SCATTERING FOR STOCHASTIC NONLINEAR SCHRÖDINGER EQUATIONS

SEBASTIAN HERR, MICHAEL RÖCKNER, AND DENG ZHANG

Abstract. We study the scattering behavior of global solutions to stochastic nonlinear Schrödinger equations with linear multiplicative noise. In the case where the quadratic variation of the noise is globally finite and the nonlinearity is defocusing, we prove that the solutions scatter at infinity in the pseudo-conformal space and in the energy space respectively, including the energy-critical case. Moreover, in the case where the noise is large, non-conservative and has infinite quadratic variation, we show that the solutions scatter at infinity with high probability for all energy-subcritical exponents.

1. Introduction and main results

We are concerned with stochastic nonlinear Schrödinger equations with linear multiplicative noise and the long-time behaviour of their solutions. More precisely, we consider

$$
\begin{aligned}
&idX = \Delta X dt + \lambda |X|^{\alpha-1} X dt - i\mu(t) X dt + i\sum_{k=1}^{N} X G_k(t) d\beta_k(t), \\
&X(0) = X_0.
\end{aligned} \tag{1.1}
$$

Throughout this paper we assume that $\alpha > 1$ and $d \geq 3$. The choice $\lambda = -1$ (resp. $\lambda = 1$) corresponds to the defocusing (resp. focusing) case [44].

The last term is taken in the sense of Itô, β_k are real-valued Brownian motions on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ with normal (in particular right-continuous) filtration $(\mathscr{F}_t)_{t\geq 0}$, $G_k(t)(x) := G_k(t,x) = g_k(t)\phi_k(x)$, g_k are real valued predictable processes, $g_k \in L^2_{loc}(\mathbb{R}^+;\mathbb{R})$ $\mathbb{P}$-a.s., and $\phi_k \in C^\infty(\mathbb{R}^d;\mathbb{C})$. μ is assumed to be of the form $\mu(t,x) = \frac{1}{2}\sum_{k=1}^N |G_k(t,x)|^2$, such that $t \mapsto |X(t)|_2^2$ is a continuous martingale. In a quantum mechanical interpretation, the β_k are the output of the continuous measurement, and the probability measure

$$
\widehat{\mathbb{P}}^T_{X_0}(d\omega) := (\mathbb{E}_{\mathbb{P}}[|X_0|_2^2])^{-1} |X(T,\omega)|_2^2 \, \mathbb{P}(d\omega)
$$

is the physical probability law of the events occurring in $[0,T]$, see [9] and references therein for more information. In particular, when $\operatorname{Re} G_k = 0$, $1 \leq k \leq N$, the mass $|X(t)|_2^2$ is pathwisely conserved, thus the quantum system has a unitary evolution in a random environment. This case also arises from molecular aggregates with thermal fluctuations, we refer to [1, 2] and references therein.

The global well-posedness of (1.1) was first studied in [18, 19] for more general linear multiplicative noise in the conservative case for a restricted range of subcritical exponents. For the full range of mass- and energy-subcritical exponents, the global well-posedness is proved in the recent papers [4, 5] in both the conservative and non-conservative cases, based on the rescaling approach and Strichartz estimates for lower order perturbations of the Laplacian. Very recently, global well-posedness for (1.1) with quite general noise

2010 *Mathematics Subject Classification.* 60H15, 35Q55, 35P25.

Key words and phrases. Stochastic nonlinear Schrödinger equation, scattering, Strichartz estimates, Wiener process.

in the L^2 case is proved in [31] by using the stochastic Strichartz estimates from [10]. In addition, we refer the reader to [22] for results on Schrödinger equations with potential perturbed by a temporal white noise, to [10, 11] for results on compact manifolds, and to [16, 21, 24] for Schrödinger equations with modulated dispersion.

In this paper, we are mainly concerned with the asymptotic behavior of solutions to stochastic nonlinear Schrödinger equations. More precisely, we focus on the scattering property of solutions, which is of physical importance and, roughly speaking, means that solutions behave asymptotically like those to linear Schrödinger equations.

There is an extensive literature on scattering in the deterministic case. Let $\alpha(d)$ denote the Strauss exponent, see (1.10) below. In the defocusing case, scattering was proved in the pseudo-conformal space (i.e., $\{u \in H^1 : |\cdot|u(\cdot) \in L^2\}$) if $\alpha \in [1+\alpha(d), 1+4/(d-2))$, $d \geq 3$. For small initial data, scattering was also obtained for $\alpha \in (1+4/d, 1+4/(d-2))$ in [45, 15]. Moreover, in the energy space H^1, the scattering property of solutions was first proved for the inter-critical case where $\alpha \in (1+4/d, 1+4/(d-2))$ in [28]. In the much more difficult energy-critical case $\alpha = 1+4/(d-2)$, global well-posedness and scattering are established in [17] for $d = 3$, in [41] for $d = 4$, and in [46] for $d \geq 5$. Moreover, global well-posedness and scattering in L^2 are proved in [25] for the mass-critical exponent $\alpha = 1+4/d$, $d \geq 3$. In the focusing case, there exists a threshold for global well-posedness, scattering and blow-up; we refer to [26, 33, 34, 36, 37] and references therein.

In the framework of stochastic mechanics, developed by E. Nelson [38], there are also several works devoted to potential scattering, in terms of diffusions instead of wave functions. See, e.g., [12, 13, 43].

However, to the best of our knowledge, there are few results on the scattering problem for stochastic nonlinear Schrödinger equations (1.1). One interesting question is that, whether the scattering property is preserved under thermal fluctuations? Furthermore, in the regime where the deterministic system fails to scatter, will the input of some large noise have the effect to improve scattering with high probability?

One major challenge here lies in establishing global-in-time Strichartz estimates for (1.1), which actually measure the dispersion and are closely related to those of time dependent and long range perturbations of the Laplacian, see (1.19) below. It is known (see e.g. [30]) that Strichartz estimates may fail for certain perturbations in the deterministic case. Because of rapid fluctuations of Brownian motions at large time, global-in-time Strichartz estimates for (1.1) in the case where $G_k(t,x) = \phi_k(x)$ (i.e. independent of time) are not expected to hold. Moreover, another key difficulty arises from the failure of conservation of several important quantities, such as the Hamiltonian and the pseudo-conformal energy. Due to the complicated formulas of these quantities, involving several stochastic integrations, it is quite difficult to derive a-priori estimates as in the deterministic case (e.g. the decay estimates of $|X(t)|_{L^{\alpha+1}}$ in the pseudo-conformal space).

In the present work, in the case where $\lambda = -1$ (defocusing) and the quadratic variation of the noise is globally bounded, we prove scattering of global solutions to (1.1) in both the pseudo-conformal and the energy space. Precisely, in the pseudo-conformal space, we prove scattering for all $\alpha \in 1+\alpha(d), 1+4/(d-2))$, $d \geq 3$, where $\alpha(d)$ is the Strauss exponent. In the energy space, we prove scattering for $\alpha \in [\max\{2, 1+4/d\}, 1+4/(d-2))$, $3 \leq d \leq 6$. Moreover, in the energy-critical case, assuming the global well-posedness of (1.1) we also obtain scattering in both spaces based on [17, 41, 46]. Thus, in dimensions $d = 3, 4$, we obtain scattering in the energy space for all exponents α of the nonlinearity ranging from the mass-critical exponent $1+4/d$ to the energy-critical exponent $1+4/(d-$

2). We also mention that, the mass-critical case is proved here based on the very recent work [25].

Furthermore, it is known from [14, Theorem 7.5.2] that the deterministic non-trivial solutions do not have any scattering states in the case where $\alpha \in (1, 1+2/d]$. This motivates our further study on the regularization effect of the noise on scattering. Actually, questions about the impact of noise on deterministic systems have attracted significant attention in the field of stochastic partial differential equations, see e.g. [7, 20, 23, 27]. Here, we prove that in the presence of a large non-conservative spatially independent noise, the stochastic solution to (1.1) exists globally and scatters at infinity with high probability for all $\alpha \in (1, 1+4/(d-2))$, both in the pseudo-conformal and the energy space. This includes the range $(1, 1+2/d]$ where scattering fails in the deterministic case.

We begin with recalling the definition of solutions to (1.1).

Definition 1.1. *Fix $T > 0$. An H^1-solution to (1.1) is an H^1-valued continuous $(\mathscr{F}_t)$-adapted process $X = X(t)$, $t \in [0,T]$, such that $|X|^\alpha \in L^1([0,T], H^{-1})$ and it satisfies*

$$\begin{aligned} X(t) =& X_0 - \int_0^t (i\Delta X(s) + \mu X(s) + \lambda i |X(s)|^{\alpha-1} X(s)) ds \\ & + \sum_{k=1}^N \int_0^t X(s) G_k(s) d\beta_k(s), \ \forall t \in [0,T], \end{aligned} \tag{1.2}$$

Here, the integral $\int_0^t X(s)G_k(s)d\beta_k(s)$ is taken in sense of Itô, and (1.2) is understood as an equation in $H^{-1}(\mathbb{R}^d)$.

As in [4, 5], we need the following assumption to assure global well-posedness of (1.1) in the energy-subcritical case discussed below.

(H0) For each $1 \leq k \leq N$, $0 < T < \infty$, $G_k(t,x) = g_k(t)\phi_k(x)$, g_k are real valued predictable processes, $g_k \in L^\infty(\Omega \times [0,T])$ and $\phi_k \in C^\infty(\mathbb{R}^d, \mathbb{C})$ such that for any muti-index γ, $1 \leq |\gamma| \leq 3$,

$$\lim_{|x|\to\infty} |x|^2 |\partial_x^\gamma \phi_k(x)| = 0. \tag{1.3}$$

Theorem 1.2. *Assume (H0). Let $1 < \alpha < 1+4/(d-2)_+$ if $\lambda = -1$, and $1 < \alpha < 1+4/d$ if $\lambda = 1$. For each $X_0 \in H^1$ and $0 < T < \infty$, there exists a unique H^1-solution X to (1.1) such that*

$$X \in L^2(\Omega; C([0,T]; H^1)) \cap L^{\alpha+1}(\Omega; C([0,T]; L^{\alpha+1})), \tag{1.4}$$

and for any Strichartz pair (ρ, γ) (see Section 5),

$$X \in L^\gamma(0,T; W^{1,\rho}), \quad \mathbb{P}-a.s.. \tag{1.5}$$

If $X_0 \in \Sigma := \{u \in H^1 : |\cdot|u(\cdot) \in L^2\}$, then $X \in L^2(\Omega; C([0,T]; \Sigma))$, and for any Strichartz pair (ρ, γ),

$$\| |\cdot| X \|_{L^\gamma(0,T;L^\rho)} < \infty, \quad \mathbb{P}-a.s.. \tag{1.6}$$

Moreover, if in addition $\lambda = -1$ and $g_k \in L^\infty(\Omega; L^2(\mathbb{R}^+))$, $1 \leq k \leq N$, then for any $X_0 \in H^1$ and any $p \geq 1$,

$$\mathbb{E} \sup_{0<t<\infty} (|X(t)|_{H^1}^p + |X(t)|_{L^{\alpha+1}}^p) \leq C(p) < \infty. \tag{1.7}$$

The proof is similar to that in [4, 5] and it is postponed to the Appendix.

In the defocusing energy-critical case, i.e., $\lambda = -1, \alpha = 1+4/(d-2)_+$, the local well-posedness for (1.1) has been proved in [5] for all $d \geq 1$, see also [19] for $d \leq 5$. However, the global well-posedness is much more difficult and remains still open. One of the main difficulties is that several important quantities such as the Hamiltonian are no longer conserved in the stochastic case. In order to consider the scattering in the energy-critical case we a-priori assume that

(H0') In the case $\lambda = -1$, $\alpha = 1 + 4/(d-2)$, for every $T > 0$ and $X_0 \in H^1$ there exists a unique H^1-solution X to (1.1) such that $X \in L^\gamma(0,T;W^{1,\rho})$, $\mathbb{P}$-a.s., for any Strichartz pair (ρ,γ). In addition, if $X_0 \in \Sigma$, then $\| |\cdot| X\|_{L^\gamma(0,T;L^\rho)} < \infty$, $\mathbb{P}$-a.s..

We first study the scattering property of global solutions in the pseudo-conformal space. In this case, the temporal functions g_k in $(H0)$ are assumed to satisfy suitable integrability and to decay to zero with appropriate speed at infinity.

(H1) For each $1 \le k \le N$,

$$\lim_{|x|\to\infty} |x|^3 |\partial_x^\gamma \phi_k(x)| = 0, \quad 1 \le |\gamma| \le 3, \tag{1.8}$$

$\operatorname{esssup}_{\omega\in\Omega} \int_0^\infty (1+s^4) g_k^2(s) ds < \infty$, and for $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$\lim_{t\nearrow 1} (1-t)^{-3} \left(\int_{\frac{t}{1-t}}^\infty g_k^2(\omega,s) ds \ln\ln \left(\int_{\frac{t}{1-t}}^\infty g_k^2(\omega,s) ds \right)^{-1} \right)^{\frac{1}{2}} = 0. \tag{1.9}$$

Let $\alpha(d)$ denote the Strauss exponent, i.e.

$$\alpha(d) = \frac{2-d+\sqrt{d^2+12d+4}}{2d}. \tag{1.10}$$

Theorem 1.3. *Let $X_0 \in \Sigma$, $\lambda = -1$ and $\alpha \in (1+\alpha(d), 1+4/(d-2)]$, $d \ge 3$. Assume $(H0)$, $(H1)$ and if $\alpha = 1+4/(d-2)$ additionally $(H0')$.*

(i). $\mathbb{P}$-a.s. there exists $v_+ \in \Sigma$, such that

$$\lim_{t\to\infty} |e^{it\Delta} e^{-\varphi_*(t)} X(t) - v_+|_\Sigma = 0 \tag{1.11}$$

with the rescaling function

$$\varphi_*(t) = -\sum_{k=1}^N \int_t^\infty G_k(s) d\beta_k(s) + \frac{1}{2} \sum_{k=1}^N \int_t^\infty \left(|G_k(s)|^2 + G_k^2(s) \right) ds. \tag{1.12}$$

(ii). Let $V(t,s)$, $s,t \in [0,\infty)$, be the evolution operators corresponding to the random equation (1.23) below in the homogeneous case where $F \equiv 0$. Then, $\mathbb{P}$-a.s. there exists $X_+ \in \Sigma$ such that

$$\lim_{t\to\infty} |V(0,t) e^{-\varphi_*(t)} X(t) - X_+|_{H^1} = 0. \tag{1.13}$$

Remark 1.4. *In Assumption $(H1)$, the $L^\infty(\Omega)$-integrability of $\int_0^\infty (1+s^4) g_k^2(s) ds$ can be relaxed to the exponential integrability with respect to Ω. See Remark 2.3 below.*

The next result is concerned with the scattering in the energy space.

Theorem 1.5. *Let $X_0 \in H^1$, $\lambda = -1$, $\alpha \in [\max\{2, 1+4/d\}, 1+4/(d-2)]$, $d \ge 3$. Assume $(H0)$, $g_k \in L^2(0,\infty)$, $\mathbb{P}$-a.s., $1 \le k \le N$, and if $\alpha = 1+4/(d-2)$ additionally $(H0')$.*

(i). $\mathbb{P}$-a.s. there exists $v_+ \in H^1$, such that

$$|e^{it\Delta} e^{-\varphi_*(t)} X(t) - v_+|_{H^1} \to 0, \quad \text{as } t \to \infty, \tag{1.14}$$

where φ_* is as in (1.12) above.

(ii). Let $V(t,s)$, $s,t\in[0,\infty)$, be the evolution operators as in Theorem 1.3. Then, $\mathbb{P}$-a.s., there exists $X_+\in H^1$ such that

$$\lim_{t\to\infty}|V(0,t)e^{-\varphi_*(t)}X(t)-X_+|_{H^1}=0. \tag{1.15}$$

Remark 1.6. *One may remove the technical condition $\alpha\geq 2$ in Theorem 1.5 by using delicate arguments as in the H^1-critical stability result in [39]. In order to keep the simplicity of exposition, we will not treat this technical problem in this paper.*

Our next result is concerned with the regularization effect of noise on scattering in the non-conservative case. We assume

(H2) For each $1\leq k\leq N$, $\phi_k\equiv v_k$ are constants, $\inf_{t>0}g_k(t)\geq c_0>0$, and $\operatorname{Re}\phi_j\neq 0$ for some $1\leq j\leq N$. Without loss of generality we may assume that $\operatorname{Re}\phi_1\neq 0$.

Theorem 1.7. *Let $X_0\in\Sigma$ (resp. H^1), $\lambda=\pm1$, $\alpha\in(1,1+\frac{4}{d-2})$, $d\geq 3$. Assume (H0) and (H2). Let g_k and v_j being fixed, $1\leq k\leq N$, $2\leq j\leq N$, and A_{v_1} denote the event that the solution X to (1.1) exists globally and scatters at infinity in Σ (resp. H^1), namely, there exists a unique $u_+\in\Sigma$ (resp. $u_+\in H^1$) such that*

$$\lim_{t\to\infty}|e^{it\Delta}e^{-\varphi(t)}X(t)-u_+|_\Sigma=0\ \ (resp.\ \lim_{t\to\infty}|e^{it\Delta}e^{-\varphi(t)}X(t)-u_+|_{H^1}=0),$$

where

$$\varphi(t,x):=\sum_{k=1}^N\int_0^t G_k(x,s)d\beta_k(s)-\frac{1}{2}\sum_{k=1}^N\int_0^t(|G_k(s,x)|^2+G_k^2(s,x))ds. \tag{1.16}$$

Then, we have

$$\mathbb{P}(A_{v_1})\to 1,\ as\ \operatorname{Re}v_1\to\infty. \tag{1.17}$$

Remark 1.8. *We emphazise that the scaling functions φ_* and φ depend on the strength of the noise (measured by the quadratic variation).*

Remark 1.9. *It is known that for $\alpha\in(1,1+2/d]$, scattering fails for solutions to deterministic nonlinear Schrödinger equations (i.e., $G_k\equiv 0$). Moreover, in the focusing case, it is well-known that solutions may blow-up in the case where $\alpha\in[1+4/d,1+4/(d-2))$. Hence, Theorem 1.7 reveals a regularizing effect of the noise on scattering. See also [7] for a regularizing effect of noise on blow-up in the non-conservative case.*

Remark 1.10. *It is interesting to consider the existence of wave operators and to raise the question whether, given any $v_+\in\Sigma$ (resp. H^1), there exists a unique solution to (1.1) such that the asymptotic behavior in Theorem 1.3 (resp. Theorem 1.5) holds? In the deterministic case, the standard proof is to solve the equation backward in time, that is, to first construct solutions on $[T,\infty)$ for T sufficiently large, and then to extend the solution to all times. However, the situation is quite different in the stochastic case. Even if one can construct solutions path by path by using the deterministic strategy, it is unclear whether the resulting solution is $\{\mathscr{F}_t\}$-adapted.*

Let us outline the proofs of Theorems 1.3-1.7. We rely on the rescaling approach recently developed in [4, 5] and on perturbative arguments. By the transformation

$$z(t,x):=e^{-\varphi(t,x)}X(t,x) \tag{1.18}$$

where φ is as (1.16), the original stochastic equation (1.1) is reduced to the random equation below

$$\partial_t z=-ie^{-\varphi(t)}\Delta(e^{\varphi(t)}z)-\lambda ie^{-\varphi(t)}F(e^{\varphi(t)}z),$$

or equivalently,

$$\partial_t z = A(t)z - \lambda i e^{-\varphi(t)} F(e^{\varphi(t)} z). \tag{1.19}$$

Here, $F(u) = |u|^{\alpha-1}u$ for $u \in \mathbb{C}$, $A(t) := -i(\Delta + b(t)\cdot\nabla + c(t))$, where

$$b(t) = 2\sum_{k=1}^{N}\int_0^t \nabla G_k(s)d\beta_k(s) - 2\int_0^t \nabla\widehat{\mu}(s)ds, \tag{1.20}$$

$$\begin{aligned} c(t) =& \sum_{j=1}^{N}\left(\sum_{k=1}^{N}\int_0^t \partial_j G_k(s)d\beta_k(s) - \int_0^t \partial_j\widehat{\mu}(s)ds\right)^2 \\ &+ \sum_{k=1}^{N}\int_0^t \Delta G_k(s)d\beta_k(s) - \int_0^t \Delta\widehat{\mu}(s)ds, \end{aligned} \tag{1.21}$$

and

$$\widehat{\mu}(t,x) := \frac{1}{2}\sum_{k=1}^{N}(|G_k(t,x)|^2 + G_k(t,x)^2) = \sum_{k=1}^{N}(\mathrm{Re}\, G_k(t,x))G_k(t,x).$$

(Note that, $\widehat{\mu} = 0$, if $\mathrm{Re}\,(G_k) = 0$ for every $1 \le k \le N$. Otherwise, $\mathrm{Re}\,(\widehat{\mu}) > 0$.)

Hence, the problem of scattering for (1.1) is reduced to that for the random equation (1.19).

This point of view proved useful for a sharp pathwise analysis of stochastic solutions and it reveals the structure of the initial stochastic equation as well. Moreover, it is very robust and applicable to several problems. We refer to [3, 6] for the applications of the rescaling approach, combined with the theory of maximal monotone operators, to stochastic partial differential equations. See also [8] for optimal bilinear control problems and [47] for pathwise Strichartz and local smoothing estimates for general stochastic dispersive equations.

The key observation here is that, by (1.20) and (1.21), the global bound on the quadratic variation of the noise implies that on the lower order coefficients, which allows us to obtain the crucial global-in-time Strichartz estimates for the time-dependent operator $A(t)$ in (1.19), see Theorem 5.1 and Corollary 5.3 below. It should be mentioned that, in the case where $G_k(t,x) = \mu_k e_k(x)$, $\mu_k \in \mathbb{C}$, $e_k \in C^\infty(\mathbb{R}^d)$, as studied in [4, 5], we have $\sup_{t\ge0}(|b(t)| + |c(t)|) = \infty$, thus the global-in-time Strichartz estimates in this case are not expected to hold, only local-in-time Strichartz estimates are available, see [4, 5, 47].

Heuristically, if $g_k \in L^2(\mathbb{R}^+)$, the rescaling function φ converges almost surely at infinity, therefore the solution to (1.19) should behave asymptotically like that to the equation

$$\partial_t z = -ie^{-\varphi(\infty)}\Delta(e^{\varphi(\infty)}z) - \lambda i e^{-\varphi(\infty)}F(e^{\varphi(\infty)}z).$$

Thus, $z_* := e^{\varphi(\infty)}z$ satisfies the equation

$$\partial_t z_* = -i\Delta z_* - \lambda i F(z_*),$$

which is actually the deterministic nonlinear Schrödinger equation. We shall mention that, the papers [14, 15, 17, 41, 46] treat the equation $\partial_t u = i\Delta u - \lambda i|u|^{\alpha-1}u$, which can be transformed to the equation above by reversing the time. Thus, similar arguments apply also to this equation. In particular, the solution z_* scatters at infinity for appropriate α.

Rigorously, we will use perturbative arguments to prove scattering properties of the random solution z to (1.19). For this purpose, we consider

$$z_*(t) := e^{\varphi(\infty)}z(t) = e^{-(\varphi(t)-\varphi(\infty))}X(t) = e^{-\varphi_*(t)}X(t), \tag{1.22}$$

where φ and φ_* are as in (1.16) and (1.12), respectively. Then, it follows from (1.19) that

$$\partial_t z_* = A_*(t)z_* - \lambda i e^{-\varphi_*(t)} F(e^{\varphi_*(t)} z_*), \tag{1.23}$$

where $A_*(t) = -i(\Delta + b_*(t)\cdot\nabla + c_*(t))$ with the coefficients

$$b_*(t) = -2\sum_{k=1}^{N}\int_t^\infty \nabla G_k(s)d\beta_k(s) + 2\int_t^\infty \nabla\widehat{\mu}(s)ds, \tag{1.24}$$

$$c_*(t) = \sum_{j=1}^{N}\left(\sum_{k=1}^{N}\int_t^\infty \partial_j G_k(s)d\beta_k(s) - \int_t^\infty \partial_j\widehat{\mu}(s)ds\right)^2$$
$$- \sum_{k=1}^{N}\int_t^\infty \Delta G_k(s)d\beta_k(s) + \int_t^\infty \Delta\widehat{\mu}(s)ds. \tag{1.25}$$

One important fact here is that the coefficients b_*, c_* are asymptotically small at large time, which suffices to yield global-in-time Strichartz estimates based on the work [35], see Section 5 below.

Next, in order to compare the solutions z_* to (1.23) and u to the deterministic nonlinear Schrödinger equation

$$\partial_t u = -i\Delta u - \lambda i F(u), \tag{1.26}$$

with $u(T) = z_*(T)$ at large time T, we set

$$v := z_* - u, \tag{1.27}$$

and obtain

$$\begin{aligned} &\partial_t v = A_*(t)v - i(b_*\cdot\nabla + c_*)u - \lambda i(e^{-\varphi_*(t)}F(e^{\varphi_*(t)}(v+u)) - F(u)),\\ &v(T) = 0. \end{aligned} \tag{1.28}$$

At this step, the proof of Theorems 1.3 and 1.5 is reduced to obtaining asymptotic estimates for the solution v to (1.28).

For the scattering in the pseudo-conformal space a key role is played by an a-priori estimate in the scale-invariant space $L^{\widetilde{q}}(0,\infty;L^{\alpha+1})$ with $\widetilde{q} = \frac{2(\alpha^2-1)}{4-(\alpha-1)(d-2)}$, which is proved in [45, 14] by the decay estimate of the $L^{\alpha+1}$-norm of solutions from the pseudo-conformal conservation law or by inhomogeneous Strichartz estimates for the non-admissible pair $(\alpha+1,\widetilde{q})$ [15, 37, 26]. However, due to the complicated nature of the evolution formula of the pseudo-conformal energy and the unavailability of Strichartz and local smoothing estimates for the non-admissible pair $(\alpha+1,\widetilde{q})$, we proceed differently. We shall compare solutions at the level of the pseudo-conformal transformations. The method of pseudo-conformal transformations has been applied successfully to prove scattering in the pseudo-conformal space, see e.g. [14, 15, 45]. It has advantage that the scattering problem of the original solutions at infinity is reduced to the Cauchy problem of their pseudo-conformal transformations at the singular time 1, which in turn can be analysed by Strichartz estimates without relying on the decay estimates of the $L^{\alpha+1}$-norm of solutions.

Finally, the regularization effect of noise on scattering in Theorem 1.7 can be proved by the rescaling transformation as well. The key point here is that, after the rescaling transformation, one obtains an exponentially decaying term $e^{(\alpha-1)\mathrm{Re}\,\varphi}$ in front of the nonlinearity, which weakens the nonlinear effect and allows random solutions to scatter at infinity even in the case $\alpha\in(1,1+2/d]$, $\lambda=\pm1$, where deterministic solutions fail to scatter. From this perspective, the noise in the non-conservative case has a damping effect to the deterministic system. We also refer the interested reader to [42] for similar phenomena for deterministic damped fractional Schrödinger equations.

The remainder of this paper is organized as follows. Sections 2 and 3 are mainly devoted to the proof of Theorems 1.3 and 1.5. Section 4 is concerned with the proof of Theorem 1.7. Finally, Section 5 contains the global-in-time Strichartz and local smoothing estimates used in this paper. The proofs of some auxiliary results are postponed to the Appendix.

2. Scattering in the pseudo-conformal space

Recall that $\Sigma = \{v \in H^1 : |\cdot| v(\cdot) \in L^2\}$ is the pseudo-conformal space. Let v be the solution to (1.28) and consider its pseudo-conformal transformation

$$\widetilde{v}(t,x) := (1-t)^{-\frac{d}{2}}\, v\left(\frac{t}{1-t}, \frac{x}{1-t}\right)\, e^{i\frac{|x|^2}{4(1-t)}}, \quad t \in [0,1),\ x \in \mathbb{R}^d. \tag{2.1}$$

By straightforward computations, we have

$$\partial_t \widetilde{v} = \widetilde{A}_*(t)\widetilde{v} - i(\widetilde{b}_* \cdot \nabla + \widetilde{c}_*)\widetilde{u} - \lambda i h(t)(e^{-\widetilde{\varphi}_*}F(e^{\widetilde{\varphi}_*}(\widetilde{v}+\widetilde{u})) - F(\widetilde{u})), \tag{2.2}$$

with $\widetilde{v}(\widetilde{T}) = 0$, $\widetilde{T} = T/(1+T) \in (0,1)$. Here, $\widetilde{\varphi}_*(t,x) = \varphi_*(\frac{t}{1-t}, \frac{x}{1-t})$, $h(t) = (1-t)^{\frac{d(\alpha-1)-4}{2}}$, and $\widetilde{A}_*(t) = -i(\Delta + \widetilde{b}_*(t)\cdot\nabla + \widetilde{c}_*(t))$ with

$$\widetilde{b}_*(t,x) =(1-t)^{-1} b_*(\frac{t}{1-t}, \frac{x}{1-t}), \tag{2.3}$$

$$\widetilde{c}_*(t,x) =(1-t)^{-2}\left(-\frac{i}{2} b_*(\frac{t}{1-t}, \frac{x}{1-t})\cdot x + c_*(\frac{t}{1-t}, \frac{x}{1-t})\right), \tag{2.4}$$

where b_*, c_* are as in (1.24) and (1.25) respectively.

Note that, under Assumptions $(H0)$ and $(H1)$, global-in-time Strichartz estimates hold for the operator $\widetilde{A}_*$ on $[0,1)$. See Section 5 below for details.

Similarly, let $\widetilde{z}_*, \widetilde{u}$ denote the pseudo-conformal transformations of z_* and u, respectively. Then, we have

$$\partial_t \widetilde{z}_* = \widetilde{A}_*(t)\widetilde{z}_* - \lambda i h(t) e^{-\widetilde{\varphi}_*}F(e^{\widetilde{\varphi}_*}\widetilde{z}_*), \tag{2.5}$$

and

$$\partial_t \widetilde{u} = -i\Delta\widetilde{u} - \lambda i h(t) F(\widetilde{u}). \tag{2.6}$$

with $\widetilde{u}(\widetilde{T}) = \widetilde{z}_*(\widetilde{T})$. Note that, $\widetilde{u}$ depends on $\widetilde{T}$ and $\widetilde{z}_*(\widetilde{T})$.

For the solution $\widetilde{z}_*$ to (2.5), we first have the estimates on any bounded interval $[0,\widetilde{T}]$ below, $0 < \widetilde{T} < 1$.

Lemma 2.1. *Assume the conditions in Theorem 1.3 to hold. For each $\widetilde{v}(0) \in \Sigma$ and $0 < \widetilde{T} < 1$, there exists a unique H^1-solution $\widetilde{z}_*$ to (2.2) on $[0,\widetilde{T}]$, such that for any Strichartz pair (ρ,γ),*

$$\|\widetilde{z}_*\|_{L^\gamma(0,\widetilde{T};W^{1,\rho})} + \|\widetilde{z}_*\|_{LS(0,\widetilde{T})} + \||\cdot|\widetilde{z}_*\|_{L^\gamma(0,\widetilde{T};L^\rho)} + \|\widetilde{z}_*\|_{C([0,\widetilde{T}];\Sigma)} < \infty, \mathbb{P}-a.s.. \tag{2.7}$$

Proof. By direct computations, for any Strichartz pair (ρ,γ) and $\widetilde{T} \in (0,1)$,

$$\|\widetilde{z}_*\|_{L^\gamma(0,\widetilde{T};L^\rho)} = \|z_*\|_{L^\gamma(0,\frac{\widetilde{T}}{1-\widetilde{T}};L^\rho)}, \tag{2.8}$$

$$\|\nabla\widetilde{z}_*\|_{L^\gamma(0,\widetilde{T};L^\rho)} \le C(1+\frac{\widetilde{T}}{1-\widetilde{T}})\|z_*\|_{L^\gamma(0,\frac{\widetilde{T}}{1-\widetilde{T}};W^{1,\rho})} + \||\cdot|z_*\|_{L^\gamma(0,\frac{\widetilde{T}}{1-\widetilde{T}};L^\rho)}. \tag{2.9}$$

Thus, in view of (1.5), (1.6), Assumption $(H0')$ and $\varphi_* \in L^\infty(0,\infty; W^{1,\infty})$, we obtain that

$$\|\widetilde{z}_*\|_{L^\gamma(0,\widetilde{T};W^{1,\rho})} < \infty, \quad a.s..$$

In particular, $\|\widetilde{z}_*\|_{C([0,\widetilde{T}];H^1)} < \infty$, a.s..

Moreover, we compute that for any $t \in [0, \widetilde{T}]$,

$$\begin{aligned}|x_j\widetilde{z}_*(t)|_2^2 =& \int (1-t)^{-d}|x_j z_*(\frac{t}{1-t}, \frac{x}{1-t})|^2 dx = \int (1-t)^2 |y_j z_*(\frac{t}{1-t}, y)|^2 dy \\ \leq& \|z_*\|^2_{C([0,\frac{\widetilde{T}}{1-\widetilde{T}}];\Sigma)} < \infty,\end{aligned}$$

which implies that

$$\|\widetilde{z}_*\|_{C([0,\widetilde{T}];\Sigma)} < \infty, \quad a.s..$$

Now, for the estimate in the local smoothing space, we take the Strichartz pair $(p,q) = (\frac{d(\alpha+1)}{d+\alpha-1}, \frac{4(\alpha+1)}{(d-2)(\alpha-1)})$. Since $\alpha \in (1, 1+4/(d-2)]$, there exist $1 < l < \infty$, $1 < \theta \leq \infty$ such that $1/q' = 1/\theta + \alpha/q$ and $1/p' = 1/l + 1/p$. Then, taking into account $\|\widetilde{\varphi}_*\|_{L^\infty(0,1;W^{1,\infty})} < \infty$ a.s., applying the Strichartz estimates to (2.5) and using the Hölder inequality we obtain

$$\begin{aligned}\|\widetilde{z}_*\|_{LS(0,\widetilde{T})} \leq& C|\widetilde{z}_*(\widetilde{T})|_2 + C\|he^{-\widetilde{\varphi}_*}F(e^{\widetilde{\varphi}_*}\widetilde{z}_*)\|_{L^{q'}(0,\widetilde{T};L^{p'})} \\ \leq& C|\widetilde{z}_*(\widetilde{T})|_2 + C|h|_{L^\theta(0,\widetilde{T})}\|\widetilde{z}_*\|^{\alpha-1}_{L^q(0,\widetilde{T};L^{(\alpha-1)l})}\|\widetilde{z}_*\|_{L^q(0,\widetilde{T};L^p)}.\end{aligned}$$

Since

$$h \in L^\theta(0,1), \ for \ \alpha \in [1+\alpha(d), 1+4/(d-2)], \tag{2.10}$$

taking into account the Sobolev imbedding $W^{1,p} \hookrightarrow L^{(\alpha-1)l}$ we obtain

$$\|\widetilde{z}_*\|_{LS(0,\widetilde{T})} \leq C|\widetilde{z}_*(\widetilde{T})|_2 + C|h|_{L^\theta(0,\widetilde{T})}\|\widetilde{z}_*\|^\alpha_{L^q(0,\widetilde{T};W^{1,p})} < \infty, \quad a.s.. \tag{2.11}$$

Concerning the weighted norm, for each $1 \leq j \leq d$, $x_j\widetilde{z}_*$ satisfies

$$\partial_t(x_j\widetilde{z}_*) = \widetilde{A}_*(t)(x_j\widetilde{z}_*) + i(2\partial_j\widetilde{z}_* + \widetilde{b}_{*,j}\widetilde{z}_*) - \lambda i h(t)|e^{\widetilde{\varphi}_*}\widetilde{z}_*|^{\alpha-1}(x_j\widetilde{z}_*). \tag{2.12}$$

Take a finite partition $\{t_k\}_{k=0}^M$ of $[0,\widetilde{T}]$ such that $\|z_*\|_{L^q(t_k,t_{k+1};W^{1,p})} \leq \varepsilon$, and let $\langle x\rangle = \sqrt{1+|x|^2}$. Arguing as above and using the fact that

$$\sup_{(t,x)\in[0,1]\times\mathbb{R}^d} \langle x\rangle^2 |\widetilde{b}_*(t,x)| < \infty,$$

(see Section 5 below) we get

$$\begin{aligned}\|x_j\widetilde{z}_*\|_{L^q(t_k,t_{k+1};L^p)} \leq& C\|\widetilde{z}_*\|_{C([0,\widetilde{T}];\Sigma)} + C\|\partial_j\widetilde{z}_* + \widetilde{b}_{*,j}\widetilde{z}_*\|_{L^1(t_k,t_{k+1};L^2)} \\ &+ C\|he^{(\alpha-1)\mathrm{Re}\,\widetilde{\varphi_*}}|\widetilde{z}_*|^{\alpha-1}(x_j\widetilde{z}_*)\|_{L^{q'}(t_k,t_{k+1};L^{p'})} \\ \leq& C\|\widetilde{z}_*\|_{C([0,\widetilde{T}];\Sigma)} + C\varepsilon^{\alpha-1}\|x_j\widetilde{z}_*\|_{L^q(t_k,t_{k+1};L^p)}.\end{aligned} \tag{2.13}$$

Thus, taking ε small enough and then summing over k we obtain $\|x_j\widetilde{z}_*\|_{L^q(0,\widetilde{T};L^p)} < \infty$, a.s.. Hence, the proof is complete. □

The following result, involving the pseudo-conformal energy, is crucial for the scattering behavior.

Lemma 2.2. *Let $\lambda = -1$, $\alpha \in (1, 1+4/(d-2)]$, $d \geq 3$. Assume (H0), (H0)' and (H1). Let X be the solution to (1.1) with $X(0) = X_0 \in \Sigma$. Define the pseudo-conformal energy*

$$E(X(s)) := \int |(y - 2i(1+s)\nabla)X(s,y)|^2 dy + \frac{8}{1+\alpha}(1+s)^2 |X(s)|_{L^{\alpha+1}}^{\alpha+1}, \quad s > 0. \tag{2.14}$$

Then, we have $\mathbb{P}$-a.s.

$$\begin{aligned} E(X(s)) =& E(X_0) + \int_0^s a(r)dr + \frac{16}{\alpha+1}(1 - \frac{d(\alpha-1)}{4}) \int_0^s (1+r)|X(r)|_{L^{\alpha+1}}^{\alpha+1} dr \\ & + \sum_{k=1}^N \int_0^s \sigma_k(r) d\beta_k(r), \quad s > 0, \end{aligned} \tag{2.15}$$

where a, σ_k are continuous $(\mathscr{F}_t)$-adapted processes, satisfying that

$$\int_0^\infty |a(r)|dr + \sup_{0 \leq s < \infty} \left| \sum_{k=1}^N \int_0^s \sigma_k(r) d\beta_k(r) \right| < \infty, \quad a.s.. \tag{2.16}$$

Remark 2.3. *In Assumption (H1), one can relax the $L^\infty(\Omega)$-integrability of $\int_0^\infty (1 + s^4) g_k^2(s) ds$ to the weaker exponential integrability, which is actually sufficient for the almost surely global bound* (2.16) *(see* (6.5) *and* (2.21) *below) and for the pathwise estimates below as well.*

2.1. **Proof of Lemma 2.2.** Let $H(X), V(X), G(X)$ be the Hamiltonian, virial and momentum functions of X as in the proof of Theorem 1.2 in the Appendix below, respectively.

Note that, for the pseudo-conformal energy given by (2.14),

$$E(X(s)) = 8(1+s)^2 H(X(s)) - 4(1+s)G(X(s)) + V(X(s)).$$

Similarly to [7, (4.11)], we have

$$\begin{aligned} G(X(s)) =& G(X_0) + 4\int_0^s H(X(r))dr + \frac{4\lambda}{\alpha+1}(1 - \frac{d(\alpha-1)}{4}) \int_0^s |X(r)|_{L^{\alpha+1}}^{\alpha+1} dr \\ & + \int_0^s a_3(r)dr + \sum_{k=1}^N \int_0^s \sigma_{3,k}(r) d\beta_{3,k}(r) \end{aligned} \tag{2.17}$$

where

$$\begin{aligned} a_3(r) =& -\sum_{k=1}^N \mathrm{Im} \int y \cdot \nabla G_k(r,y) |X(r,y)|^2 \overline{G}_k(r,y) dy, \\ \sigma_{3,k}(r) =& d \int |X(r)|^2 \mathrm{Im}\, G_k(r,y) dy - 2\mathrm{Im} \int y \cdot \nabla X(r,y) \overline{X}(r,y) \overline{G}_k(r,y) dy. \end{aligned}$$

By the Itô formulas of $V(X)$, $H(X)$, $G(X)$ in (6.3), (6.2) and (2.17), respectively,

$$\begin{aligned} E(X(s)) =& E(X_0) + \int_0^s a(r)dr - \frac{16\lambda}{\alpha+1}(1 - \frac{d(\alpha-1)}{4}) \int_0^s (1+r)|X(r)|_{L^{\alpha+1}}^{\alpha+1} dr \\ & + \sum_{k=1}^N \int_0^s \sigma_k(r) d\beta_k(r), \end{aligned} \tag{2.18}$$

where

$$a(r) = 8(1+r)^2 a_1(r) + 4(1+r) \sum_{k=1}^N \mathrm{Im} \int y \nabla G_k(r,y) |X(r,y)|^2 \overline{G}_k(r,y) dy \tag{2.19}$$

with $a_1(r)$ as in (6.2) below and

$$\begin{aligned}\sigma_k(r) =&8(1+r)^2\mathrm{Re}\ \langle\nabla(G_k(r)X(r)),\nabla X(r)\rangle_2-8\lambda(1+r)^2\int \mathrm{Re}\, G_k(r,y)|X(r,y)|^{\alpha+1}dy\\&-4(1+r)d\int |X(r,y)|^2\mathrm{Im}\, G_k(r,y)dy\\&+8(1+r)\mathrm{Im}\int y\cdot\nabla X(r,y)\overline{X}(r,y)\overline{G}_k(r,y)dy+2\int |y|^2|X(r)|^2\mathrm{Re}\, G_k(r,y)dy\end{aligned}\tag{2.20}$$

with $\sigma_{1,k},\sigma_{2,k},\sigma_{3,k}$ as in (6.2), (6.3) below and (2.17), respectively.

Regarding (2.16), we see that, by (2.19), (2.20) and (1.7),

$$\begin{aligned}\mathbb{E}\int_0^\infty |a(s)|+|\sigma_k^2(s)|ds \le& C\mathbb{E}\int_0^\infty (1+\mathcal{H}(X(s)))^2(1+s)^4g_k^2(s)ds\\ \le& C\left(\mathbb{E}\sup_{0\le s<\infty}(1+\mathcal{H}(X(s)))^4\mathbb{E}\left(\int_0^\infty(1+s)^4g_k^2(s)ds\right)^2\right)^{\frac12}<\infty,\end{aligned}\tag{2.21}$$

where $\mathcal{H}(X)=\frac12|X|_{H^1}^2+\frac{1}{\alpha+1}|X|_{L^{\alpha+1}}^{\alpha+1}$. This implies (2.16) and so finishes the proof. □

As a consequence, we have the crucial global bound for the solution $\widetilde{z}_*$ below.

Corollary 2.4. *Let $\lambda=-1$, $\alpha\in(1+\alpha(d),1+\frac{4}{d-2}]$ with $\alpha(d)$ as in (1.10), $d\ge 3$. Assume $(H0)$, $(H0)'$ and $(H1)$. Then,*

$$\|\widetilde{z}_*\|_{L^\gamma(0,1;W^{1,\rho})}+\||\cdot|\widetilde{z}_*\|_{L^\gamma(0,1;L^\rho)}+\|\widetilde{z}_*\|_{LS(0,1)}+\|\partial_j\widetilde{z}_*\|_{LS(0,1)}<\infty,\quad a.s..\tag{2.22}$$

Proof of Corollary 2.4. We consider the cases $\alpha\in[1+4/d,1+4/(d-2)]$ and $\alpha\in(1+\alpha(d),1+4/d)$ separately below.

(i) **The case $\alpha\in[1+4/d,1+4/(d-2)]$.** Let $\widetilde{X}$ be the pseudo-conformal transformation of X. Note that, if $t:=s/(1+s)$, $s\in[0,\infty)$,

$$E(X(s))=\widetilde{E}_1(\widetilde{X}(t)):=4|\nabla\widetilde{X}(t)|_2^2+\frac{8}{1+\alpha}(1-t)^{\frac d2(\alpha-1)-2}|\widetilde{X}(t)|_{L^{\alpha+1}}^{\alpha+1},\quad 0\le t<1.$$

Then, using (2.18) and the fact that $|\widetilde{X}(t)|_{L^{\alpha+1}}^{\alpha+1}=(1-t)^{-d(\alpha-1)/2}|X(s)|_{L^{\alpha+1}}^{\alpha+1}$ we get that,

$$\begin{aligned}\widetilde{E}_1(\widetilde{X}(t)) =&\widetilde{E}_1(\widetilde{X}_0)+\int_0^s a(r)dr+\sum_{k=1}^N\int_0^s\sigma_k(r)d\beta_k(r)\\&+\frac{16}{\alpha+1}(1-\frac d4(\alpha-1))\int_0^t(1-r)^{\frac d2(\alpha-1)-3}|\widetilde{X}(r)|_{L^{\alpha+1}}^{\alpha+1}dr\\ \le& C|X_0|_\Sigma^2+\sup_{0\le s<\infty}\left(\left|\int_0^s a(r)dr\right|+\sum_{k=1}^N\left|\int_0^s\sigma_k(r)d\beta_k(r)\right|\right)<\infty,\quad a.s.,\end{aligned}\tag{2.23}$$

where the last step is due to (2.16). Since $|\nabla\widetilde{X}(t)|_2^2\le\frac14\widetilde{E}_1(\widetilde{X}(t))$ and by (1.7), $\sup_{t\in[0,1)}|\widetilde{X}(t)|_2^2=\sup_{s\in[0,\infty)}|X(s)|_2^2<\infty$, a.s., we obtain that $\sup_{t\in[0,1)}|\widetilde{X}(t)|_{H^1}<\infty$, a.s..

Thus, taking into account that $\widetilde{z}_*=e^{\widetilde{\varphi}_*}\widetilde{X}$ and $\widetilde{\varphi}_*\in L^\infty(0,1;W^{1,\infty})$, we get

$$\sup_{t\in[0,1)}|\widetilde{z}_*(t)|_{H^1}<\infty,\quad a.s..\tag{2.24}$$

We claim that the estimate (2.24) is sufficient to yield that for $\widetilde{T}$ close to 1,

$$\|\widetilde{z}_*\|_{L^\gamma(\widetilde{T},1;W^{1,\rho})}+\||\cdot|\widetilde{z}_*\|_{L^\gamma(\widetilde{T},1;L^\rho)}\le C(|\widetilde{z}_*(\widetilde{T})|_\Sigma+\|\widetilde{z}_*\|_{C([0,1);H^1)})<\infty.\tag{2.25}$$

To this end, we choose the Strichartz pair (p,q), $1<\theta\le\infty$ and h as in the proof of Lemma 2.1. Similarly to (2.11), we have that for any $t\in(\widetilde{T},1)$,

$$\begin{aligned}\|\widetilde{z}_*\|_{L^q(\widetilde{T},t;W^{1,p})}\le& C|\widetilde{z}_*(\widetilde{T})|_{H^1}+C|h|_{L^\theta(\widetilde{T},t)}\|\widetilde{z}_*\|^\alpha_{L^q(\widetilde{T},t;W^{1,p})}\\ \le& C\|\widetilde{z}_*\|_{C([0,1);H^1)}+C\varepsilon(\widetilde{T})\|\widetilde{z}_*\|^\alpha_{L^q(\widetilde{T},t;W^{1,p})},\end{aligned}$$

where in the last step we also used that $\varepsilon(\widetilde{T}):=|h|_{L^\theta(\widetilde{T},1)}\to 0$ as $\widetilde{T}\to 1^-$ (see also (2.36) below), due to (2.36). Taking $\widetilde{T}$ close to 1, using [8, Lemma A.1] and then letting $t\to 1^-$ we obtain

$$\|\widetilde{z}_*\|_{L^q(\widetilde{T},1;W^{1,p})}\le C\|\widetilde{z}_*\|_{C([0,1);H^1)},\quad a.s.. \tag{2.26}$$

Moreover, similarly to (2.13), for each $1\le j\le d$,

$$\|x_j\widetilde{z}_*\|_{L^q(\widetilde{T},t;L^p)}\le C|\widetilde{z}_*(\widetilde{T})|_\Sigma+C\|\widetilde{z}_*\|_{C([0,1);H^1)}+C\varepsilon(\widetilde{T})\|x_j\widetilde{z}_*\|_{L^q(\widetilde{T},t;L^p)}.$$

Then, taking $\widetilde{T}$ close to 1 such that $C\varepsilon(\widetilde{T})\le 1/2$ and letting $t\to 1^-$ we obtain

$$\|x_j\widetilde{z}_*\|_{L^q(\widetilde{T},1;L^p)}\le C(|\widetilde{z}_*(\widetilde{T})|_\Sigma+\|\widetilde{z}_*\|_{C([0,1);H^1)}). \tag{2.27}$$

Thus, combining (2.26) and (2.27) together and using Strichartz estimates we prove (2.25), as claimed. Now, taking into account Lemma 2.1 and using Strichartz estimates to control the $LS(0,1)$-norms we obtain (2.22) in the case where $\alpha\in[1+4/d,1+4/(d-2)]$.

The case $\alpha\in(1+\alpha(d),1+4/d)$**.** In this case, let

$$\widetilde{E}_2(\widetilde{X}(t)):=(1-t)^{2-\frac{d}{2}(\alpha-1)}\widetilde{E}_1(\widetilde{X}(t))=4(1-t)^{2-\frac{d}{2}(\alpha-1)}|\nabla\widetilde{X}(t)|_2^2+\frac{8}{1+\alpha}|\widetilde{X}(t)|^{\alpha+1}_{L^{\alpha+1}}.$$

Note that, by (2.23),

$$\begin{aligned}\widetilde{E}_2(X(t))=&\widetilde{E}_2(X_0)+\int_0^s(1+r)^{\frac{d}{2}(\alpha-1)-2}a(r)dr+\sum_{k=1}^N\int_0^s(1+r)^{\frac{d}{2}(\alpha-1)-2}\sigma_k(r)d\beta_k(r)\\ &-8(1-\frac{d}{4}(\alpha-1))\int_0^t(1-r)^{1-\frac{d}{2}(\alpha-1)}|\nabla\widetilde{X}(r)|_2^2dr\\ \le& C|X_0|_\Sigma^2+\sup_{0\le s<\infty}\left(\left|\int_0^s a(r)dr\right|+\sum_{k=1}^N\left|\int_0^s\sigma_k(r)d\beta_k(r)\right|\right)<\infty,\quad a.s..\end{aligned} \tag{2.28}$$

This yields that $\sup_{0\le t<1}|\widetilde{X}(t)|^{\alpha+1}_{L^{\alpha+1}}<\infty$, and so

$$\sup_{0\le t<1}|\widetilde{z}_*(t)|^{\alpha+1}_{L^{\alpha+1}}<\infty,\quad a.s.. \tag{2.29}$$

As in the previous case, we claim that the estimate (2.25) also holds in the case where $\alpha\in(1+\alpha(d),1+4/d)$.

To this end, we choose the Strichartz pair $(p,q)=(\alpha+1,\frac{4(\alpha+1)}{d(\alpha-1)})$ and set $\widetilde{q}=\frac{2(\alpha^2-1)}{4-(\alpha-1)(d-2)}$. Note that, since $\alpha>1+\alpha(d)$, $h^{\frac{1}{\alpha-1}}\in L^{\widetilde{q}}(0,1)$, and so $\varepsilon'(\widetilde{T}):=|h^{\frac{1}{\alpha-1}}|^{\alpha-1}_{L^{\widetilde{q}}(\widetilde{T},1;L^p)}\to 0$ as $\widetilde{T}\to 1^-$. Applying Strichartz estimates to (1.23) and using Hölder's inequality and (2.29) we have for any $t\in(\widetilde{T},1)$

$$\begin{aligned}\|\widetilde{z}_*\|_{L^q(\widetilde{T},t;W^{1,p})}\le& C|\widetilde{z}_*(\widetilde{T})|_{H^1}+C\|h^{\frac{1}{\alpha-1}}z_*\|^{\alpha-1}_{L^{\widetilde{q}}(\widetilde{T},t;L^p)}\|\widetilde{z}_*\|_{L^q(\widetilde{T},t;W^{1,p})}\\ \le& C|\widetilde{z}_*(\widetilde{T})|_{H^1}+C\varepsilon'(\widetilde{T})\|\widetilde{z}_*\|_{L^q(\widetilde{T},t;W^{1,p})}.\end{aligned}$$

Then, taking $\widetilde{T}$ close to 1 such that $\varepsilon'(\widetilde{T}) = 1/(2C)$ and then letting $t \to 1^-$, we obtain

$$\|\widetilde{z}_*\|_{L^q(\widetilde{T},1;W^{1,p})} \leq C|\widetilde{z}_*(\widetilde{T})|_{H^1}. \tag{2.30}$$

This, via Strichartz estimates, yields that $\|\widetilde{z}_*\|_{C([\widetilde{T},1);H^1)} \leq C|\widetilde{z}_*(\widetilde{T})|_{H^1} < \infty$, a.s..

Similarly, for the estimate in the weighted space, we get that for each $1 \leq j \leq d$, similarly to (2.13), for any $t \in (\widetilde{T}, 1)$,

$$\begin{aligned}\|x_j\widetilde{z}_*\|_{L^q(\widetilde{T},t;L^p)} \leq & C|\widetilde{z}_*(\widetilde{T})|_\Sigma + C\|\widetilde{z}_*\|_{C([\widetilde{T},1);H^1)} + C\|h^{\frac{1}{\alpha-1}}z_*\|^{\alpha-1}_{L^{\widetilde{q}}(\widetilde{T},t;L^p)}\|x_j\widetilde{z}_*\|_{L^q(\widetilde{T},t;L^p)} \\ \leq & C|\widetilde{z}_*(\widetilde{T})|_\Sigma + C\|\widetilde{z}_*\|_{C([\widetilde{T},1);H^1)} + C\varepsilon'(T)\|x_j\widetilde{z}_*\|_{L^q(\widetilde{T},t;L^p)}.\end{aligned}$$

Thus, similar arguments as above yield that

$$\|x_j\widetilde{z}_*\|_{L^q(\widetilde{T},1;L^p)} \leq C\|\widetilde{z}_*(\widetilde{T})\|_\Sigma + C\|\widetilde{z}_*\|_{C([\widetilde{T},1);H^1)} < \infty, \ a.s.. \tag{2.31}$$

Now, we can use (2.30), (2.31) and Strichartz estimates to obtain (2.25) in the case where $\alpha \in (1+\alpha(d), 1+4/d)$, as claimed. Then, similar arguments as those below (2.27) yields (2.22).

Therefore, the proof is complete. □

It should be mentioned that, on the basis of Lemma 2.2, one can prove the scattering of X by using the Galilean operator and decay estimates as in [14, 39]. Moreover, exploring the global bound (2.22) and the equivalence of asymptotics of $\widetilde{z}_*$ at time 1 and z_* at infinity (see e.g. [14, Proposition 7.5.1]), one can also obtain the scattering (1.11). Precisely, almost surely there exists $v_+ \in \Sigma$ such that

$$\lim_{t\to\infty} |e^{it\Delta}z_*(t) - v_+|_\Sigma = \lim_{t\to\infty} |e^{it\Delta}e^{-\varphi_*(t)}X(t) - v_+|_\Sigma = 0.$$

We remark that, though Proposition 7.5.1 in [14] treats the equation $\partial_t u = i\Delta u - \lambda i|u|^{\alpha-1}u$, similar arguments apply also to the solutions z_* to (1.23) and $\widetilde{z}_*$ to (2.5) considered here. In fact, define the dilation D_β by $D_\beta z_*(x) = \beta^{\frac{d}{2}}z_*(\beta x)$, $\beta > 0$, the multiplication M_σ by $M_\sigma z_*(x) = e^{i\frac{\sigma|x|^2}{4}}z_*(x)$, $\sigma \in \mathbb{R}$, and let $\mathcal{T}(t) := e^{it\Delta}$, $t \in \mathbb{R}$. We have $\widetilde{z}_*(t) = M_{\frac{1}{1-t}}D(\frac{1}{1-t})z_*(\frac{t}{1-t})$, $t \in [0,1)$. Since $\mathcal{T}(t)D_\beta = D_\beta\mathcal{T}(\beta^2 t)$, $\mathcal{T}(t)M_\sigma = M_{\frac{\sigma}{1+\sigma t}}D_{\frac{1}{1+\sigma t}}\mathcal{T}(\frac{t}{1+\sigma t})$, we obtain $\mathcal{T}(t)\widetilde{z}_*(t) = M_1\mathcal{T}(\frac{t}{1-t})z_*(\frac{t}{1-t})$. It follows that $\mathcal{T}(s)z_*(s) = M_{-1}\mathcal{T}(\frac{s}{1+s})\widetilde{z}_*(\frac{s}{1+s})$, $s \in [0,\infty)$, which implies the equivalence of asymptotics between $\widetilde{z}_*$ and $e^{it\Delta}z_*$.

However, in the H^1 case, one can not obtain the scattering behavior of z_* directly from the uniform bound similar to (2.22). Thus, we present a different proof for scattering to illustrate the idea of comparison, which applies also to the H^1 case. Moreover, it also gives the asymptotical estimates of the solutions $\widetilde{z}_*$ to (2.5) and $\widetilde{u}$ to (2.6) (see (2.34) below) and so justifies the intuition mentioned in Section 1.

Proposition 2.5 below summarizes uniform estimates (independent of $\widetilde{T}$) for $\widetilde{u}$ used in this section.

Proposition 2.5. *Let $\lambda = -1$, $\alpha \in (1+\alpha(d), 1+\frac{4}{d-2}]$ with $\alpha(d)$ as in (1.10), $d \geq 3$. Then, for each $\widetilde{z}_*(\widetilde{T}) \in \Sigma$, there exists a unique H^1-solution $\widetilde{u}$ (depending on $\widetilde{T}$) to (2.6) on $[0,1]$ with $\widetilde{u}(\widetilde{T}) = \widetilde{z}_*(\widetilde{T})$, such that $\mathbb{P}$-a.s. $\widetilde{u} \in C([0,1];\Sigma)$, and for any Strichartz pair (γ,ρ),*

$$\|\widetilde{u}\|_{L^\gamma(\widetilde{T},1;W^{1,\rho})} + \||\cdot|\widetilde{u}\|_{L^\gamma(\widetilde{T},1;L^\rho)} + \|\widetilde{u}\|_{LS(\widetilde{T},1)} + \|\partial_j\widetilde{u}\|_{LS(\widetilde{T},1)} \leq C < \infty, \tag{2.32}$$

where $1 \le j \le d$, $LS(0,1)$ is the local smoothing space defined in Section 5, and C is independent of $\widetilde{T}$.

The proof is similar to that of Corollary 2.4, based on the pseudo-conformal energy and the global bound (2.22) of $\widetilde{z}_*(\widetilde{T})$. For simplicity, it is postponed to the Appendix.

The next lemma contains the crucial global estimates and asymptotics of the solution $\widetilde{v}$ to equation (2.2).

Lemma 2.6. *Assume the conditions in Theorem 1.3 to hold. Let $\widetilde{v}$ be the solution to (2.2) with $\widetilde{v}(\widetilde{T}) = 0$. Then, for any Strichartz pair (ρ, γ),*

$$\|\widetilde{v}\|_{L^\gamma(\widetilde{T},1;W^{1,\rho})} + \|\widetilde{v}\|_{LS(\widetilde{T},1)} + \||\cdot|\widetilde{v}\|_{L^\gamma(\widetilde{T},1;L^\rho)} + \|\widetilde{v}\|_{C([\widetilde{T},1);\Sigma)} \le C < \infty, \quad a.s., \tag{2.33}$$

where C is independent of $\widetilde{T}$. Moreover, $\mathbb{P}$-a.s., as $\widetilde{T} \to 1^-$,

$$\|\widetilde{v}\|_{L^\gamma(\widetilde{T},1;W^{1,\rho})} + \|\widetilde{v}\|_{LS(\widetilde{T},1)} + \||\cdot|\widetilde{v}\|_{L^\gamma(\widetilde{T},1;L^\rho)} + \|\widetilde{v}\|_{C([\widetilde{T},1);\Sigma)} \to 0. \tag{2.34}$$

Proof. The uniform bound (2.33) follows from (2.22) and (2.32), since $\widetilde{v} = \widetilde{z}_* - \widetilde{u}$.

Regarding (2.34), the proof is based on perturbative arguments. Let p, q, θ, l be as in the proof of Lemma 2.1. First, we consider the estimates in the spaces $L^\gamma(0,1;W^{1,\rho})$ and $LS(0,1)$, where (ρ,γ) is any Strichartz pair. Similarly to (2.11), since $\widetilde{v}(\widetilde{T}) = 0$, by (2.2) we have $\mathbb{P}$-a.s. for any $t \in (\widetilde{T}, 1)$,

$$\begin{aligned} &\|\widetilde{v}\|_{L^\gamma(\widetilde{T},t;L^\rho)} + \|\widetilde{v}\|_{C([\widetilde{T},t];L^2)} + \|\widetilde{v}\|_{LS(\widetilde{T},t)} \\ \le& C\|(\widetilde{b}_* \cdot \nabla + \widetilde{c}_*)\widetilde{u}\|_{LS'(\widetilde{T},t)} + C|h|_{L^\theta(\widetilde{T},t)}(\|\widetilde{u}\|^\alpha_{L^q(\widetilde{T},t;W^{1,p})} + \|\widetilde{v}\|^\alpha_{L^q(\widetilde{T},t;W^{1,p})}) \end{aligned} \tag{2.35}$$

where C is independent of $\widetilde{T}$ and t, $LS'(\widetilde{T},t)$ is the dual space of $LS(\widetilde{T},t)$.

Taking into account (2.10) and that $|h|_{L^\infty(\widetilde{T},t)} \le (1-\widetilde{T})^{4/(d-2)}$ if $\alpha = 1 + 4/(d-2)$, we have that

$$\varepsilon_1(\widetilde{T}) := |h|_{L^\theta(\widetilde{T},1)} \to 0 \quad as\ \widetilde{T} \to 1^-. \tag{2.36}$$

Moreover, by Assumption $(H1)$, for any $0 \le |\beta| \le 2$, $0 \le |\gamma| \le 1$, $\partial_x^\beta \widetilde{b}_*$ and $\partial_x^\gamma \widetilde{c}_*$ satisfy

$$\lim_{t\to 1} \sup_{\mathbb{R}^d} \langle x\rangle^2 \left(|\partial_x^\beta \widetilde{b}_*(t,x)| + \partial_x^\gamma \widetilde{c}_*(t,x)|\right) = 0, \quad a.s., \tag{2.37}$$

$$\lim_{|x|\to\infty} \sup_{t\in[0,1)} \langle x\rangle^2 \left(|\partial_x^\beta \widetilde{b}_*(t,x)| + \partial_x^\gamma \widetilde{c}_*(t,x)|\right) = 0. \quad a.s.. \tag{2.38}$$

(See Section 5 below.) Then, using (5.9) below (see also Remark 5.2) and the uniform bound (2.32) we get $\mathbb{P}$-a.s.

$$\|(\widetilde{b}_* \cdot \nabla + \widetilde{c}_*)\widetilde{u}\|_{LS'(\widetilde{T},t)} \le \varepsilon_2(\widetilde{T})\|\widetilde{u}\|_{LS(\widetilde{T},t)}, \tag{2.39}$$

where $\varepsilon_2(\widetilde{T}) \to 0$ as $\widetilde{T} \to 1^-$.

Thus, setting $\varepsilon(\widetilde{T}) := \varepsilon_1(\widetilde{T}) \vee \varepsilon_2(\widetilde{T})$ and using (2.35), (2.36) and (2.39) we get

$$\begin{aligned} &\|\widetilde{v}\|_{L^\gamma(\widetilde{T},t;L^\rho)} + \|\widetilde{v}\|_{C([\widetilde{T},t];L^2)} + \|\widetilde{v}\|_{LS(\widetilde{T},t)} \\ \le& C\varepsilon(\widetilde{T})(\|\widetilde{u}\|_{LS(\widetilde{T},t)} + \|\widetilde{u}\|^\alpha_{L^q(\widetilde{T},t;W^{1,p})} + \|\widetilde{v}\|^\alpha_{L^q(\widetilde{T},t;W^{1,p})}) \\ \le& C\varepsilon(\widetilde{T}), \end{aligned} \tag{2.40}$$

where C is independent of $\widetilde{T}$ and t, due to the uniform bounds (2.32) and (2.33).

Furthermore, for every $1 \le j \le d$, by (2.2), $\partial_j \widetilde{v}$ satisfies the equation

$$\partial_t(\partial_j \widetilde{v}) = \widetilde{A}_*(t)(\partial_j \widetilde{v}) + G(\widetilde{v}, \partial_j \widetilde{u}, \widetilde{u})$$

where $\partial_j \widetilde{v}(\widetilde{T}) = 0$, and

$$G(\widetilde{v}, \partial_j \widetilde{u}, \widetilde{u}) = - i(\widetilde{b}_* \nabla + \widetilde{c}_*)(\partial_j \widetilde{u}) - i(\partial_j \widetilde{b}_* \cdot \nabla + \partial_j \widetilde{c}_*)(\widetilde{v} + \widetilde{u}) \\ - \lambda i h(t)(\partial_j (e^{-\widetilde{\varphi}_*} F(e^{\widetilde{\varphi}_*}(\widetilde{v} + \widetilde{u}))) - \partial_j F(\widetilde{u})).$$

Then, similar to (2.35), we have $\mathbb{P}$-a.s.

$$\begin{aligned} &\|\partial_j \widetilde{v}\|_{L^\gamma(\widetilde{T},t;L^\rho)} + \|\partial_j \widetilde{v}\|_{C([\widetilde{T},t];L^2)} \\ \le & C(\|(\widetilde{b}_* \cdot \nabla + \widetilde{c}_*)\partial_j \widetilde{u} + (\partial_j \widetilde{b}_* \cdot \nabla + \partial_j \widetilde{c}_*)(\widetilde{v} + \widetilde{u})\|_{LS'(\widetilde{T},t)}) \\ & + C|h|_{L^\theta(\widetilde{T},t)}(\|\widetilde{v}\|^\alpha_{L^q(\widetilde{T},t;W^{1,p})} + \|\widetilde{u}\|^\alpha_{L^q(\widetilde{T},t;W^{1,p})}), \end{aligned}$$

where C is independent of $\widetilde{T}$ and t, due to the global-in-time Strichartz estimates. In view of (2.37), (2.38) and (5.9) below, the first term on the right-hand side above is bounded by

$$C\varepsilon(\widetilde{T})(\|\partial_j \widetilde{u}\|_{LS(\widetilde{T},t)} + \|\widetilde{u}\|_{LS(\widetilde{T},t)} + \|\widetilde{v}\|_{LS(\widetilde{T},t)}),$$

which along with (2.36) and the uniform bounds (2.32) and (2.33) implies that $\mathbb{P}$-a.s.

$$\|\partial_j \widetilde{v}\|_{L^\gamma(\widetilde{T},t;L^\rho)} + \|\partial_j \widetilde{v}\|_{C([\widetilde{T},t];L^2)} \le C\varepsilon(\widetilde{T}). \tag{2.41}$$

Thus, it follows from (2.40) and (2.41) that

$$\|\widetilde{v}\|_{L^\gamma(\widetilde{T},t;W^{1,\rho})} + \|\widetilde{v}\|_{C([\widetilde{T},t];H^1)} + \|\widetilde{v}\|_{LS(\widetilde{T},t)} \le C\varepsilon(\widetilde{T}) \tag{2.42}$$

where C is independent of $\widetilde{T}$ and t. Letting $t \to 1^-$ we get

$$\|\widetilde{v}\|_{L^\gamma(\widetilde{T},1;W^{1,\rho})} + \|\widetilde{v}\|_{C([\widetilde{T},1);H^1)} + \|\widetilde{v}\|_{LS(\widetilde{T},1)} \le C\varepsilon(\widetilde{T}) \to 0, \quad as\ \widetilde{T} \to 1^-,\ a.s.. \tag{2.43}$$

Next, regarding the estimate of $\||\cdot|\widetilde{v}\|_{L^\gamma(0,1;L^\rho)}$, similar to (2.12),

$$\partial_t (x_j \widetilde{v}) = \widetilde{A}_*(t)(x_j \widetilde{v}) + H(\widetilde{v}, \partial_j \widetilde{v}, \widetilde{u}),$$

where $x_j \widetilde{v}(\widetilde{T}) = 0$ and

$$H(\widetilde{v}, \partial_j \widetilde{v}, \widetilde{u}) = i(2\partial_j \widetilde{v} + \widetilde{b}_{*,j}\widetilde{v}) - i x_j (\widetilde{b}_* \cdot \nabla + \widetilde{c}_*)\widetilde{u} \\ - \lambda i h(t)(|e^{\widetilde{\varphi}_*}(\widetilde{v} + \widetilde{u})|^{\alpha-1} x_j (\widetilde{v} + \widetilde{u}) - |\widetilde{u}|^{\alpha-1}(x_j \widetilde{u})).$$

Then, similar to (2.13),

$$\begin{aligned} &\|x_j \widetilde{v}\|_{L^\gamma(\widetilde{T},t;L^\rho)} + \|x_j \widetilde{v}\|_{C([\widetilde{T},t];L^2)} \\ \le & C\Big[\|2\partial_j \widetilde{v} + \widetilde{b}_{*,j}\widetilde{v}\|_{L^1(\widetilde{T},t;L^2)} + \|x_j(\widetilde{b}_* \cdot \nabla + \widetilde{c}_*)\widetilde{u}\|_{L^1(\widetilde{T},t;L^2)} \\ & + \widetilde{\varepsilon}_1(\widetilde{T})\|\widetilde{v} + \widetilde{u}\|^{\alpha-1}_{L^q(\widetilde{T},t;W^{1,p})}(\|x_j \widetilde{u}\|_{L^q(\widetilde{T},t;L^p)} + \|x_j \widetilde{v}\|_{L^q(\widetilde{T},t;L^p)}) \\ & + \widetilde{\varepsilon}_1(\widetilde{T})\|\widetilde{u}\|^{\alpha-1}_{L^q(\widetilde{T},t;W^{1,p})}\|x_j \widetilde{u}\|_{L^q(\widetilde{T},t;L^p)}\Big], \quad \mathbb{P}-a.s., \end{aligned} \tag{2.44}$$

where C is independent of $\widetilde{T}$ and t.

Thus, by virtue of the uniform bounds (2.32) and (2.33) we obtain

$$\|x_j \widetilde{v}\|_{L^\gamma(\widetilde{T},t;L^\rho)} + \|x_j \widetilde{v}\|_{C([\widetilde{T},t];L^2)} \le C(1 - \widetilde{T}) + C\widetilde{\varepsilon}_1(\widetilde{T}) \to 0, \quad as\ \widetilde{T} \to 1^-,\ a.s.. \tag{2.45}$$

Therefore, combining (2.43) and (2.45) we obtain (2.34). The proof is finished. □

Proof of Theorem 1.3 (i). For any t_1, t_2 close to 1, we have

$$|\widetilde{z}_*(t_1) - \widetilde{z}_*(t_2)|_\Sigma \le |\widetilde{z}_*(t_1) - \widetilde{u}(t_1)|_\Sigma + |\widetilde{z}_*(t_2) - \widetilde{u}(t_2)|_\Sigma + |\widetilde{u}(t_1) - \widetilde{u}(t_2)|_\Sigma,$$

where $\widetilde{u}$ is the solution to (2.6) with $\widetilde{u}(\widetilde{T}) = \widetilde{z}_*(\widetilde{T})$, $\widetilde{T} \in (0,1)$.

Moreover, since by Proposition 2.5, $\widetilde{u}$ exists on $[0,1]$ and is continuous at time 1, we have $\mathbb{P}$-a.s.,

$$\lim_{t_1,t_2\to 1^-} |\widetilde{u}(t_1) - \widetilde{u}(t_2)|_\Sigma = 0.$$

This implies that

$$\begin{aligned}\limsup_{t_1,t_2\to\infty} |\widetilde{z}_*(t_1) - \widetilde{z}_*(t_2)|_\Sigma \leq &\limsup_{t_1\to\infty} |\widetilde{z}_*(t_1) - \widetilde{u}(t_1)|_\Sigma + \limsup_{t_2\to\infty} |\widetilde{z}_*(t_2) - \widetilde{u}(t_2)|_\Sigma \\ \leq &2\|\widetilde{z}_* - \widetilde{u}\|_{C([\widetilde{T},1);\Sigma)}.\end{aligned}$$

Then, in view of (2.34), letting $\widetilde{T} \to 1^-$ we obtain

$$\lim_{t_1,t_2\to\infty} |\widetilde{z}_*(t_1) - \widetilde{z}_*(t_2)|_\Sigma = 0.$$

This yields that $\widetilde{z}_*$ has the limit at time 1. Thus, using the equivalence of asymptotics of $\widetilde{z}_*$ at time 1 and z_* at infinity we conclude that z_* scatters at infinity in the speudo-conformal space, i.e., (1.11) holds.

Therefore, the assertion of Theorem 1.3 (i) is proved. □

Before proving Theorem 1.3 (ii), we obtain global estimates for the original solution z_* to (1.23) from those of its pseudo-conformal transformation $\widetilde{z}_*$.

Corollary 2.7. *Let $\alpha \in (1+\alpha(d), 1+\frac{4}{d-2}]$, $d \geq 3$, and z_* be the solution to* (1.23)*. Then, for any Strichartz pair (ρ, γ),*

$$\|z_*\|_{L^\gamma(0,\infty;W^{1,\rho})} < \infty, \quad a.s.. \tag{2.46}$$

Moreover, if $\widetilde{q} := \frac{2(\alpha^2-1)}{4-(\alpha-1)(d-2)}$, we have

$$\|z_*\|_{L^{\widetilde{q}}(0,\infty;L^{\alpha+1})} \leq C < \infty, \quad a.s., \tag{2.47}$$

where C is independent of $\widetilde{T}$.

Proof. We shall prove the estimates of z_* in (2.46) and (2.47) from those of the pseudo-conformal transformation $\widetilde{z}_*$ in (2.22).

For this purpose, since $(d/2 - d/\rho)\gamma - 2 = 0$, straightforward computations show that

$$\|z_*\|^\gamma_{L^\gamma(0,\infty;L^\rho)} = \int_0^1 (1-t)^{\frac{d(\rho-2)}{2\rho}\gamma-2} |\widetilde{z}_*(t)|^\gamma_{L^\rho} dt = \|\widetilde{z}_*\|^\gamma_{L^\gamma(0,1;L^\rho)} < \infty, \quad a.s., \tag{2.48}$$

where the last step is due to Corollary 2.4. Moreover, we compute that, if $s := t/(1-t)$,

$$|\nabla z_*(s)|_{L^\rho} = (1-t)^{\frac{d}{2}+1-\frac{d}{\rho}} |\nabla \widetilde{z}_*(t) - \frac{i}{2}\frac{x}{1-t}\widetilde{z}_*(t)|_{L^\rho}.$$

Since $(d/2+1)\gamma - d\gamma/\rho - 2 = \gamma$ and $d\gamma/2 - d\gamma/\rho - 2 = 0$, by Lemma 2.6, this implies that

$$\begin{aligned}\|\nabla z_*\|_{L^\gamma(0,\infty;L^\rho)} \leq &C_\gamma\bigg(\int_0^1 (1-t)^{(\frac{d}{2}+1)\gamma-\frac{d}{\rho}\gamma-2} |\nabla\widetilde{z}_*(t)|^\gamma_{L^\rho} dt \\ &+ \int_0^1 (1-t)^{\frac{d}{2}\gamma-\frac{d}{\rho}\gamma-2} ||\cdot|\widetilde{z}_*(t)|^\gamma_{L^\rho} dt\bigg)^{\frac{1}{\gamma}} \\ \leq &C_\gamma(\|\nabla\widetilde{z}_*\|_{L^\gamma(0,1;L^\rho)} + \||\cdot|\widetilde{z}_*\|_{L^\gamma(0,1;L^\rho)}) < \infty, \quad a.s..\end{aligned} \tag{2.49}$$

Thus, taking together (2.48) and (2.49) yields (2.46).

Regarding (2.47), the argument is similar to that of [15, Proposition 3.15(iv)]. Using Lemma 2.6 and the Sobolev imbedding theorem we have

$$\sup_{t\in[0,1]} |\widetilde{z}_*(t)|_{\alpha+1} \leq C\|\widetilde{z}_*\|_{C([0,1];H^1)} < \infty, \quad a.s..$$

Then, direct calculations imply that $\mathbb{P}$-a.s.

$$\|z_*\|^{\widetilde{q}}_{L^{\widetilde{q}}(0,\infty;L^{\alpha+1})} = \int_0^\infty (1+s)^{-\frac{d(\alpha-1)}{2(\alpha+1)}\widetilde{q}} |\widetilde{z}_*(\frac{s}{1+s})|^{\widetilde{q}}_{L^{\alpha+1}} ds \leq C\int_0^\infty (1+s)^{-\frac{d(\alpha-1)}{2(\alpha+1)}\widetilde{q}} ds < \infty,$$

where the last step is due to the fact that $-\frac{d(\alpha-1)}{2(\alpha+1)}\widetilde{q} = -\frac{d(\alpha-1)^2}{4-(\alpha-1)(d-2)} < -1$ for $\alpha \in (1+\alpha(d), 1+4/(d-2)]$. Thus, we obtain (2.47) and complete the proof. □

Proof of Theorem 1.7 (ii) (continued). Recall that $V(t,s)$, $s,t\geq 0$, are evolution operators corresponding to (1.23) in the homogeneous case $F\equiv 0$, i.e., for any $v_s\in H^1$, $v(t) := V(t,s)v_s$ satisfies

$$\partial_t v(t) = A_*(t)v(t), \quad v(s) = v_s. \tag{2.50}$$

We reformulate equation (1.23) in the mild form

$$z_*(t) = V(t,0)z_*(0) - \lambda i \int_0^t V(t,s)e^{-\varphi_*(s)}F(e^{\varphi_*(s)}z_*(s))ds. \tag{2.51}$$

Then, for any $0<t_1<t_2<\infty$, if $w(t) := -\lambda i\int_{t_1}^t V(t,s)e^{-\varphi_*(s)}F(e^{\varphi_*(s)}z_*(s))ds$, we have

$$V(0,t_2)z_*(t_2) - V(0,t_1)z_*(t_1) = V(0,t_2)w(t_2). \tag{2.52}$$

Moreover, under the condition (1.3) and that $g_k\in L^2(\mathbb{R}^+)$, a.s., global-in-time Strichartz estimates hold for the operator A_* in (2.50). See Section 5 below.

Below we consider two cases $\alpha\in(1+\alpha(d),1+4/(d-2))$ and $\alpha = 1+4/(d-2)$ separately.

In the case where $\alpha\in(1+\alpha(d),1+4/(d-2))$, since $V(\cdot,t_2)w(t_2)$ satisfies equation (2.50) with the final datum $w(t_2)$ at time t_2, using (2.52) and applying Corollary 5.3 (iii) with $(p,q) = (\alpha+1, \frac{4(\alpha+1)}{d(\alpha-1)})$ we obtain

$$|V(0,t_2)z_*(t_2) - V(0,t_1)z_*(t_1)|_{H^1} \leq \|V(\cdot,t_2)w(t_2)\|_{C([0,t_2];H^1)} \leq C|w(t_2)|_{H^1}, \quad a.s.,$$

where C is independent of t_1, t_2.

Moreover, since $w(\cdot)$ satisfies (1.23) with $w(t_1) = 0$, and $\varphi_*\in L^\infty(0,\infty;W^{1,\infty})$, a.s., applying Corollary 5.3 (ii) and Hölder's inequality we get that the right-hand side above is bounded by

$$\begin{aligned} C\|w\|_{C([t_1,t_2];H^1)} \leq& C\|e^{-\varphi_*(s)}F(e^{\varphi_*(s)}z_*)\|_{L^{q'}(t_1,t_2;W^{1,p'})} \\ \leq& C\|z_*\|^{\alpha-1}_{L^{\widetilde{q}}(t_1,t_2;L^p)}\|z_*\|_{L^q(t_1,t_2;W^{1,p})}, \end{aligned} \tag{2.53}$$

where $\widetilde{q} = \frac{2(\alpha^2-1)}{4-(\alpha-1)(d-2)}$ and C is independent of t_1, t_2.

Thus, in view of (2.46) and (2.47), we obtain that as $t_1,t_2\to\infty$,

$$|V(0,t_2)z_*(t_2) - V(0,t_1)z_*(t_1)|_{H^1} \to 0, \quad a.s., \tag{2.54}$$

which implies that $\mathbb{P}$-a.s. there exists $X_+\in H^1$ such that

$$|V(0,t)z_*(t) - X_+|_{H^1} \to 0, \quad as\ t\to\infty, \tag{2.55}$$

thereby proving (1.13) for the case where $\alpha\in(1+\alpha(d),1+4/(d-2))$.

The energy-critical case where $\alpha = 1 + 4/(d-2)$ can be proved similarly. Choose the Strichartz pairs $(p_1, p_1) = (2+4/d, 2+4/d)$ and $(p_2, q_2) = (\frac{2d(d+2)}{d^2+4}, \frac{2(d+2)}{d-2})$. We only need to replace (2.53) by the estimate below

$$\begin{aligned}|V(0,t_2)z_*(t_2) - V(0,t_1)z_*(t_1)|_{H^1} \leq & C\|e^{-\varphi_*(s)}F(e^{\varphi_*(s)}z_*)\|_{L^{p_1'}(t_1,t_2;W^{1,p_1'})} \\ \leq & C\|z_*\|^{\frac{4}{d-2}}_{L^{\frac{2(d+2)}{d-2}}((t_1,t_2)\times\mathbb{R}^d)}\|z_*\|_{L^{p_1}(t_1,t_2;W^{1,p_1})} \\ \leq & C\|z_*\|^{\frac{4}{d-2}}_{L^{q_2}(t_1,t_2;W^{1,p_2})}\|z_*\|_{L^{p_1}(t_1,t_2;W^{1,p_1})},\end{aligned} \tag{2.56}$$

where the last step is due to the Sobolev imbedding $W^{1,\frac{2d(d+2)}{d^2+4}} \hookrightarrow L^{\frac{2(d+2)}{d-2}}(\mathbb{R}^d)$. Thus, by virtue of (2.46) we obtain (2.54), thereby proving (1.13) in the energy-critical case where $\alpha = 1 + 4/(d-2)$, and the proof is complete. □

3. Scattering in the energy space

In this section, we use the Strichartz pairs $(p_1, p_1) := (2 + \frac{4}{d}, 2 + \frac{4}{d})$, $(p_2, q_2) := (\frac{2d(d+2)}{d^2+4}, \frac{2(d+2)}{d-2})$ and define the spaces

$$Y^0(I) := L^{p_1}(I \times \mathbb{R}^d) \cap L^{q_2}(I; L^{p_2}),$$

and

$$Y^1(I) := \{f \in Y^0(I), \nabla f \in Y^0(I)\}.$$

Let $p_1' = \frac{2(d+2)}{d+4}$. Then, $1/p_1' + 1/p_1 = 1$.

Lemma 3.1. *Let $\alpha \in [1+4/d, 1+4/(d-2)]$, $d \geq 3$, and $I \times \mathbb{R}^d$ be an arbitrary spacetime slab. Then, for any $f, g \in Y^1(I)$,*

$$\||f|^{\alpha-1}g\|_{L^{p_1'}(I\times\mathbb{R}^d)} \leq C\|f\|^{2-\frac{(\alpha-1)(d-2)}{2}}_{L^{p_1}(I\times\mathbb{R}^d)}\|f\|^{\frac{d(\alpha-1)}{2}-2}_{L^{q_2}(I;W^{1,p_2})}\|g\|_{L^{p_1}(I\times\mathbb{R}^d)}. \tag{3.1}$$

Moreover, if in addition $\alpha \geq 2$, we have

$$\||f|^{\alpha-2}\nabla f g\|_{L^{p_1'}(I\times\mathbb{R}^d)} \leq C\|f\|^{\alpha-1}_{Y^1(I)}\|g\|_{Y^1(I)}. \tag{3.2}$$

Proof. By Lemma 2.6 in [39] we know that, if $\theta_1 := \frac{2}{\alpha-1} - \frac{d-2}{2}, \theta_2 := \frac{d}{2} - \frac{2}{\alpha-1} \in [0,1]$,

$$\||f|^{\alpha-1}g\|_{L^{\frac{2(d+2)}{d+4}}(I\times\mathbb{R}^d)} \leq C\|f\|^{(\alpha-1)\theta_1}_{L^{2+\frac{4}{d}}(I\times\mathbb{R}^d)}\|f\|^{(\alpha-1)\theta_2}_{L^{\frac{2(d+2)}{d-2}}(I\times\mathbb{R}^d)}\|g\|_{L^{2+\frac{4}{d}}(I\times\mathbb{R}^d)},$$

which along with the Sobolev imbedding $W^{1,\frac{2d(d+2)}{d^2+4}} \hookrightarrow L^{\frac{2(d+2)}{d-2}}$ implies (3.1).

Regarding (3.2), using Hölder's inequality, we have

$$\||f|^{\alpha-2}\nabla f g\|_{L^{p_1'}(I\times\mathbb{R}^d)} \leq \|f\|^{\alpha-2}_{L^{\frac{(d+2)(\alpha-1)}{2}}(I\times\mathbb{R}^d)}\|\nabla f\|_{L^{2+\frac{4}{d}}(I\times\mathbb{R}^d)}\|g\|_{L^{\frac{(d+2)(\alpha-1)}{2}}(I\times\mathbb{R}^d)}.$$

Note that,

$$\|f\|_{L^{\frac{(d+2)(\alpha-1)}{2}}(I\times\mathbb{R}^d)} \leq \|f\|^{\theta_1}_{L^{2+\frac{4}{d}}(I\times\mathbb{R}^d)}\|f\|^{\theta_2}_{L^{\frac{2(d+2)}{d-2}}(I\times\mathbb{R}^d)},$$

and similar estimate holds also for g. We obtain

$$\begin{aligned}\||f|^{\alpha-2}\nabla f g\|_{L^{p_1'}(I\times\mathbb{R}^d)} \leq & \|f\|^{(\alpha-2)\theta_1}_{L^{2+\frac{4}{d}}(I\times\mathbb{R}^d)}\|f\|^{(\alpha-2)\theta_2}_{L^{\frac{2(d+2)}{d-2}}(I\times\mathbb{R}^d)}\|\nabla f\|_{L^{2+\frac{4}{d}}(I\times\mathbb{R}^d)} \\ & \cdot \|g\|^{\theta_1}_{L^{2+\frac{4}{d}}(I\times\mathbb{R}^d)}\|g\|^{\theta_2}_{L^{\frac{2(d+2)}{d-2}}(I\times\mathbb{R}^d)},\end{aligned}$$

which implies (3.2), thereby finishing the proof. □

Lemma 3.2. *Consider the situation of Theorem 1.5 and let z_* be the unique global solution to* (1.23) *with $z_*(0) = X_0$. Then, for each $X_0 \in H^1$, we have*

$$\sup_{0\leq t<\infty} |z_*(t)|_{H^1} \leq C < \infty, \quad a.s.. \tag{3.3}$$

Proof. (3.3) follows directly from (1.7) and the fact that $\varphi_* \in L^\infty(\mathbb{R}^+; W^{1,\infty})$, a.s.. □

Remark 3.3. *Unlike in the pseudo-conformal case, the uniform bound* (3.3) *of z_* does not imply immediately the scattering behavior of z_* in the energy space. Actually, in the deterministic case, more subtle interaction Morawetz estimates are needed to obtain the scattering. See, e.g.,* [17, 41, 46].

Below, we prove the scattering by using the idea of comparison as in the proof of Lemma 2.6. For this purpose, we first prove the uniform global estimates (independent of T) for the solution u to (1.26).

Proposition 3.4. *Consider $\lambda = -1$, $\alpha \in [1 + 4/d, 1 + 4/(d-2)]$, $d \geq 3$. Then, if $u(T) = z_*(T) \in H^1$, there exists a unique global H^1-solution u (depending on T) to* (1.26) *such that u scatters at infinity in H^1 and for any $1 \leq j \leq d$ and any Strichartz pair (ρ, γ),*

$$\|u\|_{L^\gamma(\mathbb{R}^+;W^{1,\rho})} + \|u\|_{LS(\mathbb{R}^+)} + \|\partial_j u\|_{LS(\mathbb{R}^+)} \leq C < \infty, \tag{3.4}$$

where $LS(\mathbb{R}^+)$ is the local smoothing space defined in Section 5, and C is independent of T.

Proof. We mainly consider the global well-posedness, scattering and global bound of $\|u\|_{L^\gamma(\mathbb{R};W^{1,\rho})}$. The global bound in the local smoothing space can be proved standardly by Strichartz estimates.

The crucial estimates used below are that, for $\alpha \in (1 + 4/d, 1 + 4/(d-2)]$ and any Strichartz pair (ρ, γ),

$$\|u\|_{L^\gamma(\mathbb{R}^+;W^{1,\rho})} \leq C(|u(T)|_2^2, H(u(T))), \tag{3.5}$$

where $H(u)$ is the Hamiltonian of u, i.e., $H(u) = \frac{1}{2}|\nabla u|_2^2 + \frac{1}{\alpha+1}|u|_{L^{\alpha+1}}^{\alpha+1}$, and for the mass-critical case $\alpha = 1 + 4/d$, if $p := 2 + 4/d$,

$$\|u\|_{L^p(\mathbb{R}^+\times\mathbb{R}^d)} \leq C(|u(T)|_2^2). \tag{3.6}$$

See [39, Subsection 5.3] for the inter-critical case $\alpha \in (1+4/d, 1+4/(d-2))$, [17, 41, 46] and [39, Subsection 4.2] for the energy-critical case, and [25] for the mass-critical case.

Now, in the case where $\alpha \in (1 + 4/d, 1 + 4/(d-2)]$, if $u(T) = z_*(T) \in H^1$, the global well-posedness and scattering of u (with T fixed) follow from [14, 17, 41, 46]. Moreover, by virtue of (3.5) and (3.3), we have for any Strichartz pair (ρ, γ), $\mathbb{P}$-a.s.,

$$\|u\|_{L^\gamma(\mathbb{R}^+;W^{1,\rho})} \leq C(|u(T)|_2^2, H(u(T))) = C(|z_*(T)|_2^2, H(z_*(T))) \leq C(|z_*(T)|_{H^1}) \leq C \tag{3.7}$$

where C is independent of T.

Regarding the mass-critical case where $\alpha = 1 + 4/d$, since $u(T) = z_*(T) \in H^1 \subset L^2$, it follows from [25] that there exists a unique global L^2-solution to (1.26). Moreover, by (3.6) and (3.3),

$$\|u\|_{L^p(\mathbb{R}^+\times\mathbb{R}^d)} \leq C(|u(T)|_2^2) \leq C(|z_*(T)|_2^2) \leq C < \infty, \quad a.s., \tag{3.8}$$

where C is independent of T. Then, we split $\mathbb{R}^+$ into $\bigcup_{j=0}^{M}(t_j, t_{j+1})$ with $t_0 = 0$, $t_{M+1} = \infty$, $M(= M(T)) < \infty$, such that $\|u\|_{L^p((t_j,t_{j+1})\times\mathbb{R}^d)} = \varepsilon$, $0 \leq j \leq M-1$, and $\|u\|_{L^p((t_M,\infty)\times\mathbb{R}^d)} \leq \varepsilon$, where ε is to be chosen later. By (3.8),

$$\sup_{0<T<\infty} M(T) \leq (\frac{C}{\varepsilon})^p \tag{3.9}$$

with C independent of T. Note that, the conservations of the Hamiltonian $H(u(T)) = H(u(t))$ and the mass $|u(T)|_2^2 = |u(t)|_2^2$, $t \in \mathbb{R}$, together with (3.3) imply that, for $0 < C < \infty$ independent of T,

(3.10)

$$|u(t)|_{H^1} \leq |u(t)|_2^2 + 2H(u(t)) = |z_*(T)|_2^2 + 2H(z_*(T)) \leq C(1 + \|z_*\|_{C(\mathbb{R}^+;H^1)}^{\alpha+1}) < \infty, \quad a.s..$$

Then, applying Strichartz estimates to (1.26) and using Hölder's inequality we get that, if $p' := 2(d+2)/(d+4)$, then

$$\begin{aligned}\|u\|_{L^p(t_j,t_{j+1};W^{1,p})} \leq& C|u(t_j)|_{H^1} + C\||u|^{\alpha-1}u\|_{L^{p'}(t_j,t_{j+1};W^{1,p'})} \\ \leq& C|u(t_j)|_{H^1} + C\|u\|_{L^p(t_j,t_{j+1};L^p)}^{\frac{4}{d}}\|u\|_{L^p(t_j,t_{j+1};W^{1,p})} \\ \leq& C + C\varepsilon^{\frac{4}{d}}\|u\|_{L^p(t_j,t_{j+1};W^{1,p})}, \quad \forall\, 0 \leq j \leq M.\end{aligned} \tag{3.11}$$

Thus, taking ε small enough (independent of T), summing over j and using the global bound (3.9) we obtain that $\|u\|_{L^p(\mathbb{R}^+;W^{1,p})} \leq C(M) \leq C < \infty$, where C is independent of T. Hence, $\|u\|_{L^\gamma(\mathbb{R}^+;W^{1,\rho})} \leq C$ for any Strichartz pair (ρ, γ) by Strichartz estimates.

In particular, this yields the scattering of u in H^1. Actually, by (1.26),

$$e^{it_2\Delta}u(t_2) - e^{it_1\Delta}u(t_1) = -\lambda i \int_{t_1}^{t_2} e^{is\Delta}F(u(s))ds, \quad \forall\, t_1, t_2 \geq T > 0.$$

Similar to (3.11), we have as $t_1, t_2 \to \infty$,

$$|e^{it_2\Delta}u(t_2) - e^{it_1\Delta}u(t_1)|_{H^1} \leq C\|u\|_{L^p(t_1,t_2;L^p)}^{\frac{4}{d}}\|u\|_{L^p(t_1,t_2;W^{1,p})} \to 0, \tag{3.12}$$

which implies the scattering in H^1 for the mass-critical exponent $\alpha = 1 + 4/d$.

Therefore, the proof is complete. □

For the solution v to (1.28), we have the crucial global estimates and asymptotics at infinity below.

Lemma 3.5. *Assume the conditions of Theorem 1.5 to hold. Let v be the solution to (1.28) with $v(T) = 0$. Then, $\mathbb{P}$-a.s., for any Strichartz pair (ρ, γ),*

$$\|v\|_{L^\gamma(T,\infty;W^{1,\rho})} \leq C < \infty, \tag{3.13}$$

where C is uniformly bounded for T large enough, and for any Strichartz pair (ρ, γ), as $T \to \infty$,

$$\|v\|_{L^\gamma(T,\infty;W^{1,\rho})} + \|v\|_{C([T,\infty);H^1)} \to 0. \tag{3.14}$$

We remark that, unlike in the pseudo-conformal case, the global estimate (3.3) dose not imply directly the global bound of z_* (and also v) in the Strichartz space $L^\gamma(T,\infty;W^{1,\rho})$. Proceeding differently, we will prove (3.13) by using the comparison arguments which simultaneously gives (3.14) as well.

Proof. As mentioned in the proof of Theorem 1.3 (ii), global-in-time Strichartz estimates hold for the operator A_* in (1.28), due to the fact that $g_k \in L^2(\mathbb{R}^+)$, $\mathbb{P}$-a.s., $1 \leq k \leq N$ (see Section 5 below).

Below, let (p_i, q_i), $i = 1, 2$, and $Y^1(I)$, $I \subset \mathbb{R}^+$, be the Strichartz pairs and Strichartz space, respectively, as in Lemma 3.1.

Applying Corollary 5.3 to (1.28) we have for any $0 < T < t < \infty$,

$$\begin{aligned}
&\|v\|_{C([T,t];H^1)} + \|v\|_{Y^1(T,t)} \\
\leq & C\|(b_* \cdot \nabla + c_*)u\|_{LS'(T,t)} + C\|\nabla((b_* \cdot \nabla + c_*)u)\|_{LS'(T,t)} \\
&+ C\|(e^{(\alpha-1)\mathrm{Re}\,\varphi_*} - 1)F(u+v)\|_{L^{p_1'}(T,t;W^{1,p_1'})} \\
&+ C\|F(u+v) - F(u)\|_{L^{p_1'}(T,t;W^{1,p_1'})}
\end{aligned} \tag{3.15}$$

where C is independent of T and t, due to the global-in-time Strichartz estimates.

Below, we estimate the four terms on the right-hand side of (3.15) respectively.

First, since $g_k \in L^\infty(\Omega; L^2(\mathbb{R}^+))$, using (1.3) we have (see Section 5 below) that $\mathbb{P}$-a.s. for $0 \leq |\beta| \leq 2$, $0 \leq |\gamma| \leq 1$,

$$\varepsilon(T) := \sup_{t\in[T,\infty),x\in\mathbb{R}^d} \langle x \rangle^2 \left(|\partial_x^\beta b_*(t,x)| + |\partial_x^\gamma c_*(t,x)|\right) \to 0, \ \textit{as}\ T \to \infty, \tag{3.16}$$

and

$$\limsup_{|x|\to\infty} \sup_{0\leq t<\infty} \langle x \rangle^2 \left(|\partial_x^\beta b_*(t,x)| + |\partial_x^\gamma c_*(t,x)|\right) = 0,$$

which implies by (5.9) below (see also Remark 5.2) and Proposition 3.4 that

$$\begin{aligned}
&\|(b_* \cdot \nabla + c_*)u\|_{LS'(T,t)} + \|\nabla((b_* \cdot \nabla + c_*)u)\|_{LS'(T,t)} \\
\leq & C\varepsilon(T)(\|u\|_{LS(T,t)} + \|\nabla u\|_{LS(T,t)}) \leq C\varepsilon(T),
\end{aligned} \tag{3.17}$$

where C is independent of T.

Moreover, using the inequality $|e^x - 1| \leq e|x|$ for $|x| \leq 1$ and the fact that $\|\varphi_*\|_{C([T,\infty);W^{1,\infty})} \leq C\varepsilon(T)$ we get

$$\begin{aligned}
\|(e^{(\alpha-1)\mathrm{Re}\,\varphi_*} - 1)F(u+v)\|_{L^{p_1'}(T,t;W^{1,p_1'})} \leq & C\varepsilon(T)(\|u\|^\alpha_{Y^1(T,t)} + \|v\|^\alpha_{Y^1(T,t)}) \\
\leq & C\varepsilon(T)(1 + \|v\|^\alpha_{Y^1(T,t)}),
\end{aligned} \tag{3.18}$$

where C is independent of T, due to Proposition 3.4.

Regarding the fourth term on the right-hand side of (3.15), we first note that

$$|F(u+v) - F(u)| \leq C(|u|^{\alpha-1} + |v|^{\alpha-1})|v|. \tag{3.19}$$

Moreover, since $\nabla F(u) = F_z(u)\nabla u + F_{\overline{z}}(u)\nabla\overline{u}$, where F_z, $F_{\overline{z}}$ are the usual complex derivatives, and similar equality holds for $\nabla F(u+v)$, we have

$$\begin{aligned}
\nabla(F(u+v) - F(u)) =& (F_z(u+v) - F_z(u))\nabla u + (F_{\overline{z}}(u+v) - F_{\overline{z}}(u))\nabla\overline{u} \\
&+ F_z(u+v)\nabla v + F_{\overline{z}}(u+v)\nabla\overline{v}.
\end{aligned}$$

Note that $|F_z(u+v)| + |F_{\overline{z}}(u+v)| \leq C|u+v|^{\alpha-1}$, and since $\alpha \geq 2$, $|F_z(u+v) - F_z(u)| + |F_{\overline{z}}(u+v) - F_{\overline{z}}(u)| \leq C(|u|^{\alpha-2} + |v|^{\alpha-2})|v|$. Hence,

$$|\nabla(F(u+v) - F(u))| \leq (|u|^{\alpha-2} + |v|^{\alpha-2})|v||\nabla u| + (|u|^{\alpha-1} + |v|^{\alpha-1})|\nabla v|, \tag{3.20}$$

Then, using (3.19), (3.20) and Lemma 3.1 we obtain

$$\begin{aligned}
\|F(u+v) - F(u)\|_{L^{p_1'}(T,t;W^{1,p_1'})} \leq & C(\|u\|_{Y^1(T,t)}\|v\|^{\alpha-1}_{Y^1(T,t)} + \|u\|^{\alpha-1}_{Y^1(T,t)}\|v\|_{Y^1(T,t)} \\
&+ \|v\|^\alpha_{Y^1(T,t)}),
\end{aligned} \tag{3.21}$$

where C is independent of T and t.

Thus, plugging (3.17), (3.18) and (3.21) into (3.15) we obtain

$$\begin{aligned}\|v\|_{C([T,t];H^1)} + \|v\|_{Y^1(T,t)} \leq & C\varepsilon(T) + C\|v\|^{\alpha}_{Y^1(T,t)} \\ & + C(\|u\|_{Y^1(T,t)}\|v\|^{\alpha-1}_{Y^1(T,t)} + \|u\|^{\alpha-1}_{Y^1(T,t)}\|v\|_{Y^1(T,t)}). \end{aligned} \tag{3.22}$$

Below we shall prove the estimate

$$\|v\|_{C([T,\infty);H^1)} + \|v\|_{Y^1(T,\infty)} \leq C'\varepsilon(T), \tag{3.23}$$

where C' is independent of T.

For this purpose, we choose $\eta, \varepsilon(T) > 0$ sufficiently small, such that

$$D_0(\eta + \eta^{\alpha-1}) \leq \frac{1}{2}, \quad 4D_0D^*(\eta)\varepsilon(T) \leq (1-\frac{1}{\alpha})(2\alpha D_0)^{-\frac{1}{\alpha-1}}, \tag{3.24}$$

where the constants $D_0, D^*(\eta)$ are as in (3.26) and (3.29) below respectively, independent of T.

For each T fixed, since $\|u\|_{Y^1(T,\infty)} \leq C_* < \infty$, we can choose $\{t_{1,j}\}_{j=0}^{M_1+1}$, $t_{1,0} = T$, $t_{1,M_1+1} = \infty$, $M_1(= M_1(T)) < \infty$, such that $\|u\|_{L^{p_1}(t_{1,j},t_{1,j+1})} = \frac{\eta}{2}$, $0 \leq j \leq M_1 - 1$, and $\|u\|_{L^{p_1}(t_{1,M_1},\infty)} \leq \frac{\eta}{2}$. Then, similarly to (3.9), $\sup_T M_1(T) \leq (\frac{2C_*}{\eta})^{p_1}$. Similarly, we choose another partition $\{t_{2,j}\}_{j=0}^{M_2}$, such that $\|u\|_{L^{q_2}(t_{2,j},t_{2,j+1};W^{1,p_2})} = \frac{\eta}{2}$, $0 \leq j \leq M_2 - 1$, $\|u\|_{L^{q_2}(t_{2,M_2},\infty;W^{1,p_2})} \leq \frac{\eta}{2}$, and so $\sup_T M_2(T) \leq (\frac{2C_*}{\eta})^{q_2}$.

Thus, we can divide $\mathbb{R}^+$ into finitely many subintervals $\{[t_j, t_{j+1}]\}_{j=0}^{M}$, satisfying that $\{t_j\}_{j=0}^{M+1} = \{t_{1,j}\}_{j=0}^{M_1+1} \cup \{t_{2,j}\}_{j=0}^{M_2+1}$, $\|u\|_{Y^1(t_j,t_{j+1})} \leq \eta$, $0 \leq j \leq M$, and

$$\sup_{0<T<\infty} M(T) \leq \sup_{0<T<\infty} (M_1(T) + M_2(T)) \leq (\frac{2C_*}{\eta})^{p_1} + (\frac{2C_*}{\eta})^{q_2} := C^*(\eta) < \infty. \tag{3.25}$$

Now, let us consider the estimate on the time interval (T, t_1). By (3.22) and the fact that $\alpha \geq 2$,

$$\begin{aligned} & \|v\|_{C([T,t_1];H^1)} + \|v\|_{Y^1(T,t_1)} \\ \leq & C\varepsilon(T) + C\|v\|^{\alpha}_{Y^1(T,t_1)} + C\eta\|v\|^{\alpha-1}_{Y^1(T,t_1)} + C\eta^{\alpha-1}\|v\|_{Y^1(T,t_1)} \\ \leq & D_0\varepsilon(T) + D_0(\eta + \eta^{\alpha-1})\|v\|_{Y^1(T,t_1)} + D_0\|v\|^{\alpha}_{Y^1(T,t_1)}, \end{aligned} \tag{3.26}$$

where $D_0 = 2C(\geq 1)$ is independent of T. Then, by the choice of η in (3.24),

$$\|v\|_{C([T,t_1];H^1)} + \|v\|_{Y^1(T,t_1)} \leq 2D_0\varepsilon(T) + 2D_0\|v\|^{\alpha}_{Y^1(T,t_1)}.$$

This, via [8, Lemma A.1] and implies that for $\varepsilon(T)$ small enough such that $2D_0\varepsilon(T) < (1-\frac{1}{\alpha})(2\alpha D_0)^{-\frac{1}{\alpha-1}}$,

$$\|v\|_{C([T,t_1];H^1)} + \|v\|_{Y^1(T,t_1)} \leq D_1\varepsilon(T) \tag{3.27}$$

with $D_1 := 2(4D_0)^{1+\alpha}(\frac{\alpha}{\alpha-1})^{\alpha}$, independent of T. In particular, $|v(t_1)|_{H^1} \leq D_1\varepsilon(T)$.

Next we use the inductive arguments. Suppose that at the j-th step $|v(t_j)|_{H^1} \leq D_j\varepsilon(T)$, $0 < j \leq M$, where $D_j(\geq 1)$ is increasing with j and is independent of T.

Then, similarly to (3.26), we have

$$\begin{aligned} & \|v\|_{C([t_j,t_{j+1}];H^1)} + \|v\|_{Y^1(t_j,t_{j+1})} \\ \leq & C|v(t_j)|_{H^1} + D_0\varepsilon(T) + D_0(\eta + \eta^{\alpha-1})\|v\|_{Y^1(t_j,t_{j+1})} + D_0\|v\|^{\alpha}_{Y^1(t_j,t_{j+1})}. \end{aligned}$$

By the choice of η in (3.24) and the inductive assumption, we get

$$\begin{aligned}\|v\|_{C([t_j,t_{j+1}];H^1)} + \|v\|_{Y^1(t_j,t_{j+1})} \leq & 2CD_j\varepsilon(T) + 2D_0\varepsilon(T) + 2D_0\|v\|^{\alpha}_{Y^1(t_j,t_{j+1})} \\ \leq & 4D_0D_j\varepsilon(T) + 2D_0\|v\|^{\alpha}_{Y^1(t_j,t_{j+1})}. \end{aligned}$$

Using (3.24) and apply [8, Lemma A.1] again, we have that for $\varepsilon(T)$ even smaller such that $4D_0D_j\varepsilon(T) < (1-\frac{1}{\alpha})(2\alpha D_0)^{-\frac{1}{\alpha-1}}$,

$$\|v\|_{C([t_j,t_{j+1}];H^1)} + \|v\|_{Y^1(t_j,t_{j+1})} \leq D_{j+1}\varepsilon(T) \tag{3.28}$$

with $D_{j+1} = 2(4D_0)^{1+\alpha}(\frac{\alpha}{\alpha-1}D_j)^\alpha = D_1D_j^\alpha$ and is independent of T. In particular, $|v(t_{j+1})|_{H^1} \leq D_{j+1}\varepsilon(T)$.

Thus, letting

$$D^*(\eta) = D_1^{\frac{\alpha^{C^*(\eta)+1}-1}{\alpha-1}} < \infty \tag{3.29}$$

with $C^*(\eta)$ as in (3.25) above, we deduce from the condition (3.24) and inductive arguments that, for each $0 \leq j \leq M \leq C^*(\eta)$, the estimate (3.28) is valid and

$$D_{j+1} = D_1^{1+\alpha+\cdots+\alpha^j} = D_1^{\frac{\alpha^{j+1}-1}{\alpha-1}} \leq D^*(\eta) < \infty,$$

provided $\varepsilon(T)$ satisfies the smallness condition in (3.24) above.

Therefore, taking the sum or maximum over $0 \leq j \leq M$ and taking into account the uniform bound (3.25) we obtain (3.23), as claimed.

Finally, (3.13) and (3.14) follow from (3.16), (3.23) and Strichartz estimates. The proof is complete. □

Proof of Theorem 1.5. (i). Note that, for any $t_1, t_2 \geq T$

$$\begin{aligned}|e^{it_1\Delta}z_*(t_1) - e^{it_2\Delta}z_*(t_2)|_{H^1} \leq & |e^{it_1\Delta}(z_*(t_1)-u(t_1))|_{H^1} + |e^{it_2\Delta}(u(t_2)-z_*(t_2))|_{H^1} \\ & + |e^{it_1\Delta}u(t_1) - e^{it_2\Delta}u(t_2)|_{H^1} \\ \leq & 2\|z_*-u\|_{C([T,\infty);H^1)} + |e^{it_1\Delta}u(t_1) - e^{it_2\Delta}u(t_2)|_{H^1},\end{aligned}$$

where u is the solution to (1.26) with $u(T) = z_*(T)$.

Since by Proposition 3.4, for each T fixed, u scatters at infinity, we have

$$\lim_{t_1,t_2\to\infty} |e^{it_1\Delta}u(t_1) - e^{it_2\Delta}u(t_2)|_{H^1} = 0.$$

Then, we get

$$\limsup_{t_1,t_2\to\infty} |e^{it_1\Delta}z_*(t_1) - e^{it_2\Delta}z_*(t_2)|_{H^1} \leq 2\|z_*-u\|_{C([T,\infty);H^1)}.$$

Thus, by virtue of (3.14), we taking T to infinity to obtain that $\{e^{it\Delta}z_*(t)\}$ is a Cauchy sequence in the space H^1, which implies the scattering of z_* at infinity specified in (1.14).

(ii). As in the proof of Theorem 1.3 (ii), Strichartz estimates and Lemma 3.1 imply that $\mathbb{P}$-a.s.,

$$\begin{aligned}&|V(0,t_2)z_*(t_2) - V(0,t_1)z_*(t_1)|_{H^1} \\ \leq & C\|e^{-\varphi_*(s)}F(e^{\varphi_*(s)}z_*)\|_{L^{\frac{2(d+2)}{d+4}}(t_1,t_2;W^{1,\frac{2(d+2)}{d+4}})} \leq C\|z_*\|^\alpha_{Y^1(t_1,t_2)},\end{aligned}$$

where $Y^1(t_1,t_2)$ is the space defined in the previous proof of Lemma 3.5 and C is independent of t_1, t_2, due to the global-in-time Strichartz estimates for A_*.

Note that, $z_* = e^{-\varphi_*}X = v + u$. We have $\|z_*\|_{Y^1(T,\infty)} < \infty$ for T large enough, due to the global bounds (3.4) and (3.13).

Thus, it follows that $|V(0,t_2)z_*(t_2) - V(0,t_1)z_*(t_1)|_{H^1} \to 0$, as $t_1, t_2 \to \infty$, $\mathbb{P}$-a.s., thereby yielding (1.15). Therefore, the proof is complete. □

4. Proof of Theorem 1.7.

We first consider the scattering in the pseudo-conformal space. As in Section 2, let $\widetilde{z}$ be the pseudo-conformal transformation of the solution z to (1.19), i.e.,

$$\widetilde{z}(t,x) := (1-t)^{-\frac{d}{2}}\ z\left(\frac{t}{1-t}, \frac{x}{1-t}\right)\ e^{i\frac{|x|^2}{4(1-t)}}, \tag{4.1}$$

where $t \in [0,1)$, $x \in \mathbb{R}^d$. We have

$$\begin{aligned} &\partial_t \widetilde{z} = -i\Delta\widetilde{z} - \lambda i h(t) e^{-\widetilde{\varphi}} F(e^{\widetilde{\varphi}}\widetilde{z}), \\ &\widetilde{z}(0) = X_0 e^{i\frac{|x|^2}{4}} \in \Sigma, \end{aligned} \tag{4.2}$$

where $h(t)$ is as in Section 2, i.e., $h(t) = (1-t)^{\frac{d(\alpha-1)-4}{2}}$, and

$$\widetilde{\varphi}(t) = \varphi(\frac{t}{1-t}) = \sum_{k=1}^{N}\left(\int_0^{\frac{t}{1-t}} v_k g_k(s)d\beta_k(s) - \int_0^{\frac{t}{1-t}} (\operatorname{Re} v_k)v_k g_k^2(s)ds\right).$$

We show that, for $\operatorname{Re} v_1$ large enough, $\widetilde{z}(=\widetilde{z}_{v_1})$ exists on $[0,1]$ with high probability.

For this purpose, set $\widetilde{\mathcal{X}}^M_\sigma := \{w \in L^\infty(0,\sigma;L^2) \cap L^q(0,\sigma;L^p) : \|w\|_{L^\infty(0,\sigma;H^1)} + \|w\|_{L^q(0,\sigma;W^{1,p})} \le M\}$ with $(p,q) = (\frac{d(\alpha+1)}{d+\alpha-1}, \frac{4(\alpha+1)}{(d-2)(\alpha-1)})$, and define $\widetilde{\Phi}$ on $\widetilde{\mathcal{X}}^M_\sigma$ by

$$\widetilde{\Phi}(w)(t) = e^{-it\Delta}(X_0 e^{i\frac{|x|^2}{4}}) - \lambda i \int_0^t e^{-i(t-s)\Delta} h(s) e^{-\widetilde{\varphi}(s)} F(e^{\widetilde{\varphi}(s)} w(s))ds, \quad w \in \widetilde{\mathcal{X}}^M_\sigma.$$

Then, similarly to (2.11), for any $w_j \in \widetilde{\mathcal{X}}^M_\sigma$, $j=1,2$,

$$\|\widetilde{\Phi}(w_j)\|_{L^\infty(0,\sigma;H^1)} + \|\widetilde{\Phi}(w_j)\|_{L^q(0,\sigma;W^{1,p})} \le C|X_0|_\Sigma + C\widetilde{\varepsilon}_\sigma^{\frac{1}{\theta}}(v_1)M^\alpha, \tag{4.3}$$

and

$$\begin{aligned} &\|\widetilde{\Phi}(w_1) - \widetilde{\Phi}(w_2)\|_{L^\infty(0,\sigma;L^2)} + \|\widetilde{\Phi}(w_1) - \widetilde{\Phi}(w_2)\|_{L^q(0,\sigma;L^p)} \\ \le & C\widetilde{\varepsilon}_\sigma^{\frac{1}{\theta}}(v_1) M^{\alpha-1}\|w_1 - w_2\|_{L^q(0,\sigma;L^p)}. \end{aligned} \tag{4.4}$$

where $\widetilde{\varepsilon}_\sigma(v_1) := |he^{(\alpha-1)\operatorname{Re}\widetilde{\varphi}}|^\theta_{L^\theta(0,\sigma)}$, $1<\theta<\infty$ is as in (2.11), and C is independent of σ and v_1. Let $M = 2C|X_0|_\Sigma$ and choose the stopping time

$$\sigma_{v_1} := \inf\{t>0, 2^\alpha C^\alpha |X_0|_\Sigma^{\alpha-1}\widetilde{\varepsilon}_t^{\frac{1}{\theta}}(v_1) > 1\} \wedge 1. \tag{4.5}$$

It follows that $\widetilde{\Phi}(\widetilde{\mathcal{X}}^M_{\sigma_{v_1}}) \subset \widetilde{\mathcal{X}}^M_{\sigma_{v_1}}$ and $\widetilde{\Phi}$ is a contraction in $L^\infty(0,\sigma_{v_1};L^2) \cap L^q(0,\sigma_{v_1};L^p)$. Hence, similar arguments as in the proof of [5, Theorem 2.1] yield that there exists a unique H^1-solution $\widetilde{z} = \widetilde{z}_{v_1}$ to (4.2) on $[0,\sigma_{v_1}]$. In particular, $\widetilde{z}_{v_1}$ exists on $[0,1]$ if $\sigma_{v_1} = 1$.

Thus, in order to show that $\widetilde{z}_{v_1}$ exists on $[0,1]$ with high probability, it suffices to prove that $\sigma_{v_1} = 1$ with high probability.

For this purpose, we consider

$$\begin{aligned} \widetilde{\varepsilon}(v_1) :=& |he^{(\alpha-1)\operatorname{Re}\widetilde{\varphi}}|^\theta_{L^\theta(0,1)} = \int_0^\infty (1+s)^{-\frac{d(\alpha-1)-4}{2}\theta-2} e^{(\alpha-1)\theta\operatorname{Re}\varphi(s)}ds \\ =& \int_0^\infty (1+s)^{-\frac{d(\alpha-1)-4}{2}\theta-2} \prod_{k=1}^N e^{(\alpha-1)\theta\operatorname{Re}\varphi_k(s)}ds, \end{aligned} \tag{4.6}$$

where

$$\operatorname{Re}\varphi_k(t) := \int_0^t \operatorname{Re} v_k g_k(s)d\beta_k(s) - \int_0^t (\operatorname{Re} v_k g_k(s))^2 ds.$$

Note that,

$$C_1 := \sup_{t>0} \prod_{k=2}^{N} \exp\{(\alpha-1)\theta \mathrm{Re}\, \varphi_k(t)\} < \infty, \quad a.s.. \tag{4.7}$$

To this end, by the theorem on time change for continuous martingales (see e.g. [32, Section 3.4]), there exists a Brownian motion $\widetilde{\beta}_k$ such that $\mathbb{P} \circ (\mathrm{Re}\, v_k \int_0^\cdot g_k d\beta_k)^{-1} = \mathbb{P} \circ \widetilde{\beta}_k^{-1}((\mathrm{Re}\, v_k)^2 \int_0^\cdot g_k^2 ds)$. Moreover, the law of the iterated logarithm for Brownian motion (see e.g. [32, Section 2.9]) implies that $\lim_{t\to\infty} \widetilde{\beta}_k(t) - t = -\infty$, a.s.. Taking into account $\int_0^t g_k^2 ds \geq c_0^2 t \to \infty$ as $t \to \infty$, we have

$$\mathbb{P}(\lim_{t\to\infty} \mathrm{Re}\, \varphi_k(t) = -\infty) = \mathbb{P}(\lim_{t\to\infty} \widetilde{\beta}_k((\mathrm{Re}\, v_k)^2 \int_0^t g_k^2 ds) - (\mathrm{Re}\, v_k)^2 \int_0^t g_k^2 ds = -\infty) = 1,$$

which implies (4.7), as claimed.

For $k = 1$, since $\inf_{t\geq 0} g_1(t) \geq c_0 > 0$, for any $\mathrm{Re}\, v_1 \geq 1$, $(\mathrm{Re}\, v_1)^2 \int_0^t g_1^2(s) ds \geq c_0^2 t \to \infty$ as $t \to \infty$ a.s.. Then, by the law of the iterated logarithm, there exist $c, t_* > 0$, such that for any $t \geq t_* > 0$ and any $\mathrm{Re}\, v_1 \geq 1$,

$$\begin{aligned}\left| \mathrm{Re}\, v_1 \int_0^t g_1(s) d\beta_1(s) \right| &\leq c \left((\mathrm{Re}\, v_1)^2 \int_0^t g_1^2(s) ds \ln\ln \left((\mathrm{Re}\, v_1)^2 \int_0^t g_1^2(s) ds \right)^{-1} \right)^{\frac{1}{2}} \\ &\leq \frac{1}{2} (\mathrm{Re}\, v_1)^2 \int_0^t g_1^2(s) ds,\end{aligned}$$

which implies that

$$\mathrm{Re}\, \varphi_1(t) \leq -\frac{1}{2} (\mathrm{Re}\, v_1)^2 \int_0^t g_1^2(s) ds \leq -\frac{1}{2} (\mathrm{Re}\, v_1)^2 c_0^2 t, \quad t \geq t_*.$$

Thus, using the dominated convergence theorem we have

$$\begin{aligned}&\int_{t_*}^{\infty} (1+s)^{-\frac{d(\alpha-1)-4}{2}\theta - 2} e^{(\alpha-1)\theta \mathrm{Re}\, \varphi_1(s)} ds \\ \leq &\int_{t_*}^{\infty} (1+s)^{-\frac{d(\alpha-1)-4}{2}\theta - 2} e^{-\frac{1}{2}(\alpha-1)\theta c_0^2 (\mathrm{Re}\, v_1)^2 s} ds \to 0,\end{aligned} \tag{4.8}$$

as $\mathrm{Re}\, v_1 \to \infty$, $\mathbb{P}$-a.s..

Moreover, $\mathbb{P}-a.s.$, for any $0 < s \leq t^*$, we see that $\widetilde{\beta}_1(\int_0^s (\mathrm{Re}\, v_1)^2 g_1^2 dr) - \int_0^s (\mathrm{Re}\, v_1)^2 g_1^2 dr \to -\infty$, as $\mathrm{Re}\, v_1 \to \infty$, which implies that

$$\widetilde{f}_{v_1}(s) := \exp\left((\alpha-1)\theta \left(\widetilde{\beta}_1(\int_0^s (\mathrm{Re}\, v_1)^2 g_1^2 dr) - \int_0^s (\mathrm{Re}\, v_1)^2 g_1^2 dr \right) \right) \to 0, \quad \mathrm{Re}\, v_1 \to \infty.$$

Taking into account

$$\sup_{\mathrm{Re}\, v_1 \geq 1} \sup_{0<s\leq t^*} \widetilde{f}_{v_1}(s) \leq \sup_{0<t<\infty} \exp\{(\alpha-1)\theta(\widetilde{\beta}_1(t) - t)\} \leq C < \infty,$$

we apply the dominated convergence theorem to obtain that

$$\int_0^{t_*} (1+s)^{-\frac{d(\alpha-1)-4}{2}\theta - 2} \widetilde{f}_{v_1}(s) ds \to 0, \quad as\ \mathrm{Re}\, v_1 \to \infty, \quad a.s.,$$

which along with the theorem on time change for continuous martingales implies that

$$\begin{aligned}
&\mathbb{P}\left(\lim_{\operatorname{Re} v_1\to\infty}\int_0^{t_*}(1+s)^{-\frac{d(\alpha-1)-4}{2}\theta-2}e^{(\alpha-1)\theta\operatorname{Re}\varphi_1(s)}ds=0\right)\\
=&\mathbb{P}\left(\lim_{\operatorname{Re} v_1\to\infty}\int_0^{t_*}(1+s)^{-\frac{d(\alpha-1)-4}{2}\theta-2}\widetilde{f}_{v_1}(s)ds=0\right)=1.
\end{aligned}\tag{4.9}$$

Hence, it follows from (4.8) and (4.9) that, as $\operatorname{Re} v_1\to\infty$,

$$\int_0^\infty(1+s)^{-\frac{d(\alpha-1)-4}{2}\theta-2}e^{(\alpha-1)\theta\operatorname{Re}\varphi_1(s)}ds\to 0,\quad a.s.. \tag{4.10}$$

Thus, plugging (4.7) and (4.10) into (4.6) we get that a.s. for any $t\geq 0$,

$$\widetilde{\varepsilon}_t(v_1)\leq\widetilde{\varepsilon}(v_1)\leq C_1\int_0^\infty(1+s)^{-\frac{d(\alpha-1)-4}{2}\theta-2}e^{(\alpha-1)\theta\operatorname{Re}\varphi_1(s)}ds\to 0,\ \operatorname{Re} v_1\to\infty,\ a.s.. \tag{4.11}$$

Then, in view of the definition of σ_{v_1} in (4.5), we obtain

$$\mathbb{P}(\sigma_{v_1}=1)\to 1,\quad as\ \operatorname{Re} v_1\to\infty, \tag{4.12}$$

which implies that $\widetilde{z}_{v_1}$ exists in the energy space H^1 on the interval $[0,1]$ with high probability if $\operatorname{Re} v_1$ is sufficiently large.

Now, we consider all random variables being evaluated at $\omega\in\{\sigma_{v_1}=1\}$. By (4.2), for all $t\in[0,1]$,

$$\int|x|^2|\widetilde{z}_{v_1}(t)|^2dx=\int|x|^2|X_0|^2dx+4\int_0^t\operatorname{Im}\int x\cdot\nabla\overline{\widetilde{z}_{v_1}}(s)\widetilde{z}_{v_1}(s)dxds.$$

Then, by Cauchy's inequality,

$$\begin{aligned}
\int|x|^2|\widetilde{z}_{v_1}(t)|^2dx\leq&|X_0|_\Sigma^2+4\int_0^t\left(\int|x|^2|\widetilde{z}_{v_1}(s)|^2dx\right)^{\frac{1}{2}}|\nabla\widetilde{z}_{v_1}|_2ds\\
\leq&(|X_0|_\Sigma^2+4\|\widetilde{z}_{v_1}\|_{C([0,1];H^1)}^2)+4\int_0^t\int|x|^2|\widetilde{z}_{v_1}(s)|^2dxds,
\end{aligned}$$

which implies by Gronwall's inequality, because $\|\widetilde{z}_{v_1}\|_{C([0,1];H^1)}<\infty$, that

$$\sup_{t\in[0,1]}\int|x|^2|\widetilde{z}_{v_1}(t)|^2dx<\infty.$$

In particular, $\widetilde{z}_{v_1}(1)\in\Sigma$.

Therefore, as in the proof of Theorem 1.3 (ii), by virtue of the equivalence between the asymptotics of $\widetilde{z}_{v_1}$ at time 1 and z_{v_1} at infinity, we conclude that the original solution z_{v_1} scatters at infinity in Σ, i.e., $\lim_{t\to\infty}|e^{it\Delta}z_{v_1}(t)-u_+|_\Sigma=0$ for some $u_+\in\Sigma$. Thus,

$$\{\sigma_{v_1}=1\}\subset A_{v_1},$$

where A_{v_1} denotes the event that the solution X to (1.1) exists globally and scatters at infinity in Σ. Taking into account (4.12) we obtain (1.17) in the pseudo-conformal space.

The noise effect on scattering in the energy space can be proved similarly. Taking into account (1.19) and that $A(t)=-i\Delta$, we set $\mathcal{X}_\tau^M:=\{w\in L^\infty(0,\tau;L^2)\cap L^q(0,\tau;L^p):\|w\|_{L^\infty(0,\tau;H^1)}+\|w\|_{L^q(0,\tau;W^{1,p})}\leq M\}$ with (p,q) as above and define Φ on $\mathcal{X}_\tau^M$ by

$$\Phi(w)=e^{-it\Delta}X_0-\lambda i\int_0^t e^{-i(t-s)\Delta}e^{-\varphi(s)}F(e^{\varphi(s)}w(s))ds,\quad w\in\mathcal{X}_\tau^M.$$

Similar to (4.3) and (4.4), for any $w_j \in \mathcal{X}^M_\tau$, $j=1,2$, if $\varepsilon_\tau(v_1) := |e^{(\alpha-1)\mathrm{Re}\,\varphi}|^\theta_{L^\theta(0,\tau)}$,

$$\|\Phi(w_j)\|_{L^\infty(0,\tau;H^1)} + \|\Phi(w_j)\|_{L^q(0,\tau;W^{1,p})} \le C|X_0|_{H^1} + C\varepsilon_\tau^{\frac{1}{\theta}}(v_1)M^\alpha,$$

and

$$\begin{aligned}&\|\Phi(w_1)-\Phi(w_2)\|_{L^\infty(0,\tau;L^2)} + \|\Phi(w_1)-\Phi(w_2)\|_{L^q(0,\tau;L^p)}\\ \le& C\varepsilon_\tau^{\frac{1}{\theta}}(v_1)M^{\alpha-1}\|w_1-w_2\|_{L^q(0,\tau;L^p)},\end{aligned}$$

where C is independent of τ and v_1. Then, taking $M = 2C|X_0|_{H^1}$ and

$$\tau_{v_1} = \inf\{t>0, 2^\alpha C^\alpha |X_0|^{\alpha-1}_{H^1}\varepsilon_t^{\frac{1}{\theta}}(v_1) > 1\}, \tag{4.13}$$

and using similar arguments as in the previous case we see that there exists a unique H^1-solution $z(=z_{v_1})$ to (1.19) on $[0,\tau_{v_1})$. In particular, z_{v_1} exists globally if $\tau_{v_1}=\infty$.

Note that, similar to (4.11), for any $t>0$,

$$\varepsilon_t(v_1) \le \varepsilon(v_1) := |e^{(\alpha-1)\mathrm{Re}\,\varphi}|^\theta_{L^\theta(0,\infty)} \to 0, \tag{4.14}$$

as $\mathrm{Re}\,v_1\to\infty$, $\mathbb{P}$-a.s., which along with (4.13) implies that

$$\mathbb{P}(\tau_{v_1}=\infty)\to 1, \quad as\ \mathrm{Re}\,v_1\to\infty. \tag{4.15}$$

Below we consider $\omega\in\{\tau_{v_1}=\infty\}$. As in (4.3), for any $t\in(0,\infty)$,

$$\|z_{v_1}\|_{L^q(0,t;W^{1,p})} \le C|X_0|_{H^1} + C\varepsilon^{\frac{1}{\theta}}(v_1)\|z_{v_1}\|^\alpha_{L^q(0,t;W^{1,p})}, \tag{4.16}$$

where C is independent of t and v_1. In view of (4.14), choosing $\mathrm{Re}\,v_1$ large enough, if necessary, and using [8, Lemma 6.1] we obtain $\|z_{v_1}\|_{L^q(0,t;W^{1,p})} \le C < \infty$, with C independent of t and v_1. Taking $t\to\infty$ we get

$$\|z_{v_1}\|_{L^q(0,\infty;W^{1,p})} < \infty. \tag{4.17}$$

Then, similar to (3.12), by virtue of (4.17) we have that for $\mathrm{Re}\,v_1$ large enough,

$$\begin{aligned}|e^{it_2\Delta}z_{v_1}(t_2) - e^{it_1\Delta}z_{v_1}(t_1)|_{H^1} \le& C\|e^{-\varphi}F(e^\varphi z_{v_1})\|_{L^{q'}(t_2,t_1;W^{1,p'})}\\ \le& C(v_1)\varepsilon^{\frac{1}{\theta}}(v_1)\|z_{v_1}\|^\alpha_{L^q(t_2,t_1;W^{1,p})} \to 0, \quad as\ t_1,t_2\to\infty,\end{aligned}$$

which implies that there exists $u_+\in H^1$ such that

$$e^{it\Delta}z_{v_1}(t)\to u_+, \quad in\ H^1,\ as\ t\to\infty,$$

thereby yielding that X scatters at infinity in H^1.

Therefore, for $\mathrm{Re}\,v_1$ large enough,

$$\{\tau_{v_1}=\infty\}\subset A_{v_1},$$

which along with (4.15) implies (1.17) in the energy space H^1. This completes the proof. □

5. Strichartz and local smoothing estimates

In this section, we summarize the Strichartz and local smoothing estimates used in this paper, mainly based on the work [35].

Let $D_0=\{|x|\le 2\}$, $D_j=\{2^j\le|x|\le 2^{j+1}\}$, and $D_{<j}=\{|x|\le 2^j\}$, $j\ge1$. Set $A_j=\mathbb{R}\times D_j$, $j\ge0$, and $A_{<j}=\mathbb{R}\times D_{<j}$, $j\ge1$. The local smoothing space is the completion of the Schwartz space with respect to the norm $\|u\|^2_{LS}=\sum_{k=-\infty}^\infty 2^k\|S_k u\|^2_{LS_k}$,

and the dual norm is $\|u\|^2_{LS'} = \sum_{k=-\infty}^{\infty} 2^{-k}\|S_k u\|^2_{LS'_k}$, where $\{S_k\}$ is a dyadic partition of unity of frequency,

$$\|u\|_{LS_k} = \|u\|_{L^2_{t,x}(A_0)} + \sup_{j>0} \| \langle x \rangle^{-\frac{1}{2}} u\|_{L^2_{t,x}(A_j)}, \quad k \geq 0,$$

$$\|u\|_{LS_k} = 2^{\frac{k}{2}}\|u\|_{L^2_{t,x}(A_{<-k})} + \sup_{j\geq -k} \|(|x| + 2^{-k})^{-\frac{1}{2}} u\|_{L^2_{t,x}(A_j)}, \quad k < 0.$$

(Since the notation X stands for the solution to (1.1), in order to avoid confusions, we use the different notation LS, instead of X in [35], for the local smoothing space.)

Similarly, for every $-\infty \leq S < T \leq \infty$, we can also define A_j, $A_{<j}$ on the time interval (S,T), and $LS(S,T)$ denotes the local smoothing space defined on (S,T).

We say that (p,q) is a Strichartz pair, if $2/q = d(1/2 - 1/p)$, $(p,q) \in [2,\infty] \times [2,\infty]$, and $(p,q,d) \neq (\infty, 2, 2)$.

We first present the Strichartz and local smoothing estimates essentially proved in [35].

Theorem 5.1. *Consider the equation*

$$i\partial_t u = (\Delta + b\cdot\nabla + c)u + f \tag{5.1}$$

with $u(0) = u_0$, $d \geq 3$. Assume that the coefficients b, c satisfy

$$\sum_{j\in\mathbb{N}} \sup_{A_j} \langle x \rangle |b(t,x)| \leq \kappa, \tag{5.2}$$

and

$$\sup_{\mathbb{R}\times\mathbb{R}^d} \langle x \rangle^2 (|c(t,x)| + |\mathrm{div}\ b(t,x)|) \leq \kappa, \tag{5.3}$$

$$\limsup_{|x|\to\infty} \langle x \rangle^2 (|c(t,x)| + |\mathrm{div}\ b(t,x)|) < \varepsilon \ll 1. \tag{5.4}$$

(i). For any $u_0 \in L^2$, $T \in (0,\infty)$ and any two Strichartz pairs (p_k, q_k), $k = 1, 2$, we have the local-in-time Strichartz estimates, i.e.,

$$\|u\|_{L^{q_1}(0,T;L^{p_1})\cap LS(0,T)} \leq C_T(|u_0|_2 + \|f\|_{L^{q'_2}(0,T;L^{p'_2})+LS'(0,T)}). \tag{5.5}$$

(ii) Assume in addition that (5.2) and (5.3) hold also for $\partial_j b$ and $\partial_j c$, $1 \leq j \leq d$. Then, for any Strichatz pairs (p_k, q_k), $1 \leq k \leq 3$,

$$\begin{aligned}\|\partial_j u\|_{L^{q_1}(0,T;L^{p_1})\cap LS(0,T)} \leq & C_T\big(|u_0|_{H^1} + \|f\|_{L^{q'_2}(0,T;L^{p'_2})+LS'(0,T)} \\ & + \|\partial_j f\|_{L^{q'_3}(0,T;L^{p'_3})+LS'(0,T)}\big),\end{aligned} \tag{5.6}$$

(iii). Assume in addition that $\kappa \leq \varepsilon \ll 1$. Then, we have the global-in-time Strichartz estimates, i.e., for any Strichartz pairs (p_k, q_k), $k = 1, 2$,

$$\|u\|_{L^{q_1}(\mathbb{R};L^{p_1})\cap LS} \leq C(|u_0|_2 + \|f\|_{L^{q'_2}(\mathbb{R};L^{p'_2})+LS'}). \tag{5.7}$$

Proof. (i). The proof is similar to that of [4, Lemma 4.1].

(ii). Estimate (5.6) can be proved similarly as in the proof of [5, Lemma 2.7]. In fact, for each $1 \leq j \leq d$, $v_j := \partial_j u$ satisfies

$$i\partial_t v_j = (\Delta + b\cdot\nabla + c)v_j + (\partial_j b\cdot\nabla + \partial_j c)u + \partial_j f \tag{5.8}$$

with $v_j(0) = \partial_j u_0$. Then, applying (5.5) to (5.8) and using

$$\|(b\cdot\nabla + c)u\|_{LS'} \leq C\kappa\|u\|_{LS} \tag{5.9}$$

(see [35, Proposition 2.3]; note that the space $\widetilde{X}$ in [35, Proposition 2.3] coincides with the local smoothing space LS defined above if $d \geq 3$) we obtain

$$
\begin{aligned}
&\|v_j\|_{L^{q_1}(0,T;L^{p_1})\cap LS(0,T)} \\
&\leq C_T(|\partial_j u_0|_2 + \|(\partial_j b\cdot\nabla + \partial_j c)u\|_{LS'(0,T)} + \|\partial_j f\|_{L^{q_3'}(0,T;L^{p_3'})+LS'(0,T)}) \\
&\leq C_T(|u_0|_{H^1} + \|u\|_{LS(0,T)} + \|\partial_j f\|_{L^{q_3'}(0,T;L^{p_3'})+LS'(0,T)}).
\end{aligned}
$$

Thus, applying again (5.5) yields immediately (5.6).

(iii). Since for the Laplacian $-\Delta$ in dimension $d \geq 3$ the associated bicharacteristic flow is not trapped and zero is not an eigenvalue or a resonance, we use [35, Theorem 1.22] and (5.9) to obtain that

$$
\begin{aligned}
\|u\|_{L^{q_1}(\mathbb{R};L^{p_1})\cap LS} \leq& C(|u_0|_2 + \|(b\cdot\nabla + c)u\|_{LS'} + \|f\|_{L^{q_2'}(\mathbb{R};L^{p_2'})+LS'}) \\
\leq& C(|u_0|_2 + \varepsilon\|u\|_{LS} + \|f\|_{L^{q_2'}(\mathbb{R};L^{p_2'})+LS'})
\end{aligned}
$$

Thus, taking ε small enough we prove (5.7). $\square$

Remark 5.2. *Using characteristic functions we see that, the estimates* (5.5)-(5.7) *and* (5.9) *are also valid on* (S,T) *for any* $-\infty \leq S < T \leq \infty$ *if the corresponding conditions hold on* (S,T).

Corollary 5.3. (i). *Consider* (5.1) *and assume that for any multi-index* $0 \leq |\beta| \leq 1$,

$$\sup_{(t,x)\in\mathbb{R}^+\times\mathbb{R}^d} \langle x\rangle^2 (|\partial_x^\beta b(t,x)| + |c(t,x)|) < \infty, \tag{5.10}$$

and

$$\limsup_{t\to\infty} \sup_{x\in\mathbb{R}^d} \langle x\rangle^2 (|\partial_x^\beta b(t,x)| + |c(t,x)|) \leq \varepsilon \ll 1, \tag{5.11}$$

$$\limsup_{|x|\to\infty} \sup_{0\leq t<\infty} \langle x\rangle^2 (|\partial_x^\beta b(t,x)| + |c(t,x)|) \leq \varepsilon \ll 1. \tag{5.12}$$

Then, for any $0 \leq S < T \leq \infty$ *and any two Strichartz pairs* (p_k, q_k), $k = 1, 2$, *we have the global-in-time Strichartz estimates*

$$\|u\|_{L^{q_1}(S,T;L^{p_1})\cap LS(S,T)} \leq C(|u(S)|_2 + \|f\|_{L^{q_2'}(S,T;L^{p_2'})+LS'(S,T)}), \tag{5.13}$$

where C *is independent of* S, T.

(ii). *Assume in addition that, for every* $1 \leq j \leq d$, $\partial_j b$ *and* $\partial_j c$ *also satisfy* (5.11) *and* (5.12). *Then, for any Strichartz pairs* (p_k, q_k), $1 \leq k \leq 3$,

$$
\begin{aligned}
\|\partial_j u\|_{L^{q_1}(S,T;L^{p_1})\cap LS(S,T)} \leq& C(|u(S)|_{H^1} + \|f\|_{L^{q_2'}(S,T;L^{p_2'})+LS'(S,T)} \\
&+ \|\partial_j f\|_{L^{q_3'}(S,T;L^{p_3'})+LS'(S,T)}),
\end{aligned} \tag{5.14}
$$

where C *is independent of* S, T.

(iii). *Assume the conditions in* (i) *(resp.* (ii)*) above to hold. Then,* (5.13) *(resp.* (5.14)*) also holds with* u_0 *replaced by the final datum* $u(T)$.

Proof. (i). It suffices to prove the assertion for $S = 0$ and $T = \infty$. By (5.12), for any $T_1 > 0$,

$$\sup_{(t,x)\in[0,T_1]\times\mathbb{R}^d} \langle x\rangle^2 (|\partial_x^\beta b(t,x)| + |c(t,x)|) \leq \kappa(T_1) < \infty, \tag{5.15}$$

$$\limsup_{|x|\to\infty} \sup_{t\in[0,T_1]} \langle x\rangle^2 (|\partial_x^\beta b(t,x)| + |c(t,x)|) \leq \varepsilon \ll 1, \tag{5.16}$$

which implies that (5.2)-(5.4) hold in the time range $[0,T_1]$. Thus, using Theorem 5.1 (i) (see also Remark 5.2) we obtain

$$\|u\|_{L^{q_1}(0,T_1;L^{p_1})\cap LS(0,T_1)} \leq C_{T_1}(|u_0|_2 + \|f\|_{L^{q_2'}(0,T_1;L^{p_2'})+LS'(0,T_1)}). \tag{5.17}$$

Moreover, for T_1 fixed and large enough, by (5.11) and (5.12),

$$\sup_{(t,x)\in[T_1,\infty)\times\mathbb{R}^d} \langle x\rangle^2 \left(|\partial_x^\beta b(t,x)| + |c(t,x)|\right) \leq 2\varepsilon \ll 1, \tag{5.18}$$

$$\limsup_{|x|\to\infty} \sup_{t\in[T_1,\infty)} \langle x\rangle^2 \left(|\partial_x^\beta b(t,x)| + |c(t,x)|\right) \leq 2\varepsilon \ll 1, \tag{5.19}$$

which implies that (5.2)-(5.4) hold on $[T_1,\infty)$. Thus, using Theorem 5.1 (iii) we get

$$\begin{aligned}\|u\|_{L^{q_1}(T_1,\infty;L^{p_1})\cap LS(T_1,\infty)} \leq& C(|u(T_1)|_2 + \|f\|_{L^{q_2'}(T_1,\infty;L^{p_2'})+LS'(T_1,\infty)}) \\ \leq& C_{T_1}(|u_0|_2 + \|f\|_{L^{q_2'}(0,\infty;L^{p_2'})+LS'(0,\infty)}),\end{aligned} \tag{5.20}$$

where in the last step we also used (5.17) with the Strichartz pair $(p_1,q_1)=(2,\infty)$, and C is independent of T_1. Combining (5.17) and (5.20) we obtain (5.13).

(ii). The argument is similar to that in the proof of Theorem 5.1 (ii).

(iii). It is sufficient to consider $S=0$. Let T_1 be as above and assume $T>T_1$ without loss of generality. Set $\widetilde{g}(t)=g(T-t)$, where $g=u,b,c,f$, $t\in[0,T]$. Then, by (5.1),

$$i\partial_t\widetilde{u} = (-1)(\Delta + \widetilde{b}(t)\cdot\nabla + \widetilde{c}(t))\widetilde{u} - \widetilde{f}.$$

Note that, (5.18) and (5.19) hold for $\widetilde{b}$ and $\widetilde{c}$ with $[T_1,\infty)$ replaced by $[0,T-T_1]$. Applying Theorem 5.1 (iii) on $[0,T-T_1]$, we get

$$\begin{aligned}\|\widetilde{u}\|_{L^{q_1}(0,T-T_1;L^{p_1})\cap LS(0,T-T_1)} \leq& C(|\widetilde{u}(0)|_2 + \|\widetilde{f}\|_{L^{q_2'}(0,T-T_1;L^{p_2'})+LS'(0,T-T_1)}) \\ \leq& C(|u(T)|_2 + \|f\|_{L^{q_2'}(T_1,T;L^{p_2'})+LS'(T_1,T)}),\end{aligned} \tag{5.21}$$

where C is independent of T.

Moreover, since (5.15) and (5.16) hold for $\widetilde{b},\widetilde{c}$ on $[T-T_1,T]$ replacing $[0,T_1]$, applying Theorem 5.1 (i) on $[T-T_1,T]$ we have

$$\begin{aligned}\|\widetilde{u}\|_{L^{q_1}(T-T_1,T;L^{p_1})\cap LS(T-T_1,T)} \leq& C_{T_1}(|\widetilde{u}(T-T_1)|_2 + \|\widetilde{f}\|_{L^{q_2'}(T-T_1,T;L^{p_2'})+LS'(T-T_1,T)}) \\ \leq& C_{T_1}(|u(T)|_2 + \|f\|_{L^{q_2'}(0,T;L^{p_2'})+LS'(0,T)})\end{aligned} \tag{5.22}$$

with C_{T_1} independent of T, where in the last step we used (5.21) with $(p_1,q_1)=(2,\infty)$.

Therefore, putting together (5.21) and (5.22) we prove (5.13) with u_0 replaced by $u(T)$. The proof for $\partial_j u$ is similar. □

In the remainder of this section we verify the global-in-time Strichartz and local smoothing estimates used in Sections 2–4.

First, consider the global-in-time Strichartz and local smoothing estimates for the operator $\widetilde{A}_*$ on $[0,1)$ in Section 2. We take $\partial_{jh}\widetilde{b}_*(t,\xi)$ for an example to verify the conditions (5.10)-(5.12) under Assumptions $(H0)$ and $(H1)$, $1\leq j,h\leq d$.

By (1.24) and (2.3),

$$\begin{aligned}\partial_{jh}\widetilde{b}_*(t,x) =& -2(1-t)^{-3}\sum_{k=1}^N \nabla\partial_{jh}\phi_k(\frac{x}{1-t})\int_{\frac{t}{1-t}}^\infty g_k(s)d\beta_k(s) \\ &+2(1-t)^{-3}\sum_{k=1}^N \nabla\partial_{jh}((\mathrm{Re}\,\phi_k)\phi_k)(\frac{x}{1-t})\int_{\frac{t}{1-t}}^\infty g_k^2(s)ds.\end{aligned}$$

Then, in view of the asymptotic flatness condition (1.8), we obtain that

$$\begin{aligned}\langle x\rangle^2 |\partial_{jh}\widetilde{b}_*(t,x)| \leq & 2\sum_{k=1}^N \langle x\rangle^2 \left(\left|\nabla\partial_{jh}\phi_k(\frac{x}{1-t})\right| + |\nabla\partial_{jh}((\operatorname{Re}\phi_k)\phi_k)(\frac{x}{1-t})\right|\Big) \\ & \cdot (1-t)^{-3}\left(\left|\int_{\frac{t}{1-t}}^{\infty} g_k(s)d\beta_k(s)\right| + \int_{\frac{t}{1-t}}^{\infty} g_k^2(s)ds \right) \\ \leq & 2\sum_{k=1}^N \widetilde{\varepsilon}_k(\frac{x}{1-t})\widetilde{r}_k(t), \end{aligned} \tag{5.23}$$

where $\widetilde{\varepsilon}_k(x) \to 0$ as $|x| \to \infty$, and $\widetilde{r}_k(t) := (1-t)^{-3}(|\int_{\frac{t}{1-t}}^{\infty} g_k(s)d\beta_k(s)| + \int_{\frac{t}{1-t}}^{\infty} g_k^2(s)ds)$. Note that

$$\sup_{x\in\mathbb{R}^d}\sup_{0\leq t<1} \widetilde{\varepsilon}_k(\frac{x}{1-t}) < \infty, \quad \lim_{|x|\to\infty}\sup_{0\leq t<1} \widetilde{\varepsilon}_k(\frac{x}{1-t}) = 0. \tag{5.24}$$

Moreover, by the law of the iterated logarithm,

$$\left|\int_{\frac{t}{1-t}}^{\infty} g_k(s)d\beta_k(s)\right| \leq \left(2\int_{\frac{t}{1-t}}^{\infty} g_k^2(s)ds \ln\ln\left(\int_{\frac{t}{1-t}}^{\infty} g_k^2(s)ds\right)^{-1}\right)^{\frac{1}{2}}, \quad a.s.,$$

which along with (1.9) implies that

$$\sup_{0\leq t<1} \widetilde{r}_k(t) < \infty, \quad \limsup_{t\nearrow 1} \widetilde{r}_k(t) = 0. \tag{5.25}$$

Thus, putting together (5.23)-(5.25) yields that the conditions (5.10)-(5.12) hold for $\partial_{jh}\widetilde{b}_*$ with ∞ and ε replaced by 1 and 0, respectively.

Similar arguments also apply to $\partial_x^\gamma\widetilde{b}_*$ and $\partial_x^\gamma\widetilde{c}_*$, $0 \leq |\gamma| \leq 1$, which, via Corollary 5.3 (see also Remark 5.2), imply global-in-time Strichartz and local smoothing estimates for $\widetilde{A}_*$ on $[0,1)$.

Next, we check that global-in-time Strichartz and local smoothing estimates for the operator A_* in (1.23) and (1.28) under the condition (1.3) and that $g_k \in L^2(\mathbb{R}^+)$, a.s., $1 \leq k \leq N$.

We illustrate this for b_* only. For any $0 \leq |\beta| \leq 2$, by (1.24),

$$\begin{aligned}\partial_x^\beta b_*(t,x) = & (-2)\sum_{k=1}^N \nabla\partial_x^\beta\phi_k(x)\int_t^\infty g_k(s)d\beta_k(s) \\ & + 2\sum_{k=1}^N \nabla\partial_x^\beta((\operatorname{Re}\phi_k)\phi_k)(x)\int_t^\infty g_k^2(s)ds,\end{aligned}$$

which implies that

$$\begin{aligned}|\langle x\rangle^2\partial_x^\beta b_*(t,x)| \leq & C\sum_{k=1}^N \langle x\rangle^2 (|\nabla\partial_x^\beta\phi_k(x)| + |\nabla\partial_x^\beta((\operatorname{Re}\phi_k)\phi_k)(x)|) \\ & \cdot \left(\left|\int_t^\infty g_k(s)d\beta_k(s)\right| + \int_t^\infty g_k^2(s)ds \right) \\ =: & C\sum_{k=1}^N \varepsilon_k(x)r_k(t).\end{aligned} \tag{5.26}$$

Note that, (1.3) implies that

$$\sup_{x\in\mathbb{R}^d} \varepsilon_k(x) < \infty, \quad \lim_{|x|\to\infty} \varepsilon_k(x) = 0. \tag{5.27}$$

Moreover, since $g_k \in L^2(\mathbb{R}^+)$, a.s., we have that $|\int_0^\infty g_k(s)d\beta_k(s)| < \infty$, a.s., and by the law of iterated logarithm, as $t\to\infty$,

$$\left|\int_t^\infty g_k(s)d\beta_k(s)\right| \leq \left(2\int_t^\infty g_k^2 ds \ln\ln(\int_t^\infty g_k^2 ds)\right)^{\frac{1}{2}} \to 0, \quad a.s..$$

Hence,

$$\sup_{0\leq t<\infty} r_k(t) < \infty, \quad \limsup_{t\to\infty} r_k(t) = 0. \tag{5.28}$$

Thus, we conclude from (5.26)-(5.28) that $\partial_x^\beta b_*$, $0\leq|\beta|\leq 2$, satisfy the conditions (5.10)-(5.12) with $\varepsilon = 0$. Similar arguments also apply to $\partial_x^\gamma c_*$, $0\leq|\gamma|\leq 1$. Thus, global-in-time Strichartz and local smoothing estimates for A_* follow from Corollary 5.3.

6. Appendix

6.1. **Proof of Theorem 1.2.** The proof is similar to that in [4, Theorem 2.2] and [5, Theorem 1.2].

First note that, under Assumptions $(H0)$, local-in-time Strichartz and local smoothing estimates hold for the operator $A(t)$ in (1.19). In fact, by Assumption $(H0)$ and the Burkholder-Davis-Gundy inequality, for any $0<T<\infty$,

$$\mathbb{E}\sup_{t\in[0,T]}\left|\int_0^t g_k d\beta_k(s)\right| \leq C\mathbb{E}\left(\int_0^T g_k^2 ds\right)^{\frac{1}{2}} \leq C\|g_k\|_{L^\infty(\Omega\times(0,T))}T^{\frac{1}{2}} < \infty,$$

which implies that

$$C_T := \sup_{t\in[0,T]}\left|\int_0^t g_k(s)d\beta_k(s)\right| + \int_0^T g_k^2(s)ds < \infty, \quad \mathbb{P}-a.s..$$

Moreover, by (1.20), for $0\leq|\beta|\leq 2$,

$$\begin{aligned}|\langle x\rangle^2 \partial_x^\beta b(t,x)| \leq & C\sum_{k=1}^N \langle x\rangle^2 \left(|\partial_x^\beta \nabla\phi_k(x)| + |\partial_x^\beta\nabla((\mathrm{Re}\,\phi_k)\phi_k)(x)|\right) \\ & \cdot\left(\left|\int_0^t g_k(s)d\beta_k(s)\right| + \int_0^t g_k^2(s)ds\right).\end{aligned}$$

Thus, by (1.3), we see that $\partial_x^\beta b$, $0\leq|\beta|\leq 2$, satisfy the conditions (5.2)–(5.4) on $[0,T]$ with b and div b replaced by $\partial_x^\beta b$. The argument for the coefficients $\partial_x^\gamma c$ is similar, $0\leq|\gamma|\leq 1$. Therefore, by Theorem 5.1 (ii), local-in-time Strichartz and local smoothing estimates hold for $A(t)$ in (1.19).

Now, the local well-posedness for (1.1) follows from similar arguments as in the proof of [5, Proposition 2.5]. As regards the global well-posedness, applying Itô's formula we have $\mathbb{P}$-a.s. for any $t\in[0,\tau^*)$, where τ^* is the maximal existing time, that

$$|X(t)|_2^2 = |X_0|_2^2 + \sum_{k=1}^N \int_0^t \sigma_{0,k}(s)d\beta_k(s), \tag{6.1}$$

and for the Hamiltonian $H(X) := \frac{1}{2}|\nabla X|_2^2 - \frac{\lambda}{\alpha+1}|X|_{L^{\alpha+1}}^{\alpha+1}$,

$$H(X(t)) = H(X_0) + \int_0^t a_1(s)ds + \sum_{k=1}^N \int_0^t \sigma_{1,k}(s)d\beta_k(s), \tag{6.2}$$

where

$$\begin{aligned} a_1(s) =& - \operatorname{Re} \int \nabla \overline{X}(s)\nabla(\mu(s)X(s))dx + \frac{1}{2}\int |\nabla(G_k(s)X(s))|^2 dx \\ & - \frac{\lambda(\alpha-1)}{2}\int (\operatorname{Re} G_k(s))^2|X(s)|^{\alpha+1}dx, \\ \sigma_{0,k}(s) =& 2\int \operatorname{Re} G_k(s)|X(s)|^2 dx, \\ \sigma_{1,k}(s) =& \operatorname{Re} \int \nabla \overline{X}(s)\nabla(G_k(s)X(s))dx - \lambda \int \operatorname{Re} G_k(s)|X(s)|^{\alpha+1}dx. \end{aligned}$$

Thus, similar arguments as in the proof of [4, Lemma 3.1] and [5, Theorem 3.1] imply that $\sup_{[0,\tau^*)} |X(t)|_{H^1}^2 < \infty$, a.s., thereby yielding that X exists globally.

Moreover, if $X_0 \in \Sigma$, we have that for the virial function $V(X(t)) := \int |x|^2|X(t,x)|^2dx$,

$$V(X(t)) = V(X_0) + 4\int_0^t G(X(s))ds + \sum_{k=1}^N \int_0^t \sigma_{2,k}(s)d\beta_k(s), \tag{6.3}$$

where $G(X) = \operatorname{Im} \int x\cdot\nabla\overline{X}Xdx$ and $\sigma_{2,k}(s) := 2\int \operatorname{Re} G_k(s)|x|^2|X(s)|^2dy$. Then, arguing as in the proof of [7, (4.6)] we obtain $X \in L^2(\Omega; C([0,T];\Sigma))$.

The proof for the estimate (1.6) in weighted spaces is similar to that of Lemma 2.1. Actually, consider the Strichartz pair (p,q) and $1 < \theta < \infty$ as in the proof of Lemma 2.1. We only need to prove that, for $z = e^{-\varphi}X$ and for each $1 \le j \le d$,

$$\|x_j z\|_{L^q(0,T;L^p)} < \infty \quad a.s.. \tag{6.4}$$

Since $\mathbb{P}$-a.s. $X \in L^q(0,T;W^{1,p})$, $\varphi \in C([0,T];W^{1,\infty})$, and $b \in C([0,T];L^\infty)$, we take a finite partition $\{(t_k,t_{k+1})\}$ of $[0,T]$, such that $\|z\|_{L^q(t_k,t_{k+1};W^{1,p})} \le \varepsilon$. Then, similar to (2.13),

$$\|x_j z\|_{L^q(t_k,t_{k+1};L^p)} \le C_T\|z\|_{C([0,T];\Sigma)} + C_T\varepsilon^{\alpha-1}\|x_j z\|_{L^q(t_k,t_{k+1};L^p)},$$

which implies (6.4) by taking ε sufficiently small and summing over k.

Regarding (1.7), we set $\mathcal{H}(X) := \frac{1}{2}|X|_{H^1}^2 - \frac{\lambda}{\alpha+1}|X|_{L^{\alpha+1}}^{\alpha+1} = H(X) + \frac{1}{2}|X|_2^2$. Applying Itô's formula and using (6.1) and (6.2) we get that for any $m \ge 2$,

$$\begin{aligned} (\mathcal{H}(X(t)))^m =& (\mathcal{H}(X(0)))^m + m\int_0^t (\mathcal{H}(X(r)))^{m-1}a_1(X(r))dr \\ & + \frac{1}{2}m(m-1)\sum_{k=1}^N \int_0^t (\mathcal{H}(X(r)))^{m-2}(\frac{1}{2}\sigma_{0,k}(r) + \sigma_{1,k}(r))^2dr + \mathcal{M}(t) \\ \le& C(m)|X_0|_{H^1}^m + C(m)\sum_{k=1}^N \int_0^t (\mathcal{H}(X(r)))^m g_k^2(r)dr + \mathcal{M}(t), \end{aligned}$$

where $\mathcal{M}(t) = \sum_{k=1}^N \int_0^t m(\mathcal{H}(X(r)))^{m-1}(\frac{1}{2}\sigma_{0,k}(r) + \sigma_{1,k}(r))d\beta_k(r)$.

Then, apply the stochastic Gronwall inequality in [40, Lemma 2.2] we obtain that for any $0 < q < p < 1$,

$$\mathbb{E} \sup_{0\leq s\leq t} (\mathcal{H}(X(s)))^{mq} \leq C(m)|X_0|_{H^1}^{mq}(\frac{p}{p-q}) \left(\mathbb{E}e^{C(m)\frac{p}{1-p}\sum_{k=1}^{N}\int_0^t g_k^2(s)ds} \right)^{\frac{1-p}{p}q}.$$

Then, taking $q = p/2 \in (0,1)$, using Fatou's lemma we obtain that for any $m \geq 2$ and $p \in (0,1)$,

$$\mathbb{E} \sup_{0\leq t<\infty} (\mathcal{H}(X(t)))^{\frac{1}{2}mp} \leq 2C(m)|X_0|_{H^1}^{\frac{1}{2}mp} \left(\mathbb{E}e^{C(m)\frac{p}{1-p}\sum_{k=1}^{N}\int_0^\infty g_k^2(t)dt} \right)^{\frac{1}{2}}, \tag{6.5}$$

which implies (1.7) because m and q can be chosen arbitrarily.

Therefore, the proof is complete. □

6.2. **Proof of Proposition 2.5.** The proof is similar to that of Corollary 2.4 based on the pseudo-conformal energy and the global bound (2.22) of $\widetilde{z}_*$.

In fact, in the case where $\alpha \in [1+4/d, 1+4/(d-2)]$, similarly to (2.23),

$$\begin{aligned}\widetilde{E}_1(\widetilde{u}(t)) =& \widetilde{E}_1(\widetilde{u}(\widetilde{T})) + \frac{16}{\alpha+1}(1-\frac{d}{4}(\alpha-1)) \int_{\widetilde{T}}^t (1-r)^{\frac{d}{2}(\alpha-1)-3}|\widetilde{X}(r)|_{L^{\alpha+1}}^{\alpha+1}dr \\ \leq& \widetilde{E}_1(z_*(\widetilde{T})) \leq C(1+\|\widetilde{z}_*\|_{C([0,1);H^1)}^{\alpha+1}) < \infty, \quad t \in [\widetilde{T}, 1),\ a.s.,\end{aligned}$$

where the last step is due to the uniform bound (2.22) and C is independent of $\widetilde{T}$. This yields that for some positive constant C independent of $\widetilde{T}$,

$$\|\widetilde{u}\|_{C([\widetilde{T},1);H^1)}^2 \leq C(1+\|\widetilde{z}_*\|_{C([0,1);H^1)}^{\alpha+1}), \quad a.s..$$

Moreover, in the case where $\alpha \in (1+\alpha(d), 1+4/d)$, similarly to (2.28), for any $t \in [\widetilde{T}, 1)$,

$$\begin{aligned}\widetilde{E}_2(\widetilde{u}(t)) =& \widetilde{E}_2(\widetilde{u}(\widetilde{T})) - 8(1-\frac{d}{4}(\alpha-1)) \int_{\widetilde{T}}^t (1-r)^{1-\frac{d}{2}(\alpha-1)}|\nabla\widetilde{X}(r)|_2^2dr \\ \leq& \widetilde{E}_2(\widetilde{z}_*(\widetilde{T})) \leq C(1+\|\widetilde{z}_*\|_{C([0,1);H^1)}^{\alpha+1}) < \infty,\ a.s.,\end{aligned}$$

which implies that for some positive constant C independent of $\widetilde{T}$,

$$\sup_{\widetilde{T}\leq t<1} |\widetilde{u}(r)|_{L^{\alpha+1}}^{\alpha+1} \leq C(1+\|\widetilde{z}_*\|_{C([0,1);H^1)}^{\alpha+1}), \quad a.s..$$

Thus, using the arguments as those below (2.24) and (2.29) we obtain (2.25) with $\widetilde{u}$ replacing $\widetilde{z}_*$. Precisely, for $\alpha \in (1+\alpha(d), 1+4/(d-2)]$ and any Strichartz pair (ρ,γ),

$$\|\widetilde{u}\|_{L^\gamma(\widetilde{T},1;W^{1,\rho})} + \||\cdot|\widetilde{u}\|_{L^\gamma(\widetilde{T},1;L^\rho)} \leq C(|\widetilde{u}(\widetilde{T})|_\Sigma + \|\widetilde{u}\|_{C([0,1);H^1)}) \leq C < \infty,$$

where C depends on $\|\widetilde{z}_*\|_{C([0,1);\Sigma)}$ and is independent of $\widetilde{T}$. This also yields uniform bound for the $LS(\widetilde{T},1)$-norms by Strichartz estimates.

Finally, using the uniform bound (2.32) and the local well-posedness arguments as in [5] we can extend the solution $\widetilde{u}$ to time 1. In particular, $\|\widetilde{u}\|_{C([0,1];\Sigma)} < \infty$, a.s..

Therefore, the proof of Proposition 2.5 is complete. □

Acknowledgements. D. Zhang is supported by NSFC (No. 11501362). Financial support by the DFG through CRC 1283 is gratefully acknowledged.

References

[1] O. Bang, P.L. Christiansen, F. If, K.O. Rasmussen, Temperature effects in a nonlinear model of monolayer Scheibe aggregates. Phys. Rev. E 49 (1994), 4627-4636.

[2] O. Bang, P.L. Christiansen, F. If, K.O. Rasmussen, Y.B. Gaididei, White noise in the two-dimensional nonlinear Schrödinger equation, Appl. Anal. 57 (1995), no. 1-2, 3-15.

[3] V. Barbu, M. Röckner, An operatorial approach to stochastic partial differential equations driven by linear multiplicative noise, *J. Eur. Math. Soc. (JEMS)* **17** (2015), no. 7, 1789–1815.

[4] V. Barbu, M. Röckner, D. Zhang, The stochastic nonlinear Schrödinger equations with multiplicative noise: the rescaling approach, *J. Nonlinear Sci.* **24** (2014), no. 3, 383–409.

[5] V. Barbu, M. Röckner, D. Zhang, Stochastic nonlinear Schrödinger equations. *Nonlinear Anal.* **136** (2016), 168–194.

[6] V. Barbu, M. Röckner, D. Zhang, The stochastic logarithmic Schrödinger equation. *J. Math. Pures Appl.* **107** (2017), no. 2, 123–149.

[7] V. Barbu, M. Röckner, D. Zhang, Stochastic nonlinear Schrödinger equations: No blow-up in the non-conservative case. *J. Differential Equations* **263** (2017), no. 11, 7919–7940.

[8] V. Barbu, M. Röckner, D. Zhang, Optimal bilinear control of nonlinear stochastic Schrödinger equations driven by linear multiplicative noise. *Ann. Probab.* (to appear)

[9] A. Barchielli, M. Gregoratti, Quantum Trajectories and Measurements in Continuous Case. The Diffusive Case, *Lecture Notes Physics* **782**, Springer Verlag, Berlin, 2009.

[10] Z. Brzézniak, A. Millet, On the stochastic Strichartz estimates and the stochastic nonlinear Schrödinger equation on a compact Riemannian manifold. *Potential Anal.* **41** (2014), no. 2, 269–315.

[11] Z. Brzézniak, F. Hornung, L. Weis, Martingale solutions for the stochastic nonlinear Schrödinger equation in the energy space. arXiv:1707.05610v1

[12] E.A. Carlen, Conservative diffusions. Comm. Math. Phys. 94 (1984), no. 3, 293-315.

[13] E.A. Carlen, Potential scattering in stochastic mechanics. Ann. Inst. H. Poincaré Phys. Théor. 42 (1985), no. 4, 407-428.

[14] T. Cazenave, Semilinear Schrödinger equations, *Courant Lecture Notes in Mathematics* **10**, New York University, Courant Institute of Mathematical Sciences, New York, 2003.

[15] T. Cazenave, F.B. Weissler, Rapidly decaying solutions of the nonlinear Schrödinger equation. *Comm. Math. Phys.* **147** (1992), no. 1, 75–100.

[16] K. Chouk, M. Gubinelli, Nonlinear PDEs with modulated dispersion I: Nonlinear Schödinger equations. *Comm. Partial Differential Equations* **40** (2015), no. 11, 2047–2081.

[17] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, T. Tao, Global well-posedness and scattering for the energy-critical nonlinear Schrödinger equation in $\mathbb{R}^3$. *Ann. of Math. (2)* **167** (2008), no. 3, 767–865.

[18] A. de Bouard, A. Debussche, A stochastic nonlinear Schrödinger equation with multiplicative noise. *Comm. Math. Phys.* **205** (1999), 161–181.

[19] A. de Bouard, A. Debussche, The stochastic nonlinear Schrödinger equation in H^1. *Stoch. Anal. Appl.* **21** (2003), 97–126.

[20] A. de Bouard, A. Debussche, Blow-up for the stochastic nonlinear Schrödinger equation with multiplicative noise. *Ann. Probab.* **33** (2005), no. 3, 1078–1110.

[21] A. de Bouard, A. Debussche, The nonlinear Schrödinger equation with white noise dispersion. *J. Funct. Anal.* **259** (2010), no. 5, 13000–1321.

[22] A. de Bouard, R. Fukuizumi, Representation formula for stochastic Schrödinger evolution equations and applications. *Nonlinearity* **25** (2012), no. 11, 2993–3022.

[23] G. Da Prato, F. Flandoli, M. Röckner, A.Y. Veretennikov, Strong uniqueness for SDEs in Hilbert spaces with nonregular drift. *Ann. Probab.* **44** (2016), no. 3, 1985–2023.

[24] A. Debussche, Y. Tsutsumi, 1D quintic nonlinear Schrödinger equation with white noise dispersion. *J. Math. Pures Appl.* **96** (2011), no. 4, 363–376.

[25] B. Dodson, Global well-posedness and scattering for the defocusing, L^2-critical nonlinear Schrödinger equation when $d \geq 3$. *J. Amer. Math. Soc.* **25** (2012), no. 2, 429–463.

[26] D.Y. Fang, J. Xie, T. Cazenave, Scattering for the focusing energy-subcritical nonlinear Schrödinger equation. *Sci. China Math.* **54** (2011), no. 10, 2037–2062.

[27] F. Flandoli, M. Gubinelli, E. Priola, Well-posedness of the transport equation by stochastic perturbation. *Invent. Math.* **180** (2010), no. 1, 1–53.

[28] J. Ginibre, G. Velo, Scattering theory in the energy space for a class of nonlinear Schrödinger equations. *J. Math. Pures Appl.* **64** (1985), no. 4, 363–401.
[29] J. Ginibre, G. Velo, Time decay of finite energy solutions of the nonlinear Klein-Gordon and Schrödinger equations. *Ann. Inst. H. Poincaré Phys. Théor.*.**43** (1985), no. 4, 399–442.
[30] M. Goldberg, L. Vega, N. Visciglia, Counterexamples of Strichartz inequalities for Schrödinger equations with repulsive potentials. *Int. Math. Res. Not.* **2006**, Art. ID 13927, 16 pp.
[31] F. Hornung, The nonlinear stochastic Schrödinger equation via stochastic Strichartz estimates. `arXiv:1611.07325`.
[32] I. Karatzas, S.E. Shreve, Brownian motion and stochastic calculus. Second edition. Graduate Texts in Mathematics, 113. Springer-Verlag, New York, 1991. xxiv+470 pp.
[33] C.E. Kenig, F. Merle, Global well-posedness, scattering and blow-up for the energy-critical, focusing, non-linear Schrödinger equation in the radial case. *Invent. Math.* **166** (2006), no. 3, 645–675.
[34] R. Killip, M. Visan, The focusing energy-critical nonlinear Schrödinger equation in dimensions five and higher. *Amer. J. Math.* **132** (2010), no. 2, 361–424.
[35] J. Marzuola, J. Metcalfe, D. Tataru, Strichartz estimates and local smoothing estimates for asymptotically flat Schrödinger equations. *J. Funct. Anal.* **255** (2008), no. 6, 1479–1553.
[36] S. Masaki, A sharp scattering condition for focusing mass-subcritical nonlinear Schrödinger equation. *Commun. Pure Appl. Anal.* **14** (2015), no. 4, 1481–1531.
[37] S. Masaki, On minimal nonscattering solution for focusing mass-subcritical nonlinear Schrödinger equation. *Comm. Partial Differential Equations* **42** (2017), no. 4, 626–653.
[38] E. Nelson, Quantum fluctuations. Princeton Series in Physics. Princeton University Press, Princeton, NJ, 1985. viii+147 pp.
[39] T. Tao, M. Visan, X.Y. Zhang, The nonlinear Schrödinger equation with combined power-type nonlinearities. *Comm. Partial Differential Equations* **32** (2007), no. 7-9, 1281-1343.
[40] M. Röckner, X.C. Zhang, Well-posedness of distribution dependent SDEs with singular drifts, arXiv: 1809.02216v1.
[41] E. Ryckman, M. Visan, Global well-posedness and scattering for the defocusing energy-critical nonlinear Schrödinger equation in $\mathbb{R}^{1+4}$. *Amer. J. Math.* **129** (2007), no. 1, 1–60.
[42] T. Saanouni, Remarks on damped fractional Schrödinger equation with pure power nonlinearity. *J. Math. Phys.* **56** (2015), no. 6, 061502, 14 pp.
[43] D.S. Shucker, Stochastic mechanics of systems with zero potential. J. Funct. Anal. 38 (1980), no. 2, 146-155.
[44] C. Sulem, P.L. Sulem, The nonlinear Schrödinger equation: self-focusing and wave collapse. *Applied Mathematical Sciences* **139**, Springer, New York, 1999.
[45] Y. Tsutsumi, Scattering problem for nonlinear Schrödinger equations. *Ann. Inst. H. Poincaré Phys. Théor.* **43** (1985), no. 3, 321–347.
[46] M. Visan, The defocusing energy-critical nonlinear Schrödinger equation in higher dimensions. *Duke Math. J.* **138** (2007), no. 2, 281–374.
[47] D. Zhang, Strichartz and local smoothing estimates for stochastic dispersive equations. `arXiv:1709.03812`

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