Quasi-Linear (Stochastic) Partial Differential Equations with Time-Fractional Derivatives [*]

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Abstract

In this paper we develop a method to solve evolution equations on Gelfand triples with time-fractional derivative based on monotonicity techniques. Applications include deterministic and stochastic quasi-linear partial differential equations with time-fractional derivatives, including time-fractional (stochastic) porous media equations (including the case where the Laplace operator is also fractional) and p-Laplace equations as special cases.

1 Introduction

Fractional Calculus has a long history and its origins can be traced back to the end of seventeenth century (cf. [52]). Although the first main steps of the theory date back to the first half of the nineteenth century, this subject became very active only over the last few decades. One of the main reasons is that scientists and engineers have developed many new models that naturally involve fractional differential equations, which have been applied successfully, e.g. in mechanics (theory of viscoelasticity and viscoplasticity)(cf. [37]), bio-chemistry (modelling of polymers and proteins)(cf. [22, 23]), electrical engineering (transmission of ultrasound waves)(cf. [21]), medicine (modelling of human tissue under mechanical loads)(cf. [19, 36]) etc. For more applications and references we refer to [3, 27, 40, 41].

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A typical example of a time-fractional equation is the time-fractional heat equation $\partial_t^\beta u = \Delta u$ with $0 < \beta < 1$, where ∂_t^β is the Caputo fractional derivative first appeared in [14] and is defined for $0 < \beta < 1$ by

$$(1.1) \qquad \partial_t^\beta f(t) = \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \int_0^t (t-s)^{-\beta}(f(s) - f(0))ds,$$

where Γ is the Gamma function defined by $\Gamma(\lambda) := \int_0^\infty t^{\lambda-1}e^{-t}dt$. Here one implicitly assumes that f is such that the integral on the right-hand side is (weakly) differentiable in t. For the precise domain of f for which $\partial_t^\beta f$ is defined and which is convenient in our case, we refer to (2.11) below.

The classical heat equation $\partial_t u = \Delta u$ describes heat propagation in homogeneous medium. The time-fractional diffusion equation $\partial_t^\beta u = \Delta u$ with $0 < \beta < 1$ has been widely used to model anomalous diffusions exhibiting subdiffusive behavior, e.g. due to particle sticking and trapping phenomena (cf. [40, 54]). While in normal diffusions (described by the heat equation or more general parabolic equations), the mean squared displacement of a diffusive particle behaves like const·t for $t \to \infty$, the time-fractional diffusion equation exhibits a behaviour like const·t^β for $t \to \infty$. This is the reason why time-fractional equations with $0 < \beta < 1$ are called subdiffusion equations in the literature, and for the case $1 < \beta < 2$ are called superdiffusion equations. In fact, there is a diverse number of real world systems which demonstrate this type of phenomena. For example, the above time-fractional equations and nonlinear variants of them are also widely used to model dynamical processes in materials with memory, e.g. the diffusion of fluids in porous media with memory (see [13]). We refer to the survey article [42] and the monographs [3, 45, 48, 49, 53, 66] for more references.

We also want to mention the interesting interplay between stochastic processes and time-fractional differential equations. A celebrated result of Einstein established a mathematical link between random walks, the diffusion equation and Brownian motion. The probability densities of the Brownian motion solve a diffusion equation (heat equation), and hence we refer to the Brownian motion as the stochastic solution to the heat equation. Similarly, the stochastic solutions of time-fractional diffusion equations are subordinated processes (e.g. iterated Brownian motion). The solutions to fractional diffusion equations are strictly related to stable densities. Indeed, the stochastic solutions of time-fractional diffusions can be realized through "time-change" by inverse stable subordinators. Just as Brownian motion is a scaling limit of simple random walks, the stochastic solutions to certain fractional diffusion equations are scaling limits of continuous time random walks, in which the $i.i.d.$ jumps are separated by $i.i.d.$ waiting times. For this subject we refer to [2, 39, 46] and the references therein.

In recent years, there has been also growing interest in stochastic time-fractional partial differential equations (see e.g. [20, 56]). For example, the authors in [18] study the L^2-theory for a class of semilinear SPDEs with time-fractional derivatives, which can be used to describe random effects on transport of particles in media with thermal memory, or particles subject to sticking and trapping. In [43, 44], the authors consider a space-time fractional stochastic heat type equation to model phenomenon with random effects with thermal memory, and they prove existence and uniqueness of mild solutions and also some intermittency property

(see also [24]). For a linear stochastic partial differential equation of fractional order both in the time and space variables with a different type of noise term we refer to [16, 26]. In [15, 17] the authors investigate linear stochastic time-fractional partial differential equations for the more general case $0 < \beta \leq 2$.

In this paper, we are mainly interested in non-linear stochastic time-fractional partial differential equations in a variational setting, which include in particular important examples of quasi-linear type as the stochastic porous media or the p-Laplace equation. They are of the following form:

$$(1.2) \qquad \partial_t^\beta X(t) + A(t, X(t)) = \partial_t^\gamma \int_0^t B(s) dW(s), \ 0 < t < T,$$

where $V \subseteq H \equiv H^* \subseteq V^*$ is a Gelfand triple, $\beta \in (0,1), \gamma \in (0, \beta + \frac{1}{2}) \cap (0,1]$, ∂_t^β is the Caputo fractional derivative, $A : [0,T] \times V \to V^*$ and $B : [0,T] \to L_{HS}(U; V)$ (here $L_{HS}(U; V)$ denotes the space of all Hilbert–Schmidt operators from U to V) are measurable.

We will establish the existence and uniqueness of solutions to (1.2) under classical monotonicity and coercivity conditions on A (see Theorems 2.1 and 2.7). The proof of the main results will be based on a functional analytic approach, which, to the best of our knowledge, is new to solve both the deterministic and stochastic time-fractional partial differential equations in the variational setting. This approach is inspired by the work of Stannat [57] on the theory of generalized Dirichlet forms, which in turn draws essential ideas from [30]. We will establish a general existence result concerning an abstract operator equation (see Theorem 2.3 below) which is central to our approach.

We should mention that time-fractional linear evolution equations in the Gelfand triple setting have first been investigated in [62]. Later on the author also proved the global solvability of a nondegerate parabolic equation with time-fractional derivative in [63] (cf. [1] for more general case). However, these results cannot be applied to quasilinear type equations like the porous media or the p-Laplace equation. In [28] the authors investigate elliptic-parabolic integro-differential equations with L^1-data. Their framework includes the time-fractional p-Laplace equation. However, the authors in [28] consider generalized solutions (*i.e.* entropy solutions). Therefore, the results of this paper generalize or complement the corresponding results in [28, 62, 63] in the setting of time-fractional quasilinear PDE with monotone coefficients. Recently, the authors in [61] derive very interesting decay estimates for the solutions of time-fractional porous medium and p-Laplace equations (by assuming the existence), and the decay behaviour is notably different from the case with usual time derivative. Hence, here we, in particular, give a positive answer to the question on the existence and uniqueness of solutions to the time-fractional porous medium equations and p-Laplace equations, which are left open in [61].

If $\beta = \gamma = 1$, then both the deterministic and stochastic equation (1.2) have been intensively investigated in the variational setting. For the deterministic case, one might refer to the monographs [4, 9, 30, 55, 64] and the references therein. But also the stochastic case (SPDE) has attracted more and more attention in recent years, we refer to some classical references [7, 29, 47] and recent works [6, 12, 25, 32, 33, 34, 35, 51, 65] (see also the references therein).

The paper is organized as follows. In Section 2 we present the main results on the existence and uniqueness of solutions to deterministic and stochastic nonlinear evolution equations with time-fractional derivatives. In Section 3 we will apply our main results to stochastic time-fractional porous medium equations and p-Laplace equations as model examples. The well-posedness of both equations have been open problems even in the deterministic case.

2 Main results

Let $(H, \langle \cdot, \cdot \rangle_H)$ be a real separable Hilbert space identified with its dual space H^* by the Riesz isomorphism. Let V be a real reflexive Banach space, continuously and densely embedded into H. Then we have the following Gelfand triple

$$V \subseteq H \equiv H^* \subseteq V^*.$$

Let $_{V^*}\langle \cdot, \cdot \rangle_V$ denote the dualization between V and its dual space V^*. Then it is easy to show that

$$_{V^*}\langle u, v \rangle_V = \langle u, v \rangle_H, \quad u \in H, v \in V.$$

Now, for $T \in [0, \infty)$ fixed, we consider the following general nonlinear evolution equation with time-fractional derivative

(2.1) $$\partial_t^\beta (u(t) - x) + A(t, u(t)) = f(t), \ 0 < t < T,$$

where

$$\partial_t^\beta (u - x) := \frac{1}{\Gamma(1 - \beta)} \frac{d}{dt} \int_0^t (t - s)^{-\beta} (u(s) - x) ds,$$

is the Riemann–Liouville fractional derivative, which coincides with the Caputo fractional derivative if $u(0) = x$, $A : [0, \infty) \times V \to V^*$ is restrictedly measurable, *i.e.* for each dt-version of $u \in L^1([0, \infty); V)$, $t \mapsto A(t, u(t))$ is V^*-measurable on $[0, \infty)$, and $f \in L^1([0, \infty); V^*)$.

Applying the Riemann–Liouville fractional integral I_t^β, defined by

$$I_t^\beta f(t) := \frac{1}{\Gamma(\beta)} \int_0^t (t - s)^{\beta - 1} f(s) ds =: (g_\beta * f)(t),$$

to equation (2.1) one obtains that

(2.2) $$u(t) = x - \frac{1}{\Gamma(\beta)} \int_0^t (t - s)^{\beta - 1} A(s, u(s)) \, ds + \frac{1}{\Gamma(\beta)} \int_0^t (t - s)^{\beta - 1} f(s) \, ds,$$

for dt-a.e. $t \in [0, \infty)$.

Here we recall that $\frac{d}{dt}$ in the definition of ∂_t^β (see (1.1)) is understood as a weak derivative. Defining $\tilde{u}$ to be equal to the right-hand side of (2.2), then this dt-version of u satisfies $\tilde{u}(0) = x$. Concerning the continuity properties of $\tilde{u}$ we refer to Theorem 2.1 below.

4

We also remark here that below we work with functions u only determined dt-a.e., so $u(0)$ can be chosen arbitrarily. Therefore, we write x in (2.1) instead of $u(0)$.

Now let us formulate the precise conditions on the coefficients in (2.1). Suppose for fixed $\alpha > 1$ that there exist constants $\delta > 0$, C and $g \in L^1([0, \infty); \mathbb{R}_+)$ such that the following conditions hold for all $t \in [0, \infty)$ and $v, v_1, v_2 \in V$.

$(H1)$ (Hemicontinuity) The map $s \mapsto {}_{V^*}\langle A(t, v_1 + sv_2), v\rangle_V$ is continuous on $\mathbb{R}$.

$(H2)$ (Monotonicity)
$$ {}_{V^*}\langle A(t, v_1) - A(t, v_2), v_1 - v_2\rangle_V \geq 0. $$

$(H3)$ (Coercivity)
$$ {}_{V^*}\langle A(t, v), v\rangle_V \geq \delta\|v\|_V^\alpha - g(t). $$

$(H4)$ (Growth)
$$ \|A(t, v)\|_{V^*} \leq g(t)^{\frac{\alpha-1}{\alpha}} + C\|v\|_V^{\alpha-1}. $$

We define the following spaces,
$$ \mathcal{V} = L^\alpha([0, \infty); V) \cap L^2([0, \infty); H), $$
$$ \mathcal{H} = L^2([0, \infty); H), $$
$$ \mathcal{V}^* = L^{\frac{\alpha}{\alpha-1}}([0, \infty); V^*) + L^2([0, \infty); H), $$

where $\|\cdot\|_\mathcal{V} := \max(\|\cdot\|_{L^\alpha([0,\infty);V)}, \|\cdot\|_\mathcal{H})$ and for $u \in \mathcal{V}^*$

$$ \|u\|_{\mathcal{V}^*} := \inf\left\{\|u_1\|_{L^{\frac{\alpha}{\alpha-1}}([0,\infty);V^*)} + \|u_2\|_\mathcal{H} : u_1 \in L^{\frac{\alpha}{\alpha-1}}([0, \infty); V^*), u_2 \in \mathcal{H} \text{ such that } u = u_1 + u_2\right\}. $$

Then for $x = 0$ (the case for general initial condition x will then follow easily as we shall see below) the original equation (2.1) can be rewritten in the following form

$$ (2.3) \qquad\qquad \partial_t^\beta u + \mathcal{A}u = f, $$

where
$$ \mathcal{A}: \mathcal{V} \to \mathcal{V}^*; (\mathcal{A}u)(t) = A(t, u(t)), \ t \in [0, \infty). $$

It is easy to see that $\mathcal{A}: \mathcal{V} \to \mathcal{V}^*$ is monotone, coercive and bounded on bounded sets.

We shall see below that ∂_t^β with domain $\{u \in \mathcal{H} \mid r \mapsto |r|^\beta \hat{u}(r) \in L^2(\mathbb{R}; H)\} \cap \mathcal{V}$ is closable as an operator from $\mathcal{V}$ to $\mathcal{V}^*$. Let $(\partial_t^\beta, \mathcal{F})$ denote its closure with norm $\|u\|_\mathcal{F} := \left(\|u\|_\mathcal{V}^2 + \|\partial_t^\beta u\|_{\mathcal{V}^*}^2\right)^{\frac{1}{2}}$, $u \in \mathcal{F}$. Here $\hat{u}$ denotes the Fourier transform of u, considered as a function from $\mathbb{R}$ to H, setting $u \equiv 0$ on $\mathbb{R} \setminus (-\infty, 0)$.

Theorem 2.1. *Suppose that $T \in [0, \infty)$ and $A \colon [0, \infty) \times V \to V^*$ satisfies $(H1)$-$(H4)$. Then for every $x \in V$ and $f \in \mathcal{V}^*$, (2.1) has a unique solution u such that $u - x\varphi \in \mathcal{F}$ for every $\varphi \in L^\alpha([0, \infty); \mathbb{R})$ with $\varphi \equiv 1$ on $[0, T+1]$. In particular,*

$$ (2.4) \qquad\qquad u - x\varphi \in L^\alpha([0, \infty); V); \ \partial_t^\beta(u - x\varphi) \in L^{\frac{\alpha}{\alpha-1}}([0, \infty); V^*) $$

5

and for dt-a.e. $t \in [0, T]$,

$$(2.5) \qquad u(t) = x - \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} A(s, u(s)) \, ds + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s) \, ds.$$

Furthermore, $t \mapsto \frac{1}{\Gamma(1-\beta)} \int_0^t (t-s)^{-\beta} (u(s) - x\varphi(s)) \, ds$ has a continuous H-valued dt-version, and if $\beta \in (\frac{\alpha-1}{\alpha}, 1)$, $t \mapsto u(t)$ has a continuous V^*-valued dt-version.

Remark 2.1. (i) As a matter of fact, the monotonicity assumption $(H2)$ of A is a sufficient condition to imply that $\mathcal{A} : \mathcal{V} \to \mathcal{V}^*$ is pseudo-monotone. The above theorem still holds if we replace $(H2)$ by assuming $\mathcal{A} : \mathcal{V} \to \mathcal{V}^*$ is pseudo-monotone. In [34, 35], a local monotonicity condition is assumed for A which yields that $A(t, \cdot) : V \to V^*$ is pseudo-monotone provided $V \subseteq H$ is a compact embedding. However, it is still not clear whether it implies that $\mathcal{A} : \mathcal{V} \to \mathcal{V}^*$ is pseudo-monotone.

(ii) It is easy to see from the proof that to have uniqueness of solutions one can replace $(H2)$ by the following weak monotonicity:

$$_{V^*}\langle A(t, v_1) - A(t, v_2), v_1 - v_2 \rangle_V + K\|v_1 - v_2\|_H^2 \geq 0,$$

where K is a positive constant.

We first recall the definition of a pseudo-monotone operator, which is a very useful generalization of a monotone operator and was first introduced by Brézis in [8]. We use the notation "$\rightharpoonup$" for weak convergence in Banach spaces.

Definition 2.1. The operator $M : \mathcal{V} \to \mathcal{V}^*$ is called pseudo-monotone if $v_n \rightharpoonup v$ in $\mathcal{V}$ as $n \to \infty$ and

$$\limsup_{n \to \infty} {}_{\mathcal{V}^*}\langle M(v_n), v_n - v \rangle_{\mathcal{V}} \leq 0$$

implies for all $u \in \mathcal{V}$

$$_{\mathcal{V}^*}\langle M(v), v - u \rangle_{\mathcal{V}} \leq \liminf_{n \to \infty} {}_{\mathcal{V}^*}\langle M(v_n), v_n - u \rangle_{\mathcal{V}}.$$

Remark 2.2. Browder introduced a slightly different definition of a pseudo-monotone operator in [11]: An operator $M : \mathcal{V} \to \mathcal{V}^*$ is called pseudo-monotone if $v_n \rightharpoonup v$ in $\mathcal{V}$ as $n \to \infty$ and

$$\limsup_{n \to \infty} {}_{\mathcal{V}^*}\langle M(v_n), v_n - v \rangle_{\mathcal{V}} \leq 0$$

implies

$$M(v_n) \rightharpoonup M(v) \quad \text{and} \quad \lim_{n \to \infty} {}_{\mathcal{V}^*}\langle M(v_n), v_n \rangle_{\mathcal{V}} = {}_{\mathcal{V}^*}\langle M(v), v \rangle_{\mathcal{V}}.$$

In particular, if M is bounded on bounded sets, then these two definitions are equivalent, we refer to [34, 35].

Lemma 2.2. *If $M : \mathcal{V} \to \mathcal{V}^*$ is pseudo-monotone, bounded on bounded sets and coercive, then M is surjective, i.e. for any $f \in \mathcal{V}^*$, the equation $Mu = f$ has a solution.*

Proof. This is a classical result due to Brézis. For the proof we refer to [8] or [64, Theorem 27.A]. □

As in [57] we consider a generator Λ, with domain $D(\Lambda, \mathcal{H})$, of a C_0-contraction semigroup of linear operators on $\mathcal{H}$ whose restrictions to $\mathcal{V}$ form a C_0-semigroup of linear operators on $\mathcal{V}$. The generator of the latter is again Λ, but with domain $D(\Lambda, \mathcal{V}) := \{u \in \mathcal{V} \mid \Lambda u \in \mathcal{V}\}$. Then $D(\Lambda, \mathcal{V})$ is dense in $\mathcal{V}$, hence so is $D(\Lambda, \mathcal{H}) \cap \mathcal{V}$.

By [57, Lemma 2.3], $\Lambda \colon D(\Lambda, \mathcal{H}) \cap \mathcal{V} \to \mathcal{V}^*$ is closable as an operator from $\mathcal{V}$ to $\mathcal{V}^*$. Denoting its closure by $(\Lambda, \mathcal{F})$, $\mathcal{F}$ is a Banach space with norm $\|u\|_{\mathcal{F}} := (\|u\|_{\mathcal{V}}^2 + \|\Lambda u\|_{\mathcal{V}^*}^2)^{\frac{1}{2}}$, $u \in \mathcal{F}$.

We will use the following abstract result in which Λ will later be taken to be $-\partial_t^\beta$ to solve equation (2.1). This abstract result is a generalization of [57, Proposition 3.2]. We replace the strong monotonicity assumption in [57, Proposition 3.2] by the classical monotonicity $(H2)$ (see also Remark 2.1) and consider a reflexive Banach space $\mathcal{V}$ in place of the Hilbert space $\mathcal{V}$ in [57].

Theorem 2.3. *For any $f \in \mathcal{V}^*$, there exists a solution $u \in \mathcal{F}$ of the equation $\mathcal{A}u - \Lambda u = f$.*

Proof. **Step 1**: For $\alpha > 0$, consider the Yosida approximation $\Lambda_\alpha \colon \mathcal{V} \to \mathcal{V}^*$ defined by

$$\Lambda_\alpha = \alpha(\alpha V_\alpha - I),$$

where $V_\alpha = (\alpha - \Lambda)^{-1}$, $\alpha > 0$, is the resolvent of $(\Lambda, D(\Lambda, \mathcal{H}))$ (on $\mathcal{H}$).

Note that $\mathcal{A} - \Lambda_\alpha$ is pseudo-monotone, coercive and bounded on bounded sets. By Lemma 2.2 there exists $u_\alpha \in \mathcal{V}$ such that $\mathcal{A}u_\alpha - \Lambda_\alpha u_\alpha = f$.

Step 2: Note that

$$_{\mathcal{V}^*}\langle \mathcal{A}u_\alpha, u_\alpha \rangle_{\mathcal{V}} \le {}_{\mathcal{V}^*}\langle \mathcal{A}u_\alpha - \Lambda_\alpha u_\alpha, u_\alpha \rangle_{\mathcal{V}} = {}_{\mathcal{V}^*}\langle f, u_\alpha \rangle_{\mathcal{V}} \le \|f\|_{\mathcal{V}^*}\|u_\alpha\|_{\mathcal{V}}.$$

Hence, by the coercivity assumption $(H3)$ we obtain that $\sup_{\alpha>0} \|u_\alpha\|_{\mathcal{V}} < \infty$, and hence $\sup_{\alpha>0} \|\mathcal{A}u_\alpha\|_{\mathcal{V}^*} < \infty$ by $(H4)$.

Since for any $v \in \mathcal{V}$

$$\begin{aligned} _{\mathcal{V}^*}\langle \Lambda_\alpha u_\alpha, v \rangle_{\mathcal{V}} &= -{}_{\mathcal{V}^*}\langle \mathcal{A}u_\alpha - \Lambda_\alpha u_\alpha, v \rangle_{\mathcal{V}} + {}_{\mathcal{V}^*}\langle \mathcal{A}u_\alpha, v \rangle_{\mathcal{V}} \\ &= -{}_{\mathcal{V}^*}\langle f, v \rangle_{\mathcal{V}} + {}_{\mathcal{V}^*}\langle \mathcal{A}u_\alpha, v \rangle_{\mathcal{V}} \\ &\le (\|f\|_{\mathcal{V}^*} + \|\mathcal{A}u_\alpha\|_{\mathcal{V}^*})\|v\|_{\mathcal{V}}, \end{aligned}$$

we have $\sup_{\alpha>0} \|\Lambda_\alpha u_\alpha\|_{\mathcal{V}^*} < \infty$.

By the apriori estimates above we know there exists a subsequence $\alpha_n \to \infty$ such that

$$\begin{aligned} u_{\alpha_n} &\rightharpoonup u \ \text{ in } \mathcal{V}; \\ \mathcal{A}u_{\alpha_n} &\rightharpoonup h \ \text{ in } \mathcal{V}^*; \\ \Lambda_{\alpha_n} u_{\alpha_n} &\rightharpoonup g \ \text{ in } \mathcal{V}^*. \end{aligned}$$

So, it is easy to see that $h - g = f$.

Notice that Λ is a linear operator and

$$\alpha_n V_{\alpha_n} u_{\alpha_n} \rightharpoonup u \quad \text{in } \mathcal{V}.$$

We have

$$\Lambda \alpha_n V_{\alpha_n} u_{\alpha_n} \rightharpoonup \Lambda u \quad \text{in } \mathcal{V}^*.$$

Since $\Lambda \alpha_n V_{\alpha_n} u_{\alpha_n} = \Lambda_{\alpha_n} u_{\alpha_n}$, we know that $\Lambda u = g$. Note that $\|\alpha_n V_{\alpha_n} u_{\alpha_n}\|_{\mathcal{V}} \leq 2C\|u_{\alpha_n}\|_{\mathcal{V}}$, hence we have

$$\sup_n \|\alpha_n V_{\alpha_n} u_{\alpha_n}\|_{\mathcal{F}} < \infty,$$

which implies that $u \in \mathcal{F}$. Recall that $\|u\|_{\mathcal{F}}^2 := \|u\|_{\mathcal{V}}^2 + \|\Lambda u\|_{\mathcal{V}^*}^2$.

Step 3: Now we only need to show $\mathcal{A}u = h$.

Since $u_{\alpha_n} \rightharpoonup u$ and $\Lambda_{\alpha_n} u \to \Lambda u$,

$$\limsup_{n \to \infty} {}_{\mathcal{V}^*}\langle \mathcal{A}u_{\alpha_n}, u_{\alpha_n} - u \rangle_{\mathcal{V}}$$
$$= \limsup_{n \to \infty} {}_{\mathcal{V}^*}\langle \Lambda_{\alpha_n} u_{\alpha_n} + f, u_{\alpha_n} - u \rangle_{\mathcal{V}}$$
$$= \limsup_{n \to \infty} {}_{\mathcal{V}^*}\langle \Lambda_{\alpha_n} u_{\alpha_n}, u_{\alpha_n} - u \rangle_{\mathcal{V}}$$
$$\leq \limsup_{n \to \infty} {}_{\mathcal{V}^*}\langle \Lambda_{\alpha_n} u, u_{\alpha_n} - u \rangle_{\mathcal{V}}$$
$$= {}_{\mathcal{V}^*}\langle \Lambda u, u - u \rangle_{\mathcal{V}} = 0,$$

where the inequality follows from ${}_{\mathcal{V}^*}\langle \Lambda_{\alpha_n}(u_{\alpha_n} - u), u_{\alpha_n} - u \rangle_{\mathcal{V}} \leq 0$.

So, we have

$$\limsup_{n \to \infty} {}_{\mathcal{V}^*}\langle \mathcal{A}u_{\alpha_n}, u_{\alpha_n} \rangle_{\mathcal{V}} \leq {}_{\mathcal{V}^*}\langle h, u \rangle_{\mathcal{V}}.$$

Hence, by the pseudomonotonicity, we have for any $w \in \mathcal{V}$

$$_{\mathcal{V}^*}\langle \mathcal{A}u, u - w \rangle_{\mathcal{V}}$$
$$\leq \liminf_{n \to \infty} {}_{\mathcal{V}^*}\langle \mathcal{A}u_{\alpha_n}, u_{\alpha_n} - w \rangle_{\mathcal{V}}$$
$$\leq \liminf_{n \to \infty} {}_{\mathcal{V}^*}\langle \mathcal{A}u_{\alpha_n}, u_{\alpha_n} \rangle_{\mathcal{V}} - {}_{\mathcal{V}^*}\langle h, w \rangle_{\mathcal{V}}$$
$$\leq {}_{\mathcal{V}^*}\langle h, u - w \rangle_{\mathcal{V}},$$

which implies $\mathcal{A}u = h$ since $w \in \mathcal{V}$ was arbitrary. $\qquad\square$

Now we want to show that we can take $\Lambda := -\partial_t^\beta$, i.e., we have to show that $-\partial_t^\beta$ generates a C_0-semigroup of contractions on $\mathcal{H}$ which can be restricted to a C_0-semigroup on $\mathcal{V}$. To this end, let us define the following "shift to the right" semigroup U_t, $t > 0$, on $\mathcal{H}$. For $f \in \mathcal{H}$, $t \geq 0$, define

$$(2.6) \qquad U_t f(r) := \mathbb{1}_{[0,\infty)}(r - t)f(r - t), \ r \in [0, \infty).$$

Then it is trivial to check that $(U_t)_{t>0}$ is a C_0-contraction semigroup on $\mathcal{H}$ and it obviously can be restricted to a C_0-semigroup on $\mathcal{V}$ (even in this case consisting also of contractions on $\mathcal{V}$). Now fix $\beta \in (0,1)$ and define for $f \in \mathcal{H}$

$$(2.7) \qquad U_t^\beta f := \int_0^\infty U_s f \, \eta_t^\beta(ds), \ t \geq 0,$$

where $(\eta_t^\beta)_{t>0}$ denotes the one-sided stable semigroup (of probability measures on $([0,\infty), \mathcal{B}([0,\infty)))$) of order β, i.e. we have for its Fourier transform

$$\hat{\eta}_t^\beta(s) = \int_0^\infty e^{isr} \, \eta_t^\beta(dr) = e^{-t(is)^\beta}, \ s \in \mathbb{R}, \ t > 0.$$

We note here that

$$(2.8) \qquad (is)^\beta = [\cos(\frac{\beta\pi}{2}\mathrm{sign}(s)) + i\sin(\frac{\beta\pi}{2}\mathrm{sign}(s))]|s|^\beta, \ s \in \mathbb{R},$$

hence $\eta_t^\beta(ds)$ is absolutely continuous with respect to Lebesgue measure ds. It is a well-known fact (see e.g. [38, Chap. II, Sect. 4b]), that $(U_t^\beta)_{t>0}$ is also a C_0-semigroup of contractions on $\mathcal{H}$ which obviously can be restricted to a C_0-semigroup on $\mathcal{V}$ (again consisting of contractions).

Proposition 2.4. *The generator $(\Lambda, D(\Lambda, \mathcal{H}))$ of $(U_t^\beta)_{t>0}$ (on $\mathcal{H}$) is given as follows*

$$D(\Lambda, \mathcal{H}) = \{u \in \mathcal{H} \mid r \mapsto |r|^\beta \hat{u}(r) \in L^2(\mathbb{R}; H)\},$$

$$(\Lambda u)^\wedge(r) = -(ir)^\beta \hat{u}(r), \ r \in \mathbb{R},$$

where $\hat{u}$ denotes the Fourier transform of u and is considered as a function from $\mathbb{R}$ to H. In particular, if $\beta > \frac{1}{2}$, then each $u \in D(\Lambda, \mathcal{H})$ has a dt-version $\tilde{u}$ such that $[0,\infty) \ni t \mapsto \tilde{u}(t) \in H$ is continuous and $\tilde{u}(0) = 0$.

Proof. Below we consider each $\eta_t^\beta(ds)$ as a measure on all of $\mathbb{R}$, by defining

$$\eta_t^\beta(A) := \eta_t^\beta(A \cap [0,\infty)), \ A \in \mathcal{B}(\mathbb{R}),$$

and any function f in $\mathcal{H}$ as a function on $\mathbb{R}$ by defining $f := 0$ on $(-\infty, 0)$.

Let $D := \{u \in \mathcal{H} \mid r \mapsto |r|^\beta \hat{u}(r) \in L^2(\mathbb{R}; H)\}$. Then for $u \in D$ we have

$$\frac{1}{t}(U_t^\beta u - u)^\wedge(r) = \frac{1}{t}\Big(\int_0^\infty U_s u \, \eta_t^\beta(ds) - u\Big)^\wedge(r)$$

$$= \frac{1}{t}\Big(\int_{\mathbb{R}} \mathbb{1}_{[0,\infty)}(\cdot - s)u(\cdot - s) \, \eta_t^\beta(ds) - u\Big)^\wedge(r)$$

$$= \frac{1}{t}(u * \eta_t^\beta - u)^\wedge(r)$$

$$= \frac{1}{t}\hat{u}(r)\Big(e^{-t(ir)^\beta} - 1\Big) \xrightarrow[t \to 0]{} -(ir)^\beta \hat{u}(r)$$

9

for dr-a.e. $r \in \mathbb{R}$. But since for all $r \in \mathbb{R}$ and $t > 0$,

$$\frac{1}{t}\left|e^{-t(ir)^\beta} - 1\right| \leq 2|r|^\beta,$$

the last convergence also holds in $\mathcal{H}$. Hence $D \subset D(\Lambda, \mathcal{H})$ and

(2.9) $$(\Lambda u)^\wedge(r) = -(ir)^\beta \hat{u}(r), \quad r \in \mathbb{R}.$$

Similarly, one checks that

(2.10) $$U_t^\beta D \subseteq D \quad \forall t > 0,$$

and that (Λ, D) is closed as an operator from $\mathcal{H}$ to $\mathcal{H}$. Since D is dense in $\mathcal{H}$, (2.10) implies (see [50, Theorem X.49]) that D is an operator core of $(\Lambda, D(\Lambda, \mathcal{H}))$. Consequently, $D = D(\Lambda, \mathcal{H})$ and Λ is given by (2.9). Furthermore, we note that for all $\beta > 0$

$$D(\Lambda, \mathcal{H}) \subset H^\beta(\mathbb{R}; H)$$

by definition of the fractional Sobolev space $H^\beta(\mathbb{R}; H)(\subset L^2(\mathbb{R}; H))$, which consists of (Hölder-) continuous functions on $\mathbb{R}$ if $\beta > \frac{1}{2}$. Therefore, if $\beta > \frac{1}{2}$, every $u \in D(\Lambda, \mathcal{H})$ has a continuous version $\tilde{u}$ on $\mathbb{R}$ which is zero on $(-\infty, 0)$, hence $\tilde{u}(0) = 0$. $\square$

Next we shall prove a representation formula for Λ in terms of the Caputo derivative ∂_t^β. To this end, we first define a domain for ∂_t^β which is convenient in our case. Let for $\beta \in (0, 1)$

(2.11) $$D(\partial_t^\beta) := \{u \in L^1([0, \infty); V^*) \mid g_{1-\beta} * u \in W^{1,1}([0, T], V^*) \text{ for all } T \in (0, \infty)\},$$

where we define for $u \in L^1([0, \infty); V^*)$

$$(g_{1-\beta} * u)(t) := \frac{1}{\Gamma(1 - \beta)} \int_0^t (t - s)^{-\beta} u(s) \, ds.$$

Obviously, for all $T \in (0, \infty)$

$$g_{1-\beta} * u = (\mathbb{1}_{[0,T]} g_{1-\beta}) * (\mathbb{1}_{[0,\infty)} u) \quad \text{on } [0, T],$$

where for $p \in [1, \infty)$ the latter function belongs to $L^p(\mathbb{R}; V^*)$ if so does $\mathbb{1}_{[0,\infty)} u$, and is in $C(\mathbb{R}; V^*)$, if in addition $\beta \in (0, \frac{p-1}{p})$.

Lemma 2.5. *Let $u \in D_0 := D(\Lambda, \mathcal{H}) \cap \mathcal{V} \cap D(\partial_t^\beta) \cap L^\infty([0, \infty); H)$. Then*

(2.12) $$\Lambda u = -\frac{d}{dt}(g_{1-\beta} * u)$$

and D_0 is an operator core for $(\Lambda, D(\Lambda, \mathcal{H}))$.

Proof. First we note that for $T \in [0, \infty)$

(2.13) $\qquad \|(g_{1-\beta} * u)(t)\|_{V^*} \leq \dfrac{\mathrm{esssup}_{s \in [0,T]} \|u(s)\|_{V^*}}{\Gamma(1-\beta)(1-\beta)} t^{1-\beta} \qquad$ for dt-a.e. $t \in [0, T]$,

and the same inequality holds with $\| \cdot \|_H$ replacing $\| \cdot \|_{V^*}$.

From now on we consider all appearing functions, originally only defined on $[0, \infty)$, as functions on all of $\mathbb{R}$ by defining them to be equal to zero on $\mathbb{R} \setminus [0, \infty)$. As in the proof of Proposition 2.4, one can check that $U_t^\beta(D_0) \subset D_0$ (for which, however, it is essential that $\eta_t^\beta(ds)$ is absolutely continuous with respect to ds) and that D_0 is dense in $\mathcal{H}$. Again applying Theorem X.49 from [50] we obtain that D_0 is an operator core of $(\Lambda, D(\Lambda, \mathcal{H}))$. Hence it remains to prove (2.12).

Let us start with calculating the Laplace transform $\mathcal{L}$ of the right hand side of (2.12) for any $u \in D(\partial_t^\beta) \cap L^\infty([0, \infty); V^*)$. So let $\lambda \in (0, \infty)$. Then integrating by parts, using (2.13) and Fubini's Theorem we obtain

$$\mathcal{L}\left(\frac{d}{dt}(g_{1-\beta} * u)\right)(\lambda) = \int_0^\infty e^{-\lambda t} \frac{d}{dt}(g_{1-\beta} * u)(t)\, dt$$

$$= \lim_{T \to \infty} \left(e^{-\lambda T}(g_{1-\beta} * u)(T) + \lambda \int_0^T e^{-\lambda t}(g_{1-\beta} * u)(t)\, dt \right)$$

$$= \frac{\lambda}{\Gamma(1-\beta)} \int_0^\infty e^{-\lambda t} \int_0^t (t-s)^{-\beta} u(s)\, ds\, dt$$

$$= \frac{\lambda}{\Gamma(1-\beta)} \int_0^\infty \int_s^\infty (t-s)^{-\beta} e^{-\lambda t}\, dt\, u(s)\, ds$$

$$= \frac{\lambda^\beta}{\Gamma(1-\beta)} \int_0^\infty e^{-\lambda s} u(s)\, ds\, \lambda \int_0^\infty (\lambda t)^{-\beta} e^{-\lambda t}\, dt$$

$$= \lambda^\beta \mathcal{L}u(\lambda), \quad \lambda \in (0, \infty).$$

For the left-hand side of (2.12) and $u \in D_0$ we find for all $h \in H$, $\lambda \in (0, \infty)$, since $\mathcal{L}(\eta_t^\beta)(\lambda) = e^{-t\lambda^\beta}$

$$\left\langle \int_0^\infty \Lambda u(r) e^{-\lambda r}\, dr, h \right\rangle_H = \lim_{t \to 0} \frac{1}{t} \int_0^\infty \left\langle U_t^\beta u(r) - u(r), h \right\rangle_H e^{-\lambda r}\, dr$$

$$= \lim_{t \to 0} \frac{1}{t} \Big(\mathcal{L}(\langle u, h \rangle_H * \eta_t^\beta) - \mathcal{L}(\langle u, h \rangle_H) \Big)(\lambda)$$

$$= \lim_{t \to 0} \frac{1}{t} (e^{-t\lambda^\beta} - 1) \mathcal{L}(\langle u, h \rangle_H)(\lambda)$$

$$= -\lambda^\beta \left\langle \mathcal{L}u(\lambda), h \right\rangle_H.$$

Hence, $\mathcal{L}(\Lambda(u))(\lambda) = -\lambda^\beta \mathcal{L}(u)(\lambda)$ and (2.12) follows. $\qquad \square$

Proposition 2.6. *Let $u \in \mathcal{F}$. Then $u \in D(\partial_t^\beta)$ and*

$$\Lambda u = -\frac{d}{dt}(g_{1-\beta} * u).$$

In particular, $g_{1-\beta} * u \in C([0, \infty), H)$.

Proof. By Lemma 2.5 there exist $u_n \in D_0$, $n \in \mathbb{N}$, such that as $n \to \infty$

$$u_n \longrightarrow u \text{ in } \mathcal{V}$$

and

$$-\partial_t^\beta u_n = \Lambda u_n \longrightarrow \Lambda u \text{ in } \mathcal{V}^*.$$

Let $T \in (0, \infty)$. Obviously, $g_{1-\beta} * u_n \longrightarrow g_{1-\beta} * u$ in $L^\alpha([0,T];V)$ as $n \to \infty$. Therefore, by completeness $g_{1-\beta} * u \in W^{1,\frac{\alpha}{\alpha-1}}([0,T];\mathcal{V}^*)$ and

$$\Lambda u = -\frac{d}{dt}(g_{1-\beta} * u) \text{ on } [0,T].$$

The last part of the assertion then follows by [4, Theorem 1.19, pp.25]. □

Proof of Theorem 2.1: Existence: Let φ be as in the assertion of the theorem. Define $\mathcal{A}_x$ as $\mathcal{A}$, but with

$$A_x(t,v) := A(t, v + x\varphi(t)), \ t > 0, \ v \in V,$$

replacing A. Then by Theorem 2.3 and Proposition 2.6 there exist $u_x \in \mathcal{F}$ such that

$$\frac{d}{dt}(g_{1-\beta} * u_x) + A_x u_x = f$$

Define $u := u_x + x\varphi$. Then $u - x\varphi(= u_x)$ satisfies (2.4) and

$$\partial_t^\beta(u - x) + \mathcal{A}u = f \quad dt\text{-a.e. on } (0,T)$$

and (2.1) is solved. The rest assertions of the Theorem follows from the last part of the assertion of Proposition 2.6 and an elementary fact about convolutions.

Uniqueness: the uniqueness proof is similar to the argument in [62, Theorem 3.1]. So, we only give a brief account of the proof here.

Recall that Riemann–Liouville fractional integral $I_t^\beta f := g_\beta * f$ and the definition of the Riemann–Liouville kernel

$$g_\beta(t) = \frac{t^{\beta-1}}{\Gamma(\beta)}, \ t > 0.$$

Suppose that u_1, u_2 are solutions to (2.1), then $u := u_1 - u_2$ is a solution to the following equation

$$\partial_t^\beta u + A(t, u_1) - A(t, u_2) = 0, \ u(0) = 0.$$

Let $g^n \in W^{1,1}$, $n \in \mathcal{N}$ be the kernel associated with the Yosida approximation of the operator ∂_t^β (cf.[60, 61]). Then by [60, Theorem 2.1] we have

$$
\begin{aligned}
\frac{d}{dt}(g^n * \|u(\cdot)\|_H^2)(t) &\leq 2\langle \frac{d}{dt}(g^n * u)(t), u(t)\rangle_H \\
&=2\ _{V^*}\langle \frac{d}{dt}(g_{1-\beta} * u)(t), u(t)\rangle_V + 2\ _{V^*}\langle \frac{d}{dt}(g^n * u)(t) - \frac{d}{dt}(g_{1-\beta} * u)(t), u(t)\rangle_V \\
&=2\ _{V^*}\langle A(t, u_1(t)) - A(t, u_2(t)), u(t)\rangle_V + 2\ _{V^*}\langle \frac{d}{dt}(g^n * u)(t) - \frac{d}{dt}(g_{1-\beta} * u)(t), u(t)\rangle_V \\
&\leq 2\ _{V^*}\langle \frac{d}{dt}(g^n * u)(t) - \frac{d}{dt}(g_{1-\beta} * u)(t), u(t)\rangle_V =: 2h^n(t).
\end{aligned}
$$

(2.14)

Note that $h^n \to 0$ in $L^1([0,T])$, which yields that $g_\beta * h^n \to 0$ in $L^1([0,T])$ (cf. [60]). Moreover, we have

$$g_\beta * \frac{d}{dt}(g^n * \|u(\cdot)\|_H^2) = \frac{d}{dt}(g^n * g_\beta * \|u(\cdot)\|_H^2) \to \frac{d}{dt}(g_{1-\beta} * g_\beta * \|u(\cdot)\|_H^2) = \|u(\cdot)\|_H^2$$

in $L^1([0,T])$ as $n \to \infty$.

Hence, by applying convolution with the kernel g_β to (2.14) we have

$$\|u(t)\|_H^2 \le 0, \ a.e.t \in [0,T],$$

which implies that $u_1 = u_2$, i.e. the solution to (2.1) is unique. $\qquad\square$

Suppose that U is a Hilbert space and $W(t)$ is a U-valued cylindrical Wiener process defined on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$. Now we consider stochastic nonlinear evolution equations with fractional time derivative of type

$$(2.15) \qquad \partial_t^\beta X(t) + A(t, X(t)) = \partial_t^\gamma \int_0^t B(s)dW(s), \ 0 < t < T,$$

where $\gamma \in (0,1)$, $B : [0,T] \to L_{HS}(U;V)$ is measurable (here $(L_{HS}(U;V), \|\cdot\|_{HS})$ denotes the space of all Hilbert-Schmidt operators from U to V).

We want to explain the reason why this form of random noise term is used in (2.15). In fact, heuristically

$$(2.16) \qquad \partial_t^\gamma \int_0^t B(s)dW(s) = I_t^{1-\gamma}[B(t)dW(t)]$$

which can be used to model systems subject to classical random noise (the case $\gamma = 1$) or random noise with certain memory effects (the case $\gamma < 1$). This form of noise will naturally appear if *e.g.* you consider the heat transfer with random effects with memory (see [18]) or use the time fractional Duhamel's principle to derive the appropriate form of stochastic time-fractional diffusion equations (see [43] for the case $\gamma = \beta$). The case $\gamma = \beta$ is investigated in many papers about stochastic heat type or wave type equations, see e.g. [15, 17, 24, 43, 44], and the case $\gamma = 1$ is studied in [16, 26].

In this work we consider the general case $\gamma < \beta + \frac{1}{2}$, this assumption is natural since the stochastic integral term in (2.16) is well-defined if and only if (*e.g.* assuming $B(\cdot)$ is bounded)

$$\int_0^T (t-s)^{2(\beta-\gamma)} \|B(s)\|_{HS}^2 ds \le C \int_0^T (t-s)^{2(\beta-\gamma)} ds < \infty,$$

which is equivalent to $\gamma < \beta + \frac{1}{2}$. The same assumption $\gamma < \beta + \frac{1}{2}$ is also used in [18] which is deduced there by a differentiability argument.

Theorem 2.7. *Suppose that $\gamma \in (0,1]$, $T \in [0,\infty)$ and that A satisfies (H1)-(H4), $B \in L^\infty([0,T], L_{HS}(U;V))$ if $\gamma < \beta + \frac{1}{2}$ or $B \in L^2([0,T], L_{HS}(U;V))$ if $\gamma \le \beta$. Then for every*

$x \in V$ (2.15) has a unique $\mathcal{F}_t$-adapted solution X such that $X - x\varphi \in \mathcal{F}$, $\mathbb{P}$-a.s. for every $\varphi \in L^\alpha([0,\infty); \mathbb{R})$ with $\varphi \equiv 1$ on $[0, T+1)$. In particular,

$$X - x\varphi \in L^\alpha([0,\infty); V); \ \partial_t^\beta(X - x\varphi) \in L^{\frac{\alpha}{\alpha-1}}([0,\infty); V^*), \ \mathbb{P}\text{-}a.s$$

and for dt-a.e. $t \in [0, T]$,

$$X(t) = x - \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} A(s, X(s)) \, ds + \frac{1}{\Gamma(1+\beta-\gamma)} \int_0^t (t-s)^{\beta-\gamma} B(s) dW(s), \ \mathbb{P}\text{-}a.s.$$

Furthermore, $t \mapsto \frac{1}{\Gamma(1-\beta)} \int_0^t (t-s)^{-\beta}(X(s) - x\varphi(s)) \, ds$ has a continuous H-valued dt-version, and $t \mapsto X(t)$ has a continuous V^*-valued dt-version if $\beta \in (\frac{\alpha-1}{\alpha}, 1)$.

Proof. Define for $t \in (0, \infty)$

$$F(t) := \frac{1}{\Gamma(1+\beta-\gamma)} \int_0^t (t-s)^{\beta-\gamma} B(s) dW(s).$$

Let $u(t) = X(t) - F(t)$, then $u(t)$ satisfies the following equation

(2.17) $$\partial_t^\beta(u(t) - x) + A(t, u(t) + F(t)) = 0, \ 0 < t < T.$$

Define

$$\tilde{A}(t, u) = A(t, u + F(t)), u \in V.$$

Since $F \in C([0, T]; V)$, $\mathbb{P}$-a.s., it is easy to see that $\tilde{A}$ still satisfies $(H1)$-$(H4)$. Hence (2.17) has a unique solution u, which implies that $u + F$ is the unique solution to (2.15).

The $\mathcal{F}_t$-adaptiveness of the solution follows by the proofs of Theorem 2.3 and Lemma 2.2. $\square$

Remark 2.3. From the above proof one can see that Theorem 2.7 also holds for the case that A and B are random coefficients (*i.e.* also depend on $\omega \in \Omega$) in a progressively measurable way [35, Section 4.1].

3   Application to Examples

In this part we will apply our main result to establish the existence and uniqueness of solutions to (stochastic) porous medium equations and p-Laplace equations with time-fractional derivative. Both are open problems even in the deterministic case. In the recent work [61], the authors derive very nice decay estimates for the solution of those equations (by assuming the existence) and the decay behaviour turns out to be notably different from the classical parabolic case. As said, here we give a positive answer to the question concerning the existence and uniqueness of solutions to the time-fractional porous medium equations and p-Laplace equations.

Both examples are just model cases which we present here motivated by [61]. There are many more examples (see e.g. [5] and [51]). In particular, fast diffusion equations are

covered as well. We are confident that we can extend our approach to the case of multi-valued operators as well.

Let $\Lambda \subset \mathbb{R}^d$ be an open bounded domain and Δ be the Laplace operator, and for $p \in [1, \infty[$ we use $L^p(\Lambda)$ and $H_0^{n,p}(\Lambda)$ to denote the pth-integrable Lebesgue space and the Sobolev space of order n in $L^p(\Lambda)$ with *Dirichlet boundary conditions*. Recall that X^* denotes the dual space of a Banach space X.

3.1 Time-fractional porous medium equation

We first introduce the porous medium operator $A(u) := \Delta\Psi(u)$. Let $\Psi : \mathbb{R} \to \mathbb{R}$ be a function having the following properties:

(Ψ1) Ψ is continuous.

(Ψ2) For all $s, t \in \mathbb{R}$
$$(t - s)(\Psi(t) - \Psi(s)) \geq 0.$$

(Ψ3) There exist $p \in [2, \infty[$, $c_1 \in]0, \infty[$, $c_2 \in [0, \infty[$ such that for all $s \in \mathbb{R}$
$$s\Psi(s) \geq c_1|s|^p - c_2.$$

(Ψ4) There exist $c_3, c_4 \in]0, \infty[$ such that for all $s \in \mathbb{R}$
$$|\Psi(s)| \leq c_3|s|^{p-1} + c_4,$$

where p is as in (Ψ3).

Now we consider the following Gelfand triple
$$V := L^p(\Lambda) \subseteq H := (H_0^{1,2}(\Lambda))^* \subseteq V^* := (L^p(\Lambda))^*$$

and stochastic time-fractional porous medium equation

(3.1) $$\partial_t^\beta X(t) = \Delta\Psi(X(t)) + \partial_t^\gamma \int_0^t B(s)\mathrm{d}W(s), \ 0 < t < T.$$

Remark 3.1. Since here H is not taken as $L^2(\Lambda)$, by the definition of the Gelfand triple, one should note that
$$V^* \neq L^{\frac{p}{p-1}}(\Lambda).$$
In fact, one can prove that (see [35, Lemma 4.1.13])
$$V^* = \Delta(L^{\frac{p}{p-1}}(\Lambda)).$$

Theorem 3.1. *Suppose that Ψ satisfies (Ψ1)-(Ψ4), $B \in L^\infty([0, T], L_{HS}(U; V))$ if $\gamma < \beta + \frac{1}{2}$ or $B \in L^2([0, T], L_{HS}(U; V))$ if $\gamma \leq \beta$. Then Theorem 2.7 applies with $A := \Delta\Psi(\cdot)$ and $\alpha := p$.*

Proof. Now we can define the *porous medium operator* $A : V \to V^*$ by

$$(3.2) \qquad A(u) := \Delta\Psi(u), \quad u \in L^p(\Lambda).$$

Note that by [35, Lemma 4.1.13] the operator A is well-defined. The conclusion follows directly from Theorem 2.7 by checking (H1)–(H4) hold for $-A$.

In fact, it is well-known that the porous medium operator satisfies the monotonicity and coercivity properties (see e.g. [35, Section 4.1]). We include the proof here for the reader's convenience.

(H1): Let $u, v, x \in V = L^p(\Lambda)$ and $\lambda \in \mathbb{R}$. Then by

$$
\begin{aligned}
{}_{V^*}\langle A(u + \lambda v), x\rangle_V &= {}_{V^*}\langle \Delta\Psi(u + \lambda v), x\rangle_V \\
(3.3) \\
&= -\int_\Lambda \Psi(u(\xi) + \lambda v(\xi)) x(\xi) \mathrm{d}\xi.
\end{aligned}
$$

By ($\Psi 4$) for $|\lambda| \le 1$ the integrand in the right-hand side of (3.3) is bounded by

$$[c_4 + c_3 2^{p-2}(|u|^{p-1} + |v|^{p-1})]|x|$$

which by Hölder's inequality is in $L^1(\Lambda)$. So, (H1) follows by ($\Psi 1$) and Lebesgue's dominated convergence theorem.

(H2): Let $u, v \in V = L^p(\Lambda)$. Then

$$
\begin{aligned}
{}_{V^*}\langle A(u) - A(v), u - v\rangle_V &= {}_{V^*}\langle \Delta(\Psi(u) - \Psi(v)), u - v\rangle_V \\
&= -\int_\Lambda [\Psi(u(\xi)) - \Psi(v(\xi))](u(\xi) - v(\xi)) \mathrm{d}\xi \\
&\le 0,
\end{aligned}
$$

where we used ($\Psi 2$) in the last step.

(H3): Let $v \in L^p(\Lambda) = V$. Then by ($\Psi 3$)

$$
\begin{aligned}
{}_{V^*}\langle A(v), v\rangle_V &= -\int_\Lambda \Psi(v(\xi)) v(\xi) \mathrm{d}\xi \\
&\le \int_\Lambda (-c_1 |v(\xi)|^p + c_2) \mathrm{d}\xi.
\end{aligned}
$$

Hence (H3) is satisfied with $\delta := c_1$, $\alpha = p$ and $g(t) = c_2 |\Lambda|$.

(H4): Let $v \in L^p(\Lambda) = V$. Then by [35, Lemma 4.1.13] and ($\Psi 4$)

$$
\begin{aligned}
\|A(v)\|_{V^*} &= \|\Delta\Psi(v)\|_{V^*} \\
&= \|\Psi(v)\|_{L^{\frac{p}{p-1}}} \\
&\le c_4 |\Lambda|^{\frac{p-1}{p}} + c_3 \left(\int |v(\xi)|^p \mathrm{d}\xi\right)^{\frac{p-1}{p}} \\
&= c_4 |\Lambda|^{\frac{p-1}{p}} + c_3 \|v\|_V^{p-1},
\end{aligned}
$$

so (H4) holds with $\alpha = p$.

16

$\square$

Remark 3.2. (*i*) For $p \in [2, \infty[$ and $\Psi(s) := s|s|^{p-2}$ we have

$$A(v) = \Delta(v|v|^{p-2}), \ v \in L^p(\Lambda),$$

which is the non-linear operator appearing in the classical porous medium equation (cf. e.g.[58, 59]), *i.e.*

$$\frac{\partial u(t)}{\partial t} = \Delta(u(t)|u(t)|^{p-2}), \quad u(0, \cdot) = u_0,$$

whose solution describes the time evolution of the density $u(t)$ of a substance in a porous medium. And as a matter of fact, the time-fractional porous medium equation was first introduced by Caputo in [13] to describe the diffusion of fluids in porous media with memory.

(*ii*) As mentioned before, our results apply to the situation in [51] where $\Delta\Psi(u)$ is replaced by $L\Psi(u)$, here L is the generator of a transient Dirichlet form on $L^2(E, \mathcal{E}, \mu)$ for abstract σ-finite measure spaces $(E, \mathcal{E}, \mu)$, so (2.15) takes the form

$$\partial_t^\beta X(t) - L\Psi(X(t)) = \partial_t^\gamma \int_0^t B(s)\mathrm{d}W(s).$$

This, in particular, includes "fractal"(*i.e.* non-local) operators L, as *e.g.* fractional Laplacian $-(-\Delta)^\alpha$, $\alpha \in (0, 1] \cap (0, \frac{d}{2})$, and the underlying domain Λ may be unbounded, e.g. $\Lambda = \mathbb{R}^d$, $d \geq 3$. Ψ may belong to the more general class described in [51], but must be monotone. In particular, $\Psi(r) := r|r|^{m-1}$ with $m \in (0, 1]$ is covered, *i.e.* the fast diffusion equation. So, equations as

$$\partial_t^\beta X(t) + (-\Delta)^\alpha \left(X(t)|X(t)|^{m-1}\right) = \partial_t^\gamma \int_0^t B(s)\mathrm{d}W(s)$$

for $m \in (0, \infty)$ are covered.

3.2 Time-fractional p-Laplace equation

Now we consider stochastic time-fractional p-Laplace equations ($p \geq 2$)

(3.4) $\qquad \partial_t^\beta X(t) = \mathrm{div}\left(|\nabla X(t)|^{p-2}\nabla X(t)\right) + \partial_t^\gamma \int_0^t B(s)\mathrm{d}W(s), \ 0 < t < T.$

We will choose the follwing Gelfand triple

$$V := H_0^{1,p}(\Lambda) \subseteq H := L^2(\Lambda) \subseteq V^* = (H_0^{1,p}(\Lambda))^*$$

and define $A : H_0^{1,p}(\Lambda) \to H_0^{1,p}(\Lambda)^*$ by

(3.5) $\qquad\qquad A(u) := \mathrm{div}(|\nabla u|^{p-2}\nabla u), \ u \in H_0^{1,p}(\Lambda);$

more precisely, given $u \in H_0^{1,p}(\Lambda)$, then we define

(3.6) $\qquad {}_{V^*}\langle A(u), v\rangle_V := -\int_\Lambda |\nabla u(\xi)|^{p-2}\langle \nabla u(\xi), \nabla v(\xi)\rangle\mathrm{d}\xi \quad \text{for all } v \in H_0^{1,p}(\Lambda).$

17

Here A is called the *p-Laplacian*, also denoted by Δ_p. Note that $\Delta_2 = \Delta$.

It is well-known that $A : V \to V^*$ is well-defined (see e.g. [35, Section 4.1]). In fact, we only need to show that the right-hand side of (3.6) defines a linear functional in $v \in V$ which is continuous with respect to $\| \ \|_V = \| \ \|_{1,p}$. First we recall that $\nabla u \in L^p(\Lambda; \mathbb{R}^d)$ for all $u \in H_0^{1,p}(\Lambda)$. Hence by Hölder's inequality

$$\int |\nabla u(\xi)|^{p-1} |\nabla v(\xi)| \mathrm{d}\xi \le \left(\int |\nabla u(\xi)|^p \mathrm{d}\xi \right)^{\frac{p-1}{p}} \left(\int |\nabla v(\xi)|^p \mathrm{d}\xi \right)^{\frac{1}{p}}$$
$$\le \|u\|_{1,p}^{p-1} \|v\|_{1,p}.$$

Since this dominates the absolute value of the right-hand side of (3.6) for all $u \in H_0^{1,p}(\Lambda)$, we have that $A(u)$ is a well-defined element of $(H_0^{1,p}(\Lambda))^*$ and that

(3.7)
$$\|A(u)\|_{V^*} \le \|u\|_V^{p-1}.$$

Theorem 3.2. *Suppose that $p \ge 2$, $B \in L^\infty([0,T], L_{HS}(U;V))$ if $\gamma < \beta + \frac{1}{2}$ or $B \in L^2([0,T], L_{HS}(U;V))$ if $\gamma \le \beta$. Then Theorem 2.7 applies with A as defined in (3.5) and $\alpha := p$.*

Proof. The conclusion follows directly from Theorem 2.7 by checking (H1)-(H4) hold for $-A$ here. Now we include the proof here also for the reader's convenience.

(H1): Let $u, v, x \in H_0^{1,p}(\Lambda)$, then by (3.6) we have to show for $\lambda \in \mathbb{R}$, $|\lambda| \le 1$

(3.8)
$$\lim_{\lambda \to 0} \int_\Lambda \Big(|\nabla(u + \lambda v)(\xi)|^{p-2} \langle \nabla(u + \lambda v)(\xi), \nabla x(\xi) \rangle$$
$$- |\nabla u(\xi)|^{p-2} \langle \nabla u(\xi), \nabla x(\xi) \rangle \Big) \mathrm{d}\xi = 0.$$

Since obviously the integrands converge to zero as $\lambda \to 0 \, \mathrm{d}\xi$-a.e., we only have to find a dominating function to apply Lebesgue's dominated convergence theorem. But obviously, since $|\lambda| \le 1$

$$|\nabla(u + \lambda v)(\xi)|^{p-2} |\langle \nabla(u + \lambda v)(\xi), \nabla x(\xi) \rangle|$$
$$\le 2^{p-2} \left(|\nabla u(\xi)|^{p-1} + |\nabla v(\xi)|^{p-1} \right) |\nabla x(\xi)|$$

and the right-hand side is in $L^1(\Lambda)$ by Hölder's inequality as we have seen above.

(H2): Let $u, v \in H_0^{1,p}(\Lambda)$. Then by (3.6)

$$
\begin{aligned}
- {}_{V^*}&\langle A(u) - A(v), u - v \rangle_V \\
&= \int_\Lambda \langle |\nabla u(\xi)|^{p-2} \nabla u(\xi) - |\nabla v(\xi)|^{p-2} \nabla v(\xi), \nabla u(\xi) - \nabla v(\xi)| > \mathrm{d}\xi \\
&= \int_\Lambda (|\nabla u(\xi)|^p + |\nabla v(\xi)|^p - |\nabla u(\xi)|^{p-2} \langle \nabla u(\xi), \nabla v(\xi) \rangle \\
&\qquad - |\nabla v(\xi)|^{p-2} \langle \nabla u(\xi), \nabla v(\xi) \rangle) \mathrm{d}\xi \\
&\geq \int_\Lambda (|\nabla u(\xi)|^p + |\nabla v(\xi)|^p - |\nabla u(\xi)|^{p-1} |\nabla v(\xi)| \\
&\qquad - |\nabla v(\xi)|^{p-1} |\nabla u(\xi)|) \mathrm{d}\xi \\
&= \int_\Lambda (|\nabla u(\xi)|^{p-1} - |\nabla v(\xi)|^{p-1})(|\nabla u(\xi)| - |\nabla v(\xi)|) \mathrm{d}\xi \\
&\geq 0,
\end{aligned}
$$

since the map $\mathbb{R}_+ \ni s \mapsto s^{p-1}$ is increasing. Hence (H2) is proved.

(H3): Because Λ is bounded by Poincaré's inequality there exists a constant $c = c(p, d, |\Lambda|) \in {]0, \infty[}$ such that

$$
(3.9) \qquad \int_\Lambda |\nabla u(\xi)|^p \mathrm{d}\xi \geq c \int_\Lambda |u(\xi)|^p \mathrm{d}\xi \quad \text{for all } u \in H_0^{1,p}(\Lambda).
$$

Hence by (3.6) for all $u \in H_0^{1,p}(\Lambda)$

$$
{}_{V^*}\langle A(u), u \rangle_V = - \int_\Lambda |\nabla u(\xi)|^p \mathrm{d}\xi \leq -\frac{\min(1, c)}{2} \|u\|_{1,p}^p.
$$

So, (H3) holds with $\alpha = p$.

(H4): This condition holds for A by (3.7) with $\alpha = p$.

$\square$

Remark 3.3. (*i*) It is easy to show that the above result also holds for the more general case where Δ_p is replaced by

$$
A(u) := \operatorname{div}\left(\Psi(|\nabla X(t)|) \nabla X(t) \right),
$$

where Ψ satisfies (Ψ1)-(Ψ4) with some $p > 1$. Moreover, one can further generalize to more general quasilinear differential operator

$$
A(u) := \sum_{|i| \leq m} (-1)^{|i|} D_i A_i(t, x, Du(t, x)), \quad \text{where} Du = (D_j u)_{|j| \leq m}.
$$

Under certain assumptions the above operator also satisfies (H1)-(H4) (cf. e.g. [64, Proposition 30.10]).

(ii) In the case of the p-Laplacian or more general operators, it is possible to add some monomials up to order p as pertubation. For example, since $H_0^{1,p}(\Lambda) \subset L^p(\Lambda)$ is continuous and dense, so

$$A(u) := \mathrm{div}(|\nabla u|^{p-2}\nabla u) - u|u|^{p-2}, \, u \in H_0^{1,p}(\Lambda),$$

still satisfies (H1)-(H4) with respect to the Gelfand triple

$$H_0^{1,p}(\Lambda) \subset L^2(\Lambda) \subset (H_0^{1,p}(\Lambda))^*.$$

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