SDES WITH CRITICAL TIME DEPENDENT DRIFTS: STRONG SOLUTIONS

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ABSTRACT. This paper is a continuation of [RZ20]. Based on a compactness criterion for random fields in Wiener-Sobolev spaces, in this paper, we prove the unique strong solvability of time-inhomogeneous stochastic differential equations with drift coefficients in critical Lebesgue spaces, which gives an affirmative answer to a longstanding open problem. As an application, we also prove a regularity criterion for solutions of a stochastic system proposed by Constantin and Iyer (Comm. Pure. Appl. Math. 61(3): 330–345, 2008), which is closely related to the Navier-Stokes equations.

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1. INTRODUCTION

Let W_t be a standard d-dimensional Brownian motion on a complete filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\in[0,T]}, \mathbf{P})$ and let b be a vector field on $\mathbb{R}^d$ satisfying the following critical Ladyzhenskaya-Prodi-Serrin (LPS) condition:

$$b \in \mathbb{L}_{q_1}^{p_1}(T) := L^{q_1}([0,T]; L^{p_1}(\mathbb{R}^d)) \text{ with } p_1, q_1 \in [2,\infty] \text{ and } \frac{d}{p_1} + \frac{2}{q_1} = 1. \tag{1.1}$$

Our primary goal is to solve the following longstanding open problem: does the stochastic differential equation (SDE)

$$X_{s,t}^x = x + \int_s^t b(r, X_{s,r}^x)\mathrm{d}r + W_t - W_s, \quad 0 \leqslant s \leqslant t \leqslant T,\ x \in \mathbb{R}^d \tag{1.2}$$

have a unique strong solution under condition (1.1)?

1.1. **Main result.** Our main result, which gives an affirmative answer to the above open problem, reads as follows. (The notation $\widetilde{\mathbb{L}}_q^p(S,T)$ appearing below is defined in (1.13))

Theorem 1.1. *Let $d \geqslant 3$. Assume b satisfies one of following two conditions*
(a) $b \in C([0,T]; L^d)$;
(b) $b \in \mathbb{L}_{q_1}^{p_1}(T)$ with $p_1, q_1 \in (2,\infty)$ and $d/p_1 + 2/q_1 = 1$.

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Then (1.2) *admits a unique strong solution such that the following estimate is valid:*

$$\sup_{x\in\mathbb{R}^d} \mathbf{E}\left(\int_s^T f(t,X^x_{s,t})\,\mathrm{d}t\right) \leqslant C\|f\|_{\widetilde{\mathbb{L}}^p_q(s,T)}, \tag{1.3}$$

where $p,q\in(1,\infty)$ *with* $\frac{d}{p}+\frac{2}{q}<2$ *and* C *is a constant independent with* f*. Moreover, the random field* $\{X^x_{s,t}\}_{\substack{x\in\mathbb{R}^d;\\ 0\leqslant s\leqslant t\leqslant T}}$ *forms a weakly differentiable stochastic flow and it satisfies the following*

(1) for any $p\in(\frac{d}{d-1},d)$ *in case (a) and* $p\in(\frac{p_1}{p_1-1},p_1)$ *in case (b),*

$$\sup_{0\leqslant s\leqslant t\leqslant T}\int_{\mathbb{R}^d}\left(\mathbf{E}|\nabla X^x_{s,t}-\mathbf{I}|^r\right)^p\mathrm{d}x<\infty,\ \ \textit{for any } r\in[2,\infty). \tag{1.4}$$

(2) for all $r\in(d,\infty)$*,* $\beta\in(0,\frac{1}{2})$*,* $R>0$*,* $x_i\in B_R$*,* $0\leqslant s_i\leqslant t_i\leqslant T$*,* $i=1,2$

$$\mathbf{E}\left|X^{x_1}_{s_1,t_1}-X^{x_2}_{s_2,t_2}\right|^r\leqslant C\left(|x_1-x_2|^{r-d}+|s_1-s_2|^{\beta(r-d)}+|t_1-t_2|^{\beta r}\right). \tag{1.5}$$

Remark 1.2. *Except for the case that* $\|b\|_{\mathbb{L}^d_\infty(T)}$ *is sufficiently small, our main approach of this paper does not work for the full endpoint case* $p_1=d$ *and* $q_1=\infty$*. However, in the later case the weak well-posedness was proved by Röckner-Zhao in* [RZ20]*, provided that the divergence of* b *satisfies an integrability condition. Therefore, we conjecture that the strong well-posedness of* (1.2) *holds when* $b\in\mathbb{L}^d_\infty(T)$ *and* $\mathrm{div} b=0$*.*

1.2. **Motivation and Previous results.** The existence of stochastic flows associated with SDEs with singular drifts and their regularity properties have various applications. For instance, in [FGP10], using the stochastic characteristics corresponding to (1.2), Flandoli-Gubinelli-Priola studied the existence and uniqueness for the stochastic transport equation in an L^∞-setting, provided that the drift b is α-Hölder continuous uniformly in t and the divergence of b satisfies some integrability condition. Later, stochastic continuity equations were also considered in [NO15] when b is divergence free and it satisfies the supercritical LPS condition

$$b\in\mathbb{L}^{p_1}_{q_1}(T)\text{ with }p_1,q_1\in(2,\infty)\text{ and }\frac{d}{p_1}+\frac{2}{q_1}<1. \tag{1.6}$$

The same SPDEs were also investigated by Fedrizzi-Flandoli in [FF13], Mohammed-Nilssen-Proske in [MNP15] and Beck-Flandoli-Gubinelli-Maurelli in [BFGM19] under different settings (see also the reference therein).

Our work is motivated by the deep connection between singular SDEs and Navier-Stokes equations. The velocity field u of an incompressible fluid not subject to an external force in $\mathbb{R}^d$ satisfies the Navier-Stokes equation

$$\partial_t u-\tfrac{1}{2}\Delta u+(\nabla u)u+\nabla P=0\ \text{ in }[0,T]\times\mathbb{R}^d, \tag{1.7a}$$

$$\mathrm{div}\,u=0, \tag{1.7b}$$

$$u(0)=\varphi. \tag{1.7c}$$

The mathematical studies of Navier-Stokes equations have a long history. In [Ler34], Leray considered (1.7a)-(1.7c) for the initial data $\varphi\in L^2$. He proved that there exists a global in time Leray-Hopf weak solution $u\in\mathbb{L}^2_\infty(T)$ with $\nabla u\in\mathbb{L}^2_2(T)$. However, to date, the problem of smoothness of Leray-Hopf weak solutions for the 3D Navier-Stokes equations remains open. Studies by Prodi [Pro59], Serrin [Ser62] and Ladyzhenskaya [Lad67] found that the interior smoothness of Leray-Hopf weak solutions to (1.7a)-(1.7b) is guaranteed, provided that $u \in \mathbb{L}_{q_1}^{p_1}(T)$, for $p_1 \in (d,\infty)$ and $d/p_1 + 2/q_1 \leqslant 1$ (see also [FJR72] and [Gig86]). These conditional regularity results and their generalizations have culminated with the work of Escauriaza-Seregin-Šverák [ESŠ03] for $d=3$ and then Dong-Du [DD09] for $d \geqslant 3$. On the other hand, in the corresponding Lagrangian description, a fluid particle motion is described by the SDE

$$\mathrm{d}X_t^x = u(t,X_t^x)\mathrm{d}t + \mathrm{d}W_t, \quad x \in \mathbb{R}^d. \tag{1.8a}$$

When u is smooth, Constantin-Iyer [CI08] presented an elegant stochastic representation for the solutions to the Navier-Stokes equation, namely

$$u(t,x) = \mathrm{P}\mathbf{E}\left[\nabla^\top (X_t^x)^{-1}\varphi\left((X_t^x)^{-1}\right)\right], \tag{1.8b}$$

where P is the Leray projection and $(X_t^x)^{-1}$ is the inverse stochastic flow of (1.8a). Conversely, if u is smooth and (u,X) solves the stochastic system (1.8a)-(1.8b), then u also solves (1.7a)-(1.7c). From then on, some researchers started to study (1.7a)-(1.7c) via investigating the corresponding stochastic Lagrangian paths, see [Rez14], [Rez16], [Zha10] and [Zha16], etc. Since the problems of the regularity of solutions to the 3D Navier-Stokes equations are very challenging, two natural questions arise: (1) If the drift term is irregular, when does (1.8a) (or (1.2)) admit a weakly differentiable stochastic flow so that the right hand side of (1.8b) can be defined? (2) Can one also obtain some conditional regularity results for the stochastic system (1.8a)-(1.8b)?

Our Theorem 1.1 shows that (1.8a) has a weakly differentiable stochastic flow if the drifts satisfy the critical LPS conditions. For the second question above, to simplify our presentation, as in [Zha10], in this paper, we study the backward stochastic system

$$\begin{cases} X_{t,s}^x = x + \displaystyle\int_t^s u\left(r, X_{t,r}^x\right)\mathrm{d}r + (W_s - W_t), & -T \leqslant t \leqslant s \leqslant 0 \\ u(t,x) = \mathrm{P}\mathbf{E}\left[\nabla^\top X_{t,0}^x \varphi\left(X_{t,0}^x\right)\right], & -T \leqslant t \leqslant 0 \end{cases} \tag{1.9}$$

corresponding to the backward Navier-Stokes equation

$$\partial_t u + \tfrac{1}{2}\Delta u + (\nabla u)u + \nabla P = 0, \quad \mathrm{div} u = 0, \quad u(0) = \varphi \tag{1.10}$$

instead of the forward one (1.8a)-(1.8b). With the help of our new estimate (1.4), we give a regularity criterion for solutions to (1.9) in Theorem 6.1 below, which can be regarded as an analogue of Serrin's regularity criterion for solutions to the 3D Navier-Stokes equations.

We close this subsection by mentioning some previous work about strong solutions to non degenerate SDEs with singular drifts. The study of strong well-posedness of non degenerate Itô equations with bounded drift coefficients dates back to [Zvo74] and [Ver80]. In [KR05], Krylov-Röckner obtained the existence and uniqueness of strong solutions to (1.2), when b satisfies the subcritical LPS condition. After that a number of papers were devoted to generalize the strong well-posedness result, as well as the following gradient estimate for X:

$$\sup_{x\in\mathbb{R}^d}\mathbf{E}\sup_{t\in[s,T]}|\nabla X^x_{s,t}|^r<\infty,\quad \forall r\geqslant 1. \tag{1.11}$$

The reader is referred to [FF11], [LT17], [MNP15], [Rez14], [XXZZ20], [Zha05], [Zha11], [Zha16] and the reference therein for more details. We also point out that when u is a Leray-Hopf solution to the Navier-Stokes equation, in [Zh19], the second named author also showed that (1.8a) also admits a unique almost everywhere stochastic flow, but the weak differentiability of the flow remained open. To the best of our knowledge, the strong solvability under the critical condition (1.1) was first touched by Beck-Flandoli-Gubinelli-Maurelli in [BFGM19], where they proved the pathwise uniqueness to SDE (1.2) in a certain class if the initial datum has a diffuse law. Recently, if b belongs to the Orlicz-critical space $L^{q_1,1}([0,T];L^{p_1})\subsetneqq\mathbb{L}^{p_1}_{q_1}(T)$ for some $p_1,q_1\in(2,\infty)$ with $d/p_1+2/q_1=1$, by Zvonkin's transformation (cf. [Zvo74]), Nam [Nam20] showed the existence and uniqueness of strong solutions for SDE (1.2). The key step in using Zvonkin type of change of variables is to construct a homeomorphism by solving the Kolmogorov equation. If, however, b only satisfies the critical LPS condition (1.1), this strategy seems impossible to implement. Very recently, Krylov [Kry20c] proved the strong well-posedness of (1.2) for the case that $b(t,x)=b(x)\in L^d(\mathbb{R}^d)$ with $d\geqslant 3$, which is a significant progress on this topic. His approach is based on his earlier work with Veretennikov [VK76] about the Wiener chaos expansion for strong solutions of (1.2), and also some new estimates obtained in [Kry20a] and [Kry20b]. It may be also possible to follow the some procedure as in [Kry20c] to study the time-inhomogeneous case, but one encounters a lot of difficulties due to the fact that there is no good PDE theory for equations with such kind of first order terms so far. In this paper, we use a very different approach from that in [Kry20c], which will be explained briefly in the next subsection.

1.3. **Approach and Structure.** The approach in this article is probabilistic, employing ideas from the Malliavin calculus coupled with some estimates for parabolic equations. In [RZ20], we obtain weak well-posedness of (1.2) under a slightly more general condition. So, to get the strong well-posedness, one only needs to show the strong existence due to a fundamental result of Cherny [Che02]. Our approach for proving strong existence is quite straightforward. Let $\{b_k\}$ be a smooth approximating sequence of the drift b in $\mathbb{L}^{p_1}_{q_1}(T)$ and $X^x_{s,t}(k)$ be the unique strong solution to (1.2) with b replaced by b_k. The main effort of the present work is to show that $X^x_{s,t}(k)$ converges to a random field $X^x_{s,t}$, which is a strong solution to equation (1.2). A key ingredient for the convergence of $X^x_{s,t}(k)$ is the fact that for each $s,t\in[0,T]$ and $R>0$, the sequence $\{X^x_{s,t}(k)\}$ is compact in $L^2(B_R\times\Omega)$. The proof for this assertion is based on a compactness criterion for L^2 random fields in Wiener spaces (see Lemma 3.1 below or [BS04]) and the following crucial estimate: for any $\alpha_i\in\{1,2,\cdots,d\}(i\in\mathbb{N}_+)$, $n\in\mathbb{N}_+$ and some $p>1$,

$$\left\|\mathbf{E}\int\cdots\int_{s\leqslant t_1\leqslant\cdots\leqslant t_n\leqslant t}\prod_{i=1}^n\partial_{\alpha_i}f_i(t_i,X^x_{s,t_i}(k))\,\mathrm{d}t_1\cdots\mathrm{d}t_n\right\|_{L^p_x}\leqslant C^{n+1}\prod_{i=1}^n\|f_i\|_{\mathbb{L}^{p_1}_{q_1}(s,t)}, \tag{1.12}$$

where C does not depend on k and α_i (see Lemma 4.2 below for the precise statement).

The framework of our work is inspired by [MNP15], but the main techniques in this work are essentially different in compare with the previous literature. For example, in [MNP15], using Girsanov's transformation, the main ingredient for the proof of the strong existence result and the gradient estimate for X was reduced to the following estimate:

$$\left\|\mathbf{E}\int\cdots\int_{0\leqslant t_1\leqslant\cdots\leqslant t_n\leqslant t}\prod_{i=1}^{n}\partial_{\alpha_i}f_i\left(t_i,x+W_{t_i}\right)\mathrm{d}t_1\cdots\mathrm{d}t_n\right\|_{L^\infty_x}\leqslant C^n t^{\frac{n}{2}}(n!)^{-\frac{1}{2}}\prod_{i=1}^{n}\|f_i\|_\infty.$$

Such a bound was first obtained by Davie in [Dav07] (cf. [Dav07, Proposition 2.2] and [Sha16, Proposition 2.1]) by proving a bound for certain block integrals. Later, Rezakhanlou [Rez14] also showed that

$$\left\|\mathbf{E}\int\cdots\int_{0\leqslant t_1\leqslant\cdots\leqslant t_n\leqslant t}\prod_{i=1}^{n}\partial_{\alpha_i}f_i\left(t_i,x+W_{t_i}\right)\mathrm{d}t_1\cdots\mathrm{d}t_n\right\|_{L^\infty_x}\leqslant C^n t^{\frac{\kappa}{2}}(n!)^{-\frac{\kappa}{2}}\prod_{i=1}^{n}\|f_i\|_{\mathbb{L}^p_q(t)},$$

provided that $\kappa:=1-\frac{d}{p}-\frac{2}{q}>0$. However, when $\kappa=0$, one can not expect to have bounds that are uniform in x, and the approach used in [MNP15] and [Rez14] seems very hard, if not impossible to deal with the critical case ($\kappa=0$). To overcome these essential difficulties, due to the fact that we are in the critical case, in this paper, we reduce the desired bounded (1.12) to a uniform in time L^p-bound on the solution to a certain parabolic equation with critical drift and a distributional valued inhomogeneous term (see the discussion before Lemma 4.2 below). To obtain such uniform bound, we investigate the PDE mentioned above in Sobolev spaces with mixed norms (see Theorem 2.3 and 2.6) with the aid of some parabolic versions of Sobolev and Morrey inequalities in mixed norm spaces, which are proved by using Sobolevskii Mixed Derivative Theorem.

The rest of this paper is organized as following: In the rest of this section, we list some notations that will be used in this paper frequently. In Section 2, we study Kolmogorov equations with inhomogeneous terms in Sobolev spaces of negative order. In Section 3, give a compactness criterion for L^2 random fields in Wiener spaces. In Section 4, we derive some crucial uniform estimates for the solutions to certain approximating SDEs. The proof of the main result is presented in Section 5. In Section 6, we apply our main result to prove a regularity criterion for solutions of a stochastic system, which is closely related to the Navier-Stokes equations.

1.4. **Notations.** We close this section by mentioning some notational conventions used throughout this paper:

- $\mathbb{N}:=\{0,1,2,\cdots\}$, $\mathbb{N}_+:=\{1,2,\cdots,\}$.
- The transpose of a matrix A is denoted by $A^\top$.
- For a differentiable map $X:\mathbb{R}^d\ni x\mapsto(X^1(x),\cdots,X^{d_1}(x))^\top\in\mathbb{R}^{d_1}$, the matrix $\nabla X(x)$ is defined by

$$\nabla X(x)=\begin{pmatrix}\partial_1X^1(x) & \partial_2X^1(x) & \cdots & \partial_dX^1(x)\\ \partial_1X^2(x) & \partial_2X^2(x) & \cdots & \partial_dX^2(x)\\ \cdots & \cdots & \cdots & \cdots\\ \partial_1X^{d_1}(x) & \partial_2X^{d_1}(x) & \cdots & \partial_dX^{d_1}(x)\end{pmatrix}.$$

- Given $S,T\in[-\infty,\infty]$, set
$$\Delta_n(S,T):=\{(t_1,\cdots,t_n)\in\mathbb{R}^n: S\leqslant t_1\leqslant\cdots\leqslant t_n\leqslant T\},\quad \Delta_n(T):=\Delta_n(0,T).$$
- Assume that for each $i\in\{1,2\}$, (X_i,Σ_i,μ_i) is a measure space. Suppose that $f:X_1\times X_2\to\mathbb{R}$, define
$$\|f\|_{L^{p_1}_{x_1}(\mu_1)L^{p_2}_{x_2}(\mu_2)}:=\left[\int_{X_1}\left(\int_{X_2}|f(x_1,x_2)|^{p_2}\mu_2(\mathrm{d}x_2)\right)^{1/p_2}\mu_1(\mathrm{d}x_1)\right]^{1/p_1}.$$
- For each $p,q\in[1,\infty]$, the space $L^q([S,T];L^p(\mathbb{R}^d))$ is denoted by $\mathbb{L}^p_q(S,T)$. For any $p,q\in(1,\infty), s\in\mathbb{R}$, define $\mathbb{H}^{s,p}_q(S,T)=L^q([S,T];H^{s,p}(\mathbb{R}^d))$, where $H^{s,p}=(1-\Delta)^{-s/2}L^p$ is the Bessel potential space.
- Throughout this paper, we fix a cutoff function
$$\chi\in C_c^\infty(\mathbb{R}^d;[0,1])\text{ with }\chi|_{B_1}=1\text{ and }\chi|_{B_2^c}=0.$$
For $r>0$ and $x\in\mathbb{R}^d$, let $\chi^z_r(x):=\chi\left(\frac{x-z}{r}\right)$. For any $p,q\in[1,\infty]$, define
$$\widetilde{L}^p:=\left\{f\in L^p_{loc}(\mathbb{R}^d):\|f\|_{\widetilde{L}^p}:=\sup_{z\in\mathbb{R}^d}\|f\chi^z_1\|_p<\infty\right\}$$
and
$$\widetilde{\mathbb{L}}^p_q(S,T):=\left\{f\in L^q([S,T];L^p_{loc}(\mathbb{R}^d)):\|f\|_{\widetilde{\mathbb{L}}^p_q(S,T)}:=\sup_{z\in\mathbb{R}^d}\|f\chi^z_1\|_{\mathbb{L}^p_q(S,T)}<\infty.\right\}.\tag{1.13}$$
The localized Bessel potential space is defined as follows:
$$\widetilde{\mathbb{H}}^{s,p}_q(S,T):=\left\{f\in L^q([S,T];H^{s,p}_{loc}(\mathbb{R}^d)):\|f\|_{\widetilde{\mathbb{H}}^{s,p}_q(S,T)}:=\sup_{z\in\mathbb{R}^d}\|f\chi^z_1\|_{\mathbb{H}^{s,p}_q(S,T)}<\infty\right\}.$$
- For simplicity, we set
$$\mathbb{L}^p_q(T):=\mathbb{L}^p_q(0,T),\ \mathbb{L}^p_q:=L^q(\mathbb{R};L^p),\ \mathbb{H}^{s,p}_q(T):=\mathbb{H}^{s,p}_q(0,T),\ \mathbb{H}^{s,p}_q:=L^q(\mathbb{R};H^{s,p})$$
and
$$\widetilde{\mathbb{L}}^p_q(T)=\widetilde{\mathbb{L}}^p_q(0,T),\quad \widetilde{\mathbb{H}}^{s,p}_q(T):=\widetilde{\mathbb{H}}^{s,p}_q(0,T).$$

2. Some auxiliary analytic results

In this section, we study the Kolmogorov equations with inhomogeneous terms in Sobolev spaces of negative order. These analytic results, which are of their own interest, will play a crucial role in proofs for the main results.

The following conclusions are variants of Theorem 1.1 and 1.2 in [Kry01].

Lemma 2.1. *Let $p,q\in(1,\infty)$ and $\alpha\in\mathbb{R}$.*

(1) Assume $\lambda>0$, $\mu\geqslant 0$. For each $u\in L^q(\mathbb{R};H^{\alpha+2,p})\cap H^{1,q}(\mathbb{R};H^{\alpha,p})$,
$$\|\partial_t u\|_{\mathbb{H}^{\alpha,p}_q}+\lambda\|\nabla^2 u\|_{\mathbb{H}^{\alpha,p}_q}+\mu\|u\|_{\mathbb{H}^{\alpha,p}_q}\leqslant C\|(\partial_t-\lambda\Delta+\mu)u\|_{\mathbb{H}^{\alpha,p}_q},\tag{2.1}$$
where C only depends on d,p,q.

(2) Assume that $f \in \mathbb{H}^{\alpha,p}_q(T)$, then the following heat equation admits a unique solution in $\mathbb{H}^{\alpha+2,p}_q(T)$:

$$\partial_t u - \tfrac{1}{2}\Delta u = f \ \text{ in } (0,T)\times\mathbb{R}^d, \quad u(0)=0.$$

Moreover,

$$\|\partial_t u\|_{\mathbb{H}^{\alpha,p}_q(T)} + \|u\|_{\mathbb{H}^{\alpha+2,p}_q(T)} \leqslant C_1\|f\|_{\mathbb{H}^{\alpha,p}_q(T)}, \tag{2.2}$$

where C_1 only depends on d,p,q,T.

Consider the following Kolmogorov equation associated with (1.2):

$$\partial_t u = \tfrac{1}{2}\Delta u + b\cdot\nabla u + f, \quad u(0)=0. \tag{2.3}$$

Throughout this paper, we fix a smooth function $\rho \in C^\infty_c(\mathbb{R}^d)$ satisfying $\rho \geqslant 0$ and $\int \rho = 1$, and set $\rho_m(\cdot) := m^d\rho(m\cdot)$.

2.1. **Case (a):** $b \in C([0,T];L^d(\mathbb{R}^d))$**.**
For any $f \in \mathbb{L}^d_\infty(T)$, define

$$K_f(m) := \sup_{t\in[0,T]} \|f(t) - f(t) *_x \rho_m\|_{L^d}. \tag{2.4}$$

Proposition 2.2. *Suppose that $f \in C([0,T];L^d)$, then $K_f(m) \to 0$ as $m\to\infty$.*

Proof. Since the map $f:[0,T]\to L^d$ is uniformly continuous, for each $\varepsilon>0$ there is a constant $\delta>0$ such that

$$\sup_{\substack{t_1,t_2\in[0,T];\\ |t_1-t_2|\leqslant\delta}} \|f(t_1)-f(t_2)\|_{L^d} < \varepsilon/2.$$

Assume that $k = \{0,1,2,\cdots,[T/\delta]\}$ and $t\in[k\delta,(k+1)\delta\wedge T]$, then

$$\begin{aligned}
&\limsup_{m\to\infty}\|f(t)-f_m(t)\|_{L^d}\\
\leqslant&\|f(t)-f(k\delta)\|_{L^d} + \limsup_{m\to\infty}\|f(k\delta)-f_m(k\delta)\|_{L^d} + \limsup_{m\to\infty}\|[f(k\delta)-f(t)]*_x\rho_m\|_{L^d}\\
\leqslant&2\|f(t)-f(k\delta)\|_{L^d} \leqslant 2\sup_{\substack{t_1,t_2\in[0,T];\\ |t_1-t_2|\leqslant\delta}}\|f(t_1)-f(t_2)\|_{L^d} < \varepsilon.
\end{aligned}$$

Thus, $\lim_{m\to\infty} K_f(m) = 0$. □

The following theorem will plays a crucial role in the proof of the first case of our main result.

Theorem 2.3. *Let $d\geqslant 3$, $\alpha\in\{0,-1\}$ and $\{a(m)\}_{m\in\mathbb{N}_+}$ be a sequence converging to zero. Assume $b\in\mathbb{L}^d_\infty(T)$ and $K_b(m)\leqslant a(m)$. Suppose that $p\in(1,d)$ and $q\in(1,\infty)$ if $\alpha=0$, or $p\in(d/(d-1),d)$ and $q\in(1,\infty)$ if $\alpha=-1$. Then for any $f\in\mathbb{H}^{\alpha,p}_q(T)$, equation (2.3) admits a unique solution in $\mathbb{H}^{\alpha+2,p}_q(T)$. Moreover,*

$$\sup_{t\in(0,T]} t^{-1}\|u\|_{\mathbb{H}^{\alpha,p}_q(t)} + \|\partial_t u\|_{\mathbb{H}^{\alpha,p}_q(T)} + \|u\|_{\mathbb{H}^{\alpha+2,p}_q(T)} \leqslant C_2\|f\|_{\mathbb{H}^{\alpha,p}_q(T)}, \tag{2.5}$$

where C_2 only depends on $d,p,q,T,\|b\|_{\mathbb{L}^d_\infty(T)},\{a(m)\}$, and is increasing in T.

Proof. Below we only give the proof for the case that $\alpha=-1$ (the case $\alpha=0$ is simpler). To prove the desired result, it suffices to show (2.5) assuming that the solution already exists, since the method of continuity is applicable.

Let $b_m=b*_x\rho_m$ and $\bar b_m=b-b_m$. By Sobolev embedding and Hölder's inequality, we have

$$\begin{aligned}\|\bar b_m\cdot\nabla u\|_{\mathbb{H}^{-1,p}_q(t)}\leqslant&C_3\|\bar b_m\cdot\nabla u\|_{\mathbb{L}_q^{\frac{dp}{p+d}}(t)}\leqslant C_3\|\bar b_m\|_{\mathbb{L}^d_\infty(t)}\|\nabla u\|_{\mathbb{L}^p_q(t)}\\ \leqslant&C_3a(m)\|u\|_{\mathbb{H}^{1,p}_q(t)},\end{aligned}\tag{2.6}$$

where $t\in[0,T]$ and C_3 only depends on d,p. Similarly,

$$\begin{aligned}\|b_m\cdot\nabla u\|_{\mathbb{H}^{-1,p}_q(t)}\leqslant&\|\mathrm{div}(b_m u)\|_{\mathbb{H}^{-1,p}_q(t)}+\|\mathrm{div}b_m\cdot u\|_{\mathbb{L}^p_q(t)}\\ \leqslant&C\|b_m\|_{L^\infty([0,T];C^2_b)}\|u\|_{\mathbb{L}^p_q(t)}\\ \leqslant&C\left(\|\rho_m\|_{d/d-1}+\|\nabla\rho_m\|_{d/d-1}\right)\|u\|_{\mathbb{L}^p_q(t)}\\ \leqslant&Cm^2\|u\|_{\mathbb{L}^p_q(t)},\end{aligned}\tag{2.7}$$

where $t\in[0,T]$ and C only depends on $d,p,\|b\|_{\mathbb{L}^d_\infty(T)}$. Thanks to Lemma 2.1, for each $t\in[0,T]$,

$$\begin{aligned}&\|\partial_t u\|_{\mathbb{H}^{-1,p}_q(t)}+\|u\|_{\mathbb{H}^{1,p}_q(t)}\\ \leqslant&C_1\left(\|b\cdot\nabla u\|_{\mathbb{H}^{-1,p}_q(t)}+\|f\|_{\mathbb{H}^{-1,p}_q(t)}\right)\\ \overset{(2.6),(2.7)}{\leqslant}&C_1\left(C_3a(m)\|\nabla u\|_{\mathbb{L}^p_q(t)}+Cm^2\|u\|_{\mathbb{L}^p_q(t)}+\|f\|_{\mathbb{H}^{-1,p}_q(t)}\right).\end{aligned}$$

Letting m be large enough such that $C_1C_3a(m)\leqslant 1/2$ and using interpolation, we obtain

$$I(t):=\|\partial_t u\|^q_{\mathbb{H}^{-1,p}_q(t)}+\|u\|^q_{\mathbb{H}^{1,p}_q(t)}\leqslant C\left(\|u\|^q_{\mathbb{H}^{-1,p}_q(t)}+\|f\|^q_{\mathbb{H}^{-1,p}_q(t)}\right),\tag{2.8}$$

where $t\in[0,T]$ and C only depends on $d,p,q,T,\|b\|_{\mathbb{L}^d_\infty(T)},\{a(m)\}$. One the other hand, for any $t\in(0,T]$, using Hölder's inequality, we have

$$\begin{aligned}\|u\|^q_{\mathbb{H}^{-1,p}_q(t)}=&\int_0^t\|u(\tau,\cdot)\|^q_{H^{-1,p}}\mathrm{d}\tau=\int_0^t\left\|\int_0^\tau\partial_t u(\sigma,\cdot)\mathrm{d}\sigma\right\|^q_{H^{-1,p}}\mathrm{d}\tau\\ \leqslant&\int_0^t\tau^{q-1}\|\partial_t u\|^q_{\mathbb{H}^{-1,p}_q(\tau)}\mathrm{d}\tau\leqslant T^{q-1}\int_0^t I(\tau)\mathrm{d}\tau\wedge\frac{t^q}{q}\|\partial_t u\|^q_{\mathbb{H}^{-1,p}_q(t)},\end{aligned}\tag{2.9}$$

and together with (2.8), we obtain

$$I(t)\leqslant C\|f\|^q_{\mathbb{H}^{-1,p}_q(T)}+C\int_0^t I(\tau)\mathrm{d}\tau$$

Gronwall's inequality yields,

$$\|\partial_t u\|_{\mathbb{H}^{-1,p}_q(T)}+\|u\|_{\mathbb{H}^{1,p}_q(T)}\leqslant C\|f\|_{\mathbb{H}^{-1,p}_q(T)}.\tag{2.10}$$

Noting that (2.9) also implies

$$\sup_{t\in(0,T]}t^{-1}\|u\|_{\mathbb{H}^{-1,p}_q(t)}\leqslant C(q)\|\partial_t u\|_{\mathbb{H}^{-1,p}_q(t)},\quad\forall t\in(0,T],$$

together with (2.10), we obtain (2.5).

$$\sup_{t\in(0,T]} t^{-1}\|u\|_{\mathbb{H}_q^{-1,p}(t)} + \|\partial_t u\|_{\mathbb{H}_q^{-1,p}(T)} + \|u\|_{\mathbb{H}_q^{1,p}(T)} \leqslant C_2\|f\|_{\mathbb{H}_q^{-1,p}(T)},$$

where C_2 only depends on $d,p,q,T,\|b\|_{\mathbb{L}^d_\infty(T)},\{a(m)\}$. So, we complete our proof. □

2.2. **Case (b):** $b\in\mathbb{L}^{p_1}_{q_1}(T)$ **with** $p_1,q_1\in(2,\infty)$ **and** $d/p_1+2/q_1=1$**.**
In this case, to obtain a result similar to Theorem 2.3, we need to prove some parabolic Morrey and Sobolev inequalities. This can be achieved by using the Mixed Derivative Theorem, which goes back to the work of Sobolevskii (cf. [Sob77]).

Let X be a Banach space and let $A: D(A)\to X$ be a closed, densely defined linear operator with dense range. Then A is called sectorial, if

$$(0,\infty)\subseteq\rho(-A) \quad \text{and} \quad \left\|\lambda(\lambda+A)^{-1}\right\|_{X\to X}\leqslant C, \quad \lambda>0,$$

where $\rho(-A)$ is the resolvent set of $-A$. Set

$$\Sigma_\phi := \{z\in\mathbb{C}\backslash\{0\} : |\arg z|<\phi\}.$$

We recall that

$$\phi_A := \inf\left\{\phi\in[0,\pi) : \Sigma_{\pi-\phi}\subseteq\rho(-A),\ \sup_{z\in\Sigma_{\pi-\phi}}\left\|z(z+A)^{-1}\right\|_{X\to X}<\infty\right\}$$

is the the spectral angle of A. For each $\theta\in(0,1)$, define

$$A^\theta x := \frac{\sin\theta\pi}{\pi}\int_0^\infty \lambda^{\theta-1}(\lambda+A)^{-1}Ax\,\mathrm{d}\lambda, \quad x\in D(A)$$

and

$$A^{-\theta}x := \frac{\sin\theta\pi}{\pi}\int_0^\infty \lambda^{-\theta}(\lambda+A)^{-1}x\,\mathrm{d}\lambda, \quad x\in X.$$

We need the following Sobolevskii Mixed Derivative Theorem (cf. [Sob77]).

Lemma 2.4 (Mixed Derivative Theorem)**.** *Let A and B be two sectorial operators in a Banach space X with spectral angles ϕ_A and ϕ_B, which are commutative and satisfy the parabolicity condition $\phi_A+\phi_B<\pi$. Then the coercivity estimate*

$$\|Ax\|_X+\lambda\|Bx\|_X\leqslant M\|Ax+\lambda Bx\|_X, \quad \forall x\in D(A)\cap D(B),\ \lambda>0$$

implies that

$$\left\|A^{(1-\theta)}B^\theta x\right\|_X\leqslant C\|Ax+Bx\|_X, \quad \forall x\in D(A)\cap D(B),\ \theta\in[0,1],$$

The following parabolic Sobolev and Morrey inequalities will be used frequently in this work.

Lemma 2.5. *Let $p,q\in(1,\infty)$, $r\in(p,\infty)$, $s\in(q,\infty)$ and $\alpha\in\mathbb{R}$. Assume $\partial_t u\in\mathbb{H}^{\alpha,p}_q(T)$, $u\in\mathbb{H}^{\alpha+2,p}_q(T)$ and $u(0)=0$.*

(1) *If* $1 < d/p + 2/q = d/r + 2/s + 1$, *then*

$$\|u\|_{\mathbb{H}_s^{\alpha+1,r}(T)} \leqslant C_4 \left(\|\partial_t u\|_{\mathbb{H}_q^{\alpha,p}(T)} + \|u\|_{\mathbb{H}_q^{\alpha+2,p}(T)} \right), \tag{2.11}$$

where C_4 *only depends on* d,p,q,r,s.

(2) *If* $2 < d/p + 2/q = d/r + 2/s + 2$, *then*

$$\|u\|_{\mathbb{H}_s^{\alpha,r}(T)} \leqslant C_5 \left(\|\partial_t u\|_{\mathbb{H}_q^{\alpha,p}(T)} + \|u\|_{\mathbb{H}_q^{\alpha+2,p}(T)} \right), \tag{2.12}$$

where C_5 *only depends on* d,p,q,r,s.

(3) *If* $0 \leqslant \theta < 1 - 1/q$, *for any* $t_1, t_2 \in [0,T]$,

$$\|u(t_1) - u(t_2)\|_{H^{\alpha+2\theta,p}} \leqslant C_6 |t_1 - t_2|^{1-1/q-\theta} \left(\|\partial_t u\|_{\mathbb{H}_q^{\alpha,p}(T)} + \|u\|_{\mathbb{H}_q^{\alpha+2,p}(T)} \right), \tag{2.13}$$

where C_6 *only depends on* d,p,q,θ.

Proof. By considering $(1-\Delta)^{\alpha/2}u$ instead of u, we see that without loss of generality we may assume $\alpha = 0$.

Let $X = L^q(\mathbb{R}; L^p(\mathbb{R}^d))$, $A = 1 + \partial_t$ and $B = 1 - \Delta$ in Lemma 2.4. It is well-known that

$$\phi_A = \frac{\pi}{2} \ \text{ and } \ \phi_B = 0.$$

Due to (2.1), for all $\lambda > 0$ we have

$$\begin{aligned}
\|Au\|_X + \lambda \|Bu\|_X =& \|u + \partial_t u\|_{\mathbb{L}_q^p} + \lambda \|u - \Delta u\|_{\mathbb{L}_q^p} \\
\leqslant& C \left(\|\partial_t u\|_{\mathbb{L}_q^p} + \lambda \|\nabla^2 u\|_{\mathbb{L}_q^p} + (1+\lambda) \|u\|_{\mathbb{L}_q^p} \right) \\
\leqslant& C \|(u + \partial_t u) + \lambda (u - \Delta u)\|_{\mathbb{L}_q^p} = C \|Au + \lambda Bu\|_X,
\end{aligned}$$

where C only depends on d,p,q. Thanks to Lemma 2.4, we obtain

$$\|A^{1-\theta} B^\theta u\|_{\mathbb{L}_q^p} \leqslant C \|\partial_t u - \Delta u + 2u\|_{\mathbb{L}_q^p} \leqslant C \left(\|\partial_t u\|_{\mathbb{L}_q^p} + \|u\|_{\mathbb{H}_q^{2,p}} \right), \tag{2.14}$$

for all $u \in H^{1,q}\left(\mathbb{R}, L^p(\mathbb{R}^d)\right) \cap L^q\left(\mathbb{R}, H^{2,p}(\mathbb{R}^d)\right)$. For any $\alpha \in (0,1), q \in (1,\infty)$ and $f \in L^q(\mathbb{R})$, we have

$$\mathscr{F}((1-\partial_{tt}^2)^{\alpha/2}(1+\partial_t)^{-\alpha} f) = \frac{(1+4\pi^2|\xi|^2)^{\alpha/2}}{(1+\mathrm{i}2\pi\xi)^\alpha} \mathscr{F}(f)(\xi) =: m(\xi)\mathscr{F}(f)(\xi),$$

where $\mathscr{F}(f)(\xi) := \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}2\pi x\xi} f(x) \mathrm{d}x$ is the Fourier transformation of f. Since $|\xi|^k m^{(k)}(\xi) \leqslant C_k < \infty$, by Mikhlin's multiplier theorem, the operator $(1-\partial_{tt}^2)^{\alpha/2}(1+\partial_t)^{-\alpha}$ is bounded on $L^q(\mathbb{R})$. Therefore,

$$\begin{aligned}
\|u\|_{H^{1-\theta,q}(\mathbb{R}; H^{2\theta,p}(\mathbb{R}^d))} =& \|(1-\partial_{tt}^2)^{\frac{1-\theta}{2}} (1-\Delta)^\theta u\|_{\mathbb{L}_q^p} \leqslant C \|A^{1-\theta} B^\theta u\|_{\mathbb{L}_q^p} \\
\overset{(2.14)}{\leqslant}& C \left(\|\partial_t u\|_{\mathbb{L}_q^p} + \|u\|_{\mathbb{H}_q^{2,p}} \right), \quad \forall \theta \in [0,1].
\end{aligned} \tag{2.15}$$

If $u \in \mathbb{H}^{2,p}_q(T)$, $\partial_t u \in \mathbb{L}^p_q(T)$ and $u(0,x)=0$, we extend u by

$$\bar{u}(t,x) := \begin{cases} u(t,x) & \text{if } t \in [0,T] \\ -3u\left(2T-t,x\right)+4u\left(\frac{3T}{2}-\frac{t}{2},x\right) & \text{if } t \in [T,2T] \\ 4u\left(\frac{3T}{2}-\frac{t}{2},x\right) & \text{if } t \in [2T,3T] \\ 0 & \text{othewise.} \end{cases}$$

By the definition of $\bar{u}$, one sees that

$$\|\partial_t \bar{u}\|_{\mathbb{L}^p_q} + \|\bar{u}\|_{\mathbb{H}^{2,p}_q} \leqslant C\left(\|\partial_t u\|_{\mathbb{L}^p_q(T)} + \|u\|_{\mathbb{H}^{2,p}_q(T)}\right). \tag{2.16}$$

Letting $\theta = \frac{1}{2} + \frac{d}{2p} - \frac{d}{2r} = 1 + \frac{1}{s} - \frac{1}{q} \in [\frac{1}{2}, 1]$, the Sobolev inequality and the above estimates imply

$$\begin{aligned} \|u\|_{\mathbb{H}^{1,r}_s(T)} \leqslant & \|\bar{u}\|_{\mathbb{H}^{1,r}_s} \leqslant \|\bar{u}\|_{H^{1-\theta,q}(\mathbb{R};H^{2\theta,p})} \\ & \overset{(2.15)}{\leqslant} C\left(\|\partial_t \bar{u}\|_{\mathbb{L}^p_q} + \|\bar{u}\|_{\mathbb{H}^{2,p}_q}\right) \overset{(2.16)}{\leqslant} C_4\left(\|\partial_t u\|_{\mathbb{L}^p_q(T)} + \|u\|_{\mathbb{H}^{2,p}_q(T)}\right). \end{aligned}$$

So, we complete our proof for (2.11). (2.12) can be proved similarly.

For (2.13), if $\theta < 1 - 1/q$, by Morrey's inequality, we have

$$\begin{aligned} \sup_{t_1,t_2 \in [0,T]} \frac{\|u(t_1)-u(t_2)\|_{H^{2\theta,p}}}{|t_1-t_2|^{1-1/q-\theta}} \leqslant & C\|\bar{u}\|_{H^{1-\theta,q}(\mathbb{R};H^{2\theta,p})} \\ \leqslant & C_6\left(\|\partial_t u\|_{\mathbb{L}^p_q(T)} + \|u\|_{\mathbb{H}^{2,p}_q(T)}\right). \end{aligned}$$

So, we complete our proof. □

For any $f \in \mathbb{L}^{p_1}_{q_1}(T)$, set

$$K'_f(m) := \|f - f\mathbf{1}_{\{|f|\leqslant m\}}\|_{\mathbb{L}^{p_1}_{q_1}(T)} \tag{2.17}$$

and

$$\omega_f(\delta) := \sup_{0\leqslant S\leqslant T-\delta} \|f\|_{\mathbb{L}^{p_1}_{q_1}(S,S+\delta)}. \tag{2.18}$$

Obviously, $K'_f(m) \to 0$, as $m \to \infty$ and $\omega_f(\delta) \to 0$, as $\delta \to 0$.

Next we give an analogue of Theorem 2.3, which is crucial in the proof of the second case of Theorem 1.1.

Theorem 2.6. *Let $d \geqslant 3$, $p_1, q_1 \in (2,\infty)$ with $d/p_1 + 2/q_1 = 1$, and $\{a(m)\}_{m\in\mathbb{N}_+}$ be a sequence converging to zero. Assume that $b \in \mathbb{L}^{p_1}_{q_1}(T)$ and $K'_b(m) \leqslant a(m)$,*

(1) if $p \in (1,p_1)$ and $q \in (1,q_1)$, then for any $f \in \mathbb{L}^p_q(T)$, equation (2.3) admits a unique solution u in $\mathbb{H}^{2,p}_q(T)$ and

$$\|\partial_t u\|_{\mathbb{L}^p_q(T)} + \|u\|_{\mathbb{H}^{2,p}_q(T)} \leqslant C\|f\|_{\mathbb{L}^p_q(T)}, \tag{2.19}$$

where C only depends on $d, p_1, q_1, p, q, T, \{a(m)\}$ and is increasing in T;

(2) *if $p\in(p_1/(p_1-1),p_1)$ and $q\in(q_1/(q_1-1),q_1)$, then for any $f\in\mathbb{H}^{-1,p}_q(T)$, equation (2.3) admits a unique solution u in $\mathbb{H}^{1,p}_q(T)$, and $u=v+w$ with v,w satisfying*

$$\|\partial_t v\|_{\mathbb{H}^{-1,p}_q(T)}+\|v\|_{\mathbb{H}^{1,p}_q(T)}\leqslant C\|f\|_{\mathbb{H}^{-1,p}_q(T)},\tag{2.20}$$

and

$$\|\partial_t w\|_{\mathbb{L}^{p'}_{q'}(T)}+\|w\|_{\mathbb{H}^{2,p'}_{q'}(T)}\leqslant C\|f\|_{\mathbb{H}^{-1,p}_q(T)},\tag{2.21}$$

where $p'=\frac{p_1p}{p_1+p}>1$, $q'=\frac{q_1q}{q_1+q}>1$, and C only depends on $d,p_1,q_1,p,q,T,\{a(m)\}$ and is increasing in T.

Proof. To prove the desired result, we only need to prove (2.19), (2.20) and (2.21) assuming that the solution already exists, since the method of continuity is applicable.

(1). Let $b_m:=b\mathbf{1}_{\{|b|\leqslant m\}}$. Rewrite (2.3) as

$$\partial_t u-\tfrac{1}{2}\Delta u=f+b_m\cdot\nabla u+(b-b_m)\cdot\nabla u.$$

Thanks to Lemma 2.1, for any $t\in[0,T]$ we have

$$\begin{aligned}&\|\partial_t u\|_{\mathbb{L}^p_q(t)}+\|u\|_{\mathbb{H}^{2,p}_q(t)}\\&\qquad\leqslant C_1\left(\|f\|_{\mathbb{L}^p_q(t)}+m\|\nabla u\|_{\mathbb{L}^p_q(t)}+\|(b-b_m)\cdot\nabla u\|_{\mathbb{L}^p_q(t)}\right),\end{aligned}$$

where $C_1=C_1(d,p,q,T)$. Letting $1/r=1/p-1/p_1$ and $1/s=1/q-1/q_1$, by (2.11) we have

$$\begin{aligned}\|(b-b_m)\cdot\nabla u\|_{\mathbb{L}^p_q(t)}\leqslant&\|(b-b_m)\|_{\mathbb{L}^{p_1}_{q_1}(t)}\|\nabla u\|_{\mathbb{L}^r_s(t)}\\\overset{(2.11)}{\leqslant}&C_4a(m)\left(\|\partial_t u\|_{\mathbb{L}^p_q(t)}+\|u\|_{\mathbb{H}^{2,p}_q(t)}\right).\end{aligned}$$

We choose m sufficiently large so that $C_1C_4a(m)\leqslant 1/2$. Thus,

$$I(t):=\|\partial_t u\|^q_{\mathbb{L}^p_q(t)}+\|u\|^q_{\mathbb{H}^{2,p}_q(t)}\leqslant C\left(\|f\|^q_{\mathbb{L}^p_q(t)}+N^q\|\nabla u\|^q_{\mathbb{L}^p_q(t)}\right).\tag{2.22}$$

Noting that

$$\begin{aligned}\|u\|^q_{\mathbb{L}^p_q(t)}=&\int_0^t\|u(\tau,\cdot)\|^q_{L^p}\mathrm{d}\tau=\int_0^t\left\|\int_0^\tau\partial_t u(\sigma,\cdot)\mathrm{d}\sigma\right\|^q_{L^p}\mathrm{d}\tau\\\leqslant&\int_0^t\tau^{q-1}\|\partial_t u\|^q_{\mathbb{L}^p_q(\tau)}\mathrm{d}\tau\leqslant C(T,q)\int_0^t I(\tau)\mathrm{d}\tau,\end{aligned}\tag{2.23}$$

and using an interpolation inequality, we obtain

$$\begin{aligned}\|\nabla u\|^q_{\mathbb{L}^p_q(t)}\leqslant&\delta\|\nabla^2 u\|^q_{\mathbb{L}^p_q(t)}+C_\delta\|u\|^q_{\mathbb{L}^p_q(t)}\\\leqslant&\delta I(t)+C_\delta\int_0^t I(\tau)\mathrm{d}\tau,\quad(\forall\varepsilon>0).\end{aligned}\tag{2.24}$$

Combing (2.22) and (2.24), we get

$$I(t)\leqslant C_7\delta m^q I(t)+C\|f\|^q_{\mathbb{L}^p_q(T)}+C_\delta m^q\int_0^t I(\tau)\mathrm{d}\tau.$$

Letting $\delta = \delta(m)$ be small enough so that $C_7\delta m^q \leqslant 1/2$, we obtain that for all $t \in [0,T]$,

$$I(t) \leqslant C\|f\|^q_{\mathbb{L}^p_q(T)} + C\int_0^t I(\tau)\mathrm{d}\tau.$$

Gronwall's inequality yields

$$\|\partial_t u\|_{\mathbb{L}^p_q(T)} + \|u\|_{\mathbb{H}^{2,p}_q(T)} \leqslant CI^{1/q}(T) \leqslant C\|f\|_{\mathbb{L}^p_q(T)}. \tag{2.25}$$

(2). Let v be the solution to

$$\partial_t v = \tfrac{1}{2}\Delta v + f, \quad v(0) = 0.$$

Again by (2.2), one sees that

$$\|\partial_t v\|_{\mathbb{H}^{-1,p}_q(T)} + \|v\|_{\mathbb{H}^{1,p}_q(T)} \leqslant C\|f\|_{\mathbb{H}^{-1,p}_q(T)}. \tag{2.26}$$

Define $w := u - v$. Then

$$\partial_t w = \tfrac{1}{2}\Delta w + b\cdot w + b\cdot\nabla v, \quad w(0) = 0.$$

Recalling that $p' = \frac{p_1 p}{p_1+p} \in (1,p_1)$ and $p' = \frac{q_1 q}{q_1+q} \in (1,q_1)$, by Hölder's inequality and (2.26), we have

$$\|b\cdot\nabla v\|_{\mathbb{L}^{p'}_{q'}(T)} \leqslant \|b\|_{\mathbb{L}^{p_1}_{q_1}(T)}\|\nabla v\|_{\mathbb{L}^p_q(T)} \leqslant C\|f\|_{\mathbb{H}^{-1,p}_q(T)}.$$

By the previous estimate (2.25), we obtain

$$\|\partial_t w\|_{\mathbb{L}^{p'}_{q'}(T)} + \|w\|_{\mathbb{H}^{2,p'}_{q'}(T)} \leqslant C\|f\|_{\mathbb{H}^{-1,p}_q(T)}. \tag{2.27}$$

Using (2.12) and noting that $\frac{d}{p'} + \frac{2}{q'} = \frac{d}{p} + \frac{d}{p_1} + \frac{2}{q} + \frac{2}{q_1} = 1 + \frac{d}{p} + \frac{2}{q}$, one sees that

$$\|w\|_{\mathbb{H}^{1,p}_q(T)} \overset{(2.12)}{\leqslant} C\left(\|\partial_t w\|_{\mathbb{L}^{p'}_{q'}(T)} + \|w\|_{\mathbb{H}^{2,p'}_{q'}(T)}\right) \overset{(2.27)}{\leqslant} C\|f\|_{\mathbb{H}^{-1,p}_q(T)}.$$

Thus, $\|u\|_{\mathbb{H}^{1,p}_q(T)} \leqslant \|v\|_{\mathbb{H}^{1,p}_q(T)} + \|w\|_{\mathbb{H}^{1,p}_q(T)} \leqslant C\|f\|_{\mathbb{H}^{-1,p}_q(T)}$. So, we complete our proof. □

Remark 2.7. *Let the assumptions in Theorem 2.3 or Theorem 2.6 hold. Suppose that $f \in \widetilde{\mathbb{H}}^{\alpha,p}_q(T)$ with $\alpha \in \{0,-1\}$. Then all the conclusions therein still hold if $\mathbb{H}^{\cdots}_{\cdots}$ and $\mathbb{L}^{\cdots}_{\cdots}$ are replaced by $\widetilde{\mathbb{H}}^{\cdots}_{\cdots}$ and $\widetilde{\mathbb{L}}^{\cdots}_{\cdots}$, respectively (cf.* [XXZZ20] *or* [RZ20]*).*

3. Compactness criterion for L^2 random fields

In this section, we give a relative compactness criterion for the random fields on the Wiener-Sobolev space, which is essentially a consequence of [BS04, Theorem 1].

Let $(\Omega,\mathscr{F},\mathbf{P})$ be a probability space and $\{W_t\}_{t\in[0,T]}$ be a d-dimensional Brownian motion on it. $\mathbf{T} = [0,T]\times\{1,2,\cdots,d\}$, μ is the product of the Lebesgue measure on $[0,T]$ times the uniform measure on $\{1,2,\cdots,d\}$. $H := L^2(\mathbf{T};\mu)$ and the scalar product is

$$\langle f,g\rangle := \sum_{i=1}^d \int_0^T f((t,i))g((t,i))\mathrm{d}t.$$

Let I_m denote the multiple stochastic integral

$$I_m(f_m) = m! \sum_{k_1,\cdots,k_m=1}^{d} \int \cdots \int_{0<t_1<\cdots<t_m<T} f_m\left((t_1,k_1),\cdots,(t_m,k_m)\right) \mathrm{d}W_{t_1}^{k_1} \ldots \mathrm{d}W_{t_m}^{k_m} \tag{3.1}$$

of $L_s^2(\mathbf{T}^m)$ (the collection of all symmetric elements in $L^2(\mathbf{T}^m)$). Let $\mathscr{H}_n$ denote the closed linear subspace of $L^2(\Omega,\mathscr{F},\mathbf{P})$ generated by the random variables $\{H_n(I_1(h)) : h \in H = L^2(\mathbf{T},\mu)\}$, where H_n is the n-th Hermite polynomial. The multiple integral I_m is a map from $L_s^2(\mathbf{T}^m)$ onto the Wiener chaos $\mathscr{H}_m$ and any $F \in L^2(\Omega,\mathscr{F},\mathbf{P})$ can be expanded into a series of multiple stochastic integrals: $F = \sum_{m=0}^{\infty} I_m(f_m)$, where $I_0(F) := \mathbf{E}F$. Let $\mathcal{S}_p$ denote the class of smooth random variables $F = f(W(h_1),\cdots,W(h_m))$ and $f \in C_p^\infty(\mathbb{R}^d)$. The Malliavin derivative of a smooth random variable F is the stochastic process $t \mapsto D_tF$ defined by

$$D_tF := \sum_{i=1}^{m} \partial_i f(W(h_1),\cdots,W(h_m))h_i(t)$$

Let $\mathbb{D}^{1,2}$ be the closure of $\mathcal{S}_p$ with respect to the norm

$$\|F\|_{\mathbb{D}^{1,2}}^2 := \mathbf{E}F^2 + \mathbf{E}\int_0^T |D_tF|^2 \mathrm{d}t.$$

Assume now $\mathcal{O}$ is a bounded domain in $\mathbb{R}^d$ with smooth boundary and F_n is a sequence of random fields in $L^2(\mathcal{O}\times\Omega)$. The following result is a variant of a compactness criteria for sequences in $L^2(\mathcal{O}\times\Omega)$ due to Bally and Saussereau [BS04].

Lemma 3.1. *Assume $K > 0$ and that the sequence $\{F_n\}_{n\in\mathbb{N}} \subseteq L^2(\mathcal{O}\times\Omega)$ satisfies the following three conditions, for all $n \in \mathbb{N}$:*

$$\mathbf{E}\|F_n\|_{H_x^1(\mathcal{O})}^2 \leqslant K, \tag{A$_1$}$$

$$\mathbf{E}\int_{\mathcal{O}}\int_0^T |D_sF_n(x)|^2 \mathrm{d}s\,\mathrm{d}x \leqslant K, \tag{A$_2$}$$

$$\mathbf{E}\int_{\mathcal{O}}\int_0^T\int_0^T \frac{|D_sF_n(x)-D_{s'}F_n(x)|^2}{|s-s'|^{1+2\beta}} \mathrm{d}s\mathrm{d}s'\,\mathrm{d}x \leqslant K, \text{ for some } \beta > 0, \tag{A$_3$}$$

then $\{F_n\}_{n\in\mathbb{N}}$ is relatively compact in $L^2(\mathcal{O}\times\Omega)$.

Proof. Since $\mathcal{O}$ is a bounded smooth domain, there exists $\{e_k\}_{k\in\mathbb{N}_+} \subseteq H_0^1(\mathcal{O})\cap C^\infty(\mathcal{O})$ and a sequence $\{\lambda_k\}_{k\in\mathbb{N}_+}$ of positive real numbers with $\lambda_k \uparrow \infty\,(k\uparrow\infty)$, such that $\Delta e_k = -\lambda_k e_k$ and $\{e_k\}_{k\in\mathbb{N}_+}$ forms a orthonormal base of $L^2(\mathcal{O})$ (cf. [Eva10]). Moreover, $e_k/\sqrt{\lambda_k}$ forms a basis of $H_0^1(\mathcal{O})$ with norm $\|f\|_{H_0^1(\mathcal{O})} := \left(\int_{\mathcal{O}}|\nabla f|^2\right)^{1/2}$. Set $\langle f,g\rangle := \int_{\mathcal{O}} fg$, then $F_n = \sum_{k=1}^{\infty}\langle F_n,e_k\rangle e_k$. Integration by parts and (A_1) yield,

$$\begin{aligned}\Big\|\sum_{k=K}^{\infty}\langle F_n,e_k\rangle e_k\Big\|^2_{L^2(\mathcal{O}\times\Omega)} &= \mathbf{E}\sum_{k=K}^{\infty}\langle F_n,e_k\rangle^2 = \mathbf{E}\sum_{k=K}^{\infty}\lambda_k^{-2}\langle F_n,\Delta e_k\rangle^2\\ =&\mathbf{E}\sum_{k=K}^{\infty}\lambda_k^{-1}\langle\nabla F_n,\nabla e_k/\sqrt{\lambda_k}\rangle^2 \leqslant \lambda_K^{-1}\mathbf{E}\sum_{k=K}^{\infty}\langle\nabla F_n,\nabla e_k/\sqrt{\lambda_k}\rangle^2\\ \leqslant&\lambda_K^{-1}\|\nabla F_n\|^2_{L^2(\mathcal{O}\times\Omega)} \leqslant C\lambda_K^{-1}\downarrow 0\ \ (K\uparrow\infty).\end{aligned}$$

Therefore, the relative compactness of the sequence $\{F_n\}_{n\in\mathbb{N}}$ in $L^2(\mathcal{O}\times\Omega)$ reduces to the relative compactness of the sequence $\{\langle F_n,e_k\rangle\}_{n\in\mathbb{N}}$ in $L^2(\Omega)$ for each $k\in\mathbb{N}_+$. By (A_1), we have

$$\mathbf{E}\langle F_n,e_k\rangle^2 \leqslant \mathbf{E}\|F_n\|^2_{L^2(\mathcal{O})}\leqslant K. \tag{3.2}$$

(A_2) and (A_3) yield for all $n\in\mathbb{N}$

$$\begin{aligned}\mathbf{E}\int_0^T|D_s\langle F_n,e_k\rangle|^2\mathrm{d}s =&\mathbf{E}\int_0^T\Big|D_s\int_{\mathcal{O}}F_n(x)e_k(x)\,\mathrm{d}x\Big|^2\mathrm{d}s\\ \leqslant&\mathbf{E}\int_{\mathcal{O}}\int_0^T|D_sF_n(x)|^2\,\mathrm{d}s\,\mathrm{d}x\leqslant K\end{aligned}\tag{3.3}$$

and

$$\begin{aligned}&\mathbf{E}\int_0^T\int_0^T\frac{|D_s\langle F_n,e_k\rangle-D_{s'}\langle F_n,e_k\rangle|^2}{|s-s'|^{1+2\beta}}\mathrm{d}s\mathrm{d}s'\\ \leqslant&\mathbf{E}\int_{\mathcal{O}}\int_0^T\int_0^T\frac{|D_sF_n(x)-D_{s'}F_n(x)|^2}{|s-s'|^{1+2\beta}}\mathrm{d}s\mathrm{d}s'\,\mathrm{d}x\leqslant K.\end{aligned}\tag{3.4}$$

By (3.2)-(3.4), Theorem 1 and Lemma 1 of [DPMN92] (with $\alpha\in(0,\beta\wedge\frac{1}{2})$ and $C=A_\alpha^{-1}$ therein), one sees that $\{\langle F_n,e_k\rangle\}_{n\in\mathbb{N}}$ is compact in $L^2(\Omega)$ for each $k\in\mathbb{N}_+$. So, we complete our proof. □

4. Estimates for the case of regular coefficients

Throughout this section, we assume $b\in L^\infty([0,T];C_b^2)$ and the unique strong solution to SDE (1.2) with $s=0$ is denoted by X_t^x. Recall that $K_f(m)$, $K_f'(m)$ and $\omega_f(m)$ are defined in (2.4), (2.17) and (2.18), respectively. The main purpose of this section is to prove

Proposition 4.1. *Let $d\geqslant 3$, $\{a(m)\}_{m\in\mathbb{N}_+}$ be a sequence converging to zero and $\ell(\delta)$ be a monotonically increasing function on $(0,T)$ with $\lim_{\delta\downarrow 0}\ell(\delta)=0$.*

(a) Assume that $b\in L^\infty([0,T];C_b^2)$ and $K_b(m)\leqslant a(m)$. Then for any $r\geqslant 2$, $p\in(\frac{d}{d-1},d)$ and $\gamma\in(0,1/2)$,

$$\|\nabla X_t^x-\mathrm{I}\|_{L_x^{pr}L_\omega^r}\leqslant Ct^{\gamma/2r},\quad \textit{for all } 0\leqslant t\leqslant T, \tag{4.1}$$

$$\|D_sX_t^x-\mathrm{I}\|_{L_x^{pr}L_\omega^r}\leqslant C(t-s)^{\gamma/2r},\quad \textit{for a.e. } s\in[0,T] \textit{ with } 0\leqslant s\leqslant t\leqslant T \tag{4.2}$$

and

$$\|D_sX_t^x-D_{s'}X_t^x\|_{L_x^{pr}L_\omega^r}\leqslant C|s-s'|^{\gamma/4r},\quad \textit{for a.e. } s,s'\in[0,T] \textit{ with } 0\leqslant s,s'\leqslant t\leqslant T, \tag{4.3}$$

where C only depends on $d,T,r,p,\gamma,\|b\|_{\mathbb{L}^d_\infty(T)}$ and $\{a(m)\}$.

(b) *Assume that* $b \in \mathbb{L}_{q_1}^{p_1}(T) \cap L^\infty([0,T];C_b^2)$ *with* $p_1, q_1 \in (2,\infty)$ *and* $d/p_1 + 2/q_1 = 1$, $K_b'(m) \leqslant a(m)$ *and* $\omega_b(\delta) \leqslant \ell(\delta)$. *Then for any* $r \geqslant 2$, *a.e.* $s, s' \in [0,T]$ *with* $0 \leqslant s, s' \leqslant t \leqslant T$, $p \in (\frac{p_1}{p_1-1}, p_1)$ *and* $\gamma \in (0, \frac{1}{2} - \frac{1}{q_1})$, *the estimates* (4.1)-(4.3) *still hold, and the constant* C *only depends on* $d, p_1, q_1, T, r, p, \gamma, \{a(m)\}$ *and* $\ell(\delta)$.

The proof of Proposition 4.1 relies on the following lemma, which contains two key estimates of this paper.

Lemma 4.2. *Let* $d \geqslant 3$, $0 \leqslant S_0 \leqslant S_1 \leqslant T$ *and* $\{a(m)\}_{m\in\mathbb{N}_+}$ *be a sequence converging to zero.*

(a) *Suppose* $b \in \mathbb{L}_\infty^d(T)$ *and* $K_b(m) \leqslant a(m)$. *Assume that* $f_i \in L^\infty([0,T];C_b^2)$ $(i \in \mathbb{N}_+)$

$$\sup_{i\in\mathbb{N}_+} \|f_i\|_{\mathbb{L}_\infty^d(T)} \leqslant N \ \ and \ \ \sup_{t\in[0,T];i\in\mathbb{N}_+} K_{f_i}(m) \leqslant d_m.$$

Then for any $p \in (\frac{d}{d-1}, d)$, $\gamma \in (0, \frac{1}{2})$, $\{\alpha_i\}_{i=1}^\infty \subseteq \{1, 2, \cdots d\}$ *and all* $n \in \mathbb{N}$

$$\begin{aligned} &\left\| \mathbf{E} \int \cdots \int_{\Delta_n(S_0,S_1)} \prod_{i=1}^n \partial_{\alpha_i} f_i \left(t_i, X_{t_i}^x\right) \mathrm{d}t_1 \cdots \mathrm{d}t_n \right\|_{L_x^p} \\ \leqslant & C^n \left(m^2 N\sqrt{S_1 - S_0} + d_m\right)^{n-1} N(S_1 - S_0)^\gamma, \end{aligned} \tag{4.4}$$

where C *only depends on* $d, p, \gamma, T, \|b\|_{\mathbb{L}_\infty^d}(T)$ *and* $\{a(m)\}$.

(b) *Suppose* $b \in \mathbb{L}_{q_1}^{p_1}(T)$ *with* $p_1, q_1 \in (2,\infty)$ *and* $d/p_1 + 2/q_1 = 1$, *and* $K_b'(m) \leqslant a(m)$. *Assume that* $f_i \in \mathbb{L}_{q_1}^{p_1}(T) \cap L^\infty([0,T];C_b^2)$ $(i \in \mathbb{N}_+)$. *Then for any* $p \in (\frac{p_1}{p_1-1}, p_1)$, $\gamma \in (0, \frac{1}{2} - \frac{1}{q_1})$, $\{\alpha_i\}_{i=1}^\infty \subseteq \{1, 2, \cdots, d\}$ *and all* $n \in \mathbb{N}$

$$\left\| \mathbf{E} \int \cdots \int_{\Delta_n(S_0,S_1)} \prod_{i=1}^n \partial_{\alpha_i} f_i \left(t_i, X_{t_i}^x\right) \mathrm{d}t_1 \cdots \mathrm{d}t_n \right\|_{L_x^p} \leqslant C^{n+1} \prod_{i=1}^n \|f_i\|_{\mathbb{L}_{q_1}^{p_1}(S_0,S_1)} (S_1 - S_0)^\gamma, \tag{4.5}$$

where C *only depends on* $d, p_1, q_1, p, \gamma, T$ *and* $\{a(m)\}$.

Proof. For fixed $n \in \mathbb{N}_+$ and $\{\alpha_i\}_{i=1}^\infty \subseteq \{1, 2, \cdots, d\}$, we set $u_{n+1} = 1$ and for any $k \in \{1, 2, \cdots, n\}$, let $g_k := (\partial_{\alpha_k} f_k) u_{k+1}$ and $u_k \in \cap_{p,q\in(1,\infty)} \widetilde{\mathbb{H}}_q^{2,p}(S_1)$ be the unique function solving equation

$$\partial_t u_k + \tfrac{1}{2}\Delta u_k + b \cdot \nabla u_k + g_k = 0 \text{ in } (S_0, S_1) \times \mathbb{R}^d, \quad u_k(S_1) = 0 \tag{4.6}$$

(cf. [XXZZ20]). Then the generalized Itô formula yields

$$-u_k(t, X_t^x) = -\int_t^{S_1} g_k(s, X_s^x) \mathrm{d}s + \int_t^{S_1} \nabla u_k(s, X_s^x) \mathrm{d}W_s, \quad \forall t \in [0, S_1]$$

which implies

$$\mathbf{E}^{\mathscr{F}_t} \int_t^{S_1} g_k(s, X_s^x) \mathrm{d}s = u_k(t, X_t^x). \tag{4.7}$$

Here the conditional expectation $\mathbf{E}(F|\mathscr{G})$ is denoted by $\mathbf{E}^{\mathscr{G}}F$. By the Markov property and (4.7),

$$
\begin{aligned}
&\mathbf{E}^{\mathscr{F}_{S_0}}\int\cdots\int_{\Delta_n(S_0,S_1)}\prod_{i=1}^{n}\partial_{\alpha_i}f_i(t_i,X^x_{t_i})\,\mathrm{d}t_1\cdots\mathrm{d}t_n\\
=&\mathbf{E}^{\mathscr{F}_{S_0}}\int\cdots\int_{\Delta_{n-1}(S_0,S_1)}\prod_{i=1}^{n-1}\partial_{\alpha_i}f_i(t_i,X^x_{t_i})\,\mathbf{E}^{\mathscr{F}_{t_{n-1}}}\left(\int_{t_{n-1}}^{S_1}\partial_{\alpha_n}f_n(t_n,X^x_{t_n})\mathrm{d}t_n\right)\mathrm{d}t_1\cdots\mathrm{d}t_{n-1}\\
\overset{(4.7)}{=}&\mathbf{E}^{\mathscr{F}_{S_0}}\int\cdots\int_{\Delta_{n-2}(S_0,S_1)}\prod_{i=1}^{n-2}\partial_{\alpha_i}f_i(t_i,X^x_{t_i})\left[\int_{t_{n-2}}^{S_1}(\partial_{\alpha_{n-1}}f_{n-1}u_n)(t_{n-1},X^x_{t_{n-1}})\,\mathrm{d}t_{n-1}\right]\mathrm{d}t_1\cdots\mathrm{d}t_{n-2}\\
=&\mathbf{E}^{\mathscr{F}_{S_0}}\int\cdots\int_{\Delta_{n-1}(S_0,S_1)}\prod_{i=1}^{n-2}\partial_{\alpha_i}f_i(t_i,X^x_{t_i})g_{n-1}(t_{n-1},X^x_{t_{n-1}})\,\mathrm{d}t_1\cdots\mathrm{d}t_{n-1}\\
=&\cdots=u_1(S_0,X^x_{S_0}).
\end{aligned}
\tag{4.8}
$$

Now let U be the solution to the following PDE:

$$
\partial_t U=\tfrac{1}{2}\Delta U+B\cdot\nabla U+G\text{ in }(0,S_1)\times\mathbb{R}^d,\quad U(0)=0,\tag{4.9}
$$

where

$$
B(t,x)=b(S_1-t,x)\mathbf{1}_{[0,S_1-S_0]}(t)+b(t+S_0-S_1,x)\mathbf{1}_{(S_1-S_0,S_1]}(t)
$$

and

$$
G(t,x)=g_1(S_1-t,x)\mathbf{1}_{[0,S_1-S_0]}(t).\tag{4.10}
$$

We note that $u_1(S_1-t)=U(t)$ for all $t\in[0,S_1-S_0]$ and that $V(t):=U(t+(S_1-S_0))$ satisfies

$$
\partial_t V=\tfrac{1}{2}\Delta V+b\cdot\nabla V\text{ in }(0,S_0)\times\mathbb{R}^d,\quad V(0,x)=U(S_1-S_0,x)=u_1(S_0,x).
$$

Therefore, for any $p\in[1,\infty)$,

$$
\begin{aligned}
&\int_{\mathbb{R}^d}\left|\mathbf{E}\int\cdots\int_{\Delta_n(S_0,S_1)}\prod_{i=1}^{n}\partial_{\alpha_i}f_i(t_i,X^x_{t_i})\,\mathrm{d}t_1\cdots\mathrm{d}t_n\right|^p\mathrm{d}x\\
\overset{(4.8)}{=}&\int_{\mathbb{R}^d}|\mathbf{E}u_1(S_0,X^x_{S_0})|^p\mathrm{d}x=\int_{\mathbb{R}^d}|V(S_0,x)|^p\mathrm{d}x=\|U(S_1)\|_p^p.
\end{aligned}
\tag{4.11}
$$

Case (a): $b\in C([0,T];\mathbb{R}^d)$**.** Set

$$
f_{k,m}(t):=f_k(t)*\rho_m,\quad \bar f_{k,m}:=f_k-f_{k,m}.
$$

Let

$$
p\in(d/(d-1),d)\ \text{ and }\ q=\gamma^{-1}\in(2,\infty).
$$

By the definitions of g_k and u_{k+1},

$$
\begin{aligned}
\|g_k\|_{\mathbb{H}^{-1,p}_q(S_0,S_1)}&=\|(\partial_{\alpha_k}f_k)u_{k+1}\|_{\mathbb{H}^{-1,p}_q(S_0,S_1)}\\
&\leqslant\|(\partial_{\alpha_k}f_{k,m})\,u_{k+1}\|_{\mathbb{H}^{-1,p}_q(S_0,S_1)}+\|(\partial_{\alpha_k}\bar f_{k,m})\,u_{k+1}\|_{\mathbb{H}^{-1,p}_q(S_0,S_1)}=:I_1+I_2.
\end{aligned}
\tag{4.12}
$$

Recalling that u_{k+1} solves (4.6) with k replaced by $k+1$, using (2.5), we get

$$
(S_1-S_0)^{-1}\|u_{k+1}\|_{\mathbb{H}^{-1,p}_q(S_0,S_1)}+\|u_{k+1}\|_{\mathbb{H}^{1,p}_q(S_0,S_1)}\leqslant C\|g_{k+1}\|_{\mathbb{H}^{-1,p}_q(S_0,S_1)}.
$$

An interpolation inequality yields

$$\begin{aligned}\|u_{k+1}\|_{\mathbb{L}^p_q(S_0,S_1)} \leqslant& C\|u_{k+1}\|^{1/2}_{\mathbb{H}^{-1,p}_q(S_0,S_1)}\|u_{k+1}\|^{1/2}_{\mathbb{H}^{1,p}_q(S_0,S_1)}\\ \leqslant& C\sqrt{S_1-S_0}\|g_{k+1}\|_{\mathbb{H}^{-1,p}_q(S_0,S_1)}.\end{aligned}$$

Thus, like the proof for (2.7), we have

$$\begin{aligned}I_1 \leqslant& C\|\partial_{\alpha_k} f_{k,m}\, u_{k+1}\|_{\mathbb{L}^p_q(S_0,S_1)}\\ \leqslant& C\|f_{k,m}\|_{L^\infty([0,T];C^2_b)}\|u_{k+1}\|_{\mathbb{L}^p_q(S_0,S_1)}\\ \leqslant& Cm^2N\sqrt{S_1-S_0}\|g_{k+1}\|_{\mathbb{H}^{-1,p}_q(S_0,S_1)}\end{aligned}\tag{4.13}$$

One the other hand, recalling that $p\in(d/(d-1),d)$, by the Sobolev embedding,

$$\begin{aligned}I_2 \leqslant& \|\partial_{\alpha_k}(\bar f_{k,m}u_{k+1})\|_{\mathbb{H}^{-1,p}_q(S_0,S_1)} + \|\bar f_{k,m}\partial_{\alpha_k}u_{k+1}\|_{\mathbb{L}_q^{\frac{dp}{d+p}}(S_0,S_1)}\\ \leqslant& C\|\bar f_{k,m}\|_{\mathbb{L}^d_\infty(S_0,S_1)}\|u_{k+1}\|_{\mathbb{L}_q^{\frac{pd}{d-p}}(S_0,S_1)} + C\|\bar f_{k,m}\|_{\mathbb{L}^d_\infty(T)}\|u_{k+1}\|_{\mathbb{H}^{1,p}_q(S_0,S_1)}\\ \leqslant& Cd_m\|g_{k+1}\|_{\mathbb{H}^{-1,p}_q(S_0,S_1)}.\end{aligned}\tag{4.14}$$

Combing (4.12)-(4.14), we get

$$\|g_k\|_{\mathbb{H}^{-1,p}_q(S_0,S_1)} \leqslant C\left(m^2N\sqrt{S_1-S_0}+d_m\right)\|g_{k+1}\|_{\mathbb{H}^{-1,p}_q(S_0,S_1)},$$

where C only depends on $d,p,\gamma,T,\|b\|_{\mathbb{L}^d_\infty(T)}$ and $\{a(m)\}$. Recalling that G is defined in (4.10), by the above estimate we obtain

$$\begin{aligned}\|G\|_{\mathbb{H}^{-1,p}_q(S_1)} \leqslant& \|g_1\|_{\mathbb{H}^{-1,p}_q(S_0,S_1)}\\ \leqslant& C^n\left(m^2N\sqrt{S_1-S_0}+d_m\right)^{n-1}\|g_n\|_{\mathbb{H}^{-1,p}_q(S_0,S_1)}\\ =& C^n\left(m^2N\sqrt{S_1-S_0}+d_m\right)^{n-1}\|f_n\|_{\mathbb{H}^{-1,p}_q(S_0,S_1)}\\ \leqslant& C^n\left(m^2N\sqrt{S_1-S_0}+d_m\right)^{n-1}N(S_1-S_0)^\gamma.\end{aligned}\tag{4.15}$$

Thus,

$$\begin{aligned}&\left\|\mathbf{E}\int\cdots\int_{\Delta_n(S_0,S_1)}\prod_{i=1}^n\partial_{\alpha_i}f_i(t_i,X^x_{t_i})\,\mathrm{d}t_1\cdots\mathrm{d}t_n\right\|_{L^p_x}\\ &\overset{(4.11)}{\leqslant} C\|U(S_1)\|_p \overset{(2.13)}{\leqslant} C\left(\|\partial_t U\|_{\mathbb{H}^{-1,p}_q(S_1)}+\|U\|_{\mathbb{H}^{1,p}_q(S_1)}\right)\\ &\qquad(\text{ taking } \alpha=-1,\theta=1/2, \text{ and noticing } q>2)\\ &\overset{(2.5)}{\leqslant} C\|G\|_{\mathbb{H}^{-1,p}_q(S_1)} \overset{(4.15)}{\leqslant} C^n\left(m^2N\sqrt{S_1-S_0}+d_m\right)^{n-1}N(S_1-S_0)^\gamma,\end{aligned}$$

where C only depends on $d,p,\gamma,\|b\|_{\mathbb{L}^d_\infty}(T)$ and $\{a(m)\}$. So, we complete the proof for (4.4).

Case (b): $b\in\mathbb{L}^{p_1}_{q_1}(T)$**.** Set

$$p\in\left(\tfrac{p_1}{p_1-1},p_1\right),\quad q=\frac{q_1}{1+\gamma q_1}\in(2,q_1),\quad p'=\frac{p_1p}{p_1+p},\quad q'=\frac{q_1q}{q_1+q}.$$

Noting that $p_1\in(d,\infty)$, $q_1\in(2,\infty)$ and $\gamma\in(0,1/2-1/q_1)$, one sees that $p\in(1,p_1), q\in(2,q_1)$ and $p'\in(1,p), q\in(1,q)$.

We claim that for each $k\in\{n,n-1,\cdots,1\}$, we have $g_k=g_k'+g_k''$ and

$$\|g_k'\|_{\mathbb{H}_q^{-1,p}(S_0,S_1)}+\|g_k''\|_{\mathbb{L}_{q'}^{p'}(S_0,S_1)}\leqslant C^{n-k+1}\prod_{i=k}^n\|f_i\|_{\mathbb{L}_{q_1}^{p_1}(S_0,S_1)}(S_1-S_0)^\gamma, \tag{4.16}$$

where C does not depend on n. By the definition of g_n, we have

$$\|g_n\|_{\mathbb{H}_q^{-1,p}(S_0,S_1)}\leqslant C\|f_n\|_{\mathbb{L}_{q_1}^{p_1}(S_0,S_1)}(S_1-S_0)^\gamma.$$

Assume $g_k=g_k'+g_k''$ and that (4.16) holds for some $k\in\{n,\cdots,2\}$. Then u_k can be decomposed as $u_k=u_k'+u_k''$, where u_k' and u_k'' solve (4.6) with g_k replaced by g_k' and g_k'', respectively. By Theorem 2.6, one sees that u_k can be further decomposed as

$$u_k=u_k'+u_k''=v_k'+w_k'+u_k'',$$

where v_k', w_k' and u_k'' satisfy

$$\|\partial_t v_k'\|_{\mathbb{H}_q^{-1,p}(S_0,S_1)}+\|v_k'\|_{\mathbb{H}_q^{1,p}(S_0,S_1)}\leqslant C\|g_k'\|_{\mathbb{H}_q^{-1,p}(S_0,S_1)}, \tag{4.17}$$

$$\|\partial_t w_k'\|_{\mathbb{L}_{q'}^{p'}(S_0,S_1)}+\|w_k'\|_{\mathbb{H}_{q'}^{2,p'}(S_0,S_1)}\leqslant C\|g_k'\|_{\mathbb{H}_q^{-1,p}(S_0,S_1)} \tag{4.18}$$

and

$$\|\partial_t u_k''\|_{\mathbb{L}_{q'}^{p'}(S_0,S_1)}+\|u_k''\|_{\mathbb{H}_{q'}^{2,p'}(S_0,S_1)}\leqslant C\|g_k''\|_{\mathbb{L}_{q'}^{p'}(S_0,S_1)}. \tag{4.19}$$

Let $r=\frac{p_1p}{p_1-p}$ and $s=\frac{1}{\gamma}=\frac{q_1q}{q-q_1}$. Recalling that $2<\frac{d}{p'}+\frac{2}{q'}=1+\frac{d}{p}+\frac{2}{q}=2+\frac{d}{r}+\frac{2}{s}$, due to (2.11), (2.12) and (4.17)-(4.19), we have

$$\begin{aligned}
\|u_k\|_{\mathbb{L}_s^r(S_0,S_1)}&\leqslant\|v_k'\|_{\mathbb{L}_s^r(S_0,S_1)}+\|w_k'\|_{\mathbb{L}_s^r(S_0,S_1)}+\|u_k''\|_{\mathbb{L}_s^r(S_0,S_1)}\\
&\overset{(2.11),(2.12)}{\leqslant}C\left(\|\partial_t v_k'\|_{\mathbb{H}_q^{-1,p}(S_0,S_1)}+\|v_k'\|_{\mathbb{H}_q^{1,p}(S_0,S_1)}\right)+C\left(\|\partial_t w_k'\|_{\mathbb{L}_{q'}^{p'}(S_0,S_1)}+\|w_k'\|_{\mathbb{H}_{q'}^{2,p'}(S_0,S_1)}\right)\\
&\quad+C\left(\|\partial_t u_k''\|_{\mathbb{L}_{q'}^{p'}(S_0,S_1)}+\|u_k''\|_{\mathbb{H}_{q'}^{2,p'}(S_0,S_1)}\right)\\
&\overset{(4.17)-(4.19)}{\leqslant}C\left(\|g_k'\|_{\mathbb{H}_{q_1}^{-1,p_1}(S_0,S_1)}+\|g_k''\|_{\mathbb{L}_{q_1}^{p_1}(S_0,S_1)}\right)
\end{aligned} \tag{4.20}$$

and

$$\|u_k\|_{\mathbb{H}_q^{1,p}(S_0,S_1)}\leqslant C\left(\|g_k'\|_{\mathbb{H}_{q_1}^{-1,p_1}(S_0,S_1)}+\|g_k''\|_{\mathbb{L}_{q_1}^{p_1}(S_0,S_1)}\right). \tag{4.21}$$

Set $g_{k-1}'=\partial_{\alpha_{k-1}}(f_{k-1}u_k)$ and $g_{k-1}''=-f_{k-1}(\partial_{\alpha_{k-1}}u_k)$. By Hölder's inequality, (4.20) and (4.21), we get

$$\begin{aligned}
&\|g_{k-1}'\|_{\mathbb{H}_q^{-1,p}(S_0,S_1)}+\|g_{k-1}''\|_{\mathbb{L}_{q'}^{p'}(S_0,S_1)}\\
\leqslant&\|f_{k-1}\|_{\mathbb{L}_{q_1}^{p_1}(S_0,S_1)}\left(\|u_k\|_{\mathbb{L}_s^r(S_0,S_1)}+\|\nabla u_k\|_{\mathbb{L}_q^p(S_0,S_1)}\right)\\
\overset{(4.20),(4.21)}{\leqslant}&C\|f_{k-1}\|_{\mathbb{L}_{q_1}^{p_1}(S_0,S_1)}\left(\|g_k'\|_{\mathbb{H}_{q_1}^{-1,p_1}(S_0,S_1)}+\|g_k''\|_{\mathbb{L}_{q_1}^{p_1}(S_0,S_1)}\right)
\end{aligned}$$

$$\leqslant C^{n-k+2}\prod_{i=k-1}^{n}\|f_i\|_{\mathbb{L}^{p_1}_{q_1}(S_0,S_1)}(S_1-S_0)^{\gamma}.$$

So, by induction (4.16) holds for all $k\in\{1,2,\cdots,n\}$. In particular,

$$\|g_1'\|_{\mathbb{H}^{-1,p}_q(S_0,S_1)}+\|g_1''\|_{\mathbb{L}^{p'}_{q'}(S_0,S_1)}\leqslant C^{n+1}\prod_{i=1}^{n}\|f_i\|_{\mathbb{L}^{p_1}_{q_1}(S_0,S_1)}(S_1-S_0)^{\gamma}.$$

Recalling that G is defined in (4.10), this can be written as

$$\begin{aligned}G(t,x)=&G'(t,x)+G''(t,x)\\ :=&g_1'(S_1-t,x)\mathbf{1}_{[0,S_1-S_0]}(t)+g_1''(S_1-t,x)\mathbf{1}_{[0,S_1-S_0]}(t)\end{aligned}$$

and

$$\begin{aligned}\|G'\|_{\mathbb{H}^{-1,p}_q(S_1)}+\|G''\|_{\mathbb{L}^{p'}_{q'}(S_1)}\leqslant& C\left(\|g_1'\|_{\mathbb{H}^{-1,p}_q(S_0,S_1)}+\|g_1''\|_{\mathbb{L}^{p'}_{q'}(S_0,S_1)}\right)\\ \leqslant& C^{n+1}\prod_{i=1}^{n}\|f_i\|_{\mathbb{L}^{p_1}_{q_1}(S_0,S_1)}(S_1-S_0)^{\gamma}.\end{aligned}\tag{4.22}$$

Assume U' and U'' solve (4.9) with G replaced by G' and G'', respectively. As in the above argument, we see that $U=U'+U''=V'+W'+U''$ and that

$$\begin{aligned}&\left(\|\partial_t V'\|_{\mathbb{H}^{-1,p}_q(S_1)}+\|V'\|_{\mathbb{H}^{1,p}_q(S_1)}\right)+\left(\|\partial_t W'\|_{\mathbb{L}^{p'}_{q'}(S_1)}+\|W'\|_{\mathbb{H}^{2,p'}_{q'}(S_1)}\right)\\ &\qquad+\left(\|\partial_t U''\|_{\mathbb{L}^{p'}_{q'}(S_1)}+\|U''\|_{\mathbb{H}^{2,p'}_{q'}(S_1)}\right)\\ \leqslant& C\left(\|G'\|_{\mathbb{H}^{-1,p}_q(S_1)}+\|G''\|_{\mathbb{L}^{p'}_{q'}(S_1)}\right)\\ \overset{(4.22)}{\leqslant}& C^{n+1}\prod_{i=1}^{n}\|f_i\|_{\mathbb{L}^{p_1}_{q_1}(S_0,S_1)}(S_1-S_0)^{\gamma},\end{aligned}\tag{4.23}$$

where the first inequality is due to Theorem 2.6.

Recalling that $q>2$, by taking $\alpha=-1$ and $\theta=1/2$ in (2.13), we get

$$\|V'\|_{\mathbb{L}^{p}_{\infty}(S_1)}\leqslant C\left(\|\partial_t V'\|_{\mathbb{H}^{-1,p}_q(S_1)}+V'\|_{\mathbb{H}^{1,p}_q(S_1)}\right).\tag{4.24}$$

Similarly, taking $q=q'>1,\alpha=0$ and $\theta=\frac{d}{2p_1}$ in inequality (2.13), and noting that $1-\frac{1}{q'}=1-\frac{1}{q_1}-\frac{1}{q}=\frac{d}{2p_1}+\frac{1}{2}-\frac{1}{q}>\frac{d}{2p_1}=\theta$ and $\frac{1}{p}=\frac{1}{p'}-\frac{2\theta}{d}$, we get

$$\begin{aligned}&\|W'\|_{\mathbb{L}^{p}_{\infty}(S_1)}+\|U''\|_{\mathbb{L}^{p}_{\infty}(S_1)}\leqslant C\left(\|W'\|_{\mathbb{H}^{2\theta,p'}_{\infty}(S_1)}+\|U''\|_{\mathbb{H}^{2\theta,p'}_{\infty}(S_1)}\right)\\ \leqslant& C\left(\|\partial_t W'\|_{\mathbb{L}^{p'}_{q'}(S_1)}+\|W'\|_{\mathbb{H}^{2,p'}_{q'}(S_1)}\right)+C\left(\|\partial_t U''\|_{\mathbb{L}^{p'}_{q'}(S_1)}+U''\|_{\mathbb{H}^{2,p'}_{q'}(S_1)}\right).\end{aligned}\tag{4.25}$$

Combining (4.23)-(4.25) with (4.11), we obtain

$$\left\|\mathbf{E}\int\cdots\int_{\Delta_n(S_0,S_1)}\prod_{i=1}^{n}\partial_{\alpha_i}f_i(t_i,X^x_{t_i})\,\mathrm{d}t_1\cdots\mathrm{d}t_n\right\|_{L^p_x}$$

$$
\begin{aligned}
&\leqslant C\|U\|_{\mathbb{L}^p_\infty(S_1)} \leqslant C\left(\|V'\|_{\mathbb{L}^p_\infty(S_1)}+\|W'\|_{\mathbb{L}^p_\infty(S_1)}+\|U''\|_{\mathbb{L}^p_\infty(S_1)}\right)\\
&\leqslant C^{n+1}\prod_{i=1}^n\|f_i\|_{\mathbb{L}^{p_1}_{q_1}(S_0,S_1)}(S_1-S_0)^\gamma.
\end{aligned}
$$

So, we complete our proof. $\square$

Note that $b\in L^\infty([0,1];C^2_b)$, the solution to the SDE (1.2) is differentiable with respect to x, and ∇X^x_t satisfies

$$
\nabla X^x_t=\mathrm{I}+\int_0^t\nabla b(s,X^x_s)\nabla X^x_s\,\mathrm{d}s.
$$

Regarding the above equation as a linear random ODE for ∇X^x_t, this equation has a unique solution and it is given by

$$
\nabla X^x_t=\mathrm{I}+\sum_{n=1}^\infty\int\cdots\int_{\Delta_n(t)}\prod_{i=1}^n\nabla b\left(t_i,X^x_{t_i}\right)\,\mathrm{d}t_1\cdots\mathrm{d}t_n, \tag{4.26}
$$

provided that this series is convergent (cf. [MNP15]). Moreover, for any $0\leqslant t_0\leqslant t\leqslant T$,

$$
\nabla X^x_t-\nabla X^x_{t_0}=\sum_{n=1}^\infty\int\cdots\int_{\Delta_n(t_0,t)}\prod_{i=1}^n\nabla b\left(t_i,X^x_{t_i}\right)\nabla X^x_{t_0}\,\mathrm{d}t_1\cdots\mathrm{d}t_n. \tag{4.27}
$$

On the other hand, the Malliavin derivative D_sX_t is the solution of the linear stochastic equation

$$
D_sX^x_t=\mathrm{I}+\int_s^t\nabla b\left(r,X^x_r\right)D_sX^x_r\,\mathrm{d}r,
$$

for a.e. $s\in[0,T]$ with $s\leqslant t$, and $D_sX_t=0$ for a.e. $s\in[0,T]$ with $s>t$. Thus, one sees that

$$
D_sX^x_t=\mathrm{I}+\sum_{n=1}^\infty\int\cdots\int_{\Delta_n(s,t)}\prod_{i=1}^n\nabla b\left(t_i,X^x_{t_i}\right)\,\mathrm{d}t_1\cdots\mathrm{d}t_n, \tag{4.28}
$$

for a.e. $s\in[0,T]$ with $s\leqslant t$, and

$$
\begin{aligned}
D_sX^x_t-D_{s'}X^x_t=&\int_s^t\nabla b\left(r,X^x_r\right)D_sX^x_r\,\mathrm{d}r-\int_{s'}^t\nabla b\left(r,X^x_r\right)D_{s'}X^x_r\,\mathrm{d}r\\
=&\int_s^{s'}\nabla b\left(r,X^x_r\right)D_sX^x_r\,\mathrm{d}r+\int_{s'}^t\nabla b\left(r,X^x_r\right)\left(D_sX^x_r-D_{s'}X^x_r\right)\,\mathrm{d}r\\
=&D_sX^x_{s'}-\mathrm{I}+\int_{s'}^t\nabla b\left(r,X^x_r\right)\left(D_sX^x_r-D_{s'}X^x_r\right)\,\mathrm{d}r
\end{aligned}
$$

for a.e. $s,s'\in[0,T]$ with $s<s'\leqslant t$. Iterating, we get

$$
\begin{aligned}
D_sX^x_t-D_{s'}X^x_t=&\left(\mathrm{I}+\sum_{n=1}^\infty\int\cdots\int_{\Delta_n(s',t)}\prod_{i=1}^n\nabla b\left(t_i,X^x_{t_i}\right)\,\mathrm{d}t_1\cdots\mathrm{d}t_n\right)\cdot\left(D_sX^x_{s'}-\mathrm{I}\right)\\
\overset{(4.28)}{=}&D_{s'}X^x_t\cdot\left(D_sX^x_{s'}-\mathrm{I}\right),
\end{aligned}
\tag{4.29}
$$

for a.e. $s,s'\in[0,T]$ with $s<s'\leqslant t$.

We are now in a position to prove our Proposition 4.1.

Proof of Proposition 4.1. **Case (a).** We only need to prove the case where r is a positive even integer. For any $n \in \mathbb{N}_+$ and $0 \leqslant S_0 \leqslant S_1 \leqslant T$, it is not hard to see that

$$\left(\int\cdots\int_{\Delta_n(S_0,S_1)} \partial_{\alpha_1} b^i(t_1, X^x_{t_1}) \cdot \partial_{\alpha_2} b^{\alpha_1}(t_2, X^x_{t_2}) \cdots \partial_j b^{\alpha_{n-1}}(t_n, X^x_{t_n})\, \mathrm{d}t_1 \mathrm{d}t_2 \cdots \mathrm{d}t_n\right)^r$$

can be written as a sum of at most r^{rn} terms of the form

$$\int\cdots\int_{\Delta_{rn}(S_0,S_1)} \partial_{\gamma_1} b^{\beta_1}(t_1, X^x_{t_1}) \cdot \partial_{\gamma_2} b^{\beta_2}(t_2, X^x_{t_2}) \cdots \partial_{\gamma_{rn}} b^{\beta_{rn}}(t_{rn}, X^x_{t_{rn}})\, \mathrm{d}t_1 \mathrm{d}t_2 \cdots \mathrm{d}t_{rn}.$$

Fix $p \in (d/(d-1), d), \gamma \in (0, 1/2)$. By the above discussion and (4.4), we have

$$\begin{aligned}
&\left\|\int\cdots\int_{\Delta_n(S_0,S_1)} \prod_{i=1}^n \nabla b\left(t_i, X^x_{t_i}\right)\, \mathrm{d}t_1 \cdots \mathrm{d}t_n\right\|_{L^{pr}_x L^r_\omega}\\
\leqslant& C\sum_{i,j=1}^d \sum_{\alpha_1,\cdots,\alpha_{n-1}=1}^d \bigg\| \int\cdots\int_{\Delta_n(S_0,S_1)} \partial_{\alpha_1} b^i(t_1, X^x_{t_1}) \cdot \partial_{\alpha_2} b^{\alpha_1}(t_2, X^x_{t_2}) \cdots\\
&\qquad\qquad \cdot \partial_j b^{\alpha_{n-1}}(t_n, X^x_{t_n})\, \mathrm{d}t_1 \mathrm{d}t_2 \cdots \mathrm{d}t_n \bigg\|_{L^{pr}_x L^r_\omega}\\
=& C\sum_{i,j=1}^d \sum_{\alpha_1,\cdots,\alpha_{n-1}=1}^d \bigg[\int_{\mathbb{R}^d}\bigg(\sum_{\beta,\gamma} \mathbf{E}\int\cdots\int_{\Delta_{rn}(S_0,S_1)} \partial_{\gamma_1} b^{\beta_1}(t_1, X^x_{t_1}) \cdot \partial_{\gamma_2} b^{\beta_2}(t_2, X^x_{t_2}) \cdots\\
&\qquad\qquad \cdot \partial_{\gamma_{rn}} b^{\beta_{rn}}(t_{rn}, X^x_{t_{rn}}) \mathrm{d}t_1 \mathrm{d}t_2 \cdots \mathrm{d}t_{rn}\bigg)^p \mathrm{d}x\bigg]^{1/pr}\\
\leqslant& C\sum_{i,j=1}^d \sum_{\alpha_1,\cdots,\alpha_{n-1}=1}^d \bigg[\sum_{\beta,\gamma} \bigg\|\mathbf{E}\int\cdots\int_{\Delta_{rn}(S_0,S_1)} \partial_{\gamma_1} b^{\beta_1}(t_1, X^x_{t_1}) \cdot \partial_{\gamma_2} b^{\beta_2}(t_2, X^x_{t_2}) \cdots\\
&\qquad\qquad \cdot \partial_{\gamma_{rn}} b^{\beta_{rn}}(t_{rn}, X^x_{t_{rn}}) \mathrm{d}t_1 \mathrm{d}t_2 \cdots \mathrm{d}t_{rn}\bigg\|_{L^p_x}\bigg]^{1/r}\\
\overset{(4.4)}{\leqslant}& (rC_8)^n \left(m^2 \|b\|_{\mathbb{L}^d_\infty(T)} \sqrt{S_1 - S_0} + a(m)\right)^{n-1/r} \|b\|^{1/r}_{\mathbb{L}^d_\infty(T)} (S_1 - S_0)^{\gamma/r}.
\end{aligned}$$

Here we also used the fact that the sum $\sum_{\beta,\gamma}$ contains at most r^{rn} terms. The constant $C_8 > 1$ only depends on $d, T, \|b\|_{\mathbb{L}^d_\infty(T)}, \{a(m)\}, r, p$ and γ. Letting m be large enough such that $C_8 r a(m) \leqslant 1/4$ and then choosing $T_r > 0$ such that $C_8 r m^2 \|b\|_{\mathbb{L}^d_\infty(T)} \sqrt{T_r} = 1/4$, we have for any $0 \leqslant S_1 - S_0 \leqslant T_r$,

$$\begin{aligned}
&\sum_{n=1}^\infty \left\|\int\cdots\int_{\Delta_n(S_0,S_1)} \prod_{i=1}^n \nabla b\left(t_i, X^x_{t_i}\right)\, \mathrm{d}t_1 \cdots \mathrm{d}t_n\right\|_{L^{pr}_x L^r_\omega}\\
\overset{(4.30)}{\leqslant}& \sum_{n=1}^\infty (C_8 r)^n \left(m^2 \|b\|_{\mathbb{L}^d_\infty(T)} \sqrt{t} + a(m)\right)^{n-1/r} \|b\|^{1/r}_{\mathbb{L}^d_\infty(T)} (S_1 - S_0)^{\gamma/r} \qquad (4.30)\\
\leqslant& 2(rC_8)^{1/r} \|b\|^{1/r}_{\mathbb{L}^d_\infty(T)} (S_1 - S_0)^{\gamma/r}.
\end{aligned}$$

Thus, by (4.26) for each $t \in [0, T_r]$,

$$\|\nabla X_t^x - \mathrm{I}\|_{L_x^{pr} L_\omega^r} \overset{(4.26)}{\leqslant} C \sum_{n=1}^{\infty} \left\| \int \cdots \int_{\Delta_n(t)} \prod_{i=1}^{n} \nabla b\left(t_i, X_{t_i}^x\right)\, \mathrm{d}t_1 \cdots \mathrm{d}t_n \right\|_{L_x^{pr} L_\omega^r} \leqslant C t^{\gamma/r}. \tag{4.31}$$

For any $t \in [T_{2r}, 2T_{2r} \wedge T]$, by (4.27), Hölder's inequality, (4.30) and (4.31) we get

$$\begin{aligned}
&\|\nabla X_t^x - \mathrm{I}\|_{L_x^{pr} L_\omega^r} \\
\leqslant &\left\|\nabla X_{T_{2r}}^x - \mathrm{I}\right\|_{L_x^{pr} L_\omega^r} + \sum_{n=1}^{\infty} \left\| \int \cdots \int_{\Delta_n(T_{2r}, t)} \nabla b\left(t_1, X_{t_1}^x\right) \cdots \nabla b\left(t_n, X_{t_n}^x\right) \mathrm{d}t_1 \cdots \mathrm{d}t_n \right\|_{L_x^{pr} L_\omega^r} \\
&+ \sum_{n=1}^{\infty} \left\| \int \cdots \int_{\Delta_n(T_{2r}, t)} \nabla b\left(t_1, X_{t_1}^x\right) \cdots \nabla b\left(t_n, X_{t_n}^x\right) \mathrm{d}t_1 \cdots \mathrm{d}t_n \right\|_{L_x^{2pr} L_\omega^{2r}} \|\nabla X_{T_{2r}}^x - \mathrm{I}\|_{L_x^{2pr} L_\omega^{2r}} \\
\leqslant &C t^{\gamma/2r}.
\end{aligned}$$

Iterating, we see that

$$\|\nabla X_t^x - \mathrm{I}\|_{L_x^{pr} L_\omega^r} \leqslant C t^{\gamma/2r}, \quad \forall t \in [0, T]. \tag{4.32}$$

Using (4.28) one sees that (4.2) can be proved in the same way as (4.1).

For (4.3). Assume $0 \leqslant s < s' \leqslant t \leqslant T$. Combing (4.29) and (4.2), we obtain

$$\begin{aligned}
&\|D_s X_t^x - D_{s'} X_t^x\|_{L_x^{pr} L_\omega^r} \\
\leqslant &\|D_s X_{s'}^x - \mathrm{I}\|_{L_x^{pr} L_\omega^r} + \|D_s X_{s'}^x - \mathrm{I}\|_{L_x^{2pr} L_\omega^{2r}} \|D_{s'} X_t^x - \mathrm{I}\|_{L_x^{2pr} L_\omega^{2r}} \\
\leqslant &C(s' - s)^{\gamma/4r},
\end{aligned}$$

for a.e. $s, s' \in [0, T]$ with $0 \leqslant s < s' \leqslant t \leqslant T$. So, we complete our proof for the first case.

Case (b). Let $\gamma \in (0, \frac{1}{2} - \frac{1}{q_1})$. By (4.5) and the argument in the previous case, one can see that for each positive even integer r,

$$\left\| \int \cdots \int_{\Delta_n(S_0, S_1)} \prod_{i=1}^{n} \nabla b\left(t_i, X_{t_i}^x\right)\, \mathrm{d}t_1 \cdots \mathrm{d}t_n \right\|_{L_x^{pr} L_\omega^r} \leqslant (rC_9)^{n+1/r} \|b\|^n_{\mathbb{L}_{q_1}^{p_1}(S_0, S_1)} (S_1 - S_0)^{\gamma/r},$$

where $C_9 > 1$ only depends on $d, p_1, q_1, T, a(m), r, p$ and γ. Since $b \in \mathbb{L}_{q_1}^{p_1}(T)$, for each even integer r there is a positive constant $T_r > 0$ depending on r, C_9 and $\ell(\delta)$ such that for any $S_0, S_1 \in [0, T]$ with $0 \leqslant S_1 - S_0 \leqslant T_r$

$$\|b\|_{\mathbb{L}_{q_1}^{p_1}(S_0, S_1)} \leqslant (2rC_9)^{-1}.$$

Thus,

$$\left\| \int \cdots \int_{\Delta_n(S_0, S_1)} \prod_{i=1}^{n} \nabla b\left(t_i, X_{t_i}^x\right)\, \mathrm{d}t_1 \cdots \mathrm{d}t_n \right\|_{L_x^{pr} L_\omega^r} \leqslant (rC_9)^{1/r} 2^{-n} (S_1 - S_0)^{\gamma/r}.$$

Our desired estimates then can be obtained by the above estimate and the same argument as in the previous case. □

5. Proof of the main result

The following lemma is a consequence of Theorem 1.1 in [RZ20].

Lemma 5.1. *Let $d \geqslant 3$. Assume that $b \in C([0,T];L^d)$ or $b \in \mathbb{L}_{q_1}^{p_1}(T)$ with $p_1, q_1 \in (2,\infty)$ and $d/p_1 + 2/q_1 = 1$. Then there is a unique weak solution to (1.2) such that for any $p, q \in (1,\infty)$ with $d/p + 2/q < 2$, the Krylov type estimate (1.3) is valid.*

Now we are in the position to prove our main result.

Proof of Theorem 1.1. **Case (a): $b \in C([0,T];L^d)$.**
Recalling that $\rho \in C_c^\infty(\mathbb{R}^d)$ satisfying $\rho \geqslant 0$ and $\int \rho = 1$, and let $b_k = b *_x \rho_k$. Since $b \in C([0,T];L^d)$, by Propostion 2.2 we have

$$\begin{aligned} \|b_k - b_k *_x \rho_m\|_{\mathbb{L}_\infty^d(T)} =& \|(b - b *_x \rho_m) *_x \rho_k\|_{\mathbb{L}_\infty^d(T)} \\ \leqslant& \|b - b *_x \rho_m\|_{\mathbb{L}_\infty^d(T)} =: a(m) \to 0 \quad (m \to \infty). \end{aligned} \tag{5.1}$$

It is well-known that for each k there is a unique continuous random field $X(k) : \Delta_2(T) \times \mathbb{R}^d \times \Omega \to \mathbb{R}^d$ such that

$$X_{s,t}^x(k) = x + \int_s^t b_k(r, X_{s,r}^x(k))\mathrm{d}r + W_t - W_s, \ \text{ for all } \ 0 \leqslant s \leqslant t \leqslant T, \ x \in \mathbb{R}^d. \tag{5.2}$$

Given $\beta \in (0,1/2)$, let

$$p \in (1,d), \ q \in (1,\infty) \ \text{ satisfying } \ \frac{d}{p} + \frac{2}{q} \in (1, 2-2\beta).$$

By estimate (5.1) and Remark 2.7, for any $s \leqslant t_1 \leqslant t_2 \leqslant T$ and $f \in \widetilde{\mathbb{L}}_q^p(T)$, there is a unique function u_k in $\widetilde{\mathbb{H}}_q^{2,p}(T)$ solving

$$\partial_t u_k + \frac{1}{2}\Delta u_k + b_k \cdot \nabla u_k + f = 0 \text{ in } (s,t_2) \times \mathbb{R}^d, \quad u_k(t_2) = 0$$

and a constant C, which does not depends on k, such that

$$\|\partial_t u_k\|_{\widetilde{\mathbb{L}}_q^p(t_1,t_2)} + \|u_k\|_{\widetilde{\mathbb{H}}_q^{2,p}(t_1,t_2)} \leqslant C\|f\|_{\widetilde{\mathbb{H}}_q^p(t_1,t_2)}. \tag{5.3}$$

By the generalized Itô formula (cf. [RZ20]),

$$-u_k(t_1, X_{s,t_1}^x(k)) = -\int_{t_1}^{t_2} f(t, X_{s,t}^x(k))\,\mathrm{d}t + \int_{t_1}^{t_2} \nabla u_k(t, X_{s,t}^x(k)) \cdot \mathrm{d}W_t.$$

Taking $\alpha = 0$ and $\theta = 1 - \frac{1}{q} - \beta > \frac{d}{2p}$ in (2.13), using Morrey's inequality and (5.3) we get

$$\begin{aligned} &\mathbf{E}\left(\int_{t_1}^{t_2} f(t, X_{s,t}^x(k))\mathrm{d}t \middle| \mathscr{F}_{t_1}\right) = \mathbf{E}\left(u_k(t_1, X_{s,t_1}^x(k)) \middle| \mathscr{F}_{t_1}\right) \\ \leqslant& \|u_k(t_1)\|_\infty \leqslant C \sup_{z \in \mathbb{R}^d} \|(u_k \chi_1^z)(t_1)\|_{H^{2\theta,p}} \\ \overset{(2.13)}{\leqslant}& C|t_2 - t_1|^\beta \left(\|\partial_t u_k\|_{\widetilde{\mathbb{L}}_q^p(t_1,t_2)} + \|u_k\|_{\widetilde{\mathbb{H}}_q^{2,p}(t_1,t_2)}\right) \\ \overset{(5.3)}{\leqslant}& C|t_2 - t_1|^\beta \|f\|_{\widetilde{\mathbb{L}}_q^p(t_1,t_2)}, \end{aligned} \tag{5.4}$$

where C only depends on $d,p,q,\beta,T,\|b\|_{\mathbb{L}^d_\infty(T)}$ and $\{a(m)\}$. Once with (5.4) in hand, it is standard to show that

$$\mathbf{E}\left|\int_{t_1}^{t_2} f(t,X^x_{s,t}(k))\mathrm{d}t\right|^r \leqslant C_r|t_2-t_1|^{\beta r}\|f\|^r_{\widetilde{\mathbb{L}}^p_q(t_1,t_2)}, \quad \forall\, f\in\widetilde{\mathbb{L}}^p_q(T),\ r>0, \tag{5.5}$$

(cf. [ZZ18]). Therefore, by noting that $p<d$, $q<\infty$ and $\beta<1/2$, we get

$$\begin{aligned}\mathbf{E}\left|X^x_{s,t_1}(k)-X^x_{s,t_2}(k)\right|^r \leqslant& C\mathbf{E}\left(\int_{t_1}^{t_2}|b_k|(t,X^x_{s,t})\mathrm{d}t\right)^r + C\mathbf{E}|W_{t_2}-W_{t_1}|^r\\ \leqslant& C|t_2-t_1|^{\beta r}\left(1+\|b\|^r_{\widetilde{\mathbb{L}}^d_\infty(T)}\right), \quad \forall r>0.\end{aligned} \tag{5.6}$$

Consequently,

$$\sup_{\substack{x\in\mathbb{R}^d;\\ 0\leqslant s\leqslant t\leqslant T}} \mathbf{E}\left|X^x_{s,t}(k)\right|^r \leqslant C, \quad \forall r>0, \tag{5.7}$$

which together with (4.1) and Morrey's inequality implies that for each $r>d$,

$$\sup_{\substack{z\in\mathbb{R}^d;\\ 0\leqslant s\leqslant t\leqslant T}} \mathbf{E}\|X^x_{s,t}(k)\|^r_{C^{1-\frac{d}{r}}_x(B_1(z))} \leqslant C \sup_{\substack{z\in\mathbb{R}^d;\\ 0\leqslant s\leqslant t\leqslant T}} \mathbf{E}\|X^x_{s,t}(k)\|^r_{H^{1,r}_x(B_1(z))} \leqslant C.$$

Thus, for any $0\leqslant s\leqslant t\leqslant T, x,y\in\mathbb{R}^d$ and $r>d$,

$$\mathbf{E}|X^x_{s,t}(k)-X^y_{s,t}(k))|^r \leqslant C|x-y|^{r-d}, \tag{5.8}$$

where C only depends on $d,p,q,r,\|b\|_{\mathbb{L}^d_\infty(T)}$ and $\{a(m)\}$.

Assume $0\leqslant s_1\leqslant s_2\leqslant t$. By the Markov property and the independence of $X^x_{s_1,s_2}(k)$ and $X^y_{s_2,t}(k)$, for each $r>d$ we obtain

$$\begin{aligned}&\mathbf{E}|X^x_{s_1,t}(k)-X^x_{s_2,t}(k)|^r\\ \leqslant& C_r\mathbf{E}\left|\int_{s_1}^{s_2} b_k\left(s,X^x_{s_1,s}(k)\right)\mathrm{d}s\right|^r + C_r\mathbf{E}\left|\int_{s_2}^{t}\left[b_k\left(s,X^x_{s_1,s}(k)\right)-b_k\left(X^x_{s_2,s}(k)\right)\right]\mathrm{d}s\right|^r\\ \overset{(5.5)}{\leqslant}& C|s_1-s_2|^{\beta r}+C\mathbf{E}\left|\int_{s_2}^{t}\left[b_k(s,X^{X^x_{s_1,s_2}(k)}_{s_2,s}(k))-b_k(s,X^x_{s_2,s}(k))\right]\mathrm{d}s\right|^r\\ \leqslant& C|s_1-s_2|^{\beta r}+C\mathbf{E}\left|X^{X^x_{s_1,s_2}(k)}_{s_2,t}(k)-X^x_{s_2,t}(k)\right|^r\\ =& C|s_1-s_2|^{\beta r}+C\mathbf{E}\left[\mathbf{E}\left|X^y_{s_2,t}(k)-X^x_{s_2,t}(k)\right|^r\Big|_{y=X^x_{s_1,s_2}(k)}\right]\\ \overset{(5.8)}{\leqslant}& C|s_1-s_2|^{\beta r}+C\mathbf{E}\left|X^x_{s_1,s_2}(k)-x\right|^{r-d}\\ \overset{(5.6)}{\leqslant}& C|s_1-s_2|^{\beta(r-d)}.\end{aligned} \tag{5.9}$$

Combing (5.6), (5.8) and (5.9), we obtain that for all $(s_i,t_i)\in\Delta_2(T)$, $i=1,2$,

$$\mathbf{E}|X^x_{s_1,t_1}(k)-X^y_{s_2,t_2}(k))|^r \leqslant C\left(|t_1-t_2|^{\beta r}+|x-y|^{r-d}+|s_1-s_2|^{\beta(r-d)}\right), \tag{5.10}$$

where $r>d$ and C only depends on d,r,T and b. On the other hand, noting that

$$\sup_k \|b_k\|_{\mathbb{L}^d_\infty(T)} \leqslant \|b\|_{\mathbb{L}^d_\infty(T)}, \quad \|b_k - b_k *_x \rho_m\|_{\mathbb{L}^d_\infty(T)} \leqslant a(m) \to 0 \ (m\to\infty),$$

by Lemma 3.1 and Proposition 4.1 one can see that for any fixed $(s,t)\in\Delta_2(T)$ and $R>0$,

$$\left\{B_R\times\Omega \ni (x,\omega)\mapsto X^x_{s,t}(k)(\omega)\in\mathbb{R}^d\right\}_{k\in\mathbb{N}_+}$$

is relatively compact in $L^2(B_R\times\Omega)$. The standard diagonal argument yields that there is a subsequence (still denoted by $X^x_{s,t}(k)$) and a countable dense subset $\mathcal{D}$ of $\mathbb{R}^d$ such that

$$X^x_{s,t}(k) \xrightarrow[k\to\infty]{L^2(\Omega)} X^x_{s,t}, \ \forall (s,t)\in\mathbb{Q}^2\cap\Delta_2(T) \text{ and } x\in\mathcal{D}.$$

By (5.7), we also have

$$X^x_{s,t}(k) \xrightarrow[k\to\infty]{L^r(\Omega)} X^x_{s,t}, \ \forall r\geqslant 1, \forall (s,t)\in\mathbb{Q}^2\cap\Delta_2(T) \text{ and } x\in\mathcal{D}.$$

Fatou's lemma and (5.10) yield that for all $(s_i,t_i)\in\mathbb{Q}^2\cap\Delta_2(T)$, $i=1,2$, and $x\in\mathcal{D}$,

$$\mathbf{E}\left|X^{x_1}_{s_1,t_1}-X^{x_2}_{s_2,t_2}\right|^r \leqslant C\left(|x_1-x_2|^{r-d}+|s_1-s_2|^{\beta(r-d)}+|t_1-t_2|^{\beta r}\right), \quad \forall r>d. \tag{5.11}$$

Therefore, $X^x_{s,t}$ can be extended to a continuous random field on $\Delta_2(T)\times\mathbb{R}^d$ satisfying (5.11) due to the Kolmogorov-Chentsov theorem, and up to a subsequence (still denoted by $X^x_{s,t}(k)$),

$$X^x_{s,t}(k,\omega) \stackrel{k\to\infty}{\longrightarrow} X^x_{s,t}(\omega), \tag{5.12}$$

for all $(s,t)\in\mathbb{Q}^2\cap\Delta_2(T)$, $x\in\mathcal{D}$ and $\mathbf{P}$-a.s. $\omega\in\Omega$. Then, since by (5.10) for $\mathbf{P}$-a.s. $\omega\in\Omega$, $X^x_{s,t}(k,\omega)$, $k\in\mathbb{N}_+$, are equicontinuous as functions of (s,t,x), (5.12) holds for all $(s,t)\in\Delta_2(T), x\in\mathbb{R}^d$ and $\omega\in\Omega_0\in\mathscr{F}$ with $\mathbf{P}(\Omega_0)=1$. Taking limits on both sides of (5.5), we get

$$\mathbf{E}\left|\int_{t_1}^{t_2} f(t,X^x_{s,t})\mathrm{d}t\right|^r \leqslant C|t_2-t_1|^{\beta r}\|f\|^r_{\mathbb{L}^p_q(t_1,t_2)}. \tag{5.13}$$

Thus, for each $x\in\mathbb{R}^d$ and $K\in\mathbb{N}_+$,

$$\begin{aligned}
&\mathbf{E}\left|\int_s^t b(\tau,X^x_{s,\tau})\mathrm{d}\tau-\int_s^t b_k(\tau,X^x_{s,\tau}(k))\mathrm{d}\tau\right|\\
\leqslant&\mathbf{E}\int_s^t |b-b_K|(\tau,X^x_{s,\tau})\mathrm{d}\tau+\mathbf{E}\int_s^t |b_K-b_k|(\tau,X^x_{s,\tau}(k))\mathrm{d}\tau\\
&+\mathbf{E}\left|\int_s^t b_K(\tau,X^x_{s,\tau})\mathrm{d}\tau-\int_s^t b_K(\tau,X^x_{s,\tau}(k))\mathrm{d}\tau\right|.
\end{aligned}$$

By our assumption on b, it holds that $b-b_k\to 0$ in $\mathbb{L}^d_\infty(T)$. So, the first and second terms on the right hand side of the above inequality converge to 0 as k goes to infinity, due to the fact that X and $X(k)$ satisfy the Krylov type estimates (5.5) and (5.13). On the other hand, by (5.12) and

Lebesgues dominated convergence theorem, the third term on the right side of the above inequality also converges to 0 as k goes to infinity. So,

$$\mathbf{E}\left|\int_s^t b(\tau,X^x_{s,\tau})\mathrm{d}\tau-\int_s^t b_k(\tau,X^x_{s,\tau}(k))\mathrm{d}\tau\right|\to 0,$$

which together with (5.12) implies

$$X^x_{s,t}-x-\int_s^t b(\tau,X^x_{s,\tau})\mathrm{d}\tau=\lim_{k\to\infty}\left(X^x_{s,t}(k)-x-\int_s^t b(\tau,X^x_{s,\tau}(k))\mathrm{d}\tau\right)=W_t,$$

i.e. the limit point $X^x_{s,\cdot}$ is a strong solution to (1.2). Hence, we obtain the strong existence of solutions to (1.2). Moreover, by the proof of Theorem of 1.1 in [RZ20], we can also see that $X^x_{s,\cdot}$ also satisfies (1.3) for any $p',q'\in(1,\infty)$ satisfying $d/p'+2/q'<2$.

Following [Che02], we next show that the limit point of $X^x_{s,t}(k)$ is the unique, and is also the unique strong solution to (1.2) satisfying (1.3). Without loss of generality we may assume $s=0$. Suppose X is a limit point of $X^x_\cdot(k)$, which is a strong solution of (1.2) with $s=0$ on $(\Omega,\mathscr{F},\{\mathscr{F}_t\}_{t\in[0,T]},\mathbf{P})$. Then there exists a measurable map $\mathcal{T}:C([0,T];\mathbb{R}^d)\to C([0,T];\mathbb{R}^d)$ such that $X_\cdot(\omega)=\mathcal{T}(W(\omega))$ for $\mathbf{P}$-a.s. ω. Let $\{\mathbf{Q}_\omega\}_{\omega\in\Omega}$ be the regular conditional expectation of X with respect $\mathscr{F}^W_T:=\sigma\{W_t:t\in[0,T]\}$. Then $\mathbf{Q}_\omega=\delta_{\mathcal{T}(W(\omega))}$ for $\mathbf{P}$-a.s. ω. Now, let Y be an another strong solution to (1.2) with $s=0$ on $(\Omega,\mathscr{F},\{\mathscr{F}_t\}_{t\in[0,T]},\mathbf{P})$ satisfying (1.3). Thanks to Lemma 5.1, we have $\mathrm{law}(X)=\mathrm{law}(Y)$, together with the fact that

$$W_t=X_t-x-\int_0^t b(r,X_r)\mathrm{d}r=Y_t-x-\int_0^t b(r,Y_r)\mathrm{d}r,$$

we obtain $\mathrm{law}(X,W)=\mathrm{law}(Y,W)$. This implies $\mathbf{Q}'_\omega$, the regular conditional expectation of Y with respect to $\mathscr{F}^W_T$, equals to $\mathbf{Q}_\omega$ for $\mathbf{P}$-a.s. ω, i.e. $\mathbf{Q}'_\omega=\delta_{\mathcal{T}(W(\omega))}$. Thus, $Y(\omega)=\mathcal{T}(W(\omega))=X(\omega)$ for $\mathbf{P}-a.s.\ \omega$.

Case (b): $b\in\mathbb{L}^{p_1}_{q_1}(T)$**.** Given $\beta\in(0,1/2)$. In this case, we take $p\in(1,p_1)$ and $q\in(1,q_1)$ such that $d/p+2/q\in(1,2-2\beta)$. Define the maximal function of $b(t,\cdot)$:

$$\mathcal{M}b(t,x):=\sup_{r>0}\fint_{B_r(x)}|b(t,y)|\,\mathrm{d}y$$

Define $b_k=(b\mathbf{1}_{|b|\leqslant k})*_x\rho_k\in L^\infty([0,T];C^2_b)$. Noting that $b_k\leqslant|b|*_x\rho_k\leqslant C\mathcal{M}b$ (cf. [DZ01, Corollary 2.8]), we have $K'_{b_k}(m)\leqslant K'_{C\mathcal{M}b}(m)$. By the basic fact that

$$\|\mathcal{M}b\|_{\mathbb{L}^{p_1}_{q_1}(T)}\asymp\|b\|_{\mathbb{L}^{p_1}_{q_1}(T)}<\infty,$$

(cf. [DZ01, Theorem 2.5]), we obtain

$$\sup_k K'_{b_k}(m)\leqslant K'_{C\mathcal{M}b}(m)=:a(m)\to 0,\ \text{as}\ m\to\infty.$$

Also we have

$$\sup_k \omega_{b_k}(\delta)\leqslant\omega_b(\delta)=:\ell(\delta)\to 0,\ \text{as}\ \delta\to 0.$$

Then our desired results in the second case can be obtained by the same procedure as for the previous case. $\square$

6. Application to Navier-Stokes equations

In [Zha10], Zhang studied the backward Navier-Stokes equation (1.10) through considering the stochastic system (1.9). As in [CI08], it was also shown in [Zha10] that the existence of smooth solutions for (1.10) and (1.9) are equivalent (see Theorem 2.3 therein). Therefore, it is quite interesting to find a regularity criterion for solutions of (1.9). Below we give one such conditional regularity result, which is similar to the Serrin criterion for the Navier-Stokes equations.

Theorem 6.1. *Let $d \geqslant 3$, $T < 0$, $p_1, q_1 \in (2,\infty)$, $q > d$ and $k, l \in \mathbb{N}$. Assume $u \in C([-T,0];L^d)$ or $u \in \mathbb{L}^{p_1}_{q_1}(-T,0)$ with $d/p_1 + 2/q_1 \leqslant 1$ and $\varphi \in H^{k,q}$. Suppose that (u,X) is a solution to the stochastic system (1.9), then $u \in \mathbb{H}^{k,q}_\infty(-T,0)$ and for any $l \leqslant k/2$, $\partial^l_t u \in \mathbb{L}^q_\infty(-T,0)$. Consequently, if $\varphi \in C^\infty_c(\mathbb{R}^d)$, then $u \in C^\infty([-T,0]\times\mathbb{R}^d)$ and it satisfies (1.10).*

Proof. Step 1. Assume that $\varphi \in L^q$, $u \in C([-T,0];L^d)$, or $u \in \mathbb{L}^{p_1}_{q_1}(-T,0)$ with $p_1, q_1 \in (2,\infty)$ and $d/p_1 + 2/q_1 \leqslant 1$. We claim that

$$\sup_{t\in[-T,0]} \|u(t)\|_q < \infty. \tag{6.1}$$

Below we only give the proof of (6.1) for the case where $u \in \mathbb{L}^{p_1}_{q_1}(-T,0)$ with $p_1, q_1 \in (2,\infty)$ and $d/p_1 + 2/q_1 = 1$, since the other cases are simpler.

Let $a \in (d,q)$ and

$$v(t,x) := \mathbf{E}\left[(\nabla^\top X^x_{t,0} - \mathrm{I})\varphi\left(X^x_{t,0}\right)\right].$$

Noting that $\frac{1}{a} - \frac{1}{q} \in (0, \frac{1}{d})$, one can always choose $r \geqslant 2$ and $p \in (\frac{p_1}{p_1-1}, p_1)$ such that $\frac{1}{pr} = \frac{1}{a} - \frac{1}{q}$. Thus, for each $t \in [-T,0]$

$$\begin{aligned}
\|v(t)\|_a \leqslant & \left\| \left\|\nabla^\top X^x_{t,0} - \mathrm{I}\right\|_{L^r_\omega} \left\|\varphi(X^x_{t,0})\right\|_{L^{r'}_\omega} \right\|_{L^a_x} \\
\leqslant & C\left\|\nabla X^x_{t,0} - \mathrm{I}\right\|_{L^{pr}_x L^r_\omega} \left\|\varphi(X^x_{t,0})\right\|_{L^q_x L^{r'}_\omega} \\
\overset{(1.4)}{\leqslant} & C\|\mathbf{E}|\varphi|^{r'}(X^x_{t,0})\|^{1/r'}_{L^{q/r'}_x} \leqslant C\|\varphi\|_q.
\end{aligned} \tag{6.2}$$

Here $r' = r/(r-1)$, and we use the fact that

$$\|\mathbf{E}f(X^\cdot_{t,0})\|_q \leqslant \|f\|_q, \quad \forall q \in [1,\infty], \tag{6.3}$$

due to the fact that u is divergence free (cf. [ZZ21, Lemma 3.2]). Recall that P is the Leray projection

$$(\mathrm{P}F)_i = F_i - \nabla(\Delta)^{-1}\mathrm{div}F = F_i - \sum_{j=1}^d R_i R_j F_j,$$

where R_i is the Riesz transformation. The L^q boundedness of R_i implies that P is a bounded map on $L^q(\mathbb{R}^d;\mathbb{R}^d)$ with $q \in (1,\infty)$. By (6.3), we have

$$\begin{aligned}
\|u\|_{\widetilde{\mathbb{L}}^a_\infty(-T,0)} = & \sup_{z\in\mathbb{R}^d} \|u\chi^z_1\|_{\mathbb{L}^a_\infty(-T,0)} \\
\leqslant & C\left(\sup_{t\in[-T,0]} \|\mathrm{P}v(t)\|_a + \sup_{t\in[-T,0]} \|\mathrm{P}\mathbf{E}\varphi(X^\cdot_{t,0})\|_q\right)
\end{aligned}$$

$$\leqslant C \sup_{t\in[-T,0]} \|v(t)\|_a + C\|\varphi\|_q \overset{(6.2)}{<} \infty.$$

Noting that $a > d$, combining the above estimate and Theorem 1.1 of [XXZZ20] we get

$$\sup_{x\in\mathbb{R}^d} \mathbf{E} \sup_{t\leqslant s\leqslant 0} |\nabla X^x_{t,s}|^r < \infty, \quad \forall r \geqslant 1. \tag{6.4}$$

Therefore, for each $t \in [-T,0]$,

$$\begin{aligned} \|u(t)\|_q \leqslant & \Big\| \left\|\nabla^\top X^x_{t,0}\right\|_{L^r_\omega} \|\varphi(X^x_{t,0})\|_{L^{r'}_\omega} \Big\|_{L^q_x} \leqslant C \left\|\nabla X^x_{t,0}\right\|_{L^\infty_x L^r_\omega} \left\|\varphi(X^x_{t,0})\right\|_{L^q_x L^{r'}_\omega} \\ \overset{(6.4)}{\leqslant} & C\|\mathbf{E}|\varphi|^{r'}(X^x_{t,0})\|^{1/r'}_{L^{q/r'}_x} \leqslant C\|\varphi\|_q < \infty. \end{aligned} \tag{6.5}$$

So, we complete our proof for (6.1).

Step 2. Now assume that $\varphi \in H^{1,q}$. By [Zha16, Lemma 7.2],

$$\partial_i u(t) = \partial_i \mathrm{P}\mathbf{E}\left[\nabla^\top X^x_{t,0}\varphi(X^x_{t,0})\right] = \mathrm{P}\mathbf{E}\left[\nabla^\top X^x_{t,0}[\nabla\varphi(X^x_{t,0}) - \nabla^\top\varphi(X^x_{t,0})]\partial_i X^x_{t,0}\right]. \tag{6.6}$$

Using Hölder's inequality and (6.4), we get

$$\begin{aligned} \|\partial_i u(t)\|_q \leqslant & C\left\|\mathbf{E}\left[\nabla^\top X^x_{t,0}[\nabla\varphi(X^x_{t,0}) - \nabla^\top\varphi(X^x_{t,0})]\partial_i X^x_{t,0}\right]\right\|_{L^q_x} \\ \leqslant & C\Big\| \|\nabla X^x_t\|_{L^{2r}_\omega} \|\nabla\varphi(X^x_{t,0})\|_{L^{r'}_\omega}\Big\|_{L^q_x} \\ \leqslant & C\|\nabla X^x_t\|_{L^\infty_x L^{2r}_\omega} \|\nabla\varphi(X^x_{t,0})\|_{L^q_x L^{r'}_\omega} \overset{(6.4)}{\leqslant} C\|\nabla\varphi\|_q < \infty. \end{aligned}$$

Hence,

$$\|u\|_{\mathbb{H}^{1,q}_\infty(-T,0)} \leqslant C\|\varphi\|_{H^{1,q}} < \infty. \tag{6.7}$$

Step 3. Assume that $\varphi \in H^{2,q}$. Following [XXZZ20], below we use a Zvonkin type change of variables to convert the first equation in (1.9) to a new SDE. Let $t \in [-T,0]$, $\lambda \geqslant 0$ and $a \in (1,\infty)$ such that $d/p + 2/a < 1$. Since $u \in \mathbb{H}^{1,q}_\infty(-T,0)$, there is a unique function U in $\mathbb{H}^{3,q}_a(-T,0)$ satisfying

$$\partial_s U + \left(\frac{\Delta}{2} - \lambda\right) U + u\cdot\nabla U + u = 0 \ \text{ in } (-T,0)\times\mathbb{R}^d, \quad U(0) = 0.$$

Moreover,

$$\lambda\|U\|_{\mathbb{H}^{1,q}_a(-T,0)} + \|\partial_t U\|_{\mathbb{H}^{1,q}_a(-T,0)} + \|\nabla^2 U\|_{\mathbb{H}^{1,q}_a(-T,0)} \leqslant C\|u\|_{\mathbb{H}^{1,q}_a(-T,0)} < \infty \tag{6.8}$$

(cf. [XXZZ20]). Since $d/p + 2/a < 1$, using (6.8), (2.13) and an interpolation inequality one can choose λ large enough so that

$$\sum_{k=0}^{2} \|\nabla^k U\|_\infty \leqslant 1/2. \tag{6.9}$$

Define

$$\Phi(s,x) := x + U(s,x).$$

By (6.9), $\Phi(s,\cdot)$ is a C^2-diffeomorphism and

$$\|\nabla\Phi\|_\infty, \|\nabla^2\Phi\|_\infty, \|\nabla\Phi^{-1}\|_\infty, \|\nabla^2\Phi^{-1}\|_\infty \leqslant C. \tag{6.10}$$

Set

$$Y^{y,k}_{t,s} := \Phi^k(s, X^{\Phi^{-1}(t,y)}_{t,s}), \quad \sigma^k_{k'}(s,y) = \partial_{k'}\Phi^k(s,\Phi^{-1}(s,y)), \quad b^k(s,y) = \lambda U^k(s,\Phi^{-1}(s,y)).$$

Then,

$$Y^y_{t,s} = y + \int_t^s b(\tau, Y^y_{t,\tau})\mathrm{d}\tau + \int_t^s \sigma(\tau, Y^y_{t,\tau})\mathrm{d}\widetilde{W}_\tau,$$

where $\widetilde{W}_\tau := W_\tau - W_t$ is a standard Brownian motion on $[t,0]$. By (6.8)-(6.10), (2.13) and the definitions of σ and b, one sees that

$$\sigma(s,y) - \mathrm{I} = \nabla U(s,\Phi^{-1}(s,y)) \in \mathbb{H}^{2,q}_a(-T,0) \cap C([-T,0]; C^2_b(\mathbb{R}^d)) \tag{6.11}$$

and

$$b \in C([-T,0]; C^2_b(\mathbb{R}^d)). \tag{6.12}$$

By the proof for [XXZZ20, Theorem 1.1], $\partial_i Y^y_{t,s}$ satisfies

$$\partial_i Y^y_{t,s} = e_i + \int_t^s \nabla b(\tau, Y^y_{t,\tau})\, \partial_i Y^y_{t,\tau}\, \mathrm{d}\tau + \int_t^s \partial_l \sigma_{k'}(\tau, Y^y_{t,\tau})\, \partial_i Y^{y,l}_{t,\tau}\, \mathrm{d}\widetilde{W}^{k'}_\tau. \tag{6.13}$$

and

$$\sup_{x\in\mathbb{R}^d} \mathbf{E} \sup_{t\leqslant s\leqslant 0} |\nabla Y^x_{t,s}|^r < \infty, \quad \forall r \geqslant 1. \tag{6.14}$$

For any $h \in \mathbb{R}^d$, define

$$\delta^h_s := \sup_{\tau\in[t,s]} |\partial_i Y^{y+h}_{t,\tau} - \partial_i Y^y_{t,\tau}|.$$

For all $m \in \mathbb{N}_+$, equation (6.13) and the Burkholder-Davis-Gundy inequality yield

$$\begin{aligned}\mathbf{E}(\delta^h_s)^{2m} \lesssim & \mathbf{E}\left|\int_t^s [\nabla b(\tau, Y^{y+h}_{t,\tau}) - \nabla b(\tau, Y^y_{t,\tau})]\partial_i Y^{y+h}_{t,\tau}\, \mathrm{d}\tau\right|^{2m} + \mathbf{E}\left|\int_t^s |\nabla b(\tau, Y^y_{t,\tau})|\, \delta^h_\tau\, \mathrm{d}\tau\right|^{2m} \\ & + \mathbf{E}\left(\int_t^s \left|\nabla\sigma(\tau, Y^{y+h}_{t,\tau}) - \nabla\sigma(\tau, Y^y_{t,\tau})\right|^2 |\partial_i Y^{y+h}_{t,\tau}|^2\, \mathrm{d}\tau\right)^m \\ & + \mathbf{E}\left(\int_t^s |\nabla\sigma(\tau, Y^y_{t,\tau})|^2 (\delta^h_\tau)^2\, \mathrm{d}\tau\right)^m =: \sum_{i=1}^4 I_i.\end{aligned} \tag{6.15}$$

Using (6.12) and (6.14) one sees that

$$I_1 \lesssim \mathbf{E}\left(\int_t^s |h| \int_0^1 |\nabla Y^{y+\theta h}_{t,\tau}|\, \mathrm{d}\theta\, |\nabla Y^{y+h}_{t,\tau}|\, \mathrm{d}\tau\right)^{2m} \lesssim |h|^{2m} \tag{6.16}$$

and

$$I_2 \lesssim \int_t^s \mathbf{E}(\delta^h_\tau)^{2m} \mathrm{d}\tau. \tag{6.17}$$

Similarly, by (6.11) we get

$$I_4 \lesssim \int_t^s \mathbf{E}(\delta^h_\tau)^{2m} \mathrm{d}\tau. \tag{6.18}$$

For I_3. Let

$$\mathcal{M}f(x) := \sup_{r>0} ⨍_{B_r(x)} |f(y)|\, \mathrm{d}y$$

be the the maximal function of f. By the fact that

$$|f(x)-f(y)| \leqslant C\,[\mathcal{M}|\nabla f|(x)+\mathcal{M}|\nabla f|(y)]\,|x-y|$$

(cf. [CDL08]), we deduce

$$\begin{aligned}
I_3 \lesssim & \mathbf{E}\left\{\int_t^s \left[(\mathcal{M}|\nabla^2\sigma|(\tau,Y_{t,\tau}^{y+h}))^2+(\mathcal{M}|\nabla^2\sigma|(\tau,Y_{t,\tau}^{y}))^2\right]\right.\\
&\left.\left(|h|\int_0^1|\nabla Y_{t,\tau}^{y+\theta h}|\mathrm{d}\theta\right)^2|\partial_i Y_{t,\tau}^{y+h}|^2\mathrm{d}\tau\right\}^m\\
\lesssim & |h|^{2m}\mathbf{E}\left\{\sup_{\tau\in[t,s]}|\nabla Y_{t,\tau}^{y+h}|^{2m}\int_0^1\sup_{\tau\in[t,s]}|\nabla Y_{t,\tau}^{y+\theta h}|^{2m}\mathrm{d}\theta\right.\\
&\left.\left(\int_t^s\left[(\mathcal{M}|\nabla^2\sigma|(\tau,Y_{t,\tau}^{y+h}))^2+(\mathcal{M}|\nabla^2\sigma|(\tau,Y_{t,\tau}^{y}))^2\right]\mathrm{d}\tau\right)^m\right\}\\
\lesssim & |h|^{2m}\left[\mathbf{E}\sup_{\tau\in[t,s]}|\nabla Y_{t,\tau}^{y+h}|^{6m}\right]^{1/3}\cdot\int_0^1\left[\mathbf{E}\sup_{\tau\in[t,s]}|\nabla Y_{t,\tau}^{y+\theta h}|^{6m}\right]^{1/3}\mathrm{d}\theta\\
&\cdot\left[\mathbf{E}\left(\int_t^s\left[(\mathcal{M}|\nabla^2\sigma|(\tau,Y_{t,\tau}^{y+h}))^2+(\mathcal{M}|\nabla^2\sigma|(\tau,Y_{t,\tau}^{y}))^2\right]\mathrm{d}\tau\right)^{3m}\right]^{1/3}\lesssim|h|^{2m}.
\end{aligned}\tag{6.19}$$

Here we used the facts that $\mathcal{M}|\nabla^2\sigma|\in\mathbb{L}_{a/2}^{q/2}(T)$ with $d/q+2/a<1$ and

$$\sup_{y\in\mathbb{R}^d}\mathbf{E}\int_t^s f(\tau,Y_{t,\tau}^y)\mathrm{d}\tau\leqslant C\|f\|_{\mathbb{L}_{a/2}^{q/2}(t,s)}$$

(cf. [XXZZ20]). Thus, combining (6.15)-(6.19), we obtain

$$\mathbf{E}(\delta_s^h)^{2m}\lesssim|h|^{2m}+\int_t^s\mathbf{E}(\delta_\tau^h)^{2m}\mathrm{d}\tau.$$

Gronwall's inequality yields

$$\mathbf{E}\sup_{\tau\in[t,s]}|\partial_i Y_{t,\tau}^{y+h}-\partial_i Y_{t,\tau}^{y}|^{2m}\lesssim|h|^{2m}$$

Follow the argument of the proof for [XXZZ20, Theorem 1.1], one sees that

$$\sup_{x\in\mathbb{R}^d}\mathbf{E}\sup_{s\in[t,0]}|\nabla^2 Y_{t,s}^x|^r<\infty,\quad\forall r\geqslant 1.$$

and

$$\begin{aligned}
\partial_{ij}Y_{t,s}^y=&\int_t^s\partial_{ll'}b(\tau,Y_{t,\tau}^y)\,\partial_i Y_{t,\tau}^{y,l}\,\partial_j Y_{t,\tau}^{y,l'}\,\mathrm{d}\tau+\int_t^s\partial_l b(\tau,Y_{t,\tau}^y)\,\partial_{ij}Y_{t,\tau}^{y,l}\,\mathrm{d}\tau\\
&+\int_t^s\partial_{ll'}\sigma_{k'}(\tau,Y_{t,\tau}^y)\,\partial_i Y_{t,\tau}^{y,l}\,\partial_j Y_{t,\tau}^{y,l'}\,\mathrm{d}\widetilde{W}_\tau^{k'}+\int_t^s\partial_l\sigma_{k'}(\tau,Y_{t,\tau}^y)\,\partial_{ij}Y_{t,\tau}^{y,l}\,\mathrm{d}\widetilde{W}_\tau^{k'}
\end{aligned}$$

Recalling that $\Phi^{-1}\in C([-T,0];C_b^2(\mathbb{R}^d))$, we see that

$$\sup_{x\in\mathbb{R}^d}\mathbf{E}\sup_{s\in[t,0]}|\nabla^2 X_{t,s}^x|^r<\infty,\quad\forall r\geqslant 1.\tag{6.20}$$

By (6.6), we have

$$\partial_{ij}u(t) = \partial_{ij}\mathbf{PE}\left[\nabla^\top X_t^x \varphi(X_t^x)\right] = \mathbf{PE}\partial_j\left[\nabla^\top X_t^x[\nabla\varphi(X_t^x) - \nabla^\top\varphi(X_t^x)]\partial_i X_t^x\right].$$

Using (6.20) and following the same procedure as in the proof for (6.7), one can verify that

$$\|u\|_{\mathbb{H}^{2,q}_\infty(-T,0))} \leqslant C\|\varphi\|_{H^{2,q}} < \infty.$$

Repeating the above process higher derivatives can be estimated similarly step by step.

Step 4. Assume $\varphi \in H^{2,q}$ and set

$$w(t) := \mathbf{E}\left[\nabla^\top X_{t,0}^x\,\varphi(X_{t,0}^x)\right].$$

By *step 3* we can see that

$$\sup_{x\in\mathbb{R}^d}\mathbf{E}\sup_{t\leqslant s\leqslant 0}|\nabla^3 X_{t,s}^x|^r < \infty \ \text{ and } \ w \in \mathbb{H}^{2,q}_\infty(-T,0).$$

Following the proof for [Zha10, Theorem 2.1], we see that w satisfies

$$\partial_t w = -\tfrac{\Delta}{2}w - (\nabla^\top w)u - (\nabla^\top u)w, \quad w(0) = \varphi. \tag{6.21}$$

Thus, $\partial_t w \in \mathbb{L}^q_\infty(-T,0)$, which also implies $\partial_t u = \partial_t \mathrm{P}w = \mathrm{P}\partial_t w \in \mathbb{L}^q_\infty(-T,0)$ due to the L^q boundedness of P.

If $\varphi \in H^{4,q}$, following the above discussion we see that $u, w \in \mathbb{H}^{4,q}_\infty(-T,0)$, which implies that the right side of (6.21) is in $\mathbb{H}^{2,q}_\infty(-T,0)$. Hence, $\partial_t w \in \mathbb{H}^{2,q}_\infty(-T,0)$ and $\partial_t u \in \mathbb{H}^{2,q}_\infty(-T,0)$. This means that $\partial_t[\frac{\Delta}{2}w + (\nabla^\top w)u + (\nabla^\top u)w] \in \mathbb{L}^q_\infty(-T,0)$, i.e. $\partial_t^2 w \in \mathbb{L}^q_\infty(-T,0)$. Repeating the same process one sees that $\partial_t^k w \in \mathbb{L}^q_\infty(-T,0)$, provided that $\varphi \in H^{2k,q}$. So, we complete our proof.

□

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