Uniqueness for nonlinear Fokker–Planck equations and weak uniqueness for McKean-Vlasov SDEs

Viorel Barbu[*] Michael Röckner[†‡]

Abstract

One proves the uniqueness of distributional solutions to nonlinear Fokker–Planck equations with monotone diffusion term and derive as a consequence (restricted) uniqueness in law for the corresponding McKean–Vlasov stochastic differential equation (SDE).

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1 Introduction

Consider the nonlinear Fokker–Planck equation

$$\begin{aligned} &u_t - \Delta\beta(u) + \operatorname{div}(b(x,u)u) = 0 \text{ in } \mathcal{D}'((0,\infty)\times\mathbb{R}^d),\\ &u(0,x) = u_0(x), \end{aligned} \tag{1.1}$$

where $\beta : \mathbb{R} \to \mathbb{R}$ and $b : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$ satisfy the following assumptions

[*]Octav Mayer Institute of Mathematics of the Romanian Academy, Iaşi, Romania. Email: vbarbu41@gmail.com

[†]Fakultät für Mathematik, Universität Bielefeld, D-33501 Bielefeld, Germany. Email: roeckner@math.uni-bielefeld.de

[‡]Academy of Mathematic and System Sciences, CAS, Beijing, China

(i) $\beta(0)=0$, $\beta\in C^1(\mathbb{R})$, *and*

$$\gamma_0|r_1-r_2|^2\le(\beta(r_1)-\beta(r_2))(r_1-r_2),\ r_1,r_2\in\mathbb{R}, \tag{1.2}$$

where $0<\gamma_0<\infty$.

(ii) $b\in C_b(\mathbb{R}^{d+1};\mathbb{R}^d)$, $b(x,\cdot)\in C^1(\mathbb{R}^d;\mathbb{R}^d)$ $\forall x\in\mathbb{R}^d$ *such that*

$$\sup\{|b^i_r(x,r)|; x\in\mathbb{R}^d, i=1,2,\ |r|\le M\}\le C_M,\ \forall M>0.$$

(ii') $b(x,0)=0$ $\forall x\in\mathbb{R}^d$, $b\in C^1(\mathbb{R}^{d+1};\mathbb{R}^d)$ *and for*

$$\delta(r):=\sup\{|b_x(x,r)|;\ x\in\mathbb{R}^d\},$$

we have $\delta\in C_b(\mathbb{R})$.

Here

$$b(x,u)=\{b^i(x,u)\}^d_{i=1}\ \text{and}\ b^i_r=\frac{\partial b^i}{\partial r}, b_x=\left\{\nabla_x b^i(x,\cdot)\right\}^d_{i=1}.$$

By a distributional solution (in the sense of Schwartz) with initial condition $u_0\in L^1$ we mean a function $u:[0,\infty)\to L^1(\mathbb{R}^d)$ such that $(u(t,\cdot)dx)_{t\in[0,T]}$ is narrowly continuous, that is,

$$\lim_{t\to s}\int_{\mathbb{R}^d}u(t,x)\psi(x)dx=\int_{\mathbb{R}^d}u(s,x)\psi(x)dx,\ \forall\psi\in C_b(\mathbb{R}^d),\ s\ge0, \tag{1.3}$$

$$\begin{aligned}\int_0^\infty\int_{\mathbb{R}^d}&(u(t,x)\varphi_t(t,x)+\beta(u(t,x))\Delta\varphi(t,x)\\&+b(x,u(t,x))u(t,x))\cdot\nabla_x\varphi(t,x))dt\,dx=0,\\&\qquad\forall\varphi\in C_0^\infty((0,\infty)\times\mathbb{R}^d).\end{aligned} \tag{1.4}$$

(In the following, we shall use the notation $b^*(x,u)=b(x,u)u$.)

In [1] it was proved, in particular, that, if (i)–(ii) hold and, in addition, for $\Phi(u)\equiv\frac{\beta(u)}{u}$, $u\in\mathbb{R}$, we have $\Phi\in C^2(\mathbb{R})$, then there is a mild solution $u\in C([0,\infty);L^1(\mathbb{R}^d))$ for each $u_0\in L^1(\mathbb{R}^d)$. The mild solution u is defined as

$$u(t)=\lim_{h\to0}u_h(t)\ \text{in}\ L^1(\mathbb{R}^d),\ \forall t\ge0,$$

where

$$\begin{aligned}&u_h(t)=u^i_h\ \text{for}\ t\in[ih,(i+h)h],\ i=0,1,...,Nh=T,\\&u^{i+1}_h-h\Delta\beta(u^{i+1}_h)+h\,\text{div}(b(x,u^{i+1}_h)u^{i+1}_h)=u^i_h\ \text{in}\ \mathcal{D}'(\mathbb{R}^d),\\&\qquad\qquad i=0,1,...,\\&u^0_h=u_0.\end{aligned} \tag{1.5}$$

Moreover, $S(t)u_0 = u(t),\ t \geq 0$, is a strongly continuous semigroup of non-expansive mappings in $L^1(\mathbb{R}^d)$.

As easily seen, any mild solution is a distributional solution but the uniqueness follows in the class of mild solutions only. Here, we shall prove the uniqueness for (1.1) in the class of distributional solutions and derive from this result the uniqueness in law of solutions to McKeen–Vlasov SDE

$$dX(t) = b(X(t), u(t, X(t)))dt + \frac{1}{\sqrt{2}} \left(\frac{\beta(u(t, X(t))}{u(t, X(t))} \right)^{\frac{1}{2}} dW(t). \tag{1.6}$$

Notation. Denote by $L^p(\mathbb{R}^d) = L^p$ the space of p-summable functions on L^p, with the norm denoted $|\cdot|_p$. By $H^k(\mathbb{R}^d) = H^k$, $k = 1, 2$, and $H^{-k}(\mathbb{R}^d) = H^{-k}$, we denote the standard Sobolev spaces on $\mathbb{R}^d$ and by $C_b(\mathbb{R}^d)$ the space of continuous and bounded functions on $\mathbb{R}^d$. By $C^k(\mathbb{R}^d)$ we denote the space of continuously differentiable functions on $\mathbb{R}^d$ of order k, by $C_b^1(\mathbb{R}^d)$ the space $\left\{ u \in C^1(\mathbb{R}^d); \frac{\partial u}{\partial y} \in C_b(\mathbb{R}^d),\ j = 1, ..., d \right\}$. The spaces of continuous and differentiable functions on $(0, T) \times \mathbb{R}^d$ are denoted in a similar way and we shall simply write

$$C_b^1(\mathbb{R}^d) = C_b^1,\ C^k(\mathbb{R}^d) = C^k,\ k = 1, 2.$$

The scalar product in L^2 is denoted $\langle \cdot, \cdot \rangle_2$ and by ${}_{H^{-1}}\langle \cdot, \cdot \rangle_{H^1}$ the pairing between H^1 and H^{-1}. Of course, on $L^2 \times L^2$ this coincides with $\langle \cdot, \cdot \rangle_2$. The scalar product $\langle \cdot, \cdot \rangle_{-1}$ on H^{-1} is taken as

$$\langle u, v \rangle_{-1} = \left\langle (I - \Delta)^{-1} u, v \right\rangle_2,\ \forall u, v \in H^{-1} \tag{1.7}$$

with the corresponding norm

$$|u|_{-1} = (\langle u, u \rangle_{-1})^{\frac{1}{2}},\ u \in H^{-1}. \tag{1.8}$$

By $\mathcal{D}'((0, \infty) \times \mathbb{R}^d)$ and $\mathcal{D}'(\mathbb{R}^d)$ we denote the space of Schwartz distributions on $(0, \infty) \times \mathbb{R}^d$ and $\mathbb{R}^d$, respectively. If $\mathcal{X}$ is a Banach space, we denote by $W^{1,2}([0, T]; \mathcal{X})$ the infinite dimensional Sobolev space $\{y \in L^2(0, T; \mathcal{X});\ \frac{dy}{dt} \in L^2(0, T; \mathcal{X})\}$, where $\frac{d}{dt}$ is taken in the sense of vectorial distributions. We also set, for each $z \in C^1(\mathbb{R}^d \times \mathbb{R})$,

$$z_r(x, r) = \frac{\partial}{\partial r}\, z(x, r), \quad z_x = \nabla_x z(x, r).$$

We shall denote the norms on $\mathbb{R}^d$ and $\mathbb{R}$ by the same symbol $|\cdot|$.

2 The main result

The next result is a uniqueness theorem for distributional solutions u to (1.1). In the special case $b \equiv 0$, such a uniqueness result for(1.1) was established earlier in [4] for continuous and monotonically nondecreasing functions β.

Theorem 2.1. *Let $T > 0$ and let conditions* (i)–(ii) *on β and b hold. For each $u_0 \in L^\infty \cap L^1$, the Fokker–Planck equation* (1.1) *has at most one distributional solution*

$$u \in L^1((0,T); L^1) \cap L^\infty((0,T) \times \mathbb{R}^d). \tag{2.1}$$

Proof. Let $u_1, u_2 \in L^1(0,T; L^1) \cap L^\infty((0,T) \times \mathbb{R}^d)$ be two distributional solutions to (1.1) and let $u = u_1 - u_2$. We have

$$\begin{aligned} &u_t - \Delta(\beta(u_1) - \beta(u_2)) + \operatorname{div}(b^*(x,u_1) - b^*(x,u_2)) = 0 \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in } \mathcal{D}'((0,\infty) \times \mathbb{R}^d) \\ &u(0,x) = 0. \end{aligned} \tag{2.2}$$

(Here, $b^*(x,r) = b(x,r)r,\ \forall x \in \mathbb{R}^d,\ r \in \mathbb{R}$.)

It should be mentioned that, by (i), (ii), it follows that $u_i, \beta(u_i), b^*(\cdot, u_i) \in L^2((0,T); L^2)$, $i = 1, 2$, and, therefore, $u \in W^{1,2}([0,T]; H^{-2})$.

Consider the operator $\Gamma : H^{-1} \to H^1$ defined by

$$\Gamma u = (1 - \Delta)^{-1} u,\ \ u \in H^{-1}(\mathbb{R}^d)$$

and note that Γ is an isomorphism of H^{-1} onto H^1 and $\Gamma \in L(H^{-2}, L^2)$. Since $u_i \in L^2(0,T; L^2)$, $i = 1, 2$, it follows that $y = \Gamma u \in L^2(0,T; H^2) \cap W^{1,2}([0,T]; L^2)$ and so, by (2.2), we have

$$\begin{aligned} &\frac{dy}{dt} - \Gamma\Delta(\beta(u_1) - \beta(u_2)) + \Gamma \operatorname{div}(b^*(x,u_1) - b^*(x,u_2)) = 0, \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{a.e. } t \in (0,T), \\ &y(0) = 0, \end{aligned} \tag{2.3}$$

where $\frac{dy}{dt} \in L^2(0,T; L^2)$. (We note that here $\frac{dy}{dt}$ is taken in the sense of L^2-valued vectorial distributions on $(0,T)$ and so $y : [0,T] \to L^2$ is absolutely continuous.)

Assume first that, in addition,

$$u \in L^2(0,T; H^1);\ \ u \in W^{1,2}([0,T]; H^{-1}). \tag{2.4}$$

Now, we take the scalar product in L^2 of (2.3) with $u = u_1 - u_2$. Taking into account that

$$\left\langle \frac{du}{dt}(t), u(t) \right\rangle_{-1} = \frac{1}{2}\,\frac{d}{dt}\,|u(t)|_{-1}^2,\ \text{ a.e. } t \in (0,T),$$

we get, by (2.3) that

$$\begin{aligned}\frac{1}{2}\,\frac{d}{dt}\,|u(t)|_{-1}^2 + \langle \beta(u_1) - \beta(u_2), u_1 - u_2\rangle_2 &= \langle \Gamma(\beta(u_1) - \beta(u_2)), u_1 - u_2\rangle_2 \\ &\quad - \langle \Gamma\,\mathrm{div}((b^*(x,u_1) - b^*(x,u_2)), u_1 - u_2\rangle_2\,,\ \text{ a.e. } t \in (0,T).\end{aligned}$$

By (1.2), this yields for a.e. $t \in (0,T)$

$$\begin{aligned}\frac{1}{2}\,\frac{d}{dt}\,|u(t)|_{-1}^2 + \gamma_0|u(t)|_2^2 &\le \langle \beta(u_1(t)) - \beta(u_2(t)), u_1(t) - u_2(t)\rangle_{-1} \\ &\quad - \langle \mathrm{div}(b^*(x,u_1(t)) - b^*(x,u_2(t))), u_1 - u_2\rangle_{-1}\,.\end{aligned} \tag{2.5}$$

We note that

$$|\Gamma f|_2 \le |f|_2,\ \forall f \in L^2. \tag{2.6}$$

We also have

$$|\mathrm{div}\,F|_{-1} \le 2|F|_2,\ \forall F \in (L^2)^d. \tag{2.7}$$

This yields

$$|\,\langle \beta(u_1) - \beta(u_2), u_1 - u_2\rangle_{-1}\,| \le |\beta(u_1) - \beta(u_2)|_2 |u|_{-1} \le \beta_M |u|_2 |u|_{-1} \tag{2.8}$$

and

$$\begin{aligned}&\left|\langle \mathrm{div}\ (b^*(x,u_1) - b^*(x,u_2)), u_1 - u_2\rangle_{-1}\right| \\ &\quad \le 2|b^*(x,u_1) - b^*(x,u_2)|_2 |u_1 - u_2|_{-1} \\ &\quad \le 2(|b|_\infty + b_M |u_1|_\infty)|u|_2 |u|_{-1},\end{aligned} \tag{2.9}$$

where $M = \max\{|u_1|_\infty, |u_2|_\infty)$ and

$$\begin{aligned}\beta_M &= \sup\{\beta'(r);\ |r| \le M\}, \\ b_M &= \sup\left\{\frac{|b(x,u_1) - b(x,u_2)|}{|r_1 - r_2|};\ x \in \mathbb{R}^d,\ |r_1|, |r_2| < M\right\} \\ &\le \sup\{|b_r(x,r)|; |r| \le M,\ x \in \mathbb{R}^d\} < \infty\end{aligned}$$

by (ii).

By (2.5)–(2.9), we get

$$\frac{1}{2}\,\frac{d}{dt}\,|u(t)|_{-1}^2+\gamma_0|u(t)|_2^2\le(\beta_M+2(|b|_\infty+b_M|u_1|_\infty)|u(t)|_2|u(t)|_{-1},\qquad \text{a.e. } t\in(0,T).$$

This yields

$$\frac{d}{dt}\,|u(t)|_{-1}^2\le C|u(t)|_{-1}^2,\ \text{ a.e. } t\in(0,T).$$

Since $u:[0,T]\to H^{-1}$ is absolutely continuous and narrowly continuous, we infer that $|u(t)|_{-1}=0$, $\forall t\in[0,T]$, and so $u\equiv 0$, as claimed. To conclude the proof, we shall prove

Claim 2.2. *If u is a distributional solution to* (1.1) *satisfying* (2.1)*, then* (2.4) *holds.*

Proof of Claim 2.2. 1°. We shall first assume that u satisfies the stronger initial condition

$$\text{essential}\ \lim_{t\to 0}\int_{\mathbb{R}^d}|u(t,x)-u_0(x)|dx=0. \tag{2.10}$$

Next, we consider the truncated function

$$\beta_M(r)=\begin{cases}\beta(r) & \text{if } |r|\le M,\\ \beta(M)+\beta'(M)(r-M) & \text{if } r>M,\\ \beta(-M)+\beta'(-M)(r+M) & \text{if } r<-M,\end{cases} \tag{2.11}$$

where $M=\|u\|_{L^\infty(0,T)\times\mathbb{R}^d)}$.

We set

$$f(t,x)\equiv b^*(x,u(t,x)),\quad f_\nu=f*\rho_\nu,\quad \nu>0,$$

where ρ_ν is a standard mollifier on $\mathbb{R}^{d+1}$ and $b^*(x,u)\equiv b(x,u)u$.

We note that in equation (1.1) one can take $\beta\equiv\beta_M$ because $|u|_\infty\le M$.

We consider the equations

$$\begin{aligned}&\widetilde{u}_t-\Delta\beta_M(\widetilde{u})+\operatorname{div}(f)=0\ \text{in}\ (0,T)\times\mathbb{R}^d,\\ &\widetilde{u}(0)=u_0,\end{aligned} \tag{2.12}$$

$$\begin{aligned}&(\widetilde{u}_\nu)_t-\Delta\beta_M(\widetilde{u}_\nu)+\operatorname{div}(f_\nu)=0\ \text{in}\ (0,T)\times\mathbb{R}^d,\\ &\widetilde{u}_\nu(0)=u_0.\end{aligned} \tag{2.13}$$

Clearly, (2.12) has a unique solution $\widetilde{u}$ satisfying (2.4), i.e.,

$$\widetilde{u} \in L^2(0,T;H^1),\ \ \widetilde{u} \in W^{1,2}([0,T];H^{-1}), \tag{2.14}$$

while $\widetilde{u}_\nu$ is more regular, that is, additionally to (2.14) it satisfies

$$\widetilde{u}_\nu \in L^\infty((0,T)\times\mathbb{R}^d)\cap C^1([0,T];L^1), \tag{2.15}$$

because the operator $u \to -\Delta\beta_M(u)$ is m-accretive in $L^1(\mathbb{R}^d)$ and $u_0 \in (L^1\cap L^\infty)(\mathbb{R}^d)$, $\operatorname{div} f_\nu \in (L^1\cap L^\infty)((0,T)\times\mathbb{R}^d)$. It is also clear that

$$\lim_{\nu\to 0}\widetilde{u}_\nu = \widetilde{u} \quad \text{strongly in } L^2(0,T;H^{-1}). \tag{2.16}$$

To prove the claim, we are going to show that

$$\widetilde{u} = u,\ \text{ a.e. in } (0,T)\times\mathbb{R}^d. \tag{2.17}$$

To this end, we shall invoke an argument due to Brezis & Crandall [4].

Namely, we subtract equations (1.1), (2.13) and get

$$(\widetilde{u}_\nu - u)_t - \Delta(\beta(\widetilde{u}_\nu) - \beta(u) = -\operatorname{div}(f_\nu - f) \text{ in } \mathcal{D}'((0,T)\times\mathbb{R}^d),$$

$$\text{essential } \lim_{t\to 0}\int_{\mathbb{R}^d}|\widetilde{u}_\nu(t,x) - u(t,x)|dx = 0.$$

We set $z_\nu = \widetilde{u}_\nu - u$, $h_\nu = \beta(\widetilde{u}_\nu) - \beta(u)$ and get

$$(z_\nu)_t - \Delta h_\nu = -\operatorname{div}(f_\nu - f) \text{ in } \mathcal{D}'((0,T)\times\mathbb{R}^d), \tag{2.18}$$

$$\text{essential } \lim_{t\to 0}\int_{\mathbb{R}^d}|z_\nu(t,x)|dx = 0. \tag{2.19}$$

Arguing as in [4] (proof of Proposition 1), we set

$$g^\nu_\varepsilon(t) = (B_\varepsilon(z_\nu), z_\nu),\ \ B_\varepsilon = (\varepsilon I - \Delta)^{-1},$$

where (,) denotes the inner product in $L^2 = L^2(\mathbb{R}^d)$, and get

$$(g^\nu_\varepsilon(t))_t = 2(\varepsilon B_\varepsilon(h_\nu) - h_\nu, z_\nu(t)) + 2(f_\nu - f, \nabla B_\varepsilon(z_\nu))$$

and, therefore (we note that $(h_\nu, z_\nu) \geq 0$),

$$g^\nu_\varepsilon(t) \leq 2\int_0^t \varepsilon(B_\varepsilon(h_\nu(s)), z_\nu(s))ds + 2\int_0^t((f_\nu - f)(s), \nabla B_\varepsilon(z_\nu(s)))ds, \tag{2.20}$$

because, by (2.19), we have, since $z_\nu \in L^\infty \cap L^1$,

$$g_\varepsilon^\nu(0^+) = \text{essential } \lim_{t\to 0}(B_\varepsilon(z_\nu(t), z_\nu(t)) = 0. \tag{2.21}$$

Recalling that, for a.e. $s \in [0,T]$,

$$\varepsilon B_\varepsilon(z_\nu(s)) - \Delta B_\varepsilon(z_\nu(s)) = z_\nu(s),$$

we get

$$g_\varepsilon^\nu(s) = (B_\varepsilon(z_\nu(s)), z_\nu(s)) = \varepsilon |B_\varepsilon(z_\nu(s))|_{L^2}^2 + |\nabla B_\varepsilon(z_\nu(s))|_{L^2}^2$$

and so, by (2.20), we have

$$g_\varepsilon^\nu(t) \le 2\int_0^t \varepsilon(B_\varepsilon h_\nu(s), z_\nu(s))ds + \int_0^t |(f_\nu - f)(s)|_{L^2}^2 ds + \int_0^t g_\varepsilon^\nu(s)ds. \tag{2.22}$$

Hence, by Gronwall's inequality,

$$g_\varepsilon^\nu(t) \le \int_0^t (2\varepsilon(B_\varepsilon(h_\nu(s)), z_\nu(s)) + |(f_\nu - f)(s)|_{L^2}^2)e^{t-s}ds. \tag{2.23}$$

On the other hand, by Lemma 1 in [4] we have for a.e. $s \in [0,T]$ and also in $L^1(0,T)$

$$\lim_{\varepsilon\to 0}(\varepsilon B_\varepsilon(h_\nu(s)), z_\nu(s)) = 0. \tag{2.24}$$

By (2.23), this yields

$$(G * z_\nu(t), z_\nu(t)) \le \int_0^t |(f_\nu - f)(s)|_{L^2}^2 e^{t-s}ds, \tag{2.25}$$

since

$$\lim_{\varepsilon\to 0} g_\varepsilon^\nu(t) = ((-\Delta)^{-1}z_\nu(t), z_\nu(t)) = (G * z_\nu(t), z_\nu(t)),$$

where G is the classical Newtonian Green function. We note that the latter inner product is well defined, since $z_\nu \in \bigcap_{p\in[1,\infty]} L^p$ and hence $G * z_\nu \in L^{\frac{2d}{d-2}}$.

Letting $\nu \to 0$ in (2.25), we find

$$\lim_{\nu\to 0}(G * z_\nu(t), z_\nu(t)) = 0.$$

In particular, $z_\nu(t) \xrightarrow[\nu\to 0]{} 0$ in H^{-1}. Since $z_\nu \xrightarrow[\nu\to 0]{} \widetilde{u}-u$ in $L^2(0,T;L^2)$ by (2.16), we have along a subsequence $z_\nu(t) \xrightarrow[\nu\to 0]{} (\widetilde{u}-u)(t)$ in L^2 for a.e. $t\in[0,T]$, hence (2.17) follows.

2°. *The proof without assuming* (2.10)*.* It follows by the argument described under (1.12) in [4] that (1.4) indeed implies (2.21). So, assumption (2.10) can be dropped.

This completes the proof. □

As mentioned earlier, under Hypotheses (i)–(ii), if $\Phi\in C^2$, where $\Phi(u)\equiv\frac{\beta(u)}{u}$, $u\in\mathbb{R}$, then the Fokker–Planck equation (1.1), for each $u_0\in L^1$, has a unique mild solution $u\in C([0,\infty);L^1)$. This mild solution is also easily checked to be a distributional solution to (1.1). As regards this solution, we also have

Proposition 2.3. *Assume that* (i), (ii), (ii') *hold, and that, for* $\Phi(u)\equiv\frac{\beta(u)}{u}$,

(iii) $\Phi\in C^2(\mathbb{R}^d)$.

Then, for each $u_0\in L^1\cap L^\infty$*, the mild solution* u *to* (1.1) *satisfies also*

$$u\in L^\infty((0,T)\times\mathbb{R}^d),\ \forall T>0. \tag{2.26}$$

Proof. We rewrite (1.1) as

$$\begin{aligned}&(u-|u_0|_\infty-\alpha(t))_t-\Delta(\beta(u)-\beta(|u_0|_\infty+\alpha(t)))\\ &\qquad+\mathrm{div}(b^*(x,u)-b^*(x,|u_0|_\infty+\alpha(t)))\\ &\qquad=-\mathrm{div}(b^*(x,|u_0|_\infty+\alpha(t)))-\alpha'(t)\le 0 \text{ in } (0,\infty)\times\mathbb{R}^d,\end{aligned} \tag{2.27}$$

where $\alpha\in C^1([0,\infty))$ is chosen in such a way that

$$\begin{aligned}&\alpha'(t)+\sup\{|b_x(x,|u_0|_\infty+\alpha(t))|;x\in\mathbb{R}^d\}(|u_0|_\infty+\alpha(t))=0,\ t\in(0,T),\\ &\alpha(0)=0.\end{aligned} \tag{2.28}$$

We may find α of the form $\alpha=\eta-|u_0|_\infty$, where η is a solution to the equation

$$\begin{aligned}&\eta'-\delta(\eta)\eta=0,\ t\ge 0,\\ &\eta(0)=|u_0|_\infty,\end{aligned} \tag{2.29}$$

$\delta(r) = \sup\{|b_x(x,r)|; x \in \mathbb{R}^d\}$, $r \in \mathbb{R}$. Clearly, (2.29) has such a solution $\eta \in C^1([0,\infty))$, $\eta \geq 0$, on $[0,\infty)$ because $\delta \in C_b(\mathbb{R})$.

Formally, if we multiply (2.27) by $\mathrm{sign}(u - |u_0|_\infty - \alpha)^+$, integrate over $\mathbb{R}^d$ and use the monotonicity of β, we get by (2.7) that

$$\frac{d}{dt}\,|(u(t) - |u_0|_\infty - \alpha(t))^+|_1 \leq 0, \text{ a.e. } t \in (0,T). \tag{2.30}$$

This yields $u(t) \leq |u_0|_\infty + \alpha(t)$, $\forall t \geq 0$, and similarly it follows that $u(t) \geq -|u_0|_\infty - \alpha(t)$. Hence, $u \in L^\infty((0,T) \times \mathbb{R}^d)$, as claimed.

The above formal argument can be made rigorous if u is a strong solution to (1.1) (which is not the case here). Then (see the detailed argument in [1])

$$\begin{aligned}
&\lim_{\delta\to 0}\frac{1}{\delta}\int_{[0<(\beta(u)-\beta(|u_0|_\infty+\alpha(t))^+)\leq\delta]} |b^*(x,u) - b^*(x,|u_0|_\infty + \alpha(t))|\,|\nabla u|dx\\
&\quad= \lim_{\delta\to 0}\frac{1}{\delta}\int_{[0<(\beta(u)-\beta(|u_0|_\infty+\alpha(t))^+)\leq\delta]} (|b(x,u) - b(x,|u_0|_\infty + \alpha(t))|\,|u|\\
&\quad+|b(x,|u_0|_\infty + \alpha(t))|\,|u - |u_0|_\infty - \alpha(t)|)|\nabla u|dx = 0,\ \forall t \in (0,T),
\end{aligned} \tag{2.31}$$

which is true if $\nabla u \in L^2(0,T;L^2)$ and $b(x,\cdot) \in \mathrm{Lip}(\mathbb{R})$ uniformly in x (which is the case if $b_u \in C_b(\mathbb{R}^d \times \mathbb{R})$). In order to be in such a situation, we approximate (1.1) by

$$\begin{aligned}
&u_t - \Delta(\beta(u) + \varepsilon\beta(u) + \mathrm{div}(b_\varepsilon(x,u)u)) = 0 \text{ in } (0,T) \times \mathbb{R}^d,\\
&u(0,x) = u_0(x),
\end{aligned} \tag{2.32}$$

where $\varepsilon > 0$ and $b_\varepsilon \in C_b^1(\mathbb{R}^d \times \mathbb{R})$ is a smooth approximation of b. (For instance, $b_\varepsilon = b * \rho_\varepsilon$, where ρ_ε is a standard mollifier.) Then, as proved earlier in [1], [2], [3], equation (2.32) has a unique solution $u_\varepsilon \in L^2(0,T;H^1) \cap C([0,T];L^1) \cap W^{1,2}([0,T];H^{-1})$ and $u_\varepsilon \to u$ in $C([0,T];L^1)$ as $\varepsilon \to 0$. An easy way to prove this is to apply the Trotter–Kato theorem to the family of m-accretive operators in L^1

$$\begin{aligned}
A_\varepsilon u &= -\Delta\beta(u) + \varepsilon\beta(u) + \mathrm{div}(b_\varepsilon(x,u)u),\\
D(A_\varepsilon) &= \{u \in L^1; -\Delta\beta(u) + \varepsilon\beta(u) + \mathrm{div}(b_\varepsilon(x,u)u) \in L^1\}.
\end{aligned}$$

(See the argument in [3].) Then, we replace (2.27) by

$$\begin{aligned}&(u_\varepsilon-|u_0|_\infty-\alpha(t))_t-\Delta(\beta(u_\varepsilon)-\beta(|u_0|_\infty+\alpha(t)))\\&+\varepsilon(\beta(u)-\beta(|u_0|_\infty+\alpha(t)))+\mathrm{div}(b^*_\varepsilon(x,u_\varepsilon)-b^*_\varepsilon(x,|u_0|_\infty+\alpha(t))\\&=-b^*_\varepsilon(x,|u_0|_\infty+\alpha(t))-\alpha'(t)-\varepsilon\beta(|u_0|_\infty+\alpha(t))\le 0,\\&\qquad\qquad\qquad\qquad\qquad\text{a.e. in } (0,T)\times\mathbb{R}^d,\end{aligned}\tag{2.33}$$

where $b^*_\varepsilon(u)=b_\varepsilon(u)u$.

Let $\mathcal{X}_\delta\in\mathrm{Lip}(\mathbb{R})$ be the following approximation of the signum function

$$\mathcal{X}_\delta(r)=\begin{cases}1 & \text{for} \quad r\ge\delta,\\ \dfrac{r}{\delta} & \text{for} \quad |r|<\delta,\\ -1 & \text{for } r<-\delta,\end{cases}$$

where $\delta>0$. If we multiply (2.33) by $\mathcal{X}_\delta((\beta(u_\varepsilon)-\beta(|u_0|_\infty+\alpha))^+)$ and integrate over $\mathbb{R}^d$, we get

$$\begin{aligned}&\int_{\mathbb{R}^d}(u_\varepsilon-|u_0|_\infty-\alpha)_t\mathcal{X}_\delta((\beta(u_\varepsilon)-\beta(|u_0|_\infty+\alpha))^+)dx\\&\quad\le\frac{1}{\delta}\int_{[0<(\beta(u_\varepsilon)-\beta(|u_0|_\infty+\alpha))^+\le\delta]}(b^*(x,u_\varepsilon)u_\varepsilon-b^*_\varepsilon(x,|u_0|_\infty+\alpha))\cdot\nabla u_\varepsilon\,dx,\\&\qquad\qquad\qquad\qquad\qquad\forall t\in(0,T),\end{aligned}$$

because β is monotonically increasing and

$$\nabla(\beta(u_\varepsilon)-\beta(|u_0|_\infty+\alpha)\cdot\nabla\mathcal{X}_\delta((\beta(u_\varepsilon)-\beta(|u_0|_\infty+\alpha))^+)\ge 0 \text{ in } (0,T)\times\mathbb{R}^d.$$

Then, by (2.31), we get, for $\delta\to 0$,

$$\int_{\mathbb{R}^d}(u_\varepsilon-|u_0|_\infty-\alpha(t))^+_t\,dx\le 0,\ \forall t\in(0,T),$$

and this yields

$$u_\varepsilon(t,x)-|u_0|_\infty-\alpha(t)\le 0,\ \text{a.e. on } (0,T)\times\mathbb{R}^d,$$

and so, $u_\varepsilon\le|u_0|_\infty+\alpha$, a.e. on $(0,T)\times\mathbb{R}^d$. Then, we pass to the limit $\varepsilon\to 0$ to get the claimed inequality. □

By Theorem 2.1 and Proposition 2.3, we therefore get the following existence and uniqueness result for (1.1).

Theorem 2.4. *Under hypotheses* (i), (ii), (ii'), (iii)*, for each* $u_0 \in L^1 \cap L^\infty$*, equation* (1.1) *has a unique distributional solution*

$$u \in L^1((0,T);L^1) \cap L^\infty((0,T)\times\mathbb{R}^d),\ \forall T>0. \tag{2.34}$$

3 Uniqueness of the linearized equation

Consider a distributional solution of the linearized equation corresponding to (1.1), that is,

$$\begin{aligned} &v_t - \Delta(\Phi(u)v + \operatorname{div}(b(x,u)v) = 0 \text{ in } \mathcal{D}'((0,\infty)\times\mathbb{R}^d,\\ &v(0,x) = v_0(x), \end{aligned} \tag{3.1}$$

where $u \in L^\infty((0,T)\times\mathbb{R}^d),\ \forall T>0$. By (i)–(ii), we have

$$b(x,u), \Phi(u) = \frac{\beta(u)}{u} \in L^\infty((0,\infty)\times\mathbb{R}^d).$$

Moreover, we have

$$\Phi(u) \geq \gamma_0 > 0,\ \text{ a.e. in } (0,\infty)\times\mathbb{R}^d. \tag{3.2}$$

In the following, we denote $\Phi(u(t,x))$ by $\Psi(t,x)$, $(t,x)\in(0,\infty)\times\mathbb{R}^d$.

Theorem 3.1. (Linearized uniqueness) *Under hypotheses* (i)–(ii)*, for each* $v_0 \in L^1 \cap L^\infty$ *and* $T>0$*, equation* (3.1) *has at most one distributional solution* $v \in L^1([0,T];L^1) \cap L^\infty((0,T)\times\mathbb{R}^d)$.

Proof. We shall proceed as in the proof of Theorem 2.1. Namely, we set $v_1 - v_2 = v$ for two solutions v_1, v_2 of (3.2) By a similar argument as in the proof of Theorem 2.1, we may assume that v satisfies (2.4). Then we get

$$\begin{aligned} &v_t - \Delta(\Psi v) + \operatorname{div}(b(x,u)v) = 0,\ \text{ a.e. } t\in(0,T),\\ &v(0) = 0. \end{aligned} \tag{3.3}$$

For $y = \Gamma v$, we get

$$\begin{aligned} &\frac{d}{dt}\,y - \Gamma\Delta(\Psi v) + \Gamma\ \operatorname{div}(b(x,u)v) = 0\\ &y(0) = 0 \end{aligned} \tag{3.4}$$

and multiplying scalarly in L^2 with v, as above we get that

$$\frac{1}{2}\,\frac{d}{dt}\,|v(t)|_{-1}^2+\gamma_0|v(t)|_2^2\le(|\Psi|_\infty+|b|_\infty)|v(t)|_2|v(t)|_{-1},\ \text{a.e. } t\in(0,T). \tag{3.5}$$

This yields

$$\frac{d}{dt}\,|v(t)|_{-1}^2\le|v(t)|_{-1}^2 \text{ a.e. } t\in(0,T),$$

and, therefore, $v\equiv 0$, as claimed.

4 Uniqueness in law of the McKean–Vlasov stochastic differential equations (SDEs)

Consider for $T\in(0,\infty)$ and $u_0\in L^1\cap L^\infty$ the McKean–Vlasov stochastic differential equation (SDE)

$$\begin{aligned} dX(t)&=b(X(t),u(t,X(t)))dt+\frac{1}{\sqrt{2}}\left(\frac{\beta(u(t,X(t)))}{u(t,X(t))}\right)^{\frac12}dW(t),\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0\le t\le T,\\ u(0,\cdot)&=\xi_0, \end{aligned}\tag{4.1}$$

on $\mathbb{R}^d$. Here, $W(t),\ t\ge 0$, is an $(\mathcal{F}_t)$-Brownian motion on a probability space $(\Omega,\mathcal{F},\mathbb{P})$ with normal filtration $\mathcal{F}_t$, $t\ge 0$, $\xi_0:\Omega\to\mathbb{R}^d$ is $\mathcal{F}_0$-measurable such that

$$\mathbb{P}\circ\xi_0^{-1}(dx)=u_0(x)dx,$$

and $u(t,x)=\frac{d\mathcal{L}_{X(t)}}{dx}(x)$ is the Lebesgue density of the marginal law $\mathcal{L}_{X(t)}=\mathbb{P}\circ X(t)^{-1}$ of the solution process $X(t)$, $t\ge 0$. Here, a solution process means an $(\mathcal{F}_t)$-adapted process with $\mathbb{P}$-a.s. continuous sample paths in $\mathbb{R}^d$ solving (4.1).

Theorem 4.1. *Let $0<T<\infty$ and let the above conditions* (i)–(ii) *on b and β hold. Let $X(t)$, $t\ge 0$, and $\widetilde{X}(t)$, $t\ge 0$, be two solutions to* (4.1) *such that, for*

$$u(t,\cdot):=\frac{d\mathcal{L}_{X(t)}}{dx},\quad \widetilde{u}(t,\cdot):=\frac{d\mathcal{L}_{\widetilde{X}(t)}}{dx},$$

we have

$$u,\widetilde{u}\in L^\infty((0,T)\times\mathbb{R}^d). \tag{4.2}$$

Then X and $\widetilde{X}$ have the same laws, i.e., $\mathbb{P}\circ X^{-1}=\mathbb{P}\circ\widetilde{X}^{-1}$.

Proof. By Itô's formula, both u and $\widetilde{u}$ satisfy the (nonlinear) Fokker–Planck equation (1.1) in the sense of Schwartz distributions. Hence, by Theorem 2.1, $u = \widetilde{u}$. Furthermore, again by Itô's formula, $\mathbb{P} \circ X^{-1}$ and $\mathbb{P} \circ \widetilde{X}^{-1}$ satisfy the martingale problem with the initial condition $u_0 dx$ for the linear Komogorov operator

$$L_u := \Phi(u)\Delta + b(\cdot, u) \cdot \nabla,$$

where $\Phi(u) = \frac{\beta(u)}{u}$, $u \in \mathbb{R}$. Hence, by Theorem 3.1, the assertion follows by Lemma 2.12 in [5].

Here, for $s \in [0, T]$, the set $\mathcal{R}_{[s,T]}$, which appears in that lemma, is chosen to be the set of all narrowly continuous, probability measure-valued solutions of (3.1) having for each $t \in [s, T]$ a density $v(t, \cdot) \in L^\infty$ with respect to Lebesgue measure such that $v \in L^\infty((0, T) \times \mathbb{R}^d)$. □

Remark 4.2. We note that, by the narrow continuity, (4.2) implies that, for every $t \in [0, T]$, $u(t, \cdot), \widetilde{u}(t, \cdot) \in L^\infty$. This fact was used in the above proof.

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