A remark on global solutions to random 3D vorticity equations for small initial data*

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Abstract

In this paper, we prove that the solution constructed in [2] satisfies the stochastic vorticity equations with the stochastic integration being understood in the sense of the integration of controlled rough path introduced in [8]. As a result, we obtain the existence and uniqueness of the global solutions to the stochastic vorticity equations in 3D case for the small initial data independent of time, which can be viewed as a stochastic version of the Kato-Fujita result (see [10]).

Keywords: stochastic vorticity equations; controlled rough path, small initial data

1 Introduction

Consider the stochastic 3D Navier-Stokes equation on $(0,\infty)\times\mathbb{R}^3$:

$$(1.1)\qquad \begin{aligned} &dX-\Delta Xdt+(X\cdot\nabla)Xdt=\sum_{i=1}^N(B_i(X)+\lambda_iX)d\beta^i(t)+\nabla\pi dt,\\ &\nabla\cdot X=0,\\ &X(0)=x,\end{aligned}$$

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where $\{\beta^i\}_{i=1}^N$ is a system of independent Brownian motions on a probability space $(\Omega, \mathcal{F}, P)$ with normal filtration $(\mathcal{F}_t)_{t\geq 0}$, and $\lambda_i \in \mathbb{R}, x : \Omega \to L^2(\mathbb{R}^3; \mathbb{R}^3)$ is a random variable. Here π denotes the pressure, Δ is the Laplacian on $L^2(\mathbb{R}^3; \mathbb{R}^3)$ and B_i are convolution operators given by

$$B_i(X)(\xi) = \int_{\mathbb{R}^3} h_i(\xi - \bar{\xi}) X(\bar{\xi}) d\bar{\xi} = (h_i * X)(\xi), \quad \xi \in \mathbb{R}^3,$$

where $h_i \in L^1(\mathbb{R}^3), i = 1, ..., N$.

Consider the vorticity field

$$U = \nabla \times X = \mathrm{curl} X$$

and apply the curl operator to equation (1.1). We obtain the transport vorticity equation on $(0, \infty) \times \mathbb{R}^3$:

$$\begin{aligned} &dU - \Delta U dt + ((X \cdot \nabla)U - (U \cdot \nabla)X)dt = \sum_{i=1}^N (h_i * U + \lambda_i U) d\beta^i(t), \\ &U_0(\xi) = (\mathrm{curl} x)(\xi), \quad \xi \in \mathbb{R}^3. \end{aligned} \tag{1.2}$$

The vorticity U is related to the velocity X by the Biot-Savart integral operator (see [4])

$$X_t(\xi) = K(U_t)(\xi) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{\xi - \bar{\xi}}{|\xi - \bar{\xi}|^3} \times U_t(\bar{\xi}) d\bar{\xi}, \quad t \in (0, \infty), \xi \in \mathbb{R}^3. \tag{1.3}$$

Then one can rewrite the vorticity equation (1.2) as

$$\begin{aligned} &dU - \Delta U dt + ((K(U) \cdot \nabla)U - (U \cdot \nabla)K(U))dt = \sum_{i=1}^N (h_i * U + \lambda_i U) d\beta_t^i, \\ &U_0(\xi) = (\mathrm{curl} x)(\xi), \quad \xi \in \mathbb{R}^3. \end{aligned} \tag{1.4}$$

In [2] using the transformation

$$U_t = \Gamma_t y_t$$

with

$$\Gamma_t = \Pi_{i=1}^N \exp\left(\beta_t^i \tilde{B}_i - \frac{t}{2} \tilde{B}_i^2\right), \quad \tilde{B}_i = B_i + \lambda_i I,$$

the authors transformed (1.4) into the following equation

$$\begin{aligned} &\frac{dy}{dt} - \Gamma_t^{-1} \Delta(\Gamma_t y_t) dt + \Gamma_t^{-1}((K(\Gamma_t y_t) \cdot \nabla)(\Gamma_t y_t) - (\Gamma_t y_t \cdot \nabla)K(\Gamma_t y_t)) = 0, \\ &y_0 = U_0. \end{aligned} \tag{1.5}$$

In [2] the authors proved that if the initial value is small enough (compared to a function depending on the paths of Brownian motions β_i), then there exists a unique solution y_t (in the mild sense) to (1.5). However, since the initial value satisfying the following condition (1.7) is not $\mathcal{F}_0$-measurable, the process y_t is not $(\mathcal{F}_t)_{t\geq 0}$-adapted. Therefore, the solution to (1.5) cannot be transformed back into (1.4).

The main aim of this paper is to obtain the stochastic version of the result of Kato-Fujita to (1.4). Let y be the solution to (1.5) obtained in [2] and define $U_t =: \Gamma_t y_t$. Since y_t is not $(\mathcal{F}_t)_{t\geq 0}$-adapted, the corresponding U_t is also not $(\mathcal{F}_t)_{t\geq 0}$-adapted. Therefore, the stochastic integral should be understood in the sense of a rough path integral or the Skorohod integral. To use the Skorohod integral and find a solution to (1.4) we have to use the shift operator (see [3], [12]), which breaks the result that there exists some $C(\omega)$ independent of time such that if $|U_0|_{3/2} \leq C(\omega)$, there exists a global solution to (1.5). Thus in this paper we understand the stochastic integral of (1.4) in the sense of a rough path integral.

Framework and main result

First we recall the main result in [2]. In the following we denote by $L^p, 1 \leq p \leq \infty$ the space $L^p(\mathbb{R}^3;\mathbb{R}^3)$ with norm $|\cdot|_p$ and by $C_b([0,\infty);L^p)$ the space of all bounded and continuous functions $u:[0,\infty)\to L^p$ with the sup-norm. We also set $D_i = \frac{\partial}{\partial \xi_i}, i = 1,2,3$. We set for $p \in (\frac{3}{2},3), q \in (1,\infty)$

$$\eta_t = \|\Gamma_t\|_{L(L^p,L^p)}\|\Gamma_t\|_{L(L^{\frac{3p}{3-p}},L^{\frac{3p}{3-p}})}\|\Gamma_t^{-1}\|_{L(L^q,L^q)}, \quad t\geq 0,$$

where $\|\cdot\|_{L(L^p,L^p)}$ is the norm of the space $L(L^p,L^p)$ of linear continuous operators on L^p.

For $p\in[1,\infty)$ we denote by $\mathcal{Z}_p$ the space of all functions $y:[0,\infty)\times\mathbb{R}^3\to\mathbb{R}^3$ such that

$$t^{1-\frac{3}{2p}}y_t \in C_b([0,\infty);L^p),$$
$$t^{\frac{3}{2}(1-\frac{1}{p})}D_i y_t \in C_b([0,\infty);L^p), \quad i=1,2,3.$$

The space $\mathcal{Z}_p$ is endowed with the norm

$$\|y\| = \sup\{t^{1-\frac{3}{2p}}|y_t|_p + t^{\frac{3}{2}(1-\frac{1}{p})}|D_i y_t|_p; t\in(0,\infty), i=1,2,3\}.$$

In the following we take $\lambda_i\in\mathbb{R}$ such that

$$|\lambda_i| > (\sqrt{12}+3)|h_i|_1, \quad i=1,2,...,N.$$

Consider the equation (1.5) in the following mild sense:

$$y_t = e^{t\Delta}U_0 + \int_0^t e^{(t-s)\Delta}\Gamma_s^{-1}M(\Gamma_s y_s)ds, \quad t\in(0,\infty), \tag{1.6}$$

where

$$M(u) = -(K(u) \cdot \nabla)(u) + (u \cdot \nabla)K(u).$$

The following is the main result in [2].

Theorem 1.1. *Let $p, q \in (1, \infty)$ such that*

$$\frac{3}{2} < p < 2, \frac{1}{q} = \frac{2}{p} - \frac{1}{3}.$$

Let $\Omega_0 = \{\sup_{t \geq 0} \eta_t < \infty\}$ and consider (1.6) for fixed $\omega \in \Omega_0$. Then $P(\Omega_0) = 1$ and there exists a positive constant C^* independent of $\omega \in \Omega_0$ such that, if $U_0 \in L^{3/2}$ satisfying

$$\sup_{t \geq 0} \eta_t |U_0|_{3/2} \leq C^*, \tag{1.7}$$

then there exists a unique solution $y \in \mathcal{Z}_p$ to (1.6). Moreover, for each $\varphi \in L^3 \cap L^{\frac{q}{q-1}}$, the function $t \to \int_{\mathbb{R}^3} y_t(\xi)\varphi(\xi)d\xi$ is continuous on $[0, \infty)$.

To formulate our first main result we introduce the following notations and definitions from rough paths theory: Fix $\frac{1}{3} < \alpha < \frac{1}{2}, 0 \leq s < t$, for $X \in C([s,t], \mathbb{R}^N)$ we define

$$\delta X_{uv} := X_v - X_u, \quad \|X\|_{\alpha,[s,t]} := \sup_{u,v \in [s,t], u \neq v} \frac{|\delta X_{uv}|}{|u-v|^\alpha}.$$

Moreover, for a tensor process $\mathbb{X} \in C([s,t]^2, \mathbb{R}^{N \times N})$ we define

$$\|\mathbb{X}\|_{2\alpha,[s,t]} := \sup_{u,v \in [s,t], u \neq v} \frac{|\mathbb{X}_{uv}|}{|u-v|^{2\alpha}}.$$

In fact, $(X, \mathbb{X})$ is an α-Hölder rough path in the sense of [7], Def.2.1 if $\|X\|_{\alpha,[s,t]} < \infty, \|\mathbb{X}\|_{2\alpha,[s,t]} < \infty$ and the following holds for every triple of times (u, v, w)

$$\mathbb{X}_{uv} - \mathbb{X}_{uw} - \mathbb{X}_{wv} = \delta X_{uw} \otimes \delta X_{wv}.$$

For an N-dimensional Brownian motion β on the probability space $(\Omega, \mathcal{F}, P)$ and $\mathbb{B}_{uv} := \int_u^v \delta\beta_{ur} \otimes d\beta_r \in \mathbb{R}^{N \times N}$, it is well known that there exists a set Ω_1 with $P(\Omega_1) = 1$ such that for $\omega \in \Omega_1$ $(\beta(\omega), \mathbb{B}(\omega))$ is an α-Hölder rough path (see [7], Prop. 3.4), where the stochastic integration is understood in the sense of Itô. In the following we consider the problem on Ω_1 ω-wise. We also introduce the following smaller space for later use: for $\varepsilon > 0$ we set

$$\mathcal{Z}_p^\varepsilon := \{y \in \mathcal{Z}_p | \sup_{s \leq u < v \leq t} u^{2\varepsilon + 1 - \frac{3}{2p}} \frac{|\delta y_{uv}|_p}{|u-v|^\varepsilon} + u^{2\varepsilon + \frac{3}{2} - \frac{3}{2p}} \frac{\sum_{j=1}^3 |\delta(D_j y)_{uv}|_p}{|u-v|^\varepsilon} < \infty, \quad \forall 0 < s < t\}.$$

Now we recall the notion of a controlled path Y relative to some reference path X due to Gubinelli [8].

Definition 1.1. Given a path $X \in C^\alpha([s,t],\mathbb{R}^N)$, we say that $Y \in C^\alpha([s,t],\mathbb{R}^N)$ is controlled by X if there exists $Y' \in C^\alpha([s,t],\mathbb{R}^{N\times N})$ so that the remainder term R, for $s \le u < v \le t$ given by the formula

$$\delta Y^\mu_{uv} = \sum_{\nu=1}^N Y'^{\mu\nu}_u \delta X^\nu_{uv} + R^\mu_{uv},$$

satisfies $\|R\|_{2\alpha,[s,t]} < \infty$. Here the superscript μ and ν relates to the coordinate.

By [8], if we are given a path Y controlled by X, then we can define the integration of Y against $(X,\mathbb{X})$, which is an extension of Young's integral (see Theorem 1 and Corollary 2 in [8]): for $0 \le s < t \le T$

$$\int_s^t Y^\mu dX^\nu := \lim_{|\mathcal{P}|\to 0} \sum_{i=0}^{n-1} (Y^\mu_{t_i}\delta X^\nu_{t_it_{i+1}} + \sum_{\mu'=1}^N Y'^{\mu\mu'}_{t_i}\mathbb{X}^{\mu'\nu}_{t_it_{i+1}}), \tag{1.8}$$

where $\mathcal{P} = \{t_0, t_1, ..., t_n\}$ is a partition of the interval $[s,t]$ such that $t_0 = s, t_n = t, t_{i+1} > t_i, |\mathcal{P}| = \sup_i |t_{i+1} - t_i|$.

Now we give the definition of solutions to equation (1.4). In the following we define the analytic weak solution to equation (1.4) and we use $\langle \cdot, \cdot \rangle$ to denote the L^2 inner product.

Definition 1.2. We say that U is a solution to equation (1.4) if $\Gamma^{-1}U \in \mathcal{Z}^\varepsilon_p$ for some $\varepsilon > 0$ and for any $\varphi \in C_c^\infty(\mathbb{R}^3;\mathbb{R}^3)$, the function $t \to \langle \Gamma_t^{-1}U_t, \varphi\rangle$ is continuous on $[0,\infty)$ and for $0 < s < t$,

$$\langle U_t - U_s, \varphi\rangle - \int_s^t [\langle U_r, \Delta\varphi\rangle - \langle M(U_r), \varphi\rangle]dr = \sum_{i=1}^N \int_s^t \langle \tilde{B}_i U_r, \varphi\rangle d\beta^i_r, \tag{1.9}$$

$$U|_{t=0} = U_0,$$

where the integral $\int_s^t \langle \tilde{B}_i U_r, \varphi\rangle d\beta^i_r$ is understood in the sense of (1.8) with respect to the rough paths $(\beta,\mathbb{B})$. Here for $0 < s < t$ $\langle \tilde{B}_i U, \varphi\rangle \in C^\alpha([s,t])$ is controlled by β in the sense of Definition 1.1 and

$$\delta(\langle \tilde{B}_i U, \varphi\rangle)_{st} = \sum_{k=1}^N \langle \tilde{B}_k\tilde{B}_i U_s, \varphi\rangle \delta\beta^k_{st} + R^i_{st}, \tag{1.10}$$

with R being the remainder term satisfying

$$\|\langle \tilde{B}_k\tilde{B}_i U, \varphi\rangle\|_{\alpha,[s,t]} < \infty, \quad \|R^i\|_{2\alpha,[s,t]} < \infty. \tag{1.11}$$

Remark 1.2. *(i) Here due to the singularity of solution U at $t = 0$, the stochastic integral defined in* (1.8) *has some problem at $t = 0$. So, in* (1.9) *we only assume $0 < s < t$. Since $\Gamma^{-1}U \in \mathcal{Z}_p$, $\int_s^t \langle M(U_r), \varphi\rangle dr$ is well-defined due to (2.35) in [2].*

(ii) In general rough paths theory, often approximations are used to give a meaning to the solution of stochastic equations (see [7], Chapter 12). However, in this case if we need the approximation equations to be well-posed for small initial data, then the conditions on the initial value might be artificial. Therefore, since our aim is to prove a stochastic version of the Kato-Fujita result (see [10]), the above definition is more suitable. We also want to mention that such kind of definition has also been used for the linear equation in [5].

The main result of this paper is the following theorem:

Theorem 1.3. *Under the condition of Theorem 1.1 and for y as obtained in Theorem 1.1, for $\omega \in \Omega_0 \cap \Omega_1$, $U_t(\omega) := \Gamma_t(\omega)y_t(\omega)$ is the unique solution to* (1.4) *in the sense of Definition 1.2.*

2 Proof of Theorem 1.3

First, we prove the following lemma.

Lemma 2.1. *(mild solution $\Leftrightarrow$ weak solution) If $y \in \mathcal{Z}_p$ is the unique solution to* (1.6)*, then for any $\varphi \in C_c^\infty(\mathbb{R}^3; \mathbb{R}^3)$*

$$\langle y_t, \varphi\rangle = \langle U_0, \varphi\rangle + \int_0^t \big[\langle y_s, \Delta\varphi\rangle + \langle \Gamma_s^{-1} M(\Gamma_s y_s), \varphi\rangle\big] ds, \quad t \in [0, \infty). \tag{2.1}$$

Conversely, if there exists $y \in \mathcal{Z}_p$ satisfying equation (2.1) *for any $\varphi \in C_c^\infty(\mathbb{R}^3; \mathbb{R}^3)$, then y is a solution to* (1.6)*.*

Proof. mild solution $\Rightarrow$ weak solution: By (1.6) we know that for $\varphi \in C_c^\infty(\mathbb{R}^3; \mathbb{R}^3), T > 0$

$$\begin{aligned}\int_0^T \langle y_t, \Delta\varphi\rangle dt = &\int_0^T \langle e^{t\Delta} U_0, \Delta\varphi\rangle dt \\ &+ \int_0^T \langle \int_0^t e^{(t-s)\Delta} \Gamma_s^{-1} M(\Gamma_s y_s) ds, \Delta\varphi\rangle dt.\end{aligned}$$

Following similar arguments as in the proof of [6], Proposition 6.4, we have

$$\int_0^T \langle e^{t\Delta} U_0, \Delta\varphi\rangle dt = \int_0^T \langle U_0, \frac{d}{dt} e^{t\Delta}\varphi\rangle dt = \langle e^{T\Delta} U_0, \varphi\rangle - \langle U_0, \varphi\rangle.$$

$$\int_0^T \langle \int_0^t e^{(t-s)\Delta} \Gamma_s^{-1} M(\Gamma_s y_s) ds, \Delta\varphi\rangle dt = \int_0^T \langle \Gamma_s^{-1} M(\Gamma_s y_s), (e^{(T-s)\Delta} - I)\varphi\rangle ds.$$

Combining the above arguments we have

$$\int_0^t \langle y_s, \Delta\varphi\rangle ds = \langle e^{t\Delta}U_0, \varphi\rangle - \langle U_0, \varphi\rangle + \int_0^t \langle e^{(t-s)\Delta}\Gamma_s^{-1}M(\Gamma_s y_s), \varphi\rangle ds$$
$$- \int_0^t \langle \Gamma_s^{-1}M(\Gamma_s y_s), \varphi\rangle ds,$$

which implies (2.1).

weak solution $\Rightarrow$ mild solution: By (2.1) and similar arguments as in the proof of [6], Proposition 6.3, we have for $\zeta \in C^1([0,T]; C_c^\infty(\mathbb{R}^3;\mathbb{R}^3)), T > 0$ and $0 < t \le T$

$$\langle y_t, \zeta_t\rangle = \langle U_0, \zeta_0\rangle + \int_0^t \big[\langle y_s, \Delta\zeta_s + \zeta_s'\rangle + \langle \Gamma_s^{-1}M(\Gamma_s y_s), \zeta_s\rangle\big] ds, \quad t \in [0,\infty). \tag{2.2}$$

Choosing $\zeta_s := e^{(t-s)\Delta}\varphi$, $\varphi \in C_c^\infty(\mathbb{R}^3;\mathbb{R}^3)$, we have

$$\langle y_t, \varphi\rangle = \langle U_0, e^{t\Delta}\varphi\rangle + \int_0^t \langle e^{(t-s)\Delta}\Gamma_s^{-1}M(\Gamma_s y_s), \varphi\rangle ds.$$

Thus (1.6) follows. □

Now we prove the following estimate for the solutions:

Lemma 2.2. *For $T > 0, \varphi \in L^{q/(q-1)} \cap L^3$ and Ω_0 being in Theorem 1.1, on Ω_0 $\sup_{t\in[0,T]} |\langle \Gamma_t y_t, \varphi\rangle| < \infty$ and $y \in \mathcal{Z}_p^\varepsilon$ for $0 < \varepsilon < \frac{1}{2} - \frac{3}{4p}$, with p, q as in Theorem 1.1.*

Proof. We have

$$y_t = e^{t\Delta}U_0 + \int_0^t e^{(t-s)\Delta}\Gamma_s^{-1}M(\Gamma_s y_s) ds.$$

Then on Ω_0

$$\begin{aligned}|\langle \Gamma_t y_t, \varphi\rangle| \le& C\|\Gamma_t\|_{L(L^{3/2},L^{3/2})}|e^{t\Delta}U_0|_{3/2} + C\|\Gamma_t\|_{L(L^q,L^q)}\int_0^t |\Gamma_s^{-1}M(\Gamma_s y_s)|_q ds\\ \le& C\|\Gamma_t\|_{L(L^{3/2},L^{3/2})}|U_0|_{3/2} + C\|\Gamma_t\|_{L(L^q,L^q)}\int_0^t \|\Gamma_s^{-1}\|_{L(L^q,L^q)}|M(\Gamma_s y_s)|_q ds\\ \le& C\|\Gamma_t\|_{L(L^{3/2},L^{3/2})}|U_0|_{3/2} + C\|\Gamma_t\|_{L(L^q,L^q)}\|y\|^2 \sup_{s\in[0,t]}\eta_s \int_0^t s^{-5/2+3/p} ds\\ <&\infty,\end{aligned}$$

where in the second inequality we used (2.15) in [2] and in the third inequality we used (2.35) in [2] and in the last inequality we used that $\|y\| \le C|U_0|_{3/2}$ by the proof of Theorem 1.1 in [2]. Now we prove $y \in \mathcal{Z}_p^\varepsilon$. We have

$$|\delta y_{uv}|_p \le |(e^{v\Delta} - e^{u\Delta})U_0|_p + |(e^{(v-u)\Delta} - 1)\int_0^u e^{(u-s)\Delta}\Gamma_s^{-1}M(\Gamma_s y_s) ds|_p$$
$$+ |\int_u^v e^{(v-s)\Delta}\Gamma_s^{-1}M(\Gamma_s y_s) ds|_p.$$

For the first term we have

$$
\begin{aligned}
&|(e^{v\Delta}-e^{u\Delta})U_0|_p=|(e^{(v-u)\Delta}-I)e^{u\Delta}U_0|_p\leq C|(e^{(v-u)\Delta}-I)e^{u\Delta}U_0|_{B^{\varepsilon}_{p,\infty}}\\
\leq &C(v-u)^{\varepsilon}|e^{u\Delta}U_0|_{B^{3\varepsilon}_{p,\infty}}\leq C(v-u)^{\varepsilon}u^{-2\varepsilon}|e^{u\Delta/2}U_0|_p\leq C(v-u)^{\varepsilon}u^{-2\varepsilon-1+\frac{3}{2p}}|U_0|_{3/2},
\end{aligned}
$$

where $B^s_{m,n}$ is the usual Besov space and we used Propositions 3.11 and 3.12 in [11]. For the second term similarly we have

$$
\begin{aligned}
&|(e^{(v-u)\Delta}-1)\int_0^u e^{(u-s)\Delta}\Gamma_s^{-1}M(\Gamma_s y_s)ds|_p\\
\leq &C(v-u)^{\varepsilon}\int_0^u |e^{(u-s)\Delta}\Gamma_s^{-1}M(\Gamma_s y_s)|_{B^{3\varepsilon}_{p,\infty}}ds\\
\leq &C(v-u)^{\varepsilon}\int_0^u (u-s)^{-2\varepsilon}|e^{(u-s)\Delta/2}\Gamma_s^{-1}M(\Gamma_s y_s)|_p ds\\
\leq &C(v-u)^{\varepsilon}\sup_{s\geq 0}\eta_s\|y\|^2\int_0^u (u-s)^{-2\varepsilon-\frac{1}{2}(\frac{3}{p}-1)}s^{-\frac{5}{2}+\frac{3}{p}}ds\\
\leq &C(v-u)^{\varepsilon}u^{-1-2\varepsilon+\frac{3}{2p}}\sup\eta_s\|y\|^2,
\end{aligned}
$$

where in the third inequality we used a similar calculation as (2.17) in [2]. For the third term we have

$$
\begin{aligned}
&|\int_u^v e^{(v-s)\Delta}\Gamma_s^{-1}M(\Gamma_s y_s)ds|_p\\
\leq &C\sup_{s\geq 0}\eta_s\|y\|^2\int_u^v (v-s)^{-\frac{1}{2}(\frac{3}{p}-1)}s^{-\frac{5}{2}+\frac{3}{p}}ds\\
= &C\sup_{s\geq 0}\eta_s\|y\|^2(v-u)^{\frac{3}{2}-\frac{3}{2p}}\int_0^1 (1-l)^{-\frac{1}{2}(\frac{3}{p}-1)}[u+l(v-u)]^{-\frac{5}{2}+\frac{3}{p}}dl\\
\leq &C\sup_{s\geq 0}\eta_s\|y\|^2(v-u)^{2\varepsilon}u^{-1-2\varepsilon+\frac{3}{2p}}\int_0^1 (1-l)^{-\frac{1}{2}(\frac{3}{p}-1)}l^{-\frac{3}{2}+\frac{3}{2p}+2\varepsilon}dl,
\end{aligned}
$$

where we used interpolation and $-1-2\varepsilon+\frac{3}{2p}<0$, $-\frac{3}{2}+\frac{3}{2p}+2\varepsilon<0$ in the last inequality. Combining the argument above we obtain that

$$
|\delta y_{uv}|_p\leq C(v-u)^{\varepsilon}u^{-2\varepsilon-1+\frac{3}{2p}}(|U_0|_{3/2}+\sup_{s\geq 0}\eta_s\|y\|^2).
$$

Similarly we have

$$
\begin{aligned}
|\delta(D_j y)_{uv}|_p\leq &|(e^{v\Delta}-e^{u\Delta})D_jU_0|_p+|(e^{(v-u)\Delta}-1)\int_0^u e^{(u-s)\Delta}D_j\Gamma_s^{-1}M(\Gamma_s y_s)ds|_p\\
&+|\int_u^v e^{(v-s)\Delta}D_j\Gamma_s^{-1}M(\Gamma_s y_s)ds|_p\\
\leq &C(v-u)^{\varepsilon}u^{-2\varepsilon-\frac{3}{2}+\frac{3}{2p}}(|U_0|_{3/2}+\sup_{s\geq 0}\eta_s\|y\|^2),
\end{aligned}
$$

where we used a similar calculation as (2.18) in [2]. Thus the second result follows. $\square$

Proof of Theorem 1.3[Existence] Now we check that $U = \Gamma y$ satisfies equation (1.9). We first calculate $\langle (\delta\Gamma y)_{uv}, \varphi\rangle$: for $0 < u < v$

$$\begin{aligned}\langle (\delta\Gamma y)_{uv}, \varphi\rangle =&\langle \delta\Gamma_{uv} y_u, \varphi\rangle + \langle \Gamma_u \delta y_{uv}, \varphi\rangle + \langle \delta\Gamma_{uv}\delta y_{uv}, \varphi\rangle\\ :=&I_1 + I_2 + I_3.\end{aligned}$$

Since $\Gamma_u\phi = \Pi_{i=1}^N \exp\left(\beta_u^i \tilde{B}_i - \frac{u}{2}\tilde{B}_i^2\right)\phi$ for $\phi \in C_c^\infty(\mathbb{R}^3;\mathbb{R}^3)$, by Taylor expansion we have

$$\delta\Gamma_{uv}\phi = \Gamma_u \sum_{i=1}^N (\delta\beta_{uv}^i \tilde{B}_i\phi - \frac{(v-u)}{2}\tilde{B}_i^2\phi + \sum_{k=1}^N \frac{1}{2}\tilde{B}_i\tilde{B}_k\phi\delta\beta_{uv}^k\delta\beta_{uv}^i) + o(|v-u|).$$

Here and in the following $o(|u-v|)$ means a higher order term of $|u-v|$. Now we recall the following result from Section 3.3 in [7]:

$$\mathbb{B}_{uv}^{ik} + \frac{1}{2}\delta^{ik}(v-u) = \mathbb{B}_{str,uv}^{ik}, \tag{2.3}$$

$$\frac{1}{2}(\mathbb{B}_{str,uv}^{ik} + \mathbb{B}_{str,uv}^{ki}) = \frac{1}{2}\delta\beta_{uv}^i\delta\beta_{uv}^k, \tag{2.4}$$

where $\delta^{ik} = 1$ if $i = k$, zero else, and $\mathbb{B}_{str,uv} := \int_u^v \delta\beta_{ur} \otimes \hat{d}\beta_r \in \mathbb{R}^{N\times N}$ with the integral in the Stratonovich sense. Then by symmetry of $\tilde{B}_i\tilde{B}_k\varphi$ with respect to i, k we have

$$\delta\Gamma_{uv}\phi = \Gamma_u \sum_{i=1}^N (\delta\beta_{uv}^i \tilde{B}_i\phi - \frac{(v-u)}{2}\tilde{B}_i^2\phi + \sum_{k=1}^N \tilde{B}_i\tilde{B}_k\phi\mathbb{B}_{str,uv}^{ik}) + o(|v-u|),$$

which by (2.3) implies that

$$I_1 = \sum_{i=1}^N \langle \Gamma_u\tilde{B}_i y_u, \varphi\rangle\delta\beta_{uv}^i + \sum_{i,k=1}^N \langle \Gamma_u\tilde{B}_k\tilde{B}_i y_u, \varphi\rangle\mathbb{B}_{uv}^{ki} + o(|u-v|).$$

Also since y satisfies equation (2.1) and $y \in \mathcal{Z}_p^\varepsilon$, we have

$$\begin{aligned}I_2 =&\langle y_u, \Delta\Gamma_u^*\varphi\rangle(v-u) + \langle \Gamma_u^{-1}M(\Gamma_u y_u), \Gamma_u^*\varphi\rangle(v-u) + o(|v-u|)\\ =&\langle \Gamma_u y_u, \Delta\varphi\rangle(v-u) + \langle M(\Gamma_u y_u), \varphi\rangle(v-u) + o(|v-u|),\end{aligned}$$

where Γ_u^* means the dual operator of Γ_u. Here in the first equality we used the following for $u < s$

$$\begin{aligned}&|\Gamma_s^{-1}M(\Gamma_s y_s) - \Gamma_u^{-1}M(\Gamma_u y_u)|_q\\ \le&\|\Gamma_s^{-1} - \Gamma_u^{-1}\|_{L(L^q,L^q)}|M(\Gamma_s y_s)|_q + \|\Gamma_u^{-1}\|_{L(L^q,L^q)}|M(\Gamma_s y_s) - M(\Gamma_u y_u)|_q\\ \le&C_u|s-u|^\varepsilon,\end{aligned} \tag{2.5}$$

where in the last inequality we used a similar calculation as Lemma 2.2 in [2]. By the above calculations we know that

$$I_3 = \langle \delta y_{uv}, \delta\Gamma^*_{uv}\varphi\rangle = o(|v-u|),$$

where $\delta\Gamma^*_{uv}$ means the dual operator of $\delta\Gamma_{uv}$. The above calculations and Lemma 2.2 and (2.35) in [2] imply that $\langle \tilde{B}_i U, \varphi\rangle$ is controlled by β in the sense of Definition 1.1 and satisfies (1.10) and (1.11). By the above calculations we also obtain that for $0 < s < t$

$$\begin{aligned}
&\langle U_t, \varphi\rangle - \langle U_s, \varphi\rangle \\
=& \sum_{[u,v]\in\mathcal{P}} \langle (\delta\Gamma y)_{uv}, \varphi\rangle \\
=& \sum_{[u,v]\in\mathcal{P}} \Bigg[\sum_{i=1}^N \langle \Gamma_u \tilde{B}_i y_u, \varphi\rangle \delta\beta^i_{uv} + \sum_{i,k=1}^N \langle \Gamma_u \tilde{B}_k \tilde{B}_i y_u, \varphi\rangle \mathbb{B}^{ki}_{uv} \\
&+ \langle \Gamma_u y_u, \Delta\varphi\rangle (v-u) + \langle M(\Gamma_u y_u), \varphi\rangle (v-u) + o(|u-v|) \Bigg],
\end{aligned}$$

where $\mathcal{P}$ is a partition of the interval $[s,t]$ similar as above. Taking the limit $|\mathcal{P}| \to 0$, by (1.8) we obtain that $U = \Gamma y$ satisfies the equation (1.9).

[Uniqueness] Now we prove the uniqueness of the solution. In fact by Theorem 1.1 we already know that the solution to (1.6) is unique, so we only need to prove that $y = \Gamma^{-1}U$ satisfies (2.1), which is equivalent to (1.6) by Lemma 2.1. We have for $0 < u < v$

$$\begin{aligned}
\langle \delta(\Gamma^{-1}U)_{uv}, \varphi\rangle =& \langle \delta\Gamma^{-1}_{uv} U_u, \varphi\rangle + \langle \Gamma^{-1}_u \delta U_{uv}, \varphi\rangle + \langle \delta\Gamma^{-1}_{uv}\delta U_{uv}, \varphi\rangle \\
:=& J_1 + J_2 + J_3.
\end{aligned}$$

Since $\Gamma^{-1}U \in \mathcal{Z}^\varepsilon_p$, we obtain the Hölder continuity of U_u when $u > 0$. Since $M(U_u) = M(\Gamma_u y_u)$, then (2.5) implies the Hölder continuity of $M(U_u)$ when $u > 0$. Then by Corollary 3 in [8] we have

$$\begin{aligned}
J_2 =& \langle \delta U_{uv}, (\Gamma^{-1}_u)^*\varphi\rangle = \langle y_u, \Delta\varphi\rangle (v-u) + \langle \Gamma^{-1}_u M(\Gamma_u y_u), \varphi\rangle (v-u) \\
&+ \sum_{k=1}^N \langle \tilde{B}_k y_u, \varphi\rangle \delta\beta^k_{uv} + \sum_{i,k=1}^N \langle \tilde{B}_i \tilde{B}_k y_u, \varphi\rangle \mathbb{B}^{ik}_{uv} + o(|u-v|),
\end{aligned}$$

where $(\Gamma^{-1}_u)^*$ means the dual operator of Γ^{-1}_u. Moreover, since

$$\Gamma^{-1}_u\varphi = \Pi_{i=1}^N \exp(-\beta^i_u \tilde{B}_i + \frac{u}{2}\tilde{B}^2_i)\varphi,$$

by Taylor expansion we have

$$\delta\Gamma^{-1}_{uv}\varphi = \Gamma^{-1}_u \sum_{i=1}^N (-\delta\beta^i_{uv}\tilde{B}_i\varphi + \frac{(v-u)}{2}\tilde{B}^2_i\varphi + \sum_{k=1}^N \frac{1}{2}\tilde{B}_i\tilde{B}_k\varphi\delta\beta^k_{uv}\delta\beta^i_{uv}) + o(|v-u|).$$

Thus, we have

$$J_1 = \langle \sum_{i=1}^{N} (-\delta\beta^i_{uv}\tilde{B}_i y_u + \frac{(v-u)}{2}\tilde{B}_i^2 y_u + \sum_{k=1}^{N}\frac{1}{2}\tilde{B}_i\tilde{B}_k y_u \delta\beta^k_{uv}\delta\beta^i_{uv}), \varphi\rangle + o(|v-u|),$$

and

$$J_3 = \langle \delta U_{uv}, (\delta\Gamma^{-1}_{uv})^* \varphi\rangle = -\sum_{k,i=1}^{N}\langle \tilde{B}_i\tilde{B}_k y_u, \varphi\rangle \delta\beta^k_{uv}\delta\beta^i_{uv} + o(|u-v|),$$

where $(\delta\Gamma^{-1}_{uv})^*$ means the dual operator of $\delta\Gamma^{-1}_{uv}$. Using (2.3) and (2.4) we obtain that

$$\begin{aligned}
&\sum_{i,k=1}^{N}\langle \tilde{B}_i\tilde{B}_k y_u, \varphi\rangle \mathbb{B}^{ik}_{uv} \\
&= \sum_{i,k=1}^{N}\langle \tilde{B}_i\tilde{B}_k y_u, \varphi\rangle \mathbb{B}^{ik}_{str,uv} - \frac{1}{2}\sum_{i=1}^{N}\langle \tilde{B}_i^2 y_u, \varphi\rangle (v-u) \\
&= \sum_{i,k=1}^{N}\langle \tilde{B}_i\tilde{B}_k y_u, \varphi\rangle [\frac{\mathbb{B}^{ik}_{str,uv} + \mathbb{B}^{ki}_{str,uv}}{2} + \frac{\mathbb{B}^{ik}_{str,uv} - \mathbb{B}^{ki}_{str,uv}}{2}] - \frac{1}{2}\sum_{i=1}^{N}\langle \tilde{B}_i^2 y_u, \varphi\rangle (v-u) \\
&= \sum_{i,k=1}^{N}\langle \tilde{B}_i\tilde{B}_k y_u, \varphi\rangle \frac{1}{2}\delta\beta^i_{uv}\delta\beta^k_{uv} - \frac{1}{2}\sum_{i=1}^{N}\langle \tilde{B}_i^2 y_u, \varphi\rangle (v-u).
\end{aligned}$$

Thus, we have that for $0 < s < t$

$$\begin{aligned}
&\langle y_t, \varphi\rangle - \langle y_s, \varphi\rangle \\
&= \sum_{[u,v]\in\mathcal{P}} \langle (\delta\Gamma^{-1}U)_{uv}, \varphi\rangle \\
&= \sum_{[u,v]\in\mathcal{P}} \Big[\langle y_u, \Delta\varphi\rangle (v-u) + \langle \Gamma_u^{-1} M(\Gamma_u y_u), \varphi\rangle (v-u) + o(|u-v|)\Big],
\end{aligned}$$

where $\mathcal{P}$ is a partition of the interval $[s,t]$. Taking the limit $|\mathcal{P}| \to 0$ we obtain that for $0 < s < t$

$$\langle y_t, \varphi\rangle = \langle y_s, \varphi\rangle + \int_s^t \big[\langle y_r, \Delta\varphi\rangle + \langle \Gamma_r^{-1} M(\Gamma_r y_r), \varphi\rangle\big] dr.$$

Now letting $s \to 0$, by the continuity of $\langle y_s, \varphi\rangle$ and $y \in \mathcal{Z}_p$ we obtain that $y = \Gamma^{-1}U$ satisfies (2.1). Thus uniqueness follows. $\square$

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