WEAK SOLUTIONS OF MCKEAN-VLASOV SDES WITH SUPERCRITICAL DRIFTS

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Abstract. Consider the following McKean-Vlasov SDE:

$$\mathrm{d}X_t = \sqrt{2}\mathrm{d}W_t + \int_{\mathbb{R}^d} K(t, X_t - y)\mu_{X_t}(\mathrm{d}y)\mathrm{d}t, \quad X_0 = x,$$

where μ_{X_t} stands for the distribution of X_t and $K(t,x) : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ is a time-dependent divergence free vector field. Under the assumption $K \in L^q_t(\widetilde{L}^p_x)$ with $\frac{d}{p} + \frac{2}{q} < 2$, where $\widetilde{L}^p_x$ stands for the localized L^p-space, we show the existence of weak solutions to the above SDE. As an application, we provide a new proof for the existence of weak solutions to 2D-Navier-Stokes equations with measure as initial vorticity.

Keywords: McKean-Vlasov system, Supercritical drift, 2D-Navier-Stokes equation, Krylov's estimate

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1. Introduction

Consider the following two dimensional Naver-Stokes equation:

$$\mathrm{d}\mathbf{u} = \Delta\mathbf{u} + \mathbf{u}\cdot\nabla u + \nabla p, \quad \mathrm{div}\mathbf{u} = 0, \tag{1.1}$$

where $\mathbf{u} = (u_1, u_2)$ stands for the velocity field, and p stands for the pressure. Let $\rho := \mathrm{curl}\mathbf{u} = \partial_1 u_2 - \partial_2 u_1$ be the vorticity of $\mathbf{u}$. It is easy to see that

$$\partial_t\rho = \Delta\rho + \mathbf{u}\cdot\nabla\rho = \Delta\rho + \mathrm{div}(\rho\cdot\mathbf{u}).$$

Moreover, by the Biot-Savart law we have (cf. [9])

$$\mathbf{u}(t,x) = \int_{\mathbb{R}^2} K_2(x-y)\rho(t,y)\mathrm{d}y =: K_2 * \rho(t,x),$$

where

$$K_2(x) := \tfrac{1}{2\pi}\Big(\tfrac{-x_2}{|x|^2}, \tfrac{x_1}{|x|^2}\Big). \tag{1.2}$$

In other words, ρ solves the following nonlinear integral-differential equation:

$$\partial_t\rho = \Delta\rho + \mathrm{div}(\rho\cdot K_2 * \rho). \tag{1.3}$$

Notice that the kernel function K_2 is of homogeneous of degree -1, and

$$\int_{\mathbb{R}^2} |K_2(x)|^p\mathrm{d}x = \infty, \quad p \in [1,\infty]. \tag{1.4}$$

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Suppose that $\rho(0,x)\geqslant 0$ and $\int_{\mathbb{R}^2}\rho(0,x)\mathrm{d}x=1$. By the maximum principle and integrating both sides of (1.3) with respect to x, we obtain that for any $t>0$,

$$\rho(t,x)\geqslant 0,\quad \int_{\mathbb{R}^2}\rho(t,x)\mathrm{d}x=\int_{\mathbb{R}^2}\rho(0,x)\mathrm{d}x=1,$$

which means that $\rho(t,\cdot)_{t\geqslant 0}$ is a family of probability measures. By the superposition principle [15], there would be a weak solution to the following McKean-Vlasov SDEs:

$$\mathrm{d}X_t=\left[\int_{\mathbb{R}^2}K_2(X_t-y)\rho(t,y)\mathrm{d}y\right]\mathrm{d}t+\sqrt{2}\mathrm{d}W_t,\ X_0=x, \tag{1.5}$$

where W is a two-dimensional standard Brownian motion and $\rho(t,\cdot)$ is the distributional density of X_t. On the other hand, if X_t solves the nonlinear SDE (1.5), by Itô's formula, the law of X_t will solve the nonlinear Fokker-Planck equation (1.3) in the distributional sense. It should be noticed that when the initial vorticity is a finite Radon measure, the existence of solutions to PDE (1.3) was obtained by Giga, Miyakawa and Osada in [5] (see also Cottet's work [2]), and the uniqueness was proven by Gallagher and Gallay in [4]. However, due to the non-integrability of K_2 (see (1.4)), it does not immediately imply the existence of weak solutions to distributional dependent SDE (1.5) by superposition principle because the following condition in [15] is not known to hold for the solution obtained in [5],

$$\int_0^T\left|\int_{\mathbb{R}^2}\left[\int_{\mathbb{R}^2}K_2(x-y)\rho(t,y)\mathrm{d}y\right]\rho(t,x)\mathrm{d}x\right|\mathrm{d}t<\infty.$$

In this paper we are concerning with the following McKean-Vlasov SDE in $\mathbb{R}^d$:

$$\mathrm{d}X_t=\left[\int_{\mathbb{R}^d}K(t,X_t,y)\mu_{X_t}(\mathrm{d}y)\right]\mathrm{d}t+\sqrt{2}\mathrm{d}W_t, \tag{1.6}$$

where $K:\mathbb{R}_+\times\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}^d$ is a measurable vector-valued function and μ_{X_t} is the law of X_t. For any $\alpha\in[0,2)$, we introduce the following index set:

$$\mathscr{I}_\alpha:=\Big\{(p,q)\in(1,\infty)^2,\ \tfrac{d}{p}+\tfrac{2}{q}<2-\alpha\Big\}.$$

Suppose that for some $(p,q)\in\mathscr{I}_1$,

$$|K(t,x,y)|\leqslant h(t,x-y),\quad h\in L^q_t(\widetilde{L}^p_x):=\cap_{T>0}L^q([0,T];\widetilde{L}^p), \tag{1.7}$$

where $\widetilde{L}^p$ is the localized L^p-space in $\mathbb{R}^d$ (see (2.2) below). Under (1.7), Röckner and the present author [12] showed the strong well-posedness to the above SDE. The integrability condition (1.7) for $\frac{d}{p}+\frac{2}{q}<1$ is usually called subcritical case in the literature; while $\frac{d}{p}+\frac{2}{q}=1$ and $\frac{d}{p}+\frac{2}{q}>1$ correspond to the critical and supercritical cases, respectively. Notice that the kernel function K_2 given in (1.2) belongs to the supercritical regime since

$$\int_{|x|<1}|K_2(x)|^p\mathrm{d}x<\infty,\quad p\in[1,2),\quad \int_{|x|<1}|K_2(x)|^2\mathrm{d}x=\infty.$$

For $\beta\geqslant 0$, let $\mathcal{P}_\beta(\mathbb{R}^d)$ be the set of all probability measures on $\mathbb{R}^d$ with finite β-order moment. The aim of this paper is to show the following existence result.

Theorem 1.1. *We suppose that in the distributional sense,*

$$\mathrm{div}K(t,\cdot,y)\leqslant 0, \tag{1.8}$$

and for some $(p,q)\in\mathscr{I}_0$,

$$|K(t,x,y)|\leqslant h(t,x-y),\quad h\in L^q_t(\widetilde{L}^p_x). \tag{1.9}$$

Let $\beta\in[0,2/(\frac{d}{p}+\frac{2}{q}))$. For any $\nu_0\in\mathcal{P}_\beta(\mathbb{R}^d)$, there exists at least one weak solution to SDE (1.6) with initial distribution ν_0. More precisely, there are stochastic basis $(\Omega,\mathscr{F},(\mathscr{F}_t)_{t\geqslant0},\mathbf{P})$ and two $\mathscr{F}_t$-adapted processes (X,W) defined on it such that

(i) $\mathbf{P}\circ X_0^{-1}=\nu_0$ and W is a d-dimensional standard $\mathscr{F}_t$-Brownian motion.

(ii) It holds that for all $t\geqslant0$,

$$X_t=X_0+\int_0^t\!\!\int_{\mathbb{R}^d}K(s,X_s,y)\mu_{X_s}(\mathrm{d}y)\mathrm{d}s+\sqrt{2}W_t,\ \mathbf{P}-a.s.,$$

where μ_{X_s} is the law of X_s.

Moreover, we have the following conclusions:

(i) For any $T>0$, there is a constant $C>0$ such that

$$\mathbf{E}\left(\sup_{t\in[0,T]}|X_t|^\beta\right)\leqslant C(\mathbf{E}|X_0|^\beta+1). \tag{1.10}$$

(ii) For Lebesgue almost all $t>0$, X_t admits a density $\rho(t,\cdot)$ with the regularity

$$\rho\in\cap_{T>0}\mathbb{H}^{\alpha,p}_q(T),\ \alpha\in[0,1],\ p,q\in(1,\infty),\ \tfrac{d}{p}+\tfrac{2}{q}>d+\alpha, \tag{1.11}$$

where $\mathbb{H}^{\alpha,p}_q(T):=L^q([0,T];H^{\alpha,p})$ and $H^{\alpha,p}$ is the usual Bessel potential space.

Recently, there are great interests to study the McKean-Vlasov or distributional dependent SDEs since it appears in the studies of propagation of chaos [14], mean-field games (cf. [1]) and nonlinear integral-partial differential equations [10, 3]. When K is bounded measurable, the existence and uniqueness of weak solutions to SDE (1.6) was proved by Li and Min [8] (see also [11] for the strong well-posedness of SDE (1.6)). As mentioned above, when K is singular and belongs to the subcritical regime, the strong existence and uniqueness was shown in [12] recently (see also [6]). We also mention that Jabin and Wang [7] showed the propagation of chaos for singular kernel K_2 above by purely analytic method. While, the existence of particle trajectories is not provided therein. Here, an open question is the uniqueness of weak solutions in the supercritical case. This is even not known for linear SDEs with supercritical drifts (cf. [17]).

As a simple application of Theorem 1.1, we have the following corollary.

Corollary 1.2. *Consider the vorticity form (1.3) of 2D-Navier-Stokes equations. Let $\beta\in[0,2)$. For any $\mu(0)\in\mathcal{P}_\beta(\mathbb{R}^2)$, there exists a continuous curve $t\mapsto\mu(t)\in\mathcal{P}_\beta(\mathbb{R}^2)$ such that for all $t\geqslant0$ and $f\in C^\infty_b(\mathbb{R}^d)$,*

$$\mu_t(f)=\mu_0(f)+\int_0^t\mu_s(\Delta f)\mathrm{d}s+\int_0^t\left[\int_{\mathbb{R}^2}\!\!\int_{\mathbb{R}^2}K_2(x-y)\cdot\nabla f(x)\mu_s(\mathrm{d}y)\mu_s(\mathrm{d}x)\right]\mathrm{d}s.$$

Moreover, for Lebesgue-almost all $t>0$, $\mu(t,\mathrm{d}x)=\rho(t,x)\mathrm{d}x$, where ρ satisfies (1.11), and for any $T>0$ and some $C>0$,

$$\mathrm{ess.}\sup_{t\in[0,T]}\int_{\mathbb{R}^2}|x|^\beta\rho(t,x)\mathrm{d}x\leqslant C\left(\int_{\mathbb{R}^2}|x|^\beta\mu_0(\mathrm{d}x)+1\right). \tag{1.12}$$

Remark 1.3. Compared with [5], the new point here is that the moment estimate (1.12) is obtained, which provides the decay estimate of the vorticity as $|x| \to \infty$. Note that the uniqueness is proven in [4], which strongly depends on the structure of K_2.

This paper is organized as following: in Section 2, we prepare necessary spaces and some well-known results about the maximum principle for the associated PDE. In Section 3, through mollifying the kernel function K, we show our main result by weak convergence method.

2. Preliminaries

We first introduce the following spaces and notations for later use. For $(\alpha, p) \in \mathbb{R}_+ \times [1, \infty]$, the Bessel potential space $H^{\alpha,p}$ is defined by

$$H^{\alpha,p} := \big\{ f \in L^1_{loc}(\mathbb{R}^d) : \|f\|_{\alpha,p} := \|(\mathbb{I} - \Delta)^{\alpha/2} f\|_p < \infty \big\},$$

where $\|\cdot\|_p$ is the usual L^p-norm, and $(\mathbb{I} - \Delta)^{\alpha/2} f$ is defined by Fourier's transform

$$(\mathbb{I} - \Delta)^{\alpha/2} f := \mathcal{F}^{-1}\big((1 + |\cdot|^2)^{\alpha/2} \mathcal{F} f\big).$$

For $T > 0$, $p, q \in [1, \infty]$ and $\alpha \in \mathbb{R}_+$, we introduce the following spaces of space-time functions,

$$\mathbb{L}^p_q(T) := L^q\big([0,T]; L^p\big), \quad \mathbb{H}^{\alpha,p}_q(T) := L^q\big([0,T]; H^{\alpha,p}\big).$$

Let $\chi \in C^\infty_c(\mathbb{R}^d)$ be a smooth function with $\chi(x) = 1$ for $|x| \leqslant 1$ and $\chi(x) = 0$ for $|x| > 2$. For $r > 0$ and $z \in \mathbb{R}^d$, define

$$\chi^z_r(x) := \chi((x - z)/r). \tag{2.1}$$

Fix $r > 0$. We introduce the following localized $H^{\alpha,p}$-space:

$$\widetilde{H}^{\alpha,p} := \Big\{ f : |\!|\!| f |\!|\!|_{\alpha,p} := \sup_z \|f \chi^z_r\|_{\alpha,p} < \infty \Big\}, \tag{2.2}$$

and the localized space-time function space

$$\widetilde{\mathbb{H}}^{\alpha,p}_q(T) := \Big\{ f : |\!|\!| f |\!|\!|_{\widetilde{\mathbb{H}}^{\alpha,p}_q(T)} := \sup_{z \in \mathbb{R}^d} \|\chi^z_r f\|_{\mathbb{H}^{\alpha,p}_q(T)} < \infty. \Big\} \tag{2.3}$$

For simplicity we shall write

$$\widetilde{\mathbb{H}}^{\alpha,p}_q := \cap_{T>0} \widetilde{\mathbb{H}}^{\alpha,p}_q(T), \quad \widetilde{\mathbb{L}}^p_q := \widetilde{\mathbb{H}}^{0,p}_q, \quad \widetilde{\mathbb{L}}^p(T) := \widetilde{\mathbb{L}}^p_p(T).$$

It should be noticed that

$$L^q([0,T]; \widetilde{L}^p) \subset \widetilde{\mathbb{L}}^p_q(T).$$

The following lemma lists some easy properties of $\widetilde{\mathbb{H}}^{\alpha,p}_q$ (see [17]).

Proposition 2.1. *Let $p, q \in (1, \infty)$, $\alpha \in \mathbb{R}_+$ and $T > 0$.*

(i) *For $r \neq r' > 0$, there is a $C = C(d, \alpha, r, r', p, q) \geqslant 1$ such that*

$$C^{-1} \sup_z \|f \chi^z_{r'}\|_{\mathbb{H}^{\alpha,p}_q(T)} \leqslant \sup_z \|f \chi^z_r\|_{\mathbb{H}^{\alpha,p}_q(T)} \leqslant C \sup_z \|f \chi^z_{r'}\|_{\mathbb{H}^{\alpha,p}_q(T)}. \tag{2.4}$$

In other words, the definition of $\widetilde{\mathbb{H}}^{\alpha,p}_q$ does not depend on the choice of r.

(ii) *Let* $(\rho_\varepsilon)_{\varepsilon\in(0,1)}$ *be a family of mollifiers in* $\mathbb{R}^d$. *For any* $f \in \widetilde{\mathbb{H}}^{\alpha,p}_q$ *and* $T > 0$, *it holds that*

$$f_\varepsilon(t,x) := f(t,\cdot) * \rho_\varepsilon(x) \in L^q_{loc}(\mathbb{R}; C^\infty_b(\mathbb{R}^d)),$$

and for some $C = C(d,\alpha,p,q) > 0$,

$$|\!|\!| f_\varepsilon |\!|\!|_{\widetilde{\mathbb{H}}^{\alpha,p}_q(T)} \leqslant C |\!|\!| f |\!|\!|_{\widetilde{\mathbb{H}}^{\alpha,p}_q(T)}, \ \forall \varepsilon \in (0,1), \tag{2.5}$$

and for any $\varphi \in C^\infty_c(\mathbb{R}^d)$,

$$\lim_{\varepsilon\to 0} \|(f_\varepsilon - f)\varphi\|_{\mathbb{H}^{\alpha,p}_q(T)} = 0. \tag{2.6}$$

We introduce the following notion about Krylov's estimate.

Definition 2.2. *Let* $p, q \in (1,\infty)$ *and* $T, \kappa > 0$. *We say a stochastic process* X *satisfies Krylov's estimate with index* p, q *and constant* κ *if for any* $f \in \widetilde{\mathbb{L}}^p_q(T)$,

$$\mathbf{E}\left(\int_0^T f(t,X_t)\mathrm{d}t\right) \leqslant \kappa |\!|\!| f |\!|\!|_{\widetilde{\mathbb{L}}^p_q(T)}. \tag{2.7}$$

The set of all such X *will be denoted by* $\mathbf{K}^{p,q}_{T,\kappa}$.

Remark 2.3. By Krylov's estimate (2.7), there is a density function $\rho^X \in \mathbb{L}^r_s(T)$ with $r = \frac{p}{p-1}$ and $s = \frac{q}{q-1}$ so that

$$\int_0^T\!\!\int_{\mathbb{R}^d} f(t,x)\rho^X_t(x)\mathrm{d}x\mathrm{d}t = \mathbf{E}\left(\int_0^T f(t,X_t)\mathrm{d}t\right) \leqslant \kappa |\!|\!| f |\!|\!|_{\widetilde{\mathbb{L}}^p_q(T)} \leqslant \kappa \|f\|_{\mathbb{L}^p_q(T)}.$$

For a space-time function $f(t,x,y) : \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ and $p_1, p_2, q_0 \in [1,\infty]$, we also introduce the norm

$$|\!|\!| f |\!|\!|_{\widetilde{\mathbb{L}}^{p_1,p_2}_{q_0}(T)} := \sup_{z,z'\in\mathbb{R}^d} \left(\int_0^T \left(\int_{\mathbb{R}^d} \mathbf{1}_{B^{z'}_1(y)} \|\mathbf{1}_{B^z_1} f(t,\cdot,y)\|^{p_2}_{p_1}\mathrm{d}y\right)^{\frac{q_0}{p_2}}\right)^{\frac{1}{q_0}}, \tag{2.8}$$

where for $z \in \mathbb{R}^d$ and $r > 0$,

$$B^z_r := \{x \in \mathbb{R}^d : |x - z| < r\}, \quad B_r := B^0_r.$$

The following lemma will be used to take the limits in the proof of the existence of weak solutions (see [12, Lemma 2.6]).

Lemma 2.4. *Let* $p_1, p_2, q_0, q_1, q_2 \in (1,\infty)$ *with* $\frac{1}{q_1} + \frac{1}{q_2} = 1 + \frac{1}{q_0}$ *and* $T, \kappa_1, \kappa_2 > 0$. *Let* $X \in \mathbf{K}^{p_1,q_1}_{T,\kappa_1}$ *and* $Y \in \mathbf{K}^{p_2,q_2}_{T,\kappa_2}$ *be two independent processes. Then for any* $f(t,x,y) \in \widetilde{\mathbb{L}}^{p_1,p_2}_{q_0}(T)$,

$$\mathbf{E}\left(\int_0^T f(t,X_t,Y_t)\mathrm{d}t\right) \leqslant \kappa_1\kappa_2 |\!|\!| f |\!|\!|_{\widetilde{\mathbb{L}}^{p_1,p_2}_{q_0}(T)}. \tag{2.9}$$

Consider the following backward PDE:

$$\partial_t u + \Delta u + b \cdot \nabla u = f, \quad u(T) \equiv 0. \tag{2.10}$$

The following maximum principle was proven in [17, Theorem 2.2].

Theorem 2.5. *Let $T>0$. Suppose that $b\in C_b^\infty([0,T]\times\mathbb{R}^d))$ satisfies* $\mathrm{div} b\leqslant 0$ *and for some $(p,q)\in\mathscr{I}_0$ and $\kappa>0$,*

$$|\!|\!| b|\!|\!|_{\widetilde{\mathbb{L}}^p_q(T)}\leqslant\kappa.$$

Let $\alpha\in[0,1]$ and $f\in C_0^\infty(\mathbb{R}^{d+1})$. For any $(\bar p,\bar q)\in\mathscr{I}_\alpha$, there is a constant $C>0$ only depending on $T,d,p,q,\alpha,\bar p,\bar q,\kappa$ such that for any smooth solution u of PDE (2.10)*,*

$$\|u\|_{L^\infty([0,T]\times\mathbb{R}^d)}\leqslant C|\!|\!| f|\!|\!|_{\widetilde{\mathbb{H}}^{-\alpha,\bar p}_{\bar q}(T)}. \tag{2.11}$$

3. Proof of Theorem 1.1

Suppose that $K(t,x,y)$ satisfies (1.8) and (1.9). Let $(\varrho^{\rm d}_n)_{n\in\mathbb{N}}$ be a family of mollifiers in $\mathbb{R}^d$ with compact supports in unit ball B_1. Define

$$K_n(t,x,y):=\int_0^\infty\!\!\int_{\mathbb{R}^{2d}}K(t',x',y')\varrho^1_n(t-t')\varrho^{\rm d}_n(x-x')\varrho^{\rm d}_n(y-y'){\rm d}t'{\rm d}x'{\rm d}y', \tag{3.1}$$

and for a probability measure μ,

$$b_n(t,x,\mu):=\int_{\mathbb{R}^d}K_n(t,x,y)\mu({\rm d}y).$$

By (ii) of Proposition 2.1 and (1.9), one sees that for each $T>0$ and $j=0,1,\cdots$,

$$\kappa^j_n:=\|\nabla^j_xK_n\|_{\mathbb{L}^\infty(T)}+\|\nabla^j_yK_n\|_{\mathbb{L}^\infty(T)}<\infty \tag{3.2}$$

and

$$\mathrm{div} b_n(t,\cdot,\mu)\leqslant 0.$$

Hence, for any $T>0$ and some $C=C(\kappa^1_n)>0$,

$$|b_n(t,x,\mu)-b_n(t,\bar x,\mu)|\leqslant C|x-\bar x|,$$

and for any two random variables X,Y,

$$|b_n(t,x,\mu_X)-b_n(t,x,\mu_Y)|\leqslant C\mathbb{E}|X-Y|.$$

Thus, there is a unique strong solution X^n_t to the following McKean-Vlasov SDE:

$${\rm d}X^n_t=b_n(t,X^n_t,\mu_{X^n_t}){\rm d}t+\sqrt{2}{\rm d}W_t,\ X^n_0\stackrel{(d)}{=}\nu_0, \tag{3.3}$$

where W is a d-dimensional Brownian motion on some probability space $(\Omega,\mathscr{F},\mathbb{P})$.

We first show the following key Krylov estimate.

Lemma 3.1. *Let $\alpha\in[0,1]$ and $(\bar p,\bar q)\in\mathscr{I}_\alpha$. For any $T>0$, there is a constant $C>0$ such that for each $n\in\mathbb{N}$,*

$$\mathbb{E}\left(\int_0^T f(t,X^n_t){\rm d}t\right)\leqslant C|\!|\!| f|\!|\!|_{\widetilde{\mathbb{H}}^{-\alpha,\bar p}_{\bar q}(T)}. \tag{3.4}$$

In particular, $X^n\in\mathbf{K}^{\bar p,\bar q}_{T,\kappa}$ for any $(\bar p,\bar q)\in\mathscr{I}_0$.

Proof. Without loss of generality, we assume that f is smooth. By the properties of convolution and $|K(t,x,y)|\leqslant h(t,x-y)$, one sees that

$$|\!|\!| b_n(t,\cdot,\mu)|\!|\!|_p\leqslant\int_{\mathbb{R}^d}|\!|\!| K_n(t,\cdot,y)|\!|\!|_p\mu({\rm d}y)\leqslant\int_0^\infty|\!|\!| h(t',\cdot)|\!|\!|_p\varrho^1_n(t-t'){\rm d}t'.$$

Hence,

$$\int_0^T |\!|\!| b_n(t,\cdot,\mu_{X_t^n}) |\!|\!|_p^q \mathrm{d}t \leqslant \int_0^T |\!|\!| h(t,\cdot) |\!|\!|_p^q \mathrm{d}t. \tag{3.5}$$

Now, consider the following backward PDE:

$$\partial u_n + \Delta u_n + b_n(t,\cdot,\mu_{X_t^n}) \cdot \nabla u_n = f, \quad u_n(T) = 0.$$

Since by (3.2), for any $j \in \mathbb{N}$,

$$\sup_{t\in[0,T]} \|\nabla^j b_n(t,\cdot,\mu_{X_t^n})\|_\infty < \infty,$$

the above PDE admits a unique smooth solution u_n with the regularities

$$\partial_t u_n, \nabla^2 u_n \in L^\infty([0,T]\times\mathbb{R}^d).$$

Moreover, by Theorem 2.5, there is a constant $C > 0$ such that for all $n \in \mathbb{N}$,

$$\|u_n\|_{L^\infty([0,T]\times\mathbb{R}^d)} \leqslant C |\!|\!| f |\!|\!|_{\widetilde{\mathbb{H}}_{\bar{q}}^{-\alpha,\bar{p}}(T)}.$$

Now by Itô's formula, we have

$$\mathbb{E} u_n(T, X_T^n) = \mathbb{E} u_n(0, X_0^n) + \mathbb{E}\int_0^T f(t, X_t^n)\mathrm{d}t,$$

which implies that

$$\mathbb{E}\int_0^T f(s, X_s^n)\mathrm{d}s \leqslant \|u_n(0,\cdot)\|_\infty \leqslant C |\!|\!| f |\!|\!|_{\widetilde{\mathbb{H}}_{\bar{q}}^{-\alpha,\bar{p}}(T)}.$$

The proof is complete. □

Let $\mathbb{C}$ be the space of all continuous functions from $\mathbb{R}_+$ to $\mathbb{R}^d$, which is endowed with the locally uniform convergence topology so that $\mathbb{C}$ is a Polish space. We also use the following convention below: The letter C with or without subscripts will denote a constant whose value may change in different places.

Lemma 3.2. *Let $\mathbb{P}_n$ be the law of $X_\cdot^n$ in $\mathbb{C}$. Then $(\mathbb{P}_n)_{n\in\mathbb{N}}$ is tight. Moreover, for any $\beta \in [0, 2/(\frac{d}{p}+\frac{2}{q}))$ and $T > 0$, there is a constant $C > 0$ such that for all $n \in \mathbb{N}$,*

$$\mathbb{E}\left(\sup_{t\in[0,T]} |X_t^n|^\beta\right) \leqslant C(\mathbb{E}|X_0^n|^\beta + 1). \tag{3.6}$$

Proof. Let $T > 0$ and $\tau \leqslant T$ be any bounded stopping time. For any $\delta > 0$ and $\gamma \in (1, 2/(\frac{d}{p}+\frac{2}{q}))$, by Hölder's inequality and (3.4) with $\alpha = 0$, we have

$$\begin{aligned}
\mathbb{E}\left(\int_\tau^{\tau+\delta} |b_n(t, X_t^n, \mu_{X_t^n})|\mathrm{d}t\right)^\gamma &\leqslant \delta^{\gamma-1}\mathbb{E}\left(\int_\tau^{\tau+\delta} |b_n(t, X_t^n, \mu_{X_t^n})|^\gamma \mathrm{d}t\right) \\
&\leqslant \delta^{\gamma-1}\mathbb{E}\left(\int_0^{T+\delta} |b_n(t, X_t^n, \mu_{X_t^n})|^\gamma \mathrm{d}t\right) \\
&\leqslant C\delta^{\gamma-1}\left[\int_0^{T+\delta} |\!|\!| |b_n(t,\cdot,\mu_{X_t^n})|^\gamma |\!|\!|_{p/\gamma}^{q/\gamma} \mathrm{d}t\right]^{\frac{\gamma}{q}} \\
&= C\delta^{\gamma-1}\left[\int_0^{T+\delta} |\!|\!| b_n(t,\cdot,\mu_{X_t^n}) |\!|\!|_p^q \mathrm{d}t\right]^{\frac{\gamma}{q}}
\end{aligned}$$

$$\overset{(3.5)}{\leqslant} C\delta^{\gamma-1}\left[\int_0^{T+\delta}|\!|\!|h(t,\cdot)|\!|\!|_p^q\mathrm{d}t\right]^{\frac{\gamma}{q}}, \tag{3.7}$$

where the constant C does not depend on n. Moreover, by Burkholder's inequality, it is easy to see that

$$\mathbb{E}\left(\sup_{s\in[0,\delta]}|W_{\tau+s}-W_\tau|^\gamma\right)\leqslant C\delta^{\gamma/2}.$$

Hence, for any $\theta\in(0,1)$, by [16, Lemma 2.7] we have

$$\sup_n\mathbb{E}\left(\sup_{s,t\in[0,T],s\neq t}|X^n_t-X^n_s|^{\theta\gamma}\right)\leqslant C\delta^{\theta(\gamma-1)}.$$

The tightness of $(\mathbb{P}_n)_{n\in\mathbb{N}}$ now follows by [13, Theorem 1.3.2]. Finally, the moment estimate (3.6) follows by (3.3) and (3.7). □

Now we can give

Proof of Theorem 1.1. Let $\mathbb{P}_n$ be the law of $X^n_\cdot$ in $\mathbb{C}$ and $\mathbb{W}$ the law of Brownian motion in $\mathbb{C}$. Consider the product probability measure $\mathbb{Q}_n:=\mathbb{P}_n\times\mathbb{P}_n\times\mathbb{W}$. By Lemma 3.2, one sees that $(\mathbb{Q}_n)_{n\in\mathbb{N}}$ is tight in $\mathbb{C}\times\mathbb{C}\times\mathbb{C}$. Let $\mathbb{Q}$ be any accumulation point. Without loss of generality, we assume that $\mathbb{Q}_n$ weakly converges to some probability measure $\mathbb{Q}$. By Skorokhod's representation theorem, there are a probability space $(\tilde\Omega,\tilde{\mathscr{F}},\tilde{\mathbf{P}})$ and random variables $(\tilde X^n,\tilde Y^n,\tilde W^n)$ and $(\tilde X,\tilde Y,\tilde W)$ defined on it such that

$$(\tilde X^n,\tilde Y^n,\tilde W^n)\to(\tilde X,\tilde Y,\tilde W),\quad \tilde{\mathbf{P}}-a.s. \tag{3.8}$$

and

$$\tilde{\mathbf{P}}\circ(\tilde X^n,\tilde Y^n,\tilde W^n)^{-1}=\mathbb{Q}_n,\quad \tilde{\mathbf{P}}\circ(\tilde X,\tilde Y,\tilde W)^{-1}=\mathbb{Q}. \tag{3.9}$$

Define $\tilde{\mathscr{F}}^n_t:=\sigma(\tilde W^n_s,\tilde X^n_s;s\leqslant t)$. We note that

$$\mathbb{P}(W_t-W_s\in\cdot|\mathscr{F}_s)=\mathbb{P}(W_t-W_s\in\cdot)$$
$$\Rightarrow\tilde{\mathbf{P}}(\tilde W^n_t-\tilde W^n_s\in\cdot|\tilde{\mathscr{F}}^n_s)=\tilde{\mathbf{P}}(\tilde W^n_t-\tilde W^n_s\in\cdot).$$

In other words, $\tilde W^n$ is an $\tilde{\mathscr{F}}^n_t$-Brownian motion. Thus, by (3.3) and (3.9) we have

$$\tilde X^n_t=\tilde X^n_0+\int_0^t b_n(s,\tilde X^n_s,\mu_{\tilde X^n_s})\mathrm{d}s+\sqrt{2}\tilde W^n_t. \tag{3.10}$$

To show the existence of a solution, the key point is to show that

$$\int_0^t b_n(s,\tilde X^n_s,\mu_{\tilde X^n_s})\mathrm{d}s\to\int_0^t b(s,\tilde X_s,\mu_{\tilde X_s})\mathrm{d}s \text{ in probability as } n\to\infty, \tag{3.11}$$

where $b(s,x,\mu):=\int_{\mathbb{R}^d}K(s,x,y)\mu(\mathrm{d}y)$. After showing this limit, we can take limits for both sides of (3.10) to obtain the existence of a solution, i.e.

$$\tilde X_t=\tilde X_0+\int_0^t b(s,\tilde X_s,\mu_{\tilde X_s})\mathrm{d}s+\sqrt{2}\tilde W_t.$$

Since $\tilde X^n$ and $\tilde Y^n$ are independent by (3.9), to prove (3.11), it suffices to show that

$$\int_0^t K_n(s,\tilde X^n_s,\tilde Y^n_s)\mathrm{d}s\to\int_0^t K(s,\tilde X_s,\tilde Y_s)\mathrm{d}s \text{ in probability as } n\to\infty.$$

The above limit will be a consequence of the following two limits: for each $m \in \mathbb{N}$,

$$\lim_{n\to\infty} \int_0^t |K_m(s, \tilde{X}_s^n, \tilde{Y}_s^n) - K_m(s, \tilde{X}_s, \tilde{Y}_s)| \mathrm{d}s = 0, \ \mathbf{P} - a.s. \tag{3.12}$$

and

$$\lim_{m\to\infty} \sup_n \mathbf{E} \int_0^t |K_m(s, \tilde{X}_s^n, \tilde{Y}_s^n) - K(s, \tilde{X}_s^n, \tilde{Y}_s^n)| \mathrm{d}s = 0. \tag{3.13}$$

Below we drop the tilde for simplicity. For fixed $m \in \mathbb{N}$, since K_m is bounded and $(x, y) \mapsto K_m(s, x, y)$ is continuous, it follows by the dominated convergence theorem and (3.8) that the limit (3.12) holds. For limit (3.13), we write

$$\mathbf{E} \int_0^t |K_m(s, X_s^n, Y_s^n) - K(s, X_s^n, Y_s^n)| \mathrm{d}s = I_{n,m}^{(1)}(R) + I_{n,m}^{(2)}(R),$$

where

$$I_{n,m}^{(1)}(R) := \mathbf{E} \int_0^t \mathbf{1}_{\{|X_s^n| \leqslant R\} \cap \{\ |Y_s^n| \leqslant R\}} |K_m(s, X_s^n, Y_s^n) - K(s, X_s^n, Y_s^n)| \mathrm{d}s,$$

$$I_{n,m}^{(2)}(R) := \mathbf{E} \int_0^t \mathbf{1}_{\{|X_s^n| > R\} \cup \{|Y_s^n| > R\}} |K_m(s, X_s^n, Y_s^n) - K(s, X_s^n, Y_s^n)| \mathrm{d}s.$$

For $I_{n,m}^{(1)}(R)$, since $(p, q) \in \mathscr{I}_0$, one can choose $\gamma > 1$ such that

$$\tfrac{d}{p} + \tfrac{2\gamma}{q} < 2.$$

Thus, by Lemma 2.4 with $p_1 = p$, $q_1 = \frac{q}{\gamma}$, $p_2 > \frac{qd}{2(\gamma-1)}$, $q_2 = \frac{q}{q+1-\gamma}$ and $q_0 = q$,

$$I_{n,m}^{(1)}(R) \leqslant C \|\!| \mathbf{1}_{B_R \times B_R} (K_m - K) \|\!|_{\widetilde{\mathbb{L}}_q^{p,p_2}(t)}, \tag{3.14}$$

where C is independent of n, m. Recalling the definition (2.8), we further have

$$I_{n,m}^{(1)}(R) \leqslant C \left(\int_0^t \left(\int_{B_{R+1}} \|\mathbf{1}_{B_{R+1}} |K_m(s, \cdot, y) - K(s, \cdot, y)| \|_p^{p_2} \mathrm{d}y \right)^{\frac{q}{p_2}} \mathrm{d}s \right)^{1/q}.$$

Since $|K(s, x, y)| \leqslant h(s, x - y)$, by (3.1) we have

$$\begin{aligned}
&\int_0^t \left(\sup_m \int_{B_{R+1}} \|\mathbf{1}_{B_{R+1}} |K_m(s, \cdot, y)| \|_p^{p_2} \mathrm{d}y \right)^{\frac{q}{p_2}} \mathrm{d}s \\
&\quad \leqslant \int_0^t \left(\int_{B_{R+2}} \|\mathbf{1}_{B_{R+2}} |h(s, \cdot - y)| \|_p^{p_2} \mathrm{d}y \right)^{\frac{q}{p_2}} \mathrm{d}s \\
&\quad \leqslant C_R \int_0^t \left(\int_{B_{2(R+2)}} |h(s, x)|^p \mathrm{d}x \right)^{\frac{q}{p}} \mathrm{d}s < \infty.
\end{aligned} \tag{3.15}$$

Hence, by the dominated convergence theorem, for each $R > 0$,

$$\lim_{m\to\infty} \sup_n I_{n,m}^{(1)}(R) = 0. \tag{3.16}$$

For $I_{n,m}^{(2)}(R)$, letting $\alpha \in (1, 2/(\frac{d}{p} + \frac{2}{q}))$, by Hölder's inequality and Lemma 2.4 again,

$$I_{n,m}^{(2)}(R) \leqslant \int_0^t (\mathbf{E}|K_m(s, X_s^n, Y_s^n) - K(s, X_s^n, Y_s^n)|^\alpha)^{\frac{1}{\alpha}}$$

$$
\begin{aligned}
&\times \mathbf{P}(\{|X^n_s|>R\}\cup\{|Y^n_s|>R\})^{1-\frac{1}{\alpha}}\mathrm{d}s\\
&\leqslant \left(\int_0^t \mathbf{E}|K_m(s,X^n_s,Y^n_s)-K(s,X^n_s,Y^n_s)|^\alpha \mathrm{d}s\right)^{\frac{1}{\alpha}}\\
&\quad\times \sup_{s\in[0,t]}\mathbf{P}(\{|X^n_s|>R\}\cup\{|Y^n_s|>R\})^{1-\frac{1}{\alpha}}\\
&\leqslant C\left(|\!|\!| K_m|\!|\!|_{\widetilde{\mathbb{L}}^{p,p_2}_q(t)}+|\!|\!| K|\!|\!|_{\widetilde{\mathbb{L}}^{p,p_2}_q(t)}\right)\sup_{s\in[0,t]}(2\mathbf{P}\{|X^n_s|>R\})^{1-\frac{1}{\alpha}}, \qquad (3.17)
\end{aligned}
$$

where C is independent of n,m and R, and p_2 is chosen being large enough as in (3.14). As in (3.15), we have

$$
\begin{aligned}
|\!|\!| K_m|\!|\!|^q_{\widetilde{\mathbb{L}}^{p,p_2}_q(T)} &= \sup_{z,z'\in\mathbb{R}^d}\int_0^T\left(\int_{\mathbb{R}^d}\mathbf{1}_{B^{z'}_1}(y)\|\mathbf{1}_{B^z_1}K_m(t,\cdot,y)\|^{p_2}_p\mathrm{d}y\right)^{\frac{q}{p_2}}\mathrm{d}t\\
&\leqslant \sup_{z,z'\in\mathbb{R}^d}\int_0^T\left(\int_{\mathbb{R}^d}\mathbf{1}_{B^{z'}_2}(y)\|\mathbf{1}_{B^z_2}h(t,\cdot-y)\|^{p_2}_p\mathrm{d}y\right)^{\frac{q}{p_2}}\mathrm{d}t\\
&\leqslant C\sup_{z,z'\in\mathbb{R}^d}\int_0^T\sup_{|y-z'|\leqslant 2}\|\mathbf{1}_{B^z_2}h(t,\cdot-y)\|^q_p\mathrm{d}t\\
&\leqslant C\int_0^T|\!|\!| h(t)|\!|\!|^q_p\mathrm{d}t<\infty. \qquad (3.18)
\end{aligned}
$$

Moreover, by (3.10) and Chebyschev's inequality, we have

$$
\begin{aligned}
\sup_{s\in[0,t]}\mathbf{P}\{|X^n_s|>R\} &\leqslant \mathbf{P}\{|X^n_0|>\tfrac{R}{3}\}+\mathbf{P}\left\{\sup_{s\in[0,t]}\sqrt{2}|W_s|>\tfrac{R}{3}\right\}\\
&\quad+\mathbf{P}\left\{\int_0^t|b_n(s,X^n_s,\mu_{X^n_s})|\mathrm{d}s>\tfrac{R}{3}\right\}\\
&\leqslant \nu\Big\{|x|>\tfrac{R}{3}\Big\}+\frac{C}{R}+\frac{3}{R}\mathbf{E}\int_0^t|b_n(s,X^n_s,\mu_{X^n_s})|\mathrm{d}s\\
&\leqslant \nu\Big\{|x|>\tfrac{R}{3}\Big\}+\frac{C}{R}, \qquad (3.19)
\end{aligned}
$$

where the constant C is independent of R and n, and the last step is due to (3.7). Combining (3.17), (3.18) and (3.19), we obtain

$$
\lim_{R\to\infty}\sup_{n,m}I^{(2)}_{n,m}(R)=0,
$$

which together with (3.16) yields (3.13). Moreover, the estimate (1.10) follows by (3.6), and the regularity estimate (1.11) follows by (3.4) and Remark 2.3. The proof is thus complete. □

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