ABSOLUTELY CONTINUOUS SOLUTIONS FOR CONTINUITY EQUATIONS IN HILBERT SPACES

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Abstract. We prove existence of solutions to continuity equations in a separable Hilbert space. We look for solutions which are absolutely continuous with respect to a reference measure γ which is Fomin–differentiable with exponentially integrable partial logarithmic derivatives. We describe a class of examples to which our result applies and for which we can prove also uniqueness. Finally, we consider the case where γ is the invariant measure of a reaction–diffusion equation and prove uniqueness of solutions in this case. We exploit that the gradient operator D_x is closable with respect to $L^p(H,\gamma)$ and a recent formula for the commutator $D_xP_t - P_tD_x$ where P_t is the transition semigroup corresponding to the reaction–diffusion equation, [10]. We stress that P_t is not necessarily symmetric in this case. This uniqueness result is an extension to such γ of that in [12] where γ was the Gaussian invariant measure of a suitable Ornstein–Uhlenbeck process.

Résumé. On démontre l'existence d'une solution de quelques équations de continuité dans un espace de Hilbert séparable. On s'interesse aux solutions absolument continues par rapport à une mesure de reference γ que l'on suppose dérivable au sens de Fomin et ayant les derivées partielles logarithmiques exponentiellement intégrables. On décrit une classe d'exemples a qui nos résultats s' appliquent et dont on peut aussi montrer l'unicité. Finalment on considère le cas où γ est la mesure invariante d'une équation de réaction–diffusion dont l'on prouve l'unicité des solutions. On utilise le fait que le gradient D_x est fermable dans $L^p(H,\gamma)$ et aussi une récente formule pour le commutateur $D_xP_t - P_tD_x$, P_t étant le sémigroupe de transitions qui corréspond à l'équation de réaction–diffusion considerée [10]. On souligne que dans ce cas P_t n'est pas nécessairement symétrique. Ce résultat d'unicité est une extension de celui obtenu dans [12] ou γ été la mesure invariante Gaussienne d'un processus de Ornstein–Uhlenbeck approprié.

1. Introduction

We are given a separable Hilbert space H (norm $|\cdot|_H$, inner product $\langle\cdot,\cdot\rangle$), a Borel vector field $F:[0,T]\times H\to H$ and a Borel probability measure ζ on H. We are concerned with the following continuity equation,

$$\int_0^T\int_H [D_tu(t,x)+\langle D_xu(t,x),F(t,x)\rangle]\,\nu_t(dx)\,dt=-\int_H u(0,x)\,\zeta(dx),\quad \forall\, u\in \mathcal{F}C^1_{b,T}, \tag{1.1}$$

where the unknown $\nu=(\nu_t)_{t\in[0,T]}$ is a probability kernel such that $\nu_0=\zeta$. Moreover, D_x represents the gradient operator and $\mathcal{F}C^1_{b,T}$ is defined as follows: let $\mathcal{F}C^k_b$ and $\mathcal{F}C^k_0$, for $k\in\mathbb{N}\cup\{\infty\}$, denote the set of all functions $f:H\to\mathbb{R}$ of the form

$$f(x)=\widetilde{f}(\langle h_1,x\rangle,\cdots,\langle h_N,x\rangle),\quad x\in H,$$

where $N\in\mathbb{N}$, $\widetilde{f}\in C^k_b(\mathbb{R}^N)$, $C^k_0(\mathbb{R}^N)$ respectively (i.e. $\widetilde{f}$ has compact support) and $h_1,\cdots,h_N\in Y$, where Y is a dense linear subspace of H to be specified later. Then $\mathcal{F}C^k_{b,T}$ is defined to be the $\mathbb{R}$–linear span of all functions $u:[0,T]\times H\to\mathbb{R}$ of the form

$$u(t,x)=g(t)f(x),\quad (t,x)\in[0,T]\times H,$$

2010 *Mathematics Subject Classification.* 35F05, 58D20, 60H07.

Key words and phrases. Continuity equations, non Gaussian measures, rank condition.

where $g \in C^1([0,T];\mathbb{R})$ with $g(T)=0$ and $f \in \mathcal{F}C_b^k$. Correspondingly, let $\mathcal{V}\mathcal{F}C_{b,T}^k$ be the set of all maps $G : [0,T] \times H \to H$ of the form

$$G(t,x) = \sum_{i=1}^N u_i(t,x) h_i, \quad (t,x) \in [0,T] \times H, \tag{1.2}$$

where $N \in \mathbb{N}$, $u_1, \cdots, u_N \in \mathcal{F}C_{b,T}^k$ and $h_1, \cdots, h_N \in Y$. Clearly, $\mathcal{F}C_{b,T}^\infty$ is dense in $L^p([0,T] \times H, \nu)$ for all finite Borel measures ν on $[0,T] \times H$ and all $p \in [1,\infty)$. $\mathcal{V}\mathcal{F}C_b^k$ denotes the set of all G as in (1.2) with $u_i \in \mathcal{F}C_{b,T}^k$ replaced by $u_i \in \mathcal{F}C_b^k$. Of course, all these spaces $\mathcal{F}C_b^k$, $\mathcal{F}C_0^k$, $\mathcal{F}C_{b,T}^k$, $\mathcal{V}\mathcal{F}C_b^k$, $\mathcal{V}\mathcal{F}C_{b,T}^k$ depend on Y. But since γ in Hypothesis 1 below will be fixed and hence the corresponding Y defined there will be fixed we do not express this dependence in the notation.

It is well known that problem (1.1) in general admits several solutions even when H is finite dimensional. So, it is natural to look for well posedness of (1.1) within the special class of measures $(\nu_t)_{t\in[0,T]}$ which are absolutely continuous with respect to a given *reference measure* γ. In this case, denoting by $\rho(t,\cdot)$ the density of ν_t with respect to γ,

$$\nu_t(dx) = \rho(t,x)\gamma(dx), \quad t \in [0,T],$$

equation (1.1) becomes

$$\begin{aligned} &\int_0^T \int_H [D_t u(t,x) + \langle D_x u(t,x), F(t,x)\rangle]\, \rho(t,x)\, \gamma(dx)\, dt \\ &= -\int_H u(0,x)\, \rho_0(x)\gamma(dx), \quad \forall\, u \in \mathcal{F}C_{b,T}^1. \end{aligned} \tag{1.3}$$

Here $\rho_0 := \rho(0,\cdot)$ is given and $\rho(t,\cdot)$, $t \in [0,T]$, is the unknown.

In this paper we prove existence and uniqueness results for solutions to (1.3).

Our basic assumption on γ is the following

Hypothesis 1. *γ is a nonnegative measure on $(H, \mathcal{B}(H))$ with $\gamma(H) < \infty$ such that there exists a dense linear subspace $Y \subset H$ having the following properties:*

For all $h \in Y$ there exists $\beta_h : H \to \mathbb{R}$ Borel measurable such that for some $c_h > 0$

$$\int_H e^{c_h|\beta_h|}\, d\gamma < \infty$$

and

$$\int_H \partial_h u\, d\gamma = -\int_H u\beta_h\, d\gamma,$$

where $\partial_h u$ denotes the partial derivative of u in the direction h.

Assume from now on that γ satisfies Hypothesis 1.

Remark 1.1. *It is well known that the operator D_x = Fréchet–derivative with domain $\mathcal{F}C_b^1$ is closable in $L^p(H,\gamma)$ for all $p \in [1,\infty)$, see e.g.* [1]. *Its closure will again be denoted by D_x and its domain will be denoted by $W^{1,p}(H,\gamma)$.*

Let $D_x^* : dom(D_x^*) \subset L^2(H,\gamma;H) \to L^2(H,\gamma)$ denote the adjoint of D_x.

Lemma 1.2. *$\mathcal{V}\mathcal{F}C_b^1 \subset dom(D_x^*)$ and for $G \in \mathcal{V}\mathcal{F}C_b^1$, $G = \sum_{i=1}^N u_i h_i$ we have*

$$D_x^* G = -\sum_{i=1}^N (\partial_{h_i} u_i + \beta_{h_i} u_i).$$

Proof. For $v \in \mathcal{F}C_b^1$ we have

$$\int_H \langle D_x v, G\rangle_H \, d\gamma = \sum_{i=1}^N \int_H \partial_{h_i} v \, u_i \, d\gamma$$

$$= \sum_{i=1}^N \int_H \partial_{h_i}(v\, u_i)\, d\gamma - \sum_{i=1}^N \int_H v\, \partial_{h_i} u_i \, d\gamma$$

$$= -\int_H v \sum_{i=1}^N (\partial_{h_i} u_i + \beta_{h_i} u_i)\, d\gamma.$$

□ □

We stress that if H is infinite dimensional, β_h is typically not bounded and not continuous. Here are some examples. For G as in Lemma 1.2, below we sometimes use the notation

$$\text{div } G := \sum_{i=1}^N \partial_{h_i} u_i.$$

Example 1.3. (i) Let Q be a symmetric positive definite operator of trace class on H and $\gamma := N(0,Q)$, i.e. the centered Gaussian measure on H with covariance operator Q. Assume that ker $Q = \{0\}$ and let Y be the linear span of all eigenvectors of Q. Then Hypothesis 1 is fulfilled with this Y and for $h \in Y$, $h = a_1 h_1 + \cdots + a_N h_N$ with $Qh_i = \lambda_i^{-1} h_i$, we have

$$\beta_h(x) = -\sum_{i=1}^N a_i \lambda_i \langle h_i, x\rangle_H, \quad x \in H.$$

This, in particular, covers the case studied in [12], where only uniqueness of solutions to (1.3) was studied.

(ii) Let $H := L^2((0,1), d\xi)$ and $A := \Delta$ with zero boundary conditions.

We recall that $N(0, \frac{1}{2}(-A)^{-1})((C([0,1];\mathbb{R})) = 1$. Define for $p \in (2,\infty)$ and $\alpha \in [0,\infty)$

$$\gamma(dx) := \frac{1}{Z}\, e^{-\frac{\alpha}{p}\int_0^1 |x(\xi)|^p d\xi}\, N(0, \tfrac{1}{2}\,(-A)^{-1})(dx),$$

where

$$Z := \int_H e^{-\frac{\alpha}{p}\int_0^1 |x(\xi)|^p d\xi}\, N(0, \tfrac{1}{2}\,(-A)^{-1})(dx).$$

Then with Y as in (i) for $Q = \frac{1}{2}\,(-A)^{-1}$ we find for $h = a_1 h_1 + \cdots + a_N h_N$ as in (i)

$$\beta_h(x) = -\sum_{i=1}^N a_i \left(\lambda_i \langle h_i, x\rangle_H + \alpha \int_0^1 h_i(\xi)\, |x(\xi)|^{p-2}\, x(\xi)\, d\xi \right) \quad \text{for } N(0, \tfrac{1}{2}\,(-A)^{-1})\text{–a.e. } x \in H \tag{1.4}$$

and obviously also the exponential integrability condition holds in Hypothesis 1.

(iii) Let H and A be as in (ii) and let γ be the invariant measure of the solution to

$$\begin{cases} dX(t) = [AX(t) + p(X(t))]dt + BdW(t), \\ \\ X(0) = x, \quad x \in H, \end{cases} \tag{1.5}$$

where p is a decreasing polynomial of odd degree equal to $N > 1$, $B \in L(H)$ with a *bounded inverse* and W is an H–valued cylindrical Wiener process on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t>0}, \mathbb{P})$

(see [11]). Then it was proved in [11, Proposition 3.5] that Hypothesis 1 holds with $Y := D(A)$, where A is as in (ii) above except that each β_h was only proved to be $L^p(L^2(0,1),\gamma)$ for every $p \geq 1$. More precisely, it was proved (see [11, eq. (3.17)]) that for all $h \in D(A)$

$$\left(\int_{L^2(0,1)} |\beta_h|^p \, d\gamma \right)^{\frac{1}{p}} \leq C_p |Ah|, \quad \forall \, p \geq 2,$$

where C_p is the constant of the Burkholder–Davis–Gundy inequality for $p \geq 2$ which (when proved by Itô's formula) can easily be seen to be smaller than $12\,p$ if $p \geq 4$. For the reader's convenience we include a proof in Appendix B below. Hence, because for all $n \in \mathbb{N}$ by Stirling's formula

$$\left(\frac{1}{n!} 12^n \, n^n \right)^{\frac{1}{n}} \leq 12n \left(\frac{1}{\sqrt{2\pi}} \, n^{-n-\frac{1}{2}} \, e^n \right)^{\frac{1}{n}} = 12e \left(\frac{1}{\sqrt{2\pi}} \right)^{\frac{1}{n}} e^{-\frac{1}{2n} \ln n} \to 12e \quad \text{as } n \to \infty,$$

we have for all $\epsilon \in (0, (12e|Ah|)^{-1}), h \in D(A) \setminus \{0\}$,

$$\int_{L^2(0,1)} e^{\epsilon |\beta_h|} \, d\gamma \leq \sum_{n=0}^{\infty} \frac{1}{n!} \epsilon^n 12^n \, n^n |Ah|^n < \infty.$$

So, for any $c_h \in (0, (12e|Ah|)^{-1})$, exponential integrability holds for $|\beta_h|$ and Hypothesis 1 is satisfied.

Define for an orthonormal basis $\{e_i, \, i \in \mathbb{N}\}$ of H consisting of elements in Y and $N \in \mathbb{N}$

$$H_N := \text{lin span} \, \{e_1, ..., e_N\}$$

and let $\Pi_N : H \to E_N$ be the orthogonal projection onto $E_N := H_N^\perp$, where $H_N^\perp$ is the orthogonal complement of H_N, i.e.

$$H = H_N \oplus E_N \equiv \mathbb{R}^N \times E_N, \tag{1.6}$$

hence, for $z \in H$, $z = (x, y)$ with unique $x \in \mathbb{R}^N$, $y \in E_N$.

Letting $\nu_N := \gamma \circ \Pi_N^{-1}$ be the image measure on $(E_N, \mathcal{B}(E_N))$ of γ under Π_N. Then we have the following well known disintegration result for γ:

Lemma 1.4. *There exists $\Psi_N : \mathbb{R}^N \times E_N \to [0, \infty)$, $\mathcal{B}(\mathbb{R}^N \times E_N)$–measurable such that*

$$\gamma(dz) = \gamma(dx \, dy) = \Psi_N^2(x, y) dx \, \nu_N(dy), \tag{1.7}$$

where dx denotes Lebesgue measure on $\mathbb{R}^N$. Furthermore, for every $y \in E_N$

$$\Psi_N(\cdot, y) \in H^{1,2}(\mathbb{R}^N, dx), \tag{1.8}$$

i.e. the Sobolev space of order 1 in $L^2(\mathbb{R}^N, dx)$.

Proof. See [2, Proposition 4.1]. □ □

We have by Hypothesis 1 that for all $1 \leq i \leq N$ there exists $c_i \in (0, \infty)$ such that

$$\begin{aligned} \infty \quad > \quad & \int_H e^{c_i |\beta_{e_i}|} \, d\gamma = \int_{E_N} \int_{\mathbb{R}^N} e^{c_i |\beta_{e_i}(x,y)|} \, \Psi_N^2(x, y) \, dx \, \nu_N(dy) \\ = \quad & \int_{E_N} \int_{\mathbb{R}^N} \exp \left[c_i \left| \frac{\partial}{\partial x_i} \, \Psi_N^2(x, y) / \Psi_N^2(x, y) \right| \right] \, \Psi_N^2(x, y) dx \, \nu_N(dy), \end{aligned} \tag{1.9}$$

where we used that for $1 \leq i \leq N$

$$\beta_{e_i}(x, y) = \frac{\partial}{\partial x_i} \, \Psi_N^2(x, y) / \Psi_N^2(x, y), \quad (x, y) \in \mathbb{R}^N \times E_N = H, \tag{1.10}$$

which is an immediate consequence of the disintegration (1.7), and the right hand side of (1.10) is defined to be zero on $\{\Psi_N = 0\}$. Hence

$$\int_{\mathbb{R}^N} \exp\left[c_i \left|\frac{\partial}{\partial x_i}\,\Psi_N^2(x,y)/\Psi_N^2(x,y)\right|\right]\,\Psi_N^2(x,y)dx < \infty \quad \text{for } \nu_N\text{-a.e., } y \in E_N \tag{1.11}$$

Define for $M, l \in \mathbb{N}$ and $(x,y) \in \mathbb{R}^N \times E_N (= H)$

$$\Psi_{N,M,l}(x,y) = \Psi_N(x,y) \quad \text{if } \Psi_N^2(\cdot,y) \text{ is } C^2\text{, strictly positive and bounded}$$

and otherwise

$$\Psi_{N,M}(x,y) := \left(\Psi_N^2(x,y) \wedge M \vee M^{-1}\right)^{1/2}, \tag{1.12}$$

$$\Psi_{N,M,l}(x,y) := \left(\Psi_{N,M}^2(\cdot,y) * \delta_l\right)^{1/2}(x), \tag{1.13}$$

where $\delta_l(x) = l^N \eta(lx)$, $x \in \mathbb{R}^N$, $\eta \in C_0^\infty(\mathbb{R}^N)$ with support in the unit ball, $\eta \geq 0$, $\eta(x) = \eta(-x)$, $x \in \mathbb{R}^N$, and $\int_{\mathbb{R}^N} \eta\,dx = 1$). We note that then clearly $\Psi_{N,M,l}(x,y) \geq M^{-1}$ for all $x \in \mathbb{R}^N$. Obviously,

$$\frac{\partial_{x_i}\,\Psi_{N,M,l}^2(\cdot,y)}{\Psi_{N,M,l}^2(\cdot,y)} \to \frac{\partial_{x_i}\,\Psi_{N,M}^2(\cdot,y)}{\Psi_{N,M}^2(\cdot,y)} \quad \text{in } L^1_{loc}(\mathbb{R}^N, dx) \text{ as } l \to \infty,\ \forall\, y \in E_N,\ 1 \leq i \leq N. \tag{1.14}$$

Concerning F in (1.1) we assume for γ and Y given as in Hypothesis 1.

Hypothesis 2. *(i) $F : [0,T] \times H \to H$ is Borel measurable and bounded.*

(ii) There exists an orthonormal basis $\{e_n,\, n \in \mathbb{N}\}$ of H consisting of elements in Y such that for every $N \in \mathbb{N}$ and ν_N a.e. $y \in E_N$

$$\frac{\partial_{x_i}\,\Psi_N^2(\cdot,y)}{\Psi_N^2(\cdot,y)} \in L^1_{loc}(\mathbb{R}^N, dx) \tag{1.15}$$

(Please see the "Note added in Proof" after the acknowledgement).

(iii) There exist $F_j : [0,T] \times H \to H$, $j \in \mathbb{N}$, such that for some $N_j \in \mathbb{N}$ increasing in j,

$$F_j(t,x) = \sum_{i=1}^{N_j} f_{ij}(t,x)e_i, \quad (t,x) \in [0,T] \times H,$$

(with e_i as in (ii)), where for $1 \leq i \leq N_j$

$$f_{ij}(t,x) = \widetilde{f}_{ij}(t, (\langle x, e_1\rangle, ..., \langle x, e_{N_j}\rangle))$$

with $\widetilde{f}_{ij} \in C_b([0,T] \times \mathbb{R}^{N_j}; \mathbb{R})$ and $\widetilde{f}_{ij}(t,\cdot) \in C_b^2(\mathbb{R}^{N_j}; \mathbb{R})$ for all $t \in [0,T]$ such that all first and all second partial derivatives are in $C([0,T] \times \mathbb{R}^{N_j}; \mathbb{R})$,

$$\begin{cases} \lim\limits_{j\to\infty} F_j = F \quad dt \otimes \gamma\text{-a.e.} \\[2ex] \sup\limits_{j\in\mathbb{N}} \|F_j\|_\infty < \infty, \\[2ex] \exists\, \delta > 0 \textit{ such that } M := \sup\limits_{j\in\mathbb{N}} C_{F_j}(\delta) < \infty, \end{cases}$$

where $C_{F_j}(\delta) := \int_{E_{N_j}} C_{F_j}(\delta, y)\,\nu_{N_j}(dy)$ and

$$C_{F_j}(\delta,y) := \sup_{M,l\in\mathbb{N}} \int_0^T \left(\int_{\mathbb{R}^{N_j}} e^{\delta(D^*_{N_j,M,l} F_j(t,x,y))^+} - 1\right) \Psi_{N_j,M,l}^2(x,y)\,dx\,dt,$$

with

$$D^*_{N_j,M,l}F_j(t,(x,y)) := -\sum_{i=1}^{N_j}\left(\partial_{e_i}f_{ij}(t,x) + f_{ij}(t,x)\,\frac{\partial}{\partial x_i}\,\Psi^2_{N_j,M,l}(x,y)/\Psi^2_{N_j,M,l}(x,y)\right). \tag{1.16}$$

Remark 1.5. We shall see in Example 2.9 below that Hypothesis 2(ii) is trivially fulfilled in Examples 1.3(i) and (ii). Whether it holds in Example 1.3(iii) is an open problem (see Remark 3.13 below) and will be a subject of further study.

Here is an abstract condition which ensures Hypothesis 2. Some concrete examples will be given later.

Proposition 1.6. *Let γ be a nonnegative measure satisfying Hypothesis 1; let $\Psi_N(x,y)$ be defined by (1.7). Let $\Lambda : H \to H$ be a positive selfadjoint Hilbert-Schmidt operator with $\Lambda e_n = \epsilon_n e_n$, for a sequence $\{\epsilon_n\}$ such that $\sum_{n=1}^{\infty}\epsilon_n^2 < \infty$. Let $F : [0,T]\times H \to H$ satisfying the conditions below. Assume:*

i) $\Psi_N(\cdot,y)$ is of class $C^2(\mathbb{R}^N)$, bounded and strictly positive for all $y \in E_N$

ii) $F = \Lambda F_0$, where $F_0 : [0,T]\times H \to H$ is uniformly continuous and bounded

iii) (divergence bounded from below) for some constant $C \geq 0$

$$\sum_{n=1}^{N}\partial_{e_n}\langle F(t,x), e_n\rangle \geq -C \qquad \text{for every } N \text{ and } x \in H$$

iv) for some constants $\delta > 0$

$$\int_H e^{\delta\sum_{n=1}^{\infty}\epsilon_n|\beta_{e_n}(x)|}\nu(dx) < \infty.$$

Then Hypothesis 2 is fulfilled.

Proof. **Step 1** (definition of F_N). In the verification of Hypothesis 2 we shall take $N_j = j$ hence, for simplicity of notations, we use N in place of j. For every $n, N \in \mathbb{N}$ with $n \leq N$ define $\widetilde{f}^0_{n,N}, \widetilde{f}_{n,N} : [0,T]\times\mathbb{R}^N \to \mathbb{R}$ as

$$\widetilde{f}^0_{n,N}(t,x_1,...,x_N) = \left\langle F_0\left(t,\sum_{i=1}^{N}x_ie_i\right), e_n\right\rangle$$

$$\widetilde{f}_{n,N}(t,x_1,...,x_N) = \left\langle F\left(t,\sum_{i=1}^{N}x_ie_i\right), e_n\right\rangle = \epsilon_n\widetilde{f}^0_{n,N}(t,x_1,...,x_N).$$

For every $N \in \mathbb{N}$, let $\theta^N : \mathbb{R}^N \to \mathbb{R}$ be a smooth probability density with support in the unit ball of center zero and for every $\delta > 0$ set

$$\theta^N_\delta(x) = \delta^{-N}\theta^N\left(\delta^{-1}x\right).$$

Let (δ_N) be an infinitesimal sequence. Define $f^0_{n,N}, f_{n,N} : [0,T]\times\mathbb{R}^N \to \mathbb{R}$ as

$$f^0_{n,N}(t,x_1,...,x_N) = \left(\theta^N_{\delta_N} * \widetilde{f}^0_{n,N}(t,\cdot)\right)(x_1,...,x_N).$$

$$f_{n,N}(t,x_1,...,x_N) = \left(\theta^N_{\delta_N} * \widetilde{f}_{n,N}(t,\cdot)\right)(x_1,...,x_N) = \epsilon_n f^0_{n,N}(t,x_1,...,x_N).$$

Then define

$$F_N(t,x) = \sum_{n=1}^{N} f_{n,N}(t,\langle x,e_1\rangle,...,\langle x,e_N\rangle)\,e_n.$$

The structure and regularity of $F_N(t,x)$ are obviously satisfied.

Step 2 (convergence of F_N). We prove here that the sequence of functions $F_N(t,x)$ converges pointwise to $F(t,x)$. Let $(t,x) \in [0,T] \times H$ be given. From the inequalities

$$\begin{aligned}
&\left|\sum_{n=1}^{N} f_{n,N}\left(t,\langle x,e_1\rangle,...,\langle x,e_N\rangle\right)e_n - \sum_{n=1}^{\infty}\langle F(t,x),e_n\rangle e_n\right|_H^2 \\
&\leq 2\sum_{n=1}^{N}\left(f_{n,N}\left(t,\langle x,e_1\rangle,...,\langle x,e_N\rangle\right) - \langle F(t,x),e_n\rangle\right)^2 + 2\sum_{n=N+1}^{\infty}\langle F(t,x),e_n\rangle^2 \\
&\leq 2\sum_{n=1}^{N}\epsilon_n^2\left(f_{n,N}^0\left(t,\langle x,e_1\rangle,...,\langle x,e_N\rangle\right) - \langle F_0(t,x),e_n\rangle\right)^2 + 2\|F_0\|_\infty^2\sum_{n=N+1}^{\infty}\epsilon_n^2
\end{aligned}$$

and the convergence of $\sum_{n=1}^{\infty}\epsilon_n^2 < \infty$ we see that it is sufficient to prove

$$\lim_{N\to\infty}\sup_{n\leq N}\left(f_{n,N}^0\left(t,\langle x,e_1\rangle,...,\langle x,e_N\rangle\right) - \langle F_0(t,x),e_n\rangle\right)^2 = 0.$$

Since (a priori we have to write $\limsup$ instead of $\lim$)

$$\begin{aligned}
&\lim_{N\to\infty}\sup_{n\leq N}\left(\left\langle F_0\left(t,\sum_{i=1}^{N}\langle x,e_i\rangle e_i\right),e_n\right\rangle - \langle F_0(t,x),e_n\rangle\right)^2 \\
&\leq \lim_{N\to\infty}\sum_{n=1}^{N}\left\langle F_0\left(t,\sum_{i=1}^{N}\langle x,e_i\rangle e_i\right) - F_0(t,x),e_n\right\rangle^2 \\
&\leq \lim_{N\to\infty}\left|F_0\left(t,\sum_{i=1}^{N}\langle x,e_i\rangle e_i\right) - F_0(t,x)\right|_H^2 = 0
\end{aligned}$$

because of the uniform continuity of F_0, we see it is sufficient to prove that

$$\lim_{N\to\infty}\sum_{n=1}^{N}\left(f_{n,N}^0\left(t,\langle x,e_1\rangle,...,\langle x,e_N\rangle\right) - \left\langle F_0\left(t,\sum_{i=1}^{N}\langle x,e_i\rangle e_i\right),e_n\right\rangle\right)^2 = 0.$$

Denote $\left\langle F_0\left(t,\sum_{i=1}^{N}\langle x,e_i\rangle e_i\right),e_n\right\rangle$ by $h_{n,N}(t,x)$. We have

$$\begin{aligned}
&\sum_{n=1}^{N}\left|f_{n,N}^0\left(t,\langle x,e_1\rangle,...,\langle x,e_N\rangle\right) - h_{n,N}(t,x)\right|^2 \\
&= \sum_{n=1}^{N}\left|\left(\theta_{\delta_N}^N * \widetilde{f}_{n,N}^0(t,\cdot)\right)\left(\langle x,e_1\rangle,...,\langle x,e_N\rangle\right) - h_{n,N}(t,x)\right|^2 \\
&\leq \int_{\mathbb{R}^N}\theta_{\delta_N}^N\left(...,\langle x,e_j\rangle - x_j',...\right)\sum_{n=1}^{N}\left|\left\langle F_0\left(t,\sum_{i=1}^{N}x_i'e_i\right),e_n\right\rangle - h_{n,N}(t,x)\right|^2 dx_1'...dx_N' \\
&\leq \int_{\mathbb{R}^N}\theta_{\delta_N}^N\left(...,\langle x,e_j\rangle - x_j',...\right)\left\|F_0\left(t,\sum_{i=1}^{N}x_i'e_i\right) - F_0\left(t,\sum_{i=1}^{N}\langle x,e_i\rangle e_i\right)\right\|^2 dx_1'...dx_N'.
\end{aligned}$$

Since θ^N has support in the unit ball of center zero, $\theta^N_{\delta_N}$ has support in the ball or radius δ_N and center zero. Denoting by η_N the numbers (related to modulus of continuity)

$$\eta_N = \sup_{\left|\sum_{i=1}^N x_i' e_i - \sum_{i=1}^N \langle x, e_i\rangle e_i\right|_H \le \delta_N} \left| F_0\left(t, \sum_{i=1}^N x_i' e_i\right) - F_0\left(t, \sum_{i=1}^N \langle x, e_i\rangle e_i\right)\right|$$

we have

$$\sum_{n=1}^N \left| f^0_{n,N}\left(t, \langle x, e_1\rangle, ..., \langle x, e_N\rangle\right) - h_{n,N}(t,x)\right|^2 \le \eta_N^2.$$

Since $\delta_N \to 0$ and F_0 is uniformly continuous, we deduce $\eta_N^2 \to 0$ and the proof is complete. The proof of the equi–boundedness of the family $F_N(t,x)$ is similar (we only sketch the main steps):

$$\begin{aligned}|F_N(t,x)|_H^2 &= \sum_{n=1}^N \left(f_{n,N}\left(t, \langle x, e_1\rangle, ..., \langle x, e_N\rangle\right)\right)^2 \\ &= \sum_{n=1}^N \epsilon_n^2 \left(f^0_{n,N}\left(t, \langle x, e_1\rangle, ..., \langle x, e_N\rangle\right)\right)^2 \le \|F_0\|_\infty^2 \sum_{n=1}^\infty \epsilon_n^2.\end{aligned}$$

Step 3 (exponential bound). Finally, let us check the last condition of Hypothesis 2. Since $\Psi_N(\cdot, y)$ is of class $C^2\left(\mathbb{R}^N\right)$ and bounded, we can take $\Psi_{N,M,l}(x,y) = \Psi_N(\cdot, y)$. If $G_N(x) = \sum_{n=1}^N u_n(x) e_n$, then, with the notations used above,

$$D^*_{N,M,l} G_N(x,y) = -\sum_{n=1}^N \left(\partial_{e_n} u_n(x) + u_n(x)\beta_{e_n}(x,y)\right).$$

Hence

$$\begin{aligned}&D^*_{N,M,l} F_N(t,(x,y)) \\ &= -\sum_{n=1}^N \left(\partial_{e_n} f_{n,N}\left(t, \langle x, e_1\rangle, ..., \langle x, e_N\rangle\right) + f_{n,N}\left(t, \langle x, e_1\rangle, ..., \langle x, e_N\rangle\right)\beta_{e_n}(x,y)\right)\end{aligned}$$

$$\le -\left(\theta^N_{\delta_N} * \sum_{n=1}^N \partial_{e_n} \widetilde{f}_{n,N}(t,\cdot)\right)\left(\langle x, e_1\rangle, ..., \langle x, e_N\rangle\right) + \sum_{n=1}^N \epsilon_n \left|f^0_{n,N}\left(t, \langle x, e_1\rangle, ..., \langle x, e_N\rangle\right)\right| \left|\beta_{e_n}(x,y)\right|.$$

But

$$\sum_{n=1}^N \partial_{e_n} \widetilde{f}_{n,N}(t, x_1, ..., x_N) = \sum_{n=1}^N \partial_{e_n}\left\langle F\left(t, \sum_{i=1}^N x_i e_i\right), e_n\right\rangle \ge -C$$

hence

$$-\left(\theta^N_{\delta_N} * \sum_{n=1}^N \partial_{e_n} \widetilde{f}_{n,N}(t,\cdot)\right)\left(\langle x, e_1\rangle, ..., \langle x, e_N\rangle\right) \le C.$$

And

$$\begin{aligned}\left|f^0_{n,N}\left(t, \langle x, e_1\rangle, ..., \langle x, e_N\rangle\right)\right| &\le \left|\left(\theta^N_{\delta_N} * \widetilde{f}^0_{n,N}(t,\cdot)\right)\left(\langle x, e_1\rangle, ..., \langle x, e_N\rangle\right)\right| \\ \le \int_{\mathbb{R}^N} \theta^N_{\delta_N}\left(..., \langle x, e_j\rangle - x_j', ...\right) &\left|\left\langle F_0\left(t, \sum_{i=1}^N x_i e_i\right), e_n\right\rangle\right| dx_1' ... dx_N' \le \|F_0\|_\infty.\end{aligned}$$

Summarizing,

$$D^*_{N,M,l} F_N(t,(x,y)) \le C + \|F_0\|_\infty \sum_{n=1}^N \epsilon_n \left|\beta_{e_n}(x,y)\right|$$

and thus, finally,

$$\sup_{N\in\mathbb{N}}\int_{E_N}\sup_{M,l\in\mathbb{N}}\left(\int_0^T\int_{\mathbb{R}^N}e^{\delta D^*_{N,M,l}F_N(t,(x,y))}\Psi^2_{N,M,l}\left(x,y\right)dxdt\right)\nu_N\left(dy\right)dt$$
$$\leq T\int_H e^{\delta\left[C+\|F_0\|_\infty\sum_{n=1}^\infty\epsilon_n|\beta_{e_n}(x)|\right]}\nu\left(dx\right)<\infty$$

for some $\delta > 0$. □ □

Definition 1.7. *Let $\rho_0 \in L^1(H,\gamma))$. A* solution *of the continuity equation* (1.3) *is a function $\rho \in L^1(0,T;L^1(H,\gamma)$ such that $\rho(0,\cdot) = \rho_0$ and* (1.3) *is fulfilled.*

If $\rho_0 \ln \rho_0 \in L^1(H,\gamma)$, in Section 2, we shall prove existence of a solution of (1.3) by introducing the following approximating equation, where F is replaced by (F_j) (fulfilling Hypothesis 2) and ρ_0 by $\rho_{j,0}$, where $(\rho_{j,0})$ is a sequence in $\mathcal{F}C^1_b$, converging to ρ_0 in $L^1(H,\gamma)$:

$$\begin{aligned}&\int_0^T\int_H\left[D_tu(t,x)+\langle D_xu(t,x),F_j(t,x)\rangle\right]\,\rho_j(t,x)\,\gamma(dx)\,dt\\&=-\int_H u(0,x)\,\rho_{j0}(x)\gamma(dx),\quad\forall\,u\in\mathcal{F}C^1_{b,T},\end{aligned}\tag{1.17}$$

which has a solution ρ_j since F_j is regular. Then we shall show that a subsequence of (ρ_j) converges weakly to a solution of (1.3). In Section 3 we prove uniqueness of solutions to (1.3) for a whole class of (non–Gaussian) reference measures γ based on an infinite dimensional analogue of DiPerna–Lions type commutator estimates (see [14]).

We present a whole explicit class of examples to which our results apply, i.e. for which we have both existence and uniqueness of solutions to (1.3) (see Example 2.9 below).

To our knowledge, earliest existence (and uniqueness) results for equation (1.3) concern the case where H is finite dimensional and the reference measure is the Lebesgue measure, see the seminal papers [14] and [3]. If H is infinite dimensional and γ is a Gaussian measure, problem (1.1) has been studied in [4], [16] and [12]. In [17] also non–Gaussian measures, γ, e.g. Gibbs measures were studied. However, only in the case where F does not depend on t. A very general approach in metric spaces has been presented in [5], but under the assumption $\operatorname{div}_\gamma F$ is bounded. Our assumptions for getting existence of solutions, however, do not require $\operatorname{div}_\gamma F$ to be bounded and our uniqueness results include cases where the reference measure γ is not Gaussian and not even Gibbsian, i.e. the smoothing semigroup P_ϵ is not symmetric on $L^2(H,\gamma)$.

We finish this section with some notations and preliminaries. $\mathcal{B}(H)$ denotes the set of all Borel subsets and $\mathcal{P}(H)$ the set of all Borel probabilities on H. A *probability kernel* in $[0,T]$ is a mapping $[0,T] \to \mathcal{P}(H)$, $t \mapsto \mu_t$, such that the mapping $[0,T] \to \mathbb{R}$, $t \mapsto \mu_t(I)$ is measurable for any $I \in \mathcal{B}(H)$. $L(H)$ is the set of all linear bounded operators in H, $C_b(H)$, $C_b(H;H)$ the space of all real continuous and bounded mappings $\varphi\colon H \to \mathbb{R}$ and $\varphi\colon H \to H$ respectively, endowed with the sup norm

$$\|\varphi\|_\infty = \sup_{x\in H}|\varphi(x)|,$$

whereas $C^k_b(H)$, $k > 1$, will denote the space of all real functions which are continuous and bounded together with their derivatives of order less or equal to k. $B_b(H)$ will represent the space of all real, bounded and Borel mappings on H. Moreover, we shall denote by $\|\cdot\|_p$ the norm in $L^p(H,\gamma)$, $p \in [1,\infty]$. For any $x, y \in H$ we denote either by $\langle x,y\rangle$ or by $x \cdot y$ the scalar product between x and y. Finally, if (e_h) is an orthonormal basis in H we set $x_h = \langle x, e_h\rangle$ for all $x \in H$ and $G_h = \langle G, e_h\rangle$, $h \in \mathbb{N}$, for all $G \in L^2(H,\nu;H)$. Finally, we state a lemma, needed in what follows, whose straightforward proof is left to the reader.

Lemma 1.8. *Assume, besides Hypothesis 1, that $F \in$ dom (D_x^*) and $\varphi \in C_b^1(H)$. Then $\varphi F \in$ dom (D_x^*) and we have*

$$D_x^*(\varphi F) = \varphi\, D_x^*(F) - \langle D_x\varphi, F\rangle. \tag{1.18}$$

2. The main existence result

First we notice that if $F \in \mathrm{dom}\,(D_x^*)$ then a regular solution ρ to (1.3) solves the equation

$$\begin{cases} D_t\rho + \langle F, D_x\rho\rangle - D_x^*F\,\rho = 0, \\ \\ \rho(0,\cdot) = \rho_0, \end{cases} \tag{2.1}$$

and vice-versa. In fact, since for all $u \in \mathcal{F}C_{b,T}^1$

$$\int_0^T D_t u(t,x)\,\rho(t,x)\,dt = -\int_0^T u(t,x)\,D_t\rho(t,x)\,dt - u(0,x)\rho(0,x), \quad x \in H, \tag{2.2}$$

and (thanks to Lemma 1.8)

$$\begin{aligned} &\int_H \langle D_x u(t,x), F(t,x)\rangle\,\rho(t,x)\,\gamma(dx) = \int_H \langle D_x u(t,x), \rho(t,x)F(t,x)\rangle\,\gamma(dx) \\ &= \int_H u(t,x)\,D_x^*(\rho F)(t,x)\,\gamma(dx) = \int_H u(t,x)\,\rho(t,x)\,D_x^*F(t,x)\,\gamma(dx) \\ &- \int_H u(t,x)\,\langle D_x\rho(t,x), F(t,x)\rangle\,\gamma(dx). \end{aligned} \tag{2.3}$$

Clearly (2.2) and (2.3) imply that (1.3) is equivalent to

$$\begin{cases} \displaystyle\int_0^T\int_H u(t,x)\,[-D_t\rho(t,x) + D_x^*F(t,x)\rho(t,x) - \langle D_x\rho(t,x), F(t,x)\rangle\,]\,\gamma(dx)\,dt = 0, \\ \\ \rho(0,\cdot) = \rho_0, \end{cases} \tag{2.4}$$

for all $u \in \mathcal{F}C_{b,T}^1$. By the density of $\mathcal{V}\mathcal{F}C_{b,T}^1$ in $L^2([0,T]\times H, dt\otimes d\gamma)$ we obtain (2.1).

Theorem 2.1. *Assume that Hypotheses 1 and 2 hold. Let $\zeta := \rho_0\cdot\gamma$ be a probability measure on $(H, \mathcal{B}(H))$ such that*

$$\int_H \rho_0\,\ln\rho_0\,d\gamma < \infty. \tag{2.5}$$

Then there exists $\rho : [0,T]\times H \to \mathbb{R}_+$, $\mathcal{B}([0,T]\times H)$–measurable such that $\nu_t(dx) = \rho(t,x)\gamma(dx)$, $t \in [0,T]$, are probability measures on $(H, \mathcal{B}(H))$ such that (1.1) *(equivalently* (1.3)*) holds. In addition,*

$$\int_0^T\int_H \rho(t,x)\,\ln\rho(t,x)\,\gamma(dx)\,dt < \infty. \tag{2.6}$$

Proof. By disintegration we shall reduce the proof to the case $H = \mathbb{R}^N$ and by regularization to Corollary A.2 in Appendix A. Let $\{e_n,\, n \in \mathbb{N}\}$ be the orthonormal basis from Hypothesis 2(ii)

Case 1. Suppose first that $F : [0,T]\times H \to H$ is as an F_j from Hypothesis 2(iii), $\rho_0 \in \mathcal{F}C_0^1$, $\rho_0 \geq 0$.

Hence for some $N \in \mathbb{N}$ (which we fix below and shall no longer explicitly express in the notation below, i.e. write $\Psi_{N,M,l}$ as $\Psi_{M,l}$, E instead of E_N, etc.)

$$F(t,x) = \sum_{i=1}^{N} f_i(t,x)\, e_i, \quad (t,x) \in [0,T] \times H, \tag{2.7}$$

where for $1 \leq i \leq N$,

$$f_i(t,x) = \widetilde{f}_i(t, \langle e_1, x\rangle, ..., \langle e_N, x\rangle)$$

and

$$\rho_0(x) = \widetilde{\rho}_0(\langle e_1, x\rangle, ..., \langle e_N, x\rangle)$$

with $\widetilde{\rho}_0 \in C_0^1(\mathbb{R}^N)$ and $\widetilde{f}_i$ as in Hypothesis 2(iii).

Then by Corollary A.2 applied with $\Psi = \Psi^2_{M,l}(\cdot, y)$, we know that

$$\rho_{M,l}(t,(x,y)) := \rho_0(\xi(T, T-t, x))\, e^{\int_0^t D^*_{M,l}\, F(T-u, (\xi(T-u, T-t, x), y))\, du}, \quad (t,x) \in [0,T] \times \mathbb{R}^N, \tag{2.8}$$

where (see Lemma1.2 and (1.16))

$$D^*_{M,l} F_j(r,(x,y)) := -\sum_{i=1}^{N} \left(\partial_{e_i} f_{ij}(t,x) + f_{ij}(t,x)\, \frac{\partial}{\partial x_i}\, \Psi^2_{M,l}(x,y)/\Psi^2_{M,l}(x,y) \right), \tag{2.9}$$

$r \in [0,T],\ x \in \mathbb{R}^N$, solves

$$\begin{cases} D_t \rho_{M,l}(t,(x,y)) + \langle F(t,x), D_x \rho_{M,l}(t,(x,y))\rangle - D^*_{M,l} F(t,(x,y)) \rho_{M,l}(t,(x,y)) = 0, \\ \\ \rho_{M,l}(0,(x,y)) = \rho_0(x). \end{cases} \tag{2.10}$$

Since $\widetilde{\rho}_0$ has compact support in $\mathbb{R}^N$ and since F is bounded, we see from (2.8) that there exists a closed ball $K_R \subset \mathbb{R}^N$, centred at zero and radius $R \geq 1$, such that

$$\rho_{M,l}(t,(\cdot, y)) = 0 \quad \text{on}\, \mathbb{R}^N \setminus K_R \text{ for all}\, (t,y) \in [0,T] \times E;\ M, l \in \mathbb{N}. \tag{2.11}$$

Furthermore, rewriting (2.8) as (2.1) one easily sees that for all $t \in [0,T]$

$$\int_{\mathbb{R}^N} \rho_{M,l}(t,(x,y))\, \Psi^2_{M,l}(x,y)\, dx = \int_{\mathbb{R}^N} \rho_0(x)\, \Psi^2_{M,l}(x,y)\, dx. \tag{2.12}$$

Below all statements are claimed to hold for ν-a.e., $y \in E$.

We need a few further lemmas of which the first is the most crucial, to prove Case 1.

Lemma 2.2. *Let $\epsilon > 0$. Then for all $1 \leq i \leq N$, $l, M \in \mathbb{N}$*

$$\begin{aligned} &\int_{\mathbb{R}^N} \left(\exp\left[\epsilon \left| \left(\frac{\partial \Psi^2_{M,l}}{\partial x_i} \,/ \Psi^2_{M,l} \right) (x,y) \right| \right] - 1 \right)\, \Psi^2_{M,l}(x,y)\, dx \\ &\leq \int_{\mathbb{R}^N} \left(\exp\left[\epsilon \left| \left(\frac{\partial \Psi^2_{M}}{\partial x_i} / \Psi^2_{M} \right) (x,y) \right| \right] - 1 \right)\, \Psi^2_{M}(x,y)\, dx \\ &\leq \int_{\mathbb{R}^N} \left(\exp\left[\epsilon\, |\beta_{e_i}(x,y)| \right] - 1 \right)\, \Psi^2(x,y)\, dx. \end{aligned} \tag{2.13}$$

Proof. Obviously, the left hand side of (2.13) is equal to

$$\int_{\mathbb{R}^N} \left(\exp\left[\epsilon \left| \int_{\mathbb{R}^N} \left(\frac{\partial \Psi^2_M}{\partial x_i} \,/ \Psi^2_M \right) (\tilde{x}, y)\, \Psi^2_M(\tilde{x}, y)\, \delta_l(x - \tilde{x})\, d\tilde{x}\, (\Psi^2_{M,l}(x,y))^{-1} \right| \right] - 1 \right) \Psi^2_{M,l}(x,y)) dx. \tag{2.14}$$

Taking the modulus under the integral and applying Jensen's inequality for fixed $x \in \mathbb{R}^N$ to the probability measure

$$\Psi^2_{M,l}(x,y))^{-1}\,\Psi^2_M(\tilde{x},y)\,\delta_l(x-\tilde{x})\,d\tilde{x}$$

and the convex function $r \mapsto e^{\epsilon r}-1$, $r \geq 0$, we obtain that (2.14) is dominated by

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\left(\exp\left[\epsilon\left(\left|\frac{\partial \Psi^2_M}{\partial x_i}\right|/\Psi^2_M\right)(\tilde{x},y)\right]-1\right)\,\Psi^2_M(\tilde{x},y)\,\delta_l(x-\tilde{x})\,d\tilde{x}\,dx.$$

By Young's inequality and since $\|\delta_l\|_{L^1(\mathbb{R}^N)}=1$, the latter is dominated by

$$\int_{\mathbb{R}^N}\left(\exp\left[\epsilon\left(\left|\frac{\partial \Psi^2_M}{\partial x_i}\right|/\Psi^2_M\right)(x,y)\right]-1\right)\,\Psi^2_M(x,y)\,dx. \tag{2.15}$$

Hence the fist inequality in (2.13) is proved. To show the second we note that

$$\frac{\partial \Psi^2_M}{\partial x_i}(\cdot,y)=\mathbb{1}_{\{M^{-1}<\Psi^2(\cdot,y)<M\}}\,\frac{\partial \Psi^2}{\partial x_i}(\cdot,y),\quad dx\text{–a.s..}$$

Hence the integral in (2.15) is dominated by

$$\int_{\mathbb{R}^N}\mathbb{1}_{\{M^{-1}<\Psi^2(\cdot,y)<M\}}\left(\exp\left[\epsilon\left(\left|\frac{\partial \Psi^2}{\partial x_i}\right|/\Psi^2\right)(x,y)\right]-1\right)\,\Psi^2(x,y)\,dx,$$

which in turn by (1.10) is dominated by the last integral in (2.13) □ □

Lemma 2.3. *For $\delta>0$ let $C_F(\delta)$ and $C_F(\delta,y)$ be as in Hypothesis 2(iii). Then for*

$$\delta := \inf_{1\leq i\leq N}\frac{c_i}{N(\|f_i\|_\infty+1)},$$

we have

$$C_F(\delta,y)\leq \sup_{M,l\in\mathbb{N}}\int_0^T\int_{\mathbb{R}^N}\left(\exp\left[-\delta\sum_{i=1}^N\partial_{e_i}f(t,x)\right]^+\right.$$
$$\left.\times\exp\left[\delta\sum_{i=1}^N\|f_i\|_\infty\left(\left|\frac{\partial\Psi^2_{M,l}}{\partial x_i}\right|\Psi^2_{M,l}\right)(x,y)\right]-1\right)\Psi^2_{M,l}(x,y)\,dx\,dt<\infty$$

and $C_F(\delta)<\infty$.

Proof. By (1.10), (1.11) and convexity of the function $r\mapsto be^{ar}-1$, $r\geq 0$, for $a,b>0$, this follows immediately from Lemma 2.2 and (1.9). □ □

Lemma 2.4. *(i) We have for all $M\in\mathbb{N}$, $t\in[0,T]$*

$$\lim_{l\to\infty}D^*_{M,l}F(t,(x,y))=-\sum_{i=1}^N\left[\partial_{e_i}f_i(t,x)+f_i(t,x)\left(\frac{\partial\Psi^2_M}{\partial x_i}/\Psi^2_M\right)(x,y)\right]=:D^*_MF(t,(x,y)),$$

and

$$\lim_{M\to\infty}D^*_MF(t,(x,y))=-\sum_{i=1}^N\left[\partial_{e_i}f_i(t,x)+f_i(t,x)\beta_{e_i}(x,y)\right]=D^*_xF(t,(x,y)),$$

in $L^1_{loc}(\mathbb{R}^N,dx)$.

(ii) Let ρ_M and ρ be defined as $\rho_{M,l}$ with $D^*_{M,l}F$ replaced by D^*_MF and D^*_xF respectively.

Then there exist subsequences $(l_k)_{k\in\mathbb{N}}$, $(M_k)_{k\in\mathbb{N}}$ such that we have for dx–a.e. $x\in\mathbb{R}^N$, for all $M\in\mathbb{N}$

$$\lim_{k\to\infty}\rho_{M,l_k}(t,(x,y))=\rho_M(t,(x,y)),\quad\forall\,t\in[0,T]$$

and
$$\lim_{k\to\infty} \rho_{M_k}(t,(x,y)) = \rho(t,(x,y)), \quad \forall\, t \in [0,T].$$

Proof. (i) Obviously, for all $M \in \mathbb{N}$ by (1.14)
$$\lim_{l\to\infty} D^*_{M,l}F(t,(\cdot,y)) = D^*_M F(t,(\cdot,y)), \quad \text{in } L^1_{loc}(\mathbb{R}^N,dx),\, \forall\, t\in[0,T].$$
The second assertion follows, because
$$\left(\frac{\partial \Psi^2_M}{\partial x_i} / \Psi^2_M\right)(x,y) = 1\!\!1_{(M^{-1},M)}\left(\Psi^2(x,y)\right)\left(\frac{\partial \Psi^2}{\partial x_i} / \Psi^2\right)(x,y). \tag{2.16}$$

(ii) Fix $t\in[0,T]$. Then for all $u\in[0,t]$
$$x \mapsto \xi(T-u,T-t,x)$$
is a C^1–diffeomorphism on $\mathbb{R}^N$. Let $\phi_{u,t} : \mathbb{R}^N \to \mathbb{R}^N$ be its inverse (i.e. just the corresponding backward flow). Then for every $K\subset\mathbb{R}^N$, K compact, and $\Delta D^*_{M,l}F := |D^*_M F - D^*_{M,l}F|$ we have
$$\int_K \int_0^t \Delta D^*_{M,l}F(T-u,(\xi(T-u,T-t,x),y)\,du\,dx$$
$$= \int_0^t \int_{\xi(T-u,T-t,K)} \Delta D^*_{M,l}F(T-u,(x,y))\,|\det D\phi_{u,t}(x)|\,dx\,du.$$
Since F is bounded, there exists a ball $B_R(0)$ so that for large enough $R>0$, $\xi(T-u,T-t,K)\subset B_R(0)$ for all $t\in[0,T]$. Hence by Fubini's Theorem the above integral is dominated by
$$\int_{B_R(0)} \int_0^t |\det D\phi_{u,t}(x)| \Delta D^*_{M,l}F(T-u,(x,y))\ dx\,du. \tag{2.17}$$
The specific dependence of F on $T-u$ and the well known explicit formula of $\det D\phi_{u,t}$ (recall $\phi_{u,t}$ is a flow) implies that
$$x \mapsto \int_0^t |\det D\phi_{u,t}(x)|\,\widetilde{f}_i(T-u,x)\,du$$
is locally bounded on $\mathbb{R}^N$, so that (1.14) can be applied to show that the term in (2.17) converges to zero as $l\to\infty$. So, the first assertion follows. Then also the second assertion follows by (1.15), (2.16) and the same arguments. □ □

Lemma 2.5. *Let $l, M\in\mathbb{N}$. Then for all $t\in[0,T]$ and $\delta>0$*
$$\int_{\mathbb{R}^N} \rho_{M,l}(t,(x,y))\left(\ln\rho_{M,l}(t,(x,y)) - 1\right)\Psi^2_{M,l}(x,y)\,dx$$
$$\le e^{t/\delta}\Bigg[\int_{\mathbb{R}^N} \rho_0(x)|\ln\rho_0(x)-1|\Psi^2_{M,l}(x,y)\,dx + C_F(\delta,y) \tag{2.18}$$
$$+\frac{t}{\delta}|\ln\delta|\int_{\mathbb{R}^N}\rho_0(x)\Psi^2_{M,l}(x,y)\ dx + \frac{t}{M}|K_{R+1}| + t\int_{\mathbb{R}^N}\Psi^2(x,y)\,dx\Bigg]$$
where $C_F(\delta,y)$ is as defined in Hypothesis 2(iii) and $|K_{R+1}|$ denotes the Lebesgue measure of the ball $K_{R+1}\subset\mathbb{R}^N$, centred at 0 and radius $R+1$, where R is as in (2.11).

Proof. Since $\rho_{M,l}(t,(\cdot,y))$ has compact support in $\mathbb{R}^N$ for all $(t,y) \in [0,T]\times E$ by the regularity properties of $\rho_{M,L}$ stated in Corollary A.2 of Appendix A, all integrals below are well defined. Since $M,l\in\mathbb{N}$ and $y\in E$ are fixed, for simplicity of notation we denote the maps $x\mapsto \rho_{M,l}(t,(x,y))$ and $x\mapsto \Psi_{M,l}(x,y)$ by $\rho(t)$, Ψ respectively. Then for $t\in[0,T]$,

$$\int_{\mathbb{R}^N} \rho(t)(\ln\rho(t)-1)\,\Psi^2 dx$$

$$= \int_{\mathbb{R}^N} \rho_0(\ln\rho_0 - 1)\,\Psi^2 dx + \int_{\mathbb{R}^N}\int_0^t \tfrac{d}{ds}[\rho(s)(\ln\rho(s)-1)]\,ds\,\Psi^2 dx$$

$$= \int_{\mathbb{R}^N} \rho_0(\ln\rho_0 - 1)\,\Psi^2 dx + \int_{\mathbb{R}^N}\int_0^t \ln\rho(s) D_s\rho(s)\,ds\,\Psi^2 dx$$

$$= \int_{\mathbb{R}^N} \rho_0(\ln\rho_0 - 1)\,\Psi^2 dx - \int_0^t\int_{\mathbb{R}^N} \langle F(s,x), D_x(\rho(s)(\ln\rho(s)-1))\rangle\,\Psi^2 dx\,ds$$

$$+ \int_0^t\int_{\mathbb{R}^N} D^*_{M,l}\,F(s,(\cdot,y))\rho(s)\,\ln\rho(s)\,\Psi^2 dx\,ds$$

$$= \int_{\mathbb{R}^N} \rho_0(\ln\rho_0 - 1)\,\Psi^2 dx + \int_0^t\int_{\mathbb{R}^N} D^*_{M,l}\,F(s,(\cdot,y))\rho(s)\,\Psi^2 dx\,ds$$

$$\leq \int_{\mathbb{R}^N} \rho_0(\ln\rho_0 - 1)\,\Psi^2 dx + \int_0^t\int_{\mathbb{R}^N} \Big[e^{\delta(D^*_{M,l}\,F(s,(\cdot,y))^+} - 1 + \tfrac{1}{\delta}\,\rho(s)\,(\ln(\tfrac{1}{\delta}\,\rho(s)) - 1)\Big]\;\Psi^2 dx\,ds$$

$$+t\int_{K_R} \Psi^2(x,y)\,dy,$$

where in the third equality we used (2.10), in the fourth equality we used Fubini's theorem and the definition of $D^*_{M,l}$ and finally, in the last inequality we used (2.11) and that $ab\leq e^a + b(\ln b - 1)$ for $a,b\geq 0$. Now the assertion follows by Gronwall's lemma, since by (2.12)

$$\int_{\mathbb{R}^N} \rho_{M,l}(t,(x,y))\,\Psi^2_{M,l}(x,y)\,dx = \int_{\mathbb{R}^N} \rho_0(x)\,\Psi^2_{M,l}(x,y)\,dx, \quad \forall\ \in[0,T], \tag{2.19}$$

and since

$$\int_{K_R} \Psi^2_{M,l}(x,y)\,dx \leq \frac{1}{M}\,|K_{R+1}| + \int_{\mathbb{R}^N} \Psi^2(x,y)\,dx.$$

□ □

Lemma 2.6. *Let $M\in\mathbb{N}$, $\rho_{M,l,y}(t,x) := \rho_{M,l}(t,(x,y))$, $t\in[0,T]$, $x\in\mathbb{R}^N$, and $\Psi_{M,l,y}(x) := \Psi_{M,l}(x,y)$, $x\in\mathbb{R}^N$. Then $\{\rho_{M,l,y}\cdot\Psi^2_{M,l,y} : l\in\mathbb{N}\}$ is uniformly integrable with respect to the measure $\chi(x)\,dx\,dt$, where χ is the indicator function of an arbitrary compact set in $\mathbb{R}^N$.*

Proof. Let $c\in(1,\infty)$. Then for all $l\in\mathbb{N}$ and $\rho_l := \rho_{M,l,y}$, $\Psi_l := \Psi_{M,l,y}$,

$$\int_0^T\int_{\mathbb{R}^N} \mathbb{1}_{\{\rho_l\Psi_l^2\geq c\}}\,\rho_l\Psi_l^2\,\chi\,dx\,dt \leq \tfrac{1}{\ln c}\int_0^T\int_{\mathbb{R}^N} \mathbb{1}_{\{\rho_l\Psi_l^2\geq c\}}\,(\ln\rho_l + \ln\Psi_l^2)\rho_l\Psi_l^2\,\chi\,dx\,dt$$

$$\leq \tfrac{1}{\ln c}\int_0^T\int_{\mathbb{R}^N} |\rho_l\ln\rho_l|\Psi_l^2\,dx\,dt + \tfrac{\ln(M+1)}{\ln c}\int_0^T\int_{\mathbb{R}^N} \rho_l\Psi_l^2\,dx\,dt.$$

Since $r \ln r - r \geq -1$, $r \in [0, \infty)$, it follows by Lemma 2.5 and (2.19), that both integrals on the right hand side of the last inequality are uniformly bounded in l and the assertion follows. $\square$ $\square$

Now we proceed with the proof of Case 1 of Theorem 2.1. It follows by (2.10) (analogously to (2.1)–(2.4) above) that for all

$$u(t,x) := g(t)f(x), \quad t \in [0,T],\, x \in \mathbb{R}^N, \tag{2.20}$$

$g \in C^1([0,T];\mathbb{R})$ with $g(T) = 0$ and $f \in C_0^1(\mathbb{R}^N)$ that

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^N} [D_t u(t,x) + \langle D_x u(t,x), F(t,x)\rangle]\, \rho_{M,l}(t,(x,y))\, \Psi^2_{M,l}(x,y)\, dx\, dt \\ &= -\int_{\mathbb{R}^N} u(0,x)\rho_0(x)\, \Psi^2_{M,l}(x,y)\, dx. \end{aligned} \tag{2.21}$$

By Lemma 2.4(ii) and Lemma 2.6 we can pass to the limit in (2.21) along the subsequence $(l_k)_{k\in\mathbb{N}}$ from Lemma 2.4 to conclude that for such u

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^N} [D_t u(t,x) + \langle D_x u(t,x), F(t,x)\rangle]\, \rho_M(t,(x,y))\, \Psi^2_M(x,y)\, dx\, dt \\ &= -\int_{\mathbb{R}^N} u(0,x)\rho_0(x)\, \Psi^2_M(x,y)\, dx. \end{aligned} \tag{2.22}$$

We can also pass to the limit in (2.19) to get

$$\int_{\mathbb{R}^N} \rho_M(t,(x,y))\, \Psi^2_M(x,y)\, dx = \int_{\mathbb{R}^N} \rho_0(x)\, \Psi^2_M(x,y)\,, dx, \quad \forall\, t \in [0,T]. \tag{2.23}$$

Furthermore, by Lemma 2.4(ii) and Lemma 2.5 we deduce from (2.18) by Fatou's lemma that for all $t \in [0,T]$, $\delta > 0$

$$\begin{aligned} &\int_{\mathbb{R}^N} \rho_M(t,(x,y))(\ln \rho_M(t,(x,y)) - 1)\, \Psi^2_M(x,y)\, dx \\ &\leq e^{t/\delta}\Big[\int_{\mathbb{R}^N} \rho_0(x)|\ln \rho_0(x) - 1|\, \Psi^2_M(x,y)\, dx + C_F(\delta,y) + \tfrac{t}{\delta}|\ln\delta| \int_{\mathbb{R}^N} \rho_0(x)\, \Psi^2_M(x,y)\, dx \\ &+ \frac{t}{M}\,|K_{R+1}| + t\int_{\mathbb{R}^N} \Psi^2(x,y)\, dx\Big]. \end{aligned} \tag{2.24}$$

Taking now the subsequence $(M_k)_{k\in\mathbb{N}}$ from Lemma 2.4 instead of M and using exactly analogous arguments as above, we can pass to the limit in (2.22), (2.23) and (2.24) to obtain that for all u as in (2.20)

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^N} [D_t u(t,x) + \langle D_x u(t,x), F(t,x)\rangle]\, \rho(t,(x,y))\, \Psi^2(x,y)\, dx\, dt \\ &= -\int_{\mathbb{R}^N} u(0,x)\rho_0(x)\, \Psi^2(x,y)\, dx, \end{aligned} \tag{2.25}$$

and for all $t \in [0,T]$

$$\int_{\mathbb{R}^N} \rho(t,(x,y))\, \Psi^2(x,y)\, dx = \int_{\mathbb{R}^N} \rho_0(x)\, \Psi^2(x,y)\, dx, \tag{2.26}$$

and for all $t \in [0,T]$, $\delta > 0$

$$
\begin{aligned}
&\int_{\mathbb{R}^N} \rho(t,(x,y))(\ln \rho(t,(x,y)) - 1)\, \Psi^2(x,y)\, dx \\
&\le e^{t/\delta}\Big[\int_{\mathbb{R}^N} \rho_0(x)|\ln \rho_0(x) - 1|\, \Psi^2(x,y)\, dx + C_F(\delta,y) + \tfrac{t}{\delta}|\ln \delta| \int_{\mathbb{R}^N} \rho_0(x)\, \Psi^2(x,y)\, dx \qquad (2.27)\\
&+ t\int_{\mathbb{R}^N} \Psi^2(x,y)\, dx\Big].
\end{aligned}
$$

Taking the special δ from Lemma 2.3 and $C_F(\delta,y)$ as in Lemma 2.4 in the situation of Case 1 the assertion of Theorem 2.1 now follows easily from the disintegration formula (1.7), integrating (2.25) with respect to ν and by approximating the functions u in (1.1) in the obvious way. From (2.27) we get (2.6) after integrating over y with respect to ν. □ □

Remark 2.7. (i) We here emphasize that in the situation of Case 1 we have an explicit formula for the solution density in (2.25) given by

$$\rho(t,(x,y)) = \rho_0(\xi(T,T-t,x))e^{-\int_0^t D_x^* F(T-u,\xi(T-u,T-t,y))du} \tag{2.28}$$

for $t \in [0,T]$ and dx–a.e. $x \in \mathbb{R}^N$ with ξ given as in Corollary A.2 of Appendix A.

(ii) Integrating (2.27) over $y \in E$ with respect to ν, from Lemma 1.4 we obtain that for all $t \in [0,T]$, $\delta > 0$

$$\int_H \rho(t,x)(\ln \rho(t,x) - 1)\, \gamma(dx) \le e^{t/\delta}\left[\int_H \rho_0|\ln \rho_0 - 1|\, d\gamma + C_F(\delta) + \tfrac{t}{\delta}|\ln \delta| \int_H \rho_0\, d\gamma + t\gamma(H)\right] \tag{2.29}$$

and likewise from (2.26) that for all $t \in [0,T]$

$$\int_H \rho(t,x)\, \gamma(dx) = \int_H \rho_0(x)\, \gamma(dx) = 1. \tag{2.30}$$

Case 2. Let F_j, $j \in \mathbb{N}$, be as in Hypothesis 2. Choose nonnegative $\rho_{0,j} \in \mathcal{F}C_0^1$ such that

$$\lim_{j\to\infty} \rho_{0,j} = \rho_0 \quad \text{in } L^1(H,\gamma) \tag{2.31}$$

and

$$\sup_{j\in\mathbb{N}} \int_H \rho_{0,j}\, \ln \rho_{0,j}\, d\gamma < \infty. \tag{2.32}$$

For existence of such $\rho_{0,j}$, $j \in \mathbb{N}$, see Corollary C.3 in Appendix C below.

Let ρ_j be the corresponding solutions to (1.1) with F_j replacing F and $\zeta := \rho_0 \cdot \gamma$, which exist by Case 1. Then by (2.29) with $\rho_j, F_j, \rho_{0,j}$ replacing ρ, F and ρ_0 respectively, Hypothesis 2 and (2.30) imply that

$$\sup_{j\in\mathbb{N}} \sup_{t\in[0,T]} \int_H \rho_j(t,x)\, \ln \rho_j(t,x)\, \gamma(dx) < \infty. \tag{2.33}$$

By Case 1 we have for all $u \in \mathcal{F}C_{b,T}^1$

$$
\begin{aligned}
&\int_0^T \int_H \left[\frac{d}{dt}\, u(t,x) + \langle D_x u(t,x), F_j(t,x)\rangle_H\right] \rho_j(t,x)\, \gamma(dx)\, dt \\
&= -\int_H u(0,x)\, \rho_{0,j}(x)\gamma(dx).
\end{aligned} \tag{2.34}
$$

So, by (2.31) we only have to consider the convergence of the left hand side of (2.34), more precisely only the part of it involving F_j. But

$$\begin{aligned}&\left|\int_0^T \int_H (\langle D_x u, F_j\rangle_H\, \rho_j - \langle D_x u, F\rangle_H\, \rho)\, d\gamma\, dt\right| \\ &\leq \|Du\|_\infty \int_0^T \int_H |F_j - F|_H\, \rho_j\, d\gamma\, dt + \left|\int_0^T \int_H \langle F, Du\rangle\, (\rho_j - \rho)\, d\gamma\, dt\right|\end{aligned} \tag{2.35}$$

Because of the boundedness of $\langle F, Du\rangle$ the second term on the right hand side of (2.35) converges to 0 if $j \to \infty$. Let $\epsilon > 0$. Then, by Young's inequality, the first term on the right hand side of (2.35) is up to a constant dominated by

$$\int_0^T \int_H e^{\frac{1}{\epsilon}|F_j - F|_H}\, d\gamma\, dt + \epsilon \int_0^T \int_H \rho_j \ln(\epsilon\rho_j)\, d\gamma\, dt,$$

of which the first summand converges to zero as $j \to \infty$, since F_j, F are uniformly bounded, while the second summand is dominated by

$$\epsilon \int_0^T \int_H \rho_j \ln \rho_j\, d\gamma\, dt + \epsilon \ln \epsilon,$$

which can be made arbitrarily small uniformly in j because of (2.33). Hence putting all this together we conclude that the right hand side of (2.35) converges to 0 as $j \to \infty$. (2.6) then follows by weak lower semi–continuity. Finally from (2.30) and (2.31) it follows that $\nu_t(dx) := \rho(t, x)\, \gamma(dx)$ is a probability measure for all $t \in [0, T]$. Thus Theorem 2.1 is completely proved.

Remark 2.8. Though the finite entropy condition in the initial measure ρ_0 is crucial in the proof of Theorem 2.1, it could be replaced by a corresponding assumption with $r \mapsto r(\ln r - 1)$ replaced by another Young fnction (see Appendix C below) and adjusting Hypothesis 2(ii) accordingly. In particular, we can take e. g. $r \mapsto r^p$, $r \geq 0$, $p > 1$. Then the exponential integrability condition on $D_x^* F$ in Hypothesis 2(iii) can be replaced by an $L^{p'}$-integrability condition with $p' = \frac{p}{p-1}$. Hence the solution ρ to (1.3) would be in $L^p([0, T] \times H, dt \otimes \gamma)$, provided $\rho_0 \in L^p(H, \gamma)$. Therefore, we get existence of solutions also in the situation of Section 3, provided B in (3.1) is the identity operator (see Corollary 3.12 below). Likewise, e.g. for the Young $r \to r^p$, $r \geq 0$, $p > 1$, one can relax the assumption on exponential integrability on β_h, $h \in Y$, in Hypothesis 1 by $L^p(H, \gamma)$ integrability.

Example 2.9. Let us discuss Hypothesis 2(ii) for γ as in Example 1.3(ii). In this case we choose $\{e_n : n \in \mathbb{N}\}$ to be the eigenbasis of A given by

$$e_n(\xi) := \sqrt{\frac{2}{\pi}}\, \sin(n\pi\xi), \quad \xi \in [0, 1],\ n \in \mathbb{N}.$$

Then for $A_n e_n = -\lambda_n e_n$ with $\lambda_n := \pi^2\, n^2$, $n \in \mathbb{N}$. Now consider the corresponding disintegration (1.7). Then $N(0, \frac{1}{2}\,(-A)^{-1})$ is by independence equal to the convolutions of his projections on H_N and E_N respectively. Hence

$$\Psi_N^2(x, y) = \frac{1}{(2\pi\lambda_1 \cdots \lambda_N)^{N/2} Z} \exp\left(-\frac{\alpha}{p} \int_0^1 |x(\xi) + y(\xi)|^p\, d\xi\, - \frac{1}{4} \sum_{i=1}^N \lambda_i^{-1} \langle e_i, x\rangle^2\right)$$

where $y \in E_N$ and $x(\xi) = \langle x, e_1\rangle e_1(\xi) + \cdots + \langle x, e_n\rangle e_n(\xi)$. So, obviously for ν_N–a.e. $y \in E_N$, $x \mapsto \Psi_N^2(x, y)$ is continuous and strictly positive on H_N, since $x + y \in L^p(0, 1) =: L^p$, because $N(0, \frac{1}{2}(-A)^{-1})(C([0, 1]; \mathbb{R})) = 1$. Thus (1.15) holds. Unfortunately so far we do not know whether (1.15) holds in case of γ as in 1.3–(iii). Now consider again the situation of 1.3–(ii). We are now going to present a class $F : [0, T] \times H \to H$ for which Theorem 2.1 applies: Let $f \in C_b([0, T] \times \mathbb{R}; \mathbb{R})$

such that $f(t,\cdot) \in C^1(\mathbb{R};\mathbb{R})$ for every $t \in [0,T]$ and there exist $K \in (0,\infty)$, $\delta \in (0,p)$ such that for $f'(t,r) = f_r(t,r)$

$$f'(t,r) \geq -K(1+|r|^2+\alpha|r|^{p-\delta}), \quad \forall\,(t,r) \in [0,T]\times\mathbb{R}.$$

Define $F_0 : [0,T]\times L^2(0,1) \to L^2(0,1)$ by

$$F_0(t,x)(\xi) := f(t,x(\xi)), \quad \xi \in (0,1),\ t \in [0,T]]$$

and $F : [0,T]\times L^2(0,1) \to L^2(0,1)$ by

$$F(t,x) := (-A)^{-1}F_0(t,x), \quad x \in L^2(0,1),\ t \in [0,T]. \tag{2.36}$$

Now we want to check Hypothesis 2 for this type of F.

Claim 1. For every $\epsilon > 0$ there exists $C_\epsilon \in (0,\infty)$ such that

$$\sum_{i=1}^{N} \partial_{e_i} F^i(t,x) \geq -C_\epsilon - \epsilon(|x|^2_{L^2} + \alpha|x|^p_{L^p]}), \quad x \in L^p(0,1),\ t \in [0,T],\ N \in \mathbb{N},$$

where

$$F^i(t,x) := \langle e_i, F(t,x)\rangle.$$

Proof of Claim 1. Let $x \in L^p(0,1),\ t \in [0,T]$. Then

$$\begin{aligned}
\sum_{i=1}^{N} \partial_{e_i} F^i(t,x) &= \sum_{i=1}^{N} \lambda_i^{-1}\partial_{e_i}\int_0^1 e_i(\xi)\, f(t,x(\xi))\, d\xi \\
&= \sum_{i=1}^{N} \lambda_i^{-1}\int_0^1 e_i^2(\xi)\, f'(t,x(\xi))\, d\xi \\
&\geq -K\sum_{i=1}^{\infty} \lambda_i^{-1}\int_0^1 e_i^2(\xi)\,(1+|x(\xi)|^2+\alpha|x(\xi)|^{p-\delta})\, d\xi \\
&\geq -C_\epsilon - \epsilon(|x(\xi)|^2_{L^2} + \alpha|x(\xi)|^p_{L^p})
\end{aligned}$$

by Youngs inequality. □

Claim 2. For every $\epsilon > 0$ there exists $C_\epsilon \in (0,\infty)$ such that

$$\sum_{i=1}^{N} \beta_i(x)F^i(t,x) \geq -C_\epsilon - \epsilon(|x(\xi)|^2_{L^2} + \alpha|x(\xi)|^p_{L^p}), \quad \forall\, x \in L^p(0,1),\, t \in [0,T],\, N \in \mathbb{N}.$$

Proof of Claim 2. Let $x \in L^p(0,1),\ t \in [0,T]$. Then by (1.4)

$$\begin{aligned}
\sum_{i=1}^{N} \beta_i(x) F^i(t,x) &\geq -\sum_{i=1}^{N} \int_0^1 e_i(\xi)\, x(\xi)\, d\xi \int_0^1 e_i(\xi)\, f(t, x(\xi))\, d\xi \\
&\quad -\alpha \sum_{i=1}^{N} \lambda_i^{-1} \int_0^1 e_i(\xi)\, |x(\xi)|^{p-2} x(\xi)\, d\xi \int_0^1 e_i(\xi)\, f(t, x(\xi))\, d\xi \\
&\geq -\langle P_N F_0(t,x), P_N x\rangle - \alpha \sum_{i=1}^{\infty} \lambda_i^{-1} |f|_\infty \sqrt{\tfrac{2}{\pi}}\, |e_i|_{L^p} \left| |x|^{p-1} \right|_{L^{p/(p-1)}} \\
&\geq -|F_0(t,x)|_{L^2}\, |x|_{L^2} - \alpha \sum_{i=1}^{\infty} \lambda_i^{-1} |f|_\infty \tfrac{2}{\pi} |x|_{L^p}^{p-1} \\
&\geq -C_\epsilon - \epsilon(|x|_{L^2}^2 + \alpha |x|_{L^p}^p),
\end{aligned}$$

where P_N denotes the orthogonal projection in $L^2(0,1)$ onto H_N, i.e. the linear span of $\{e_1, ..., e_N\}$. $\square$

We note that C_ϵ can be taken in both claims to be a function only on δ, K and $|f|_\infty$ which is increasing in K and $|f|_\infty$, while decreasing in δ.

Now let us prove that by Claim 1 and Claim 2 that Hypothesis 2 is satisfied. To avoid a further regularization procedure let us additionally assume that $f(t,\cdot) \in C^2(\mathbb{R})$ for all $t \in [0,T]$ and $\frac{\partial}{dr} f(t,\cdot), \frac{\partial^2}{dr^2} f(t,\cdot) \in C([0,T]\times\mathbb{R})$. Define for $j \in \mathbb{N}$, $x \in H$, $t \in [0,T]$

$$F_j(t,x) := P_j F(t, P_j x) = \sum_{i=1}^{j} \left(\lambda_i^{-1} \int_0^1 e_i(\xi)\, f(t, (P_j x)(\xi))\, d\xi \right) e_i, \tag{2.37}$$

where P_j is the orthogonal projection onto the linear span of $\{e_1, ..., e_j\}$ in $H = L^2(0,1)$. Then obviously F_j is as in Hypothesis 2(iii) with $N_j = j$ and

$$\widetilde{f}_i(t, x_1, ..., x_j) = \lambda_i \int_0^1 e_i(\xi) f\left(t, \sum_{l=1}^{j} x_l e_l(\xi)\right) d\xi,$$

for $(x_1, ..., x_j) \in \mathbb{R}^j$. Now let us consider the corresponding $C_{F_j}(\delta)$ from Hypothesis 2(iii) and Ψ_N defined in Lemma 2.3. Note that $\Psi_N^2(\cdot, y)$ above is C^2 and strictly positive on $H_j = \mathbb{R}^j$ for ν_N-a.e. $y \in E$. Hence by definition $\Psi_{N,M,l}^2 = \Psi_N^2$ for all $M, l \in \mathbb{N}$. Hence for $(x,y) \in H_j \oplus H_j^\perp$, $t \in [0,T]$ by Claim 1 and Claim 2

$$D_{N_j,M,l}^* F_j(t,(x,y)) \leq C_\epsilon + \epsilon(|(x,y)|_{L^2}^2 + \alpha |(x,y)|_{L^p}^p).$$

Here we used that $\|P_j\|_{L^p \to L^p} \leq c_p \in (0,\infty)$ which is independent of j (see e.g. [19, Section 2C16]). Hence obviously for $\delta \in (0,1)$

$$\sup_{j \in \mathbb{N}} C_{F_j}(\delta) < \infty.$$

Hence by Theorem 2.1 we have a solution

$$\nu_t(dx) = \rho(t,x)\gamma(dx), \quad t \in [0,T],$$

with γ as above, for equation (1.1) for F as above with initial condition $\rho_0\,\gamma$ with ρ_0 in $L \log L$ with respect to γ.

Now we shall prove that this solution is also unique provided $\alpha > 0$, so γ is not Gaussian. We shall, however, apply a uniqueness result for the Gaussian reference measure $N(0, \frac{1}{2}(-A)^{-1})$ proved in [12], because ν_t has the density

$$\bar{\rho}(t,x) = \rho(t,x)\frac{1}{Z}\, e^{-\frac{\alpha}{p}|x|^p_{L^p}}, \quad (t,x) \in [0,T] \times H$$

with respect to $N(0, \frac{1}{2}(-A)^{-1})$. Let us first show that $\bar{\rho}$ is bounded in (t,x). To this end we first note that because $\sum_{i=1}^{\infty} \lambda_i^{-1} < \infty$,

$$R := \sup_{j\in\mathbb{N}} \||F_j|_{L^p}\|_{\infty} < \infty.$$

Hence the corresponding flows ξ_j from (A.1) with F_j replacing F will all stay in the L^p ball $B^p_{TR}(x)$ for all times in $[0,T]$ when started at x in $L^p(0,1)$. This implies by Claim 1 and 2 that the exponent of the density ρ^j in (2.28) with F_j replacing F will also have an upper bound of type

$$C_\epsilon + \epsilon(|x|^2_{L^2} + \alpha|x|^p_{L^p}), \quad \forall\, x \in L^p(0,1)$$

independent of j. Hence it follow that

$$\bar{\rho}^j(t,x) := \rho^j(t,x)\frac{1}{Z}\, e^{-\frac{\alpha}{p}|x|^p_{L^p}}, \quad (t,x) \in [0,T] \times H$$

is $N(0, \frac{1}{2}(-A)^{-1})$–essentially bounded, uniformly in j, hence so is its a.e. limit $\bar{\rho}$.

Now we can apply Theorem 2.3 in [12] for $p = \infty$ (which by a misprint there, seems to be excluded, but is in fact included in that theorem) to conclude uniqueness if we can prove the following properties (a)–(c) of F defined above. For this we additionally assume:

$$\textit{There exists } C, M \in (0,\infty) \textit{ such that } |f'(t,r)| \le C(1 + |r|^M), \quad r \in \mathbb{R}. \tag{2.38}$$

(a) $F([0,T] \times H) \subset (-A)^{-1/2}(H)$.

(b) There exists $s \in (1,\infty)$ such that

$$\int_0^T \int_H |(-A)^{1/2}F(t,x)|^s_H\, \gamma_0(dx)\, dt < \infty,$$

(c) $F \in L^2(0,T; W^{1,s}(H; H, \gamma_0))$, which is defined as the closure of all vector fields $F([0,T]\times H) \to H$ of type (2.7) with respect to the nom

$$\|F\|_{1,s,T} := \left(\int_0^T \int_H (\|DF(t,x)\|^s_{\mathcal{L}_2(H)} + |F(t,x)|^2_H)\, \gamma_0(dx)\, dt\right)^{1/s},$$

where $\|\cdot\|_{\mathcal{L}_2(H)}$ denotes the Hilbert–Schmidt norm and $\gamma_0 = N(0, \frac{1}{2}(-A)^{-1})$.

By the definition of F in (2.36) property (a) obviously holds. (b) holds for all $s \in (1,\infty)$ since

$$|(-A)^{1/2}F(t,x)|_H = |(-A)^{-1/2}F_0(t,x)|_H \le \textit{const.}\|f\|_\infty.$$

So, let us check (c): Let F_N be as in (2.37). Then for $1 \le i,j \le N$

$$\partial_{e_j}\langle e_i, F_N(t,x)\rangle = \frac{1}{\lambda_i}\int_0^1 e_i(\xi)\, e_j(\xi) f'(t,(P_N x)(\xi))\, d\xi, \quad (t,x) \in [0,T] \times H.$$

Hence by (2.38) for some constant $c_1 \in (0,\infty)$

$$\|DF_N(t,x)\|^2_{\mathcal{L}_2(H)} = \sum_{i=1}^N \frac{1}{\lambda_i}\int_0^1 e_i^2(\xi)\, |f'(t,(P_N x)(\xi))|^2\, d\xi$$

$$\le C_1 \sum_{i=1}^\infty \frac{1}{\lambda_i} \sup_{N\in\mathbb{N}} \|P_N\|^{2M}_{L^{2M}\to L^{2M}} \left(1 + |x|^{2M}_{L^{2M}}\right).$$

Hence $F_N(t,x))$, $N\in\mathbb{N}$, is bounded in the norm $\|\cdot\|_{1,2,T}$. Since $\sup_{n\in\mathbb{N}}\|F_N\|_\infty<\infty$ and $F_N\to F$ $dt\otimes\gamma_0$–a.e., (c) follows for $s=2$, because the operator D is closable.

3. Uniqueness

In Example 2.9 of previous section we proved uniqueness for (1.3) using the uniqueness result from [12] for Gaussian reference measures γ. For non–Gaussian, reference measures γ uniqueness for (1.3) is much more difficult to prove. In this section we do that for a whole class of non Gaussian, reference measures γ.

3.1. **Notations and preliminaries.** In this section, we take as reference measure γ the invariant measure of the following reaction–diffusion equation in $H:=L^2(0,1)$,

$$\begin{cases} dX(t)=[AX(t)+p(X(t))]dt+BdW(t),\\[2mm] X(0)=x,\quad x\in H,\end{cases}\tag{3.1}$$

where A is the realisation of the Laplace operator D_ξ^2 equipped with Dirichlet boundary conditions,

$$Ax=D_\xi^2x,\quad x\in D(A),\quad D(A)=H^2(0,1)\cap H_0^1(0,1),$$

p is a decreasing polynomial of odd degree equal to $N>1$, $B\in L(H)$ with a *bounded inverse* and W is an H–valued cylindrical Wiener process on a filtered probability space $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t>0},\mathbb{P})$. Let us recall the definition of solution of (3.1).

Definition 3.1. *(i). Let $x\in L^{2N}(0,1)$; we say that $X\in C_W([0,T];H)$* (1) *is a* mild *solution of problem* (3.1) *if $X(t)\in L^{2N}(0,1)$ for all $t\ge 0$ and fulfills the following integral equation*

$$X(t)=e^{tA}x+\int_0^t e^{(t-s)A}p(X(s))ds+\int_0^t e^{(t-s)A}dW(s),\quad t\ge 0.\tag{3.2}$$

(ii). Let $x\in H$; we say that $X\in C_W([0,T];H)$ is a generalized *solution of problem* (3.1) *if there exists a sequence $(x_n)\subset L^{2N}(0,1)$, such that*

$$\lim_{n\to\infty}x_n=x\quad\text{in } L^2(0,1),$$

and

$$\lim_{n\to\infty}X(\cdot,x_n)=X(\cdot,x)\quad\text{in } C_W([0,T];H).$$

It is convenient to introduce the following approximating problem

$$\begin{cases} dX_\alpha(t)=(AX_\alpha(t)+p_\alpha(X_\alpha(t))dt+B\,dW(t),\\[2mm] X_\alpha(0)=x\in H,\end{cases}\tag{3.3}$$

where for any $\alpha\in(0,1]$, p_α are the Yosida approximations of p, that is

$$p_\alpha(r)=\frac{1}{\alpha}\,(r-J_\alpha(r)),\ J_\alpha(r)=(1-\alpha p(\cdot))^{-1}(r),\quad r\in\mathbb{R}.$$

Notice that, since p_α is Lipschitz continuous, then for any $\alpha>0$, and any $x\in H$, problem (3.3) has a unique solution $X_\alpha(\cdot,x)\in C_W([0,T];H)$.

The following result is proved in [8, Theorem 4.8]

Proposition 3.2. *Let $T>0$, then*

(i) *If $x\in L^{2N}(0,1)$, problem* (3.1) *has a unique mild solution $X(\cdot,x)$.*

(1) By $C_W([0,T];H)$ we mean the set of H–valued stochastic processes continuous in mean square and adapted to the filtration $(\mathcal{F}_t)$.

(ii) *If* $x \in L^2(0,1)$*, problem* (3.1) *has a unique generalized solution* $X(\cdot,x)$.

In both cases $\lim_{\alpha\to 0} X_\alpha(\cdot,x) = X(\cdot,x)$ *in* $C_W([0,T];H)$.

Let us introduce now the transition semigroups P_t and P_t^α, setting

$$P_t\varphi(x) = \mathbb{E}[\varphi(X(t,x))], \quad \varphi \in B_b(H) \tag{3.4}$$

and

$$P_t^\alpha\varphi(x) = \mathbb{E}[\varphi(X_\alpha(t,x))], \quad \varphi \in B_b(H).$$

This definition extends to vector fields: if $G : H \to H$ is measurable bounded, we call $(\mathbf{P}_t G)(x)$ the element of H such that

$$\langle (\mathbf{P}_t G)(x), h\rangle_H = \mathbb{E}\left[\langle G(X(t,x)), h\rangle_H\right]$$

for every $h \in H$. It exists since

$$\left|\mathbb{E}\left[\langle G(X(\epsilon,x)), h\rangle_H\right]\right| \le \mathbb{E}\left[|G(t,x)|_H\right] |h|_H \le C_G |h|_H$$

where C_G bounds G. In the sequel we shall use the notation

$$\left(\frac{I-\mathbf{P}_t}{t}\right) G(t,x)$$

for $\frac{G(t,x)-(\mathbf{P}_t G(t,\cdot))(x)}{t}$ and for analogous expressions. We shall use similar notations for the semigroups associate to the Yosida regularizations, P_t^α and $\mathbf{P}_t^\alpha$.

Denote by $L_2(H)$ (resp. $\mathcal{L}(H)$) the Hilbert-Schmidt norm (resp. operator norm) of operators in H.

The sequence (e_j)

$$e_j(\xi) = \sqrt{\tfrac{2}{\pi}}\ \sin(j\pi\xi), \quad \xi \in [0,1],\ j \in \mathbb{N}, \tag{3.5}$$

is an orthonormal basis in H and it results

$$Ae_j = -\alpha_j e_j, \quad \forall\, j \in \mathbb{N}, \tag{3.6}$$

where

$$\alpha_j := \pi^2 j^2, \quad \forall\, j \in \mathbb{N}.$$

Lemma 3.3. *For every* $\theta_0 > 1/4$ *we have* $(-A)^{-\theta_0} \in L_2(H)$.

Proof. We have in fact

$$|(-A)^{-\theta_0}|^2_{L_2(H)} = \sum_{j\in\mathbb{N}} |(-A)^{-\theta_0} e_j|_H^2 = \sum_{j\in\mathbb{N}} |j|^{-4\theta_0} < \infty.$$

□ □

In the sequel we denote by θ_0 any number in $(\frac{1}{4},\frac{1}{2})$. We need $\theta_0 < \frac{1}{2}$ for the results on stochastic convolution.

Remark 3.4. When B is equal to the identity, (3.1) is a gradient system and the corresponding transition semigroup P_t is symmetric whereas if $B \neq I$, P_t is not symmetric.

For P_t^α the following Bismut-Elworthy-Li formula holds, see [15] and [13].

$$\langle D_x P_t^\alpha \varphi(x), h\rangle = \frac{1}{t}\,\mathbb{E}\left[\varphi(X_\alpha(t,x)) \int_0^t \langle B^{-1}\eta_\alpha^h(s,x), dW(s)\rangle\right], \quad h \in H, \tag{3.7}$$

where for any $h \in H$, $\eta_\alpha^h(t,x) =: D_x X_\alpha(t,x)\cdot h$ is the differential of $X_\alpha(t,x)$ with respect to x in the direction h. $\eta_\alpha^h(t,x)$ is the solution of the following equation with random coefficients

$$D_t\eta_\alpha^h(t,x) = A\eta_\alpha^h(t,x) + D_x p_\alpha(X_\alpha(t,x))\eta_\alpha^h(t,x), \quad \eta_\alpha^h(0,x) = h. \tag{3.8}$$

The proof of the following lemma is a straightforward consequence of the dissipativity of $p(\cdot)$.

Lemma 3.5. *It results*

$$|\eta_\alpha^h(t,x)|_H \le |h|_H, \quad \forall\, t \ge 0,\, x,h \in H,\, \alpha \in (0,1]. \tag{3.9}$$

Proposition 3.6. *Semigroups P_t and P_t^α have unique invariant measures γ, γ^α respectively. Moreover γ^α is weakly convergent to γ and for any $N \in \mathbb{N}$ there exists $c_N > 0$ such that*

$$\int_H |x|^{2N}_{L^{2N}(0,1)} \gamma^\alpha(dx) \le c_N, \quad \int_H |x|^{2N}_{L^{2N}(0,1)} \gamma(dx) \le c_N. \tag{3.10}$$

(see [8, Proposition 4.20] and [10, Proposition 15]).

Corollary 3.7. *Let $h(x) \in D(A)$–ν–a.e. $x \in H$, and $Ah \in L^4(H),\gamma)$. Then there exists $K > 0$ such that*

$$\int_H |Dp_\alpha(x)h(x)|^2\, \gamma(dx) \le K\|Ah\|^2_{L^4(H,\gamma)}, \quad \forall\, \alpha \in (0,1]. \tag{3.11}$$

Proof. Let $h(x) \in D(A)$. Then there is $K_1 > 0$ such that

$$|p'(x)h(x)|^2 \le K_1 |x^{N-1}|^2\, |h(x)|^2_{D(A)} \le K_1 |x|^{2N-2}_{L^{2N-2}}\, |h(x)|^2_{D(A)}.$$

Integrating with respect to γ over H and using Hölder's inequality, yields

$$\begin{aligned}\int_H |p'(x)h(x)|^2\, \gamma(dx) &\le K_1 \int_H |x|^{2N-2}_{L^{2N-2}}\, |Ah(x)|^2\, \gamma(dx) \\ &\le K_1 \int_H |x|^{4N-4}_{L^{2N-2}}\, \gamma(dx)\, \|Ah\|^2_{L^4(H,\gamma)}.\end{aligned}$$

Now the conclusion follows from (3.10). □ □

Let us finally recall the elementary identity, see [10]

$$\langle P_t^\alpha D_x\varphi, h\rangle = \langle D_x P_t^\alpha \varphi, h\rangle - \int_0^t P^\alpha_{t-s}[\langle Ah + D_x p^\alpha(x)h, D_x P_s^\alpha \varphi\rangle]\, ds. \tag{3.12}$$

where $h \in D(A)$ and $\varphi \in C_b^1(H)$.

3.2. **The range condition.** Let us consider the Kolmogorov operator

$$\mathcal{K}u(t,x) = D_t u(t,x) + \langle F(t,x), D_x u(t,x)\rangle, \tag{3.13}$$

defined for all $u \in \mathcal{F}C^1_{b,T}$, the space of all functions u defined in Section 1 with $Y = D(A)$.

Now the continuity equation (1.3) can be written as

$$\int_0^T \int_H \mathcal{K}u(t,x)\, \rho(t,x)\, \gamma(dx)dt = -\int_H u(0,x)\rho_0(x)\, \gamma(dx), \quad u \in \mathcal{F}C^1_b. \tag{3.14}$$

The following result has be proven in [12].

Proposition 3.8. *Assume that for $p \in [1,\infty)$ the following range condition is fulfilled*

$$\mathcal{K}(\mathcal{F}C^1_{b,T}) \text{ is dense in } L^p([0,T];L^p(H,\gamma)). \tag{3.15}$$

Then if ρ_1 and ρ_2 are two solutions of (3.14) in $L^{p'}([0,T];L^{p'}(H,\gamma))$, with $p' = \frac{p}{p-1}$, $p = \frac{p}{p-1}$, we have $\rho_1 = \rho_2$.

Let now consider the approximating equation

$$\begin{cases} D_t u_j(t,x) + \langle F_j(t,x), D_x u_j(t,x)\rangle = f(t,x), \\ \\ u_j(T,\cdot) = 0, \end{cases} \tag{3.16}$$

where (F_j) where defined in Hypothesis 2 and $f \in \mathcal{F}C^1_{b,T}$. Problem (3.16) has a unique classical solution given by

$$u_j(t,x) = -\int_t^T f(s, \xi_j(s,t,x))ds. \tag{3.17}$$

where ξ_j is the solution to

$$\frac{d}{dt}\,\xi_j(t) = F_j(t, \xi_j(t)), \quad \xi_j(s) = x. \tag{3.18}$$

Let us consider a further approximation $P_\epsilon u_j(t,x)$ of $u(t,x)$, where P_ϵ is the transition semigroup defined in (3.4) and $\epsilon \in (0,1]$. Applying P_ϵ to both sides of equation (3.16) we have

$$D_t(P_\epsilon u_j) + \langle F, D_x P_\epsilon u_j\rangle = P_\epsilon f + \langle F - F_j, D_x P_\epsilon u_j\rangle + B_\epsilon(F_j, u_j),$$

where $B_\epsilon(F_j, u_j)$ is the *DiPerna–Lions* commutator defined for $\epsilon \in (0,1]$ as

$$B_\epsilon(u,F)(t,x) := \langle D_x P_\epsilon u(t,x), F(t,x)\rangle - P_\epsilon(\langle D_x u(t,x), F(t,x)\rangle), \quad \forall\, u \in \mathcal{F}C^1_{b,T}\,, F \in \mathcal{V}\mathcal{F}C^1_{b,T}. \tag{3.19}$$

Now the range condition follows provided

$$\lim_{\epsilon\to 0}\lim_{j\to\infty} B_\epsilon(u_j, F_j) = 0 \quad \text{in } u \in L^1([0,T], L^1(H,\gamma)). \tag{3.20}$$

As shown in [12], the basic tool to show (3.20) is provided by an estimate for the integral

$$\int_0^T \int_H |B_\epsilon(u,F)|\, dt\, d\gamma, \quad \epsilon \in (0,1], \quad \forall\, u \in \mathcal{F}C^1_{b,T}\,, F \in \mathcal{V}\mathcal{F}C^1_{b,T},$$

in terms of $\|u\|_\infty$ independent of ϵ.

3.3. **Main result.** To express the main result of this section we need some definitions.

Definition 3.9. *We call $\mathcal{V}(H,\gamma)$ the space of all measurable functions $\phi : H \to \mathbb{R}$ such that*

$$\|\phi\|^2_{\mathcal{V}(H,\gamma)} := \sup_{\epsilon\in(0,1)} \int_H \phi(x) \left(\frac{I - P_\epsilon}{\epsilon}\right) \phi(x)\, \gamma(dx)$$

is finite and we endow $\mathcal{V}(H,\gamma)$ by the norm $\|\phi\|_{\mathcal{V}(H,\gamma)}$. Similarly we call $\mathcal{V}(H,H,\gamma)$ the space of all measurable vector fields $G : H \to \mathbb{R}$ such that

$$\|G\|^2_{\mathcal{V}(H,H,\gamma)} := \sup_{\epsilon\in(0,1)} \int_H \left\langle \left(\frac{I - \mathbf{P}_\epsilon}{\epsilon}\right) G(x)\,, G(x) \right\rangle_H \gamma(dx)$$

is finite and we endow $\mathcal{V}(H,H,\gamma)$ by the norm $\|G\|_{\mathcal{V}(H,H,\gamma)}$.

We note that in the symmetric case $(B = I)$, $\mathcal{V}(H,\gamma)$ coincides with $D((-\mathcal{L})^{1/2})$.

Lemma 3.10. *The space $\mathcal{F}C^2_b(H)$ is contained in $\mathcal{V}(H,\gamma)$. Similar result holds for every vector field G of the form $G = \sum_{h=1}^n G_h e_h$, with $G_h \in \mathcal{F}C^2_b$ for all $h = 1, ..., n$.*

Proof. We have

$$(I - P_\epsilon)\,\phi(x) = \int_0^\epsilon P_s \mathcal{L}\phi(x)\, ds$$

where $\mathcal{L}$ is the infinitesimal generator of P_t. One can check that $\mathcal{L}\phi$ is a bounded continuous function; in particular this is true for the term $\langle p(x)\,, D_x\phi(x)\rangle$ because the argument of ϕ is in the space of continuous functions. Hence $\left(\frac{I-P_\epsilon}{\epsilon}\right)\phi$ is also bounded and thus $\phi \in \mathcal{V}(H,\gamma)$. □ □

Finally, we have our main estimate. Given $\theta_0 \in (\frac{1}{4}, \frac{1}{2})$ and $\theta \in \left(\theta_0, \frac{1}{2}\right)$, we define

$$\|F\|_{p,q,\gamma,T} := \left\|(-A)^{\theta_0} F\right\|_{L^{\frac{p}{p-1}}(0,T;\mathcal{V}(H,H,\gamma))} + \left\|(-A)^{1/2+\theta} F\right\|_{L^{\frac{p}{p-1}}(0,T;L^q(H,\gamma))} + \|\operatorname{div} F\|_{L^{\frac{p}{p-1}}\left(0,T;L^{\frac{p}{p-1}}(H,\gamma)\right)}.$$

Theorem 3.11. *For every p, q satisfying*

$$p \in (2,\infty), \quad \frac{1}{p} + \frac{1}{q} < 1,$$

for every vector field $F : [0,T] \times H \to D\left((-A)^{1/2+\theta}\right)$ such that $\|F\|_{p,q,\gamma,T}$ is finite, there is at most one solution of the continuity equation in $L^{q'}([0,T]; L^{p'}(H,\gamma))$, with $p' = \frac{p}{p-1}$, $q' = \frac{q}{q-1}$.

Proof. The conclusion of the theorem follows from the rank condition proved in Theorem 3.19 below, and Proposition 3.8. □ □

Corollary 3.12. *If B in* (3.1) *is the identity, then under the conditions of Theorem 3.11 there exists a unique solution of the continuity equation in $L^{q'}([0,T]; L^{q'}(H,\gamma))$*

Proof. The existence follows by Theorem 2.1 and Remark 2.8. □ □

Remark 3.13. As already mentioned in Remark 1.5, so far we cannot prove whether Hypothesis 2(ii) holds for γ as in Example 1.3(iii), if B in (1.5), (3.1) is not the identity operator. In this case it was proved in [7], [9] that γ has a density f with respect to $\gamma_0 := N(0, \frac{1}{2}(-A)^{-1})$ such that $\sqrt{f} \in W^{1,2}(H,\gamma_0)$, i.e. the Sobolev space of order 1 in $L^2(H,\gamma_0)$. To verify Hypothesis 2(ii) it would be enough to show that $x \mapsto f(x,y), (x,y) \in H_N \oplus E_N$, is continuous and strictly positive on H_N, for all $N \in \mathbb{N}$ and ν_N–a.e. $y \in E_N$, where A, H_N, E_N and ν_N are as in Example 2.9. However, so far we did not succeed to prove this. If this could be shown, Corollary 3.12 would hold for any B in (1.5), (3.1).

3.4. **Estimating the commutator.** We first express the DiPerna–Lions commutator $B_\epsilon(u,F)$ using the identity (3.12). It is convenient to introduce the approximating commutator

$$B^\alpha_\epsilon(u,F)(t,x) := D_x P^\alpha_\epsilon u(t,x) \cdot F(t,x) - P^\alpha_\epsilon (D_x u(t,x) \cdot F(t,x)), \quad \forall\, u \in \mathcal{F}C^1_{b,T}(H), F \in \mathcal{V}\mathcal{F}C^1_{b,T}(H) \tag{3.21}$$

for any $\alpha \in (0,1]$.

Lemma 3.14. *Assume that $F = \sum_{h=1}^n F^h e_h$, with $F^h \in \mathcal{V}\mathcal{F}C^1_{b,T}(D(A))$, $h = 1, ..., n$. Then we have*

$$\begin{aligned}
B^\alpha_\epsilon(u,F) &= \frac{1}{\epsilon}\,\mathbb{E}\left[u(t,X_\alpha(\epsilon,x))(F(t,x) - F(t,X_\alpha(\epsilon,x)) \cdot \int_0^\epsilon (D_x X_\alpha(\eta,x))^* \pi_n (B^{-1})^* dW(\eta)\right] \\
&+ \int_0^\epsilon P^\alpha_{\epsilon-\eta}\Bigg\{\frac{1}{\eta}\mathbb{E}\Big[u(t,X_\alpha(\eta,x)) \\
&\quad \times \Big\langle F(t,X_\alpha(\eta,x)), \int_0^\eta (A + Dp_\alpha(x)))(D_x X_\alpha(\lambda,x))^* \pi_n (B^{-1})^*\, dW(\lambda)\Big\rangle\Big]\Bigg\} d\eta \\
&\quad + P^\alpha_\epsilon(u \operatorname{div} F),
\end{aligned} \tag{3.22}$$

where π_n is the orthogonal projector on $(e_1, ..., e_n)$.

Proof. Taking into account (3.12), we write

$$
\begin{aligned}
P^\alpha_\epsilon(Du\cdot F) &= \sum_{h=1}^n P^\alpha_\epsilon(D_h u\, F^h) = \sum_{h=1}^n P^\alpha_\epsilon(D_h(uF^h)) - P^\alpha_\epsilon(u\,\mathrm{div}\,F)\\
&= \sum_{h=1}^n D_h P^\alpha_\epsilon(uF^h) - \sum_{h=1}^n \int_0^\epsilon P^\alpha_{\epsilon-\eta}\left[D_x P^\alpha_\eta(uF_h)\cdot(Ae_h + Dp_\alpha e_h)\right] d\eta - P^\alpha_\epsilon(u\,\mathrm{div}\,F).
\end{aligned}
\tag{3.23}
$$

Therefore

$$
\begin{aligned}
B^\alpha_\epsilon(u,F) &= \sum_{h=1}^n [D_h P^\alpha_\epsilon(u)F_h - D_h(P^\alpha_\epsilon(uF_h)]\\
&+\sum_{h=1}^n \int_0^\epsilon P^\alpha_{\epsilon-\eta}[D_x P^\alpha_\eta(uF_h)\cdot(Ae_h + D_x p_\alpha(x)e_h)]d\eta + P^\alpha_\epsilon(u\,\mathrm{div}\,F)\\
&=: I_1 + I_2 + I_3.
\end{aligned}
\tag{3.24}
$$

Let us write I_1 and I_2 in a more compact way. Recalling the Bismut-Elworthy-Li formula (3.7) we have

$$
\begin{aligned}
I_1 &= \frac{1}{\epsilon}\sum_{h=1}^n \mathbb{E}\left[u(t,X_\alpha(\epsilon,x))(F_h(t,x) - F_h(t,X_\alpha(\epsilon,x))\int_0^\epsilon D_x X_\alpha(\eta,x)e_h)\cdot \pi_n(B^{-1})^* dW(\eta)\right]\\
&= \frac{1}{\epsilon}\mathbb{E}\left[u(t,X_\alpha(\epsilon,x)(F(t,x) - F(t,X_\alpha(\epsilon,x))\cdot\int_0^\epsilon (D_x X_\alpha(\eta,x))^*\pi_n(B^{-1})^* dW(\eta)\right]
\end{aligned}
\tag{3.25}
$$

(the last integral is well defined because obviously $\pi_n(B^{-1})^*(X_x(\eta,x))^*$ is Hilbert–Schmidt.) As for I_2 we have, using again (3.7)

$$
\begin{aligned}
I_2 &= \sum_{h=1}^n \int_0^\epsilon P^\alpha_{\epsilon-\eta}[D_x P^\alpha_\eta(uF_h)\cdot(Ae_h + D_x p_\alpha(x)e_h)]d\eta\\
&= \sum_{h=1}^n \int_0^\epsilon P^\alpha_{\epsilon-\eta}\Bigg\{\frac{1}{\eta}\mathbb{E}\Big[u(t,X_\alpha(\eta,x))\,F_h(t,X_\alpha(\eta,x)))\\
&\qquad\times\int_0^\eta \langle B^{-1}D_x X_\alpha(\lambda,x)(A\pi_n e_h + D_x p_\alpha \pi_n e_h), dW(\lambda)\rangle\Big]\Bigg\}d\eta \qquad (3.26)\\
&= \int_0^\epsilon P^\alpha_{\epsilon-\eta}\Bigg\{\frac{1}{\eta}\mathbb{E}\Big[u(t,X_\alpha(\eta,x))\,F(t,X_\alpha(\eta,x)))\\
&\qquad\cdot\int_0^\eta (A + D_x p_\alpha(x))((D_x X_\alpha(\eta,x))^*\,\pi_n(B^{-1})^* dW(\lambda)\Big]\Bigg\}d\eta.
\end{aligned}
$$

So, (3.22) follows. □ □

The following corollary is a consequence of Lemma 3.14 taking into account the invariance of γ_α.

Corollary 3.15. *Assume that* $F = \sum_{h=1}^n F^h e_h$, *with* $F^h \in \mathcal{F}C_b^1(D(A))$, $h = 1, ..., n$. *Then we have,*

$$
\begin{aligned}
&\int_H |B_\epsilon^\alpha(u, F)| d\gamma_\alpha \\
&\leq \frac{1}{\epsilon} \int_H \mathbb{E} \left| u(t, X_\alpha(\epsilon, x))(F(t, x) - F(t, X_\alpha(\epsilon, x)) \cdot \int_0^\epsilon (D_x X_\alpha(\eta, x))^* \pi_n (B^{-1})^* dW(\eta) \right| d\gamma_\alpha \\
&+ \int_H \int_0^\epsilon \frac{1}{\eta} \mathbb{E} \Big| u(X_\alpha(\eta, x))\, F(X(\eta, x))) \\
&\qquad\qquad \cdot \int_0^\eta (A + D_x p_\alpha(x)))(D_x X_\alpha(\eta, x))^* \pi_n (B^{-1})^* dW(\lambda) \Big| d\eta\, d\gamma_\alpha \\
&+ \int_H |u \operatorname{div} F|\, d\gamma_\alpha =: J_1 + J_2 + J_3.
\end{aligned} \tag{3.27}
$$

To estimate $\int_H |B_\epsilon(u, F)|\, d\gamma$ we need some preliminary results.

Proposition 3.16. *For every* $p \in (2, \infty]$ *there is a constant* $C_p > 0$, *independent of* α *and* ϵ, *such that*

$$
\begin{aligned}
&\frac{1}{\epsilon} \int_H \mathbb{E} \left[\left| u(t, x) \left\langle F(t, x) - F(t, X_\alpha(\epsilon, x)), \int_0^\epsilon (D_x X_\alpha(\eta, x))^* \left(B^{-1}\right)^* dW(\eta) \right\rangle \right| \right] \gamma_\alpha(dx) \\
&\leq C_{A,B,p} \left(\int_H \left\langle \left(\frac{I - \mathbf{P}_\epsilon^\alpha}{\epsilon} \right) (-A)^{\theta_0} F(t, x), (-A)^{\theta_0} F(t, x) \right\rangle_H \gamma_\alpha(dx) \right)^{1/2} \left(\int_H |u(t, x)|^p \gamma_\alpha(dx) \right)^{1/p}
\end{aligned}
$$

where $C_{A,B,p} = C_p \left\| (-A)^{-\theta_0} \right\|_{L_2(H)} \left\| B^{-1} \right\|_{\mathcal{L}(H)}$ *for some constant* $C_p > 0$.

Proof. Call I the integral we have to estimate. To shorten the notations, call I' the stochastic integral

$$
I' := \int_0^\epsilon (-A)^{-\theta_0} (D_x X_\alpha(\eta, x))^* \left(B^{-1}\right)^* dW(\eta).
$$

We have

$$
\begin{aligned}
I &= \frac{1}{\epsilon} \int_H \mathbb{E} \left[u(t, x) \left\langle (-A)^{\theta_0} F(t, x) - (-A)^{1/2} F(t, X_\alpha(\epsilon, x)), I' \right\rangle \right] \gamma_\alpha(dx) \\
&\leq \frac{1}{\epsilon} \left(\int_H \mathbb{E} \left[\left\| (-A)^{\theta_0} F(t, x) - (-A)^{\theta_0} F(t, X_\alpha(\epsilon, x)) \right\|_H^2 \right] \gamma_\alpha(dx) \right)^{1/2} \\
&\quad \cdot \left(\int_H |u(t, x)|^p \gamma_\alpha(dx) \right)^{1/p} \left(\int_H \mathbb{E} \left[\left\| I' \right\|_H^{r(p)} \right] \gamma_\alpha(dx) \right)^{1/r(p)}
\end{aligned}
$$

with $\frac{1}{p} + \frac{1}{2} + \frac{1}{r(p)} = 1$ namely $r(p) = \frac{p-2}{2p}$ and in particular with the condition

$$
p \in (2, \infty].
$$

By the Burkholder-Davies–Gundy inequality,

$$\begin{aligned}
&\mathbb{E}\left[\left|I'\right|_H^{r(p)}\right] \leq C_p \mathbb{E}\left[\left(\int_0^\epsilon \left\|(-A)^{-\theta_0}\left(D_x X_\alpha(\eta,x)\right)^*\left(B^{-1}\right)^*\right\|_{L_2(H)}^2 d\eta\right)^{r(p)/2}\right] \\
&\leq C_p \left\|(-A)^{-\theta_0}\right\|_{L_2(H)}^{r(p)} \left\|B^{-1}\right\|_{\mathcal{L}(H)}^{r(p)} \mathbb{E}\left[\left(\int_0^\epsilon \left\|D_x X_\alpha(\eta,x)\right\|_{\mathcal{L}(H)}^2 d\eta\right)^{r(p)/2}\right] \\
&\leq C_p \left\|(-A)^{-\theta_0}\right\|_{L_2(H)}^{r(p)} \left\|B^{-1}\right\|_{\mathcal{L}(H)}^{r(p)} \left(\sqrt{\epsilon}\right)^{r(p)}
\end{aligned}$$

because, by dissipativity of the reaction diffusion system,

$$\left\|D_x X_\alpha(\eta,x)\right\|_{\mathcal{L}(H)} \leq 1.$$

Therefore

$$I \leq \frac{C}{\sqrt{\epsilon}}\left(\int_H \mathbb{E}\left[\left|(-A)^{\theta_0} F(t,x) - (-A)^{\theta_0} F(t, X_\alpha(\epsilon,x))\right|_H^2\right]\gamma_\alpha(dx)\right)^{1/2}\left(\int_H |u(t,x)|^p \gamma_\alpha(dx)\right)^{1/p}$$

where $C = C_p^{1/r(p)}\left\|(-A)^{-\theta_0}\right\|_{L_2(H)}\left\|B^{-1}\right\|_{\mathcal{L}(H)}$. Finally, writing $G(t,x) = (-A)^{\theta_0} F(t,x)$,

$$\begin{aligned}
&\int_H \mathbb{E}\left[\left|(-A)^{1/2} F(t,x) - (-A)^{1/2} F(t, X_\alpha(\epsilon,x))\right|_H^2\right]\gamma_\alpha(dx) \\
&= \int_H \left(|G(t,x)|_H^2 - 2\mathbb{E}\left[\langle G(t,x), G(t, X_\alpha(\epsilon,x))\rangle_H\right] + \mathbb{E}\left[|G(t, X_\alpha(\epsilon,x))|_H^2\right]\right)\gamma_\alpha(dx).
\end{aligned}$$

Now

$$\mathbb{E}\left[\langle G(t,x), G(t, X_\alpha(\epsilon,x))\rangle_H\right] = \langle G(t,x), \mathbb{E}\left[G(t, X_\alpha(\epsilon,x))\right]\rangle_H = \langle G(t,x), \left(\mathbf{P}_\epsilon^\alpha G(t,\cdot)\right)(x)\rangle_H$$

$$\begin{aligned}
\int_H \mathbb{E}\left[|G(t, X_\alpha(\epsilon,x))|_H^2\right]\gamma_\alpha(dx) &= \int_H \left(P_\epsilon^\alpha |G(t,\cdot)|_H^2\right)(x)\,\gamma_\alpha(dx) \\
&= \int_H |G(t,x)|_H^2\,\gamma_\alpha(dx)
\end{aligned}$$

because γ_α is invariant for P_ϵ^α, hence

$$\begin{aligned}
&\int_H \mathbb{E}\left[\left|(-A)^{1/2} F(t,x) - (-A)^{1/2} F(t, X_\alpha(\epsilon,x))\right|_H^2\right]\gamma_\alpha(dx) \\
&= 2\int_H \left(|G(t,x)|_H^2 - \langle G(t,x), \left(\mathbf{P}_\epsilon^\alpha G(t,\cdot)\right)(x)\rangle_H\right)\gamma_\alpha(dx) \\
&= 2\int_H \langle G(t,x), G(t,x) - \left(\mathbf{P}_\epsilon^\alpha G(t,\cdot)\right)(x)\rangle_H\,\gamma_\alpha(dx).
\end{aligned}$$

Collecting these facts, we have proved the proposition. □ □

Proposition 3.17. *Under the assumptions of Theorem 3.11 there exist constants $C_{A,B,p}$ (given by Proposition 3.16) and $C_{A,B,p,q,\theta}$, both independent of α and ϵ, such that*

$$\begin{aligned}
&\int_H \left|B_\epsilon^\alpha\left(u,F\right)\left(t,x\right)\right|\gamma_\alpha\left(dx\right) \\
&\leq C_{A,B,p}\left\|u\left(t,\cdot\right)\right\|_{L^p(H,\gamma_\alpha)}\left(\int_H\left\langle\left(\frac{I-\mathbf{P}_\epsilon^\alpha}{\epsilon}\right)\left(-A\right)^{\theta_0}F\left(t,x\right),\left(-A\right)^{\theta_0}F\left(t,x\right)\right\rangle_H\gamma_\alpha\left(dx\right)\right)^{1/2} \\
&+C_{A,B,p,q,\theta}\left\|u\left(t,\cdot\right)\right\|_{L^p(H,\gamma_\alpha)}\left\|\left(-A\right)^{1/2+\theta}F\left(t,\cdot\right)\right\|_{L^q(H,\gamma_\alpha)} \\
&+\left\|u\left(t,\cdot\right)\right\|_{L^p(H,\gamma_\alpha)}\left\|\operatorname{div}F\left(t,\cdot\right)\right\|_{L^{\frac{p}{p-1}}(H,\gamma_\alpha)}
\end{aligned}$$

for all functions $u\in\mathcal{F}C_{b,T}^1(H)$ and vector field F of the form $F=\sum_{h=1}^n F_he_h$, with $F_h\in\mathcal{F}C_{b,T}^2(H)$ for all $h=1,...,n$.

Proof. **Step 1**. We know

$$\int_H\left|B_\epsilon^\alpha\left(u,F\right)\left(t,x\right)\right|\gamma_\alpha\left(dx\right)\leq J_1+J_2+J_3$$

where

$$J_1=\frac{1}{\epsilon}\int_H\mathbb{E}\left[\left|u\left(t,x\right)\left\langle F\left(t,x\right)-F\left(t,X_\alpha\left(\epsilon,x\right)\right),\int_0^\epsilon\left(D_xX_\alpha\left(\eta,x\right)\right)^*\left(B^{-1}\right)^*dW\left(\eta\right)\right\rangle\right|\right]\gamma_\alpha\left(dx\right)$$

$$J_2=\int_H\int_0^\epsilon\frac{1}{\eta}\mathbb{E}\left[\left|u\left(t,X_\alpha\left(\eta,x\right)\right)\left\langle F\left(t,X_\alpha\left(\eta,x\right)\right),J_2'\right\rangle\right|\right]d\eta d\gamma_\alpha\left(x\right)$$

$$J_3=\int_H u\left(t,x\right)\operatorname{div}F\left(t,x\right)\gamma_\alpha\left(dx\right).$$

where for shortness we wrote

$$J_2'=\int_0^\eta\left(A+D_xp_\alpha\left(x\right)\right)^*\left(D_xX_\alpha\left(\lambda,x\right)\right)^*\left(B^{-1}\right)^*dW\left(\lambda\right).$$

The estimate for J_1 has been made above and the estimate for J_3 is trivial. We need only to estimate J_2. Let $r>0$ be such that

$$\frac{1}{p}+\frac{1}{q}+\frac{1}{r}=1.$$

Then

$$\begin{aligned}
J_2&\leq\int_0^\epsilon\frac{1}{\eta}d\eta\left(\int_H\mathbb{E}\left[\left|u\left(t,X_\alpha\left(\eta,x\right)\right)\right|^p\right]\gamma_\alpha\left(dx\right)\right)^{1/p} \\
&\cdot\left(\int_H\mathbb{E}\left[\left|\left(-A\right)^{1/2+\theta}F\left(t,X_\alpha\left(\eta,x\right)\right)\right|_H^q\right]\gamma_\alpha\left(dx\right)\right)^{1/q}\left(\int_H\mathbb{E}\left[\left|J_2'\right|_H^r\right]\gamma_\alpha\left(dx\right)\right)^{1/r}
\end{aligned}$$

$$\begin{aligned}
&\leq\int_0^\epsilon\frac{1}{\eta}d\eta\left(\int_H\left(P_\eta^\alpha\left(\left|u\left(t,\cdot\right)\right|^p\right)\right)\left(x\right)\gamma_\alpha\left(dx\right)\right)^{1/p} \\
&\cdot\left(\int_H\left(P_\eta^\alpha\left(\left|\left(-A\right)^{1/2+\theta}F\left(t,\cdot\right)\right|_H^q\right)\right)\left(x\right)\gamma_\alpha\left(dx\right)\right)^{1/q} \\
&\cdot\left(\int_H\mathbb{E}\left[\left(\int_0^\eta\left\|\left(-A\right)^{-1/2-\theta}\left(A+D_xp_\alpha\left(x\right)\right)^*\left(D_xX_\alpha\left(\lambda,x\right)\right)^*\left(B^{-1}\right)^*\right\|_{L_2(H)}^2d\lambda\right)^{r/2}\right]\gamma_\alpha\left(dx\right)\right)^{1/r}
\end{aligned}$$

and using invariance of γ_α for P_η^α and the fact that B^{-1} is bounded,

$$J_2 \leq \left\|B^{-1}\right\|_{\mathcal{L}(H)} C(\epsilon,\theta,r) \left\|u(t,\cdot)\right\|_{L^p(H,\gamma_\alpha)} \left\|(-A)^{1/2+\theta} F(t,\cdot)\right\|_{L^q(H,\gamma_\alpha)}$$

where $C(\epsilon,\theta,r)$ and $g(x)$ are given respectively by:

$$\int_0^\epsilon \frac{1}{\eta} d\eta \left(\int_H \mathbb{E}\left[\left(\int_0^\eta \left\|(-A)^{-1/2-\theta}(A+D_x p_\alpha(x))^* (D_x X_\alpha(\lambda,x))^*\right\|_{L_2(H)}^2 d\lambda\right)^{r/2}\right]\gamma_\alpha(dx)\right)^{1/r}$$

$$\leq \int_0^\epsilon \frac{1}{\eta} d\eta \left(\int_H \mathbb{E}\left[\left(\int_0^\eta \left\|(-A)^{1/2-\theta}(D_x X_\alpha(\lambda,x))^*\right\|_{L_2(H)}^2 d\lambda\right)^{r/2}\right] g(x)\gamma_\alpha(dx)\right)^{1/r}$$

$$g(x) := \left\|(-A)^{-1/2-\theta}(A+D_x p_\alpha(x))^*(-A)^{-1/2+\theta}\right\|_{\mathcal{L}(H)}^r.$$

It remains to estimate $C(\epsilon,\theta,r,\theta)$ (which a priori may be infinite).

Step 2. From [11, Corollary 2.3], we have, for $\delta \in (0,1-\alpha)$,

$$\int_0^\eta \left|(-A)^{(1-\alpha-\delta)/2} D_x X_\alpha(t,x) h\right|_H^2 dt \leq C(T)\Delta_T(x)\eta^\delta \|h\|_{D\left((-A)^{-\alpha/2}\right)}^2$$

where

$$\Delta_T(x) = 1 + \sup_{t\in[0,T]} \|D_x p_\alpha(X_\alpha(t,x))\|_\infty^2$$

(it is a random variable). In particular, choosing δ very small and $\alpha = 1-2\delta < 1-\delta$, since the H norm is bounded by any $D((-A)^\varepsilon)$-norm for $\varepsilon > 0$, we get

$$\int_0^\eta |D_x X_\alpha(t,x)h|_H^2 dt \leq C(T)\Delta_T(x)\eta^\delta |h|_{D\left((-A)^{-1/2+\delta}\right)}^2.$$

Hence, for $\delta = \theta - \theta_0$ (all constants denoted by $C, C(T)$ below, different from line to line, may depend on T but not on α),

$$\begin{aligned}
&\int_0^\eta \left\|(-A)^{1/2-\theta}(D_x X_\alpha(\lambda,x))^*\right\|_{L_2(H)}^2 d\lambda \\
&= \int_0^\eta \left\|D_x X_\alpha(\lambda,x)(-A)^{1/2-\theta}\right\|_{L_2(H)}^2 d\lambda \\
&= \sum_k \int_0^\eta \left|D_x X_\alpha(\lambda,x)(-A)^{1/2-\theta} e_k\right|_H^2 d\lambda \\
&\leq C(T)\Delta_T(x)\eta^{2(\theta-\theta_0)} \sum_k \left|(-A)^{1/2-\theta} e_k\right|_{D\left((-A)^{-1/2+(\theta-\theta_0)}\right)}^2 \\
&= C(T)\Delta_T(x)\eta^{\theta-\theta_0} \sum_k \left|(-A)^{-\theta_0} e_k\right|_H^2 \\
&= C(T)\Delta_T(x)\eta^{\theta-\theta_0} \left\|(-A)^{-\theta_0}\right\|_{L_2(H)}^2.
\end{aligned}$$

Hence

$$\begin{aligned}
C(\epsilon,\theta,r) &\leq \int_0^\epsilon \frac{1}{\eta} d\eta \left(\int_H \mathbb{E}\left[\left(C(T)\Delta_T(x)\eta^{\theta-\theta_0}\left\|(-A)^{-\theta_0}\right\|_{L_2(H)}^2\right)^{r/2}\right] g(x)\gamma_\alpha(dx)\right)^{1/r} \\
&= C(T)^{1/2}\left\|(-A)^{-\theta_0}\right\|_{L_2(H)} \int_0^\epsilon \frac{\eta^{r(\theta-\theta_0)/2}}{\eta} d\eta \left(\int_H \mathbb{E}\left[\Delta_T(x)^{r/2}\right] g(x)\gamma_\alpha(dx)\right)^{1/r}
\end{aligned}$$

It remains to bound

$$
\begin{aligned}
&\int_H \mathbb{E}\left[\Delta_T(x)^{r/2}\right] g(x)\,\gamma_\alpha(dx) \\
&= \int_H \mathbb{E}\left[\Delta_T(x)^{r/2}\right] \left\|(-A)^{-1/2-\theta}\left(A + D_x p_\alpha(x)\right)^*(-A)^{-1/2+\theta}\right\|_{\mathcal{L}(H)}^r \gamma_\alpha(dx) \\
&\le C\int_H \mathbb{E}\left[\Delta_T(x)^{r/2}\right]\left(1 + \left\|(-A)^{-1/2+\theta} D_x p_\alpha(x)(-A)^{-1/2-\theta}\right\|_{\mathcal{L}(H)}^r\right)\gamma_\alpha(dx) \\
&\le C\left(\int_H \mathbb{E}\left[\Delta_T(x)^r\right]\gamma_\alpha(dx)\right)^{1/2}\cdot \\
&\cdot\left(\int_H\left(1 + \left\|(-A)^{-1/2+\theta} D_x p_\alpha(x)(-A)^{-1/2-\theta}\right\|_{\mathcal{L}(H)}^{2r}\right)\gamma_\alpha(dx)\right)^{1/2}
\end{aligned}
$$

renaming the constants. We have

$$\Delta_T(x) \le 1 + C\sup_{t\in[0,T]}\|X_\alpha(t,x)\|_\infty^{N-1}$$

and thus, by [11, Theorem 4.8 (iii)],

$$\mathbb{E}\left[\Delta_T(x)^r\right] \le C + C|x|_H^{r(N-1)}$$

which implies

$$\int_H \mathbb{E}\left[\Delta_T(x)^r\right]\gamma_\alpha(dx) \le C.$$

Finally, since

$$\left(D_x p_\alpha(x)h\right)(\xi) = p'(J_\alpha(x(\xi)))h(\xi)$$

we have

$$|D_x p_\alpha(x)h|_H \le C\|x\|_\infty^{N-1}\|h\|_H$$

namely

$$\|D_x p_\alpha(x)\|_{\mathcal{L}(H)} \le C\|x\|_\infty^{N-1}$$

and therefore, being both $(-A)^{-1/2+\theta}$ and $(-A)^{-1/2-\theta}$ bounded in H (recall that $\theta < \frac{1}{2}$),

$$\left\|(-A)^{-1/2+\theta} D_x p_\alpha(x)(-A)^{-1/2-\theta}\right\|_{\mathcal{L}(H)} \le \|D_x p_\alpha(x)\|_{\mathcal{L}(H)} \le C\|x\|_\infty^{N-1}$$

which implies

$$\int_H\left(1 + \left\|(-A)^{-1/2+\theta} D_x p_\alpha(x)(-A)^{-1/2-\theta}\right\|_{\mathcal{L}(H)}^{2r}\right)\gamma_\alpha(dx) \le C.$$

$\square$ $\square$

Corollary 3.18. *Under the assumption of Theorem 3.11 there exist constants $C_{A,B,p}$, $C_{A,B,p,q,\theta}$, independent of ϵ, such that*

$$
\begin{aligned}
&\int_H |B_\epsilon(u,F)(t,x)|\,\gamma(dx) \le C_{A,B,p}\|u(t,\cdot)\|_{L^p(H,\gamma)}\left\|(-A)^{\theta_0}F(t,\cdot)\right\|_{\mathcal{V}(H,H,\gamma)} \\
&+ C_{A,B,p,q,\theta}\|u(t,\cdot)\|_{L^p(H,\gamma)}\left\|(-A)^{1/2+\theta}F(t,\cdot)\right\|_{L^q(H,\gamma)} + \|u(t,\cdot)\|_{L^p(H,\gamma)}\|\mathrm{div}\,F(t,\cdot)\|_{L^{\frac{p}{p-1}}(H,\gamma)}
\end{aligned}
$$

for all functions $u \in \mathcal{F}C_{b,T}^1$ and vector field F of the form $F = \sum_{h=1}^n F_h e_h$, with $F_h \in \mathcal{F}C_{b,T}^2$ for all $h = 1, ..., n$.

Proof. Let us consider term by term the main inequality of Proposition 3.17. Since $x \mapsto u(t,\cdot)$ is bounded continuous function,

$$\lim_{\alpha\to 0}\|u(t,\cdot)\|^p_{L^p(H,\gamma_\alpha)} = \lim_{\alpha\to 0}\int_H |u(t,x)|^p\,\gamma_\alpha(dx) = \int_H |u(t,x)|^p\,\gamma(dx)$$

because γ_α converges weakly to γ. The same argument applies to the terms $\left\|(-A)^{1/2+\theta}F(t,\cdot)\right\|_{L^q(H,\gamma_\alpha)}$ and $\|\operatorname{div} F(t,\cdot)\|_{L^{\frac{p}{p-1}}(H,\gamma_\alpha)}$.

We have to prove that

$$\lim_{\alpha\to 0}\int_H |B^\alpha_\epsilon(u,F)(t,x)|\,\gamma_\alpha(dx) = \int_H |B_\epsilon(u,F)(t,x)|\,\gamma(dx).$$

We have

$$\left|\int_H |B^\alpha_\epsilon(u,F)(t,x)|\,\gamma_\alpha(dx) - \int_H |B_\epsilon(u,F)(t,x)|\,\gamma(dx)\right| \le I_1 + |I_2|$$

where

$$I_1 = \int_H \left||B^\alpha_\epsilon(u,F)(t,x)| - |B_\epsilon(u,F)(t,x)|\right|\gamma_\alpha(dx)$$

$$I_2 = \int_H |B_\epsilon(u,F)(t,x)|\,\gamma_\alpha(dx) - \int_H |B_\epsilon(u,F)(t,x)|\,\gamma(dx).$$

Recall that ϕ bounded continuous implies $x \mapsto (P^\alpha_\epsilon\phi)(x)$ continuous and bounded by $\|\phi\|_\infty$. One can prove that when ϕ has also bounded continuous derivatives, $x \mapsto (D_x P^\alpha_\epsilon\phi)(x)$ is also continuous and uniformly bounded in α. The same is true without α. Then $|B_\epsilon(u,F)(t,x)|$ is bounded continuous. It follows that $|I_2| \to 0$ as $\alpha \to 0$, because γ_α converges weakly to γ. Moreover, since the family $\{\gamma_\alpha\}$ is tight, given $\eta > 0$ there is a compact set $K_\eta \subset H$ such that $\gamma_\alpha(K_\eta) \ge 1-\eta$ for all α; and for what we have just said, outside K_η we may use the fact that $|B^\alpha_\epsilon(u,F)(t,x)|$ is uniformly bounded in α. Then we rewrite

$$I_1 \le \int_{K_\eta} \left||B^\alpha_\epsilon(u,F)(t,x)| - |B_\epsilon(u,F)(t,x)|\right|\gamma_\alpha(dx) + C\eta.$$

Recall that, when ϕ is bounded continuous, $P^\alpha_\epsilon\phi$ converges to $P_\epsilon\phi$ as $\alpha \to 0$ uniformly on bounded sets of H; and when ϕ has also bounded continuous derivatives, also $D_x P^\alpha_\epsilon\phi$ converges to $D_x P_\epsilon\phi$ as $\alpha \to 0$, uniformly on bounded sets of H. Hence $\left||B^\alpha_\epsilon(u,F)(t,x)| - |B_\epsilon(u,F)(t,x)|\right|$ converges to zero uniformly on K_η.

With the same argument, given $\phi \in \mathcal{F}C^2_b$, for every ϵ, we have

$$\lim_{\alpha\to 0}\int_H \phi(x)\left(\frac{I-P^\alpha_\epsilon}{\epsilon}\right)\phi(x)\,\gamma_\alpha(dx) = \int_H \phi(x)\left(\frac{I-P_\epsilon}{\epsilon}\right)\phi(x)\,\gamma(dx).$$

Then, for every ϵ,

$$\lim_{\alpha\to 0}\int_H \phi(x)\left(\frac{I-P^\alpha_\epsilon}{\epsilon}\right)\phi(x)\,\gamma_\alpha(dx) \le \|\phi\|^2_{\mathcal{V}(H,\gamma)}.$$

We apply this inequality in the vector case to $(-A)^{\theta_0}F(t,\cdot)$. □ □

Finally, we have our main estimate.

Theorem 3.19. *Under the assumptions of Theorem 3.11 there exist constants* $C_{A,B,p}$,$C_{A,B,p,q,\theta}$ *such that*

$$\int_0^T\int_H |B_\epsilon(u,F)(t,x)|\,\gamma(dx)\,dt \le C_{A,B,p,\theta}\,\|u\|_{L^p(0,T;L^p(H,\gamma))}\,\|F\|_{p,q,\gamma,T}$$

for all functions $u \in L^p(0,T;L^p(H,\gamma))$ *and vector fields* $F:[0,T]\times H \to D\left((-A)^{1/2+\theta}\right)$ *such that* $\|F\|_{p,q,\gamma,T}$ *is finite. Moreover, for such* (u,F)*,*

$$\lim_{\epsilon\to 0}\int_0^T\int_H |B_\epsilon(u,F)(t,x)|\,\gamma(dx)\,dt = 0.$$

Under these conditions, the rank condition follows.

Proof. The proof is similar to [12]. □ □

Appendix

Appendix A. Deterministic Feynman–Kac formula and the solution of (2.1) for sufficiently regular F

Consider the equation

$$\begin{cases} \dfrac{d}{dt}\,\xi(t) = \widetilde{F}(t,\xi(t)), \\ \\ \xi(s) = x, \quad x \in \mathbb{R}^d, \end{cases} \tag{A.1}$$

with $\widetilde{F}$ regular, namely it belongs to the class $\mathcal{VF}C_b^1(H)$. Let $V\colon [0,T]\times\mathbb{R}^d \to \mathbb{R}$ be also regular. We want to solve

$$\begin{cases} v_s(s,x) \quad + \quad \langle D_x v(s,x), \widetilde{F}(s,x)\rangle + V(s,x)v(s,x) = 0, \quad 0 \le s < T, \\ \\ v(T,x) \quad = \quad \varphi(x), \quad x \in H. \end{cases} \tag{A.2}$$

The following result is well known, see e.g. [20]. We present, however, a proof for the reader's convenience.

Proposition A.1. *Assume $\widetilde{F} \in C_b([0,T]\times\mathbb{R}^d;\mathbb{R}^d)$ such that $\widetilde{F}(t,\cdot) \in C^1(\mathbb{R}^d,\mathbb{R}^d)$ for all $t \in [0,T]$ and let $V \in C([0,T]\times\mathbb{R}^d)$ such that $V(t,\cdot) \in C^1(\mathbb{R}^d)$ for all $t \in [0,T]$ such that $D_xV : [0,T]\times\mathbb{R}^d \to \mathbb{R}^d$ is continuous. Let $\varphi \in C^1(\mathbb{R}^d)$. Then the solution to (A.2) is given by*

$$v(s,x) = \varphi(\xi(T,s,x))e^{\int_s^T V(u,\xi(u,s,x))du}, \qquad (s,x) \in [0,T]\times\mathbb{R}^d, \tag{A.3}$$

where for $s \le t$, $\xi(t,s,x)$ denotes the solution to (A.1) at time t when started at time s at $x \in \mathbb{R}^d$. In particular, $v(\cdot,x) \in C^1([0,T])$ for every $x \in \mathbb{R}^d$ and $D_tv \in C([0,T]\times\mathbb{R}^d)$.

Proof. We only present the main steps. We shall check that v defined by (A.3) is a solution to (A.2).

For any decomposition $\{s = s_0 < s_1 < \cdots < s_n = T\}$ of $[s,T]$ we write

$$v(s,x) - \varphi(x) = -\sum_{k=1}^n [v(s_k,x) - v(s_{k-1},x)],$$

which is equivalent to,

$$\begin{aligned} v(s,x) - \varphi(x) &= -\sum_{k=1}^n [v(s_k,x) - v(s_k,\xi(s_k,s_{k-1},x))] \\ &- \sum_{k=1}^n [v(s_k,\xi(s_k,s_{k-1},x)) - v(s_{k-1},x)] =: J_1 - J_2. \end{aligned} \tag{A.4}$$

Concerning J_1 we write thanks to Taylor's formula

$$\begin{aligned} J_1 &\sim \sum_{k=1}^n \langle D_x v(s_k,x), \xi(s_k,s_{k-1},x) - x\rangle \sim \sum_{k=1}^n \langle D_x v(s_k,x), \widetilde{F}(s_k,x)\rangle (s_k - s_{k-1}) \\ &\to \int_s^T \langle D_x v(r,x), \widetilde{F}(r,x)\rangle dr. \end{aligned} \tag{A.5}$$

Concerning J_2 we write [2]

$$
\begin{aligned}
J_2 &= \sum_{k=1}^{n} v(s_k, \xi(s_k, s_{k-1}, x)) - v(s_{k-1}, x)) \\
&= \sum_{k=1}^{n} \varphi(\xi(T, s_k, \xi(s_k, s_{k-1}, x)))e^{\int_{s_k}^T V(u,\xi(u,s_k,\xi(s_k,s_{k-1},x)))du} \\
&- \sum_{k=1}^{n} \varphi(\xi(T, s_{k-1}, x))e^{\int_{s_{k-1}}^T V(u,\xi(u,s_{k-1},x))du} \\
&= \sum_{k=1}^{n} \varphi(\xi(T, s_{k-1}, x)) \left[e^{\int_{s_k}^T V(u,\xi(u,s_{k-1},x))du} - e^{\int_{s_{k-1}}^T V(u,\xi(u,s_{k-1},x))du}\right] \\
&= \sum_{k=1}^{n} v(s_{k-1}, x)) \left(e^{-\int_{s_{k-1}}^{s_k} V(u,\xi(u,s_{k-1},x))du} - 1\right) \\
&\sim -\sum_{k=1}^{n} v(s_{k-1}, x)V(s_{k-1}, x)(s_k - s_{k-1}) \to -\int_s^T v(r, x)V(r, x)dr.
\end{aligned}
\tag{A.6}
$$

Replacing J_1 and J_2 given by (A.5) and (A.6) respectively in (A.4), yields

$$v(s, x) = \varphi(x) + \int_s^T \langle D_x v(r, x), \widetilde{F}(r, x)\rangle dr + \int_s^T v(r, x)V(r, x)dr$$

and the claim is proved. □ □

As a trivial consequence we obtain

Corollary A.2. *Let $\Psi \in C^2(\mathbb{R}^d)$, Ψ bounded and strictly positive. Let $F \in C_b([0, T] \times \mathbb{R}^d; \mathbb{R}^d)$ such that $F(t, \cdot) \in C^1(\mathbb{R}^d; \mathbb{R}^d)$ and define*

$$D_x^* F(t, \cdot) := -\mathrm{div}\, F(t, \cdot) - \langle F(t, \cdot), D_x\Psi/\Psi\rangle_{\mathbb{R}^d}.$$

Assume that $D_x^* F(t, \cdot) \in C^1(\mathbb{R}^d)$ for all $t \in [0, T]$, and $D_x^* F \in C([0, T] \times \mathbb{R}^d)$, $D_x D_x^* F \in C([0, T] \times \mathbb{R}^d; \mathbb{R}^d)$. Then for every $\rho_0 \in C^1(\mathbb{R}^d)$, $\rho_0 \geq 0$,

$$\rho(t, x) := \rho_0(\xi(T, T - t, x))e^{\int_0^t D_x^* F(T-u,\xi(T-u,T-t,x))du}$$

is a solution of (2.1), *where $\xi(\cdot, s, x)$ is the solution to* (A.1) *started at time s at $x \in \mathbb{R}^d$, with $\widetilde{F}(t, x) := -F(T - t, x)$, $(t, x) \in [0, T] \times \mathbb{R}^d$. Furthermore, $\rho(\cdot, x) \in C^1([0, T])$ for every $x \in \mathbb{R}^d$ and $D_t\rho \in C([0, T] \times \mathbb{R}^d)$.*

Proof. Apply Proposition A.1 with $\widetilde{F}$ as in the assertion above,

$$V(t, x) = D_x^* F(T - t, x), \quad (t, x) \in [0, T] \times \mathbb{R}^d$$

and $\varphi := \rho_0$. □ □

---

[2] In the second line below we use that $\xi(T, s_k, \xi(s_k, s_{k-1}, x)) = \xi(T, s_{k-1}, x)$

Appendix B. A remark on the Burkholder–Davis–Gundy inequality

Our aim in this section is to prove the following proposition.

Proposition B.1. *Let $p \geq 4$. Then for every $t \geq 0$,*

$$\mathbb{E} \sup_{s\in[0,t]} \left| \int_0^t \Phi(s) dW(s) \right|^p \leq c_p \left[\mathbb{E} \left(\int_0^t \|\Phi(s)\|^2_{L_2^0} \, ds \right)^{p/2} \right], \tag{B.1}$$

where $c_p := 12^p \, p^p$.

Proof. Set

$$Z(t) = \int_0^t \Phi(s) dW(s), \quad t \geq 0,$$

and apply Itô's formula to $f(Z(\cdot))$ where $f(x) = |x|^p, \; x \in H$. Since

$$f_{xx}(x) = p(p-2)|x|^{p-4} x \otimes x + p|x|^{p-2} I, \quad x \in H,$$

we have

$$\|f_{xx}(x)\| \leq p(p-1)|x|^{p-2},$$

therefore

$$|\mathrm{Tr}\; \Phi^*(t) f_{xx}(Z(t)) \Phi(t) Q| \leq p(p-1)|Z(t)|^{p-2} \|\Phi(t)\|^2_{L_2^0}.$$

By taking expectation in the identity

$$|Z(t)|^p = p \int_0^t |Z(s)|^{p-2} \langle Z(s), dZ(s) \rangle + \frac{1}{2} \int_0^t \mathrm{Tr}\; [\Phi^*(s) f_{xx}(Z(s)) \Phi(s) Q] ds,$$

we obtain by the Burkholder–Davis–Gundy inequality for $p = 1$

$$\begin{aligned}
&\mathbb{E} \sup_{s\in[0,t]} |Z(s)|^p \leq \frac{p(p-1)}{2} \, \mathbb{E} \left(\int_0^t |Z(s)|^{p-2} \|\Phi(s)\|^2_{L_2^0} \, ds \right) \\
&+ 3p\mathbb{E} \left[\left(\int_0^t \|\Phi(s)\|^2_{L_2^0} \, |Z(s)|^{2p-2} ds \right)^{1/2} \right] \\
&\leq \frac{p(p-1)}{2} \, \mathbb{E} \left(\sup_{s\in[0,t]} |Z(s)|^{p-2} \int_0^t \|\Phi(s)\|^2_{L_2^0} ds \right) \\
&+ 3p\mathbb{E} \left[\sup_{s\in[0,t]} |Z(s)|^{p-1} \left(\int_0^t \|\Phi(s)\|^2_{L_2^0} ds \right)^{1/2} \right] \\
&\leq \frac{p(p-1)}{2} \, \left[\mathbb{E} \left(\sup_{s\in[0,t]} |Z(s)|^p \right) \right]^{\frac{p-2}{p}} \, \left[\mathbb{E} \left(\int_0^t \|\Phi(s)\|^2_{L_2^0} ds \right)^{\frac{p}{2}} \right]^{\frac{2}{p}} \\
&+ 3p\mathbb{E} \left[\sup_{s\in[0,t]} |Z(s)|^p \right]^{\frac{p-1}{p}} \, \left[\mathbb{E} \left(\int_0^t \|\Phi(s)\|^2_{L_2^0} ds \right)^{\frac{p}{2}} \right]^{\frac{1}{p}} \\
&:= J_1 + J_2.
\end{aligned} \tag{B.2}$$

For J_1 we use Young's inequality with exponents $\frac{p}{p-2}$ and $\frac{p}{2}$ and find

$$J_1 \le \frac{1}{4}\,\mathbb{E}\left[\sup_{s\in[0,t]} |Z(s)|^p\right] + 2^{p-1}p^p\,\mathbb{E}\left(\int_0^t \|\Phi(s)\|^2_{L_2^0} ds\right)^{\frac{p}{2}}$$

For J_2 we use Young's inequality with exponents $\frac{p}{p-1}$ and p and find

$$J_2 \le \frac{1}{4}\,E\left[\sup_{s\in[0,t]} |Z(s)|^p\right] + \frac{1}{2}\,12^p\,p^p\,\mathbb{E}\left(\int_0^t \|\Phi(s)\|^2_{L_2^0} ds\right)^{\frac{p}{2}}.$$

Now (B.1) with $c_p := 12^p\,p^p$ follows. □ □

Appendix C. Density of $\mathcal{F}C_b^1$ in Orlicz spaces

Let $N : \mathbb{R} \to [0,\infty)$ be continuous and a Young function, i.e. convex, even and $N(0) = 0$.

Consider the measure space $(H, \mathcal{B}(H), \gamma)$, where H is as before a separable real Hilbert space with Borel σ–algebra $\mathcal{B}(H)$ and γ a nonnegative finite measure on $(H, \mathcal{B}(H))$. We recall that the Orlicz space L_N corresponding to N is defined as

$$L_N := L_N(H,\gamma) := \{f : H \to \mathbb{R} :\ f \textit{ is } \mathcal{B}(H)\textit{–measurable and } \int_H N(af)d\gamma < \infty \textit{ for some } a > 0\}$$

or equivalently

$$L_N := \{f : H \to \mathbb{R} :\ f \textit{ is } \mathcal{B}(H)\textit{–measurable and } \|f\|_{L_N} < \infty\},$$

where

$$\|f\|_{L_N} := \inf\left\{\lambda > 0 : \int_H N(f/\lambda)\,d\gamma \le 1\right\}.$$

$(L_N, \|\cdot\|_{L_N})$ is a Banach space (see e.g. [21]).

Proposition C.1. *$\mathcal{F}C_b^1$ is dense in $((L_N, \|\cdot\|_{L_N})$, where $\mathcal{F}C_b^1$ is defined as in Section 1. Furthermore, if $f \in L_N$, $f \ge 0$, then there exist nonnegative $f_n \in \mathcal{F}C_b^1$, $n \in \mathbb{N}$, such that*

$$\lim_{n\to\infty} \|f - f_n\|_{L_N} = 0.$$

Both assertions remain true, if $\mathcal{F}C_b^1$ is replaced by $\mathcal{F}C_0^1$

Proof. We need the following lemma whose proof is straightforward, see e.g. [18, Lemma 1.16]

Lemma C.2. *Let $f_n \in L_N$, $n \in \mathbb{N}$. Then the following assertions are equivalent:*

(i) $\lim_{n\to\infty} \|f_n\|_{L_N} = 0$

(ii) For all $a \in (0,\infty)$

$$\limsup_{n\to\infty} \int_H N(af_n)\,d\gamma \le 1$$

(iii) For all $a \in (0,\infty)$

$$\lim_{n\to\infty} \int_H N(af_n)\,d\gamma = 0.$$

Proof of Proposition C.1.

We shall use a monotone class argument. Define

$$\mathcal{M} := \Big\{ f : H \to \mathbb{R} : \ f \textit{ bounded, } \mathcal{B}(H)\text{–}\textit{measurable such that}$$

$$\lim_{n\to\infty} \|f - f_n\|_{L_N} = 0, \textit{ for some } f_n \in \mathcal{F}C_b^1,\ n \in \mathbb{N} \Big\}.$$

Obviously, $\mathcal{M}$ is a linear space, $\mathcal{F}C_b^1 \subset \mathcal{M}$ and $\mathcal{F}C_b^1$ is closed under multiplication and contains the constant function 1. Furthermore, if $0 \le u_n \in \mathcal{M}$, $n \in \mathbb{N}$, such that $u_n \uparrow u$ as $n \to \infty$ for some bounded $u : H \to [0,\infty)$, then for each $n \in \mathbb{N}$ there exists $f_n \in \mathcal{F}C_b^1$ such that

$$\|u_n - f_n\|_{L_N} \le \frac{1}{n}. \tag{C.1}$$

But since N is continuous on $\mathbb{R}$, hence locally bounded, we have that for every $a \in (0,\infty)$, $N(a(u - u_n))$, $n \in \mathbb{N}$, are uniformly bounded. Consequently, by Lebesgue's dominated convergence theorem and Lemma C.2, we conclude that

$$\lim_{n\to\infty} \|u - u_n\|_{L_N} = 0. \tag{C.2}$$

(C.1) and (C.2) imply that $u \in \mathcal{M}$, and therefore $\mathcal{M}$ is a monotone vector space and thus by the monotone class theorem $\mathcal{M}$ is equal to the set of all bounded $\sigma(\mathcal{F}C_b^1)$–measurable functions on H. But $\sigma(\mathcal{F}C_b^1) = \mathcal{B}(H)$, since the weak and norm–Borel σ–algebra on a separable Banach space coincide. Hence $\mathcal{M}$ is equal to all bounded $\mathcal{B}(H)$–measurable functions on H. Since by Lemma C.2 and the same arguments as above every f in L_N can be approximated in the norm $\|\cdot\|_{L_N}$ by bounded $\mathcal{B}(H)$–measurable functions, the first assertion of the proposition is proved.

Now let $f \in L_N$, $f \ge 0$. By the argument above we may assume that f is bounded. Then by what we have just proved we can find $f_n \in \mathcal{F}C_b^1$ such that

$$\lim_{n\to\infty} \|f - f_n\|_{L_N} = 0.$$

Since $|f - f_n^+| = |f^+ - f_n^+| \le |f - f_n|$ for all $n \in \mathbb{N}$ and N is even and increasing on $[0,\infty)$ (because N is convex and $N(0) = 0$), Lemma C.2 immediately implies that

$$\lim_{n\to\infty} \|f - f_n^+\|_{L_N} = 0.$$

Fix $n \in \mathbb{N}$ and for $\epsilon > 0$ take an increasing function $\chi_\epsilon \in C^1(\mathbb{R})$, $\chi_\epsilon(s) = s, \forall\, s \in [0,\infty)$ and $\chi_\epsilon(s) = -\epsilon$ if $s \in (-\infty, -2\epsilon)$. Then for each $n \in \mathbb{N}$

$$\lim_{m\to\infty} \left\| f_n^+ - \left(\chi_{\frac{1}{m}}(f_n) + \frac{1}{m} \right) \right\|_\infty = 0.$$

So, again by Lemma C.2 and Lebesgue's dominated convergence theorem it follows that

$$\lim_{m\to\infty} \left\| f_n^+ - \left(\chi_{\frac{1}{m}}(f_n) + \frac{1}{m} \right) \right\|_{L_N} = 0.$$

But obviously, $\chi_{\frac{1}{m}}(f_n) + \frac{1}{m} \in \mathcal{F}C_b^1$, $m \in \mathbb{N}$, and each such function is nonnegative. Hence the second part of the assertion follows. The third part of the assertion then follows by similar arguments and multiplying by a sequence of suitable localizing functions. $\square$ $\square$

Corollary C.3. *Let $\rho \ge 0$, $\mathcal{B}(H)$–measurable such that*

$$\int_H \rho \log \rho \, d\gamma < \infty.$$

Then there exist nonnegative $\rho_n \in \mathcal{F}C_b^1$, $n \in \mathbb{N}$, *such that*

$$\lim_{n\to\infty} \rho_n = \rho \quad \text{in } L^1(H,\gamma)$$

and

$$\sup_{n\in\mathbb{N}} \int_H \rho_n \log \rho_n \, d\gamma < \infty.$$

Proof. Let $N(s) := (|s|+1)\ln(|s|+1) - |s|$, $s \in \mathbb{R}$. Then it is easy to check that N is a continuous Young function. Hence by Proposition C.1 we can find $\rho_n \in \mathcal{F}C_b^1$, $\rho_n \geq 0$, $n \in \mathbb{N}$, such that

$$\lim_{n\to\infty} \|\rho - \rho_n\|_{L_N} = 0. \tag{C.3}$$

Since $L_N \subset L^1(H,\gamma)$ continuously (see [18, Proposition 1.15]), the first assertion follows. Furthermore, we have for all $s \in (0,\infty)$

$$s\ln s - s \leq s\ln(s+1) \leq (s+1)\ln(s+1) - s = N(s)$$

and hence for $n \in \mathbb{N}$ by the convexity of N and every $a \in (0,\infty)$

$$\begin{aligned}
&\int_H \rho_n \ln \rho_n \, d\gamma = \frac{1}{a}\int_H a\rho_n \ln(a\rho_n)\, d\gamma - \ln a \int_H \rho_n \, d\gamma \\
&\leq \frac{1}{a}\int_H N(a\rho_n)\, d\gamma + |1-\ln a| \int_H \rho_n \, d\gamma \\
&\leq \frac{1}{2a}\int_H N(2a(\rho_n - \rho))\, d\gamma + \frac{1}{2a}\int_H N(2a\rho)\, d\gamma + |1-\ln a| \int_H \rho_n \, d\gamma.
\end{aligned}$$

Hence by the first part of the assertion, (C.3) and Lemma C.2, it follows that

$$\limsup_{n\to\infty} \int_H \rho_n \ln \rho_n \, d\gamma \leq \frac{1}{2a}\int_H N(2a\rho)\, d\gamma + |1-\ln a| \int_H \rho \, d\gamma.$$

But since $\rho \in L_N$ we can find $a > 0$ such that the right hand side is finite. Hence the second part of the assertion also follows. □ □

Acknowledgments

G. Da Prato and F. Flandoli are partially supported by GNAMPA from INdAM. M. Röckner is supported by SFB 1283 through the DFG. We also would like to thank an anonymous referee for his comments which led to an improvement of this paper.

Note added in Proof

After this paper had been accepted for publication by JMPA in final form, we noticed that as a simple consequence of Proposition 6.4.1 in [6], our Hypothesis 2(ii) is in fact a consequence of our Hypothesis 1, Lemma 1.4 and (1.9). Hypothesis 2(ii) can hence be dropped. In particular, our results therefore also apply to our Example 1.3(iii) and Remarks 1.5 and 3.13 can be dropped as well. We would like to thank Alexander Shaposhnikov for pointing out this particular result in the above reference to us.

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