**The Poisson equation and estimates for distances between stationary distributions of diffusions**

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**Abstract**

We estimate distances between stationary solutions to Fokker–Planck–Kolmogorov equations with different diffusion and drift coefficients. To this end we study the Poisson equation on the whole space. We have obtained sufficient conditions for stationary solutions to satisfy the Poincaré and logarithmic Sobolev inequalities.

**Keywords:** Fokker–Planck–Kolmogorov equation, Poisson equation, logarithmic Sobolev inequality, Poincaré inequality, Kantorovich distance, total variation distance

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## 1. Introduction

Let us consider two Borel probability measures $\mu$ and $\sigma$ on $\mathbb{R}^d$ satisfying the stationary Fokker–Planck–Kolmogorov equations $L_\mu^*\mu = 0$ and $L_\sigma^*\sigma = 0$, where $L_\mu^*$ and $L_\sigma^*$ are formally adjoint operators to second order elliptic operators

$$L_\mu u = \mathrm{tr}(A_\mu D^2 u) + \mathrm{div}(b_\mu u) \quad \text{and} \quad L_\sigma u = \mathrm{tr}(A_\sigma D^2 u) + \mathrm{div}(b_\sigma u).$$

Below we explain in which sense the equations are understood. The indices $\mu$ and $\sigma$ in the notation for coefficients do not mean any dependence on the measures, but only serve for distinguishing two different equations satisfied by two given measures. The diffusion matrices $A_\mu(x)$, $A_\sigma(x)$ are assumed throughout to be symmetric and positive definite. The main problem this paper is concerned with is obtaining bounds on distances between the measures $\mu$ and $\sigma$ through certain distances between the diffusion coefficients $A_\mu$ and $A_\sigma$ and the drift coefficients $b_\mu$ and $b_\sigma$. In obtaining such bounds an important role is played by the Poisson equation

$$L_\mu u = \psi,$$

so considerable attention will be paid to investigation of solutions to such equations on the whole space $\mathbb{R}^d$. We shall give a brief survey of already known methods of obtaining estimates and give a number of new results, which concern not only estimates for stationary distributions, but also the Poisson equation itself.

Investigation of dependence of solutions on the coefficients of the Fokker–Planck–Kolmogorov equation is important for the whole number of nonlinear problems: existence and uniqueness of solutions to stationary McKean–Vlasov equations, continuity and differentiability of distributions of diffusion processes with respect to a parameter, and optimal control. If, for example, in the case $A_\mu = A_\sigma = I$ we have obtained an estimate $\|\mu - \sigma\|_{TV} \le C\|b_\mu - b_\sigma\|_{L^2(\sigma)}$, where $\|\cdot\|_{TV}$ is the total variation norm of a signed measure (defined as the sum of values on the whole space of its positive and negative parts), then with its aid we can derive an existence and uniqueness theorem for the nonlinear equation

$$\Delta\mu - \mathrm{div}(b(x,\mu)\mu) = 0,$$

in which the drift $b_\mu$ now depends on the solution $\mu$, in the following way. Let us consider a mapping $F$ on the space of probability measures defined as follows: $F(\sigma) = \mu$ if $L_\sigma^*\mu = 0$, where $L_\sigma$ is the operator with the drift $b_\sigma$. If $|b(x,\mu) - b(x,\sigma)| \le qC^{-1}\|\mu - \sigma\|_{TV}$, where $0 < q < 1$, then the mapping $F$ satisfies the estimate

$$\|F(\sigma_1) - F(\sigma_2)\|_{TV} \le q\|\sigma_1 - \sigma_2\|_{TV},$$

i.e., $F$ is a contracting mapping (see [8] and [9]).

In the paper [8], in the case of locally bounded coefficients $b_\mu$, $b_\sigma$ and $A_\mu = A_\sigma = I$, an estimate for the $L^2(\mu)$-norm of the gradient of the function $\sqrt{d\sigma/d\mu}$ via the $L^2(\sigma)$-norm of the difference $b_\mu - b_\sigma$ was obtained. Under the additional assumption that the measure $\mu$ satisfies the logarithmic Sobolev inequality, this estimate yields estimates for the entropy of $\sigma$ with respect to $\mu$, for the Kantorovich distance between $\mu$ and $\sigma$, and for the total variation norm of the difference $\sigma - \mu$. An example of the Fokker–Planck–Kolmogorov equation whose probability solution satisfies the logarithmic Sobolev inequality is (see [12, Theorem 5.6.36]) the equation with the unit diffusion matrix and a drift $b$ satisfying the monotonicity condition

$$\langle b(x) - b(y), x - y\rangle \le -\kappa |x-y|^2.$$

In this paper, we generalize this result to the case of nonconstant and different diffusion matrices and show that if in place of the logarithmic Sobolev inequality we require that $\mu$ satisfies the Poincaré inequality, then one can obtain an estimate for the Hellinger integral, from which an estimate for the total variation of the difference $\sigma - \mu$ follows. Note that a sufficient condition ensuring the Poincaré inequality for the solution $\mu$ is (see [2, Theorem 1.4], [16, Theorem 1.1]) the symmetry of the operator $L_\mu$ on $L^2(\mu)$ and the existence of a Lyapunov function, in particular, no monotonicity of $b$ is required. However, the symmetry of $L_\mu$ is a very substantial restriction (see also [19] and [25]). Say, for the unit diffusion matrix it is fulfilled only when the drift coincides with the logarithmic gradient of the density of the solution. There is also the whole number of papers (see [2], [3], [4], [5], [15], [16], [17], [18], [24], [26], [31], [32], [33], [34], and [35]), in which the conditions for the validity of inequalities of log-Sobolev type are derived in terms of certain curvatures connected with the coefficients of the equation (such as the Bakry $\Gamma_2$-condition). However, these criteria require higher smoothness of the coefficients and are difficult to verify through the coefficients of the equation. With the aid of the Poisson equation we obtain new sufficient conditions under which the solution satisfies the Poincaré and logarithmic Sobolev inequalities.

In the paper [9], another approach has been considered to obtaining estimates for distances between $\mu$ and $\sigma$ based on the properties of solutions to the Poisson equations $L_\mu u - \lambda u = \psi$ and $L_\mu u = \psi$. As above, two typical situations have been studied there: 1) $b$ satisfies the monotonicity condition and 2) there exists a Lyapunov function. In the first situation estimates on the gradient for solutions to the Poisson equation have been obtained with the aid of the maximum principle (it is here that the monotonicity of $b$ is needed). In the second situation estimates of solutions to the Poisson equation from the paper [29] have been used, which imposes very restrictive conditions on the coefficients (this case is commented in more detail below). Such conditions are connected with the method of constructing and studying solutions to the Poisson equation based on analysis of the formula expressing the solution $u$ to the equation $L_\mu u = \psi$ as the expectation of the variable $\int_0^\infty \psi(X_t)\,dt$ with the solution $X_t$ to the corresponding stochastic equation; in addition, this method employs estimates of the rate of convergence of transition probabilities of the diffusion process $X_t$ to the stationary distribution $\mu$. Moreover, the estimate for the gradient in [29, Theorem 1 and Theorem 2] was actually obtained under the assumption of the global boundedness of the coefficients.

In this paper, we suggest an alternative approach (not using probability representations) to the study of the Poisson equation, which enables us to obtain more general results under much less restrictive conditions on the coefficients. Finally, we apply these results for obtaining new bounds on the total variation of the difference $\sigma - \mu$ and the Poincaré and logarithmic Sobolev inequalities for the solution. Note that for transition probabilities, i.e., solutions to parabolic

Fokker–Planck–Kolmogorov equations, analogous questions have been studied in the paper [14]. Close estimates for the maximum of the difference between two transition probability densities and their applications are studied in [22] and [23]. A survey of the theory of Fokker–Planck–Kolmogorov equations is given in [12] and [11], for uniqueness questions see also [13].

Let $W^{p,k}(\mathbb{R}^d)$ denote the Sobolev class of functions belonging to $L^p(\mathbb{R}^d)$ along with their generalized partial derivatives up to order $k$. The Sobolev norm $\|f\|_{p,k}$ is defined as the sum of the $L^p$-norms of the aforementioned functions. Let $W^{p,k}_{loc}(\mathbb{R}^d)$ denote the class of functions $f$ such that $\zeta f \in W^{p,k}(\mathbb{R}^d)$ for all functions $\zeta$ from the class $C_0^\infty(\mathbb{R}^d)$ of infinitely differentiable functions with compact support.

The inner product and norm in $\mathbb{R}^d$ are denoted by $\langle x, y\rangle$ and $|x|$, respectively. Let $B(0,R)$ denote the closed ball of radius $R$ centered at zero.

Let

$$L_{A,b}u = \operatorname{tr}(AD^2u) + \langle b, \nabla u\rangle.$$

Let $p > d$. Throughout, if the otherwise is not stated explicitly, we assume that the coefficients $A$ and $b$ satisfy the following conditions:

($\mathrm{H_a}$) the matrix $A(x) = (a^{ij}(x))_{1\le i,j\le d}$ is symmetric, positive definite and its elements $a^{ij}$ belong to the Sobolev class $W^{p,1}_{loc}(\mathbb{R}^d)$ (we always pick continuous versions of the functions $a^{ij}$);

($\mathrm{H_b}$) $b(x) = (b^i(x))_{1\le i\le d}$ is a Borel vector field on $\mathbb{R}^d$ and $b^i \in L^p_{loc}(\mathbb{R}^d)$.

We shall say that a Borel probability measure $\mu$ on $\mathbb{R}^d$ (i.e., $\mu \ge 0$ and $\mu(\mathbb{R}^d) = 1$) satisfies the stationary Fokker–Planck–Kolmogorov equation

$$L^*_{A,b}\mu = 0 \tag{1.1}$$

if $a^{ij}, b^i \in L^1_{loc}(\mu)$ and

$$\int_{\mathbb{R}^d} L_{A,b}u\, d\mu = 0 \quad \forall u \in C_0^\infty(\mathbb{R}^d).$$

In the sense of distributions equation (1.1) can be written in the form (the so-called "double divergence form")

$$\partial_{x_i}\partial_{x_j}(a^{ij}\mu) - \partial_{x_i}(b^i\mu) = 0$$

with summation with respect to the repeated indices. Under our conditions on the coefficients the measure $\mu$ has a continuous strictly positive density $\varrho$ with respect to Lebesgue measure and $\varrho \in W^{p,1}_{loc}(\mathbb{R}^d)$. For this density the equation can be written in the divergence form

$$\partial_{x_i}(a^{ij}\partial_{x_j}\varrho) + \partial_{x_i}((\partial_{x_j}a^{ij} - b^i)\varrho) = 0.$$

Note that a sufficient condition for the existence and uniqueness of a probability measure $\mu$ satisfying equation (1.1) is the existence of a function $V \in C^2(\mathbb{R}^d)$ and a number $C > 0$ such that $\lim\limits_{|x|\to+\infty} V(x) = +\infty$ and $L_{A,b}V(x) \le -C$ outside some ball.

This paper consists of the introduction and three sections. Section 2 is concerned with estimates obtained under the assumption that the solution satisfies the logarithmic Sobolev or the Poincaré inequality. In Section 3 we discuss bounds obtained with the aid of the Poisson equation and consider sufficient conditions for the validity of Poincaré-type and logarithmic Sobolev inequalities. In the last section we study the Poisson equation itself.

## 2. Estimates on the basis of the logarithmic Sobolev inequality and the Poincaré inequality

Suppose that measures $\mu = \varrho_\mu\, dx$ and $\sigma = \varrho_\sigma\, dx$ are probability solutions to the equations $L^*_{A_\mu,b_\mu}\mu = 0$ and $L^*_{A_\sigma,b_\sigma}\sigma = 0$, the coefficients of which satisfy conditions ($\mathrm{H_a}$) and ($\mathrm{H_b}$), which will be assumed throughout.

Let us introduce the following notation:

$$h_\mu = (h_\mu^i)_{i=1}^d, \quad h_\mu^i = b_\mu^i - \sum_{j=1}^d \partial_{x_j} a_\mu^{ij},$$

$$h_\sigma = (h_\sigma^i)_{i=1}^d, \quad h_\sigma^i = b_\sigma^i - \sum_{j=1}^d \partial_{x_j} a_\sigma^{ij}.$$

Set

$$\Phi = \frac{(A_\mu - A_\sigma)\nabla\varrho_\sigma}{\varrho_\sigma} + h_\sigma - h_\mu.$$

We observe that $\Phi = b_\sigma - b_\mu$ if $A_\mu = A_\sigma$.

Further for shortness of notation in place of $L_{A_\mu,b_\mu}$ and $L_{A_\sigma,b_\sigma}$ we write $L_\mu$ and $L_\sigma$, respectively.

Let

$$v(x) = \frac{\varrho_\sigma(x)}{\varrho_\mu(x)}. \tag{2.1}$$

Let $W^{p,1}(\mu)$ denote the weighted Sobolev class obtained by completing $C_0^\infty(\mathbb{R}^d)$ with respect to the Sobolev norm $\|f\|_{p,1,\mu}$, which differs from the usual one by the measure $\mu$ used in place of Lebesgue measure. By the indicated properties of the density of the measure, functions from this class do not differ locally from functions of class $W^{p,1}_{loc}(\mathbb{R}^d)$. Hence $W^{p,1}(\mu)$ consists of functions of class $W^{p,1}_{loc}(\mathbb{R}^d)$ with finite norm $\|\cdot\|_{p,1,\mu}$.

The next assertion generalizes Theorem 1 from [8], where the matrix $A$ was constant.

**Theorem 2.1.** *Suppose that $|A_\mu^{-1/2}\Phi| \in L^2(\sigma)$ and at least one of the following conditions is fulfilled:*

(i) $(1+|x|)^{-2}a_\mu^{ij},\ (1+|x|)^{-1}|b_\mu| \in L^1(\mu)$,

(ii) *there exists a function $V \in C^2(\mathbb{R}^d)$ such that $V \geq 0$, $\lim_{|x|\to+\infty} V(x) = +\infty$,*

$$L_\mu V(x) \leq MV(x)$$

*for all $x$ and some number $M > 0$ and*

$$\langle \Phi, \nabla V\rangle (1+V)^{-1} \in L^1(\sigma).$$

*Then we have*

$$\int_{\mathbb{R}^d} \frac{|A_\mu^{1/2}\nabla v|^2}{v}\,d\mu \leq \int_{\mathbb{R}^d} |A_\mu^{-1/2}\Phi|^2\,d\sigma.$$

*If $A_\mu \geq \alpha I$, then this estimate yields that $\sqrt{v} \in W^{2,1}(\mu)$.*

Since the proof repeats the justification of Theorem 1 from [8] with minor technical changes, we confine ourselves to an informal reasoning.

*Proof.* We observe that the function $v$ given by (2.1) satisfies the equation

$$L_\mu^*(\varrho_\mu v) = \mathrm{div}(\Phi\varrho_\sigma).$$

For smooth functions $u, w$ and $f$ there hold the equalities

$$L_\mu^*(uw) = wL_\mu^* u + uL_\mu^* w + 2\langle A\nabla u, \nabla w\rangle + uw\mathrm{div}h_\mu,$$

$$L_\mu^* f(u) = f'(u)L_\mu^* u + f''(u)|A^{1/2}\nabla u|^2 + (uf'(u) - f(u))\mathrm{div}h_\mu.$$

Let $f \in C^2((0,+\infty))$ and $f'' \geq 0$. Then

$$L_\mu^*\big(f(v)\varrho_\mu\big) = \varrho_\mu L_\mu^* f(v) + 2f'(v)\langle A\nabla v, \nabla\varrho_\mu\rangle + f(v)\varrho_\mu\mathrm{div}h_\mu,$$

which on account of the expression above for $L_\mu^*(f(v))$ gives the equality

$$f''(v)|A_\mu^{1/2}\nabla v|^2\varrho_\mu = L_\mu^*\bigl(f(v)\varrho_\mu\bigr) - f'(v)\mathrm{div}(\Phi\varrho_\sigma).$$

Multiplying this equality by a nonnegative function $\psi \in C_0^\infty(\mathbb{R}^d)$ and integrating by parts we obtain

$$\int_{\mathbb{R}^d} f''(v)|A_\mu^{1/2}\nabla v|^2\varrho_\mu\psi\,dx = \int_{\mathbb{R}^d} f(v)\varrho_\mu L_\mu\psi\,dx + \\ + \int_{\mathbb{R}^d} f''(v)\langle\nabla v,\Phi\rangle v\varrho_\mu\psi\,dx + \int_{\mathbb{R}^d} f'(v)\langle\nabla\psi,\Phi\rangle v\varrho_\sigma\,dx.$$

The hypothesis of the theorem ensures the existence of a sequence of functions $\psi_N$ such that $\psi_N \to 1$, $|\nabla\psi_N| \to 0$, $L_\mu\psi_N \to 0$ and the integrals of the form indicated above with the functions $\nabla\psi_N$ and $L\psi_N$ tend to zero (for a rigorous reasoning, see [8]). Substituting $\psi_N$ in place of $\psi$ and letting $N$ go to infinity, we obtain

$$\int_{\mathbb{R}^d} f''(v)|A_\mu^{1/2}\nabla v|^2\varrho_\mu\,dx = \int_{\mathbb{R}^d} f''(v)\langle\nabla v,\Phi\rangle v\varrho_\mu\,dx.$$

Applying the inequality

$$\langle\nabla v,\Phi\rangle v \le \frac{1}{2}|A^{1/2}\nabla v|^2 + \frac{1}{2}|A^{-1/2}\Phi|^2v^2,$$

we arrive at the estimate

$$\int_{\mathbb{R}^d} f''(v)|A_\mu^{1/2}\nabla v|^2\varrho_\mu\,dx \le \int_{\mathbb{R}^d} f''(v)|A^{-1/2}\Phi|^2v^2\varrho_\mu\,dx.$$

After substitution $f(v) = v\log v$ we obtain the assertion of the theorem. □

Given a number $C_S$ and a Borel measurable matrix-valued mapping $A$, we shall say that a probability measure $\mu$ satisfies the logarithmic Sobolev inequality with the constant $C_S$ and the matrix $A$ if

$$\mathrm{Ent}_\mu f^2 := \int_{\mathbb{R}^d} f^2\log(f^2)\,d\mu - \int_{\mathbb{R}^d} f^2\,d\mu\log\int_{\mathbb{R}^d} f^2\,d\mu \le C_S\int_{\mathbb{R}^d}|A^{1/2}\nabla f|^2\,d\mu$$

for every function $f \in C_0^\infty(\mathbb{R}^d)$. Under our assumptions this inequality extends to all functions $f \in W^{2,1}(\mu)$ if $A$ is bounded.

We recall (see [10]) that on the set of probability measures with finite moment of order $p$ the Kantorovich $p$-metric is defined by

$$W_p(\mu,\sigma) = \inf\biggl(\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|x-y|^p\,\pi(dx\,dy)\biggr)^{1/p},$$

where inf is taken over all probability measures $\pi$ on $\mathbb{R}^d\times\mathbb{R}^d$ with projections $\mu$ and $\sigma$ on the factors.

**Corollary 2.2.** *If in addition to the conditions of Theorem* 2.1 *it is known that the measure $\mu$ satisfies the logarithmic Sobolev inequality with the constant $C_S$ and the matrix $A_\mu$, then the following assertions are true.*

(i) *We have the entropy estimate*

$$\mathrm{Ent}_\mu v \le \frac{C_S}{4}\int_{\mathbb{R}^d}|A_\mu^{-1/2}\Phi|^2\,d\sigma.$$

(ii) *If the measures $\mu$ and $\sigma$ have finite second moments, then we have the estimate on the Kantorovich 2-metric*

$$W_2(\mu,\sigma)^2 \le \frac{C_S^2}{4}\int_{\mathbb{R}^d} |A_\mu^{-1/2}\Phi|^2\, d\sigma.$$

(iii) *We have the estimate for the total variation*

$$\|\mu-\sigma\|_{TV}^2 \le \frac{C_S}{2}\int_{\mathbb{R}^d} |A_\mu^{-1/2}\Phi|^2\, d\sigma.$$

*Proof.* Assertion (i) follows from Theorem 2.1 and the logarithmic Sobolev inequality. Since the logarithmic Sobolev inequality implies the so-called transport inequality (see [10])

$$W_2(\mu,\sigma)^2 \le 2^{-1}C_S \mathrm{Ent}_\mu v,$$

assertion (i) yields assertion (ii). Finally, assertion (iii) follows from assertion (i) and the known Pinsker–Kulback–Csiszár inequality $\|\mu-\sigma\|_{TV}\le\sqrt{2\mathrm{Ent}_\mu v}$ (see, for example, [7, Theorem 2.12.24]). □

**Example 2.3.** Let $A_\mu = A_\sigma = I$ and suppose that for some positive $\kappa$ we have the inequality

$$\langle b_\mu(x)-b_\mu(y), x-y\rangle \le -\kappa|x-y|^2.$$

Then the measure $\mu$ satisfies the logarithmic Sobolev inequality with the constant $2/\kappa$ (see [12, Theorem 5.6.36]) and with this constant in place of $C_S$ there hold the aforementioned estimates in (i)–(iii), moreover, the integral in the right-hand side in this case equals $\|b_\mu - b_\sigma\|_{L^2(\sigma)}^2$.

**Example 2.4.** Let us consider the partial case of the previous example with $\mu = e^{-V}\,dx$ and $b_\mu = -\nabla V$. The monotonicity condition on $b_\mu$ becomes the convexity condition on $V$ with the estimate $D^2V \ge \kappa I$. Then we obtain that the entropy of $\sigma$ with respect to $\mu$ is estimated by $\|\nabla V + b_\sigma\|_{L^2(\sigma)}$. Let now $\sigma = e^{-H}\,dx$ and $b_\sigma = -\nabla H$. Applying a generalization of the transport inequality (see [10, Theorem 3.3.1]), we obtain

$$\int_{\mathbb{R}^d} |\nabla\Phi_V - \nabla\Phi_H|^2 e^{-H}\,dx \le \frac{K}{2\kappa}\int_{\mathbb{R}^d} |\nabla V-\nabla H|^2 e^{-H}\,dx,$$

where $\nabla\Phi_V$ and $\nabla\Phi_H$ are 2-optimal mappings taking the measure $e^{-V}\,dx$ and $e^{-H}\,dx$ to a given measure $m = e^{-P}\,dx$ such that $D^2P \ge K\cdot I$ and $K>0$. We recall that the 2-optimal mapping taking the measure $e^{-V}\,dx$ to the measure $m = e^{-P}\,dx$ is a Borel transformation of the first measure to the second one minimizing the integral

$$\int_{\mathbb{R}^d} |T(x)-x|^2\, e^{-V}\,dx$$

over all transformations $T$ of the first measure to the second one. It is known that such a minimizing mapping exists, is unique and has the form $\nabla\Phi_V$ with some convex function $\Phi_V$.

Given a number $C_P$ and a Borel measurable matrix-valued mapping $A$, we shall say that a probability measure $\mu$ satisfies the Poincaré inequality with the constant $C_P$ and the matrix $A$ if

$$\int_{\mathbb{R}^d}\Bigl|f-\int_{\mathbb{R}^d} f\,d\mu\Bigr|^2 d\mu \le C_P\int_{\mathbb{R}^d} |A^{1/2}\nabla f|^2\,d\mu$$

for every function $f\in C_0^\infty(\mathbb{R}^d)$. Under our assumptions this inequality extends to all functions $f\in W^{2,1}(\mu)$ if $A$ is bounded.

We recall that the Hellinger integral (see [7, p. 300]) is the quantity

$$H(\mu,\sigma) = \int_{\mathbb{R}^d}\sqrt{\varrho_\mu\varrho_\sigma}\,dx.$$

**Corollary 2.5.** *If in addition to the conditions of Theorem* 2.1 *it is known that the measure $\mu$ satisfies the Poincaré inequality with the constant $C_P$ and the matrix $A_\mu$, then the following estimates are valid*:

$$1 - H(\mu,\sigma)^2 \le \frac{C_P}{4}\int_{\mathbb{R}^d} |A_\mu^{-1/2}\Phi|^2\, d\sigma,$$

$$\|\mu-\sigma\|_{TV}^2 \le C_P \int_{\mathbb{R}^d} |A_\mu^{-1/2}\Phi|^2\, d\sigma.$$

*Proof.* By the Poincaré inequality

$$\int_{\mathbb{R}^d}\biggl|\sqrt{v} - \int_{\mathbb{R}^d}\sqrt{v}\,d\mu\biggr|^2 d\mu \le \frac{C_P}{4}\int_{\mathbb{R}^d}\frac{|A_\mu^{1/2}\nabla v|^2}{v}\,d\mu.$$

For the proof of the first inequality it suffices to observe that

$$\int_{\mathbb{R}^d}\biggl|\sqrt{v} - \int_{\mathbb{R}^d}\sqrt{v}\,d\mu\biggr|^2 d\mu = 1 - H(\mu,\sigma)^2.$$

The second inequality follows from the first one and the inequality $\|\mu-\sigma\|_{TV} \le 2\sqrt{1-H(\mu,\sigma)^2}$ (see [7, Theorem 4.7.36]). □

**Example 2.6.** Let $A_\mu = A_\sigma = I$ and $b_\mu = -\nabla V$, where $V \in C^2(\mathbb{R}^d)$, the function $V$ is bounded from below and the function $e^{-V}$ is a probability density. It is known (see [12, Theorem 4.1.11]) that the measure $\mu = e^{-V}\,dx$ is a unique probability solution to the equation $L_\mu^*\mu = 0$. Suppose that there exist a function $W \in C^2(\mathbb{R}^d)$ with $W \ge 1$ and positive numbers $\theta$, $\beta$, and $R$ such that

$$L_\mu W \le -\theta W + \beta I_{B(0,R)}.$$

According to [2, Theorem 1.4] the measure $\mu$ satisfies the Poincaré inequality with the unit matrix and constant $C_P = \theta^{-1}(1+\beta\cdot\beta_R)$, where $\beta_R$ is the constant from the Poincaré inequality for the measure $\mu$ restricted to $B(0,R)$. Thus, all assertions of Corollary 2.5 are fulfilled with this constant $C_P$, moreover, the integral in the right-hand side in this case equals $\|\nabla V + b_\sigma\|_{L^2(\sigma)}^2$.

**Example 2.7.** There are other conditions on the measure $\mu = e^{-V}\,dx$ under which the Poincaré inequality holds, for example, it is proved in [6] that for this it suffices that $V$ be convex. Using this result, one can show that

$$\|e^{-V} - \sigma\|_{TV} \le C_V \|\nabla V + b_\sigma\|_{L^2(\sigma)}$$

for every convex function $V \in C^2(\mathbb{R}^d)$ such that $e^{-V}\,dx$ is a probability measure and for every probability measure $\sigma$ satisfying the equation $L_\sigma^*\sigma = 0$ (without any restrictions on $b_\sigma$, except for condition ($\mathrm{H_b}$)). In particular, if $\sigma = e^{-H}\,dx$ and $b_\sigma = -\nabla H$, then

$$\|e^{-V} - e^{-H}\|_{TV} \le C_V \biggl(\int_{\mathbb{R}^d} |\nabla H - \nabla V|^2 e^{-H}\,dx\biggr)^{1/2}.$$

Let us observe that the last inequality can be easily derived directly without using estimates for distances between solutions to Fokker–Planck–Kolmogorov equations.

Finding conditions ensuring that a probability solution to the stationary FPK equation satisfies the logarithmic Sobolev inequality or the Poincaré inequality is a difficult problem, such conditions are not easily expressed in terms of the coefficients of the equation, especially in the case of a nonconstant diffusion matrix. Hence an alternative approach is necessary. In addition, as we shall see below, the estimates obtained with the aid of the Poisson equation enable us to derive some analogs of Sobolev and Poincaré inequalities for probability solutions to the Fokker–Planck–Kolmogorov equation.

## 3. Estimates on the basis of the properties of solutions to the Poisson equation

The idea of this approach is very simple. Suppose that for every bounded smooth function $\psi$ we are able to solve the Poisson equation

$$L_\mu u = \widetilde{\psi}$$

with the right-hand side

$$\widetilde{\psi} := \psi - \int_{\mathbb{R}^d} \psi \, d\mu.$$

Then for the solution $u$ we have

$$\int_{\mathbb{R}^d} \psi \, d(\mu - \sigma) = \int_{\mathbb{R}^d} \langle \nabla u, \Phi \rangle \, d\sigma.$$

In the last equality we write $\psi$ in place of $\widetilde{\psi}$, since the integral of any constant against the measure $\mu - \sigma$ is zero. If, for example, we know that the boundedness of $\psi$ yields the boundedness of $|\nabla u|$, then we immediately obtain the estimate

$$\|\mu - \sigma\|_{TV} \le C \|\Phi\|_{L^1(\sigma)}.$$

If for the boundedness of $|\nabla u|$ the boundedness of $|\nabla \psi|$ is needed (this is the case if we obtain this estimate by differentiating the equation $Lu = \psi$ and applying the maximum principle), then we obtain the estimate

$$W_1(\mu, \sigma) \le C \|\Phi\|_{L^1(\sigma)}.$$

Estimates of this form have been obtained in [9], we recall the formulations of the corresponding assertions at the end of this section. However, it is possible to use the equation $Lu = \psi$ only for estimating $|u|$, and to derive estimates on the gradient $|\nabla u|$ from the equality

$$\int_{\mathbb{R}^d} |A_\mu^{1/2} \nabla u|^2 \, d\sigma = -\int_{\mathbb{R}^d} [u\psi + u\langle \Phi, \nabla u \rangle] \, d\sigma.$$

In this case we obtain an estimate on $\|\mu - \sigma\|_{TV}$ via $\|\Phi\|_{L^2(\sigma)}$. This is the approach we discuss in this section.

The next result is a partial case of Theorem 4.7 from the next section (here $f(s) = s$, $\Theta = 1$),, but we mention this case here for using in the proof of the theorem below.

**Proposition 3.1.** *Suppose that $A$ and $b$ satisfy conditions* ($\mathrm{H_a}$) *and* ($\mathrm{H_b}$) *and there exist a positive function $V \in C^2(\mathbb{R}^d)$, a positive number $\gamma$ and a ball $Q$ of radius $R$ centered at zero such that*

$$L_{A,b} V(x) \le -\gamma V(x) \quad \text{for all} \quad x \in \mathbb{R}^d \setminus Q.$$

*Then, for every smooth function $\psi$ such that $\sup |\psi/V| < \infty$ and*

$$\int_{\mathbb{R}^d} \psi \, d\mu = 0,$$

*there exists a solution $u \in W^{p,2}_{loc}(\mathbb{R}^d)$ to the equation $L_{A,b} u = \psi$ such that*

$$\sup_{\mathbb{R}^d} \Bigl| \frac{u}{V} \Bigr| \le C \sup_{\mathbb{R}^d} \Bigl| \frac{\psi}{V} \Bigr|.$$

*The number $C$ depends on $d$, $\gamma$, $R$, $\|a^{ij}\|_{W^{p,1}(Q_1)}$, $\|b^i\|_{L^p(Q_1)}$, and $\sup_{x \in Q_1} \|A(x)^{-1}\|$, and also on the minimum of the function $V$ on $Q_1$ and the maximum of the function $V$ and absolute values of its first and second derivatives on $Q_1$, where $Q_1$ is the ball of radius $R+1$ centered at zero.*

**Theorem 3.2.** *Suppose that there exist a positive function $V \in C^2(\mathbb{R}^d)$, a positive number $\gamma$ and a ball $Q$ of radius $R$ centered at zero such that*

$$L_{A_\mu, b_\mu} V(x) \le -\gamma V(x) \quad \textit{whenever} \quad x \in \mathbb{R}^d \setminus Q.$$

*Suppose also that the functions*

$$V^2, \quad (1+|x|)^{-1}|\Phi|V^2, \quad |A_\mu^{-1/2}\Phi|^2 V^2, \quad (1+|x|)^{-1}|b_\mu^i|V^2, \quad (1+|x|)^{-2}|a_\mu^{ij}|V^2$$

*are integrable with respect to the measure $\sigma$ on all of $\mathbb{R}^d$. Then one has the estimate*

$$\|V(\mu-\sigma)\|_{TV} \le C\Bigl(\int_{\mathbb{R}^d} |A_\mu^{-1/2}\Phi|^2\, d\sigma\Bigr)^{1/2} \Bigl(\int_{\mathbb{R}^d} [V^2 + |A_\mu^{-1/2}\Phi|^2 V^2]\, d\sigma\Bigr)^{1/2},$$

*where $C$ depends on the quantities listed above in Proposition* 3.1.

*Proof.* Let $\psi \in C_0^\infty(\mathbb{R}^d)$ and $|\psi| \le V$. We observe that $\|V\|_{L^1(\mu)} = C_1 < \infty$ and the function

$$\widetilde{\psi} = \psi - \int_{\mathbb{R}^d} \psi\, d\mu$$

satisfies the inequality $|\widetilde{\psi}| \le V + C_1 \le C_2 V$ (we recall that $V$ is a positive continuous function tending to $+\infty$). By Proposition 3.1 there exists a solution $u$ to the equation $Lu = \widetilde{\psi}$ such that $|u| \le C_3 V$.

Let $\zeta_N = \zeta(x/N)$, where $\zeta \in C_0^\infty(\mathbb{R}^d)$, $\zeta \ge 0$, $\zeta(x) = 1$ if $|x| < 1$ and $\zeta(x) = 0$ if $|x| > 2$. Since $L_\mu^*(\sigma) = \operatorname{div}(\Phi\sigma)$, we have

$$\int_{\mathbb{R}^d} L_\mu(u^2\zeta_N) + \langle \Phi, \nabla(u^2\zeta_N)\rangle\, d\sigma = 0.$$

The expression $J := 2^{-1}L_\mu(u^2\zeta_N) + 2^{-1}\langle \Phi, \nabla(u^2\zeta_N)\rangle$ equals

$$\zeta_N u\widetilde{\psi} + \zeta_N |A_\mu^{1/2}\nabla u|^2 + u\langle A_\mu \nabla\zeta_N, \nabla u\rangle + \frac{u^2}{2} L_\mu \zeta_N + u\langle \Phi, \nabla u\rangle \zeta_N + \frac{u^2}{2}\langle \Phi, \nabla\zeta_N\rangle.$$

Applying the inequality $xy \le 4^{-1}x^2 + y^2$ to the third and fifth terms, we obtain

$$J \ge \zeta_N u\widetilde{\psi} + 2^{-1}\zeta_N |A_\mu^{1/2}\nabla u|^2 - u^2|A_\mu^{1/2}\nabla\zeta_N|^2 + \frac{u^2}{2} L_\mu\zeta_N - u^2|A_\mu^{-1/2}\Phi|^2\zeta_N + \frac{u^2}{2}\langle \Phi, \nabla\zeta_N\rangle.$$

We now apply the estimates $|u| \le C_3 V$ and $|\widetilde{\psi}| \le C_2 V$:

$$J \ge -\zeta_N C_2 C_3 V^2 + 2^{-1}\zeta_N |A_\mu^{1/2}\nabla u|^2 - C_3^2 V^2 |A_\mu^{1/2}\nabla\zeta_N|^2 \\ - \frac{C_3^2}{2} V^2 |L_\mu\zeta_N| - C_3^2 V^2 |A_\mu^{-1/2}\Phi|^2\zeta_N - \frac{C_3^2}{2} V^2 |\Phi|\,|\nabla\zeta_N|.$$

Integrating this inequality with respect to $\sigma$ and letting $N \to \infty$, we arrive at the estimate

$$\int_{\mathbb{R}^d} |A_\mu^{1/2}\nabla u|^2\, d\sigma \le C_4 \int_{\mathbb{R}^d} [V^2 + |A_\mu^{-1/2}\Phi|^2 V^2]\, d\sigma.$$

Taking into account this estimate and acting similarly with the expression

$$\int_{\mathbb{R}^d} L_\mu(u\zeta_N)\, d\sigma = -\int_{\mathbb{R}^d} \langle \Phi, \nabla(u\zeta_N)\rangle\, d\sigma,$$

we arrive at the inequality

$$\int_{\mathbb{R}^d} \widetilde{\psi}\, d\sigma \le \int_{\mathbb{R}^d} |A_\mu^{-1/2}\Phi|\,|A_\mu^{1/2}\nabla u|\, d\sigma.$$

Applying the Cauchy–Bunyakovskii inequality, we estimate the right-hand side by

$$C\Bigl(\int_{\mathbb{R}^d} |A_\mu^{-1/2}\Phi|^2\,d\sigma\Bigr)^{1/2}\Bigl(\int_{\mathbb{R}^d}\bigl[V^2+|A_\mu^{-1/2}\Phi|^2V^2\bigr]\,d\sigma\Bigr)^{1/2}.$$

It remains to recall the definition of $\widetilde{\psi}$ and write the left-hand side of the inequality as follows:

$$\int_{\mathbb{R}^d}\psi\,d(\sigma-\mu).$$

Since $\psi$ was an arbitrary function from $C_0^\infty(\mathbb{R}^d)$ satisfying the bound $|\psi|\le V$, the obtained inequalities yield the assertion of the theorem. □

**Example 3.3.** Let $A_\mu = A_\sigma = I$ and

$$\langle b_\mu(x), x\rangle \le -\gamma|x|^2, \quad \text{where } \gamma>0.$$

Suppose that $|b_\mu(x)|+|b_\sigma(x)|\le C_0(1+|x|)^m$ for some numbers $m\ge 1$ and $C_0>0$. If the measure $\sigma$ has a finite moment of order $2m+2$, then the conditions of Theorem 3.2 are fulfilled and the estimate

$$\|(1+|x|)^2(\mu-\sigma)\|_{TV}\le C\|b_\mu-b_\sigma\|_{L^2(\sigma)}\bigl(\|(1+|x|)^2\|_{L^2(\sigma)}+\|(1+|x|)^2(b_\mu-b_\sigma)\|_{L^2(\sigma)}\bigr)$$

holds, where $C$ depends on $\gamma$, $C_0$, and $m$.

The next result was obtained in [9, Theorem 2.2] in a somewhat different form, but in its formulation a condition was omitted that was necessary for applying in the proof a theorem from [29] (more precisely, Theorem 2 from [29], where in the formulation the restriction on the drift coefficient $b$ is also omitted, without which the reasoning from [29] for estimating the gradient of the solution is not applicable; actually, the justification given there is valid only for a bounded drift coefficient $b$). We now give a corrected stronger assertion.

**Theorem 3.4.** *Suppose that $A_\mu$ and $A_\sigma$ satisfy condition* (H$_\text{a}$) *and $A_\mu$ is uniformly bounded along with $A_\mu^{-1}$ and let the functions $a_\mu^{ij}$ be uniformly continuous with a modulus of continuity $\omega$. Suppose that for some positive numbers $\kappa$, $\gamma_1$, $\gamma_2$, $\gamma_3$ and $m$ there hold the inequalities*

$$\langle b_\mu(x),x\rangle\le\gamma_1-\gamma_2|x|^\kappa,\quad |b_\mu(x)|\le\gamma_3(1+|x|)^m.$$

*If $|\Phi|(1+|x|)^m\in L^1(\sigma)$, then*

$$\|\mu-\sigma\|_{TV}\le C\int_{\mathbb{R}^d}(1+|x|)^m|\Phi|\,d\sigma,$$

*where $C$ depends on $d$, $\kappa$, $\gamma_1$, $\gamma_2$, $\gamma_3$, $m$, the modulus of continuity $\omega$, and $\|a^{ij}\|_{W^{p,1}(B(0,R))}$, where $R$ depends only on $\gamma_1$, $\gamma_2$, $\kappa$, and the* sup-*norms of $A_\mu$, $A_\mu^{-1}$.*

*Proof.* Let $\psi\in C_0^\infty(\mathbb{R}^d)$ and $|\psi|\le 1$. According to Example 4.9(iii) (see below) there exists a bounded solution $u$ to the equation $L_\mu u=\widetilde{\psi}$, where

$$\widetilde{\psi}=\psi-\int_{\mathbb{R}^d}\psi\,d\mu.$$

By Proposition 4.10 we have the estimate

$$|\nabla u(x)|\le C(1+|x|)^m.$$

As above, using the sequence of functions $\zeta_k$, we justify the inequality

$$\int_{\mathbb{R}^d}\widetilde{\psi}\,d\sigma\le\int_{\mathbb{R}^d}|\nabla u|\,|\Phi|\,d\sigma,$$

from which the assertion of the theorem follows. □

We now formulate yet another result (see [9, Theorem 2.1]) based on estimates of the gradient obtained by means of differentiating the equation and applying the maximum principle.

**Theorem 3.5.** *Suppose that $A_\mu$ and $A_\sigma$ satisfy the Lipschitz condition with some constant $\Lambda > 0$, are bounded and $A_\mu \geq \alpha I$, $A_\sigma \geq \alpha I$ with some constant $\alpha > 0$. Let $b_\mu$ and $b_\sigma$ satisfy condition* $(\mathrm{H_b})$. *Suppose that $b_\mu \in L^1(\mu+\sigma)$, $\Phi \in L^1(\sigma)$, $|x| \in L^1(\sigma)$ and there exists a number $\kappa > d^2\Lambda^2/(4\alpha^2)$ such that for all $x$, $y$ we have*

$$\langle b_\mu(x) - b_\mu(y), x-y\rangle \leq -\kappa|x-y|^2.$$

*Then the measure $\mu$ has a finite first moment and*

$$W_1(\mu,\sigma) \leq \frac{1}{m}\int_{\mathbb{R}^d} |\Phi|\, d\sigma, \quad m = \kappa - \frac{d^2\Lambda^2}{4\alpha^2}.$$

*In addition, there exists a number $C > 0$ depending only on $\Lambda$, $\alpha$, $d$ and $\kappa$ such that*

$$\|\mu - \sigma\|_{TV} \leq C\int_{\mathbb{R}^d} |\Phi|\, d\sigma.$$

Let us observe that in the case of a constant matrix $A_\mu = A_\sigma$ the estimate on the Kantorovich metric from the last theorem is fulfilled with $m = \kappa$, which agrees with the bounds obtained in [8]. In addition, in this case the constant $C$ in the estimate for the total variation does not depend on the dimension $d$. This can be easily deduced from the justification of this estimate given in [9] and the fact that the estimate on the gradient of the solution to the Poisson equation from [30, Theorem 3.11] used in this justification does not depend on dimension.

Theorem 3.5 enables us to prove some analogs of the transport inequality and the Poincaré inequality for solutions to the stationary Fokker–Planck–Kolmogorov equation.

**Corollary 3.6.** *Suppose that the coefficients $A_\mu$ and $b_\mu$ satisfy the conditions of Theorem* 3.5. *Let*

$$F = \frac{A_\mu \nabla \varrho_\mu}{\varrho_\mu} - h_\mu.$$

*Then for every probability measure $f \cdot \mu$ with a smooth positive density $f$, the inequality*

$$W_1(\mu, f\mu) \leq \frac{1}{m}\int_{\mathbb{R}^d} |A_\mu \nabla f|\, d\mu + \frac{1}{m}\int_{\mathbb{R}^d} |F| f\, d\mu$$

*holds, and in the case where $\mu = e^{-V}\,dx$, $b_\mu = -\nabla V$, $D^2 V \geq \kappa I$ and $A_\mu = I$ this inequality has the form*

$$W_1(\mu, f\mu) \leq \frac{1}{\kappa}\int_{\mathbb{R}^d} |\nabla f|\, d\mu.$$

*Proof.* It suffices to apply the previous theorem to the measures $\mu$ and $\sigma = f\cdot\mu$, where $\sigma$ satisfies the equation with the matrix $A_\mu$ and drift

$$b_\sigma = \frac{A_\mu \nabla f}{f} + (F + b_\mu),$$

which is verified directly. □

Modifying the reasoning used to derive the estimate of the total variation of the difference $\mu - \sigma$, we can obtain a Poincaré-type inequality.

**Theorem 3.7.** *Let $A_\mu$ and $A_\mu^{-1}$ be uniformly bounded. Suppose that for some positive numbers $\kappa$, $\gamma_1$, $\gamma_2$, $\gamma_3$ and $m$ we have*

$$\langle b_\mu(x), x\rangle \leq \gamma_1 - \gamma_2|x|^\kappa, \quad |b_\mu(x)| \leq \gamma_3(1+|x|)^m.$$

*Then, for every function $f \in C_0^\infty(\mathbb{R}^d)$,*

$$\int_{\mathbb{R}^d}\Bigl|f-\int_{\mathbb{R}^d} f\,d\mu\Bigr|\,d\mu \le C\int_{\mathbb{R}^d}\bigl(1+|x|^m+|b_\mu-\beta_\mu|\bigr)|\nabla f|\,d\mu,$$

*where*

$$\beta_\mu^i=\sum_{j=1}^d \frac{\partial_{x_j}(a_\mu^{ij}\varrho_\mu)}{\varrho_\mu}$$

*and the number $C$ depends on the quantities mentioned in Theorem* 3.4.

*Proof.* Let $\psi \in C_0^\infty(\mathbb{R}^d)$. Set

$$\widetilde{\psi}=\psi-\int_{\mathbb{R}^d}\psi\,d\mu.$$

We observe that

$$\int_{\mathbb{R}^d}\Bigl(f-\int_{\mathbb{R}^d} f\,d\mu\Bigr)\psi\,d\mu=\int_{\mathbb{R}^d} f\widetilde{\psi}\,d\mu.$$

Let $u$ be the solution to the equation $Lu=\widetilde{\psi}$. Integrating by parts we obtain the equality

$$\int_{\mathbb{R}^d}\Bigl(f-\int_{\mathbb{R}^d} f\,d\mu\Bigr)\psi\,d\mu=\int_{\mathbb{R}^d}-\langle A_\mu\nabla u,\nabla f\rangle\,d\mu+\int_{\mathbb{R}^d}\langle\nabla u,q\rangle f\,dx,$$

where

$$q^j=b_\mu^j\varrho_\mu-\sum_{i=1}^d\partial_{x_i}\bigl(a_\mu^{ij}\varrho_\mu\bigr),\quad \mathrm{div}q=0.$$

Since $\mathrm{div}q=0$, we have

$$\int_{\mathbb{R}^d}\langle\nabla u,q\rangle f\,dx=-\int_{\mathbb{R}^d}\langle\nabla f,q\rangle u\,dx.$$

Let $|\psi|\le 1$. Then $|u(x)|\le C_1$ and $|\nabla u(x)|\le C_1(1+|x|)^m$ for all $x$ and some number $C_1$. Applying these estimates, we obtain the inequality

$$\int_{\mathbb{R}^d}\Bigl(f-\int_{\mathbb{R}^d} f\,d\mu\Bigr)\psi\,d\mu\le C\int_{\mathbb{R}^d}\bigl(|1+|x|^m+|b_\mu-\beta_\mu|\bigr)|\nabla f|\,d\mu,$$

which yields the assertion of the theorem. □

**Remark 3.8.** Suppose in addition to the hypotheses of the previous theorem that the matrix $A_\mu$ is Lipschitz. Applying the Cauchy–Bunyakovskii inequality to the right-hand side of the inequality from Theorem 3.7, we obtain

$$\int_{\mathbb{R}^d}\Bigl|f-\int_{\mathbb{R}^d} f\,d\mu\Bigr|\,d\mu\le C\Bigl(\int_{\mathbb{R}^d}\bigl(|1+|x|^m+|b_\mu-\beta_\mu|\bigr)^2\,d\mu\Bigr)^{1/2}\Bigl(\int_{\mathbb{R}^d}|\nabla f|^2\,d\mu\Bigr)^{1/2}.$$

According to [12, Theorem 3.1.2], we have the estimate

$$\int_{\mathbb{R}^d}\Bigl|\frac{\nabla\varrho_\mu}{\varrho_\mu}\Bigr|^2\,d\mu\le C_1+C_1\|b_\mu\|_{L^2(\mu)^2}.$$

Thus, we arrive at the inequality

$$\int_{\mathbb{R}^d}\Bigl|f-\int_{\mathbb{R}^d} f\,d\mu\Bigr|\,d\mu\le\widetilde{C}\Bigl(\int_{\mathbb{R}^d}|\nabla f|^2\,d\mu\Bigr)^{1/2},$$

where $\widetilde{C}$ is expressed through $C$, $\||x|^m\|_{L^2(\mu)}$, $\|b_\mu\|_{L^2(\mu)}$, the Lipschitz constant and the sup-norms of $A_\mu$ and $A_\mu^{-1}$.

**Theorem 3.9.** *Suppose that $A_\mu$ satisfies* (H$_\mathrm{a}$) *and the coefficient $b_\mu$ is locally bounded and there exists a function $V \in C^2(\mathbb{R}^d)$ such that $V > 0$, $\lim\limits_{|x|\to+\infty} V(x) = +\infty$ and, for some number $\gamma > 0$, the inequality*

$$L_\mu V(x) + \frac{|A_\mu(x)^{1/2}\nabla V(x)|^2}{V} \le -\gamma V(x)$$

*holds outside some ball $Q$ of radius $R$ centered at zero. Let $Q_1$ be the ball of radius $R+1$ centered at zero. Then for every function $f \in C_0^\infty(\mathbb{R}^d)$ we have the inequality*

$$\int_{\mathbb{R}^d}\Bigl|f - \int_{\mathbb{R}^d} f\,d\mu\Bigr|^2 d\mu \le C\int_{\mathbb{R}^d} |A_\mu^{1/2}\nabla f|^2(1 + |A_\mu^{-1/2}(b_\mu - \beta_\mu)|)^2\,d\mu,$$

*where $C$ depends on $d$, $\|a_\mu^{ij}\|_{W^{p,1}(Q_1)}$, $\sup_{x\in Q_1}|b_\mu^i(x)|$, $\sup_{x\in Q_1}\|A_\mu(x)^{-1}\|$, $V$, $\gamma$, and $R$.*

*In the partial case where $b_\mu = \beta_\mu$ and $A_\mu$ is bounded this inequality coincides with the classical Poincaré inequality*

$$\int_{\mathbb{R}^d}\Bigl|f - \int_{\mathbb{R}^d} f\,d\mu\Bigr|^2 d\mu \le C\int_{\mathbb{R}^d} |\nabla f|^2\,d\mu.$$

*Proof.* Let $\psi \in C_0^\infty(\mathbb{R}^d)$. Set

$$\widetilde{\psi} = \psi - \int_{\mathbb{R}^d}\psi\,d\mu.$$

Arguing as in the previous theorem, we arrive at the equality

$$\int_{\mathbb{R}^d}\Bigl(f - \int_{\mathbb{R}^d} f\,d\mu\Bigr)\psi\,d\mu = \int_{\mathbb{R}^d} -\langle A_\mu\nabla u, \nabla f\rangle\,d\mu + \int_{\mathbb{R}^d}\langle \nabla f, q\rangle u\,dx,$$

where the vector field $q$ is defined in the proof of the previous theorem. The right-hand side of the last equality is estimated by the expression

$$\Bigl(\int_{\mathbb{R}^d}|A_\mu^{1/2}\nabla u|^2\,d\mu\Bigr)^{1/2}\Bigl(\int_{\mathbb{R}^d}|A_\mu^{1/2}\nabla f|^2(1 + |A_\mu^{-1/2}(b_\mu - \beta_\mu)|)^2\,d\mu\Bigr)^{1/2}.$$

Applying estimate (4.4) from Example 4.5 (where we replace $B$ by $\mathbb{R}^d$), we finally obtain the inequality

$$\int_{\mathbb{R}^d}\Bigl(f - \int_{\mathbb{R}^d} f\,d\mu\Bigr)\psi\,d\mu \le C\|\psi\|_{L^2(\mu)}\Bigl(\int_{\mathbb{R}^d}|A_\mu^{1/2}\nabla f|^2(1 + |A_\mu^{-1/2}(b_\mu - \beta_\mu)|)^2\,d\mu\Bigr)^{1/2},$$

which yields the assertion of the theorem. □

**Corollary 3.10.** *Suppose that in addition to the conditions of the theorem it is known that $a_\mu^{ij} \in C_b^3(\mathbb{R}^d)$ and $b_\mu \in C^2(\mathbb{R}^d)$. Set*

$$W(x) = \max_{|x-y|\le 1}\bigl(1 + |b_\mu(y)| + |Db_\mu(y)| + |D^2 b_\mu(y)|\bigr).$$

*Then, for every function $f \in C_0^\infty(\mathbb{R}^d)$,*

$$\int_{\mathbb{R}^d}\Bigl|f - \int_{\mathbb{R}^d} f\,d\mu\Bigr|^2 d\mu \le C\int_{\mathbb{R}^d}|\nabla f|^2 W^2\,d\mu,$$

*where $C$ depends on $d$, $\|a_\mu^{ij}\|_{C_b^3(\mathbb{R}^d)}$, $\sup_{x\in Q_1}|b_\mu^i(x)|$, $\sup_{x\in\mathbb{R}^d}\|A_\mu(x)^{-1}\|$, $V$, $\gamma$, and $R$.*

*Proof.* It suffices to apply the estimate

$$\Bigl|\frac{\nabla\varrho(x)}{\varrho(x)}\Bigr| \le C_1 W(x)$$

from [27, Theorem 5.2]. □

**Example 3.11.** Suppose that $A_\mu$ and $A_\mu^{-1}$ are uniformly bounded, $a_\mu^{ij} \in C_b^3(\mathbb{R}^d)$ and $b_\mu \in C^2(\mathbb{R}^d)$. Suppose also that there exist positive numbers $\gamma_1$, $\gamma_2$, $\gamma_3$ and $m$ such that

$$\langle b_\mu(x), x\rangle \le \gamma_1 - \gamma_2|x|^2, \quad |b_\mu(x)| + |Db_\mu(x)| + |D^2 b_\mu(x)| \le \gamma_3(1+|x|)^m.$$

Then the probability solution $\mu$ to the equation $L_\mu^*\mu = 0$ satisfies the following Poincaré-type inequality:

$$\int_{\mathbb{R}^d} \Bigl| f - \int_{\mathbb{R}^d} f\,d\mu \Bigr|^2 d\mu \le C \int_{\mathbb{R}^d} |\nabla f|^2 (1+|x|)^{2m}\,d\mu.$$

Indeed, for justification it suffices to apply the last corollary.

Let us show that under the conditions of the last example there is an analog of the logarithmic Sobolev inequality.

**Theorem 3.12.** *Suppose that $A_\mu$ and $A_\mu^{-1}$ are uniformly bounded, $a_\mu^{ij} \in C_b^3(\mathbb{R}^d)$, $b_\mu \in C^2(\mathbb{R}^d)$. Suppose also that there exist positive numbers $\gamma_1$, $\gamma_2$, $\gamma_3$ and $m$ such that*

$$\langle b_\mu(x), x\rangle \le \gamma_1 - \gamma_2|x|^{m+1}, \quad |b_\mu(x)| + |Db_\mu(x)| + |D^2 b_\mu(x)| \le \gamma_1(1+|x|)^m.$$

*Then for the probability solution $\mu$ to the equation $L_\mu^*\mu = 0$ there exists a number $C$ such that for every function $f \in C_0^\infty(\mathbb{R}^d)$ we have*

$$\int_{\mathbb{R}^d} f^2 \log(f^2)\,d\mu - \int_{\mathbb{R}^d} f^2\,d\mu \log\Bigl(\int_{\mathbb{R}^d} f^2\,d\mu\Bigr) \le C\int_{\mathbb{R}^d} |\nabla f|^2(1+|x|)^{m+4}\,d\mu + C\int_{\mathbb{R}^d} |f|^2(1+|x|)^m\,d\mu, \quad (3.1)$$

$$\int_{\mathbb{R}^d} f^2 \log(f^2)\,d\mu - \int_{\mathbb{R}^d} f^2\,d\mu \log\Bigl(\int_{\mathbb{R}^d} f^2\,d\mu\Bigr) \le C\int_{\mathbb{R}^d} |\nabla f|^2(1+|x|)^{m+4}\,d\mu + C. \quad (3.2)$$

*In particular, the inclusion $|\nabla f|^2 |\log|\nabla f||^{1-\delta} \in L^1(\mu)$, where $\delta$ is a number from $(0,1)$, yields the inclusion $f^2\log(f^2) \in L^1(\mu)$.*

*Proof.* The reasoning practically repeats part of the justification of an analogous (and even more general) inequality from the paper [1]. An important role is played by the following estimates from [27]:

$$\Bigl|\frac{\nabla\varrho_\mu(x)}{\varrho_\mu(x)}\Bigr| \le C_1(1+|x|)^m, \quad \exp\bigl(-C_2(1+|x|)^{m+1}\bigr) \le \varrho(x) \le \exp\bigl(-C_3(1+|x|)^{m+1}\bigr).$$

Suppose first that $\|f\|_{L^2(\mu)} = 1$. We have

$$\int_{\mathbb{R}^d} f^2\log(f^2)\,d\mu = \int_{\mathbb{R}^d} f^2\Bigl(\log(f^2) + \log\varrho_\mu\Bigr)\,d\mu - \int_{\mathbb{R}^d} f^2\log\varrho_\mu\,d\mu. \quad (3.3)$$

Let us estimate the second term in the right-hand side. Let

$$\psi = -\log\varrho_\mu + \int_{\mathbb{R}^d}\log\varrho_\mu\,d\mu.$$

The bounds on $\varrho_\mu$ given above yield that $|\psi(x)| \le C_4(1+|x|)^{m+1}$. According to Example 4.9(iv) with $\alpha = 1$ and the estimate from Proposition 4.10 there exists a solution $u$ to the equation $Lu = \psi$ such that

$$|u(x)| \le C_5(1+|x|)^2, \quad |\nabla u(x)| \le C_5(1+|x|)^{m+2}.$$

Since $\|f\|_{L^2(\mu)} = 1$, we have

$$-\int_{\mathbb{R}^d} f^2\log\varrho_\mu\,d\mu = \int_{\mathbb{R}^d} f^2 Lu\,d\mu - \int_{\mathbb{R}^d}\log\varrho_\mu\,d\mu,$$

where the second term is estimated by some constant and the first term after integration by parts takes the following form:

$$\int_{\mathbb{R}^d} -2\langle A_\mu \nabla u, \nabla f\rangle f\, d\mu + \int_{\mathbb{R}^d} 2\langle \nabla f, b_\mu - \beta_\mu\rangle u f\, d\mu.$$

Applying to this expression the estimates for the functions $u$, $|b_\mu|$ and $|\beta_\mu|$, we arrive at the inequality

$$\int_{\mathbb{R}^d} f^2 Lu\, d\mu \le C_6 \int_{\mathbb{R}^d} |f||\nabla f|(1+|x|)^{m+2}\, d\mu.$$

From the elementary estimate

$$|f||\nabla f|(1+|x|)^{m+2} \le 2^{-1}|f|^2(1+|x|)^m + 2^{-1}|\nabla f|^2(1+|x|)^{m+4}$$

we obtain

$$-\int_{\mathbb{R}^d} f^2 \log \varrho_\mu\, d\mu \le C_7 + C_7\int_{\mathbb{R}^d} |\nabla f|^2(1+|x|)^{m+4}\, d\mu + C_7\int_{\mathbb{R}^d} |f|^2(1+|x|)^m\, d\mu.$$

Let us consider the first term in the right-hand side of (3.3). Since for $\delta > 0$ we have $t \le C_\delta e^{\delta t}$, the inequality

$$\int_{\mathbb{R}^d} f^2\Bigl(\log(f^2) + \log\varrho_\mu\Bigr)\, d\mu \le C_\delta \int_{\mathbb{R}^d} |f|^{2+2\delta}\varrho_\mu^{1+\delta}\, dx$$

holds. Applying Hölder's inequality, one can estimate the integral in the right-hand side by the expression

$$\|f^2\varrho_\mu\|_{L^{d/(d-1)}(\mathbb{R}^d)}\biggl(\int_{\mathbb{R}^d} |f|^{2\delta d}\varrho_\mu^{\delta d}\, dx\biggr)^{1/d}.$$

Let $\delta = 1/d$. Then the second multiplier equals 1. Let us apply to $\|f^2\varrho_\mu\|_{L^{d/(d-1)}(\mathbb{R}^d)}$ the Sobolev inequality

$$\|f^2\varrho_\mu\|_{L^{d/(d-1)}(\mathbb{R}^d)} \le 2C(d)\int_{\mathbb{R}^d} |f||\nabla f|\, d\mu + C(d)\int_{\mathbb{R}^d} \frac{|\nabla\varrho_\mu|}{\varrho_\mu}|f|^2\, d\mu.$$

Since $|f||\nabla f| \le 2^{-1}|f|^2 + 2^{-1}|\nabla f|^2$ and $|\nabla\varrho_\mu(x)|\varrho_\mu(x)^{-1} \le C_2(1+|x|)^m$, we have

$$\int_{\mathbb{R}^d} f^2\Bigl(\log(f^2) + \log\varrho_\mu\Bigr)\, d\mu \le C_8\int_{\mathbb{R}^d} |\nabla f|^2\, d\mu + C_8\int_{\mathbb{R}^d} |f|^2(1+|x|)^m\, d\mu.$$

Combining the obtained estimates and replacing $f$ by $f/\|f\|_{L^2(\mu)}$, we arrive at the inequality

$$\int_{\mathbb{R}^d} f^2\log(f^2)\, d\mu - \int_{\mathbb{R}^d} f^2\, d\mu \log\biggl(\int_{\mathbb{R}^d} f^2\, d\mu\biggr)$$
$$\le C_9\int_{\mathbb{R}^d} |\nabla f|^2(1+|x|)^{m+4}\, d\mu + C_9\int_{\mathbb{R}^d} |f|^2(1+|x|)^m\, d\mu.$$

Thus, inequality (3.1) is proved. Applying to the product $|f|^2(1+|x|)^m$ the elementary inequality $ab \le \varepsilon a\log a + e^{b/\varepsilon}$ with a sufficiently small $\varepsilon$ and taking into account that the function $\exp(M|x|^m)$ belongs to $L^1(\mu)$ for every $M$ (see [12, Section 2.3]), we obtain inequality (3.2). The last assertion of the theorem is verified similarly. □

## 4. The Poisson equation

Let $A$ and $b$ satisfy conditions ($\mathrm{H_a}$) and ($\mathrm{H_b}$) with $p > d$. Suppose that there exist a positive function $V \in C^2(\mathbb{R}^d)$ and a number $\gamma > 0$ such that

$$\lim_{|x|\to+\infty} V(x) = +\infty \quad \text{and } L_{A,b}V \le -\gamma \text{ outside some ball } Q.$$

Then there exists a unique probability solution $\mu$ to the equation $L^*_{A,b}\mu = 0$, moreover, $\mu = \varrho\,dx$, where $\varrho \in W^{p,1}_{loc}(\mathbb{R}^d)$.

Let us also fix a larger ball $Q_1 \supset Q$. There exists a sufficiently large number $c$ for which the bounded domain $B = \{x\colon V(x) < c\}$ contains the closure of $Q_1$ and has the boundary $\partial B = \{x\colon V(x) = c\}$ of finite perimeter. The latter is true for almost every number $c$, as shown in [20, § 5.5], moreover, for $B$ and $\partial B$ the usual formula of reducing the volume integral to the surface integral is valid.

Let $\psi \in C_0^\infty(\mathbb{R}^d)$ and

$$\int_{\mathbb{R}^d} \psi\varrho\,dx = 0.$$

For the sequel the following observation will be useful: if a function $u \in W^{p,2}(B)$ is such that $u - M \in W^{p,1}_0(B)$ for some constant $M$ (we shall say that $u$ is constant on $\partial B$), then

$$\int_B L_{A,b}u\,\varrho\,dx = \int_{\partial B} \langle A\nabla u, \nu\rangle \varrho\,ds, \tag{4.1}$$

where $\nu$ is the outer normal to $\partial B$. In addition, if $L_{A,b}u = \psi$ on $B$, the support of $\psi$ is contained in $B$ and the function $u$ is constant on $\partial B$, then

$$\int_{\partial B} \langle A\nabla u, \nu\rangle \varrho\,ds = 0.$$

Let us consider on $B$ the Dirichlet problem

$$L_{A,b}u = \psi, \quad u|_{\partial B} = 0.$$

It is known (see [28, §5.6]) that the solution $u$ exists and belongs to $W^{p,2}(B) \cap W^{p,1}_0(B)$, where $W^{p,1}_0(B)$ is the closure of $C_0^\infty(B)$ with respect to the Sobolev norm $\|\cdot\|_{p,1}$.

**Lemma 4.1.** *We have the estimate*

$$\int_B |\sqrt{A}\nabla u|^2 \varrho\,dx = -\int_B \psi u\varrho\,dx.$$

*Proof.* Since $u^2$ and $\nabla(u^2)$ vanish on $\partial B$, by the integration by parts we obtain the equality

$$\int_B L_{A,b}(u^2)\varrho\,dx = 0.$$

It remains to observe that

$$L_{A,b}(u^2) = 2uL_{A,b}u + 2|\sqrt{A}\nabla u|^2 = 2u\psi + 2|\sqrt{A}\nabla u|^2.$$

The lemma is proved. □

Let $f \in C^2((0, +\infty))$, $f > 0$, $f' > 0$. Set

$$H(V) = \frac{f'(V)}{f(V)}, \quad G(V) = \frac{f'(V)}{f(V)}LV + \frac{f''(V)}{f(V)}|\sqrt{A}\nabla V|^2,$$

$$\Psi = \frac{\psi}{f(V)}.$$

Suppose that there exists a positive continuous function $\Theta$ such that

$$G(V)+|\sqrt{A}\nabla V|^2H(V)^2\le -\Theta \quad \text{on } \mathbb{R}^d\backslash Q. \tag{4.2}$$

This function $\Theta$ will be involved in the hypotheses of several results below.

Let us subtract from the function $u$ the quantity $|Q|^{-1}\int_Q u\,dx$ and retain the same symbol. Now

$$\int_Q u\,dx=0$$

and the function $u$ is constant on $\partial B$. Note that for the new function $u$ the estimate from Lemma 4.1 holds.

The function

$$w:=f(V)^{-1}u$$

satisfies the equation

$$L_{A,b}w+2\langle A\nabla w,\nabla V\rangle H(V)+wG(V)=\Psi$$

with the right-hand side $\Psi$ defined above and is constant on $\partial B$.

**Lemma 4.2.** *For every $m\ge 1$, we have the estimate*

$$\int_B w^{2m}\Theta\varrho\,dx\le 2m\int_Q w^{2m}\bigl(\Theta+G(V)+|\sqrt{A}\nabla V|^2H(V)^2\bigr)\varrho\,dx+\int_B \Theta^{1-2m}\Psi^{2m}\varrho\,dx.$$

*Proof.* According to equality (4.1) we have

$$\int_B L_{A,b}(w^{2m})\varrho\,dx=\int_{\partial B}2mw^{2m-1}\langle A\nabla w,\nu\rangle\varrho\,ds,$$

where $\nu=\nabla V/|\nabla V|$, since $\partial B=\{V=c\}$. Let $u=u_c$ on $\partial B$. Then $w=w_c=u_cf(c)^{-1}$ on $\partial B$. Since

$$\nabla w=\frac{\nabla u}{f(c)}-H(c)w_c\nabla V \quad \text{on } \partial B,$$

we have

$$\begin{aligned}\int_{\partial B}&2mw^{2m-1}\langle A\nabla w,\nu\rangle\varrho\,ds\\ &=2mw_c^{2m-1}f(c)^{-1}\int_{\partial B}\langle A\nabla u,\nu\rangle\varrho\,ds-2mw_c^{2m}H(c)\int_{\partial B}\langle A\nabla V,\nabla V\rangle|\nabla V|^{-1}\varrho\,dx.\end{aligned}$$

The first integral in the right-hand side equals zero and the second integral is nonnegative. Therefore, one has

$$\int_B L_{A,b}(w^{2m})\varrho\,dx\le 0.$$

Let us write $L_{A,b}(w^{2m})$ in more detail:

$$\begin{aligned}(2m)^{-1}L_{A,b}(w^{2m})&=w^{2m-1}L_{A,b}w+(2m-1)w^{2m-2}|\sqrt{A}\nabla w|^2\\ &=-2w^{2m-1}\langle A\nabla w,\nabla V\rangle H(V)-w^{2m}G(V)+w^{2m-1}\Psi+(2m-1)w^{2m-2}|\sqrt{A}\nabla w|^2.\end{aligned}$$

Since

$$2w^{2m-1}\langle A\nabla w,\nabla V\rangle H(V)\le|\sqrt{A}\nabla w|^2w^{2m-2}+|\sqrt{A}\nabla V|^2H(V)^2w^{2m},$$

we have

$$(2m)^{-1}L_{A,b}(w^{2m})\ge -w^{2m}\bigl(G(V)+|\sqrt{A}\nabla V|^2H(V)^2\bigr)+w^{2m-1}\Psi.$$

Multiplying the last inequality by $\varrho$ and integrating over $B$, we arrive at the estimate

$$-\int_B w^{2m}\bigl(G(V)+|\sqrt{A}\nabla V|^2 H(V)^2\bigr)\varrho\,dx \le -\int_B w^{2m-1}\Psi\varrho\,dx.$$

Since

$$G(V)+|\sqrt{A}\nabla V|^2 H(V)^2 \le -\Theta \quad \text{outside } Q,$$

we have

$$\int_B w^{2m}\Theta\varrho\,dx \le \int_Q w^{2m}\bigl(\Theta+G(V)+|\sqrt{A}\nabla V|^2 H(V)^2\bigr)\varrho\,dx - \int_B w^{2m-1}\Psi\varrho\,dx.$$

Let us estimate the expression $w^{2m-1}\Psi$ with the aid of Young's inequality:

$$|w^{2m-1}\Psi| \le (1-(2m)^{-1})w^{2m}\Theta + (2m)^{-1}\Theta^{1-2m}\Psi^{2m}.$$

Applying this estimate to $\displaystyle\int_B w^{2m-1}\Psi\varrho\,dx$, we obtain the assertion of the lemma. □

**Corollary 4.3.** *We have the estimate*

$$\sup_B\Bigl|\frac{u}{f(V)}\Bigr| \le \sup_Q\Bigl|\frac{u}{f(V)}\Bigr| + \sup_B\Bigl|\frac{\psi}{f(V)\Theta}\Bigr|$$

*Proof.* According to Lemma 4.2 we have

$$\Bigl(\int_B w^{2m}\Theta\varrho\,dx\Bigr)^{1/2m}$$
$$\le (2m)^{1/2m}\max_Q|w|\Bigl(\int_Q\bigl|\Theta+G(V)+|\sqrt{A}\nabla V|^2H(V)^2\bigr|\varrho\,dx\Bigr)^{1/2m} + \Bigl(\int_B\Bigl|\frac{\Psi}{\Theta}\Bigr|^{2m}\Theta\varrho\,dx\Bigr)^{1/2m}.$$

Letting $m\to\infty$, we obtain the desired estimate on $\sup_B|uf(V)^{-1}|$. □

We observe that the condition $L_{A,b}V\le-\gamma$ outside the ball $Q$, indicated at the beginning of the section, along with Harnack's inequality enables us to estimate the value $\varrho(0)$ from below, hence also estimate $\varrho$ from below on any ball centered at the origin. Indeed, by Harnack's inequality there exists a number $C_1$ such that on the ball $Q_1$ we have

$$C_1^{-1}\varrho(0)\le\varrho(x)\le C_1\varrho(0).$$

The condition $L_{A,b}V\le-\gamma$ outside $Q$ can be written as $LV\le-\gamma+(LV)I_Q$ everywhere on $\mathbb{R}^d$. It is known (see [12, Theorem 2.3.2]) that

$$\gamma\le\int_Q|LV|\varrho\,dx,$$

which along with the previous estimates gives the two-sided estimate

$$\gamma C_1^{-1}\Bigl(\int_Q|LV|\,dx\Bigr)\le\varrho(0)\le C_1\varrho(x)\quad\text{for all}\quad x\in Q_1.$$

In addition, the known a priori estimate (see [12, Theorem 1.7.4])

$$\|\varrho\|_{W^{p,1}(Q)}\le C_2\|\varrho\|_{L^1(Q_1)}$$

and the equality $\|\varrho\|_{L^1(\mathbb{R}^d)}=1$ yield the existence of a constant $C_3$ for which

$$\sup_Q\varrho\le C_3.$$

The constants $C_1$, $C_2$ and $C_3$ depend only on $d$, $\|a^{ij}\|_{W^{p,1}(Q_1)}$, $\|b^i\|_{L^p(Q_1)}$, and $\sup_{x\in Q_1}\|A(x)^{-1}\|$.

**Corollary 4.4.** *Suppose (only in this corollary) that the coefficient $b$ is locally bounded. Then we have the estimate*

$$\int_B \Big|\frac{u}{f(V)}\Big|^2 \Theta\varrho\,dx \le C\int_B \frac{|\psi|^2 f(V)^2}{\Theta}\varrho\,dx,$$

*where $C$ depends on $d$, $\|a^{ij}\|_{W^{p,1}(Q_1)}$, $\sup_{x\in Q_1}|b^i(x)|$, $\sup_{x\in Q_1}\|A(x)^{-1}\|$, $\Theta$, $f$, $V$, $\gamma$, and $c_1$.*

*Proof.* By Lemma 4.2 with $m=1$ we have

$$\int_B \Big|\frac{u}{f(V)}\Big|^2 \Theta\varrho\,dx \le C_1\int_Q u^2\varrho\,dx + \int_B \frac{|\psi|^2 f(V)^2}{\Theta}\varrho\,dx. \tag{4.3}$$

According to the remark made above there exists a constant $C_2$ such that $C_2^{-1}\le\varrho\le C_2$ on $Q$. Applying these inequalities and the Poincaré inequality for estimating the $L^2$-norm of $u$, we obtain

$$\int_Q u^2\varrho\,dx \le C_3\int_Q |A^{1/2}\nabla u|^2\varrho\,dx.$$

With the aid of Lemma 4.1 we obtain

$$\int_Q u^2\varrho\,dx \le C_3\int_B |\psi|\,|u|\varrho\,dx,$$

where the right-hand side can be estimated as follows:

$$C_3\int_B |\psi|\,|u|\varrho\,dx \le \frac{1}{2C_1}\int_B \Big|\frac{u}{f(V)}\Big|^2\Theta\varrho\,dx + \frac{1}{2}C_1C_3^2\int_B \frac{|\psi|^2 f(V)^2}{\Theta}\varrho\,dx.$$

Applying these estimates to the right-hand side of (4.3), we complete the proof. □

**Example 4.5.** Suppose that the coefficient $b$ is locally bounded and there exists a function $V\in C^2(\mathbb{R}^d)$ such that $\lim\limits_{|x|\to+\infty} V(x)=+\infty$ and for some number $\gamma>0$ one has

$$L_{A,b}V(x) + |\sqrt{A(x)}\nabla V(x)|^2 \le -\gamma \quad \text{if } x\notin Q.$$

Then

$$\int_B u^2\varrho\,dx \le C\int_B \psi^2\varrho\,dx,$$

where $C$ is a constant of the type indicated in Corollary 4.4. Indeed, it suffices to apply Corollary 4.4 with the functions $\Theta=\gamma$ and $f(t)=\operatorname{arctg}(\log(1+t))$. Combining this estimate with the assertion of Lemma 4.1, we obtain the inequality

$$\int_B |\sqrt{A}\nabla u|^2\varrho\,dx \le C\int_B \psi^2\varrho\,dx. \tag{4.4}$$

**Corollary 4.6.** *We have the estimate*

$$\sup_B\Big|\frac{u}{f(V)}\Big| \le C\int_B |\psi| f(V)\varrho\,dx + C\sup_B\Big|\frac{\psi}{f(V)\Theta}\Big|,$$

*where $C$ depends on $d$, $\|a^{ij}\|_{W^{p,1}(Q_1)}$, $\|b^i\|_{L^p(Q_1)}$, $\sup_{x\in Q_1}\|A(x)^{-1}\|$, $\Theta$, $f$, $V$, $\gamma$, and $c_1$.*

*Proof.* The known estimate (see [12, Theorem 1.7.4])

$$\|u\|_{W^{p,1}(Q)} \le C_1\big(\|u\|_{L^2(Q_1)} + \|\psi\|_{L^p(Q_1)}\big)$$

with some number $C_1$ and the Sobolev embedding theorem yield the estimate

$$\max_Q |u| \le C_2\|u\|_{L^2(Q_1)} + C_2\sup_{Q_1}|\psi|. \tag{4.5}$$

The Poincaré inequality gives

$$\|u\|_{L^2(Q_1)} \le C_P \|\nabla u\|_{L^2(Q_1)}.$$

According to what has been said above, there exists $C_H > 0$ such that $\varrho(x) \ge C_H$ if $x \in Q_1$. Therefore,

$$\sup_Q |u| \le C_3 \|\|\sqrt{A}\nabla u\|_{L^2(\varrho\, dx, B)} + C_3 \sup_{Q_1} |\psi|,$$

where $C_3$ depends on $C_P$, $C_H$, $C_2$ and $\sup_{x\in Q_1} \|A(x)^{-1}\|$. By Lemma 4.1 we have

$$\|\sqrt{A}\nabla u\|^2_{L^2(\varrho\, dx)} \le \int_B |\psi|\, |u| \varrho\, dx.$$

Recall that $u = wV$. We arrive at the inequality

$$\sup_Q |u| \le C_3 \Bigl(\int_B |\psi|\, |u|\varrho\, dx\Bigr)^{1/2} + C_3 \sup_{Q_1} |\psi|,$$

which yields the estimate

$$\sup_Q |u| \le C_3 \Bigl(\sup_B |uf(V)^{-1}|\Bigr)^{1/2} \Bigl(\int_B |\psi| f(V)\varrho\, dx\Bigr)^{1/2} + C_3 \sup_{Q_1} |\psi|.$$

Substituting this in the inequality from Corollary 4.3, we obtain

$$\sup_B |uf(V)^{-1}| \le C_4 \Bigl(\sup_B |uf(V)^{-1}|\Bigr)^{1/2} \Bigl(\int_B |\psi| f(V)\varrho\, dx\Bigr)^{1/2} + C_4 \sup_B |\psi f(V)^{-1}\Theta^{-1}|.$$

Estimating the first term in the right-hand side by means of the inequality

$$C_4 \Bigl(\sup_B |uf(V)^{-1}|\Bigr)^{1/2} \Bigl(\int_B |\psi| f(V)\varrho\, dx\Bigr)^{1/2} \le 2^{-1} \sup_B |uf(V)^{-1}| + 2^{-1} C_4^2 \int_B |\psi| f(V)\varrho\, dx,$$

we arrive at the desired estimate with some number $C$. □

Now we can prove an existence theorem for the equation $L_{A,b}u = \psi$. Recall that $V$ is the function mentioned at the beginning of this section and the function $\Theta$ is introduced in (4.2).

**Theorem 4.7.** *Suppose that condition* (4.2) *is fulfilled. Then, for every function* $\psi \in L^1(\mu)$ *such that*

$$\sup_{\mathbb{R}^d} \Bigl|\frac{\psi}{f(V)\Theta}\Bigr| < \infty \quad \textit{and} \quad \int_{\mathbb{R}^d} \psi\, d\mu = 0,$$

*there exists a function* $u \in W^{2,p}_{loc}(\mathbb{R}^d)$ *satisfying the equation* $L_{A,b}u = \psi$ *and the inequality*

$$\sup_{\mathbb{R}^d} \Bigl|\frac{u}{f(V)}\Bigr| \le C \int_{\mathbb{R}^d} |\psi| f(V)\varrho\, dx + C \sup_{\mathbb{R}^d} \Bigl|\frac{\psi}{f(V)\Theta}\Bigr|,$$

*where* $C$ *depends only on* $d$, $\|a^{ij}\|_{W^{p,1}(Q_1)}$, $\|b^i\|_{L^p(Q_1)}$, $\sup_{x\in Q_1} \|A(x)^{-1}\|$, $\Theta$, $f$, $\gamma$, $c_1$, *and also on the minimum of the function* $V$ *on* $Q_1$ *and the maximum of the function* $V$ *and absolute values of its first and second derivatives on* $Q_1$.

*Proof.* We first consider the case where $\psi \in C_0^\infty(\mathbb{R}^d)$. Let $\{c_n\}$ be a sequence of numbers increasing to $+\infty$ and let $B_n = \{x\colon V(x) < c_n\}$ be bounded domains with sufficiently smooth boundaries $\partial B_n = \{x\colon V(x) = c_n\}$. We shall consider only those numbers $n$ for which the support of $\psi$ is contained in $B_n$. Let $u_n$ be the solution to the Dirichlet problem

$$L_{A,b}u_n = \psi, \quad u_n|_{\partial B_n} = 0.$$

Let us subtract from $u_n$ the quantity $|Q|^{-1}\int_Q u_n\,dx$ and denote the new function again by $u_n$. As proved above, for $u_n$ one has the estimate (with a constant independent of $n$)

$$\sup_{B_n}\Bigl|\frac{u_n}{f(V)}\Bigr| \le C\int_{\mathbb{R}^d}|\psi| f(V)\varrho\,dx + C\sup_{\mathbb{R}^d}\Bigl|\frac{\psi}{f(V)\Theta}\Bigr|.$$

Thus, the sequence $\{u_n\}$ is uniformly bounded on every ball (starting from a number depending on this ball), which implies the uniform boundedness of the Sobolev norms $\|u_n\|_{W^{2,p}}$ on every ball, because $u_n$ satisfies the equation $L_{A,b}u_n = \psi$ (see [28, Theorem 5.6.2]). Therefore, passing to a subsequence, we can assume that the functions $u_n$ converge uniformly and in $W^{2,p}$ on every ball to some solution $u$ of the equation $L_{A,b}u = \psi$. It is clear that for this solution the indicated estimate is fulfilled. The case of a general function $\psi$ reduces to the considered one as follows. Let $\zeta_n \in C_0^\infty$ and $\zeta_n \to \psi$ in $L^1(\mu)$. Let us fix some nonnegative function $\zeta \in C_0^\infty(\mathbb{R}^d)$ with $\|\zeta\|_{L^1(\mu)} = 1$. Set

$$\psi_n = \zeta_n - \zeta\int_{\mathbb{R}^d}\zeta_n\,d\mu.$$

We now solve the equation with the right-hand side $\psi_n$ and choose in the constructed sequence of solutions a convergent subsequence. □

Let us also give a sufficient condition for the uniqueness of a solution, which is a simple application of the maximum principle.

**Theorem 4.8.** *Suppose that there exist positive functions $V_1$ and $V_2$ and a ball $Q$ such that*

$$L_{A,b}V_1(x) \le 0 \quad \textit{if} \quad x \notin Q \quad \textit{and} \quad \lim_{|x|\to+\infty}\frac{V_1(x)}{V_2(x)} = +\infty.$$

*Then the equation $L_{A,b}u = 0$ has no nonconstant solutions in the class of functions $u$ for which*

$$\sup_{\mathbb{R}^d}\Bigl|\frac{u}{V_2}\Bigr| < \infty.$$

*Proof.* Let us consider the function $u - \varepsilon V_1$. We have

$$L_{A,b}(u - \varepsilon V_1) \ge 0 \quad \text{if} \quad x \notin Q.$$

Since $u$ is growing not faster than $V_2$ and $\lim\limits_{|x|\to+\infty} V_1(x)V_2(x)^{-1} = +\infty$, outside some ball the function $u - \varepsilon V_1$ is negative and by the maximum principle

$$u(x) \le \varepsilon V_1(x) + \sup_Q(u - \varepsilon V_1).$$

Letting $\varepsilon \to 0$, we obtain $u \le \sup_Q u$. This means that the maximum of the solution is attained in the interior of the domain and by the strong maximum principle the solution is constant. □

Let us consider some examples.

Let $V(x) = 1 + |x|^2$, let the matrix $A$ be uniformly bounded, and let

$$\langle b(x), x\rangle \le C_1 - C_2|x|^\gamma.$$

Let $\alpha < \gamma$. Since $L_{A,b}(e^{C|x|^\alpha}) < 0$ outside some ball, by Theorem 4.8 any two solutions growing not faster than $e^{C|x|^\alpha}$ with some $C$ differ by a constant. Thus, the solutions considered in the following examples illustrating Theorem 4.7 are unique in the class of functions which grow not faster than the indicated exponent, i.e., satisfy the condition

$$\sup_{|x|>1}\frac{\log(1 + |u(x)|)}{|x|^\alpha} < \infty. \tag{4.6}$$

**Example 4.9.** (i) Let $f(t) = \log t$. The condition of Theorem 4.7 is fulfilled with the function

$$\Theta(x) = M(1 + |x|^{\gamma-2})(\log(1 + |x|))^{-1}$$

with a suitable number $M$. Thus, for a function $\psi$ such that

$$|\psi(x)| \le C_2 + C_2|x|^{\gamma-2},$$

there exists a solution $u$ to the equation $L_{A,b}u = \psi$ which is unique in the class of functions with condition (4.6) and for this solution we have

$$|u(x)| \le C + C \log |x|.$$

(ii) Let $f(t) = \operatorname{arctg}(\log t)$. The condition of Theorem 4.7 is fulfilled with the function

$$\Theta(x) = M(1 + |x|^{\gamma-2})(\log(1 + |x|))^{-2}$$

with a suitable number $M$. For a function $\psi$ such that

$$|\psi(x)| \le C_2 + C_2(\log(1 + |x|))^{-2}|x|^{\gamma-2},$$

there exists a solution $u$ to the equation $L_{A,b}u = \psi$ bounded on $\mathbb{R}^d$ and this solution is unique in the class of functions with condition (4.6).

(iii) Let now $f(t) = e^t$ and $\gamma > 2$. The conditions of Theorem 4.7 are fulfilled with $\Theta(x) = M(1 + |x|)^{\gamma}$ with a suitable number $M$. Thus, for a function $\psi$ such that

$$|\psi(x)| \le C_2(1 + |x|)^{-\gamma}e^{|x|^2},$$

there exists a solution $u$ to the equation $L_{A,b}u = \psi$ which is unique in the class of functions with condition (4.6) and for this solution we have

$$|u(x)| \le Ce^{|x|^2}.$$

(iv) Let $f(t) = t^{\alpha}$. Then the condition of the theorem is fulfilled with $\Theta(x) = (1 + |x|)^{\gamma-1}$. Thus, for a function $\psi$ such that

$$|\psi(x)| \le C_2(1 + |x|)^{2\alpha+\gamma-1},$$

there exists a unique solution $u$ to the equation $L_{A,b}u = \psi$ such that

$$|u(x)| \le C(1 + |x|)^{2\alpha}.$$

In the next proposition, using an idea from [29] and taking into account the growth of coefficients, we derive an estimate on the growth of the gradient of the solution.

**Proposition 4.10.** *Let $A$ and $A^{-1}$ be uniformly bounded and let the matrix elements $a^{ij}$ be uniformly continuous. Then there exists a constant $C$, depending on the coefficients of the equation, such that the solution $u$ to the equation $L_{A,b}u = \psi$ satisfies the pointwise estimate*

$$|\nabla u(x)| \le C(1 + \sup_{y\in B(x,1)} |b(y)|) \sup_{y\in B(x,1)} |u(y)| + C(1 + \sup_{y\in B(x,1)} |b(y)|)^{-1} \sup_{y\in B(x,1)} |\psi(y)|.$$

*In particular, if*

$$|\psi(x)| \le C_1(1 + |x|)^k, \quad |u(x)| \le C_1(1 + |x|)^m, \quad |b(x)| \le C_1(1 + |x|)^t,$$

*then*

$$|\nabla u(x)| \le C_2\Big((1 + |x|)^{m+t} + C_2(1 + |x|)^k\Big).$$

*Proof.* We shall assume that $x = 0$. The general case reduces to this one by a shift. Let $\lambda = 1 + \sup_{B(0,1)} |b|$. Set $u(x) = v(\lambda x)$. Then the function $v$ satisfies the equation with the new coefficients

$$a^{ij}_\lambda(y) = a^{ij}(y/\lambda),\ b^i_\lambda(y) = \lambda^{-1} b(y/\lambda),\ \psi_\lambda(y) = \lambda^{-2}\psi(y/\lambda).$$

We observe that $|b^i_\lambda| \le 1$ and the modulus of continuity of the function $a^{ij}_\lambda(y)$ is estimated by the modulus of continuity of the function $a^{ij}$. Let $q > d$. According to [21, Theorem 9.11] one has the estimate

$$\|v\|_{W^{2,q}(B(0,1/2))} \le C(\|v\|_{L^q(B(0,1))} + \|\psi_\lambda\|_{L^q(B(0,1))}).$$

Estimating the norms in the right-hand side by $\sup |v|$ and $\sup |\psi_\lambda|$, and applying to the left-hand side the Sobolev embedding theorem, we arrive at the estimate

$$\sup_{B(0,1/2)} |\nabla v| \le C'(\sup_{B(0,1)} |v| + \sup_{B(0,1)} |\psi_\lambda|).$$

Returning to the function $u$ and coordinates $x$, we obtain the desired inequality. □

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