_n)dt, \\ & Z_{n+1}(0) = x. \end{aligned}$$

By (3.17) and (3.12), we also get that

$$\begin{aligned} & \lambda A(t)X_{n+1}(t) + J(X_{n+1}(t)) - \lambda\nu X_{n+1}(t) \\ & \qquad = J(Z_{n+1}(t)) + \lambda A(t)X_n(t) - \lambda\nu X_n(t),\ t \in (0,T), \end{aligned}$$

which are just equations (2.2), (2.3). Moreover, by (3.11), we see that (2.5) holds.

Assume now that $A(t) : V \to V'$ is odd. Then so is $\mathcal{A}^* : \mathcal{V} \to \mathcal{V}$ and also the operator G defined by (3.15). Then, according to a result of J. Baillon [1], the sequence $\{v_n\}$ defined by (3.10), that is $v_{n+1} = G(v_n)$, is strongly convergent in $\mathcal{V}$. Recalling (3.9), we infer that so is the sequence $\{y_n\}$ and, consequently, (2.6) holds. This completes the proof of Theorem 2.1.

Remark 3.1. One might expect that a similar splitting scheme can be constructed for nonlinear monotone operators $A(t) : V \to V'$, where V is a reflexive Banach space and $A(t)$ are demicontinuous coercive and with polynomial growth as in [3]. In fact, in this case, one might replace (2.2) by

$$\lambda dZ_{n+1} + Z_{n+1}dt + \lambda\nu Z_{n+1}dt = Z_{n+1}dW - \lambda A_H(t)X_n dt + \lambda\nu X_n dt + X_n dt, \quad t \in (0,T),$$

$$\lambda A_H(t)X_{n+1} + X_{n+1} - \lambda\nu X_{n+1} = Z_{n+1} + \lambda A_H(t)X_n - \lambda\nu X_n,$$

where $A_H(t)u = A(t)u \cap H$. This question will be addressed in Section 5 below (see Remark 5.2).

4 Examples

We shall illustrate here the splitting algorithm (2.2)–(2.3) for a few parabolic stochastic differential equations.

Example 4.1. *Nonlinear stochastic parabolic equations.*

Consider the reaction-diffusion stochastic equation in $\mathcal{O} \subset \mathbb{R}^d$,

$$\begin{aligned} &dX - \mathrm{div}(a(t,\xi,\nabla X))dt + \nu X dt + \psi(X)dt = XdW \text{ in } (0,T)\times\mathcal{O},\\ &X = 0 \text{ on } (0,T)\times\partial\mathcal{O}, \quad X(0) = x \text{ in } \mathcal{O}. \end{aligned} \tag{4.1}$$

Here, $a : (0,T)\times\mathcal{O}\times\mathbb{R}^d \to \mathbb{R}^d$ is measurable in (t,ξ,r) continuous in r on $\mathbb{R}^d$, $a(t,\xi,0) = 0$. (The more general case, when $a : (0,T)\times\mathcal{O}\times\Omega\times\mathbb{R}^d) \to \mathbb{R}^d$ is progressively measurable, could also be considered.) We assume also that

$$\begin{aligned} &(a(t,\xi,r_1) - a(t,\xi,r_2))\cdot(r_1 - r_2) \ge 0, && \forall r_1, r_2 \in \mathbb{R}^d,\ (t,\xi) \in (0,T)\times\mathcal{O},\\ &a(t,\xi,r)\cdot r \ge a_1|r|_d^2 + a_2, && \forall r \in \mathbb{R}^d,\ (t,\xi) \in (0,T)\times\mathcal{O},\\ &|a(t,\xi,r)|_d \le c_1|r|_d + c_2, && \forall r \in \mathbb{R}^d,\ (t,\xi) \in (0,T)\times\mathcal{O}, \end{aligned}$$

where $a_1, c_1, \nu > 0,\ a_2, c_2 \in \mathbb{R}$, are independent of (t,ξ), and $\psi : \mathbb{R} \to \mathbb{R}$ is a continuous and monotonically nondecreasing function such that $\psi(0) = 0$ and $|\psi(r)| \le C(|r|^{\frac{2d}{d+2}} + 1)$, $\forall r \in \mathbb{R}$. Here $\mathcal{O} \subset \mathbb{R}^d$ is a bounded open subset with smooth boundary $\partial\mathcal{O}$, and $|\cdot|_d$ is the Euclidean norm of $\mathbb{R}^d$.

If $H = L^2(\mathcal{O})$, $V = H_0^1(\mathcal{O})$, $V' = H^{-1}(\mathcal{O})$ and , for $t \in (0,T)$, the operator $A(t) : V \to V'$ is defined by

$${}_{V'}\langle A(t)y,\varphi\rangle_V = \int_{\mathcal{O}} (a(t,\xi,\nabla y)\cdot\nabla\varphi + \psi(y)\varphi)d\xi,\ \forall\varphi \in H_0^1(\mathcal{O}),\ y \in H_0^1(\mathcal{O}),$$

then Hypotheses (i)–(iii) are satisfied. As regards the Wiener process W, we assume here that, besides (1.6), the following condition holds:

$$\sum_{j=1}^{\infty} \mu_j^2 |\nabla e_j|_\infty^2 < \infty.$$

Then, by Theorem 2.1, where H, V and $A(t)$ are defined above and $J = -\Delta$ with Dirichlet homogeneous boundary conditions, if $x \in H_0^1(\mathcal{O})$, the solution

$$X \in L^2(\Omega; C([0,T]; L^2(\mathcal{O})) \cap L^2((0,T) \times \Omega; H_0^1(\mathcal{O})))$$

to (4.1) can be obtained as

$$X = w - \lim_{n\to\infty} X_n \ \text{ in } L^2((0,T) \times \Omega; H_0^1(\mathcal{O})), \tag{4.2}$$

where $(X_n, Z_n) \in L^2((0,T)\times\Omega; H_0^1(\mathcal{O}))$ is the solution to the system (we take $\lambda = 1$)

$$\begin{aligned}
&dZ_{n+1} - \Delta Z_{n+1}dt + \nu Z_{n+1}dt = Z_{n+1}dW + \operatorname{div}(a(t,\xi,\nabla X_n))dt - \Delta X_n dt \\
&\hspace{25em} \text{in } (0,T) \times \mathcal{O}, \\
&Z_{n+1}(0) = x \ \text{ in } \mathcal{O}, \\
&Z_{n+1} = 0 \quad \text{in } (0,T) \times \partial\mathcal{O}, \\
&\operatorname{div} a(\nabla X_{n+1}) + \Delta X_{n+1} = \Delta Z_{n+1} + \operatorname{div}(a(t,\xi,\nabla X_n)) \text{ in } (0,T) \times \mathcal{O},
\end{aligned} \tag{4.3}$$

where $X_0 \in L^2((0,T) \times \Omega; H_0^1(\mathcal{O}))$ is arbitrary but $\mathcal{F}_t$-adapted. Moreover, if $a(t,\xi,-r) \equiv -a(t,\xi,r)$, $\forall r \in \mathbb{R}^d$, then the convergence (4.2) is strong in $L^2((0,T) \times \Omega; H_0^1(\mathcal{O}))$.

Example 4.2. *Stochastic porous media equations.*

Consider the stochastic equation

$$\begin{aligned}
&dX - \Delta\psi(t,\xi,X)dt - \nu\Delta X dt = X dW && \text{in } (0,T) \times \mathcal{O}, \\
&X(0,\xi) = x(\xi) && \text{in } \mathcal{O}, \\
&\psi(t,\xi,X(t,\xi)) = 0 && \text{on } (0,T) \times \partial\mathcal{O},
\end{aligned} \tag{4.4}$$

where $\mathcal{O}$ is a bounded domain in $\mathbb{R}^d$, $\nu > 0$, the function $\psi : [0,T] \times \overline{\mathcal{O}} \times \mathbb{R} \to \mathbb{R}$ is continuous, $r \to \psi(t,\xi,r)$ is monotonically increasing in r, and there exist

$a \in (0, \infty)$ and $c \in [0, \infty)$ such that

$$\begin{aligned} r\psi(t,\xi,r) &\geq a|r|^2 - c, \quad \forall r \in \mathbb{R},\ (t,\xi,r) \in [0,T] \times \overline{\mathcal{O}}, \\ |\psi(t,\xi,r)| &\leq c(1+|r|), \quad \forall r \in \mathbb{R},\ (t,\xi,r) \in [0,T] \times \overline{\mathcal{O}}. \end{aligned} \tag{4.5}$$

We shall write equation (4.4) under the form (1.1) with $H = H^{-1}(\mathcal{O})$. Namely, we take $V = L^2(\mathcal{O}),\ H = H^{-1}(\mathcal{O})$, and V' is the dual of V with the pivot space $H^{-1}(\mathcal{O})$. Then, $V \subset H \subset V'$ and

$$V' = \{\theta \in \mathcal{D}'(\mathcal{O}) : \theta = -\Delta v,\ v \in L^2(\mathcal{O})\},$$

where Δ is taken in the sense of distributions on $\mathcal{O}$. (Here $\mathcal{D}'(\mathcal{O})$ is the space of Schwartz distributions on $\mathcal{O}$.) The duality ${}_{V'}\langle\cdot,\cdot\rangle_V$ is defined as

$${}_{V'}\langle\theta, u\rangle_V = \int_{\mathcal{O}} \widetilde{\theta} u\, d\xi, \quad \widetilde{\theta} = (-\Delta)^{-1}\theta,$$

where Δ is the Laplace operator with homogeneous Dirichlet boundary conditions on $\partial\mathcal{O}$. The duality mapping $J : V \to V'$ is just the operator $-\Delta$ defined from $L^2(\mathcal{O})$ to $V' \subset \mathcal{D}'(\mathcal{O})$ by

$$\Delta u(\varphi) = \int_{\mathcal{O}} u\Delta\varphi\, d\xi,\ \forall \varphi \in H_0^1(\mathcal{O}) \cap H^2(\mathcal{O}).$$

The operator $A(t) : V \to V'$ is defined by

$${}_{V'}\langle A(t)y, v\rangle_V = \int_{\mathcal{O}} \psi(t,\xi,y)v\, d\xi, \quad \forall y, v \in V = L^2(\mathcal{O}),\ t \in [0,T].$$

Then, Hypotheses (i)–(iv) hold and so, if $x \in L^2(\mathcal{O})$, by Theorem 2.1, the solution $X \in L^2(\Omega; C([0,T]; H^{-1}(\mathcal{O})) \cap L^2((0,T) \times \Omega; L^2(\mathcal{O})))$ to (4.4) is given by

$$X = w - \lim_{n\to\infty} X_n \ \text{ in } L^2((0,T) \times \Omega; L^2(\mathcal{O})),$$

where

$$\begin{gathered} dZ_{n+1} - \Delta Z_{n+1}dt + \nu Z_{n+1}dt = Z_{n+1}dW - \Delta\psi(t,\cdot,X_n)dt - \Delta Z_n dt \\ \text{in } (0,T) \times \mathcal{O}, \\ Z_{n+1}(0) = x \in L^2(\mathcal{O}),\ n = 0,1,..., \\ \Delta\psi(t,\cdot,X_{n+1}) + X_{n+1} = Z_{n+1} + \Delta\psi(t,\cdot,X_n), \quad \text{in } \mathcal{O}, \\ \psi(t,\cdot,X_{n+1}(t,\cdot)) = 0 \ \text{ on } \partial\mathcal{O}, \\ n = 0,1,...,X_0 \in L^2(0,T; L^2(\Omega; L^2(\mathcal{O}))). \end{gathered} \tag{4.6}$$

If $\psi(t,\xi,r) = -\psi(t,\xi,-r),\ \forall r \in \mathbb{R}$, then the convergence of the sequence $\{X_n\}$ is strong in $L^2((0,T)\times\Omega; L^2(\mathcal{O}))$.

5 The case where $A(t)$ is maximal monotone in $H \times H$

Consider now equation (1.1) under the following assumptions on A:

(j) *$A : [0,T] \times H \times \Omega \to H$ is progressively measurable and, for each $(t,\omega) \times [0,T] \times \Omega$ the operator $u \to A(t,\omega,u)$ is maximal monotone in $H \times H$. Moreover, there is $f \in L^2((0,T)\times\Omega; H)$ such that*

$$(I + A(t))^{-1} f(t) \in L^2((0,T)\times\Omega; H). \tag{5.1}$$

We assume also that condition (2.1) holds.

It should be noted that, if $A(t) : V \to V'$ satisfies assumptions (i)-(ii), where V is a reflexive Banach space, then the operator $A(t) : H \to H$, defined by

$$A(t)_H u = A(t)u \cap V,$$

satisfies assumption (j). However, the class of the operators A satisfying (j) is considerably larger.

We consider the splitting scheme (which is well defined by strong monotonicity of $\mathcal{A}_1^* + \mathcal{B}_1^*$)

$$\begin{aligned} &\lambda dY_{n+1} + (1+\lambda\nu)Y_{n+1}dt = \lambda Y_{n+1}dW \\ &\qquad\qquad + (V_n - ((1-\lambda\nu)I - \lambda A(t))^{-1}V_n)dt, \\ &Y_{n+1}(0) = x \text{ in } (0,T), \\ &V_{n+1} = Y_{n+1} + V_n - ((1-\lambda\nu)I - \lambda A(t))^{-1}V_n, \end{aligned} \tag{5.2}$$

where $V_0 \in L^2((0,T)\times\Omega; H)$ is an $(\mathcal{F}_t)_{t\ge 0}$-adapted process such that $A(t)V_0 \in L^2((0,T)\times\Omega; H)$. We have

Theorem 5.1. *Assume that $x \in H$ and that equation* (1.1) *has a solution $X \in L^2(\Omega; C([0,T]; H))$ such that $A(t)X \in L^2((0,T)\times\Omega; H)$.*

Then, for $n \to \infty$,

$$V_n \to V \text{ weakly in } L^2((0,T)\times\Omega; H), \tag{5.3}$$

where $X = ((1-\lambda\nu)I + A(t))^{-1}V$ *is the solution to* (1.1).

If $A(t)$ *is odd, then the convergence* (5.3) *is strong.*

Proof. The operators $\mathcal{A}_1^*$ and $\mathcal{B}_1^*$ defined by

$$\begin{aligned}(\mathcal{A}_1^* u)(t) &= e^{-W}A(t)(e^W u) - \nu u, \quad \forall u \in D(\mathcal{A}_1^*),\\ (\mathcal{B}_1^* u)(t) &= \frac{du}{dt} + (\mu+\nu)u, \qquad\quad \forall u \in D(\mathcal{B}_1^*),\end{aligned}$$

with the domains

$$\begin{aligned}D(\mathcal{A}_1^*) &= \{u \in \mathcal{H};\ e^{-W}A(t)(e^W u) - \nu u \in \mathcal{H}\},\\ D(\mathcal{B}_1^*) &= \{u \in \mathcal{H};\ u \in W^{1,2}([0,T];H)\cdot \mathbb{P}\text{-a.s.},\ u(0)=x\}\end{aligned}$$

are, by the above hypotheses, maximal monotone in $\mathcal{H}\times\mathcal{H}$ (see also [4]). Moreover, there is at least one solution y^* to the equation

$$\mathcal{A}_1^* y^* + \mathcal{B}_1^* y^* = 0. \tag{5.4}$$

Then, again by [8], it follows that the sequence $\{v_n\} \subset \mathcal{H}$ defined by

$$\begin{aligned}v_{n+1} = (I+\lambda\mathcal{B}_1^*)^{-1}(2(I+\lambda\mathcal{A}_1^*)^{-1}v_n - v_n) + v_n - (I+\lambda\mathcal{A}_1^*)^{-1}v_n,\\ n=0,1,...\end{aligned} \tag{5.5}$$

is weakly convergent in $\mathcal{H}$ to v^*, where $(1+\lambda\mathcal{A}_1^*)^{-1}v^* = y^*$ is the solution to equation (5.4).

We set

$$\widetilde{z}_{n+1} = v_{n+1} - v_n + (I+\lambda\mathcal{A}_1^*)^{-1}v_n \tag{5.6}$$

and, by (5.5), we have

$$\widetilde{z}_{n+1} + \lambda\mathcal{B}_1^* z_{n+1} = v_n - (I+\lambda\mathcal{A}_1^*)^{-1}v_n. \tag{5.7}$$

Then, if $Y_n = e^W\widetilde{z}_n$ and $V_n = e^W v_n$, we can rewrite (5.6)-(5.7) as (5.2) and get (5.3), as claimed.

Remark 5.2. The convergence of the splitting algorithm (5.1)-(5.2) does not require conditions of the form (ii)-(iii) for the operator $A(t)$ but in change it requires the existence of a sufficiently regular solution X for equation (1.1)

$(A(t)x \in L^2((0,T)\times\Omega; H))$ which is not the case for Examples 4.1, 4.2. Such a condition holds, however, for the stochastic reaction-diffusion equation

$$\begin{aligned}
&dX - \Delta X\, dt + \Psi(X)dt = X\, dW \text{ in } (0,T)\times\mathcal{O},\\
&X = 0 \text{ on } (0,T)\times\partial\mathcal{O},\\
&X(0) = x,
\end{aligned}$$

if $x \in H_0^1(\mathcal{O})$ and $\Psi : \mathbb{R} \to \mathbb{R}$ is continuous and monotonically increasing and for other stochastic parabolic equations as well.

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