Exponential Ergodicity for Singular Reflecting McKean-Vlasov SDEs *

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Abstract

By refining a recent result of Xie and Zhang [23], we prove the exponential ergodicity under a weighted variation norm for singular SDEs with drift containing a local integrable term and a coercive term (Theorem 2.1). This result is then extended to singular reflecting SDEs as well as singular McKean-Vlasov SDEs with or without reflection (Theorems 2.3, 2.4). We also present a general result deducing the uniform ergodicity of McKean-Vlasov SDEs from that of classical SDEs (Lemma 3.3). As an application, the L^1-exponential convergence is derived for a class of non-symmetric singular granular media equations (Example 2.1).

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1 Introduction

Let $D \subset \mathbb{R}^d$ be a connected open domain including the global situation $D = \mathbb{R}^d$, and let $\mathscr{P}$ denote the space of probability measures on $\bar{D}$, the closure of D. Consider the following distribution dependent (i.e. McKean-Vlasov) SDE on $\bar{D}$ with reflection if $D \neq \mathbb{R}^d$:

$$(1.1) \qquad \mathrm{d}X_t = b(X_t, \mathscr{L}_{X_t})\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t + \mathbf{n}(X_t)\mathrm{d}l_t, \quad t \geq 0,$$

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where $(W_t)_{t\geq 0}$ is an m-dimensional Brownian motion on a complete filtration probability space $(\Omega, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$, $\mathscr{L}_{X_t}$ is the distribution of X_t,

$$b : D \times \mathscr{P} \to \mathbb{R}^d, \sigma : D \times \mathscr{P} \to \mathbb{R}^d \otimes \mathbb{R}^m$$

are measurable, and when $D \neq \mathbb{R}^d$, $\mathbf{n}$ is the inward unit normal vector field of the boundary ∂D, and l_t is an adapted continuous increasing process which increases only when $X_t \in \partial D$.

In the case that $D = \mathbb{R}^d$, we have $l_t = 0$ so that (1.1) becomes the McKean-Vlasov SDE

$$\mathrm{d}X_t = b(X_t, \mathscr{L}_{X_t})\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t, \quad t \geq 0. \tag{1.2}$$

If moreover $b(x, \mu) = b(x)$ does not depend on μ, it reduces to the classical Itô's SDE

$$\mathrm{d}X_t = b(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t, \quad t \geq 0. \tag{1.3}$$

In the recent work [21], the well-posedness and regularity estimates have been studied for solutions to (1.1) with b containing a locally integrable term and a Lipchitz continuous term. However, the ergodicity was only investigated under monotone or Lyapunov conditions excluding this singular situation. See also [4, 8, 9, 10, 11, 13, 16, 20] and references within for results on the ergodicity of McKean-Vlasov SDEs without reflection under monotone or Lyapunov conditions. On the other hand, by using Zvokin's transform, the exponential ergodicity was proved by Xie and Zhang [23] for the singular SDE (1.3). In this paper, we aim to refine the result of [23] and make extensions to singular SDEs with reflection and distribution dependent drift.

When the SDE (1.1) is well-posed, let $P_t^*\nu = \mathscr{L}_{X_t}$ for the solution with initial distribution $\nu \in \mathscr{P}$. We will study the exponential convergence of P_t^* under the weighted variation distance induced by a positive measurable function V:

$$\|\mu - \nu\|_V := |\mu - \nu|(V) = \sup_{|f|\leq V} |\mu(f) - \nu(f)|, \quad \mu, \nu \in \mathscr{P},$$

where $|\mu-\nu|$ is the total variation of $\mu-\nu$ and $\mu(f) := \int f\mathrm{d}\mu$ for a measure μ and $f \in L^1(\mu)$. When $V = 1$, $\|\cdot\|_V$ reduces to the the total variation norm $\|\cdot\|_{var}$.

We will consider $b(x, \mu) = b^{(0)}(x) + b^{(1)}(x, \mu)$, where $b^{(0)}$ is the singular term satisfying

$$\sup_{z\in\mathbb{R}^d} \int_{B(z,1)\cap D} |b^{(0)}(x)|^p(\mathrm{d}x) < \infty \tag{1.4}$$

for some $p > d \vee 2$, and $b^{(1)}(\cdot, \mu)$ is a coercive term such that

$$\limsup_{x\in\bar{D},|x|\to\infty} \sup_{\mu\in\mathscr{P}} \langle b^{(1)}(x, \mu), \nabla V(x)\rangle = -\infty$$

holds for some compact function $V \in C^2(\mathbb{R}^d)$ (i.e. $\{V \leq r\}$ is compact for any $r > 0$). The later condition is trivial for bounded D by taking $V = 1$ and the convention that $\sup \emptyset = -\infty$.

To conclude this section, we present below an example for the L^1-exponential convergence of non-symmetric singular granular media equations, see [5, 9, 13] for the study of regular and symmetric models for $D = \mathbb{R}^d$.

Example 2.1. Let $D = \mathbb{R}^d$ or be a bounded $C^{2,L}$-domain (see Definition 2.1 below). Consider the following nonlinear PDE for probability density functions on $\bar{D}$:

$$(1.5) \qquad \partial_t \varrho_t = \Delta \varrho_t - \mathrm{div}\big\{\varrho_t b + \varrho_t (W * \varrho_t)\big\}, \quad \nabla_{\mathbf{n}} \varrho_t|_{\partial D} = 0 \text{ if } \partial D \neq \emptyset,$$

where

(i) W is a bounded measurable function on $\bar{D} \times \bar{D}$, and

$$(W * \varrho_t)(x) := \int_{\mathbb{R}^d} W(x,z)\varrho_t(z)\mathrm{d}z;$$

(ii) $b = b^{(0)} + b^{(1)}$ is a vector field such that (1.4) holds for some $p > d \vee 2$, and $b^{(1)}$ is locally bounded with $b^{(1)}(x) = -\phi(|x|^2)x$ for larger $|x|$ and some increasing function $\phi : [0,\infty) \to [1,\infty)$ with $\int_1^\infty \frac{\mathrm{d}s}{s\phi(s)} < \infty$.

In physics, ρ_t stands for the distribution density of particles, W describes the interaction among particles, and b refers to the potential of individual particles. When b and W are not of gradient type, the associated mean field particle systems are non-symmetric.

To characterize (1.5) using (1.1), let

$$b(x,\mu) = b(x) + (W * \mu)(x), \quad \sigma(x) = \sqrt{2}\mathbf{I}_d,$$

where $\mathbf{I}_d$ is the $d \times d$ identity matrix, and $(W * \mu)(x) := \int_{\bar{D}} W(x,z)\mu(\mathrm{d}z)$.

By (i) and (ii), **(A1)** holds for $V(x) := |x|^2$ when $D = \mathbb{R}^d$, while **(A2)** holds for $V = 1$ when D is a bounded $C_b^{2,L}$ domain. So, by Theorem 2.4, (1.1) is well-posed, and by Itô's formula, $\rho_t(x) := \frac{\mathrm{d}P_t^*\nu}{\mathrm{d}x}$ solves (1.5) for $\rho_0(x) := \frac{\mathrm{d}\nu}{\mathrm{d}x}$, see Subsection 1.2 in [21]. On the other hand, when $D = \mathbb{R}^d$ the superposition principle in [2] says that a solution of (1.5) is the distribution density of a weak solution to (1.1), such that (1.5) is well-posed as well.

Therefore, by Theorem 2.4, when $\|W\|_\infty$ is small enough, P_t^* has a unique invariant probability measure μ satisfying (2.14), so that the solution $\rho_t := \frac{\mathrm{d}P_t^*\nu}{\mathrm{d}x}$ of (1.5) satisfies

$$\|\rho_t - \rho\|_{L^1} = \|P_t^*\nu - \mu\|_{var} \le c\mathrm{e}^{-\lambda t}\|\rho_0 - \rho\|_{L^1}, \quad t \ge 0$$

for some constants $c, \lambda > 0$, where ρ is the density function of μ.

In the remainder of the paper, we first state our main results in Section 2, then present some lemmas in Section 3, and finally prove the main results in Section 4.

2 Main results

To measure the singularity of the SDE, we introduce some functional spaces used in [22]. For any $p \ge 1$, let L^p be the class of measurable functions f on D such that

$$\|f\|_{L^p} := \left(\int_D |f(x)|^p \mathrm{d}x\right)^{\frac{1}{p}} < \infty.$$

For any $\epsilon > 0$ and $p \geq 1$, let $H^{\epsilon,p} := (1-\Delta)^{-\frac{\epsilon}{2}} L^p$ with

$$\|f\|_{H^{\epsilon,p}} := \|(1-\Delta)^{\frac{\epsilon}{2}} f\|_{L^p} < \infty, \quad f \in H^{\epsilon,p},$$

where Δ is the (Neumann if $\partial D \neq \emptyset$) Laplacian. For any $z \in \mathbb{R}^d$ and $r > 0$, let

$$B(z,r) := \{x \in \mathbb{R}^d : |x - z| \leq r\}$$

be the closed ball centered at z with radius r. We will simply denote $B_r = B(0,r)$ for $r > 0$. We write $f \in \tilde{L}^p$ if

$$\|f\|_{\tilde{L}^p} := \sup_{z \in \bar{D}} \|1_{B(z,1)} f\|_{L^p} < \infty.$$

Moreover, let $g \in C_0^\infty(\bar{D})$ with $g|_{B_1} = 1$ and the Neumann boundary condition $\nabla_{\mathbf{n}} g|_{\partial D} = 0$ if ∂D exists. We denote $f \in \tilde{H}^{\epsilon,p}$ if

$$\|f\|_{\tilde{H}^{\epsilon,p}} := \sup_{z \in \bar{D}} \|g(z + \cdot) f\|_{H^{\epsilon,p}} < \infty.$$

We note that the space $\tilde{H}^{\epsilon,p}$ does not depend on the choice of g. If a vector or matrix valued function has components in one of the above introduced spaces, then it is said in the same space with norm defined as the sum of components' norms.

In the following we state our main results in different situations.

2.1 Singular SDEs

We will prove the ergodicity of SDE (1.3) under the following assumption.

(A1) σ is weakly differentiable, $\sigma\sigma^*$ is invertible, and $b = b^{(0)} + b^{(1)}$ such that the following conditions hold.

(1) There exists $p > d \vee 2$ such that

$$\|\sigma\|_\infty + \|(\sigma\sigma^*)^{-1}\|_\infty + \|b^{(0)}\|_{\tilde{L}^p} + \|\nabla\sigma\|_{\tilde{L}^p} < \infty.$$

(2) $b^{(1)}$ is locally bounded, there exist constants $K > 0, \varepsilon \in (0,1)$, some compact function $V \in C^2(\mathbb{R}^d; [1,\infty))$, and a continuous increasing function $\Phi : [1,\infty) \to [1,\infty)$ with $\Phi(n) \to \infty$ as $n \to \infty$, such that

$$\begin{aligned} &\langle b^{(1)}, \nabla V\rangle(x) + \varepsilon |b^{(1)}(x)| \sup_{B(x,\varepsilon)} |\nabla^2 V| \leq K - \varepsilon(\Phi \circ V)(x), \\ &\lim_{|x|\to\infty} \sup_{B(x,\varepsilon)} \frac{\|\nabla^2 V\| + |\nabla V|}{V(x) \wedge (\Phi \circ V)(x)} = 0. \end{aligned} \tag{2.1}$$

Theorem 2.1. *Assume* **(A1)**. *Then* (1.3) *is well-posed, the associated Markov semigroup P_t has a unique invariant probability measure μ such that $\mu(\Phi(\varepsilon_0 V)) < \infty$ for some $\varepsilon_0 \in (0,1)$, and*

$$\lim_{t\to\infty} \|P_t^* \nu - \mu\|_{var} = 0, \quad \nu \in \mathscr{P}. \tag{2.2}$$

Moreover:

(1) *If* $\Phi(r) \geq \delta r$ *for some constant* $\delta > 0$ *and all* $r \geq 0$, *then there exist constants* $c > 1, \lambda > 0$ *such that*

$$\|P_t^*\mu_1 - P_t^*\mu_2\|_V \leq c\mathrm{e}^{-\lambda t}\|\mu_1 - \mu_2\|_V, \quad \mu_1, \mu_2 \in \mathscr{P}, t \geq 0. \tag{2.3}$$

In particular,

$$\|P_t^*\nu - \mu\|_V \leq c\mathrm{e}^{-\lambda t}\|\nu - \mu\|_V, \quad \nu \in \mathscr{P}, t \geq 0.$$

(2) *Let* $H(r) := \int_0^r \frac{\mathrm{d}s}{\Phi(s)} < \infty$ *for* $r \geq 0$. *If* Φ *is convex, then there exist constants* $k > 1, \lambda > 0$ *such that*

$$\|P_t^*\delta_x - \mu\|_V \leq k\big\{1 + H^{-1}(H(V(x)) - k^{-1}t)\big\}\mathrm{e}^{-\lambda t}, \quad x \in \mathbb{R}^d, t \geq 0, \tag{2.4}$$

where H^{-1} *is the inverse of* H *with* $H^{-1}(r) := 0$ *for* $r \leq 0$. *Consequently, if* $H(\infty) < \infty$ *then there exist constants* $c, \lambda, t^* > 0$ *such that*

$$\|P_t^*\mu_1 - \mu_2\|_V \leq c\mathrm{e}^{-\lambda t}\|\mu_1 - \mu_2\|_{var}, \quad t \geq t^*, \mu_1, \mu_2 \in \mathscr{P}. \tag{2.5}$$

To illustrate this result, we present below a consequence which covers the situation of [23, Theorem 2.10] where

$$\langle b^{(1)}(x), x\rangle \leq c_1 - c_2|x|^{1+p}, \quad |b^{(1)}(x)| \leq c_1(1 + |x|)^p$$

holds for some constants $c_1, c_2 > 0$ and $p \geq 1$. Indeed, Corollary 2.2 implies the exponential ergodicity under the weaker condition

$$\langle b^{(1)}(x), x\rangle \leq c_1 - c_2|x|^{1+p}, \quad |b^{(1)}(x)| \leq c_1(1 + |x|)^{p+1} \tag{2.6}$$

for some constants $p, c_1, c_2 > 0$ (p may smaller than 1, $|b^{(1)}|$ may have higher order growth), since in this case, (2.7) and (2.8) hold for $\phi(r) := (1 + r)^{\frac{1+p}{2}}$, and (2.9) holds for $\psi(r) := (1 + r^2)^q$ for any $q > 0$ when $p \geq 1$.

Corollary 2.2. *Assume* **(A1)**(1) *and let* $b^{(1)}$ *satisfy*

$$\langle b^{(1)}(x), x\rangle \leq c_1 - c_2\phi(|x|^2), \quad |b^{(1)}(x)| \leq c_1\phi(|x|^2), \quad x \in \mathbb{R}^d \tag{2.7}$$

for some constants $c_1, c_2 > 0$ *and increasing function* $\phi : [0, \infty) \to [1, \infty)$ *with*

$$\alpha := \liminf_{r\to\infty} \frac{\log \phi(r)}{r} > \frac{1}{2}. \tag{2.8}$$

Then

(1) (1.3) *is well-posed,* P_t *has a unique invariant probability measure* μ *such that* $\mu(V) < \infty$ *and* (2.3) *hold for* $V := \mathrm{e}^{(1+|\cdot|^2)^\theta}$ *with* $\theta \in ((1 - \alpha)^+, \frac{1}{2})$. *In general, for any increasing function* $1 \leq \psi \in C^2([1, \infty))$ *satisfying*

$$\liminf_{r\to\infty} \frac{\psi'(r)\phi(r)}{\psi(r)} > 0, \quad \lim_{r\to\infty} \frac{\psi''(r)r}{\psi(r)} = 0, \tag{2.9}$$

$\mu(V) < \infty$ *and* (2.3) *hold for* $V := \psi(|\cdot|^2)$.

(2) *If* $\int_0^\infty \frac{\mathrm{d}s}{\phi(s)} < \infty$, *then* (2.5) *holds* $V := (1+|\cdot|^2)^q (q > 0)$ *and some constants* $c, \lambda, t^* > 0$.

Remark 2.1. We have the following assertions on the invariant probability measure μ and the ergodicity in Wasserstein distance and relative entropy.

(1) According to [3, Corollary 1.6.7 and Theorem 3.4.2], **(A1)** implies that μ has a strictly positive density function $\rho \in H^{1,p}_{loc}$, the space of functions f such that $fg \in H^{1,2}$ for all $g \in C_0^\infty(\mathbb{R}^d)$. Moreover, by [3, Theorem 3.1.2], when σ is Lipschitz continuous and $\mu(|b|^2) < \infty$, we have $\sqrt{\rho} \in H^{1,2}$. So, when (2.7) holds for $\phi(r) \sim r^p$ for some $p > \frac{1}{2}$ and large $r > 0$, Corollary 2.2(1) implies that μ has density with $\sqrt{\rho} \in H^{1,2}$. See also [18] and [19] for different type global regularity estimates on ρ under integrability conditions.

(2) Let $V := (1 + |\cdot|^2)^{\frac{p}{2}}$ for some $p \geq 1$. By [15, Theorem 6.15], there exists a constant $c(p) > 0$ such that

$$\mathbb{W}_p(\mu, \nu)^p \leq c(p)\|\mu - \nu\|_V,$$

where

$$\mathbb{W}_p(\mu_1, \mu_2) := \inf_{\pi \in \mathscr{C}(\mu_1,\mu_2)} \left(\int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^p \pi(\mathrm{d}x, \mathrm{d}y) \right)^{\frac{1}{p}}$$

for $\mathscr{C}(\mu_1, \mu_2)$ being the set of couplings for μ_1 and μ_2. So, by Corollary 2.2, if **(A1)** holds with $\Phi(r) \geq \delta r$ for some $\delta > 0$, then there exist constants $c, \lambda > 0$ such that

$$\mathbb{W}_p(P_t^*\nu, \mu)^p \leq c(1 + \nu(|\cdot|^p))\mathrm{e}^{-\lambda t}, \quad t \geq 0, \nu \in \mathscr{P};$$

and if moreover Φ is convex with $\int_0^\infty \frac{\mathrm{d}s}{\Phi(s)} < \infty$, then there exist constants $c, \lambda, t^* > 0$ such that

$$\mathbb{W}_p(P_t^*\nu, \mu)^p \leq c\mathrm{e}^{-\lambda t}\|\mu - \nu\|_{var}, \quad t \geq t^*, \nu \in \mathscr{P}.$$

(3) When $b^{(1)}$ is Lipschitz continuous, the log-Harnack inequality in [24, Theorem 4.1] implies

$$\mathrm{Ent}(P_t^*\nu|\mu) \leq \frac{c'}{1 \wedge t}\mathbb{W}_2(\nu, \mu)^2, \quad \nu \in \mathscr{P}, t > 0$$

for some constant $c' > 0$, where $\mathrm{Ent}(\nu|\mu)$ is the relative entropy. Thus, by Corollary 2.2, if **(A1)** holds for $V(x) := 1 + |x|^2$ and $\Phi(r) \geq \delta r$ for some constant $\delta > 0$, then there exist constants $c, \lambda > 0$ such that

$$\mathrm{Ent}(P_t^*\nu|\mu) \leq c(1 + \nu(|\cdot|^2))\mathrm{e}^{-\lambda t}, \quad t \geq 1, \nu \in \mathscr{P};$$

and if moreover Φ is convex with $\int_0^\infty \frac{\mathrm{d}s}{\Phi(s)} < \infty$, then there exist $c, \lambda, t^* > 0$ such that

$$\mathrm{Ent}(P_t^*\nu|\mu) \leq c\mathrm{e}^{-\lambda t}\|\mu - \nu\|_{var}, \quad t \geq t^*, \nu \in \mathscr{P}.$$

2.2 Singular reflecting SDEs

Consider the following reflecting SDE on $D \neq \mathbb{R}^d$:

$$\mathrm{d}X_t = b(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t + \mathbf{n}(X_t)\mathrm{d}l_t, \quad t \geq 0, \tag{2.10}$$

where $\partial D \in C_b^{2,L}$ which is defined as follows.

Definition 2.1. Let ρ_∂ be the distance function to ∂D. For any $k \in \mathbb{N}$, we write $\partial D \in C_b^k$ if there exists a constant $r_0 > 0$ such that the polar coordinate around ∂D

$$\partial D \times [-r_0, r_0] \ni (\theta, r) \mapsto \theta + r\mathbf{n}(\theta) \in B_{r_0}(\partial D) := \{x \in \mathbb{R}^d : \rho_\partial(x) \le r_0\}$$

is a C^k-diffeomorphism. We write $\partial D \in C_b^{k,L}$, if it is C_b^k with $\nabla^k \rho_\partial$ being Lipschitz continuous on $B_{r_0}(\partial D)$.

We also need heat kernel estimates for the Neumann semigroup $\{P_t^\sigma\}_{t\ge 0}$ generated by

$$L^\sigma := \frac{1}{2}\mathrm{tr}\big(\sigma_t\sigma_t^*\nabla^2\big).$$

For any $\varphi \in C_b^2(\bar{D})$, let $P_t^\sigma\varphi$ be the solution of solve the PDE

$$\partial_t u_t = L^\sigma u_t, \quad \nabla_{\mathbf{n}} u_t|_{\partial D} = 0 \text{ for } s > 0, u_0 = \varphi. \tag{2.11}$$

We will prove the exponential ergodicity of (2.10) under the following assumption.

(A2) $\partial D \in C_b^{2,L}$ and the following conditions hold.

(1) **(A1)** holds for $\bar{D}$ replacing $\mathbb{R}^d$, and there exists $r_0 > 0$ such that

$$\nabla_{\mathbf{n}(x)} V(y) \le 0, \quad x \in \partial D, |y - x| \le r_0. \tag{2.12}$$

(2) For any $\varphi \in C_b^2(\bar{D})$, the PDE (2.11) has a unique solution $P_t^\sigma\varphi \in C_b^{1,2}(\bar{D})$, such that for some constant $c > 0$ we have

$$\|\nabla^i P_t^\sigma\varphi\|_\infty \le c(1 \wedge t)^{-\frac{1}{2}}\|\nabla^{i-1}\varphi\|_\infty, \quad t > 0,\ i = 1, 2, \varphi \in C_b^2(\bar{D}),$$

where $\nabla^0\varphi := \varphi$.

As explained in [21, Remark 2.2(2)] that, **(A2)**(2) holds if D is bounded and σ is Hölder continuous. Moreover, (2.12) is trivial when ∂D is bounded, since in this case we may take $1 \le \tilde{V} \in C^2(\mathbb{R}^d)$ such that $\tilde{V} = 1$ on $\partial_{r_0}(\partial D)$ and $\tilde{V} = V$ outside a compact set, so that (2.1) remains true for $\tilde{V}$ replacing V. Similarly, (2.12) holds for $V(x_1, x_2) := V_1(x_1) + V_2(x_2)$ and $D = D_1 \times \mathbb{R}^l$ where $l \in \mathbb{N}$ is less than d, $\partial D_1 \subset \mathbb{R}^{d-l}$ is bounded, and $V_1 = 1$ in a neighborhood of ∂D_1.

Theorem 2.3. *Assume* **(A2)**. *Then all assertions in Theorem 2.1 hold for the reflecting SDE* (2.10).

2.3 Singular McKean-Vlasov SDEs with or without reflection

We now consider the SDE (1.1) for $D = \mathbb{R}^d$ or D being a $C_b^{2,L}$ domain, where in the first case we set $l_t = 0$.

Theorem 2.4. *Assume that for any $\nu \in \mathscr{P}$, $(\sigma, b(\cdot, \nu))$ satisfies* **(A1)** *for $D = \mathbb{R}^d$ or* **(A2)** *for $D \ne \mathbb{R}^d$, and that*

$$|b(x, \mu_1) - b(x, \mu_2)| \le \kappa\|\mu_1 - \mu_2\|_{var}, \quad x \in \bar{D}, \mu_1, \mu_2 \in \mathscr{P} \tag{2.13}$$

holds for some constant $\kappa > 0$. Then:

(1) (1.1) *is well-posed for any initial value.*

(2) *If* $\kappa > 0$ *is small enough and* Φ *is convex with* $\int_0^\infty \frac{\mathrm{d}s}{\Phi(s)} < \infty$*, then* P_t^* *has a unique invariant probability measure* μ*,* $\mu(\Phi(\varepsilon_0 V)) < \infty$ *holds for some constant* $\varepsilon_0 > 0$*, and there exist constants* $c, \lambda > 0$ *such that*

$$\|P_t^*\nu - \mu\|_{var} \le c\mathrm{e}^{-\lambda t}\|\mu - \nu\|_{var}, \quad t \ge 0, \nu \in \mathscr{P}. \tag{2.14}$$

Remark 2.2. According to the proof of (2.3), when $b(x,\mu) = b(x)$ does not depend on μ, (2.14) implies

$$\|P_t^*\mu_1 - P_t^*\mu_2\|_{var} \le 2c\mathrm{e}^{-\lambda t}\|\mu_1 - \mu_2\|_{var}, \quad t \ge 0, \mu_1, \mu_2 \in \mathscr{P}.$$

However, the argument is no longer valid when $b(x,\mu)$ depends on μ, since in this case P_t^* is nonlinear, so that the following formula fails:

$$P_t^*\big\{\varepsilon\mu_1 + (1-\varepsilon)\mu_2\big\} = \varepsilon P_t^*\mu_1 + (1-\varepsilon)P_t^*\mu_2, \quad \varepsilon \in (0,1), \mu_1, \mu_2 \in \mathscr{P}.$$

3 Some lemmas

We first present a result for time dependent reflecting SDEs, then solve the elliptic equation with Neumann boundary condition when $D \ne \mathbb{R}^d$ which will be used to make Zvonkin's transform, and finally establish a general result on the uniform ergodicity of distribution dependent SDEs.

3.1 The time dependent setting

We first consider the following time dependent SDE with reflection when ∂D exists:

$$\mathrm{d}X_t = b_t(X_t)\mathrm{d}t + \sigma_t(X_t)\mathrm{d}W_t + \mathbf{n}(X_t)\mathrm{d}l_t, \quad t \ge 0. \tag{3.1}$$

For any $T > 0$ and $p, q > 1$, let $\tilde{L}_q^p(T)$ denote the class of measurable functions f on $[0,T] \times \bar{D}$ such that

$$\|f\|_{\tilde{L}_q^p(T)} := \sup_{z \in \bar{D}} \bigg(\int_0^T \|1_{B(z,1)} f_t\|_{L^p}^q \mathrm{d}t\bigg)^{\frac{1}{q}} < \infty.$$

For any $\epsilon > 0$, let $\tilde{H}_q^{\epsilon,p}(T)$ be the space of $f \in \tilde{L}_q^p$ with

$$\|f\|_{\tilde{H}_q^{\epsilon,p}(T)} := \sup_{z \in \bar{D}} \bigg(\int_0^T \|f_t\|_{\mathbb{H}^{\epsilon,p}}^q \mathrm{d}t\bigg)^{\frac{1}{q}} < \infty.$$

We will study the well-posedness, strong Feller property and irreducibility under the following assumptions for $D = \mathbb{R}^d$ and $D \ne \mathbb{R}^d$ respectively.

(A3) Let $T > 0, D = \mathbb{R}^d$, $a_t(x) := (\sigma_t\sigma_t^*)(x)$ and $b_t(x,\mu) = b_t^{(0)}(x) + b_t^{(1)}(x,\mu)$.

(1) a is invertible with $\|a\|_\infty + \|a^{-1}\|_\infty < \infty$ and

$$\lim_{\varepsilon\to 0} \sup_{|x-y|\le\varepsilon, t\in[0,T]} \|a_t(x) - a_t(y)\| = 0.$$

(2) There exist $(p_0, q_0), (p_1, q_1) \in \mathscr{K} := \{(p,q) : p, q \in (2,\infty), \frac{d}{p} + \frac{2}{q} < 1\}$ such that

$$|b^{(0)}| \in \tilde{L}_{q_0}^{p_0}, \quad \|\nabla\sigma\| \in \tilde{L}_{q_1}^{p_1}.$$

(3) There exist constants $K, \varepsilon > 0$, increasing $\phi \in C^1([0,\infty);[1,\infty))$ with $\int_0^\infty \frac{\mathrm{d}s}{r+\phi(s)} = \infty$, and a compact function $V \in C^2(\mathbb{R}^d;[1,\infty))$ such that

$$\sup_{B(x,\varepsilon)} \big\{|\nabla V| + \|\nabla^2 V\|\big\} \le KV(x),$$
$$\langle b_t^{(1)}(x), \nabla V(x)\rangle + \varepsilon|b_t^{(1)}(x)| \sup_{B(x,\varepsilon)} \|\nabla^2 V\| \le K\phi(V(x)), \quad (t,x) \in [0,T]\times\mathbb{R}^d.$$

When $D \neq \mathbb{R}^d$, we consider the following time dependent differential operator on $\bar{D}$:

$$L_t^\sigma := \frac{1}{2}\mathrm{tr}\big(\sigma_t\sigma_t^*\nabla^2\big), \quad t \in [0,T]. \tag{3.2}$$

Let $\{P_{s,t}^\sigma\}_{T\ge t_1\ge t\ge s\ge 0}$ be the Neumann semigroup on $\bar{D}$ generated by L_t^σ; that is, for any $\varphi \in C_b^2(\bar{D})$, and any $t \in (0,T]$, $(P_{s,t}^\sigma\varphi)_{s\in[0,t]}$ is the unique solution of the PDE

$$\partial_s u_s = -L_s^\sigma u_s, \quad \nabla_{\mathbf{n}} u_s|_{\partial D} = 0 \text{ for } s \in [0,t), u_t = \varphi. \tag{3.3}$$

For any $t > 0$, let $C_b^{1,2}([0,t]\times\bar{D})$ be the set of functions $f \in C_b([0,t]\times\bar{D})$ with bounded and continuous derivatives $\partial_t f, \nabla f$ and $\nabla^2 f$.

(A4) $D \in C_b^{2,L}$, **(A3)** holds with V satisfying (2.12) holds for some $r_0 > 0$. Moreover, for any $\varphi \in C_b^2(\bar{D})$ and $t \in (0,T]$, the PDE (2.11) has a unique solution $P_{\cdot,t}^\sigma\varphi \in C_b^{1,2}([0,t]\times\bar{D})$, such that for some constant $c > 0$ we have

$$\|\nabla^i P_{s,t}^\sigma\varphi\|_\infty \le c(t-s)^{-\frac{1}{2}}\|\nabla^{i-1}\varphi\|_\infty, \quad 0 \le s < t \le T, \ i = 1,2, \varphi \in C_b^2(\bar{D}). \tag{3.4}$$

We have the following result, where the well-posedness for $D = \mathbb{R}^d$ has been addressed in [12].

Lemma 3.1. *Assume* **(A3)** *for* $D = \mathbb{R}^d$ *and* **(A4)** *for* $D \neq \mathbb{R}^d$. *Then* (3.1) *is well-posed up to time* T. *Moreover, for any* $t \in (0,T]$,

$$\lim_{\bar{D}\ni y\to x} \|P_t^*\delta_x - P_t^*\delta_y\|_{var} = 0, \quad t \in (0,T], x \in \bar{D}, \tag{3.5}$$

and P_t *has probability density (i.e. heat kernel)* $p_t(x,y)$ *such that*

$$\inf_{x,y\in\bar{D}\cap B_N,\ \rho_\partial(y)\ge N^{-1}} p_t(x,y) > 0, \quad N > 1, t \in (0,T], \tag{3.6}$$

where $\inf\emptyset := \infty$.

Proof. (a) The well-posedness. For any $n \geq 1$, let

$$b^n := 1_{B_n} b^{(1)} + b^{(0)}.$$

Since $b^{(1)}$ is locally bounded, by [22, Theorem 1.1] for $D = \mathbb{R}^d$ and [21, Theorem 2.2] for $D \neq \mathbb{R}^d$, for any $x \in \bar{D}$, the following SDE is well-posed:

$$\mathrm{d}X_t^{x,n} = b^n(X_t^{x,n})\mathrm{d}t + \sigma(X_t^{x,n})\mathrm{d}W_t + \mathbf{n}(X_t^{x,n})\mathrm{d}l_t^{x,n}, \quad X_0^{x,n} = x.$$

Let $\tau_n^x := \inf\{t \geq 0 : |X_t^{x,n}| \geq n\}$. Then $X_t^{x,n}$ solves (1.3) up to time τ_n^x, and by the uniqueness we have

$$X_t^{x,n} = X_t^{x,m}, \quad t \leq \tau_n^x \wedge \tau_m^x, n, m \geq 1.$$

So, it suffices to prove that $\tau_n^x \to \infty$ as $n \to \infty$.

Let $L_t^0 := L_t^\sigma + \nabla_{b_t^{(0)}}$. By [22, Theorem 3.1] for $D = \mathbb{R}^d$ and [21, Lemma 2.6] for $D \neq \mathbb{R}^d$, **(A3)** implies that for any $\lambda \geq 0$, the PDE

$$(\partial_t + L_t^0)u_t = \lambda u_t - b_t^{(0)}, \quad t \in [0,T], u_T = 0, \nabla_{\mathbf{n}} u_t|_{\partial D} = 0 \tag{3.7}$$

has a unique solution $u \in \tilde{H}_{q_0}^{p_0}(T)$, and there exist constants $\lambda_0, c, \theta > 0$ such that

$$\lambda^\theta(\|u\|_\infty + \|\nabla u\|_\infty) + \|\partial_t u\|_{\tilde{L}_{q_0}^{p_0}(T)} + \|\nabla^2 u\|_{\tilde{L}_{q_0}^{p_0}(T)} \leq c, \quad \lambda \geq \lambda_0. \tag{3.8}$$

So, we may take $\lambda \geq \lambda_0$ such that

$$\|u\|_\infty + \|\nabla u\|_\infty \leq \varepsilon, \tag{3.9}$$

where we take $\varepsilon \leq r_0$ when ∂D exists. Let $\Theta_t(x) = x + u_t(x)$. By (2.12) and (3.9) for $\varepsilon \leq r_0$ when ∂D exists, we have

$$\langle \nabla V(Y_t^{x,n}), \mathbf{n}(X_t^{x,n})\rangle \mathrm{d}l_t^{x,n} \leq 0.$$

So, by Itô's formula, $Y_t^{x,n} := \Theta_t(X_t^{x,n})$ satisfies

$$\mathrm{d}Y_t^{x,n} = \big\{1_{B_n} b_t^{(1)} + \lambda u_t + 1_{B_n}\nabla_{b_t^{(1)}} u_t\big\}(X_t^{x,n})\mathrm{d}t + \{(\nabla\Theta_t)\sigma_t\}(X_t^{x,n})\mathrm{d}W_t + \mathbf{n}(X_t^n)\mathrm{d}l_t^n. \tag{3.10}$$

By (3.9) and **(A3)**(3) with (2.12) when $\partial D \neq \emptyset$, there exists a constant $c_0 > 0$ such that for some martingale M_t,

$$\begin{aligned}
&\mathrm{d}\{V(Y_t^{x,n}) + M_t\} \\
&\leq \Big[\big\langle\{b^{(1)} + \nabla_{b^{(1)}} u_t\}(X_t^{x,n}), \nabla V(Y_t^{x,n})\big\rangle + c_0(|\nabla V(Y_t^{x,n})| + \|\nabla^2 V(Y_t^{x,n})\|)\Big]\mathrm{d}t \\
&\leq \Big\{\langle b^{(1)}(X_t^{x,n}), \nabla V(X_t^{x,n})\rangle + \varepsilon|b^{(1)}(X_t^{x,n})| \sup_{B(X_t^{x,n},\varepsilon)} \|\nabla^2 V\| + c_0 K V(Y_t^{x,n})\Big\}\mathrm{d}t \\
&\leq \big\{K\phi(V(X_t^{x,n})) + c_0 K V(Y_t^{x,n})\big\}\mathrm{d}t \leq K\big\{\phi((1+\varepsilon K)V(Y_t^{x,n})) + c_0 V(Y_t^{x,n})\big\}\mathrm{d}t, \quad t \leq \tau_n^x.
\end{aligned}$$

Letting $H(r) := \int_0^r \frac{\mathrm{d}s}{r + \phi((1+\varepsilon K)s)}$, by Itô's formula and noting that $\phi' \geq 0$, we find a constant $c_1 > 0$ such that

$$\mathrm{d}H(V(Y_t^{x,n})) \leq c_1 \mathrm{d}t + \mathrm{d}\tilde{M}_t, \quad t \in [0, \tau_n^x]$$

holds for some martingale $\tilde{M}_t$. Thus,

$$\mathbb{E}[(H\circ V)(Y^{x,n}_{t\wedge\tau^x_n})] \le V(x+u(x)) + c_1 t,\quad t\ge 0, n\ge 1.$$

Since (3.9) and $|z|\ge n$ imply $|\Theta_t(z)|\ge |z|-|u(z)|\ge n-\varepsilon$, we derive

$$\mathbb{P}(\tau^x_n\le t)\le \frac{V(x+\Theta_0(x))+c_1t}{\inf_{|y|\ge n-\varepsilon} H(V(y))} =: \varepsilon_{t,n}(x),\quad t>0. \tag{3.11}$$

Since $\lim_{|x|\to\infty} H(V)(x) = \int_0^\infty \frac{\mathrm{d}s}{s+\phi((1+\varepsilon K)s)} = \infty$, we obtain $\tau^x_n\to\infty (n\to\infty)$ as desired.

(b) Proof of (3.5). By [17, Proposition 1.3.8], the log-Harnack inequality

$$P_t\log f(y)\le \log P_t f(x) + c|x-y|^2,\quad x,y\in\bar{D}, 0<f\in\mathscr{B}_b(\bar{D})$$

for some constant $c>0$ implies the gradient estimate

$$|\nabla P_t f|^2\le 2cP_t|f|^2,\quad f\in\mathscr{B}_b(\bar{D}),$$

and hence

$$\lim_{y\to x}\|P_t^*\delta_x - P_t^*\delta_y\|_{var}=0,\quad x\in\bar{D}.$$

Let P^n_t be the Markov semigroup associated with X^n_t. Thus, by the log-Harnack inequality in [24, Theorem 4.1] for $D=\mathbb{R}^d$ and in [21, Theorem 4.1] for $D\ne\mathbb{R}^d$, we have

$$\lim_{y\to x}\|(P^n_t)^*\delta_x - (P^n_t)^*\delta_y\|_{var}=0,\quad t\in(0,T]. \tag{3.12}$$

On the other hand, by (3.11) and $X_t=X^n_t$ for $t\le\tau_n$, we obtain

$$\begin{aligned}&\lim_{n\to\infty}\sup_{y\in\bar{D}\cap B(x,1)}\|P_t^*\delta_y-(P^n_t)^*\delta_y\|_{var} = \lim_{n\to\infty}\sup_{|f|\le 1, y\in\bar{D}\cap B(x,1)}|P_tf(y)-P^n_tf(y)|\\ &\le 2\lim_{n\to\infty}\sup_{y\in\bar{D}\cap B(x,1)}\mathbb{P}(\tau^y_n\le t)=0.\end{aligned}$$

Combining this with (3.12) and the triangle inequality, we prove (3.5).

(c) Finally, let $L_t := L^\sigma_t + \nabla_{b_t}$. By Itô's formula, for any $f\in C^2_0((0,T)\times D)$ we have

$$\mathrm{d}f_t(X_t) = (\partial_t + L_t)f_t(X_t)\mathrm{d}t + \mathrm{d}M_t$$

for some martingale M_t, so that $f_0=f_T=0$ yields

$$\int_{(0,T)} P_t\{(\partial_t + L)f_t\}\mathrm{d}t = 0,\quad f\in C^\infty_0((0,T)\times D).$$

By the Harnack inequality as in [1, Theorem 3] (see also [14]), for any $0<s<t\le T$ and $N>1$ with

$$\tilde{B}_N := \big\{x\in\bar{D}\cap B_N:\ \rho_\partial(x)\ge N^{-1}\big\}$$

having positive volume, there exists a constant $c(s,t,N) > 0$ such that the heat kernel $p_t(x,y)$ of P_t satisfies

$$\sup_{\tilde B_N} p_s(x,\cdot) \le c(s,t,N) \inf_{\tilde B_N} p_t(x,\cdot), \quad x \in \bar D. \tag{3.13}$$

Since $\int_{\tilde B_N} p_s(x,y)\mathrm{d}y \to 1$ as $N\to\infty$, this implies $p_t(x,y) > 0$ for any $(t,x,y) \in (0,T]\times\bar D\times D$. In particular, $P_t 1_{\tilde B_N} > 0$. On the other hand, (3.5) implies that $P_t 1_{\tilde B_N}$ is continuous, so that

$$\inf_{x\in\bar D\cap B_N} P_t 1_{\tilde B_N}(x) > 0, \quad t\in(0,T].$$

This together with (3.13) gives

$$\inf_{(\bar D\cap B_N)\times\tilde B_N} p_t \ge \frac{1}{c(s,t,N)} \inf_{x\in\bar D\cap\bar B_N} P_s 1_{\tilde B_N}(x) > 0, \quad 0<s<t\le T.$$

Therefore, (3.6) holds.

□

3.2 Elliptic equation

(A5) $D=\mathbb{R}^d$, σ and b satisfy the following conditions.

(1) $a := \sigma\sigma^*$ is invertible and uniformly continuous with $\|a\|_\infty + \|a^{-1}\|_\infty < \infty$.

(2) $b = b^{(0)} + b^{(1)}$, where $|b^{(0)}| \in \tilde L^p$ for some $p > d$, and $b^{(1)}$ is Lipschitz continuous.

(A6) $\partial D \in C_b^{2,L}$, **(A5)** holds for $\bar D$ replacing $\mathbb{R}^d$, and **(A2)**(2) holds for $L := L^\sigma + \nabla_{b^{(1)}}$ replacing L^σ.

The following lemma extends Theorem 2.10 in [23] for $D=\mathbb{R}^d$ and $b^{(1)}=0$.

Lemma 3.2. *Assume* **(A5)** *for $D=\mathbb{R}^d$ and* **(A6)** *for $D\ne\mathbb{R}^d$. There exist constants $\lambda_0>0$ increasing in $\|b^{(0)}\|_{\tilde L^p}$ such that for any $\lambda\ge\lambda_0$ and any $f\in\tilde L^k$ for some $k\in(1,\infty)$, the elliptic equation*

$$(L-\lambda)u = f,\ \nabla_{\mathbf n} u|_{\partial D} = 0 \text{ if } D\ne\mathbb{R}^d \tag{3.14}$$

has a unique solution $u\in\tilde H^{2,k}$. Moreover, for any $p'\in[k,\infty]$ and $\theta\in[0,2-\frac{d}{k}+\frac{d}{p'})$, there exists a constant $c>0$ increasing in $\|b^{(0)}\|_{\tilde L^p}$ such that

$$\lambda^{\frac12(2-\theta+\frac{d}{p'}-\frac{d}{k})}\|u\|_{\tilde H^{\theta,p'}} + \|u\|_{\tilde H^{2,k}} \le c\|f\|_{\tilde L^k}, \quad f\in\tilde L^k. \tag{3.15}$$

Proof. (a) Let us verify the priori estimate (3.15) for a solution u to (3.14), which in particular implies the uniqueness, since the difference of two solutions solves the equation with $f=0$.

For $u\in\tilde H^{2,k}$ solving (3.14), let

$$\bar u_t = u(1-t), \quad t\in[0,1].$$

By (3.14) we have

$$(\partial_t + L - \lambda)\bar u_t = f(1-t) - u, \quad t\in[0,1], \bar u_1 = 0, \ \nabla_{\mathbf n}\bar u_t|_{\partial D} = 0 \text{ if } D\ne\mathbb R^d.$$

By Theorem 2.1 with $q=q'=2$ in [24] for $D=\mathbb R^d$, and Lemma 2.6 in [21] for $D\ne\mathbb R^d$, there exists a constant $\lambda_1, c_1>1$ increasing in $\|b^{(0)}\|_{\tilde L^p}$ such that

$$\lambda^{\frac12(2-\theta+\frac{d}{p'}-\frac{d}{k})}\|\bar u\|_{\tilde H_2^{\theta,p'}} + \|\bar u\|_{\tilde H_2^{2,k}} \le c_1\|f(1-t)-u\|_{\tilde L_2^k} \le c_1\|f\|_{\tilde L^k} + c_1\|u\|_{\tilde L^k}. \tag{3.16}$$

Taking $\theta=0, p=p'$ noting that $c' := \|1-g\|_{L^2([0,1])} = \frac{1}{\sqrt3}$ for $g_t := 1-t$, we obtain

$$\lambda^{\frac12(2-\theta)}\|u\|_{\tilde L^k} \le \sqrt3\, c_1\big(\|f\|_{\tilde L^k} + \|u\|_{\tilde L^k}\big), \quad \lambda\ge\lambda_1.$$

By letting $\lambda_0>\lambda_1$ such that

$$\lambda_0^{\frac12(2-\theta)} \ge 2\sqrt3\, c_1,$$

we obtain

$$\|u\|_{\tilde L^k} \le \|f\|_{\tilde L^k}, \quad \lambda\ge\lambda_0.$$

Combining this with (3.16) implies (3.15) for some constant $c>0$.

(b) Existence of solution for bounded f. Since the map

$$\mathscr B_b(\mathbb R^d)\ni g\mapsto H(g) := f + \int_0^1 \mathrm e^{-\lambda s} g\mathrm ds$$

is contractive under the uniform norm, there exists a unique $g\in\mathscr B_b(\mathbb R^d)$ such that

$$g = f + \int_0^1 \mathrm e^{-\lambda s} g\mathrm ds. \tag{3.17}$$

By [24, Theorem 2.1] for $D=\mathbb R^d$ and [21, Lemma 2.6] for $D\ne\mathbb R^d$, for any $\lambda\ge0$ the PDE

$$(\partial_t + L)\bar u_t = \lambda\bar u_t + g, \quad t\in[0,1], \bar u_1 = 0, \ \nabla_{\mathbf n}\bar u_t|_{\partial D} = 0 \text{ if } \partial D\ne\emptyset \tag{3.18}$$

has a unique solution $\bar u\in\tilde H_n^{2,n}(1)\cap\tilde H_\infty^{1,\infty}(1)$ and $(\partial_t+\nabla_{b^{(1)}})\bar u\in\tilde L_n^n(1)$ for any $n\ge1$. Thus,

$$u := \int_0^1 \mathrm e^{-\lambda s}\bar u_s\mathrm ds \in \tilde H^{2,k}.$$

Since for each $i=1,2$, ∇^i is a continuous linear operator from $\tilde H^{i,k}$ to $\tilde L^k$, (3.17) and (3.18) imply a.e.

$$Lu = \int_0^1 \mathrm e^{-\lambda s} L\bar u_s\mathrm ds = g + \lambda u - \int_0^1 \mathrm e^{-\lambda s}\partial_s\bar u_s\mathrm ds = g+\lambda u + \bar u_0. \tag{3.19}$$

On the other hand, let X_t solve (1.3). By Itô's formula,

$$\mathrm d\bar u_t(X_t) = (\partial_t + L)\bar u(X_t)\mathrm dt + \mathrm dM_t = \{\lambda\bar u_t + g\}(X_t)\mathrm dt + \mathrm dM_t, \quad t\in[0,1]$$

holds for some martingale M_t. This together with $\bar u_1=0$ gives

$$0=\bar u_0 \mathrm{e}^{\lambda}+\int_0^1 \mathrm{e}^{\lambda(1-s)}P_s g\mathrm{d}s,$$

so that $\bar u_0=-\int_0^1 \mathrm{e}^{-\lambda s}P_s g\mathrm{d}s$. Combining this with (3.17) and (3.19), we conclude that u solves (3.14).

(c) Existence of solution for $f\in\tilde L^k$. Let $\{f_n\}_{n\ge1}\subset\mathscr{B}_b(\bar D)$ such that $\|f_n-f\|_{\tilde L^k}\to0$ as $n\to\infty$. Let u^n solves (3.14) for f_n replacing f. Then

$$L(u_n-u_m)=f_n-f_m,\quad n,m\ge1.$$

By (3.15),

$$\lim_{n,m\to\infty}\big\{\|u_n-u_m\|_{\tilde H^{\theta,p'}}+\|\nabla^2(u_n-u_m)\|_{\tilde L^k}\big\}=0,$$

so that $u:=\lim_{n\to\infty}u_n$ exists in $\tilde H^{\theta,p'}\cap\tilde H^{2,k}$, which solves (3.14). □

3.3 Uniform ergodicity for distribution dependent SDEs

We now consider (1.1). For any $\gamma\in\mathscr{P}$, consider the following SDE with fixed distribution parameter:

$$\mathrm{d}X_t^\gamma=b(X_t^\gamma,\gamma)+\sigma(X_t^\gamma)\mathrm{d}W_t+\mathbf{n}(X_t^\gamma)\mathrm{d}l_t^\gamma. \tag{3.20}$$

The following result says that if (3.20) is uniformly ergodic uniformly in γ, and if the dependence of $b(x,\mu)$ on μ is weak enough, then (1.1) is uniformly ergodic.

Lemma 3.3. *Assume that for each $\gamma\in\mathscr{P}$ the SDE (3.20) is well-posed, the associated Markov semigroup P_t^γ satisfies*

$$\|(P_t^\gamma)^*\mu_1-(P_t^\gamma)^*\mu_2\|_{var}\le c\mathrm{e}^{-\lambda t}\|\mu_1-\mu_2\|_{var},\quad t\ge0,\gamma,\mu_1,\mu_2\in\mathscr{P} \tag{3.21}$$

for some constants $c,\lambda>0$. Then (1.1) is well-posed. Moreover:

(1) *If (2.13) holds for some $\kappa\in\big(0,\frac{\sqrt{\lambda}}{2\sqrt{\log(2c)}}\big)$, then P_t^* associated with (1.1) has a unique invariant probability measure μ.*

(2) *If (2.13) holds for some $\kappa\in(0,\hat\kappa)$, where*

$$\hat\kappa:=\sup\Big\{\kappa>0:\frac{(c\kappa)^2(2c)^{\frac{2\kappa^2}{\lambda}}}{\lambda+\kappa^2}<\frac12\Big\}>0,$$

then there exists a constant $c'>0$ such that

$$\|P_t^*\nu-\mu\|_{var}\le c'\mathrm{e}^{-\lambda' t}\|\nu-\mu\|_{var},\quad t\ge0,\nu\in\mathscr{P} \tag{3.22}$$

holds for

$$\lambda':=-\frac{\lambda}{\log(2c)}\log\Big(\frac12+\frac{(c\kappa)^2(2c)^{\frac{2\kappa^2}{\lambda}}}{\lambda+\kappa^2}\Big)>0.$$

Proof. The well-posedness follows from that of (3.20) and [21, Theorem 3.2] for $k=0$.

(a) Existence and uniqueness of μ. For any $\gamma\in\mathscr{P}$, (3.21) implies that P_t^γ has a unique invariant probability measure μ_γ. It suffices to prove that the map $\gamma\mapsto\mu_\gamma$ has a unique fixed point μ, which is the unique invariant probability measure of P_t^*.

For $\gamma_1,\gamma_2\in\mathscr{P}$, (3.20) implies

$$\|(P_t^{\gamma_1})^*\mu_{\gamma_2}-\mu_{\gamma_1}\|_{var}\le c\mathrm{e}^{-\lambda t}\|\mu_{\gamma_2}-\mu_{\gamma_1}\|_{var},\quad t\ge 0. \tag{3.23}$$

On the other hand, let (X_t^1,X_t^2) solve the SDEs

$$\mathrm{d}X_t^i=b(X_t^i,\gamma_i)+\sigma(X_t^i)\mathrm{d}W_t+\mathbf{n}(X_t^i)\mathrm{d}l_t^i,\quad i=1,2$$

with $X_0^1=X_0^2$ having distribution μ_{γ_2}. Since μ_{γ_2} is $(P_t^{\gamma_2})^*$-invariant, we have

$$\mathscr{L}_{X_t^2}=(P_t^{\gamma_2})^*\mu_{\gamma_2}=\mu_{\gamma_2},\quad \mathscr{L}_{X_t^1}=(P_t^{\gamma_1})^*\mu_{\gamma_2},\quad t\ge 0. \tag{3.24}$$

By (2.13),

$$R_t=\mathrm{e}^{\int_0^t\langle\{\sigma^*(\sigma\sigma^*)^{-1}[b(\cdot,\gamma_2)-b(\cdot,\gamma_1)]\}(X_s^1),\mathrm{d}W_s\rangle-\frac{1}{2}\int_0^t|\{\sigma^*(\sigma\sigma^*)^{-1}[b(\cdot,\gamma_2)-b(\cdot,\gamma_1)]\}(X_s^1)|^2\mathrm{d}s},\quad t\ge 0$$

is a martingale, and by Girsanov's theorem, for any $t>0$,

$$\tilde W_r:=W_r-\int_0^r\{\sigma^*(\sigma\sigma^*)^{-1}[b(\cdot,\gamma_2)-b(\cdot,\gamma_1)]\}(X_s^1)\mathrm{d}s,\quad r\in[0,t]$$

is a Brownian motion under $\mathbb{Q}_t:=R_t\mathbb{P}$. Reformulating the SDE for X_r^1 as

$$\mathrm{d}X_r^1=b(X_r^1,\gamma_2)\mathrm{d}r+\sigma(X_r^1)\mathrm{d}\tilde W_r+\mathbf{n}(X_r^1)\mathrm{d}l_r^1,\quad r\in[0,t],$$

by $X_0^1=X_0^2$ and the weak uniqueness, the law of X_t^1 under $\mathbb{Q}_t$ satisfies

$$\mathscr{L}_{X_t^1|\mathbb{Q}_t}=\mathscr{L}_{X_t^2}=(P_t^{\gamma_2})^*\mu_{\gamma_2}.$$

Combining this with (3.24) and Pinsker's inequality, we obtain

$$\begin{aligned}&\|(P_t^{\gamma_1})^*\mu_{\gamma_2}-\mu_{\gamma_2}\|_{var}^2=\|(P_t^{\gamma_1})^*\mu_{\gamma_2}-(P_t^{\gamma_2})^*\mu_{\gamma_2}\|_{var}^2\\&=\sup_{|f|\le 1}\big|\mathbb{E}[f(X_t^1)]-\mathbb{E}[f(X_t^1)R_t]\big|^2\le\big(\mathbb{E}|R_t-1|\big)^2\le 2\mathbb{E}[R_t\log R_t]\\&=2\mathbb{E}_{\mathbb{Q}_t}[\log R_t]=\mathbb{E}_{\mathbb{Q}_t}\int_0^t\big|\{\sigma^*(\sigma\sigma^*)^{-1}[b(\cdot,\gamma_2)-b(\cdot,\gamma_1)]\}(X_s^1)\big|^2\mathrm{d}s.\end{aligned} \tag{3.25}$$

Thus, (2.13) implies

$$\|(P_t^{\gamma_1})^*\mu_{\gamma_2}-\mu_{\gamma_2}\|_{var}^2\le\kappa^2\int_0^t\|\gamma_1-\gamma_2\|_{var}^2\mathrm{d}s=\kappa^2t\|\gamma_1-\gamma_2\|_{var}^2.$$

Combining this with (3.23) and taking $t=\frac{\log(2c)}{\lambda}$, we derive

$$\|\mu_{\gamma_1}-\mu_{\gamma_2}\|_{var}\le\|(P_t^{\gamma_1})^*\mu_{\gamma_1}-\mu_{\gamma_1}\|_{var}+\|(P_t^{\gamma_1})^*-\mu_{\gamma_2}\|_{var}$$

$$\leq \Big\{\kappa\sqrt{t}+c\mathrm{e}^{-\lambda t}\Big\}\|\gamma_1-\gamma_2\|_{var} = \bigg\{\frac{1}{2}+\frac{\kappa\sqrt{\log(2c)}}{\sqrt{\lambda}}\bigg\}\|\gamma_1-\gamma_2\|_{var} =: \delta\|\gamma_1-\gamma_2\|_{var}.$$

When $\kappa < \kappa_0 := \frac{\sqrt{\lambda}}{2\sqrt{\log(2c)}}$, we have $\delta < 1$ so that μ_γ is contractive in γ, hence it has a unique fixed point.

(b) Uniform ergodicity. Let μ be the unique invariant probability measure of P_t^*, and for any $\nu \in \mathscr{P}$ let $(\bar{X}_0, X_0)$ be $\mathscr{F}_0$-measurable such that

$$\mathbb{P}(\bar{X}_0 \neq X_0) = \frac{1}{2}\|\mu-\nu\|_{var}, \quad \mathscr{L}_{\bar{X}_0} = \mu, \quad \mathscr{L}_{X_0} = \nu.$$

Let $\bar{X}_t$ and X_t solve the following SDEs with initial values $\bar{X}_0$ and X_0 respectively:

$$\begin{aligned}
\mathrm{d}\bar{X}_t &= b(\bar{X}_t,\mu)\mathrm{d}t + \sigma(\bar{X}_t)\mathrm{d}W_t + \mathbf{n}(\bar{X}_t)\mathrm{d}\bar{l}_t,\\
\mathrm{d}X_t &= b(X_t,P_t^*\nu)\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t + \mathbf{n}(X_t)\mathrm{d}l_t.
\end{aligned}$$

Since μ is P_t^*-invariant, we have

$$\mathscr{L}_{\bar{X}_t} = (P_t^\mu)^*\mu = P_t^*\mu = \mu. \tag{3.26}$$

Moreover, $\mathscr{L}_{X_t} = P_t^*\nu$ by the definition of P_t^*. Let

$$\bar{R}_t := \mathrm{e}^{\int_0^t \langle \{\sigma^*(\sigma\sigma^*)^{-1}[b(\cdot,\mu)-b(\cdot,P_s^*\nu)]\}(X_s^1),\mathrm{d}W_s\rangle - \frac{1}{2}\int_0^t |\{\sigma^*(\sigma\sigma^*)^{-1}[b(\cdot,\mu)-b(\cdot,P_s^*)]\}(X_s^1)|^2\mathrm{d}s}.$$

Similarly to (3.25), by (2.13), Girsanov's theorem and Pinsker's inequality, we obtain

$$\|(P_t^\mu)^*\nu - P_t^*\nu\|_{var}^2 = \sup_{|f|\leq 1}\big|\mathbb{E}[f(X_t)\bar{R}_t] - \mathbb{E}[f(X_t)]\big|^2 \leq \kappa^2\int_0^t \|\mu - P_s^*\nu\|_{var}^2\mathrm{d}s, \quad t\geq 0.$$

This together with (3.23) for $\gamma_1 = \mu$ and (3.26) gives

$$\begin{aligned}
\|P_t^*\nu-\mu\|_{var}^2 &\leq 2\|P_t^*\nu - (P_t^\mu)^*\nu\|_{var}^2 + 2\|(P_t^\mu)^*\nu-\mu\|_{var}^2\\
&\leq 2\kappa^2\int_0^t \|\mu-P_s^*\nu\|_{var}^2\mathrm{d}s + 2c^2\mathrm{e}^{-2\lambda t}\|\nu-\mu\|_{var}^2, \quad t\geq 0.
\end{aligned}$$

By Gronwall's inequality we obtain

$$\begin{aligned}
\|P_t^*\nu-\mu\|_{var}^2 &\leq \|\mu-\nu\|_{var}^2\bigg(2c^2\mathrm{e}^{-2\lambda t} + 2\kappa^2c^2\int_0^t \mathrm{e}^{-2\lambda s+2\kappa^2(t-s)}\mathrm{d}s\bigg)\\
&\leq \Big\{2c^2\mathrm{e}^{-2\lambda t} + \frac{(c\kappa)^2\mathrm{e}^{2\kappa^2 t}}{\lambda+\kappa^2}\Big\}\|\mu-\nu\|_{var}^2, \quad t\geq 0.
\end{aligned}$$

Taking $t = \hat{t} := \frac{\log(2c)}{\lambda}$, we arrive at

$$\|P_{\hat{t}}^*\nu-\mu\|_{var}^2 \leq \delta_\kappa\|\mu-\nu\|_{var}^2, \quad \nu\in\mathscr{P}$$

for

$$\delta_\kappa := \Big(\frac{1}{2} + \frac{(c\kappa)^2(2c)^{\frac{2\kappa^2}{\lambda}}}{\lambda+\kappa^2}\Big) < 1, \quad \kappa < \hat{\kappa}.$$

So, (3.22) holds for some constant $c' > 0$ due to the semigroup property $P_{t+s}^* = P_t^*P_s^*$. □

To verify condition (3.21), we present below a Harris type theorem on the uniform ergodicity for a family of Markov processes.

Lemma 3.4. *Let (E,ρ) be a metric space and let $\{(P_t^i)_{t\ge 0} : i\in I\}$ be a family of Markov semigroups on $\mathscr{B}_b(E)$. If there exist $t_0>0$ and measurable set $B\subset E$ such that*

$$\alpha := \inf_{i\in I, x\in E} P_{t_0}^i 1_B(x) > 0, \tag{3.27}$$

$$\beta := \sup_{i\in I, x,y\in B} \|(P_{t_1}^i)^*\delta_x - (P_{t_1}^i)^*\delta_y\|_{var} < 2, \tag{3.28}$$

then there exists $c>0$ such that

$$\sup_{i\in I, x,y\in E} \|(P_t^i)^*\delta_x - (P_t^i)^*\delta_y\|_{var} \le c\mathrm{e}^{-\lambda t}, \quad t\ge 0 \tag{3.29}$$

holds for $\lambda := \frac{1}{t_0+t_1}\log\frac{2}{2-\alpha^2(2-\beta)} > 0$.

Proof. The proof is more or less standard. By the semigroup property, we have

$$\begin{aligned}
&\|(P_{t_0+t_1}^i)^*\delta_x - (P_{t_0+t_1}^i)^*\delta_y\|_{var}\\
&= \sup_{|f|\le 1}\bigg|\int_{E\times E}\big(P_{t_1}^i f(x') - P_{t_1}^i f(y')\big)\{(P_{t_0}^i)^*\delta_x\}(\mathrm{d}x')\{(P_{t_0}^i)^*\delta_y\}(\mathrm{d}y')\bigg|\\
&\le \int_{B\times B}\|(P_{t_1}^i)^*\delta_{x'} - (P_{t_1}^i)^*\delta_{y'}\|_{var}\{(P_{t_0}^i)^*\delta_x\}(\mathrm{d}x')\{(P_{t_0}^i)^*\delta_y\}(\mathrm{d}y')\\
&\quad + 2\int_{(B\times B)^c}\{(P_{t_0}^i)^*\delta_x\}(\mathrm{d}x')\{(P_{t_0}^i)^*\delta_y\}(\mathrm{d}y')\\
&\le \beta\{P_{t_0}^i 1_B(x)\}P_{t_0}^i 1_B(y) + 2\big[1 - \{P_{t_0}^i 1_B(x)\}P_{t_0}^i 1_B(y)\big] \le 2-\alpha^2(2-\beta).
\end{aligned}$$

Thus, for $\delta := \frac{2-\alpha^2(2-\beta)}{2} < 1$, we have

$$\|(P_{t_0+t_1}^i)^*\delta_x - (P_{t_0+t_1}^i)^*\delta_y\|_{var} \le \delta\|\delta_x-\delta_y\|_{var}, \quad x,y\in E.$$

Combining this with the semigroup property, we find constants $c>0$ such that (3.29) holds for the claimed $\lambda>0$. □

4 Proofs of main results

Proofs of Theorems 2.1 and 2.3. Obviously, (2.1) implies **(A3)**(3) for any $T>0$ and $\phi(r)=1$, so that by Lemma 3.1, **(A1)** and **(A2)** imply the well-posedness, strong Feller property and irreducibility of (1.3) and (2.10) respectively. According to [6, Theorem 4.2.1], the strong Feller property and the irreducibility imply the uniqueness of invariant probability measure. So, it remains to prove the existence of the invariant probability measure μ and the claimed assertions on the ergodicity.

(a) Let u solve (3.14) for $b = -b^{(0)}$ and large enough $\lambda > 0$ such that (3.15) implies (3.9). Moreover, for $\Theta(x) := x + u(x)$, let $\hat{P}_t$ be the Markov semigroup associated with $Y_t := \Theta(X_t)$, so that

$$\hat{P}_t f(x) = \{P_t(f \circ \Phi)\}(\Phi^{-1}(x)), \quad t \geq 0, x \in \mathbb{R}^d, f \in \mathscr{B}_b(\mathbb{R}^d). \tag{4.1}$$

Since $\lim_{|x|\to\infty} \sup_{|y-x|\leq\varepsilon} \frac{|\nabla V(y)|}{V(x)} = 0$, by (3.9) and $V \geq 1$ we find a constant $\theta \in (0,1)$ such that

$$\theta V(y) \leq V(x) \leq \theta^{-1} V(y), \quad y = \Theta(x), \ x \in \bar{D}. \tag{4.2}$$

Thus, it suffices to prove the desired assertions for $\hat{P}_t$ replacing P_t, where the unique invariant probability measure $\hat{\mu}$ of $\hat{P}_t$ and that μ of P_t satisfies

$$\hat{\mu} = \mu \circ \Theta^{-1}. \tag{4.3}$$

(b) Let X_t^n, Y_t^n and τ_n be in the proof of Lemma 3.1 for the present time-homogenous setting. Since $Y_t^n = Y_t$ and $1_{B_n}(X_t^n) = 1$ for $t \leq \tau_n$, and since $\tau_n \to \infty$ as $n \to \infty$, (3.10) implies

$$\mathrm{d}Y_t = \big\{b^{(1)} + \lambda u + \nabla_{b^{(1)}} u\big\}(X_t)\mathrm{d}t + \{(\nabla\Theta)\sigma\}(X_t)\mathrm{d}W_t + \mathbf{n}(X_t)\mathrm{d}l_t,$$

so that for any $\varepsilon \in (0, 1 \wedge r_0)$, where $r_0 > 0$ is in (2.12) when $\partial D \neq \emptyset$, by Itô's formula and (2.12), we find a constant $c_\varepsilon > 0$ such that

$$\begin{aligned}\mathrm{d}\{V(Y_t) + M_t\} &\leq \Big\{\big\langle \{b^{(1)} + \nabla_{b^{(1)}} u\}(X_t), \nabla V(Y_t)\big\rangle + c_\varepsilon(|\nabla V(Y_t)| + \|\nabla^2 V(Y_t)\|)\Big\}\mathrm{d}t \\ &\leq \Big\{\langle b^{(1)}(X_t), \nabla V(X_t)\rangle + \varepsilon |b^{(1)}(X_t)| \sup_{B(X_t,\varepsilon)} \|\nabla^2 V\| + c_\varepsilon \sup_{B(X_t,\varepsilon)} (|\nabla V| + \|\nabla^2 V\|)\Big\}\mathrm{d}t.\end{aligned}$$

Combining this with (2.1) and (2.12) for $D \neq \mathbb{R}^d$, when $\varepsilon > 0$ is small enough we find constants $c_1, c_2 > 0$ such that

$$\mathrm{d}\{V(Y_t) + M_t\} \leq \{c_1 - c_2\Phi(V(X_t))\}\mathrm{d}t.$$

By (4.2), this implies that for some constant $c_4 > 0$,

$$\mathrm{d}V(Y_t) \leq \big\{c_4 - c_2\Phi(\theta V(Y_t))\big\}\mathrm{d}t - \mathrm{d}M_t. \tag{4.4}$$

Thus,

$$\int_0^t \mathbb{E}\Phi(\theta V(Y_s))\mathrm{d}s \leq \frac{c_4 + V(x)}{c_2} < \infty, \quad t > 0, Y_0 = x \in \Theta(\bar{D}).$$

Since $\Phi(\theta V)$ is a compact function, this implies the existence of invariant probability $\hat{\mu}$ according to the standard Bogoliov-Krylov's tightness argument. Moreover, (4.4) implies $\hat{\mu}(\Phi(\theta V)) < \infty$, so that by (4.2) and (4.3), $\mu(\Phi(\varepsilon_0 V)) < \infty$ holds for $\varepsilon_0 = \theta^2$.

(c) By (3.6), (4.1) and (4.2), any compact set $\mathbf{K} \subset \Theta(\bar{D})$ is a petite set of $\hat{P}_t$, i.e. there exit $t > 0$ and a nontrivial measure ν such that

$$\inf_{x\in\mathbf{K}} \hat{P}_t^* \delta_x \geq \nu.$$

Let $\hat{L}$ be the generator of $\hat{P}_t$. When $\Phi(r) \geq kr$ for some constant $k > 0$, (4.4) implies

$$\hat{L}V(x) \leq k_1 - k_2 V(x), \quad t \geq 0, x \in \Theta(\bar{D}) \tag{4.5}$$

for some constants $k_1, k_2 > 0$. Since $\lim_{|x|\to\infty} V(x) = \infty$ and as observed above that any compact set is a petite set for $\hat{P}_t$, by Theorem 5.2(c) in [7], we obtain

$$\|\hat{P}_t^* \delta_x - \hat{\mu}\|_V \leq c\mathrm{e}^{-\lambda t} V(x), \quad x \in \Theta(\bar{D}), t \geq 0$$

for some constants $c, \lambda > 0$. Thus,

$$\|\hat{P}_t^* \delta_x - \hat{P}_t^* \delta_y\|_V \leq c\mathrm{e}^{-\lambda t}(V(x) + V(y)), \quad t \geq 0, x, y \in \Theta(\bar{D}).$$

Therefore, for any probability measures μ_1, μ_2 on $\Theta(\bar{D})$,

$$\begin{aligned}
&\|\hat{P}_t^* \mu_1 - \hat{P}_t^* \mu_2\|_V = \|\hat{P}_t^*(\mu_1 - \mu_2)^+ - \hat{P}_t^*(\mu_1 - \mu_2)^-\|_V \\
&= \frac{1}{2}\|\mu_1 - \mu_2\|_{var} \Big\| \hat{P}_t^* \frac{2(\mu_1 - \mu_2)^+}{\|\mu_1 - \mu_2\|_{var}} - \hat{P}_t^* \frac{2(\mu_1 - \mu_2)^-}{\|\mu_1 - \mu_2\|_{var}} \Big\|_V \\
&\leq \frac{c}{2}\mathrm{e}^{-\lambda t}\|\mu_1 - \mu_2\|_{var} \Big(\frac{2(\mu_1 - \mu_2)^+}{\|\mu_1 - \mu_2\|_{var}} + \frac{2(\mu_1 - \mu_2)^-}{\|\mu_1 - \mu_2\|_{var}} \Big)(V) \\
&\leq c\mathrm{e}^{-\lambda t}\|\mu_1 - \mu_2\|_V.
\end{aligned}$$

This together with (4.1) and (4.2) implies (2.3) for some constants $c, \lambda > 0$.

(d) Let Φ be convex. By Jensen's inequality and (4.4), $\gamma_t := \theta \mathbb{E}[V(Y_t)]$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\gamma_t \leq \theta c_4 - \theta c_2 \Phi(\gamma_t), \quad t \geq 0. \tag{4.6}$$

Let

$$H(r) := \int_0^r \frac{\mathrm{d}s}{\Phi(s)}, \quad r \geq 0.$$

We aim to prove that for some constant $k > 1$

$$\gamma_t \leq k + H^{-1}(H(\gamma_0) - tk^{-1}), \quad t \geq 0, \tag{4.7}$$

where $H^{-1}(r) := 0$ for $r \leq 0$. We prove this estimate by considering three situations.

(1) Let $\Phi(\gamma_0) \leq \frac{c_4}{c_2}$. Since (4.6) implies $\gamma_t' \leq 0$ for $\gamma_t \geq \Phi^{-1}(\frac{c_4}{c_2})$, so

$$\gamma_t \leq \Phi^{-1}(c_4/c_2), \quad t \geq 0. \tag{4.8}$$

(2) Let $\frac{c_4}{c_2} < \Phi(\gamma_0) \leq \frac{2c_4}{c_2}$. Then (4.6) implies $\gamma_t' \leq 0$ for all $t \geq 0$ so that

$$\gamma_t \leq \Phi^{-1}(2c_4/c_2), \quad t \geq 0. \tag{4.9}$$

(3) Let $\Phi(\gamma_0) > \frac{2c_4}{c_2}$. If

$$t \le t_0 := \inf\Big\{t \ge 0 : \Phi(\gamma_t) \le \frac{2c_4}{c_2}\Big\},$$

then (4.6) implies

$$\frac{\mathrm{d}H(\gamma_t)}{\mathrm{d}t} \le -2c_4,$$

so that

$$H(\gamma_t) \le H(\gamma_0) - 2c_4 t, \quad t \in [0, t_0], \tag{4.10}$$

which implies

$$\gamma_t \le H^{-1}(H(\gamma_0) - 2c_4 t), \quad t \in [0, t_0].$$

Noting that when $t > t_0$, $(\gamma_t)_{t \ge t_0}$ satisfies (4.6) with γ_{t_0} satisfies $\frac{c_4}{c_2} < \Phi(\gamma_{t_0}) \le \frac{2c_4}{c_2}$, so that (4.8) holds, i.e.

$$\gamma_t \le \Phi^{-1}(2c_4/c_2).$$

In conclusion, we obtain

$$\gamma_t \le \Phi^{-1}(2c_4/c_2) + H^{-1}(H(\gamma_0) - 2c_4 t), \quad t \ge 0.$$

Combining this with (1) and (2), we prove (4.7) for some constant $k > 1$.

(e) Since $1 \le \Phi(r) \to \infty$ as $r \to \infty$, when Φ is convex we find a constant $\delta > 0$ such that $\Phi(r) \ge \delta r, r \ge 0$. So, by step (b), (2.3) holds. Combining this with (4.7) and applying the semigroup property, we derive

$$\begin{aligned} \|\hat{P}_t^* \delta_x - \hat{\mu}\|_V &= \sup_{|f| \le V} |\hat{P}_{t/2}(\hat{P}_{t/2} f - \hat{\mu}(f))(x)| \\ &\le c\mathrm{e}^{-\lambda t/2} \hat{P}_{t/2} V(x) \le c\big\{k + H^{-1}(H(\theta V(x)) - (2k)^{-1} t)\big\}\mathrm{e}^{-\lambda t/2}. \end{aligned}$$

Combining this with (4.1), (4.2) and (4.3), we prove (2.4) for some constants $k, \lambda > 0$.

Finally, if $H(\infty) < \infty$, we take $t^* = kH(\infty)$ in (2.4) to derive

$$\sup_{x \in \bar{D}} \|P_t \delta_x - \mu\|_V \le c\mathrm{e}^{-\lambda t}, \quad t \ge t^*$$

for some constants $c, \lambda > 0$, which implies (2.5) by the argument leading to (2.3) in step (c).

Proof of Corollary 2.2. By (2.8), for any $\theta \in ((1-\alpha)^+, \frac{1}{2})$ there exists a constant $c_3 > 0$ such that

$$\phi(r) \ge c_3(1 + r)^{1-\theta}, \quad r \ge 0.$$

Then (2.1) holds for $V := \mathrm{e}^{(1+|\cdot|^2)^\theta}$ and $\Phi(r) = r$. So the first assertion in (1) follows from Theorem 2.1(1).

Next, (2.7) and (2.9) imply (2.1) for $V := \psi(|\cdot|^2)$ and $\Phi(r) = r$, so that the second assertion in (1) holds by Theorem 2.1(1).

Finally, if $\int_0^\infty \frac{\mathrm{d}s}{\phi(s)} < \infty$, then for any $q > 0$, (2.1) holds for $V := (1 + |\cdot|^2)^q$ and $\Phi(r) = (1+r)^{1-\frac{1}{q}} \phi(r^{\frac{1}{q}})$, so that $\int_0^\infty \frac{\mathrm{d}s}{\Phi(s)} < \infty$. Then the proof is finished by Theorem 2.1(2).

Proof Theorem 2.4. According to Theorems 2.1-2.3 and Lemma 3.3, it suffices to verify (3.21). By Lemma 3.4, we only need to prove (3.27) and (3.28) for the family $\{P_t^\gamma : \gamma \in \mathscr{P}\}$.

(a) Proof of (3.28). Let us fix $\gamma \in \mathscr{P}$, and let $X_t^{x,\gamma}$ solve (3.20) with $X_0^\gamma = x$. For any $\nu \in \mathscr{P}$, by Girsanov's theorem we have

$$P_t^\nu f(x) = \mathbb{E}[f(X_t^{x,\gamma}) R_t^{x,\gamma,\nu}], \quad t \geq 0,$$

where

$$R_t^{x,\gamma,\nu} := \mathrm{e}^{\int_0^t \langle \{\sigma^*(\sigma\sigma^*)^{-1}[b(\cdot,\nu)-b(\cdot,\gamma)]\}(X_s^{x,\gamma}), \mathrm{d}W_s\rangle - \frac{1}{2}|\{\sigma^*(\sigma\sigma^*)^{-1}[b(\cdot,\nu)-b(\cdot,\gamma)]\}(X_s^{x,\gamma})|^2 \mathrm{d}s}.$$

So, (2.13) and Pinsker's inequality imply

$$\|(P_t^\gamma)^*\delta_z - (P_t^\nu)^*\delta_z\|_{var}^2 \leq (\mathbb{E}|R_t^{\gamma,\nu} - 1|)^2 \leq \kappa^2 t \|\gamma - \nu\|_{var}^2 \leq 4\kappa^2 t, \quad t \geq 0, z \in \bar{D}, \nu \in \mathscr{P}.$$

Taking $t_1 = \frac{1}{16\kappa^2}$, we obtain

$$\sup_{\nu \in \mathscr{P}} \|(P_{t_1}^\gamma)^*\delta_z - (P_{t_1}^\nu)^*\delta_z\|_{var} \leq \frac{1}{2}, \quad z \in \bar{D}, \nu \in \mathscr{P}. \tag{4.11}$$

On the other hand, by (3.5), there exists $x_0 \in D$ and a constant $\varepsilon > 0$ such that $B(x_0, \varepsilon) \subset D$ and

$$\|(P_{t_1}^\gamma)^*\delta_x - (P_{t_1}^\gamma)^*\delta_y\|_{var} \leq \frac{1}{4}, \quad x, y \in B(x_0, \varepsilon).$$

Combining this with (4.11) we derive

$$\sup_{\nu \in \mathscr{P}} \|(P_{t_1}^\nu)^*\delta_x - (P_{t_1}^\nu)^*\delta_y\|_{var} \leq \frac{3}{2} < 2, \quad x, y \in B(x_0, \varepsilon).$$

So, (3.28) holds for $B = B(x_0, \varepsilon)$.

(b) Let u solve (3.14) for $f = -b^{(0)}$ and large $\lambda > 0$ such that (3.9) holds, and let $\Theta(x) = x + u(x)$. Since $(\sigma, b(\cdot, \gamma))$ satisfies **(A1)**, by (2.13) we prove (4.4) with $Y_t^{x,\nu} := \Theta(X_t^{x,\nu})$ replacing Y_t for some constants $c_1, c_2 > 0$ and all $\nu \in \mathscr{P}$. So, by $H(\infty) < \infty$ and the argument leading to (4.7), we obtain

$$\sup_{\nu \in \mathscr{P}, x \in \bar{D}} \mathbb{E}[V(Y_t^{x,\nu})] \leq \theta^{-1} k, \quad t \geq kH(\infty) =: t_2.$$

This together with (4.2) implies

$$\sup_{\nu \in \mathscr{P}, x \in \bar{D}} \mathbb{E}[V(X_t^{x,\nu})] \leq \theta^{-2} k, \quad t \geq t_2.$$

Letting $\mathbf{K} := \{V \leq 2\theta^{-2} k\}$, we derive

$$\inf_{\nu \in \mathscr{P}, x \in \bar{D}} P_{t_2}^\nu 1_{\mathbf{K}}(x) \geq \frac{1}{2}. \tag{4.12}$$

On the other hand, by Girsanov's theorem and Schwartz's inequality, we find a constant $c_0 > 0$ such that

$$P_1^\nu 1_{B(x_0,\varepsilon)}(x) = \mathbb{E}\big[1_{B(x_0,\varepsilon)}(X_1^{x,\gamma})R_1^{x,\gamma,\nu}\big] \ge \frac{\{\mathbb{E}1_{B(x_0,\varepsilon)}(X_1^{x,\gamma})\}^2}{\mathbb{E}R_1^{x,\gamma,\nu}} \ge c_0(P_1^\gamma 1_{B(x_0,\varepsilon)}(x))^2.$$

Since $\mathbf{K}$ is bounded, combining this with Lemma 3.1 for P_t^γ, we find a constant $c_1 > 0$ such that

$$\inf_{\nu\in\mathscr{P},x\in\mathbf{K}} P_1^\nu 1_{B(x_0,\varepsilon)}(x) \ge c_1.$$

This together with (4.12) and the semigroup property yields

$$P_{t_2+1}^\nu 1_{B(x_0,\varepsilon)}(x) \ge P_{t_2}^\nu\big\{1_{\mathbf{K}}P_1^\nu 1_{B(x_0,\varepsilon)}\big\}(x) \ge c_1 P_{t_2}^\nu 1_{\mathbf{K}}(x) \ge \frac{c_1}{2} > 0, \quad x\in\bar{D}, \nu\in\mathscr{P}.$$

Therefore, (3.27) holds for $t_0 = t_2 + 1$.

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