# Nonlinear Fokker–Planck equations driven by Gaussian linear multiplicative noise

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**Abstract**

Existence of a strong solution in $H^{-1}(\mathbb{R}^d)$ is proved for the stochastic nonlinear Fokker–Planck equation

$$dX - \operatorname{div}(DX)dt - \Delta\beta(X)dt = X\,dW \text{ in } (0,T)\times\mathbb{R}^d,\ X(0)=x,$$

respectively, for a corresponding random differential equation. Here $d \geq 1$, $W$ is a Wiener process in $H^{-1}(\mathbb{R}^d)$, $D \in C^1(\mathbb{R}^d,\mathbb{R}^d)$ and $\beta$ is a continuous monotonically increasing function satisfying some appropriate sublinear growth conditions which are compatible with the physical models arising in statistical mechanics. The solution exists for $x \in L^1 \cap L^\infty$ and preserves positivity. If $\beta$ is locally Lipschitz, the solution is unique, pathwise Lipschitz continuous with respect to initial data in $H^{-1}(\mathbb{R}^d)$. Stochastic Fokker-Planck equations with nonlinear drift of the form $dX - \operatorname{div}(a(X))dt - \Delta\beta(X)dt = X\,dW$ are also considered for Lipschitzian continuous functions $a:\mathbb{R}\to\mathbb{R}^d$.

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**Keywords:** Wiener process, Fokker–Planck equation, random differential equation, $m$-accretive operator.

## 1 Introduction

We first consider the stochastic partial differential equation

$$\begin{aligned} &dX - \operatorname{div}(DX)dt - \Delta\beta(X)dt = X\,dW \text{ in } (0,T)\times\mathbb{R}^d,\ T>0,\\ &X(0,\xi)=x(\xi),\ \xi\in\mathbb{R}^d,\ 1\leq d<\infty, \end{aligned} \tag{1.1}$$

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where $W$ is a Wiener process in $H^{-1} := H^{-1}(\mathbb{R}^d)$ over a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ with normal filtration $(\mathcal{F}_t)_{t\geq 0}$ of the form

$$W = \sum_{j=1}^{N} \mu_j e_j \beta_j. \tag{1.2}$$

Here $\{e_1, ..., e_N\}$ is an orthonormal system in $H^{-1}(\mathbb{R}^d)$ belonging to $C_b^2(\mathbb{R}^d) \cap W^{2,1}(\mathbb{R}^d)$, $\mu_j \in \mathbb{R}$ and $\{\beta_j\}_{j=1}^{\infty}$ are independent $(\mathcal{F}_t)$-Brownian motions on $(\Omega, \mathcal{F}, \mathbb{P})$. As regards the functions $D : \mathbb{R}^d \to \mathbb{R}^d$ and $\beta : \mathbb{R} \to \mathbb{R}$, we assume that

(i) $D \in C_b^1(\mathbb{R}^d; \mathbb{R}^d);\ |D| \in L^1(\mathbb{R}^d),\ \operatorname{div} D \in L^2(\mathbb{R}^d)$.

(ii) $\beta \in C(\mathbb{R}) \cap C^2(\mathbb{R} \setminus \{0\})$ *is monotonically nondecreasing,* $\beta(0) = 0$, *and there are* $m \in [0, 1]$, $a_i \in (0, \infty)$, $i = 1, 2, 3$, *such that*

$$\begin{aligned} |\beta(r)| &\leq a_1 |r|^m,\ \forall r \in \mathbb{R}, & (1.3)\\ |\beta''(r) r^2| + \beta'(r)|r| &\leq a_2 |\beta(r)|,\ \forall r \in \mathbb{R} \setminus \{0\}, & (1.4)\\ \beta'(r) &\neq 0 \text{ and } \operatorname{sign} r\, \beta''(r) \leq 0,\ \forall r \in \mathbb{R} \setminus \{0\}. & (1.5) \end{aligned}$$

(iii) *There exists a decreasing function* $\varphi : (0, 1] \to (0, \infty)$ *such that*

$$\beta'(\lambda r) \leq \varphi(\lambda) \beta'(r),\ \forall r \in \mathbb{R} \setminus \{0\},\ \lambda \in (0, 1]. \tag{1.6}$$

We note here that since, by (1.5), $\beta'$ is decreasing on $(0, \infty)$ and increasing on $(-\infty, 0)$, we also have

$$\beta'(r) \leq \beta'(\lambda r),\ \forall r \in \mathbb{R} \setminus \{0\},\ \lambda \in (0, 1]. \tag{1.7}$$

A typical example is $\beta(r) \equiv a_1 r |r|^{m-1}$, where $a_1 > 0$.

It should be said that $e^{\pm W}$ is a linear multiplier in the spaces $L^p$ and $H^1$ and this fact will be frequently used in the sequel.

Equation (1.1), which in the linear, deterministic case (that is, for $\beta(r) \equiv ar$, $W = 0$) reduces to the classical Fokker–Planck equation, describes the particle transport dynamics in disordered media driven by highly irregular or stochastic field forces. This is the so called anomalous diffusion dynamics (see, e.g., [15], [16]) in contrast to the normal diffusion processes governed by the linear Fokker–Planck equation.

The stochastic version (1.1) considered here can be viewed as a Fokker-Planck equation in a random environment or a generalized mean field Fokker-Planck equation ([10], [11], [12]).

The case considered here, that is hypothesis (1.3) with $0 \le m \le 1$ is that of a fast diffusion (see, e.g., [4]) which, for $D \equiv 0$ is relevant in plasma physics and the kinetic theory of gas. It should be said that in statistical physics, the deterministic Fokker–Planck equation (1.1) is related to the so-called correspondence principle (see, e.g., [16], [21]) in statistical mechanics which associates this equation to the entropy function

$$S(u) = \int_{\mathbb{R}} \Phi(u) d\xi,$$

where the function $\Phi \in C(\mathbb{R}) \cap C^2(\mathbb{R} \setminus \{0\})$ satisfies

$$\Phi'' < 0,\ \ \Phi' \ge 0,\ \ \Phi'(0) = +\infty, \tag{1.8}$$

and $\beta$ is defined by

$$\beta(r) = \Phi(r) - r\Phi'(r),\ \ \forall r \ge 0. \tag{1.9}$$

For instance, if $\beta(r) \equiv a$ sign $(r) \log(1 + |r|)$, $a > 0$, and $\Phi(u) = -u \log u +$ $(1 + u) \log(1 + u)$, then (1.1) is the classical boson equation in the Bose–Einstein statistics (see, e.g., [16]), while for $\beta(r) \equiv a|r|^{m-1}r$, one gets the so-called Plastino and Plastino model [21] in statistical mechanics.

We note that in both cases $\beta$ satisfies (ii) and (iii) above, and in the first case $\beta$ is locally Lipschitz.

Assumption (ii) leaves out the low diffusion case $m > 1$ which is relevant in porous media dynamics of low diffusion processes. (See, e.g., [4].) However, for the examples in statistical mechanics mentioned above, the case $m > 1$ is not relevant. In fact, the entropy function corresponding to $\beta(u) = u^m$ is by (1.9) formally given in $1 - D$ by

$$S(u) = \frac{1}{1-m} \int_{\mathbb{R}} (u^m - u) d\xi,\ \ \Phi(u) = \frac{1}{1-m} (u^m - u),$$

for which the entropic conditions (1.8) are not satisfied if $m > 1$.

For vanishing drift $D$, equation (1.1) reduces to the fast diffusion stochastic porous media equation studied in [8] (see, also, [4]).

By the transformation

$$X(t) = e^{W(t)} y(t),\ t \geq 0, \tag{1.10}$$

equation (1.1) reduces, via Itô's formula, to the random differential equation (see, e.g., [5], [6], [7])

$$\begin{aligned} &\frac{\partial y}{\partial t} - e^{-W} \mathrm{div}(e^W Dy) - e^{-W} \Delta\beta(e^W y) + \frac{1}{2}\,\mu y = 0 \text{ in } (0,T) \times \mathbb{R}^d, \\ &y(0,\xi) = x(\xi),\ \xi \in \mathbb{R}^d, \end{aligned} \tag{1.11}$$

where

$$\mu = \sum_{j=1}^{N} \mu_j^2 e_j^2. \tag{1.12}$$

Here, without loss of generality, we assume that $t \mapsto W(t)(\omega) \in H^{-1}$ is continuous for all $\omega \in \Omega$.

The purpose of this work is to show that, under hypotheses (i)-(iii), for every $\omega \in \Omega$, $1 \leq d < \infty$, and $x$ in a suitable space, the Cauchy problem (1.11) has at least one strong solution which is unique if, in addition, $\beta$ is locally Lipschitz on $\mathbb{R}$. By a strong solution to (1.11) we mean an absolutely continuous function $y : [0,T] \to H^{-1}(\mathbb{R}^d)$ such that $\mathrm{div}(e^W Dy)(t) \in H^{-1}$, a.e. $t \in (0,T)$, and (1.11) holds on $(0,T)$. Of course, if $y$ is $(\mathcal{F}_t)_{t\geq 0}$-adapted (which we shall show), then $X = e^W y$ is a strong solution to (1.1). A nice feature of the random differential equation (1.11) and its version with a nonlinear function in its divergence part (see equation (4.2) below) is that, though it is not of accretive type in any of the spaces $H^{-1}(\mathbb{R}^d)$ or $L^1(\mathbb{R}^d)$, which are naturally associated with nonlinear parabolic equations of this type, it turns out to be accessible by the theory of nonlinear semigroups of contractions in $L^1(\mathbb{R}^d)$, by a modification of the Crandall-Liggett discretization scheme for perturbed nonlinear accretive equations (see Appendix).

However, the general existence theory for the nonlinear accretive Cauchy problem in a Banach space is not directly applicable to equation (1.1) because $W$ is not smooth. So, the first step was to approximate $W$ by a family of smooth random functions $\{W_\varepsilon(t)\}_{\varepsilon>0}$ and so equation (1.11) too by a family of nonlinear evolution equation with smooth time-dependent coefficients (see equation (3.2) below). Afterward, one passes to the limit $\varepsilon \to 0$ in the corresponding equation by combining sharp $H^{-1}$-energetic and $L^1$-techniques.

This approach which will lead to existence of a strong solution $y$ to (1.11) is one of the main novelty of this work.

In [5], the authors studied equation (1.11) for $m \in (1,5)$ and $1 \le d \le 3$, on a bounded domain in the special case of a vanishing drift term $D$. It should be said, however, that the treatment in $\mathbb{R}^d$ developed here is quite different and requires specific techniques to be made precise below. (Under related hypotheses on $\beta$, the existence for the stochastic equation (1.1) with $D \equiv 0$ was also studied in [8].)

In [17], the following parabolic-hyperbolic quasilinear stochastic equation was recently studied on $T^d$ in the framework of kinetic solutions

$$dX - \mathrm{div}(B(X))dt - \mathrm{div}(A(X)\nabla X)dt = \Phi(X)dW, \tag{1.13}$$

where $B \in C^2(\mathbb{R}, \mathbb{R}^{d\times d})$ and $A \in C^1(\mathbb{R}; \mathbb{R}^{d\times d})$. (Along these lines, see also [18].) It should be said, however, that there is no overlap with our work as far as conditions (i) on the nonlinear diffusion term $\beta$ is concerned for which one assumes here different conditions to cover fast diffusions. In fact, the results of [18], though obtained in a more general context, apply to low diffusion equations (that is, $\beta(r) \approx ar^m$, $m \ge 2$, $a(r) \approx r^k$, $k > 1$). In addition, the rescaling technique used here is different from that used in [18] and its main advantage is that it leads to sharper regularity results for solutions by fully exploiting the parabolic nature of the resulting random differential equation.

# 2 Notation and the main results

We shall denote the norm of the space $\mathbb{R}^d$ by $|\cdot|$ and by $\langle\ ,\ \rangle$ the Euclidean inner product. Let $L^p(\mathbb{R}^d) = L^p$, $1 \le p \le \infty$, denote the standard real $L^p$ space on $\mathbb{R}^d$ with Lebesgue measure. The scalar product of $L^2$ is denoted by $(\cdot,\cdot)_2$. The norm of $L^p$ will be denoted by $|\cdot|_p$. $H^1(\mathbb{R}^d)$, briefly denoted $H^1$, is the Sobolev space $\left\{u \in L^2;\ \frac{\partial u}{\partial \xi_i} \in L^2,\ i{=}1,2,...,d\right\}$ with the standard norm $\|u\|_{H^1} = \left(\int_{\mathbb{R}^d}(u^2 + |\nabla u|^2)d\xi\right)^{\frac{1}{2}}$. The dual space of $H^1$ will be denoted by $H^{-1}$ and its norm by $|\cdot|_{-1}$. Likewise, $W^{r,p} = W^{r,p}(\mathbb{R}^d)$, $r \in \mathbb{N}$, $p \in [1,\infty]$, denote the usual Sobolev spaces. Denote by $\Delta$ the Laplace operator on $\mathbb{R}^d$. By $W^{1,p}([0,T]; H^{-1})$ we denote the space of all absolutely continuous $u : [0,T] \to H^{-1}$ such that $u, \frac{du}{dt} \in L^p(0,T; H^{-1})$. Given a Banach space $X$, let $L^p(0,T;X)$ denote the space of $X$-valued Bochner $L^p$-integrable functions on $(0,T)$. By $C([0,T];X)$, we denote the space of continuous functions $u$ :

$[0,T] \to X$ and by $C^1([0,T];X)$ the corresponding space of continuously differentiable functions.

We set

$$D_0 = \{x \in L^1 \cap L^\infty \cap H^1;\ \beta(x) \in H^1,\ \Delta x \in L^1,\ \Delta\beta(x) \in L^1\}.$$

**Lemma 2.1.** *Let $p \in [1,\infty)$ and $x \in L^1 \cap L^\infty$. Then there exist $u_n \in D_0$, $n \in \mathbb{N}$, such that $u_n \to x$ in $L^p$ and $\{x_n;\ n \in \mathbb{N}\}$ is bounded in $L^1 \cap L^\infty$. In particular,*

$$\overline{D}_0^{L^p} = L^p, \quad \overline{D}_0^{H^{-1}} = H^{-1},$$

*where the left hand sides denote the closures of $D_0$ in the respective spaces.*

**Proof.** Because $L^2$ is dense in $H^{-1}$, it suffices to prove

$$L^1 \cap L^\infty \subset \overline{D}_0^{L^p}.$$

So, let $x \in L^1 \cap L^\infty$ and define

$$u(\xi) = \varphi(\xi)e^{-\delta|\xi|^2},\ \xi \in \mathbb{R}^d, \tag{2.1}$$

where $\varphi \in C_b^2(\mathbb{R}^d), |\varphi| \geq \varepsilon,\ \varepsilon,\delta \in (0,1)$. Then, by (1.3), $\beta(u) \in L^1 \cap L^\infty$ and

$$\nabla\beta(u) = \frac{1}{\varphi}\,\beta'(u)u(\nabla\varphi - 2\delta\varphi\xi),$$

which is in $L^1 \cap L^\infty$ by (1.3), (1.4). So, $\beta(u) \in H^1$. Furthermore, obviously, $\Delta u \in L^1 \cap L^\infty$, and

$$\begin{aligned}\Delta\beta(u) &= \frac{1}{\varphi}\,\beta'(u)u[\Delta\varphi - (2d\delta - 4\delta^2|\xi|^2)\varphi - 4\delta\xi\cdot\nabla\varphi] \\ &\quad + \frac{1}{\varphi^2}\,\beta''(u)u|\nabla\varphi - 2\delta\varphi\xi|^2.\end{aligned}$$

Since $|\varphi| \geq \varepsilon$, it follows by (1.3) and (1.4) that $\Delta\beta(u) \in L^1 \cap L^\infty$. We have

$$x = \lim_{\delta\to 0}\lim_{\varepsilon\to\infty}(x^+ \vee \varepsilon - x^- \wedge (-\varepsilon))e^{-\delta|\xi|^2},$$

where both limits are in $L^p$ and, obviously, each function on the right under the limits for fixed $\varepsilon,\delta \in (0,1)$ can be approximated by functors of type (2.1) in $L^p$. ∎

Theorem 2.2 is the main result.

**Theorem 2.2.** *Under Hypotheses* (i)–(iii), *for each* $x \in D_0$, *equation* (1.11) *has, for each* $\omega \in \Omega$, *at least one strong solution*

$$y \in W^{1,2}([0,T];H^{-1}) \cap L^\infty((0,T)\times\mathbb{R}^d) \cap L^\infty(0,T;L^1), \tag{2.2}$$

$$y \in L^2(0,T;H^1), \tag{2.3}$$

$$\beta(e^W y) \in L^2(0,T;H^1). \tag{2.4}$$

*Moreover, if* $x \geq 0$, *a.e. on* $\mathbb{R}^d$, *then* $y \geq 0$, *a.e. on* $(0,T)\times\mathbb{R}^d$.

*If* $\beta$ *is locally Lipschitz on* $\mathbb{R}$ *and assumptions* (i)–(iii) *hold, then there is a unique strong solution* $y$ *to* (1.11). *This solution is* $(\mathcal{F}_t)$*–adapted, the map* $D_0 \ni x \to y(t,x)$ *is Lipschitz from* $H^{-1}$ *to* $C([0,T];H^{-1})$ *on balls in* $L^1 \cap L^\infty$ *and* $y$ *extends by density to a strong solution to* (1.11), *satisfying* (2.2), (2.4), *for all* $x \in L^1 \cap L^\infty$.

Now, coming back to equation (1.1), we recall (see, e.g., [4], [5], [8]) that a continuous $(\mathcal{F}_t)_{t\geq 0}$-adapted process $X:[0,T]\to H^{-1}$ is called strong solution to (1.1) if the following conditions hold:

$$X \in L^2([0,T];L^2), \qquad \mathbb{P}\text{-a.s.}, \tag{2.5}$$

$$\beta(X) \in L^2(0,T;H^1), \qquad \mathbb{P}\text{-a.s.}, \tag{2.6}$$

$$X(t) - \int_0^t \operatorname{div}(DX(s))ds - \int_0^t \Delta\beta(X(s))ds = x + \int_0^t X(s)dW(s), \quad \forall t \in [0,T],\ \mathbb{P}\text{-a.s.} \tag{2.7}$$

We note here that, by (2.5) and (3.6) below,

$$\operatorname{div}(DX) \in L^2(0,T,H^{-1}),\ \mathbb{P}\text{-a.s.}$$

The stochastic (Itô-) integral in (2.6) is the standard one (see [14], [19], [22]). In fact, in the terminology of these references, $W$ is a $Q$-Wiener process $W^Q$ on $H^{-1}$, where $Q: H^{-1}\to H^{-1}$ is the symmetric trace class operator defined by

$$Qh := \sum_{k=1}^N \mu_k (e_k,h)_{-1} e_k,\ h \in H^{-1}.$$

**Theorem 2.3.** *If* $\beta$ *is locally Lipschitz on* $\mathbb{R}$ *and assumptions* (i)–(iii) *hold, then, for every* $x \in D_0$, *equation* (1.1) *has a unique strong solution* $X = e^W y$, *which satisfies*

$$Xe^{-W} \in W^{1,2}([0,T];H^{-1}),\ \mathbb{P}\text{-}a.s., \tag{2.8}$$

*and $X \geq 0$, a.e. on $(0,T) \times \mathbb{R}^d \times \Omega$ if $x \geq 0$, a.e. on $\mathbb{R}^d$. Moreover, the map $x \mapsto X(t,x)$ is $H^{-1}$-Lipschitz from balls in $L^1 \cap L^\infty$ to $C([0,T];H^{-1})$.*

The argument used to show that $X$ is a strong solution to (1.1) is standard up to a stopping time argument and very similar to that from the works [6], [7] and so it will be omitted.

It should be said that assumptions of Theorem 2.3 (that is, (i)-(iii) and $\beta$ locally Lipschitz) hold for the boson equation

$$dX - \operatorname{div}(DX)dt - \Delta(\log(1+|X|))dt = X\,dW$$

and for other significant models in statistical mechanics. However, it leaves out the Plastino & Plastino model [13] for which all we can prove is the existence of a strong solution to the corresponding random differential equation (1.11).

A result as Theorem 2.3 was previously proved in [8] for equation (1.1) in the special case of vanishing drift $D$ by a direct approximation approach to the stochastic equation (1.1). The approach used here, based on the random differential equation (1.11), is completely different and leads to sharper results. Indeed, by (2.2), it follows that besides (2.5) the solution $X$ to (1.1) satisfies also (2.8), which is, of course, a new result.

It should be emphasized that the random differential equation (1.11) has an interest in itself as a model for particles dynamics driven by random transport and diffusion coefficients (see, e.g., [10]). In particular, the convergence of this solution to a stationary state or, more generally, the existence of a random attractor is a problem of utmost importance for its physical significance related to the so-called Boltzmann $H$-theorem (see [16], [23]). We note here that, if our solution is unique for every fixed $\omega$, which is proved in this paper if $\beta$ is locally Lipschitz, then, since it solves a deterministic PDE with random coefficients, it satisfies the strict cocycle property, so gives rise to a random dynamical system. This is the first and a fundamental ingredient to prove the existence of a random attractor. However, the uniqueness of solutions $y$ to (1.11) under assumptions (i)-(iii) remains an open problem.

# 3 Proof of Theorem 2.2

Below we fix $\omega \in \Omega$, but do not express it in the notation.

Let $\beta_j^\varepsilon \subset C^1([0,T];\mathbb{R})$, $1 \le j \le N$, be defined by $\beta_j^\varepsilon(t) = (\mathbf{1}_{[0,\infty)}\beta_j * \rho_\varepsilon)(t)$, where $\rho_\varepsilon(t) \equiv \frac{1}{\varepsilon}\,\rho\left(\frac{t}{\varepsilon}\right)$ is a standard mollifier with $\rho \in C_0^\infty(\mathbb{R})$, $\rho \ge 0$. We set

$$W_\varepsilon(t,\xi) = \sum_{j=1}^N \mu_j e_j(\xi)\beta_j^\varepsilon(t),\ \ t \ge 0,\ \ \xi \in \mathbb{R}^d.$$

Then we have for its time derivative

$$(W_\varepsilon)_t \in C([0,T] \times \mathbb{R}^d)$$

and

$$W_\varepsilon(t,\xi) \to W(t,\xi) \text{ uniformly in } (t,x) \in [0,T] \times \mathbb{R}^d$$

as $\varepsilon \to 0$.

For each $\varepsilon \in (0,1]$, consider the approximating equation of (1.11)

$$\begin{aligned} &\frac{\partial y_\varepsilon}{\partial t} - e^{-W_\varepsilon}\mathrm{div}(e^{W_\varepsilon}Dy_\varepsilon) - e^{-W_\varepsilon}\Delta(\beta(e^{W_\varepsilon}y_\varepsilon) + \varepsilon e^{W_\varepsilon}y_\varepsilon) \\ &\qquad\qquad + \varepsilon e^{-W_\varepsilon}\beta(e^{W_\varepsilon}y_\varepsilon) + \frac{1}{2}\,\mu y_\varepsilon = 0 \text{ in } (0,T) \times \mathbb{R}^d, \\ &y_\varepsilon(0,\xi) = x(\xi),\ \ \xi \in \mathbb{R}^d. \end{aligned} \tag{3.1}$$

Setting $z_\varepsilon = e^{W_\varepsilon}y_\varepsilon$, we get the equation

$$\begin{aligned} &\frac{\partial z_\varepsilon}{\partial t} - \Delta(\beta(z_\varepsilon) + \varepsilon z_\varepsilon) - \mathrm{div}(Dz_\varepsilon) + \varepsilon\beta(z_\varepsilon) \\ &\qquad + \left(\frac{1}{2}\,\mu - (W_\varepsilon)_t\right) z_\varepsilon = 0 \text{ in } (0,T) \times \mathbb{R}^d, \\ &z_\varepsilon(0,\xi) = x(\xi),\ \ \xi \in \mathbb{R}^d. \end{aligned} \tag{3.2}$$

We have

**Lemma 3.1.** *Assume that $x \in H^1$ such that $\beta(x) \in H^1$. Then, for each $\varepsilon \in (0,1]$, equation (3.1) considered on $H^{-1}$ has a unique strong solution $y_\varepsilon$ (see the Appendix) satisfying*

$$y_\varepsilon \ \in \ W^{1,\infty}([0,T];H^{-1}) \cap L^\infty(0,T;H^1). \tag{3.3}$$

*Moreover, if $x \in D(A_1)$ with $D(A_1)$ defined as in the claim following (3.14) below, then $y_\varepsilon \in C([0,T];L^1)$ and $z_\varepsilon = e^{W_\varepsilon}y_\varepsilon$, obtained as the limit of the finite difference scheme (5.11), is a mild solution to (3.2) in the space $L^1$.*

**Proof.** It suffices to prove that equation (3.2) has a unique solution

$$z_\varepsilon \in W^{1,\infty}([0,T];H^{-1}) \cap L^\infty(0,T;H^1), \tag{3.4}$$

and $\beta(z_\varepsilon):[0,T]\to H^1$ is right continuous.

Let us first prove existence and uniqueness of a solution to (3.2) considered as an equation on $H^{-1}$. Define the operator $A : D(A) \to H^{-1}$ by

$$Az = -\Delta(\beta(z)+\varepsilon z) + \varepsilon\beta(z) - \operatorname{div}(Dz) + \frac{\mu}{2}\,z, \tag{3.5}$$

with the domain $D(A) = \{z \in H^1 : \beta(z) \in H^1\}$. We endow the space $H^{-1}$ with the scalar product

$$\langle y,z\rangle_{-1,\varepsilon} = {}_{H^1}\left\langle(\varepsilon I-\Delta)^{-1}y,z\right\rangle_{H^{-1}}\ \ y,z\in H^{-1},$$

and with the corresponding norm $\|y\|_{-1,\varepsilon} = (\langle y,y\rangle_{-1,\varepsilon})^{\frac12}$. Taking into account that

$$\|\operatorname{div}(Dz)\|_{-1,\varepsilon} \le \frac{1}{\sqrt{\varepsilon}}\,|D|_\infty|z|_2,\ \forall z\in L^2, \tag{3.6}$$

we see that, for all $z,\bar z\in D(A)$,

$$\langle (A+\alpha I)z-(A+\alpha I)\bar z, z-\bar z\rangle_{-1,\varepsilon} \ge 0,$$

if

$$\alpha_\varepsilon = \frac{1}{\varepsilon}\,(|D|_\infty + \frac12\,|\mu|_\infty). \tag{3.7}$$

This means that $(A+\alpha I)$ is accretive in $H^{-1}$. Moreover, $A$ is quasi-$m$-accretive, that is, $R(\lambda+\alpha_\varepsilon)I+A) = H^{-1}$ for all $\lambda>0$. Indeed, for $f\in H^{-1}$, the equation

$$(\alpha_\varepsilon+\lambda)z - \Delta(\beta(z)+\varepsilon z) + \varepsilon\beta(z) - \operatorname{div}(Dz) + \frac{\mu}{2}\,z = f, \tag{3.8}$$

or, equivalently,

$$\begin{aligned}(\alpha_\varepsilon+\lambda)(\varepsilon I-\Delta)^{-1}z+\beta(z)+\varepsilon z-(\varepsilon I-\Delta)^{-1}\left(\operatorname{div}(Dz)+\varepsilon^2 z-\frac{\mu}{2}\,z\right)\\ = (\varepsilon I-\Delta)^{-1}f\end{aligned} \tag{3.9}$$

has, for $\lambda>0$, a unique solution $z\in L^2$. Indeed, equation (3.9) is of the form

$$\varepsilon z + B(z) + \Gamma z = (\varepsilon I-\Delta)^{-1}f \in H^1,$$

where the operators $B : L^2 \to L^2$ and $\Gamma : L^2 \to L^2$ are given by

$$\begin{aligned} B(z)(\xi) &= \beta(z(\xi)), \text{ a.e. in } \mathbb{R}^d, \\ \Gamma(z) &= (\alpha_\varepsilon + \lambda)(\varepsilon I - \Delta)^{-1} z - (\varepsilon I - \Delta)^{-1} \left( \mathrm{div}(Dz) + \varepsilon^2 z - \frac{\mu}{2}\, z \right). \end{aligned}$$

Since $B$ is $m$-accretive and $\Gamma$ is accretive and continuous in $L^2$, it follows that $R(\varepsilon I + B + \Gamma) = L^2$ and so there is a solution $z \in L^2$ to (3.9). Since, by (3.9), $\beta(z) + \varepsilon z \in H^1$, since the inverse of $r \mapsto \beta(r) + \varepsilon r$ is Lipschitz and equal to zero at $r = 0$, it follows that $z \in D(A)$, as claimed.

Now, we shall apply Lemma 5.1 and Corollary 5.2 in the Appendix, where $X = H^{-1}$, $A$ is the operator (3.5) and $\Lambda(t) \in L(H^{-1}, H^{-1})$, $\forall t \in [0, T]$ defined by

$$\Lambda(t)u = -(W_\varepsilon)_t u, \ \forall u \in H^{-1}, \tag{3.10}$$

and get a strong solution $z_\varepsilon$ to (3.2) satisfying

$$z_\varepsilon \in W^{1,\infty}([0, T]; H^{-1}). \tag{3.11}$$

But, indeed, also

$$z_\varepsilon \in L^\infty(0, T; H^1),$$

i.e., (3.4) holds. This can be seen as follows.

By Corollary 5.2, it immediately follows that

$$\beta(z_\varepsilon) + \varepsilon z_\varepsilon - (\varepsilon z_\varepsilon - \Delta)^{-1} \mathrm{div}(D z_\varepsilon) \in L^\infty(0, T; H^1). \tag{3.12}$$

An elementary consideration shows that, for $\varepsilon \in (0, 1)$,

$$|(\varepsilon I - \Delta)^{-1} \mathrm{div}(Dz)|_{L^2} \le c|z|_{-1,\varepsilon}, \ \forall z \in L^2, \tag{3.13}$$

where $c$ is a constant (only depending on $|D|_{C_b^1}$ and $d$). Since $z_\varepsilon$ is a strong solution, we have $z_\varepsilon \in D(A) \subset H^1 (\subset L^2) dt$-a.e. Hence, it follows by (3.11)-(3.13) that

$$\beta(z_\varepsilon) + \varepsilon z_\varepsilon \in L^\infty(0, T; L^2),$$

hence also $z_\varepsilon \in L^\infty(0, T; L^2)$. So by (3.6) we conclude

$$(\varepsilon I - \Delta)^{-1} \mathrm{div}(D z_\varepsilon) \in L^\infty(0, T; H^1).$$

Hence (3.12) implies that $\beta(z_\varepsilon) + \varepsilon z_\varepsilon \in L^\infty(0, T; H^1)$ and thus $z_\varepsilon \in L^\infty(0, T; H^1)$.

We are now going to construct the realization of the operator $A$ in $L^1$. We consider the operator $A_0$ defined by

$$\begin{aligned} &A_0 z = -\Delta(\beta(z)+\varepsilon z)+\varepsilon\beta(z)-\operatorname{div}(Dz)+\tfrac{\mu}{2}\,z,\\ &z\in D(A_0)=D(A)\cap\{z\in L^1;\ \beta(z),\Delta(\beta(z)+\varepsilon z)\in L^1\}. \end{aligned} \tag{3.14}$$

**Claim.** *Its closure $A_1=\overline{A}_0$ in $L^1\times L^1$ is quasi m-accretive.*

Indeed, since $\operatorname{div} D\in L^\infty$, $D\in L^1\cap L^\infty\subset L^2$, we have for all $z\in H^1\cap L^1$

$$\int_{\mathbb{R}^d}\operatorname{div}(Dz)\operatorname{sign} z\,d\xi=\int_{\mathbb{R}^d}\operatorname{div} D|z|d\xi+\int D\cdot\nabla|z|d\xi=0. \tag{3.15}$$

But, by [1], Theorem 3.5, also $D(A_0)\ni z\mapsto\Delta(\beta(z)+\varepsilon z)$ is accretive on $L^1$; hence, since $\beta$ is accretive, $A_0$ is accretive on $L^1$ and hence so is $\overline{A}_0$. But we also have, for $\alpha>\alpha_\varepsilon$,

$$R(\alpha I+A_0)\supset H^{-1}\cap L^1, \tag{3.16}$$

because, for $f\in H^{-1}\cap L^1$, as we have seen above, there exists $z\in D(A)$ such that $\alpha z+Az=f$. But, indeed, $z\in L^1$. This can be seen as follows: for $\delta>0$, define for $r\in\mathbb{R}$

$$\mathcal{X}_\delta(r):=\begin{cases} 1 & \text{if}\quad r>\delta,\\ \dfrac{r}{\delta} & \text{if}\quad r\in[-\delta,\delta],\\ -1 & \text{if}\quad r<\delta. \end{cases} \tag{3.17}$$

Then $\mathcal{X}_\delta(z)\in H^1$ and, applying ${}_{H^1}\langle\mathcal{X}_\delta(z),\cdot\rangle_{H^{-1}}$ to (3.8), we find

$$\begin{aligned} \alpha&\int_{\mathbb{R}^d}\mathcal{X}_\delta(z)z\,d\xi+\int_{\mathbb{R}^d}\mathcal{X}'_\delta(z)|\nabla z|^2(\beta'(z)+\varepsilon)d\xi\\ &+\varepsilon\int_{\mathbb{R}^d}\mathcal{X}_\delta(z)\beta(z)d\xi-\int_{\mathbb{R}^d}\operatorname{div} D\ \mathcal{X}_\delta(z)z\,d\xi\\ &-\int_{\mathbb{R}^d}\langle D,\nabla z\rangle\,\mathcal{X}_\delta(z)d\xi+\frac{1}{2}\int_{\mathbb{R}^d}\mathcal{X}_\delta(z)\mu\,z\,d\xi=\int_{\mathbb{R}^d}\mathcal{X}_\delta(z)f\,d\xi. \end{aligned}$$

Hence, dropping the second, third and sixth term (which are nonnegative) on the left hand side and then letting $\delta\to 0$, because $D$, $\operatorname{div} D\in L^2$ we obtain

$$\alpha|z|_1\le|f|_1.$$

But then it follows from (3.9) that $\beta(z) \in L^1$ and hence, by (3.8), that $z \in D(A_0)$ and (3.16) is proved. Taking $L^1$-closure, we conclude that

$$\overline{R(\alpha I + A_0)}^{L^1} = L^1.$$

This implies that $\overline{A}_0$ is quasi-$m$-accretive, because for $\alpha$ large enough

$$R(\alpha I + \overline{A}_0) \supset \overline{R(\alpha I + A_0)}^{L^1},$$

and the claim is proved.

Then, again by Lemma 5.1 and Corollary 5.2, applied to $X = L^1$ and to the operator $A_1$, it follows that for $x \in L^1$ equation (3.2) has a unique mild solution $\widetilde{z}_\varepsilon \in C([0,T]; L^1)$ and $\widetilde{y}_\varepsilon = e^{-W_\varepsilon}\widetilde{z}_\varepsilon$ is the mild solution to (3.1).

Let us note that $\widetilde{z}_\varepsilon = z_\varepsilon$ (and $\widetilde{y}_\varepsilon = y_\varepsilon$, respectively) for $x \in D(A_0)$. Indeed, as seen in Lemma 5.1, both $z_\varepsilon$ and $\widetilde{z}_\varepsilon$ are limits of finite difference scheme as (5.10), where $A$ is given by (3.5) and by $A_1 = \overline{A}_0^{L^1}$, respectively. But, by (3.16),

$$(I + hA_0)^{-1}y = (I + hA)^{-1}y, \ \forall y \in H^{-1} \cap L^1, \ \forall h \in (0, \alpha_\varepsilon^{-1}).$$

The solutions $u_1 \in L^1$ and $u \in H^1$ respectively of

$$u_1 + h(A_1 + \Lambda(ih))u_1 = y \tag{3.18}$$

and

$$u + h(A + \Lambda(ih))u = y \tag{3.19}$$

for small enough $h$ are obtained by iterating the strict contractions $B_1 : L^1 \to L^1$, $B : H^{-1} \to H^{-1}$, defined by

$$B_1 v := (1 + hA_1)^{-1}(y - h\Lambda(ih)v), \ v \in L^1,$$

and

$$Bv := (1 + hA)^{-1}(y - h\Lambda(ih)v), \ v \in H^{-1}.$$

Here $\Lambda(t)$ is given by (3.10), hence $\Lambda(ih)$ leaves both $L^1$ and $H^{-1}$ invariant. Therefore, starting the iteration in a point $v_0 \in H^{-1} \cap L^1$, we obtain by (3.16) that

$$B_1^n v_0 = B^n v_0 \in D(A_0), \ \forall n \in \mathbb{N},$$

and that this sequence converges both in $L^1$ and $H^{-1}$.

This implies that

$$(I+h(A_0+\Lambda(ih)))^{-1}y=(I+h(A+\Lambda(ih)))^{-1}y,\ i=0,1,...,\ \forall y\in H^{-1}\cap L^1.$$

This means that the finite difference schemes (5.11) in Lemma 5.1, applied separately in the spaces $L^1$ and $H^{-1}$, lead for $x\in D(A)\cap D(A_1)$ to the same values $u^h=z_\varepsilon^h$ ($\widetilde{u}^h=\widetilde{z}_\varepsilon^h$, respectively) and so, for the limit $h\to 0$, we get $z_\varepsilon=\widetilde{z}_\varepsilon$ for initial data $x\in D(A)\cap D(A_1)$. Hence $y_\varepsilon=\widetilde{y}_\varepsilon$, if $x\in D(A)\cap D(A_1)$. ∎

To get rigorous estimates for solutions $y_\varepsilon$ to equation (3.1), it is convenient to approximate it by the solution $y_\varepsilon^\lambda$ to the equation

$$\begin{aligned}&\frac{\partial y_\varepsilon^\lambda}{\partial t}-e^{-W_\varepsilon}\mathrm{div}(e^{W_\varepsilon}Dy_\varepsilon^\lambda)-e^{-W_\varepsilon}\Delta(\beta_\lambda(e^{W_\varepsilon}y_\varepsilon^\lambda)\\&\qquad+\varepsilon e^{W_\varepsilon}y_\varepsilon^\lambda)+\varepsilon e^{-W_\varepsilon}\beta_\lambda(e^{W_\varepsilon}y_\varepsilon^\lambda)+\frac{1}{2}\,\mu y_\varepsilon^\lambda=0,\\&y_\varepsilon^\lambda(0)=x,\end{aligned}\tag{3.20}$$

where $\beta_\lambda=\beta((I+\lambda\beta)^{-1})=\frac{1}{\lambda}\,(I-(I+\lambda\beta)^{-1})$ is the Yosida approximation of $\beta$. We recall that $\beta_\lambda$ is monotonically increasing, Lipschitzian and

$$\lim_{\lambda\to 0}\beta_\lambda(r)=\beta(r)\text{ uniformly on compacts in }\mathbb{R}.$$

We have

**Lemma 3.2.** *For $\lambda\to 0$, we have, for each $\varepsilon\in(0,1)$,*

$$y_\varepsilon^\lambda\to y_\varepsilon\ \textit{in}\ C([0,T];H^{-1}).$$

**Proof.** It suffices to prove the convergence for the solution $z_\varepsilon^\lambda$ to equation (3.2) with $\beta$ replaced by $\beta_\lambda$. If we subtract the corresponding equation, we get

$$\begin{aligned}&\frac{\partial}{\partial t}\,(z_\varepsilon-z_\varepsilon^\lambda)+(\varepsilon-\Delta)((\beta(z_\varepsilon)-\beta_\lambda(z_\varepsilon^\lambda))+\varepsilon(z_\varepsilon-z_\varepsilon^\lambda))\\&\qquad-\mathrm{div}(D(z_\varepsilon-z_\varepsilon^\lambda))+\frac{1}{2}\,(\mu-\varepsilon^2-(W_\varepsilon)_t)(z_\varepsilon-z_\varepsilon^\lambda)=0,\\&(z_\varepsilon-z_\varepsilon^\lambda)(0)=0.\end{aligned}$$

Applying $\left\langle z_\varepsilon - z_\varepsilon^\lambda, \cdot\right\rangle_{-1,\varepsilon}$ to this equation and integrating on $(0,t)$, we get

$$\|(z_\varepsilon - z_\varepsilon^\lambda)(t)\|^2_{-1,\varepsilon} + \int_0^t \int_{\mathbb{R}^d} (\beta(z_\varepsilon) - \beta_\lambda(z_\varepsilon^\lambda) + \varepsilon(z_\varepsilon - z_\varepsilon^\lambda))(z_\varepsilon - z_\varepsilon^\lambda) ds\, d\xi$$
$$\le C_\varepsilon \int_0^t \|z_\varepsilon(s) - z_\varepsilon^\lambda(s)\|^2_{-1,\varepsilon} ds + \int_0^t \left\langle \operatorname{div} D(z_\varepsilon - z_\varepsilon^\lambda), z_\varepsilon - z_\varepsilon^\lambda \right\rangle_{-1,\varepsilon} ds$$
$$\le C_\varepsilon \int_0^t \|z_\varepsilon(s) - z_\varepsilon^\lambda(s)\|^2_{-1,\varepsilon} ds + C_\varepsilon^1 \int_0^t |z_\varepsilon(s) - z_\varepsilon^\lambda(s)|_2 \|z_\varepsilon(s) - z_\varepsilon^\lambda(s)\|_{-1,\varepsilon} ds.$$

This yields

$$\|z_\varepsilon(t) - z_\varepsilon^\lambda(t)\|^2_{-1,\varepsilon}$$
$$\le C_2^\varepsilon \left( \int_0^t \|z_\varepsilon(s) - z_\varepsilon^\lambda(s)\|^2_{-1,\varepsilon} ds + \int_0^t \int_{\mathbb{R}^d} |\beta(z_\varepsilon) - \beta_\lambda(z_\varepsilon^\lambda)|^2 d\xi ds \right)$$

Taking into account that, as easily seen for each $\varepsilon \in (0,1)$, $\{z_\varepsilon^\lambda\}$ is bounded in $L^2((0,T)\times\mathbb{R}^d)$ and $\beta_\lambda(z_\varepsilon) \to \beta(z_\varepsilon)$, a.e. in $(0,T)\times\mathbb{R}^d$ as $\lambda \to 0$, and $|\beta_\lambda(z_\varepsilon^\lambda)| \le |\beta(z_\varepsilon^\lambda)| \le K(1+|z_\varepsilon^\lambda|)$, we infer by Lebesgue's dominated convergence theorem that, for $\lambda \to 0$,

$$z_\varepsilon^\lambda(t) \to z_\varepsilon(t) \text{ in } H^{-1} \text{ uniformly on } [0,T],$$

as claimed. ∎

**Lemma 3.3.** *Let $x \in D(A) \cap D(A_1)$. Then $y_\varepsilon \in L^\infty((0,T)\times\mathbb{R}^d) \cap L^\infty(0,T;L^1)$ and*

$$\sup_{\varepsilon\in(0,1)} \{|y_\varepsilon|_{L^\infty((0,T)\times\mathbb{R}^d)}\} \le C(1+|x|_\infty), \tag{3.21}$$

$$\sup\nolimits_{\varepsilon\in(0,1)} |y_\varepsilon|_{L^\infty(0,T;L^1)} \le C(|x|_1+1), \tag{3.22}$$

*where $C$ is independent of $x$.*

**Proof.** Let $M = |x|_\infty + 1$ and $\alpha \in C^1[0,T]$ be such that $\alpha(0)=0$, $\alpha' \ge 0$. Since $y_\varepsilon$ is a strong solution of (3.1) in $H^{-1}$, we have

$$\begin{aligned}
&\frac{\partial}{\partial t}(y_\varepsilon - M - \alpha(t)) - e^{-W_\varepsilon}\Delta\left(\beta(e^{W_\varepsilon}y_\varepsilon) + \varepsilon e^{W_\varepsilon}y_\varepsilon\right)\\
&\quad + e^{-W_\varepsilon}\Delta(\beta(e^{W_\varepsilon}(M+\alpha(t))) + \varepsilon e^{W_\varepsilon}(M+\alpha(t)))\\
&\quad + \varepsilon e^{-W_\varepsilon}(\beta(e^{W_\varepsilon}y_\varepsilon) - \beta(e^{W_\varepsilon}(M+\alpha(t))))\\
&\quad - e^{-W_\varepsilon}\operatorname{div}\left(e^{W_\varepsilon}D(y_\varepsilon - M - \alpha(t))\right)\\
&\quad + \frac{1}{2}\mu(y_\varepsilon - M - \alpha(t)) = F_\varepsilon - \alpha',
\end{aligned} \tag{3.23}$$

where

$$
\begin{aligned}
F_\varepsilon = {} & e^{-W_\varepsilon}\mathrm{div}(De^{W_\varepsilon}(M+\alpha(t))) - \varepsilon e^{-W_\varepsilon}\beta(e^{W_\varepsilon}(M+\alpha(t))) \\
& -\frac{1}{2}\,\mu(M+\alpha(t)) + e^{-W_\varepsilon}\Delta\beta(e^{W_\varepsilon}(M+\alpha(t))) \\
& +\varepsilon(M+\alpha(t))e^{-W_\varepsilon}\Delta(e^{W_\varepsilon}),
\end{aligned}
\tag{3.24}
$$

and $\alpha$ will be chosen below, so that

$$F_\varepsilon - \alpha' \le 0.$$

To make clear the argument, we shall first prove (3.21) under the condition

$$\frac{\partial y_\varepsilon}{\partial t}, \beta(e^{W_\varepsilon}y_\varepsilon), \Delta(\beta(e^{W_\varepsilon}y_\varepsilon) + \varepsilon e^{W_\varepsilon}y_\varepsilon) \in L^1((0,T)\times\mathbb{R}^d). \tag{3.25}$$

Now, we multiply (3.23) by $\mathrm{sign}(y_\varepsilon - M - \alpha(t))^+$ and integrate over $(0,t)\times\mathbb{R}^d$. We note here that, by (1.4), (1.5), we have that $e^{-W_\varepsilon}\Delta(\beta(e^{W_\varepsilon}(M+\alpha(t)))) \in L^1$ and that, after this multiplication, all terms on the left hand side of (3.23) become integrable, because of (3.25) and, since $\beta$ is increasing, and satisfies (1.3)–(1.4). By the monotonicity of $\beta$, and by the elementary inequality

$$\int_{\mathbb{R}^d} \Delta z\,\mathrm{sign}\,(z-M_1)^+ d\xi \le 0,\ \ \forall z \in H^1 \text{ with } \Delta z \in L^1(\mathbb{R}^d),\ \ M_1 \ge 0. \tag{3.26}$$

we have, because

$$
\begin{aligned}
\mathrm{sign}(y_\varepsilon - M - \alpha(t))^+ &= \mathrm{sign}(\beta(e^{W_\varepsilon}y_\varepsilon) - \beta(e^{W_\varepsilon}(M+\alpha(t))))^+ \\
&= \mathrm{sign}((\beta+\varepsilon I)(e^{W_\varepsilon}y_\varepsilon) - (\beta+\varepsilon I)(e^{W_\varepsilon}(M+\alpha(t))))^+,
\end{aligned}
$$

where $I(r) = r$, $r \in \mathbb{R}$,

$$
\begin{aligned}
&J(t) := -\int_{\mathbb{R}^d} e^{-W_\varepsilon}[\Delta(\beta(e^{W_\varepsilon}y_\varepsilon)+\varepsilon e^{W_\varepsilon}y_\varepsilon)\\
&-\Delta(\beta(e^{W_\varepsilon}(M+\alpha(t)))+\varepsilon e^{W_\varepsilon}(M+\alpha(t)))\\
&-\varepsilon(\beta(e^{W_\varepsilon}y_\varepsilon)-\beta(e^{W_\varepsilon}(M+\alpha(t))))]\mathrm{sign}(y_\varepsilon-M-\alpha(t))^+d\xi\\
&\geq -2\int_{\mathbb{R}^d} e^{-W_\varepsilon}\nabla[\beta(e^{W_\varepsilon}y_\varepsilon)+\varepsilon e^{W_\varepsilon}y_\varepsilon-\beta(e^{W_\varepsilon}(M+\alpha(t)))\\
&-\varepsilon e^{W_\varepsilon}(M+\alpha(t))]\cdot\nabla W_\varepsilon\,\mathrm{sign}(y_\varepsilon-M-\alpha(t))^+d\xi\\
&+\int_{\mathbb{R}^d}\Delta(e^{-W_\varepsilon})(\beta(e^{W_\varepsilon}y_\varepsilon)-\beta(e^{W_\varepsilon}(M+\alpha(t)))\\
&+\varepsilon e^{W_\varepsilon}(y_\varepsilon-M-\alpha(t)))\mathrm{sign}(y_\varepsilon-M-\alpha(t))^+d\xi\\
&=-\int_{\mathbb{R}^d}\Delta(e^{-W_\varepsilon})((\beta+\varepsilon I)(e^{W_\varepsilon}y_\varepsilon)-(\beta+\varepsilon I)(e^{W_\varepsilon}(M+\alpha(t))))^+d\xi\\
&\geq -(\beta'(e^{-\|W\|_\infty}M)+1)e^{\|W\|_\infty}\|e^{W_\varepsilon}\Delta(e^{-W_\varepsilon})\|_\infty\int_{\mathbb{R}^d}(y_\varepsilon-M-\alpha(t))^+d\xi,
\end{aligned}
\tag{3.27}
$$

where, in the last step, we used that on $\{y_\varepsilon - M - \alpha(t) > 0\}$ by the mean value theorem and (1.5), we have

$$
\begin{aligned}
&\beta(e^{W_\varepsilon}y_\varepsilon)-\beta(e^{W_\varepsilon}(M+\alpha(t)))\leq\beta'(e^{W_\varepsilon}(M+\alpha(t)))\cdot e^{W_\varepsilon}(y_\varepsilon-M-\alpha(t))\\
&\leq\beta'(e^{-\|W\|_\infty}M)e^{\|W\|_\infty}(y_\varepsilon-M-\alpha(t)).
\end{aligned}
$$

This yields

$$
\begin{aligned}
\int_0^t J(s)ds \geq -(\beta'(e^{-\|W\|_\infty}M)+1)e^{\|W\|_\infty}(\|\Delta W\|_\infty+\|\nabla W\|^2_\infty)\\
\cdot\int_0^t|(y_\varepsilon-(M+\alpha(s)))^+|_1)ds,
\end{aligned}
\tag{3.28}
$$

where $\|\cdot\|_\infty$ is the norm of $L^\infty((0,T)\times\mathbb{R}^d)$. (Here and everywhere in the following we shall denote by $C$ several positive constants independent of $W$ and $\varepsilon$.) We also have, since $\partial_i f\,\mathrm{sign}\,f^+ = \partial_i f^+$,

$$\int_0^t\int_{\mathbb{R}^d} e^{-W_\varepsilon}\mathrm{div}(e^{W_\varepsilon}D(y_\varepsilon-M-\alpha(s)))\mathrm{sign}\,(y_\varepsilon-M-\alpha(s))^+ds\,d\xi = \int_0^t\int_{\mathbb{R}^d}\nabla W_\varepsilon\cdot D(y_\varepsilon-M-\alpha(s))^+ds\,d\xi. \tag{3.29}$$

Taking into account that

$$\int_{\mathbb{R}^d}\frac{\partial}{\partial t}\,(y_\varepsilon(t,\xi)-M-\alpha(t))\mathrm{sign}(y_\varepsilon(t,\xi)-M-\alpha(t))^+d\xi = \frac{d}{dt}\,|(y_\varepsilon(t)-M-\alpha(t))^+|_1,\ \text{a.e. } t\in(0,T),$$

after some calculations involving (3.23)–(3.29), assuming that $F_\varepsilon\le\alpha'$, we obtain that

$$\begin{aligned}|(y_\varepsilon(t)-M-\alpha(t))^+|_1 &\le \int_0^t\int_{\mathbb{R}^d}((L+1)(\|\Delta W_\varepsilon\|_\infty \\ &+\|\nabla W_\varepsilon\|_\infty^2)+\nabla W_\varepsilon\cdot D)(y_\varepsilon-M-\alpha(s))^+ds\,d\xi \\ &\le(\beta'(e^{-\|W\|_\infty}M)+1)e^{\|W\|_\infty}(\|\Delta W_\varepsilon\|_\infty+\|\nabla W_\varepsilon\|_\infty^2) \\ &+\|\nabla W_\varepsilon\|_\infty\|D\|_\infty\int_0^t|(y_\varepsilon(s)-M-\alpha(s))^+|_1)ds.\end{aligned} \tag{3.30}$$

By (3.30), it follows that

$$|(y_\varepsilon(t)-M-\alpha(t))^+|_1=0 \tag{3.31}$$

if $F_\varepsilon\le\alpha'$, a.e. in $(0,T)\times\mathbb{R}^d$. To find $\alpha$ so that this holds, we set

$$C:=e^{\|W\|}(\|\mathrm{div}\,D\|_\infty+\|D\|_\infty^2+\|\mu\|_\infty+2+a_1 +a_1a_2)(\|\Delta W\|_\infty+\|\nabla W\|_\infty^2+1).$$

Then, by assumptions (1.3), (1.4), and an elementary calculation, we have

$$F_\varepsilon\le C(M+\alpha(t))=\alpha'(t),$$

if $\alpha(t)=M(\exp(Ct)-1)$, and so (3.31) holds. Hence

$$y_\varepsilon(t)\le M+\alpha(t)\le M+\alpha(T)<\infty,\ \forall t\in[0,T].$$

Since the function $r \mapsto -\beta(-r)$, $r \in \mathbb{R}$, enjoys the same properties as $\beta$, by a symmetric argument we get

$$y_\varepsilon(t) \geq -M - \alpha(t),\ \forall t \in [0,T],$$

and so (3.21) follows.

To remove condition (3.25), we are going to approximate (3.23) by the finite difference scheme (3.38) below. To this end, let us first recall that $A_1$ is the $L^1$-closure of

$$A_0 z = -\Delta(\beta(z) + \varepsilon z) + \varepsilon\beta(z) - \operatorname{div}(Dz) + \frac{1}{2}\,\mu z,\ z \in D(A_0)$$

(see (3.14)). Moreover, by (3.16) for each $f \in L^1 \cap L^\infty$ ($\subset H^{-1}$) and $\lambda > \lambda_0$, the equation

$$\lambda z + A_0 z = f \tag{3.32}$$

has a unique solution $z \in D(A_0) \cap L^\infty \subset L^1 \cap H^1 \cap L^\infty$ and $z, \beta(z) \in H^1 \cap L^1$. To see that indeed we also have that $z \in L^\infty$, we first note that, for all $M \in (0,\infty)$, $z \in H^1$, $(z-M)^+ = z - z \wedge M \in H^1$ and that it is easy to see that (cf. (3.26))

$$\int \Delta(z-M)\operatorname{sign}(z-M)^+ d\xi \leq 0. \tag{3.33}$$

Choosing $M = |f|_\infty$ and $\lambda \in (0,\infty)$ large enough, we have for the solution $z$ of (3.32) that

$$\begin{aligned}\lambda(z-M) - \Delta(\beta(z) - \beta(M) + \varepsilon(z-M)) - \operatorname{div}(D(z-M))\\ +\frac{\mu}{2}\,(z-M) = f - \lambda M + M \operatorname{div} D - \frac{\mu}{2}\, M \leq 0.\end{aligned}$$

Multiplying by $\operatorname{sign}(z-M)^+$ and integrating over $\mathbb{R}^d$ by (3.32), it follows that

$$\lambda \int_{\mathbb{R}^d} (z-M)^+ d\xi + \frac{1}{2} \int \mu(z = M)^+ d\xi \leq 0,$$

hence $z \leq M$. Since $r \mapsto -\beta(-r)$, $r \in \mathbb{R}$, enjoys the same properties as $\beta$, by symmetry we get $z \geq -M$, so $z \in L^\infty$. Hence

$$(\lambda I + A_1)^{-1}(L^1 \cap L^\infty) \subset D(A_0) \cap L^\infty \subset L^1 \cap H^1 \cap L^\infty,\ \forall \lambda > \lambda_0. \tag{3.34}$$

Now, let us show that the solution $z_\varepsilon$ constructed in Lemma 3.1 is also the limit of another, for our purpose more convenient finite difference scheme. To this end, define for $h \in (0,1)$ and $0 \leq i \leq N-1$, with $N := \left[\frac{T}{h}\right]$,

$$\nu_i^h := \frac{1}{h}(e^{-W_{i+1}} - e^{-W_i}) + (W_\varepsilon)_t(ih)e^{-W_i},$$

where $W_i := W_\varepsilon(ih)$. Now, consider the finite difference approximation scheme (again setting $\widetilde{u}_i := \widetilde{u}_i^h$)

$$\begin{aligned}&\frac{1}{h}(\widetilde{u}_{i+1} - \widetilde{u}_i) + A_1\widetilde{u}_{i+1} + \Lambda(ih)\widetilde{u}_{i+1} + \nu_i^h\widetilde{u}_{i+1} = 0,\\&\widetilde{u}_0^h = u_0 = x.\end{aligned} \tag{3.35}$$

If $u_i := u_i^h$ is as in (5.11), then

$$\frac{1}{h}(u_{i+1} - u_i) + A_1u_{i+1} + \Lambda(ih)u_{i+1} + \nu_i^h u_{i+1} + \eta_i(h) = 0,$$

where $\eta_i(h) = -\nu_i^h \to 0$ in $L^1$, uniformly on $[0, T]$ as $h \to 0$. Hence, by the same arguments to prove that the schemes (5.10) and (5.11) in the proof of Lemma 5.1 render the same limit, we obtain that

$$\lim_{h\to 0} \widetilde{u}^h = z_\varepsilon \text{ in } L^1 \text{ and } H^{-1} \text{ uniformly on } [0, T].$$

Setting $y_i := y_i^h = e^{-W_i}\widetilde{u}_i$, we conclude that

$$\lim_{h\to 0} y_\varepsilon^h = y_\varepsilon \text{ in } L^1 \text{ and } H^{-1} \text{ uniformly on } [0, T], \tag{3.36}$$

and, for $0 \le i \le N - 1$, $N := \left[\frac{T}{h}\right]$,

$$\begin{aligned}&\frac{1}{h}(y_{i+1} - y_i) + e^{-W_{i+1}}A_1(e^{W_{i+1}}y_{i+1}) = 0,\\&y_0 = x_0,\end{aligned} \tag{3.37}$$

where $y_\varepsilon^h(t) := y_i$ for $t \in [ih, (i + 1)h)$. Since $x \in L^1 \cap L^\infty$, by (3.34) we have that $e^{W_i}y_i \in D(A_0) \cap L^\infty$, $0 \le i \le N$. So, in (3.37) we may replace $A_1$ by $A_0$.

Now, the approximating scheme (3.37) can be written as

$$\begin{aligned}&\frac{1}{h}\left(y_{i+1} - y_i - (\alpha(ih) - \alpha((i - 1)h))\right) + e^{-W_{i+1}}(A_0(e^{W_{i+1}}y_{i+1})\\&\quad -A(e^{W_{i+1}}(M + \alpha(ih)))) = F_\varepsilon^i - \frac{1}{h}\left(\alpha(ih) - \alpha((i - 1)h)\right) \le 0,\end{aligned} \tag{3.38}$$

where

$$F_\varepsilon^i = e^{-W_{i+1}}\mathrm{div}(De^{W_{i+1}}(M+\alpha(ih))) - \varepsilon e^{-W_{i+1}}\beta(e^{W_{i+1}}(M+\alpha(ih)))$$
$$-\frac{1}{2}\mu(M+\alpha(ih)) + e^{-W_{i+1}}\Delta\beta(e^{W_{i+1}}(M+\alpha(ih)))$$
$$+\varepsilon(M+\alpha(ih))e^{-W_{i+1}}\Delta e^{W_{i+1}},$$

where $A(e^{W_{i+1}}(M+\alpha(ih)))$ is "algebraically" defined as if $A = A_0$, but the argument is not in the domain of $D(A_0)$ (and not even in $D(A_1)$). We note that choosing $\alpha$ as above, again by (1.3), (1.4) and an elementary calculation, we indeed have that the right hand side of (3.38) is negative. By (3.34) we see that $\beta(e^{W_{i+1}}y_{i+1}), \Delta(\beta(e^{W_{i+1}}y_{i+1}) + \varepsilon e^{W_{i+1}}y_{i+1})$ are in $L^1(\mathbb{R}^d)$.

Now, we multiply (3.38) by $\mathrm{sign}(y_{i+1}-M-\alpha(ih))^+$ and take into account that

$$\frac{1}{h}\int_{\mathbb{R}^d}(y_{i+1}-y_i-(\alpha(ih)-\alpha((i-1)h)))\mathrm{sign}(y_{i+1}-M-\alpha(ih))^+d\xi$$
$$\geq \frac{1}{h}(|(y_{i+1}-M-\alpha(ih))^+|_1 - |(y_i - M - \alpha((i-1)h))^+|_1). \quad (3.39)$$

Arguing as in (3.27)-(3.28), we get by (3.26)

$$I_1^i := -\int_{\mathbb{R}^d} e^{-W_{i+1}}[\Delta(\beta(e^{W_{i+1}}y_{i+1}) + \varepsilon e^{W_{i+1}}y_{i+1}) - \Delta(\beta(e^{W_{i+1}}(M+\alpha(ih)))$$
$$+\varepsilon e^{W_{i+1}}(M+\alpha(ih)))]\mathrm{sign}(y_{i+1}-(M+\alpha(ih)))^+d\xi \quad (3.40)$$
$$\geq \int_{\mathbb{R}^d}\Delta(e^{-W_{i+1}})((\beta+\varepsilon I)(e^{W_{i+1}}y_{i+1}) - (\beta+\varepsilon I)(e^{W_{i+1}}(M+\alpha(ih))))^+d\xi$$
$$\geq -(\beta'(e^{-\|W\|_\infty}M)+1)e^{\|W\|_\infty}(\|\Delta W\|_\infty + \|\nabla W\|_\infty^2)\int_{\mathbb{R}^d}(y_{i+1}-M-\alpha(ih))^+d\xi.$$

Similarly, we have

$$I_2^i = \int_{\mathbb{R}^d} e^{-W_{i+1}}\mathrm{div}(e^{W_{i+1}}D(y_{i+1}-M-\alpha(ih)))\mathrm{sign}(y_{i+1}-M-\alpha(ih))^+d\xi$$
$$= \int_{\mathbb{R}^d}(D(y_{i+1}-M-\alpha(ih)))\cdot\nabla W_{i+1}\,\mathrm{sign}(y_{i+1}-M-\alpha(ih))^+d\xi.$$

This yields

$$I_2^i \leq \|D\|_\infty\|\nabla W\|_\infty\int_{\mathbb{R}^d}(y_{i+1}-M-\alpha(ih))^+d\xi. \quad (3.41)$$

Combining estimates (3.38), (3.39), (3.40), (3.41) and the facts that $\mu \geq 0$ and $\beta$ is increasing, we get the discrete analogue of (3.30), that is, for $C := (\beta'(e^{-\|W\|_\infty}M) + 1)e^{\|W\|_\infty}(\|\Delta W\|_\infty + \|\nabla W\|_\infty^2) + \|D\|_\infty \|\nabla W\|_\infty$,

$$\begin{aligned}&\frac{1}{h}\left(|(y_{i+1} - M - \alpha(ih))^+|_1 - |(y_1 - M - \alpha((i-1)h))^+|_1\right)\\&\leq C|(y_{i+1} - M - \alpha(ih))^+|_1.\end{aligned}$$

Summing up from $i = 0$ to $k$, we get

$$\frac{1}{h}\,|(y_{k+1} - M - \alpha(ik))^+|_1 \leq C \sum_{i=0}^{k} |(y_{i+1} - M - \alpha(ih))^+|_1,$$

which implies, for all $t \in [0, T]$,

$$|(y_\varepsilon^h(t) - M - \alpha^h(t))^+|_1 = 0,$$

where $\alpha^h(t) = ih$ on $[ih, (i+1)h[$, $0 \leq i \leq N-1$. Letting $h \to 0$ as above, we get (3.21).

To obtain estimate (3.22), we multiply equation (3.37) by $\operatorname{sign} y_{i+1}$ and integrate over $(0,t) \times \mathbb{R}^d$. Then, similarly as above we find, since $\mu \geq 0$ and $\beta$ is increasing, that

$$\begin{aligned}\frac{1}{h}\left(|y_{i+1}^+|_1 - |y_i^+|_1\right) &\leq \frac{1}{h}\int_{\mathbb{R}^d} (y_{i+1} - y_i)\operatorname{sign} y_{i+1}^+ d\xi\\&\leq \int_{\mathbb{R}^d} e^{-W_{i+1}}\Delta((\beta + \varepsilon I)(e^{W_{i+1}}y_{i+1}))\operatorname{sign} y_{i+1}^+ d\xi\\&+ \int_{\mathbb{R}^d} e^{-W_{i+1}}\operatorname{div}(De^{W_{i+1}}y_{i+1})\operatorname{sign} y_{i+1}^+ d\xi\\&\leq \int_{\mathbb{R}^d} \Delta e^{-W_{i+1}}(\beta + \varepsilon I)(e^{W_{i+1}}y_{i+1})\operatorname{sign} y_{i+1}^+ d\xi\\&+ \int_{\mathbb{R}^d} \nabla W_{i+1} \cdot Dy_{i+1}^+ d\xi\\&\leq C|y_{i+1}^+|_1,\end{aligned} \tag{3.42}$$

where

$$C := (\|\Delta W\|_\infty + \|\nabla W\|_\infty^2)(\beta'(e^{-\|W\|_\infty}M) + 1)e^{\|W\|_\infty} + \|\nabla W\|_\infty \|D\|_\infty.$$

Hence, summing from $i=0$ to $k$, we obtain

$$|y_{i+1}^+|_1 \le |x|_1 + Ch\sum_{i=0}^{k}\int_{\mathbb{R}^d} y_{i+1}^+ d\xi.$$

Since $r \mapsto -\beta(-r)$, $r \in \mathbb{R}$, also fulfills all our assumptions on $\beta$, by a symmetry argument we find

$$|y_{i+1}^-|_1 \le |x|_1 + Ch\sum_{i=0}^{k}\int_{\mathbb{R}^d} y_{i+1}^- d\xi.$$

This implies that $\forall t \in [0,T]$

$$|y_\varepsilon^h(t)|_1 \le |x|_1 e^{CT}$$

and (3.22) follows letting $h \to 0$. ■

**Lemma 3.4.** *Let $x \in D(A)\cap D(A_1)$. Then there exists an increasing function $C:[0,\infty)\to(0,\infty)$ such that*

$$\sup_{t\in[0,T]} |y_\varepsilon(t)|_2^2 + \int_0^T\int_{\mathbb{R}^d} |\nabla\beta(e^{W_\varepsilon}y_\varepsilon)|^2 ds\, d\xi \le C(|x|_\infty + |x|_1),\ \forall \varepsilon \in (0,1], \quad (3.43)$$

*for a constant $C>0$, independent of $\varepsilon \in (0,1]$.*

**Proof.** Clearly, by Lemma 3.3 we only have to prove the bound in (3.43) for the integral on the left hand side. To this end, we multiply (3.1) by $\beta(y_\varepsilon)$ and integrate over $(0,t)\times\mathbb{R}^d$. Taking into account that (see [1], Lemma 4.4)

$$\int_{\mathbb{R}^d} j(y_\varepsilon(t))d\xi = \int_0^t {}_{H^{-1}}\left\langle \frac{dy_\varepsilon}{ds}(s), \beta(y_\varepsilon(s))\right\rangle_{H^1} ds + \int_{\mathbb{R}^d} j(x)d\xi,$$

where $j(r) = \int_0^r \beta(s)ds$, $r \in \mathbb{R}$, and that

$${}_{H^{-1}}\left\langle \Delta\beta(e^{W_\varepsilon}y_\varepsilon), e^{-W_\varepsilon}\beta(y_\varepsilon)\right\rangle_{H^1} = -\int_{\mathbb{R}^d} \nabla\beta(e^{W_\varepsilon}y_\varepsilon)\cdot\nabla(e^{-W_\varepsilon}\beta(y_\varepsilon))d\xi,$$

we get from (3.1) that

$$\begin{aligned}
&\int_{\mathbb{R}^d} j(y_\varepsilon(t))d\xi \\
&\qquad + \int_0^t\int_{\mathbb{R}^d}[(\nabla\beta(e^{W_\varepsilon}y_\varepsilon) + \varepsilon\nabla(e^{W_\varepsilon}y_\varepsilon))\cdot\nabla(\beta(y_\varepsilon)e^{-W_\varepsilon})]d\xi\, ds \qquad (3.44)\\
&\qquad \le \int_{\mathbb{R}^d} j(x)d\xi + \int_0^t\int_{\mathbb{R}^d} e^{-W_\varepsilon}\mathrm{div}(e^{W_\varepsilon}Dy_\varepsilon)\beta(y_\varepsilon)d\xi\, ds.
\end{aligned}$$

Let us denote the first and second term on the left side of (3.44) $I_1$ and $I_2$, respectively, and the two on the right $I_3$ and $I_4$. Then

$$\begin{aligned} I_4 &= \int_0^t \int_{\mathbb{R}^d} yD\cdot(\beta'(y)\nabla y - \beta\nabla W)d\xi\, ds \\ &\leq \|\nabla W\|_\infty \|D\|_\infty \int_0^t \int_{\mathbb{R}^d} |y_\varepsilon|\,|\beta(y_\varepsilon)|d\xi\, ds \\ &+ \|D\|_\infty \int_0^t \int_{\mathbb{R}^d} |y_\varepsilon|\beta'(y_\varepsilon)|\nabla y_\varepsilon| d\xi\, ds. \end{aligned} \tag{3.45}$$

Obviously, the first integral in the preceding line by Lemma 3.3 and (1.4) is bounded by $C_1(1+|x|_\infty^2)$ with a constant $C_1>0$ independent of $\varepsilon$. Since by (1.6), (1.7) we have

$$\begin{aligned} \beta'(y_\varepsilon) &\leq \beta'(e^{-\|W\|_\infty + W_\varepsilon} y_\varepsilon) \\ &\leq \varphi(e^{-\|W\|_\infty})\beta'(e^{W_\varepsilon} y_\varepsilon), \end{aligned} \tag{3.46}$$

the second integral in the r.h.s. of (3.45), again by Lemma 3.3 can be bounded by

$$\delta \int_0^t \int_{\mathbb{R}^d} |\nabla\beta(e^{W_\varepsilon} y_\varepsilon)|^2 d\xi\, ds + \frac{C_2}{\delta}\,(1+|x|_\infty^2),\ \forall \delta > 0,$$

where $C_2>0$ is a constant independent of $\varepsilon$. So, altogether

$$I_4 \leq \delta \int_0^t \int_{\mathbb{R}^d} |\nabla\beta(e^{W_\varepsilon} y_\varepsilon)|^2 d\xi\, dx + \left(\frac{C_2}{\delta} + C_1\right)(1+|x|_\infty^2),\ \forall \delta > 0. \tag{3.47}$$

Clearly, by (1.3),

$$I_3 \leq \sup_{r\in[-x|_\infty, |x|_\infty]} |\beta(r)|\ |x|_1 \leq C(1+|x|_\infty)|x|_1 (<\infty). \tag{3.48}$$

Furthermore,

$$I_2 = \int_0^t \int_{\mathbb{R}^d} \nabla\beta(e^{W_\varepsilon} y_\varepsilon)\cdot\nabla(\beta(y_\varepsilon)e^{-W_\varepsilon})d\xi\, ds + \varepsilon \widetilde{I}_2,$$

where

$$
\begin{aligned}
\widetilde{I}_2 &:= \int_0^t \int_{\mathbb{R}^d} \nabla(e^{W_\varepsilon} y_\varepsilon) \cdot [\beta'(y_\varepsilon) e^{-W_\varepsilon} \nabla y_\varepsilon - \beta(y_\varepsilon) e^{-W_\varepsilon} \nabla W_\varepsilon] d\xi\, ds \\
&= \int_0^t \int_{\mathbb{R}^d} \nabla(e^{W_\varepsilon} y_\varepsilon) \cdot [e^{-2W_\varepsilon} \beta'(y_\varepsilon) \nabla(e^{W_\varepsilon} y_\varepsilon) \\
&\quad - e^{-2W_\varepsilon} \beta'(y_\varepsilon) y_\varepsilon e^{W_\varepsilon} \nabla W_\varepsilon - \beta(y_\varepsilon) e^{-W_\varepsilon} \nabla W_\varepsilon] d\xi\, ds \\
&\geq - \int_0^t \int_{\mathbb{R}^d} [|\nabla(e^{W} y_\varepsilon)| \beta'(y_\varepsilon) |y_\varepsilon| + |\nabla(e^{W_\varepsilon} y_\varepsilon)|\, |\beta(y_\varepsilon)|] e^{-W_\varepsilon} |\nabla W_\varepsilon| d\xi\, ds.
\end{aligned}
$$

Since, by Lemma 3.3, we have $\sup_{\varepsilon \in (0,1)} \|y_\varepsilon\|_\infty < \infty$, it follows by (1.5) that

$$
\beta'(e^{W_\varepsilon} y_\varepsilon) \geq \beta'(e^{\|W\|_\infty} \operatorname{sign} y_\varepsilon \sup_{\varepsilon \in (0,1)} \|y_\varepsilon\|_\infty) \ (> 0). \tag{3.49}
$$

Combining this with (3.46), we conclude that, for some increasing functions $\widetilde{C}_3, C_3 : [0, \infty) \to (0, \infty)$, independent of $\varepsilon$

$$
\begin{aligned}
\widetilde{I}_2 \ &\geq \ - \int_0^t \int_{\mathbb{R}^d} [\varphi(e^{-\|W\|_\infty}) |\nabla \beta(e^{W_\varepsilon} y_\varepsilon)|\, |y_\varepsilon| \qquad (3.50) \\
&+ \ (\beta'(e^{\|W\|_\infty} \operatorname{sign} y_\varepsilon \sup_{\varepsilon \in (0,1)} \|y_\varepsilon\|_\infty))^{-1} |\nabla \beta(e^{W_\varepsilon} y_\varepsilon)|\, |\beta(y_\varepsilon)|] e^{-W_\varepsilon} |\nabla W_\varepsilon| d\xi\, ds \\
&\geq \ -\delta \int_0^t \int_{\mathbb{R}^d} |\nabla \beta(e^{W_\varepsilon} y_\varepsilon)|^2 d\xi\, ds \qquad (3.51) \\
&- \ \frac{\widetilde{C}_3(|x|_\infty)}{\delta} \int_0^t \int_{\mathbb{R}^d} (|y_\varepsilon|^2 + |\beta(y_\varepsilon)|^2) |\nabla W_\varepsilon|^2 d\xi\, ds \\
&\geq \ -\delta \int_0^t \int_{\mathbb{R}^d} |\nabla \beta(e^{W_\varepsilon} y_\varepsilon)|^2 d\xi\, ds - \frac{C_3(|x|_\infty)}{\delta}, \ \forall \delta > 0,
\end{aligned}
$$

where in the last step we used that, by (1.3), the second integral in the previous to the last line in (3.50) by Lemma 3.3 is up to a constant, independent of $\varepsilon$, bounded by

$$
\int_0^t \int_{\mathbb{R}^d} |\nabla W|^2 d\xi\, ds < \infty.
$$

Furthermore,

$$\begin{aligned}
\widetilde{\widetilde{I}}_2 :&= \int_0^t \int_{\mathbb{R}^d} \nabla\beta(e^{W_\varepsilon} y_\varepsilon)\cdot\nabla(\beta(y_\varepsilon)e^{-W_\varepsilon})d\xi\,ds \\
&= \int_0^t \int_{\mathbb{R}^d} \nabla\beta(e^{W_\varepsilon} y_\varepsilon)\cdot(e^{-W_\varepsilon}\beta'(y_\varepsilon)\nabla y_\varepsilon - \beta(y_\varepsilon)e^{-W_\varepsilon}\nabla W_\varepsilon)d\xi\,ds \\
&= \int_0^t \int_{\mathbb{R}^d} \beta'(e^{W_\varepsilon} y_\varepsilon)\beta'(y_\varepsilon)e^{-2W_\varepsilon}\nabla(e^{W_\varepsilon} y_\varepsilon)\cdot(\nabla(e^{W_\varepsilon} y_\varepsilon) - y_\varepsilon e^{W_\varepsilon}\nabla W_\varepsilon)d\xi\,ds \\
&- \int_0^t \int_{\mathbb{R}^d} \nabla\beta(e^{W_\varepsilon} y_\varepsilon)\beta(y_\varepsilon)e^{-W_\varepsilon}\nabla W_\varepsilon\,d\xi\,ds.
\end{aligned}$$

Since, by (1.6), (1.7),

$$\begin{aligned}
\beta'(y_\varepsilon) &\geq \beta'(e^{\|W_\varepsilon\|_\infty} y_\varepsilon) = \beta'(e^{\|W_\varepsilon\|_\infty - W_\varepsilon} e^{W_\varepsilon} y_\varepsilon) \\
&\geq (\varphi(e^{-\|W_\varepsilon\|_\infty + W_\varepsilon}))^{-1}\beta'(e^{W_\varepsilon} y_\varepsilon) \\
&\geq \varphi(e^{-2\|W\|_\infty})^{-1}\beta'(e^{W_\varepsilon} y_\varepsilon),
\end{aligned} \tag{3.52}$$

we obtain that, for some constant $C_4 > 0$ and an increasing function $C_5 : [0,\infty) \to (0,\infty)$, independent of $\varepsilon$,

$$\begin{aligned}
\widetilde{\widetilde{I}}_2 &\geq C_4 \int_0^t \int_{\mathbb{R}^d} |\nabla\beta(e^{W_\varepsilon} y_\varepsilon)|^2 d\xi\,ds \\
&\quad - \int_0^t \int_{\mathbb{R}^d} |\nabla\beta(e^{W_\varepsilon} y_\varepsilon)| e^{-W_\varepsilon} |\nabla W_\varepsilon| [|y_\varepsilon|\beta'(y_\varepsilon) + |\beta(y_\varepsilon)|] d\xi\,ds \\
&\geq (C_4 - \delta)\int_0^t \int_{\mathbb{R}^d} |\nabla\beta(e^{W_\varepsilon} y_\varepsilon)|^2 d\xi\,ds - \frac{C_5(|x|_\infty)}{\delta},\ \forall\delta > 0,
\end{aligned} \tag{3.53}$$

where in the last step we used that

$$e^{-\|W\|_\infty} \sup_{\varepsilon\in(0,1)} (\|y_\varepsilon\beta'(y_\varepsilon)\|_\infty + \|\beta(y_\varepsilon)\|_\infty) \int_0^t \int_{\mathbb{R}^d} |\nabla W|^2 d\xi\,ds < \infty,$$

because of (1.3), (1.4) and Lemma 3.3. Finally, we note that, by Lemma 3.3,

$$\begin{aligned}
I_1 &\geq -\sup\left\{\beta(r) | r \in \Big[-\sup_{\varepsilon\in(0,1)} \|y_\varepsilon\|_\infty, \sup_{\varepsilon\in(0,1)} \|y_\varepsilon\|_\infty\Big]\right\} \sup_{\varepsilon\in(0,1)} |y_\varepsilon|_{L^\infty(0,T;L^1)} \\
&= -C_6(|x|_\infty + |x|_1),
\end{aligned} \tag{3.54}$$

for some increasing function $C_6 : [0,\infty) \to (0,\infty)$.

Recalling that $I_2 = \widetilde{\widetilde{I}}_2 + \varepsilon \widetilde{I}_2$ and combining (3.47), (3.48), (3.50), (3.53) and (3.54), the assertion of Lemma 3.4 is proved. ■

**Proof of existence.** Assume first that Hypotheses (i)–(iii) hold. Let $x \in D_0$. It follows, by Lemmas 3.3 and 3.4, that $\{\beta(e^{W_\varepsilon} y_\varepsilon)\}$ is bounded in $L^2(0,T;H^1)$, $\{y_\varepsilon\}$ is bounded in $L^\infty(0,T;L^2) \cap L^\infty((0,T)\times\mathbb{R}^d)$ and $\left\{\frac{dy_\varepsilon}{dt}\right\}$ is bounded in $L^2(0,T;H^{-1})$.

Moreover, taking into account that $\nabla\beta(e^{W_\varepsilon} y_\varepsilon) = \beta'(e^{W_\varepsilon} y_\varepsilon)\nabla(e^{W_\varepsilon} y_\varepsilon)$ and that by assumption (1.5) and estimate (3.21),

$$\beta'(e^{W_\varepsilon} y_\varepsilon) \ge \rho > 0, \text{ a.e. in } (0,T)\times\mathbb{R}^d, \tag{3.55}$$

it follows that $\{y_\varepsilon\}$ is bounded in $L^2(0,T;H^1)$. As a matter of fact, we have

$$\begin{aligned}\sup_{\varepsilon\in(0,1]} \Bigg\{ & \|y_\varepsilon\|_\infty + \|y_\varepsilon\|_{L^\infty(0,T;L^1)} + \|e^{W_\varepsilon} y_\varepsilon\|_{L^2(0,T;H^1)} + \|y_\varepsilon\|_{L^2(0,T;H^1)} \\ & + \|\beta(e^{W_\varepsilon} y_\varepsilon)\|_{L^2(0,T;H^1)} + \left\|\frac{dy_\varepsilon}{dt}\right\|_{L^2(0,T;H^{-1})} \Bigg\} \le C^*(\omega),\ \omega\in\Omega,\end{aligned} \tag{3.56}$$

where $C^*$ is $\mathcal{F}$–measurable and $0 < C^*(\omega) \le Ce^{\|W\|_\infty}(\exp(\|\nabla W\|_\infty + \|\Delta W\|_\infty + 1)$, $\forall \omega \in \Omega$. Then, by the Aubin compactness theorem (see, e.g., [1], p. 26), $\{y_\varepsilon\}$ is compact in each $L^2(0,T;L^2(B_R))$ where $B_R = \{\xi \in \mathbb{R}^d;\ |\xi| \le R\}$. Hence, for fixed $\omega\in\Omega$ along a subsequence, again denoted $\{\varepsilon\}$, we have

$$\begin{array}{rcll} y_\varepsilon & \longrightarrow & y & \text{strongly in } L^2((0,T);L^2_{\rm loc}(\mathbb{R}^d)) \\ & & & \text{weak-star in } L^\infty((0,T)\times\mathbb{R}^d) \\ & & & \text{weakly in } L^2(0,T;H^1), \\ \beta(e^{W_\varepsilon} y_\varepsilon) & \longrightarrow & \eta & \text{weakly in } L^2(0,T;H^1) \\ \dfrac{dy_\varepsilon}{dt} & \longrightarrow & \dfrac{dy}{dt} & \text{weakly in } L^2(0,T;H^{-1}) \\ W_\varepsilon & \longrightarrow & W & \text{in } C([0,T]\times\mathbb{R}^d), \end{array} \tag{3.57}$$

and so, letting $\varepsilon\to 0$ in equation (3.1), we see that

$$\begin{aligned} & \frac{dy}{dt} - e^{-W}{\rm div}(De^W y) - e^{-W}\Delta\eta + \frac{1}{2}\,\mu y = 0 \text{ in } (0,T)\times\mathbb{R}^d, \\ & y(0) = x \ \text{ on } \mathbb{R}^d. \end{aligned} \tag{3.58}$$

Here, we recall that $\Delta : H^1 \to H^{-1}$ is continuous and, by (3.6), $u \mapsto \mathrm{div}(Du)$ is continuous from $L^2$ to $H^{-1}$, while $\frac{d}{dt}$ is considered in sense of $H^{-1}$-valued distributions on $(0,T)$ or, equivalently, a.e. on $(0,T)$. Clearly, estimate (3.56) remains true for $y$.

To show that $y$ is a solution to (1.11), it remains to be proven that $\eta = \beta(e^W y)$, a.e. in $(0,T) \times \mathbb{R}^d$. Since the map $z \to \beta(z)$ is maximal monotone in each $L^2((0,T) \times B_R)$, it is closed and so the latter follows by (3.57). Moreover, if the solution $y$ to (1.11) is unique (we shall see later on that this happens if $\beta$ is locally Lipschitz), it follows that the sequence $\{y_\varepsilon\}$ arising in (3.57) is independent of $\omega \in \Omega$, and so $y$ is $(\mathcal{F}_t)_{t\geq 0}$–adapted.

**Uniqueness.** Assume that, besides assumptions (i)-(iii), $\beta$ is locally Lipschitz on $\mathbb{R}$ and consider $y_1, y_2$ to be two solutions to equation (1.11) satisfying (2.2)–(2.4) and let $z = y_1 - y_2$. We have

$$\begin{aligned} &\frac{\partial z}{\partial t} - e^{-W}\mathrm{div}(e^W Dz) - e^{-W}\Delta(\beta(e^W y_1) - \beta(e^W y_2)) + \frac{1}{2}\,\mu z = 0 \\ &\hspace{20em} \text{in } (0,T) \times \mathbb{R}^d, \\ &z(0) = 0 \quad \text{in } \mathbb{R}^d. \end{aligned} \tag{3.59}$$

Equivalently,

$$\begin{aligned} \frac{\partial z}{\partial t} + (I - \Delta)(z\eta) &= e^{-W}\,\mathrm{div}(e^W Dz) - e^W \Delta(e^{-W}) z\eta \\ &\quad - 2\nabla(e^{-W}) \cdot \nabla(e^W z\eta) - \frac{1}{2}\,\mu z + z\eta, \end{aligned} \tag{3.60}$$

where

$$\eta = \begin{cases} \dfrac{\beta(e^W y_1) - \beta(e^W y_2)}{e^W z} & \text{on } [(\xi,t); z(t,\xi) \neq 0], \\ 0 & \text{on } [(\xi,t); z(t,\xi) = 0]. \end{cases}$$

We note that, by Hypothesis (ii) (1.5), we have, for some $\alpha_i = C^i(|x|_1 + |x|_\infty)$, $i = 0,1$, where $C^i : [0,\infty) \to (0,\infty)$ are increasing continuous functions,

$$0 < \alpha_0 \leq \eta \leq \alpha_1, \ \text{a.e. in } (0,T) \times \mathbb{R}^d, \tag{3.61}$$

because $\beta$ is locally Lipschitz on $\mathbb{R}$.

We have $z \in L^2(0,T; H^1(\mathbb{R}^d))$ and $\frac{\partial z}{\partial t} \in L^2(0,T; H^{-1})$. We multiply (3.60) by $(I - \Delta)^{-1} z$ and integrate over $\mathbb{R}^d$ to get

$$
\begin{aligned}
&\frac{1}{2}\,|z(t)|^2_{-1}+\int_0^t\int_{\mathbb{R}^d}\eta z^2 ds\,d\xi\\
&=\frac{1}{2}\,|z(0)|^2_{-1}+\int_0^t\int_{\mathbb{R}^d}e^{-W}\mathrm{div}(e^W Dz)(I-\Delta)^{-1}z ds\,d\xi\\
&-\int_0^t\int_{\mathbb{R}^d}e^W\Delta(e^{-W})z\eta(I-\Delta)^{-1}z ds\,d\xi\\
&-2\int_0^t\int_{\mathbb{R}^d}\nabla(e^{-W})\cdot\nabla(e^W z\eta)(I-\Delta)^{-1}z ds\,d\xi\\
&+\int_0^t\int_{\mathbb{R}^d}\left(-\frac{1}{2}\,\mu+\eta\right)z(I-\Delta)^{-1}z ds\,d\xi\\
&=\frac{1}{2}\,|z(0)|^2_{-1}+\int_0^t(I_1+I_2+I_3+I_4)ds.
\end{aligned}
\tag{3.62}
$$

By the right hand side of (3.61), we get the following estimates

$$
\begin{aligned}
|I_1| &\le C|z|_2|z|_{-1},\\
|I_1| &\le C\int_{\mathbb{R}^d}|\beta(e^W y_1)-\beta(e^W y_2)|\,|(I-\Delta)^{-1}z|d\xi\\
&\le C|\beta(e^W y_1)-\beta(e^W y_2)|_2|z|_{-1}\le \widetilde{C}|z|_2|z|_{-1}\\
|I_3| &\le \left|2\int_{\mathbb{R}^d}(\beta(e^W y_1)-\beta(e^W y_2))\mathrm{div}(\nabla(e^{-W})(I-\Delta)^{-1}z)d\xi\right|\\
&\le C|\beta(e^W y_1)-\beta(e^W y_2)|_2|z|_{-1}\le \widetilde{C}|z|_2|z|_{-1}\\
|I_4| &\le C(|z|^2_{-1}+|z|_1|z|_{-1}).
\end{aligned}
$$

We note also that, by (3.61), we have

$$
|z|_2|z|_{-1}\le|\sqrt{\eta}\,z|_2(\sqrt{\alpha_0})^{-1}|z|_{-1}\le\frac{1}{2}\,|\sqrt{\eta}\,z|_2^2+\frac{1}{2}\,\alpha_0^{-1}|z|^2_{-1}.
$$

Then, by (3.62), we obtain that

$$
\frac{d}{dt}\,|z(t)|^2_{-1}\le C_2|z(t)|^2_{-1},\ \text{a.e. } t>0,
$$

which implies $z \equiv 0$, as claimed.

Note also that, by (3.59) and (3.62) it follows also that there exist increasing $C_1, C_2 : [0,\infty) \to (0,\infty)$ such that, for all $x, \bar{x} \in x \in D_0$, one has

$$|y(t,x) - y(t,\bar{x})|_{-1} \le C_1(|x|_{L^1\cap L^\infty} + |\bar{x}|_{L^1\cap L^\infty})|x - \bar{x}|_{-1},\ \forall t \in [0,T], \quad (3.63)$$
$$|y(t,x) - y(t,\bar{x})|_1 \le C_2(|x|_{L^1\cap L^\infty} + |\bar{x}|_{L^1\cap L^\infty})|x - \bar{x}|_1,\ \forall t \in [0,T]. \quad (3.64)$$

Indeed, if one applies (3.62) for $z(t) = y(t,x) - y(t,\bar{x})$ and uses the above estimates on $I_i$, $i = 1,2,3,4$, and (3.61), one gets (3.63). To get (3.64), we multiply (3.60) by $\operatorname{sgn} z$ (or, more exactly, by $\mathcal{X}_\delta(\tau)$, where $\mathcal{X}_\delta$ is given by (3.17)) and integrate over $\mathbb{R}^d$.

By (3.21), (3.22) and Lemma 3.4, we have

$$|y(t)|_\infty + |y(t)|_1 + \int_0^T \int_{\mathbb{R}^d} |\nabla\beta(y(t,x)(\xi))|^2 dt\, d\xi \le C_3(|x|_\infty + |x|_1), \quad \forall t \in [0,T], \quad (3.65)$$

for some increasing functions $C_i : [0,\infty) \to (0,\infty)$, $i = 1,2,3$. This means that, by Lemma 2.1, for all $x \in L^1 \cap L^\infty$, $y = y(t,x)$ extends by density to a strong solution to (1.11). The map $L^1 \cap L^\infty \ni x \mapsto y(t,x)$ is then Lipschitz on balls in $L^1 \cap L^\infty$. Such a function $y$ satisfies equation (1.11), a.e. on $(0,T) \times \mathbb{R}^d$, and by (3.65) we have

$$y \in W^{1,2}([0,T]; H^{-1}) \cap L^\infty((0,T) \times \mathbb{R}^d), \quad \beta(e^W y) \in L^2(0,T; H^1).$$

This completes the proof of Theorem 2.2. ∎

**Remark 3.5.** By (3.63) and Lemma 2.1, it follows also that, for $x \in L^1 \cap L^\infty$, there is a unique mild (generalized) solution $y \in L^\infty(0,T; L^1) \cap L^\infty((0,T) \times \mathbb{R}^d)$ defined as the limit of mild solutions, that is,

$$y = \lim_{n\to\infty} y(\cdot, x_n) \text{ in } L(0,T; L^1)$$

for $x_n \to x$ in $L^1$, where $\{x_n\} \subset D_0$ and is bounded in $L^1 \cap L^\infty$.

It should be said also that, in the case where $\beta$ is not locally Lipschitz, we do not know whether we have uniqueness. So, the sequence $\{y_\varepsilon\}$ arising in (3.57) might depend on the fixed $\omega \in \Omega$ and so we cannot conclude that the limit $y$ is $(\mathcal{F}_t)_{t\ge 0}$–adapted.

# 4 The stochastic equation with nonlinear drift

We consider here the equation

$$
\begin{aligned}
&dX - \operatorname{div}(a(X))dt - \Delta\beta(X)dt = X\,dW \text{ in } (0,T)\times\mathbb{R}^d,\\
&X(0,\xi) = x(\xi),\ \xi\in\mathbb{R}^d,
\end{aligned}
\tag{4.1}
$$

where $\beta$ and $W$ are as in Section 1, while $a:\mathbb{R}\to\mathbb{R}^d$ satisfies the following assumption

(iv) $a$ is Lipschitzian and $a(0)=0$.

The strong solution $X$ to equation (4.1) is defined as for equation (1.1).

For simplicity, we shall use the notations

$$u_\xi = \nabla u,\quad u_{\xi\xi} = \Delta u.$$

By transformation (1.10), we reduce the stochastic equation (4.1) to the equation (see (1.11))

$$
\begin{aligned}
&\frac{\partial y}{\partial t} - e^{-W}\operatorname{div}(a(e^W y)) - e^{-W}(\beta(e^W y))_{\xi\xi} + \frac{1}{2}\,\mu y = 0 \text{ in } (0,T)\times\mathbb{R}^d,\\
&y(t,\xi) = x(\xi).
\end{aligned}
\tag{4.2}
$$

We have

**Theorem 4.1.** *If assumptions* (ii), (iii), (iv) *hold and $\beta$ is locally Lipschitz, for each $x\in D_0$, there is a unique strong solution $y$ to equation* (4.2) *satisfying* (2.2)–(2.4). *Moreover, the process $y$ is $(\mathcal{F}_t)_{t\ge0}$-adapted and, if $x\ge0$, a.e. on $\mathbb{R}^d$, then $y\ge0$, a.e. on $(0,T)\times\mathbb{R}^d$, and the map $D_0\ni x\to y(\cdot,x)$ is Lipschitz from $H^{-1}$ to $C([0,T],H^{-1})$ on balls in $L^1\cap L^\infty$ and extends to a strong solution to* (4.1) *satisfying* (2.2), (2.4), *for all $x\in L^1\cap L^\infty$.*

**Proof.** Since the proof is essentially the same as that of Theorem 2.2, we only sketch it, by emphasizing, however, the points where arise major differences in the argument.

We consider the approximating equation (see (3.1))

$$
\begin{aligned}
&\frac{\partial y_\varepsilon}{\partial t} - e^{-W_\varepsilon}\operatorname{div}(a(e^{W_\varepsilon} y_\varepsilon)) - e^{-W_\varepsilon}\beta(e^{W_\varepsilon} y_\varepsilon) - \varepsilon e^{-W_\varepsilon}(e^{W_\varepsilon} y_\varepsilon)_{\xi\xi}\\
&\qquad + \varepsilon e^{-W_\varepsilon}\beta(e^{W_\varepsilon} y_\varepsilon) + \frac{1}{2}\,\mu y_\varepsilon = 0 \text{ in } (0,T)\times\mathbb{R}^d,\\
&y_\varepsilon(0,\xi) = x(\xi),\ \xi\in\mathbb{R}^d,
\end{aligned}
\tag{4.3}
$$

which, by the same argument as that in the proof of Lemma 3.1, has a unique solution $y_\varepsilon$ which satisfies (3.3)–(3.4).

We note that Lemmas 3.1, 3.3 and 3.4 remain valid in this case too. Indeed, we note that, instead of (3.23) and (3.24), we have

$$\begin{aligned}&\frac{\partial}{\partial t}(y_\varepsilon-M-\alpha(t))-e^{-W_\varepsilon}(\beta(e^{W_\varepsilon}y_\varepsilon)+\varepsilon e^{W_\varepsilon}y_\varepsilon)\\&-(\beta(e^{W_\varepsilon}(M+\alpha(t))-\varepsilon e^{W_\varepsilon}(M+\alpha(t))))_{\xi\xi}\\&+\varepsilon e^{-W_\varepsilon}(\beta(e^{W_\varepsilon}y_\varepsilon)-\beta(e^{\|W_\varepsilon\|}(M+\alpha(t))))\\&-e^{-W_\varepsilon}(\operatorname{div}(a(e^{W_\varepsilon}y_\varepsilon)-a(e^{W_\varepsilon}(M+\alpha(t)))))\\&-\frac{1}{2}\,\mu(y_\varepsilon-M-\alpha(t))=F_\varepsilon-\alpha'(t),\end{aligned}\tag{4.4}$$

where

$$\begin{aligned}F_\varepsilon=e^{-W_\varepsilon}\operatorname{div}a(e^{W_\varepsilon}(M+\alpha(t)))-\frac{1}{2}\,(M+\alpha(t))+e^{-W_\varepsilon}(\beta(e^{W_\varepsilon}(M+\alpha(t))))_{\xi\xi}\\-\varepsilon e^{-W_\varepsilon}\beta(e^{W_\varepsilon}(M+\alpha(t)))+\varepsilon(M+\alpha(t))e^{-W_\varepsilon}(e^{W_\varepsilon})_{\xi\xi}\end{aligned}$$

(or it discretized analogue (3.38)).

In order to treat the term in $a$ arising in (4.3), we note that

$$\begin{aligned}-\int_0^t\!\!\int_{\mathbb{R}^d}&e^{-W_\varepsilon}\operatorname{div}(a(e^{W_\varepsilon}y_\varepsilon)-a(e^{W_\varepsilon}(M+\alpha(s)))\operatorname{sign}(y_\varepsilon-(M+\alpha(s)))^+)ds\,d\xi\\-&\int_0^t\!\!\int_{\mathbb{R}^d}(\operatorname{div}(a(e^{W_\varepsilon}y_\varepsilon)-a(e^{W_\varepsilon}(M+\alpha(s)))e^{-W_\varepsilon})\\&\qquad\qquad\operatorname{sign}(e^{W_\varepsilon}y_\varepsilon-e^{W_\varepsilon}(M+\alpha(s))e^{-W_\varepsilon})^+)ds\,d\xi\\+&\int_0^t\!\!\int_{\mathbb{R}^d}(e^{-W_\varepsilon})_\xi\cdot(a(e^{W_\varepsilon}y_\varepsilon)-a(e^{W_\varepsilon}(M+\alpha(s)))\\&\qquad\qquad\operatorname{sign}(e^{W_\varepsilon}y_\varepsilon-e^{W_\varepsilon}(M+\alpha(s)))^+ds\,d\xi\\\le&\,L\int_0^t\int_{\mathbb{R}^d}|(e^{-W_\varepsilon})_\xi|(y_\varepsilon-M-\alpha(s))^+ds\,d\xi,\end{aligned}$$

because $a$ is Lipschitz and

$$\int_{\mathbb{R}^d}(e^{-W_\varepsilon}(a(u)-a(v)))_\xi\operatorname{sign}(u-v)d\xi=0\tag{4.5}$$

for $u=e^{W_\varepsilon}y_\varepsilon$ and $v=e^{W_\varepsilon}(M+\alpha(t))$. To prove (4.5), we consider the approximation $\mathcal{X}_\delta$ of the signum function defined by (3.17). We have

$$\begin{aligned}
H_\delta(t) &= \int_{\mathbb{R}^d} \operatorname{div}(e^{-W_\varepsilon}(a(u)-a(v)))\mathcal{X}_\delta(u-v)d\xi \\
&= -\int_{\mathbb{R}^d} e^{-W_\varepsilon}(a(u)-a(v))\cdot(u-v)_\xi \mathcal{X}_\delta'(u-v)d\xi \\
&= -\frac{1}{\delta}\int_{[|u-v|\le\delta]} e^{-W_\varepsilon}(a(u)-a(v))\cdot(u-v)_\xi d\xi.
\end{aligned}$$

For $\delta\to 0$, we get

$$\lim_{\delta\to 0} H_\delta(t) = \int_{\mathbb{R}^d} e^{-W_\varepsilon}\operatorname{div}(a(u(t,\xi))-a(v(t,\xi)))\operatorname{sign}(u(t,\xi)-v(t,\xi))d\xi$$

while

$$|H_\delta(t)| \le \operatorname{Lip}(a)\int_{[|u-v|\le\delta]} e^{-W_\varepsilon}|(u-v)_\xi|d\xi.$$

This yields

$$\limsup_{\delta\to 0}|H_\delta(t)| \le \int_{[|u-v|=0]} e^{-W_\varepsilon}|(u-v)_\xi|d\xi = 0,$$

because $(u-v)_\xi=0$ on $\{\xi;\ (u-v)(\xi)=0\}$. (We recall that $u,v\in H^1$.)

Then estimate (3.29) with $a$ in place of $D$ remains true in this case.

Multiplying (4.4) by $\operatorname{sign}(y_\varepsilon-M-\alpha(t))^+$ and integrating on $(0,t)\times\mathbb{R}^d$, we get by (3.28) an estimate of the form (3.30) from which we infer that

$$|(y_\varepsilon(t)-M-\alpha(t))^+|_1=0,\ t\in(0,T),$$

for $\alpha$ chosen as in the proof of Lemma 3.3 and so

$$y_\varepsilon\le M+\alpha(t),\ \text{a.e. in } (0,T)\times\mathbb{R}^d,$$

and, similarly,

$$y_\varepsilon\ge -M-\alpha(t),\ \text{a.e. in } (0,T)\times\mathbb{R}^d.$$

Taking into account that

$$\begin{aligned}
\int_0^t\int_{\mathbb{R}^d} e^{-W_\varepsilon}\operatorname{div}(a(e^{W_\varepsilon}y_\varepsilon))_\xi\beta(y_\varepsilon)ds\,d\xi &= -\int_0^t\int_{\mathbb{R}^d} a(e^{W_\varepsilon}y_\varepsilon)\cdot(e^{-W_\varepsilon}\beta(y_\varepsilon))_\xi ds\,d\xi \\
&\le C\int_0^t\int_{\mathbb{R}^d}(|e^{W_\varepsilon}y_\varepsilon|(|y_\varepsilon|^m|(e^{-W_\varepsilon}))_\xi| + e^{-W_\varepsilon}\beta'(y_\varepsilon)|\nabla y_\varepsilon|)ds\,d\xi,
\end{aligned}$$

and, recalling that $\sup_{\varepsilon>0}\{|y_\varepsilon|_\infty\} < \infty$, it follows as in the proof of Lemma 3.4 that estimate (3.43) holds in this case too. Hence, there is $y \in C([0,T];L^2_{\rm loc})\cap L^\infty((0,T)\times\mathbb{R}^d)\cap L^2(0,T;H^1)$ such that (3.57) holds. Moreover, we have, for $\varepsilon\to 0$,

$$a(e^{W_\varepsilon}y_\varepsilon)\to a(e^W y) \text{ in } L^2((0,T);L^2_{\rm loc})$$

and so, for $\varepsilon\to 0$

$$\mathrm{div}(a(e^{W_\varepsilon}y_\varepsilon))\to \mathrm{div}(a(e^W y)) \text{ in } L^2([0,T);H^{-1}_{\rm loc}).$$

Then letting $\varepsilon\to 0$ in (4.3), we see that $y$ is a solution to equation (4.2) satisfying (2.2)-(2.4). Moreover, multiplying (4.3) by sign$y_\varepsilon$ and taking into account that, as seen earlier,

$$\int_{\mathbb{R}^d} e^{-W_\varepsilon}\mathrm{div}(a(e^{W_\varepsilon}y_\varepsilon))\mathrm{sign}\, y_\varepsilon d\xi \le C\int_{\mathbb{R}^d}|e^{W_\varepsilon}y_\varepsilon|d\xi,$$

we get as in the proof of Lemma 3.3 that

$$|y_\varepsilon(t)|_1\le C|x|_1,\ \forall t\in[0,T],$$

where $C$ is independent of $\varepsilon$.

**Uniqueness.** If $\beta$ is locally Lipschitz and $y_1,y_2$ are solutions to (4.1), for $z=y_1-y_2$, we get (see (3.59))

$$\begin{aligned}&\frac{\partial z}{\partial t}-\mathrm{div}(a(e^W y_1)-a(e^W y_2))-e^{-W}(\beta(e^W y_1)-\beta(e^W y_2))_{\xi\xi}+\frac{1}{2}\,\mu z=0\\ &z(0)=0,\end{aligned}$$

and, arguing as in the proof of uniqueness in Theorem 2.2, we get $z\equiv 0$. If $\beta\in L^1_{\rm loc}(\mathbb{R})$, then, multiplying scalarly in $L^2$ by $(I-\Delta)^{-1}z$ and using the local Lipschitzianity of $\beta$ and $a$, we get as above the estimates (3.63)–(3.65). ∎

By Theorem 4.1, we have

**Corollary 4.2.** *If assumptions* (ii), (iii), (iv) *hold and $\beta$ is locally Lipschitz, then for each $x\in D_0$ there is a unique strong solution $X$ to the stochastic equation* (4.1)*, which satisfies*

$$Xe^{-W}\in W^{1,2}([0,T];H^{-1}),\ \mathbb{P}\text{-}a.s., \tag{4.6}$$

*and $X\ge 0$, a.e. on $(0,T)\times\mathbb{R}^d\times\Omega$ if $x\ge 0$, a.e. on $\mathbb{R}^d$. Moreover, the map $x\mapsto X(t,x)$ is $H^{-1}$-Lipschitz from balls in $L^1\cap L^\infty$ to $C([0,T];H^{-1})$.*

**Remark 4.3.** If $a$ is not Lipschitz, one cannot expect a strong solution for equation (4.1). In the deterministic case, if $\beta \equiv 0$, equation (4.1) reduces to a first order quasilinear equation previously studied by S. Kruzkov [20] (see, also, [9], [13]), who introduced and proved existence of a generalized solution involving the so-called "entropy" conditions. (See also [2] for the case where $\beta$ is present.) So, also in this case, one might expect to have a generalized solution in sense of Kruzkov, but this remains to be done.

# 5 Appendix

Here, we shall briefly review a few definitions and results pertaining the nonlinear Cauchy problem in Banach spaces for quasi-$m$-accretive operators.

Let $X$ be a Banach space with the norm denoted $\|\cdot\|_X$. A nonlinear operator $A : D(A) \subset X \to X$ (possibly multivalued) is said to be accretive if

$$\|x_1 - x_2 + \lambda(y_1 - y_2)\|_X \geq \|x_1 - x_2\|_X, \ \forall \lambda > 0, \ \forall y_i \in Ax_i, \ i = 1, 2,$$

and quasi-accretive if $A + \alpha I$ is accretive for some $\alpha > 0$. Equivalently,

$$_X(y_1 - y_2, \eta)_{X'} \geq 0, \text{ for some } \eta \in J(x_1 - x_2),$$

where $J : X \to X'$ is the duality map of the space $X$. (Here, $X'$ is the dual of $X$.) The operator $A$ is said to be *m-accretive* if the range $R(\lambda I + A)$ of $\lambda I + A$ is all of $X$ for all $\lambda > 0$ and quasi $m$-accretive if $R(\lambda I + A) = X$ for $\lambda > \lambda_0 > 0$.

If $A$ is quasi $m$-accretive, $u_0 \in \overline{D(A)}$ and $g \in C([0, T]; X)$, then the Cauchy problem

$$\begin{aligned} &\frac{du}{dt} + Au \ni g \text{ in (0,T)}, \\ &u(0) = u_0, \end{aligned} \tag{5.1}$$

has a unique mild solution $u \in C([0, T]; X)$ defined by

$$u(t) = \lim_{h \to 0} u^h(t) \text{ strongly in } X \text{ and uniformly on } [0, T], \tag{5.2}$$

$$\begin{aligned} &u^h(t) = u_i^h \text{ for } t \in [ih, (i+1)h], \\ &\frac{1}{h}\,(u_{i+1}^h - u_i^h) + Au_{i+1} \ni \frac{1}{h} \int_{ih}^{(i+1)h} g(t)dt, \\ &\qquad i = 0, 1, ..., N-1, \text{ with } N = \left[\tfrac{T}{h}\right], \\ &u_0^h = u_0. \end{aligned} \tag{5.3}$$

(See, e.g., [1], Section 4.1, Corollary 4.2.) (For $g \equiv 0$, this is just the Crandall-Liggett exponential formula.) Moreover, if the space $X$ is reflexive and $g \in W^{1,1}([0,T];X)$, then $u$ is a strong absolutely continuous solution to (5.1), that is, it satisfies a.e. (5.1) and

$$u \in W^{1,\infty}([0,T];X),\ \ Au \in L^\infty(0,T;X). \tag{5.4}$$

Finally, if $X$ is uniformly convex, then $\frac{d}{dt}\,u(t)$ is continuous from the right.

We consider now the Cauchy problem

$$\begin{aligned} &\frac{du}{dt}\,(t) + Au(t) + \Lambda(t)u(t) = 0,\ \ \forall t \in (0,T),\\ &u(0) = u_0, \end{aligned} \tag{5.5}$$

where $A$ is quasi-$m$-accretive, $u_0 \in \overline{D(A)}$ and $\Lambda \in C([0,T];L(X,X))$. Since it is enough for the applications in this paper, let us for simplicity assume that $A$ is single-valued. We have

**Lemma 5.1.** *The Cauchy problem* (5.5) *has a unique mild solution* $u \in C([0,T];X)$ *and* $u$ *is given as the limit in* (5.9) *of the finite difference scheme* (5.11) *below. Moreover, if* $u_0 \in D(A)$ *and*

$$\|\Lambda(t) - \Lambda(s)\|_{L(X,X)} \le L|t-s|,\ \ \forall s,t \in [0,T], \tag{5.6}$$

*then* $u : [0,T] \to X$ *is Lipschitz.*

**Proof.** Consider the operator $\mathcal{A} : D(\mathcal{A}) \subset L^1(0,T;X) \to L^1(0,T;X)$ defined by $\mathcal{A}u = g$ if $u \in C([0,T];X)$ is the mild solution to (5.1). By the existence theory for (5.1), it follows that $R(\lambda I + \mathcal{A}) = L^1(0,T;X)$, $\forall \lambda > 0$, and by (5.3) we see that $\mathcal{A}$ is quasi-accretive. Indeed, if $\lambda_0 \ge 0$ such that $A + \lambda_0 I$ is $m$-accretive, then by [1], Theorem 4.1 and Proposition 3.7(iv), we have for solutions $u, \bar{u}$ for (5.1) with $g, \bar{g}$, respectively, on the right hand side

$$\|u(t) - \bar{u}(t)\|_X \le \int_0^t e^{\lambda_0(t-s)}\|g(s) - \bar{g}(s)\|_X ds,\ \ \forall \lambda > 0,\ \ g, \bar{g} \in L^1(0,T;X),$$

which yields

$$\|u - \bar{u}\|_{L^1(0,T;X)} \le \frac{e^{\lambda_0 T}}{\lambda_0}\,\|g - \bar{g}\|_{L^1(0,T;X)}.$$

Hence $\mathcal{A}$ is quasi-$m$-accretive.

The operator $\widetilde{\Lambda} : L^1(0,T;X) \to L^1(0,T;X)$ defined by

$$(\widetilde{\Lambda} u)(t) = \Lambda(t)u(t),\ t \in [0,T], \tag{5.7}$$

is linear continuous and this implies that $\mathcal{A} + \widetilde{\Lambda}$ is quasi $m$-accretive in $L^1(0,T;X)$. Hence there is $\lambda_0 > 0$ such that $R(\lambda I + \mathcal{A} + \widetilde{\Lambda}) = L^1(0,T;X)$ for $\lambda > \lambda_0 > 0$.

This means that, for every $g \in C([0,T],X)$, the equation

$$\begin{aligned} &\frac{du}{dt} + Au + \lambda u = g(t) - \Lambda(t)u,\ t \in (0,T), \\ &u(0) = u_0 \end{aligned} \tag{5.8}$$

has a unique mild solution for $\lambda > \lambda_0$.

Now, let us show that this implies that also (5.5) has a unique mild solution. This is well known, but we include the proof for the reader's convenience. So, fix $\lambda > \lambda_0$ and let $u, \bar{u}$ be the unique mild solutions of (5.8) with $\lambda g$ and $\lambda \bar{g}$ replacing $g$ on its right hand side, where $g, \bar{g} \in \mathcal{X} := C([0,T];X)$, equipped with the norm $\|\cdot\|_{\mathcal{X}} := \|\cdot\|_{\mathcal{X},T}$, where for $t \in [0,T]$

$$\|g\|_{\mathcal{X},t} := \sup\{e^{-\alpha s}\|g(s)\|_X;\ s \in [0,t]\}$$

and $\alpha > 0$ will be chosen later. Then, by [1], Theorem 4.1 and Proposition 3.7(iv), for all $t \in [0,T]$, it follows that

$$\|u - \bar{u}\|_{\mathcal{X},t} \le \int_0^t e^{-(\lambda - \lambda_0 + \alpha)(t-s)}(\lambda\|g - \bar{g}\|_{\mathcal{X},s} ds + C\|u - \bar{u}\|_{\mathcal{X},s})ds,$$

where $C := \sup\limits_{t \in [0,T]} \|\Lambda(t)\|_{L(X,X)}$. Hence, by Gronwall's lemma,

$$\|u - \bar{u}\|_{\mathcal{X}} \le \frac{\lambda e^{CT}}{\lambda - \lambda_0 + \alpha}\|g - \bar{g}\|_{\mathcal{X}}.$$

Now, choosing $\alpha$ large enough, it follows that the map which maps $g$ to the solution $u$ of (5.8) with $\lambda g$ replacing $g$ on its right hand side, is a strict contraction on $\mathcal{X}$. Hence, by Banach's fixed point theorem, (5.5) has a unique mild solution, $u$.

Moreover, as a mild solution to (5.5), by (5.2)and (5.3) where $g(t) = \Lambda(t)u(t)$, $u$ satisfies

$$u = \lim_{h \to 0} u^h(t) \text{ strongly in } X \text{ and uniformly on } [0,T], \tag{5.9}$$

where, for $h > 0$,

$$\begin{aligned}
&u^h(t) = u_i^h \text{ for } t \in [ih, (i+1)h),\\
&\frac{1}{h}\,(u_{i+1}^h - u_i^h) + Au_{i+1}^h + \frac{1}{h}\int_{ih}^{(i+1)h} \Lambda(t)u(t)dt = 0,\\
&\qquad\qquad\qquad i = 0, 1, ..., N-1, \text{ with } N = \left[\tfrac{T}{h}\right],\\
&u_0^h = u_0.
\end{aligned} \tag{5.10}$$

As easily seen, we may replace (5.10) by

$$\frac{1}{h}\,(u_{i+1}^h - u_i^h) + Au_{i+1}^h + \Lambda(ih)u_{i+1}^h = 0. \tag{5.11}$$

Indeed, setting $u_i := u_i^h$, we may rewrite (5.10) as

$$\frac{1}{h}\,(u_{i+1} - u_i) + Au_{i+1} + \Lambda(ih)u_{i+1} + \eta_i(h) = 0, \tag{5.12}$$

where $\|\eta_i\| \leq \delta(h),\ \forall i$, and $\delta(h) \to 0$ uniformly on $[0, T]$ as $h = \frac{T}{N}$ goes to zero.

Now, if $v = v_i,\ i = 0, 1, ...N-1$, is the solution to (5.11), subtracting the equation (5.11) from (5.12), we get for $y_i = u_i - v_i$ the equation

$$y_{i+1} + h(Au_{i+1} - Av_{i+1}) + h\Lambda(ih)y_{i+1} = y_i - h\eta_i(h)$$

and, by the quasi-accretivity of $A$, this yields

$$\|y_{i+1}\| \leq \mu h\|y_{i+1}\| + \|y_i\| + h\delta(h),\ \forall i = 0, 1, ..., n-1,$$

where $\mu = \lambda + \sup\limits_{t\in[0,T]} \|\Lambda(t)\|_{L(X;X)}$. This yields for small enough $h$

$$\|y_{i+1}\| \leq (1 - \mu h)^{-1}(\|y_i\| + h\delta(h))$$

and, taking into account that $y_0 = 0$ and that $h = \frac{T}{N}$, we get that for $h$ small enough

$$\|y_{i+1}\| \leq h\delta(h)(1 - \mu h)^{-1} \sum_{1\leq j\leq i} (1 - \mu h)^{-j} \leq \frac{\delta(h)}{\mu}\left(1 - \frac{T\mu}{N}\right)^{-N}.$$

Hence $y_i = y_i^h$ goes to zero in $X$ as $h$ goes to zero and this completes the proof of the equivalence of (5.11) and (5.10).

Now, we shall prove that, if $u_0 \in D(A)$ and (5.6) holds, then $u$ is Lipschitz.

By (5.6), we have

$$\begin{aligned} &\|\Lambda(t)u(t) - \Lambda(s)u(s)\|_X \\ &\qquad \le L|t-s|\|u\|_{C([0,T];X)} + \|\Lambda(t)\|_{L(X,X)}\|u(t)-u(s)\|_X \\ &\qquad \le C_1(|t-s| + \|u(t)-u(s)\|_X),\ \forall s,t \in [0,T]. \end{aligned} \tag{5.13}$$

We consider now the equation

$$\begin{aligned} &\frac{du_\lambda}{dt} + A_\lambda u_\lambda + \Lambda(t)u = 0,\ t \in [0,T], \\ &u_\lambda(0) = u_0, \end{aligned} \tag{5.14}$$

where $A_\lambda = \lambda^{-1}(I - (I+\lambda A)^{-1})$ is the Yosida approximation of $A$. The Cauchy problem has a unique differentiable solution $u_\lambda : [0,T] \to X$ and, since $A_\lambda$ is $\widetilde{\lambda}_0$-accretive for some $\widetilde{\lambda}_0 > 0$, we have by (5.14)

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|u_\lambda(t+h) - u_\lambda(t)\|_X^2 \le \|\Lambda(t+h)u(t+h) - \Lambda(t)u(t)\|_X \\ &\|u_\lambda(t+h) - u_\lambda(t)\|_X + \widetilde{\lambda}_0\|u_\lambda(t+h) - u_\lambda(t)\|_X^2,\ t, t+h \in [0,T]. \end{aligned}$$

By (5.13), this yields

$$\begin{aligned} &\|u_\lambda(t+h) - u_\lambda(t)\|_X \le e^{(\widetilde{\lambda}_0 + C_1)t}\|u_\lambda(h) - u_\lambda(0)\|_X \\ &\qquad + C\int_0^t e^{(\widetilde{\lambda}_0 + C_1)(t-s)}(h + \|u(s+h) - u(s)\|_X)ds. \end{aligned} \tag{5.15}$$

On the other hand, by (5.14) we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|u_\lambda(t) - u_0\|_X^2 &\le \widetilde{\lambda}_0\|u_\lambda(t) - u_0\|_X^2 + \|A_\lambda u_0\|_X\|u_\lambda(t) - u_0\|_X \\ &+ \|\Lambda(t)u(t)\|_X\|u_\lambda(t) - u_0\|_X,\ \forall t \in [0,T]. \end{aligned}$$

Hence

$$\begin{aligned} \|u_\lambda(t) - u_0\|_X &\le \int_0^t e^{\widetilde{\lambda}_0(t-s)}(\|A_\lambda u_0\|_X + \|\Lambda(s)u(s)\|_X)ds \\ &\le C_2(\|Au_0\|_X + 1),\ \forall t \in [0,T]. \end{aligned}$$

Substituting into (5.15), yields

$$
\begin{aligned}
&\|u_\lambda(t+h) - u_\lambda(t)\|_X \\
&\quad \le C_3 \left( h + \int_0^t e^{(\widetilde{\lambda}_0 + C_1)(t-s)} (h + \|u(s+h) - u(s)\|_X) \right) ds, \qquad (5.16) \\
&\quad \forall \lambda > 0,\ t, t+h \in [0,T].
\end{aligned}
$$

On the other hand, since for each $\varepsilon > 0$

$$\lim_{\lambda\to 0} (I + \varepsilon A_\lambda)^{-1} x = (I + \varepsilon A)^{-1} x,\ \forall x \in H,$$

by the Trotter-Kato theorem for nonlinear semigroups of contractions, we have (see [1], Corollary 4.5)

$$u_\lambda \longrightarrow v \ \text{ in } C([0,T];X) \ \text{ as } \lambda \to 0,$$

where $v$ is the solution to

$$
\begin{aligned}
&\frac{dv}{dt} + Av + \Lambda(t)u = 0, \\
&v(0) = u_0.
\end{aligned}
$$

By the quasi-accretivity of $A$, it follows that $v = u$. Then, letting $\lambda \to 0$ in (5.16), we get

$$\|u(t+h) - u(t)\|_X \le C_3 \left( h + \int_0^t (h + \|u(s+h) - u(s)\|_X) ds \right)$$

and by Gronwall's inequality, we get

$$\|u(t+h) - u(t)\|_X \le C_4 h,\ \forall t, t+h \in [0,T],$$

as claimed. This completes the proof. ∎

If the space $X$ is reflexive, we infer that, under the conditions of Lemma 5.1, $u \in W^{1,\infty}([0,T];X)$ is a.e. differentiable, and satisfies equation (5.5), a.e. on $(0,T)$. We have, therefore,

**Corollary 5.2.** *If the space $X$ is reflexive, $u_0 \in D(A)$, and $\Lambda$ satisfies* (5.6)*, then the mild solution $u$ to* (5.5) *is a strong absolutely continuous solution, which satisfies* (5.4)*.*

It should be mentioned that the latter case applies to $X = H^{-1}$, but not to $X = L^1$. In the latter case, the solution $u$ is only continuous.

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