Stochastic Heat Equations with Values in a Riemannian Manifold*

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Abstract

The main result of this note is the existence of martingale solutions to the stochastic heat equation (SHE) in a Riemannian manifold by using suitable Dirichlet forms on the corresponding path/loop space. Moreover, we present some characterizations of the lower bound of the Ricci curvature by functional inequalities of various associated Dirichlet forms.

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1 Introduction

sect1

This work is motivated by Tadahisa Funaki's pioneering work [8] for regular noise and Martin Hairer's recent construction [12] with singular noise of a natural evolution on the loop space over a Riemannian manifold (M, g). Both consider the formal Langevin dynamics associated to the energy

$$E(u) = \frac{1}{2}\int_{S^1} g_{u(x)}(\partial_x u(x), \partial_x u(x))\mathrm{d}x,$$

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for smooth functions $u : S^1 \to M$. One would like to build a Markov process u taking values in loops over M with invariant (even symmetrizing) measure formally given by $\exp(-2E(u))Du$. A natural way of interpreting $\exp(-2E(u))Du$ is to think of it as the Brownian bridge measure on M. See [1] for proofs that natural approximations of $\exp(-2E(u))Du$ do indeed converge to Wiener measure on $C([0,1];M)$.

Processes with invariant (even symmetrizing) measure given by Wiener measure on $C([0,1];M)$ were first constructed in the nineties by using the Dirichlet form given by the Malliavin gradient on path and loop spaces over Riemannian manifolds, see [7, 2]. In this case, we call the associated Dirichlet form **O-U Dirichlet form**. For an alternative approach, not based on Dirichlet forms, see [15]. After that there were several follow-up papers concentrating on non-compact Riemannian manifold, see [5, 19]. In particular, when $M = \mathbb{R}^d$ these processes correspond to the Ornstein-Uhlenbeck processes from Malliavin calculus. When $M = \mathbb{R}^d$ the stochastic heat equation also admits Wiener measure as the invariant measure. To construct the solution to the stochastic heat equation on Riemannian manifold, in [12] Martin Hairer wrote the equation in local coordinates informally as:

eq1.1

$$\dot{u}^\alpha = \partial_x^2 u^\alpha + \Gamma^\alpha{}_{\beta\gamma}(u)\partial_x u^\beta \partial_x u^\gamma + \sigma_i^\alpha(u)\xi_i, \tag{1.1}$$

where Einstein's convention of summation over repeated indices is applied and $\Gamma^\alpha{}_{\beta\gamma}$ are the Christoffel symbols for the Levi-Civita connection of (M,g), σ_i^α are the local coordinates for the vector fields σ_i on M satisfying $g_u(h,\bar{h}) = \sum_i g_u(h,\sigma_i)g_u(\bar{h},\sigma_i)$ for $h,\bar{h} \in T_uM$, and ξ_i is a collection of independent space-time white noises. Equation (1.1) may be considered as some kind of a multi-component version of the KPZ equation. By regularity structure theory, recently developed in [11, 3, 4], local well-posedness of (1.1) has been obtained in [12].

In this note, we construct a new Dirichlet form (L^2**-Dirichlet form**) such that the associated Markov process solves the stochastic heat equation (SHE) with values in a Riemannian manifold. Moreover, we obtain some new characterizations of the lower bound of the Ricci curvature in terms of $\mathbf{L^2}$-gradient and functional inequalities associated to the above Dirichlet form. In addition, we also prove the logarithmic Sobolev inequality holds on the path space over a Riemannian manifold with lower bounded Ricci curvature. As a consequence, for the process we have L^2-exponential ergodicity, recurrent irreducibility and the strong law of large numbers.

In Sections 2 and 3 below, we present and discuss these results in detail and explain the framework. We also sketch some proofs. The details of the proofs are contained in [16].

2 A Diffusion Process on Path Space

sect2

Throughout this article, suppose that M is a complete and stochastically complete Riemannian manifold with dimension d, and ρ be the Riemannian distance on M. Fix

$o \in M$ and $T > 0$. The based path space $W_o(M) = \{\gamma \in C([0,1]; M) : \gamma(0) = o\}$, which is a Polish space under the uniform distance

$$d_\infty(\gamma, \sigma) := \sup_{t \in [0,1]} \rho(\gamma(t), \sigma(t)), \quad \gamma, \sigma \in W_o(M).$$

In order to construct Dirichlet forms associated to stochastic heat equations on Riemannian path space, we need to introduce the following L^1-distance, which is a smaller distance than the above uniform distance d_∞ on $W_o(M)$:

$$\tilde{d}(\gamma, \eta) := \int_0^1 \rho(\gamma_s, \eta_s) \mathrm{d}s, \quad \gamma, \eta \in W_o(M).$$

Let E denote the closure of $W_o(M)$ in $\{\eta : [0,1] \to M; \int_0^1 \rho(o, \eta_s) ds < \infty\}$ with respect to the distance $\tilde{d}$. Then E is a Polish space.

Let $O(M)$ be the orthonormal frame bundle over M, and let $\pi : O(M) \to M$ be the canonical projection. Choosing a standard othornormal basis $\{H_i\}_{i=1}^d$ of horizontal vector fields on $O(M)$, and consider the following SDE,

eq2.1 (2.1)
$$\begin{cases} dU_t = \sum_{i=1}^d H_i(U_t) \circ \mathrm{d}B_t^i, \quad t \geq 0 \\ U_0 = u_o, \end{cases}$$

where u_o is a fixed orthonormal basis of $T_o M$ and $B_t^1, \cdots, B_t^d$ are independent Brownian motions on $\mathbb{R}$. Then $x_t := \pi(U_t)$, $t \geq 0$ is the Brownian motion on M with initial point o, and $U_.$ is the (stochastic) horizontal lift along $x_.$. Let μ_o be the distribution of $x_{[0,1]}$, then μ_o is a probability measure on $W_o(M)$.

Let $\mathscr{F}C_b^1$ be the space of bounded Lipschitz continuous cylinder functions on $W_o(M)$, i.e. for every $F \in \mathscr{F}C_b^1$, there exist some $m \geq 1$, $g_i \in \mathrm{Lip}(M)$, $m \in \mathbb{N}$, $f \in C_b^1(\mathbb{R}^m)$ such that

eq2.2 (2.2)
$$F(\gamma) = f\left(\int_0^1 g_1(s, \gamma_s) \mathrm{d}s, \int_0^1 g_2(s, \gamma_s) \mathrm{d}s, ..., \int_0^1 g_m(s, \gamma_s) \mathrm{d}s\right), \quad \gamma \in W_o(M),$$

where

$$\mathrm{Lip}(M) := \{g : [0,1] \times M \to \mathbb{R}, |g(s, \eta_s) - g(s, \gamma_s)| \leq C\rho(\eta_s, \gamma_s), s \in [0,1], \eta, \gamma \in E\}.$$

For any $F \in \mathscr{F}C_b^1$ with (2.2) form and $h \in \mathbf{H} := L^2([0,1]; \mathbb{R}^d)$, the directional derivature of F with respect to h is given by

$$D_h F(\gamma) = \sum_{j=1}^m \hat{\partial}_j f(\gamma) \int_0^1 \left\langle U_s^{-1}(\gamma) \nabla g_j(s, \gamma_s), h_s \right\rangle_{\mathbb{R}^d} \mathrm{d}s, \quad \gamma \in W_o(M),$$

where

$$\hat{\partial}_j f(\gamma) := \partial_j f\bigg(\int_0^1 g_1(s,\gamma_s)\mathrm{d}s, \int_0^1 g_2(s,\gamma_s)\mathrm{d}s, ..., \int_0^1 g_m(s,\gamma_s)\mathrm{d}s\bigg),$$

and for $\gamma \in E\backslash W_o(M)$ we define $D_hF(\gamma) = 0$. By Riesz's representation theorem, there exists a gradient operator $DF(\gamma) \in \mathbf{H}$ such that $\langle DF(\gamma), h\rangle_{\mathbf{H}} = D_hF(\gamma), \gamma \in E, h \in \mathbf{H}$. In particular, for $\gamma \in W_o(M)$, $DF(\gamma) = \sum_{j=1}^m \hat{\partial}_j f(\gamma) U_s^{-1}(\gamma)\nabla g_j(s,\gamma_s)$. We call DF the $\mathbf{L^2}$-gradient of F on path space. Denote by $\mathbb{H}$ the Cameron-Martin space:

$$\mathbb{H} := \left\{ h \in C^1([0,1];\mathbb{R}^d) \middle| h(0) = 0, \|h\|_{\mathbb{H}}^2 := \int_0^1 \|h'(s)\|^2 \mathrm{d}s < \infty \right\}.$$

Taking $\{e_k\} \subset \mathbb{H}$ such that it is an orthonormal basis in $\mathbf{H}$, consider the following symmetric quadratic form

$$\mathscr{E}(F,G) := \frac{1}{2}\int_E \langle DF, DG\rangle_{\mathbf{H}} \mathrm{d}\mu_o = \frac{1}{2}\sum_{k=1}^{\infty}\int_E D_{e_k}F D_{e_k}G \mathrm{d}\mu_o; \quad F, G \in \mathscr{F}C_b^1.$$

T2.1 **Theorem 2.1.** *The quadratic form $(\mathscr{E}, \mathscr{F}C_b^1)$ is closable and its closure $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ is a quasi-regular Dirichlet form on $L^2(E;\mu_o) = L^2(W_o(M);\mu_o)$.*

Sketch of the proof: For the compact Riemannian manifold, we can derive the closability of $(\mathscr{E}, \mathscr{F}C_b^1)$ by the integration by parts formula in [7] along each e_k. By a localization technique, the integration by parts formula in [7] also can be extended to the general Riemannian manifolds, which implies the closability in the general case. The quasi-regularity of the Dirichlet form follows essentially by the same argument as in [13]. □

By using the theory of Dirichlet forms (refer to [13]), we obtain:

T2.2 **Theorem 2.2.** *There exists a conservative (Markov) diffusion process $M = (\Omega, \mathscr{F}, \mathscr{M}_t, (X(t))_{t\geq 0}, (\mathbf{P}^z)_{z\in E})$ on E* properly associated with *$(\mathscr{E}, \mathscr{D}(\mathscr{E}))$, i.e. for $u \in L^2(E;\mu_o) \cap \mathscr{B}_b(E)$, the transition semigroup $P_t u(z) := \mathbf{E}^z[u(X(t))]$ is a $\mathscr{E}$-quasi-continuous version of $T_t u$ for all $t > 0$, where T_t is the semigroup associated with $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$.*

Here for the notion of $\mathscr{E}$-quasi-continuity we refer to [13, ChapterIII,Definition 3.2]. By Fukushima's decomposition we have

T2.3 **Theorem 2.3.** *There exists a* properly $\mathscr{E}$-exceptional set *$S \subset E$, i.e. $\mu_o(S) = 0$ and $\mathbf{P}^z[X(t) \in E \setminus S, \forall t \geq 0] = 1$ for $z \in E\backslash S$, such that $\forall z \in E\backslash S$ under $\mathbf{P}^z$, the sample paths of the associated process $M = (\Omega, \mathscr{F}, \mathscr{M}_t, (X(t))_{t\geq 0}, (\mathbf{P}^z)_{z\in E})$ on E satisfy the following: for $u \in \mathscr{D}(\mathscr{E})$*

eq2.3 (2.3)
$$u(X_t) - u(X_0) = M_t^u + N_t^u \quad \mathbf{P}^z - a.s.,$$

where M^u is a martingale with quadratic variation process given by $\int_0^t |Du(X_s)|^2_{\mathbf{H}} ds$ and N^u is a zero quadratic variation process. In particular, for $u \in D(L)$, $N_t^u = \int_0^t Lu(X_s)ds$, where L is the generator of $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$.

r2.4 **Remark 2.4.** *(a) If we choose $u(\gamma) = \int_{r_1}^{r_2} u^\alpha(\gamma_s)ds \in \mathcal{F}C_b^1$, with local coordinates u^α on M, then the quadratic variation process for M^u is the same as that for the martingale part in (1.1).*

(b) Theorems 2.2-2.3 still hold if the path space is replaced by the loop space(or the free path and free loop cases) and Wiener measure is replaced by the associated measure under some suitable conditions.

3 Properties of SHE

sect3

In this section, we will study properties of $X_t, t \geq 0$, constructed in Section 2. First we present the logarithmic Sobolev inequality for the damped gradient $\tilde{D}F$ assuming M is stochastically complete, which implies the logarithmic Sobolev inequality for the Dirichlet form considered in Section 2.

For any $F \in \mathscr{F}C_b^1$, we define the damped gradient $\tilde{D}F$ of F by

$$\tilde{D}F(t) = M_t^{-1} \int_t^1 M_s(DF(s))\mathrm{d}s,$$

where M_t is the solution of the equation

$$\frac{\mathrm{d}}{\mathrm{d}t} M_t + \frac{1}{2} M_t Ric_{U_t} = 0, \quad M_0 = I.$$

Suppose that $\mathrm{Ric} \geq -K$ for $K \in \mathbb{R}$. Define the quadratic form corresponding to $\tilde{D}F$ by

$$\tilde{\mathscr{E}}(F,G) = \frac{1}{2}\int_E \langle \tilde{D}F, \tilde{D}G\rangle_{\mathbf{H}} \mathrm{d}\mu_o, \quad F, G \in \mathscr{F}C_b^1.$$

T3.1 **Theorem 3.1.** *[Log-Sobolev inequality] Suppose that* Ric $\geq -K$ *for* $K \in \mathbb{R}$. *The log-Sobolev inequality holds for $(\tilde{\mathscr{E}}, \mathscr{D}(\tilde{\mathscr{E}}))$, i.e.,*

$$\mu_o(F^2 \log F^2) \leq 2\tilde{\mathscr{E}}(F,F), \qquad F \in \mathscr{F}C_b^1, \ \mu_o(F^2) = 1.$$

In particular, we have

$$\mu_o(F^2 \log F^2) \leq 2C(K)\mathscr{E}(F,F), \qquad F \in \mathscr{F}C_b^1, \ \mu_o(F^2) = 1$$

where $C(K) = \frac{e^K - 1 - K}{K^2} \wedge C_0(K)$ with

$$C_0(K) = \begin{cases} \frac{4}{K^2}\left(1 - \sqrt{2e^{\frac{K}{2}} - e^K}\right), & \text{if } K < 0, \\ \frac{2}{K^2}\left(e^K - 2e^{\frac{K}{2}} + 1\right), & \text{if } K > 0. \end{cases}$$

r3.2 **Remark 3.2.** *(i) In fact, Theorem 3.1 had first been proved in [10]. Compared to the results in there, our constant $C(K)$ is smaller. By comparing the classical O-U Dirichlet form and the L^2-Dirichlet form, we note that the the LSI associated to the two Dirichlet forms are essentially different, the former requires upper and lower bounds of the Ricci curvature of M, and the latter only needs a lower bound for the Ricci curvature.*

(ii) According to [17], the log-Sobolev inequality implies hypercontractivity of the associated semigroup P_t, in particular, the L^2-exponential ergodicity of the process: $\|P_t f - \int f d\mu_o\|_{L^2}^2 \le e^{-t/C(K)}\|F\|_{L^2}^2$.

(iii) The log-Sobolev inequality also implies the irreducibility of the Dirichlet form $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$. It is obvious that the Dirichlet form $(\mathscr{E}, \mathscr{D}(\mathscr{E}))$ is recurrent. Combining these two results, by [FOT94, Theorem 4.7.1], for every nearly Borel non-exceptional set B,

$$\mathbf{P}^x(\sigma_B \circ \theta_n < \infty, \forall n \ge 0) = 1, \quad \text{for q.e. } x \in X.$$

Here $\sigma_B = \inf\{t > 0 : X_t \in B\}$, θ is the shift operator for the Markov process X, and for the definition of nearly Borel non-exceptional set we refer to [FOT94]. Moreover by [FOT94, Theorem 4.7.3] we obtain the following strong law of large numbers: for $f \in L^1(E, \mu_o)$

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t f(X_s)\mathrm{d}s = \int f \mathrm{d}\mu_o, \quad \mathbf{P}^x - a.s.,$$

for q.e. $x \in E$.

Sketch of the proof of Theorem 3.1: The proof follows from the following martingale representation: for $F \in L^2(\mu_o)$,

$$F = \mathbb{E}(F) + \int_0^1 \left\langle \mathbb{E}\left[M_s^{-1} \int_s^1 M_\tau(DF(\tau))\mathrm{d}\tau \middle| \mathscr{F}_s \right], \mathrm{d}W_s \right\rangle,$$

and some delicate estimates. Here W is the anti-development of γ and $\{\mathscr{F}_s\}$ is the filtration generated by W. □

Upper and lower bounds of the Ricci curvature on a Riemannian manifold were well characterized by the diffusion process associated to the O-U Dirichlet form given by the Malliavin gradient in [14]. If the O-U Dirichlet form is replaced by our L^2-Dirichlet form, then we can only obtain the following characterizations for the lower bound of the Ricci curvature. This further indicates that these two processes have essential differences.

In fact, the results in Section 2 and Theorem 3.1 also hold when we change 1 to any $T > 0$. To state our results, let us first introduce some notations: For any point

$y \in M$ and $T > 0$, let $x_{y,[0,T]}$ be the Brownian motion starting from $y \in M$ up to time T, and $\mu_{T,y}$ be the distribution of Brownian motion $x_{y,[0,T]}$ on $W_y^T(M) := \{\gamma \in C([0,T];M) | \gamma(0) = y\}$. For any $n \geq 1$ and $G \in \mathscr{F}C_b^T$ with $\mathscr{F}C_b^T$ defined as in (2.2) with 1 replaced by T, define

$$\begin{aligned}\mathcal{E}_{T,n,y}^K(G,G) &= (1+n)C_1(K)\int_{W_y^T(M)}\int_0^{T-\frac{1}{n}} |DG(\gamma)(s)|_{\mathbb{R}^d}^2 \mathrm{d}s\mathrm{d}\mu_{T,y}(\gamma)\\ &\quad + \left(\frac{1}{n}+\frac{1}{n^2}\right) C_{2,n}(K)\int_{W_y^T(M)}\int_{T-\frac{1}{n}}^T |DG(\gamma)(s)|_{\mathbb{R}^d}^2 \mathrm{d}s\mathrm{d}\mu_{T,y}(\gamma).\end{aligned}$$

where

$$C_1(K) = \left[\frac{1}{K^2}\left(TKe^{KT} - e^{KT} + 1\right)\right] \bigvee \frac{T^2}{2}, \quad C_{2,n}(K) = \frac{e^{KT}-1}{K}\left(1 \vee e^{-\frac{K}{n}}\right).$$

Let p_t be the Markov semigroup of the process x_y given by $p_t f(y) = \mathbb{E}[f(x_{t,y})], y \in M, f \in \mathscr{B}_b(M), t \geq 0$. Denote by $C_0^\infty(M)$ the set of all smooth functions with compact support on M.

T3.3 **Theorem 3.3.** *For $K \in \mathbb{R}$, the following statements are equivalent:*

(1) $\mathrm{Ric} \geq -K$.

(2) *For every $f \in C_0^\infty(M), T > 0$ and $y \in M$, we have*

$$\left|\int_0^T \nabla p_s f(y) ds\right| \leq \int_0^T \mathrm{e}^{\frac{Ks}{2}} p_s |\nabla f|(y) \mathrm{d}s.$$

(3) *For every $y \in M, T > 0$, the following log-Sobolev inequality holds for every $n \in \mathbb{N}$:*

$$\mu_{T,y}(F^2 \log F^2) \leq 2\mathscr{E}_{T,n,y}^K(F,F), \quad F \in \mathscr{F}C_b^T,\ \mu_{T,y}(F^2) = 1.$$

(4) *For every $y \in M, T > 0$, the following Poincaré-inequality holds for every $n \in \mathbb{N}$:*

$$\mu_{T,y}(F^2) \leq \mathscr{E}_{T,n,y}^K(F,F), \quad F \in \mathscr{F}C_b^T,\ \mu_{T,y}(F) = 0.$$

Sketch of the proof: 1)$\Longrightarrow$ 2) follows from the gradient formula. Conversely, taking $F(\gamma) := \int_0^T f(\gamma_s)ds$ for some function $f \in C_0^1(M)$ with

eq3.1 (3.1) $$f \in C_0^\infty(M), \quad |\nabla f|(y) = 1, \quad \mathrm{Hess}_f(y) = 0,$$

and applying F into 2), 1) can be derived from the following formula in [18]

$$\frac{1}{2}\mathrm{Ric}(\nabla f, \nabla f)(y) = \lim_{T\downarrow 0} \frac{p_T|\nabla f|(y) - |\nabla p_T f|(y)}{T}.$$

1)⇒ 3) follows similarly as in the proof of Theorem 3.1.

3)⇒ 4) is standard.

4)⇒ 1): For each $k \geq 1$, take $F(\gamma) = k \int_{T-1/k}^{T} f(\gamma_s)ds$ for some f as (3.1). Then using this formula

$$\frac{1}{2}\mathrm{Ric}(\nabla f, \nabla f)(y) = \lim_{T\downarrow 0} \frac{1}{T}\left(\frac{p_T f^2(y) - (p_T f)^2(y)}{2T} - |\nabla p_T f(y)|^2\right),$$

it is not difficult to obtain 1). □

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