MULTI SOLITARY WAVES TO STOCHASTIC NONLINEAR SCHRÖDINGER EQUATIONS

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Abstract. In this paper, we present a pathwise construction of multi-soliton solutions for focusing stochastic nonlinear Schrödinger equations with linear multiplicative noise, in both the L^2-critical and subcritical cases. The constructed multi-solitons behave asymptotically as a sum of K solitary waves, where K is any given finite number. Moreover, the convergence rate of the remainders can be of either exponential or polynomial type, which reflects the effects of the noise in the system on the asymptotical behavior of the solutions. The major difficulty in our construction of stochastic multi-solitons is the absence of pseudo-conformal invariance. Unlike in the deterministic case [49, 56], the existence of stochastic multi-solitons cannot be obtained from that of stochastic multi-bubble blow-up solutions in [56, 59]. Our proof is mainly based on the rescaling approach in [41], relying on two types of Doss-Sussman transforms, and on the modulation method in [16, 46], in which the crucial ingredient is the monotonicity of the Lyapunov type functional constructed by Martel, Merle and Tsai [47]. In our stochastic case, this functional depends on the Brownian paths in the noise.

1. Introduction and formulation of main results

1.1. **Introduction.** In this paper we consider the following type of focusing stochastic nonlinear Schrödinger equations (SNLS for short) with linear multiplicative noise:

$$\begin{cases} dX(t) = i\Delta X(t)dt + i|X(t)|^{p-1}X(t)dt - \mu(t)X(t)dt + \sum_{k=1}^{N} X(t)G_k(t)dB_k(t), \\ X(T_0) = X_0 \in H^1(\mathbb{R}^d). \end{cases} \tag{1.1}$$

Here, $1 < p \leq 1 + \frac{4}{d}$, $d \geq 1$, $T_0 \geq 0$, $\{B_k\}$ are the standard N-dimensional real valued Brownian motions on a normal stochastic basis $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, \mathbb{P})$, $G_k(t,x) = i\phi_k(x)g_k(t)$, $x \in \mathbb{R}^d$, $t \geq 0$, $\{\phi_k\} \subseteq C_b^\infty(\mathbb{R}^d, \mathbb{R})$, $\{g_k\} \subseteq C^\alpha(\mathbb{R}^+, \mathbb{R})$, $\alpha \in (\frac{1}{3}, \frac{1}{2})$, are $\{\mathscr{F}_t\}$-adapted processes that are controlled by $\{B_k\}$, and $X(t)G_k(t)dB_k(t)$ is taken in the sense of controlled rough paths (see Definition 1.2 below). The term μ is of form

$$\mu(t,x) = \frac{1}{2}\sum_{k=1}^{N} \phi_k(x)^2 g_k(t)^2, \quad x \in \mathbb{R}^d, \ t \geq 0, \tag{1.2}$$

such that the conservation law of mass is satisfied. In particular, if the processes are $\{\mathscr{F}_t\}$-adapted, then the rough integration coincides with the usual Itô integration ([39, Chapter 5]), and $-\mu X dt + \sum_{k=1}^{N} XG_k(t)dB_k(t)$ is exactly the standard Stratonovich differential. For convenience, we focus on the case $N < \infty$, but the infinite case $N = \infty$ can also be treated under suitable summability conditions of the spatial functions $\{\phi_k\}$.

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Nonlinear Schrödinger equations have various applications in contunuum mechanics, plasma physics and optics. In crystals the noise corresponds to scattering of excitons by phonons, due to thermal vibrations of the molecules, and its effect on the coherence of the ground state solitary solutions was investigated in the two-dimensional L^2-critical case in [3] (see also [2]). The influence of noise on the collapse was also studied in [55] for the L^2-critical case in dimensions $d = 1, 2$. Another important application can be found in open quantum systems, where the noise is of non-conservative type and $\{\|X(t)\|_{L^2}^2\}$ is a continuous martingale such that the mean $\mathbb{E}\|X(t)\|_{L^2}^2$ is conserved and the "physical" probability law can be defined. We refer to [9, Section 2] for more physical interpretations. We also refer to [14, 22, 27, 52, 53] for the numerical experiments to investigate the dynamics of stochastic solutions.

It is known that SNLS is H^1 globally well-posed in the L^2-subcritical case $1 < p < 1 + \frac{4}{d}$, and is locally well-posed in the critical case $p = 1 + \frac{4}{d}$. See, e.g., [19, 11, 6] and references therein.

The large time behavior of solutions, however, are more delicate. Different phenomena have been exhibited in the defocusing and focusing cases.

As a matter of fact, for the canonical nonlinear Schrödinger equation (NLS for short)

$$\begin{cases} du = i\Delta u dt + \lambda i|u|^{p-1}udt, \\ u(T_0) = u_0 \in H^1(\mathbb{R}^d), \end{cases} \tag{1.3}$$

in the defocusing L^2-critical case (i.e., $\lambda = -1, p = 1 + \frac{4}{d}$), solutions exist globally and even scatter at infinity, i.e., solutions behave asymptotically as free linear solutions. See the works by Dodson [28, 30, 31]. The scattering phenomena are also exhibited in the stochastic case. We refer to [41] for the H^1-subcritical and critical cases, and [35, 36, 37, 38, 64] for the L^2-critical case.

However, in the focusing L^2-critical case (i.e., $\lambda = 1$, $p = 1 + \frac{4}{d}$) different dynamics appear. An important role is played by the mass of the *ground state*, which is the unique positive radial solution to the nonlinear elliptic equation

$$\Delta Q - Q + Q^p = 0, \quad Q \in H^1(\mathbb{R}^d). \tag{1.4}$$

By [10, Theorem 1] (see also [13, Theorem 8.1.1]), Q is smooth and decays at infinity exponentially fast, i.e., there exist $C, \delta_0 > 0$ such that for any multi-index $|\upsilon| \leq 3$,

$$|\partial_x^\upsilon Q(x)| \leq Ce^{-\delta_0|x|}, \;\; x \in \mathbb{R}^d. \tag{1.5}$$

On one hand, in the subcritical mass regime $\|u_0\|_{L^2}^2 < \|Q\|_{L^2}^2$, solutions exist globally and scatter at infinity, see [29]. On the other hand, in the (super)critical mass regime $\|u_0\|_{L^2}^2 \geq \|Q\|_{L^2}^2$, solutions may form singularities in finite time or do not scatter at infinity.

One typical blow-up dynamics in the critical mass regime is the *pseudo-conformal blow-up solution*

$$S_T(t, x) = (w(T-t))^{-\frac{d}{2}} Q\left(\frac{x - x^*}{w(T-t)}\right) e^{-\frac{i}{4}\frac{|x-x^*|^2}{T-t} + \frac{i}{w^2(T-t)} + i\vartheta}, \tag{1.6}$$

where $T \in \mathbb{R}$, $w > 0$, $x^* \in \mathbb{R}^d$ and $\vartheta \in \mathbb{R}$. We note that, $\|S_T\|_{L^2}^2 = \|Q\|_{L^2}^2$ and S_T blows up at time T with speed $\|\nabla S_T(t)\| \sim (T-t)^{-1}$. A remarkable result proved by Merle [50] is that, the pseudo-conformal blow-up solution is the unique H^1 blow-up solution to L^2-critical NLS with critical mass, up to the symmetries of the equation. Very recently, Dodson [32, 33] has proved that S_T is indeed the unique L^2 finite time blow-up solution to L^2-critical NLS with critical mass, up to symmetries.

Another important dynamics is the *solitary wave*

$$R(t, x) := Q_w\left(x - v^*t - x^0\right) e^{i(\frac{1}{2}v^*\cdot x - \frac{1}{4}|v^*|^2 t + w^{-2}t + \vartheta)}. \tag{1.7}$$

where the parameters $x^0 \in \mathbb{R}^d$, and $w \in \mathbb{R}^+$, $v^* \in \mathbb{R}^d$, $\vartheta \in \mathbb{R}$, correspond to the frequency, propagation speed and phase, respectively, and

$$Q_w(x) = w^{-\frac{2}{p-1}} Q\left(\frac{x}{w}\right), \tag{1.8}$$

satisfying the nonlinear elliptic equation

$$\Delta Q_w - w^{-2} Q_w + Q_w^p = 0. \tag{1.9}$$

In contrast to the above scattering solutions and pseudo-conformal blow-up solutions, the solitary wave exists globally but does not scatter at infinity. An important underlying relationship is that, the solitary wave and the pseudo-conformal blow-up solution can be transformed into each other through the *pseudo-conformal transform*:

$$S_T(t,x) = C_T(R)(t,x) := \frac{1}{(T-t)^{\frac{d}{2}}} R\left(\frac{1}{T-t}, \frac{x}{T-t}\right) e^{-i\frac{|x|^2}{4(T-t)}}, \ \ t \neq T, \ x^* = v^* + (T-t)x^0. \tag{1.10}$$

Furthermore, according to the famous *soliton resolution conjecture*, global solutions to nonlinear dispersive equations are expected to behave asymptotically as a sum of solitary waves plus a dispersive part. One particular global solution is the *multi-soliton* (or, *multi-solitary wave solution*), which is defined on $[T_0, \infty)$ for some $T_0 \in \mathbb{R}$ and satisfies

$$\|u(t) - \sum_{k=1}^{K} R_k(t)\|_{H^1} \to 0, \ \ as \ t \to \infty, \tag{1.11}$$

where $K \in \mathbb{N} \setminus \{0\}$, R_k is the solitary wave of form

$$R_k(t,x) := Q_{w_k^0}\left(x - v_k t - x_k^0\right) e^{i(\frac{1}{2}v_k \cdot x - \frac{1}{4}|v_k|^2 t + (w_k^0)^{-2} t + \theta_k^0)}, \tag{1.12}$$

with parameters $w_k^0 \in \mathbb{R}^+$, $v_k, x_k^0 \in \mathbb{R}^d$, $\theta_k^0 \in \mathbb{R}$, and where $Q_{w_k^0}$ satisfies equation (1.9) with w_k^0 replacing w, $1 \leq k \leq K$. That is, the multi-soliton behaves exactly as a sum of solitons without loss of mass by dispersion.

Multi-solitons have attracted significant interest in the literature. The construction of multi-solitons to NLS in the non-integrable case was initiated by Merle [49] in the L^2-critical case. The proof in [49] is based on the construction of multi-bubble pseudo-conformal blow-up solutions and on the pseudo-conformal invariance. Afterwards, multi-solitons in the L^2-subcritical and super-critical cases were constructed, respectively, by Martel and Merle [46] and by Côte, Martel and Merle [17]. The method in [46, 17] is quite different from that of [49]. It relies on the modulation analysis and the monotonicity of functionals adapted to multi-solitons. This method has also been applied in the study of the stability problem of multi-solitons, [47]. Recently, for quite general nonlinearities, the smoothness and conditional uniqueness of multi-solitons were studied by Côte and Friederich [16] in both the L^2-subcritical and critical cases. The uniqueness issue of multi-solitons to L^2-critical NLS, particularly in the low asymptotical regime, was recently studied in [12].

Multi-solitons are also exhibited in various models. For the generalized Korteweg-de Vries (gKdV) equations, we refer to the pioneering work by Martel [45], where the construction and uniqueness of multi-solitons were proved in the subcritical and critical cases. The construction and classification in the supercritical case were obtained by Cômbet [15]. We also refer to [44] for the classification of dynamics near solitons for the L^2-critical gKdV equation with a saturated perturbation. For other dispersive equations, see, e.g., [18] for the Klein-Gordon equation, [43] for the Hartree equation and [54] for the water-waves system.

In the stochastic case, for the one dimensional cubic SNLS, the small noise asymptotic of the tails of mass and timing jitter in soliton transmission was studied by Debussche and Gautier [26]. Moreover, the influence of noise on the propagation of standing waves was studied by de Bouard and Fukuizumi [23] for the Bose-Einstein condensation, where the trapping potential varies randomly in time. Quite interestingly, it was proved in [23] that the solution decomposes into the sum of a randomly modulated standing wave and a small remainder, and the first order of the remainder converges to a Gaussian process, as the amplitudes of noise tends to zero. For the stochastic KdV equations we refer to [20] for the random modulation of solitons, and [21, 25] for the exist problem from a neighborhood of solitons

The main interest of this paper is to understand the quantitative properties of soliton dynamics for SNLS.

Recently, several typical blow-up dynamics have been constructed for SNLS. Critical mass blow-up solutions were constructed in [58], which yields that the mass of the ground state is exactly the threshold of the global existence and blow-up for SNLS. The loglog blow-up solutions and the multi-bubble pseudo-conformal blow-up solutions were constructed in [34] and [59], respectively. Very recently, in [56] we also constructed the multi-bubble Bourgain-Wang type solutions, which behave asymptotically as a sum of pseudo-conformal blow-up solutions and a smooth residue. This, in particular, provides examples for the mass quantization conjecture ([51]). Another interesting outcome is the existence of non-pure multi-solitons to L^2-critical NLS, which behave as a sum of multi-solitons plus a scattering part, predicted by the soliton resolution conjecture.

It should be mentioned that, one major difficulty in our construction of stochastic multi-solitons is the absence of pseudo-conformal invariance. Unlike in the deterministic case [49, 56], the existence of stochastic multi-solitons cannot be obtained from that of stochastic multi-bubble blow-up solutions in [56, 59].

In the present work, we provide path-by-path constructions of stochastic multi-solitons to SNLS. More precisely, in both the L^2-subcritical and critical cases $1 < p \leq 1 + \frac{4}{d}$, for $\mathbb{P}$-a.e. $\omega \in \Omega$, the multi-solitons to (1.1) are constructed and, up to a random phase transformation, behave asymptotically as a sum of K solitary waves, where K is any given finite number. Quite interestingly, the decay rate of the corresponding asymptotical behavior can be of either exponential or polynomial type, which is closely related to that of the spatial functions $\{\phi_l\}$ and temporal functions $\{g_l\}$ in the noise. To the best of our knowledge, this provides the first explicit constructions of multi-solitons to SNLS.

Our strategy of proof is mainly based on the rescaling approach and the modulation method.

The rescaling approach in [41] relies on two types of Doss-Sussman transforms, which enable us to study the large time behavior of solutions by transforming the original equation to random Schrödinger equations, for which the sharper *pathwise analysis* can be performed. This method is actually quite robust for many other stochastic partial differential equations, see, e.g., [1, 4, 24] and references therein. The solvability relationship between two equations via the transform is indeed nontrivial in infinite dimensional spaces. An interesting outcome here is, that we extend the solvability in the critical case for dimensions $d = 1, 2$ in [58] to the entire (sub)critical regime for all dimensions.

Let us mention that, the pathwise analysis in [41] is based on the stability of scattering which, however, is quite difficult for the multi-solitons in the subcritical case ([47]) and even fails in the critical case. Instead, we construct multi-solitions in a direct way by using the modulation method and analysing the Lyapunov type functional constructed by Martel, Merle and Tsai [47].

It is also worth noting that, in order to treat the subcritical and critical cases in a uniform manner, the soliton profiles in the geometrical decomposition here exhibit a quite unified structure in both cases, which is different from the works [16, 46]. The unstable direction $\mathrm{Re}\langle\widetilde{R}_k, \varepsilon\rangle$ (see Corollary 4.2 below) is not involved in the geometrical decomposition. Instead, it will be controlled by the almost conservation of the local mass. This permits to fix the frequency $w_k \equiv w_k^0$ in the subcritical case and, in particular, simplifies the derivation of the time-independent main part of the Lyapunov type functional. In the critical case, though the frequency w_k varies with time, the main part keeps still time independent, due to the scaling invariance and the key Pohozaev identity.

Notations. For any $x = (x_1, \cdots, x_d) \in \mathbb{R}^d$ and any multi-index $\nu = (\nu_1, \cdots, \nu_d)$, let $|\nu| = \sum_{j=1}^d \nu_j$, $\langle x\rangle = (1 + |x|^2)^{1/2}$, $\partial_x^\nu = \partial_{x_1}^{\nu_1} \cdots \partial_{x_d}^{\nu_d}$ and $\langle\nabla\rangle = (I - \Delta)^{1/2}$.

For $1 \le p \le \infty$, $L^p = L^p(\mathbb{R}^d)$ is the space of p-integrable (complex-valued) functions endowed with the norm $\|\cdot\|_{L^p}$, and $W^{s,p}$ denotes the standard Sobolev space, $s \in \mathbb{R}$. In particular, $L^2(\mathbb{R}^d)$ is the Hilbert space endowed with the inner product $\langle v, w\rangle = \int_{\mathbb{R}^d} v(x)\bar{w}(x)dx$, and $H^s := W^{s,2}$. As usual, $L^q(0, T; L^p)$ means the space of all integrable L^p-valued functions $f : (0, T) \to L^p$ with the norm $\|\cdot\|_{L^q(0,T;L^p)}$, and $C([0, T]; L^p)$ denotes the space of all L^p-valued continuous functions on $[0, T]$ with the sup norm over t. For any Hölder continuous function $f \in C^\alpha(I)$, $\alpha > 0$ and $I \subseteq \mathbb{R}^+$, we write $\delta f_{st} := f(t) - f(s)$, $s, t \in I$, and $\|f\|_{\alpha,I} := \sup_{s,t\in I, s\neq t} \frac{|\delta f_{st}|}{|s-t|^\alpha}$. Let C_c^∞ be the space of all compactly supported smooth functions on $\mathbb{R}^d$. We also set $\dot{g} := \frac{d}{dt}g$ for any C^1 functions.

The symbol $u = O(v)$ means that $|u/v|$ stays bounded, and $v_n = o(1)$ means $|v_n|$ tends to zero as $n \to \infty$. Throughout this paper, we use C, δ for various constants that may change from line to line.

1.2. **Formulation of main results.** To begin with, we first recall some basic notions of controlled rough paths. For more details of the theory of (controlled) rough path we refer to [39, 40] and the references therein.

Given a path $X \in C^\alpha([0, T]; \mathbb{R}^N)$, $0 < T < \infty$, we say that $Y \in C^\alpha([0, T]; \mathbb{R}^N)$ is controlled by X if there exists $Y' \in C^\alpha([0, T]; \mathbb{R}^{N\times N})$ such that the remainder term R^Y implicitly given by

$$\delta Y_{j,st} = \sum_{k=1}^N Y'_{jk}(s)\delta X_{k,st} + \delta R^Y_{j,st}$$

satisfies $\|R^Y_j\|_{2\alpha,[s,t]} < \infty$, $1 \le j \le N$. This defines the controlled rough path $(Y, Y') \in \mathscr{D}_X^{2\alpha}([0, T]; \mathbb{R}^N)$, and Y' is the so called Gubinelli's derivative.

One typical example is that, the N-dimensional Brownian motions $B = (B_j)_{j=1}^N$ can be enhanced to a rough path $\mathbf{B} = (B, \mathbb{B})$, where $\mathbb{B}_{jk,st} := \int_s^t \delta B_{j,sr} dB_k(r)$ with the integration taken in the sense of Itô, $\delta B_{j,st} = B_j(t) - B_j(s)$. It is known that $\|B\|_{\alpha,[0,T]} < \infty$, $\|\mathbb{B}\|_{2\alpha,[s,t]} < \infty$, $\mathbb{P}$-a.s., where $\frac{1}{3} < \alpha < \frac{1}{2}$ (see [39, Section 3.2]).

Given a path Y controlled by the N-dimensional Brownian motion, i.e., $Y \in \mathscr{D}_B^{2\alpha}([S, T]; \mathbb{R}^N)$, $0 < S < T < \infty$, we can define the rough integration of Y against $\mathbf{B} = (B, \mathbb{B})$ as follows (see [39, Theorem 4.10]), for each $1 \le k \le N$,

$$\int_S^T Y_k(r)dB_k(r) := \lim_{|\mathscr{P}|\to 0} \sum_{i=0}^{n-1} \left(Y_k(t_i)\delta B_{k,t_it_{i+1}} + \sum_{j=1}^N Y'_{kj}(t_i)\mathbb{B}_{jk,t_it_{i+1}} \right), \tag{1.13}$$

where $\mathscr{P} := \{t_0, t_1, \cdots, t_n\}$ is a partition of $[S, T]$ so that $t_0 = S$, $t_n = T$, $|\mathscr{P}| := \max_{0\le i\le n-1} |t_{i+1} - t_i|$.

Throughout this paper we assume that

(A0) For every $1 \leq l \leq N$,

$$\lim_{|x|\to\infty} |x|^2 |\partial_x^\upsilon \phi_l(x)| = 0, \quad \upsilon \neq 0. \tag{1.14}$$

(A1) For every $1 \leq l \leq N$, $\{g_l\}$ are $\{\mathscr{F}_t\}$-adapted continuous processes and controlled by the Brownian motions $\{B_l\}$, i.e., $\{g_l\} \subseteq \mathscr{D}_B^{2\alpha}(\mathbb{R}^+;\mathbb{R}^N)$ with the Gubinelli derivative $\{g'_{lj}\}_{j,l=1}^N$. In addition, ϕ_l and g_l satisfy one of the following two cases:

Case (I): $g_l \in L^2(\mathbb{R}^+)$, $\mathbb{P} - a.s.$, and there exists $c_l > 0$ such that

$$\sum_{|\nu|\leq 4} |\partial^\nu \phi_l(x)| \leq C e^{-c_l|x|}. \tag{1.15}$$

Case (II): $\mathbb{P}$-a.s., $g_l \in L^2(\mathbb{R}^+)$ and there exists $c^* > 0$ such that for t large enough,

$$\int_t^\infty g_l^2 ds \log\left(\int_t^\infty g_l^2 ds\right)^{-1} \leq \frac{c^*}{t^2}. \tag{1.16}$$

In addition, let $\upsilon_* \in \mathbb{N}$. ϕ_l satisfies that for $|x| > 1$

$$\sum_{|\upsilon|\leq 4} |\partial^\upsilon \phi_l(x)| \leq C|x|^{-\upsilon_*}. \tag{1.17}$$

Remark 1.1. Case (I) *and* Case (II) *correspond, respectively, to the exponential and polynomial decay rates of the spatial functions of noise. Actually, because the mutlti-soliton is the asymptotically approximate solution to NLS, it would be necessary to consider noise that decays to 0 as $t \to \infty$. The $L^2(\mathbb{R}^+)$-integrability of $\{g_l\}$ then corresponds to the case of globally finite quadratic variation of the noise, and thus to the smallness of W_* defined in* (1.21)*. The asymptotics* (1.16) *is closely related to the Levy Hölder continuity of Brownian motions, see the proof of* (5.10) *below. One may close the bootstrap arguments in the proof below by assuming even stronger temporal decay of the noise with the exponential rate $B_*(t) \sim e^{-\delta t}$ or with the polynomial rate $B_*(t) \sim t^{-p}$ with p sufficiently large, where B_* is defined in* (3.7)*. However, in order to avoid the strong temporal decay conditions, it is more convenient to consider the spatial decay conditions in* Case (I) *and* Case (II)*, which turn out to be efficient to study the soliton behavior in the stochastic case. Interestingly, this spatial behavior can be transferred to that of the temporal behavior as shown in* (1.20) *and* (1.22) *below.*

For simplicity, we mainly focus on the case $c_l = 1$, $1 \leq l \leq N$, and denote by ϕ the decay functions in (1.15) and (1.17), i.e., for $|x| > 1$,

$$\phi(x) := \begin{cases} e^{-|x|}, & \text{in Case (I);} \\ |x|^{-\upsilon_*}, & \text{in Case (II).} \end{cases} \tag{1.18}$$

The solution to (1.1) is taken in the sense of controlled rough path.

Definition 1.2. *Let $1 < p \leq 1 + \frac{4}{d}$, $d \geq 1$. We say that X is a solution to* (1.1) *on $[T_0, \tau^*)$, where $T_0, \tau^* \in (0, \infty]$ are random variables, if $\mathbb{P}$-a.s. for any $\varphi \in C_c^\infty$, $t \mapsto \langle X(t), \varphi\rangle$ is continuous on $[T_0, \tau^*)$ and for any $T_0 \leq s < t < \tau^*$,*

$$\langle X(t) - X(s), \varphi\rangle - \int_s^t \langle iX, \Delta\varphi\rangle + \langle i|X|^{p-1}X, \varphi\rangle - \langle \mu X, \varphi\rangle dr = \sum_{k=1}^N \int_s^t \langle i\phi_k g_k X, \varphi\rangle dB_k(r).$$

Here the integral $\int_s^t \langle i\phi_k g_k X, \varphi\rangle dB_k(r)$ *is taken in the sense of controlled rough path with respect to the rough paths* $(B, \mathbb{B})$*, that is,* $\langle i\phi_k g_k X, \varphi\rangle \in C^\alpha([s,t])$*,*

$$\delta(\langle i\phi_k X, \varphi\rangle)_{st} = \sum_{j=1}^{N} \langle -\phi_j \phi_k g_j(s) g_k(s) X(s) + i\phi_k g'_{kj}(s) X(s), \varphi\rangle \delta B_{j,st} + \delta R_{k,st}, \tag{1.19}$$

and $\|\langle \phi_j \phi_k g_j g_k X, \varphi\rangle\|_{\alpha,[s,t]} + \|\langle \phi_k g'_{kj} X, \varphi\rangle\|_{\alpha,[s,t]} < \infty$, $\|R_k\|_{2\alpha,[s,t]} < \infty$, $\alpha \in (\frac{1}{3}, \frac{1}{2})$.

The H^1 local solvability of (1.1) can be proved by using the fixed point arguments as in [6, Theorems 1.2 and 2.1]. The key ingredients are the Strichartz and local smoothing estimates for Schrödinger equations with lower order perturbations, due to the asymptotical flatness condition (1.14). See, e.g., [48, 63]. It also relies on Theorem 1.5 below, which relates equations (1.1) and (1.29) through the Doss-Sussman type transform (1.23).

The main result of this paper is formulated in Theorem 1.3 below, concerning the large time soliton dynamics of (1.1) in both the L^2-subcritical and critical cases.

Theorem 1.3. *Consider* (1.1) *with* $1 < p \leq 1 + \frac{4}{d}$*,* $d \geq 1$*. Let* $w_k^0 > 0$*,* $\theta_k^0 \in \mathbb{R}$*,* $x_k^0 \in \mathbb{R}^d$*,* $v_k \in \mathbb{R}^d \setminus \{0\}$*,* $1 \leq k \leq K$*, such that* $v_j \neq v_k$ *for any* $j \neq k$*. Assume* $(A0)$ *and* $(A1)$ *with* υ_* *sufficiently large in* Case (II)*. Then, for* $\mathbb{P}$*-a.e.* $\omega \in \Omega$*, there exists* $T_0 = T_0(\omega)$ *sufficiently large and* $X_*(\omega) \in H^1$*, such that there exists an* H^1 *solution* $X(\omega)$ *to* (1.1) *on* $[T_0, \infty)$ *satisfying* $X(\omega, T_0) = X_*(\omega)$ *and*

$$\|e^{-W_*(t)} X(t) - \sum_{k=1}^{K} R_k(t)\|_{H^1} \leq C \int_t^\infty s\phi^{\frac{1}{2}}(\delta s) ds, \quad t \geq T_0. \tag{1.20}$$

Here,

$$W_*(t,x) = -\sum_{l=1}^{N} \int_t^\infty i\phi_l(x) g_l(s) dB_l(s), \tag{1.21}$$

$\{R_k\}$ *are the solitary waves given by* (1.12)*,* ϕ *is the decay function in* (1.18) *and* $C, \delta > 0$*. In particular, for* $t \geq T_0$*,*

$$\|X(t) - \sum_{k=1}^{K} R_k(t)\|_{H^1} \leq C \sum_{l=1}^{N} \left(\int_t^\infty g_l^2(s) ds \log\left(\int_t^\infty g_l^2(s) ds \right)^{-1} \right)^{\frac{1}{2}} + C \int_t^\infty s\phi^{\frac{1}{2}}(\delta s) ds. \tag{1.22}$$

Moreover, in the L^2*-subcritical case* $1 < p < 1 + \frac{4}{d}$*, there exists a solution* X *to* (1.1) *on the whole time regime* $[0, \infty)$*, satisfying the asymptotic behavior* (1.20)*.*

Remark 1.4. (i)*. To the best of our knowledge, Theorem 1.3 provides the first quantitative construction of multi-solitons to* (1.1) *in the stochastic case. It would be also interesting to note that, the decay rate in* (1.20) *can be of either exponential or polynomial type in* Cases (I) *and* (II)*, respectively, and the first term in the upper bound* (1.22) *is related to the asymptotical behavior of noise* W_**. These reflect the noise effects on the soliton dynamics.*

(ii)*. In comparison with the scattering results in* [41, 64]*, where the solutions behave asymptotically like a free Schrödinger flow in the defocusing (sub)critical cases, the asymptotics* (1.20) *in Theorem 1.3 gives a different asymptotic behavior in the focusing case, namely, the solutions do not scatter at infinity and may even propagate as any finitely many decoupled solitary waves.*

(iii) *Let us mention that in the critical case, although the equation is unstable under the perturbation of the initial data, by applying the Doss-Sussman transformations and the modulation method, we have that the structure of equation is stable for the noise satisfying either* Case (I) *or* Case (II)*. In particular, the constants in the estimates of the modulation equations and functionals* *are uniformly bounded for the large times, which permits to perform the bootstrap arguments to construct multi-solitons.*

Moreover, in order to control the extra unstable direction of the linearized operator in the critical case, we introduce a new modulation parameter $w_k(t)$ and an extra orthogonal condition in the geometrical decomposition. The geometrical decomposition is taken in a quite unified formulation in both the critical and subcritical cases, which simplifies the previous work [46] *in the subcritical case and is new in the critical case. In view of the critical scaling invariance and the Pohozaev identity, the modulation parameter w_k does not affect the main part of the Lyapunov type functional, and the coercivity of the second order terms $H(\varepsilon)$ still holds in the critical case. This also helps to construct multi-solitons in the critical case.*

The strategy of proof is mainly based on the rescaling approach in [41] which relies on two types of Doss-Sussman transformations, and on the modulation method in [16, 46, 47].

One of the main advantages of Doss-Sussman type transform is, that the sharper pathwise analysis can be performed to the resulting random solutions, which is quite robust in the study of stochastic partial differential equations. We refer to, e.g., [1] for the stochastic Camassa-Holm equation and [4] for the stochastic porous media equation. For the case of SNLS, we refer to [24] for SNLS with potentials multiplied by a temporal real-valued white noise, [7] for the stochastic logarithmic Schrödinger equation. See also [8, 62] for optimal control problems, [64] for the defocusing critical case, and [34, 56, 58, 59] for the construction of (multi-bubble) blow-up solutions.

Here we first apply the Doss-Sussman type transform

$$v := e^{-W}X \tag{1.23}$$

with

$$W(t,x) := \sum_{k=1}^{N}\int_0^t i\phi_k(x)g_k(s)dB_k(s). \tag{1.24}$$

to reduce (1.1) to an equation with random lower order perturbations

$$\begin{cases} i\partial_t v + (\Delta + b\cdot\nabla + c)v + |v|^{p-1}v = 0, \\ v(T_0) = e^{-W(T_0)}X_0, \end{cases} \tag{1.25}$$

where the coefficients of low order perturbations

$$b(t,x) = 2\nabla W(t,x) = 2i\sum_{l=1}^{N}\int_0^t \nabla\phi_l(x)g_l(s)dB_l(s), \tag{1.26}$$

$$\begin{aligned} c(t,x) &= \sum_{j=1}^{d}(\partial_j W(t,x))^2 + \Delta W(t,x), \\ &= -\sum_{j=1}^{d}\left(\sum_{l=1}^{N}\int_0^t \partial_j\phi_l(x)g_l(s)dB_l(s)\right)^2 + i\int_0^t \Delta\phi_l(x)g_l(s)dB_l(s). \end{aligned} \tag{1.27}$$

It should be mentioned that, the solvability between two equations via the Doss-Sussman type transform is indeed nontrivial in infinite dimensional spaces.

The H^1 local solvability of equation (1.25) can be proved as in [6, Theorem 2.1 and Proposition 2.5], relying on the Strichartz and local smoothing estimates for the Laplacian with lower order perturbations, due to Assumption $(A0)$.

Furthermore, the solvability of equation (1.1) can be inherited from that of equation (1.25) by Theorem 1.5 below, which in particular extends the L^2-critical result for dimensions $d = 1, 2$ in [58] to the whole L^2-(sub)critical regime for all dimensions.

Theorem 1.5. *Let* $1 \leq p \leq 1 + \frac{4}{d}$, $d \geq 1$. *Let* v *be the solution to* (1.25) *on* $[T_0, \tau^*)$ *with* $v(T_0) = v_0 \in H^1$*, where* $T_0, \tau^* \in (0, \infty]$ *are random variables. Then,* $\mathbb{P}$*-a.s.,* $X := e^W v$ *is the solution to equation* (1.1) *on* $[T_0, \tau^*)$ *in the sense of Definition 1.2 above.*

In order to study the large time behavior of solutions, we use the ideas from [41] to perform a second Doss-Sussman type transform.

To be precise, using the theorem on time change for continuous martingales (cf. [42], Section 3.4) we regard $\int_0^t g_l(s)dB_l(s)$ as a time-changed Brownian motion $\widetilde{B}_l(s(t))$ with $s(t) = \int_0^t g_l^2(r)dr$, $\mathbb{P}$-a.s.. Then, by the L^2-integrability $g_l \in L^2(\mathbb{R}^+)$, we infer that as time goes to infinity, $s(t)$ converges to $\int_0^\infty g_l^2(r)dr$, and thus $\int_0^t g_l(s)dB_l(s) \to \int_0^\infty g_l(s)dB_l(s)$ and $W(t) \to W(\infty)$, $\mathbb{P}$-a.s., $1 \leq l \leq N$.

Then, as in [41], we apply a second transformation

$$u = e^{W(\infty)}v = e^{-W_*(t)}X(t), \tag{1.28}$$

where W_* is given by (1.21), and derive from (1.25) a new equation

$$\begin{cases} iu_t + (\Delta + b_* \cdot \nabla + c_*)u + |u|^{p-1}u = 0, \\ u(T_0) = e^{-W_*(T_0)}X_0, \end{cases} \tag{1.29}$$

where the coefficients of lower order perturbations

$$b_*(t,x) = 2\nabla W_*(t,x) = -2i\sum_{l=1}^N \int_t^\infty \nabla\phi_l(x)g_l(s)dB_l(s), \tag{1.30}$$

$$\begin{aligned} c_*(t,x) &= \sum_{j=1}^d (\partial_j W_*(t,x))^2 + \Delta W_*(t,x), \\ &= -\sum_{j=1}^d \left(\sum_{l=1}^N \int_t^\infty \partial_j\phi_l(x)g_l(s)dB_l(s)\right)^2 - \sum_{l=1}^N \int_t^\infty i\Delta\phi_l(x)g_l(s)dB_l(s). \end{aligned} \tag{1.31}$$

The proof of Theorems 1.3 now can be reduced to that of the following result.

Theorem 1.6. *Consider equation* (1.29) *with* $1 < p \leq 1 + \frac{4}{d}$, $d \geq 1$. *Let* $w_k^0 > 0$, $\theta_k^0 \in \mathbb{R}$, $x_k^0 \in \mathbb{R}^d$, $v_k \in \mathbb{R}^d \setminus \{0\}$, $1 \leq k \leq K$, *such that* $v_j \neq v_k$ *for any* $j \neq k$. *Assume* (A0) *and* (A1) *with* υ_* *sufficiently large in* Case (II). *Then, for* $\mathbb{P}$*-a.e.* $\omega \in \Omega$*, there exists* $T_0 = T_0(\omega)$ *large enough and* $u_*(\omega) \in H^1$*, such that there exists a unique solution* $u(\omega) \in C([T_0, \infty); H^1)$ *to* (1.29) *satisfying* $u(\omega, T_0) = u_*(\omega)$ *and*

$$\|u(t) - \sum_{k=1}^K R_k(t)\|_{H^1} \leq C\int_t^\infty s\phi^{\frac{1}{2}}(\delta s)ds, \quad t \geq T_0, \tag{1.32}$$

where $C, \delta > 0$ *and* $\{R_k\}$ *are the solitary waves given by* (1.12).

As mentioned above, the absence of the pseudo-conformal symmetry causes a major difficulty in the construction of multi-solitons in the stochastic case. Hence, unlike in the deterministic case, the stochastic multi-solitons to (1.29) cannot be obtained from the multi-bubble blow-up solutions constructed in [56, 59].

To overcome this problem, we construct the multi-solitons in a direct way by using the modulation method in [16, 46, 47].

To be precise, we first obtain the geometrical decomposition of solutions into the soliton profiles rescaled by different parameters and a remainder, moduling suitable orthogonality conditions corresponding to the coercivity of linearized operators around the ground state. Unlike in [16, 46], the structure of soliton profiles is of a quite unified form in the subcritical and critical cases, which enables us to treat both cases in a uniform manner.

It is worth noting that, the unstable direction (i.e., $\mathrm{Re}\langle\widetilde{R}_k, \varepsilon\rangle$ in (4.17) below) is not involved in the geometrical decomposition, instead it will be controlled by the almost conservation law of the local mass. This enables us to fix the frequency parameter $w_k \equiv w_k^0$ in the geometrical decomposition in the subcritical case, and in particular to simplify the proof in the subcritical case. The other geometrical parameters will be controlled by the modulation equations under the orthogonality conditions.

Concerning the control of the remainder, the crucial ingredient is the monotonicity of the Lyapunov type functional adapted to multi-solitons, which was first constructed by Martel, Merle and Tsai [47] in the study of stability problem of multi-solitons. The analysis of the Lyapunov functional will be based on several controls of the local quantities. We note that, these functionals depend on Brownian paths in the stochastic case. Moreover, the conservation law of energy is also destroyed by the presence of noise. Again, the rescaling approach enables us to perform the sharp analysis with Brownian paths fixed, and thus to obtain the quantitative controls of the variation of functionals in terms of Brownian paths.

Consequently, together with the coercivity of linearized operators and bootstrap arguments, the noise effects on the exponential or polynomial decay rate of the remainder are derived, which lead to the desirable stochastic multi-solitons to SNLS (1.1) by using compactness arguments.

The remainder of this paper is structured as follows. We first prove Theorem 1.5 in Section 2 which relates the solvability between two equations (1.1) and (1.25). Then, Section 3 contains the geometrical decomposition and the estimate of modulation equations. Section 4 is mainly concerned with the control of several important functionals, including the local mass, local momentum, energy and the crucial Lyapunov type functional. Then, Section 5 is devoted to the proof of the main results, the crucial ingredients there are the uniform estimate of remainder and modulation parameters, based on bootstrap arguments, and the compactness arguments. At last, Section 6, i.e., the Appendix, contains the coercivity of linearized operators, the decoupling lemma for solitary waves with distinct velocities and several technical proofs.

2. Rescaled random equations

This section is mainly concerned with the proof of Theorem 1.5 which permits to relate both equations (1.1) and (1.25).

Proof of Theorem 1.5. Let us fix any $T \in (T_0, \tau^*)$ and recall that $B_k \in C^\alpha([T_0, T])$ for any $\alpha \in (\frac{1}{3}, \frac{1}{2})$, $1 \le k \le N$, $\mathbb{P}$-a.s.. For any $\varphi \in C_c^\infty$ and any $T_0 \le s < t \le T$,

$$\langle \delta X_{st}, \varphi\rangle = \langle (\delta e^W)_{st} v(s), \varphi\rangle + \langle e^{W(s)}\delta v_{st}, \varphi\rangle + \langle (\delta e^W)_{st}\delta v_{st}, \varphi\rangle. \tag{2.1}$$

Below we treat each term on the R.H.S. above separately.

(*i*) *Estimate of* $\langle (\delta e^W)_{st} v(s), \varphi\rangle$. Since $(g_k) \in \mathcal{D}_B^{2\alpha}([T_0, T]; \mathbb{R}^N)$, by (1.24),

$$\delta W_{st} = \sum_{k=1}^N \int_s^t i\phi_k g_k(r) dB_k(r) = \sum_{k=1}^N i\phi_k g_k(s)\delta B_{k,st} + \sum_{j,k=1}^N i\phi_k g'_{kj}(s)\mathbb{B}_{jk,st} + o(t-s). \tag{2.2}$$

Then, by Taylor's expansion,

$$(\delta e^W)_{st} = e^{W(s)}\left(\sum_{k=1}^N i\phi_k g_k(s)\delta B_{k,st} - \frac{1}{2}\sum_{j,k=1}^N \phi_j\phi_k g_j(s)g_k(s)\delta B_{j,st}\delta B_{k,st} + \sum_{j,k=1}^N i\phi_k g'_{kj}(s)\mathbb{B}_{jk,st}\right) + o(t-s). \tag{2.3}$$

Taking into account (see [39, Section 3.3], [57, p.9])

$$\delta B_{j,st}\delta B_{k,st} = \mathbb{B}_{jk,st} + \mathbb{B}_{kj,st} + \delta_{jk}(t-s), \tag{2.4}$$

we thus obtain

$$\begin{aligned}(\delta e^W)_{st} =& e^{W(s)}\Big(-\mu(t-s) + \sum_{k=1}^N i\phi_k g_k(s)\delta B_{k,st} \\ &+ \sum_{j,k=1}^N \left(-\phi_j\phi_k g_j(s)g_k(s) + i\phi_k g'_{kj}(s)\right)\mathbb{B}_{jk,st}\Big) + o(t-s),\end{aligned} \tag{2.5}$$

which yields that

$$\begin{aligned}\langle(\delta e^W)_{st}v(s),\varphi\rangle =& \langle-\mu(e^{W(s)}v(s)),\varphi\rangle(t-s) + \sum_{k=1}^N\langle i\phi_k g_k(s)(e^{W(s)}v(s)),\varphi\rangle\delta B_{k,st} \\ &+ \sum_{j,k=1}^N\langle\left(-\phi_j\phi_k g_j(s)g_k(s) + i\phi_k g'_{kj}(s)\right)(e^{W(s)}v(s)),\varphi\rangle\mathbb{B}_{jk,st} + o(t-s).\end{aligned} \tag{2.6}$$

(ii) *Estimate of* $\langle e^{W(s)}\delta v_{st},\varphi\rangle$. Let $f(v) := |v|^{p-1}v$. We claim that

$$\langle e^{W(s)}\delta v_{st},\varphi\rangle = \langle i\Delta(e^{W(s)}v(s)),\varphi\rangle(t-s) + \langle if(e^{W(s)}v(s)),\varphi\rangle(t-s) + o(t-s). \tag{2.7}$$

In order to prove (2.7), using equation (1.25) we have

$$\begin{aligned}\langle e^{W(s)}\delta v_{st},\varphi\rangle =& \langle e^{W(s)}\int_s^t ie^{-W(r)}\Delta(e^{W(r)}v(r))dr,\varphi\rangle + \langle e^{W(s)}\int_s^t if(v(r))dr,\varphi\rangle \\ =&: K_1 + K_2.\end{aligned} \tag{2.8}$$

Note that,

$$\begin{aligned}K_1 =& \langle i\Delta(e^{W(s)}v(s)),\varphi\rangle(t-s) + \int_s^t\langle v(r)-v(s),(-i)\Delta(e^{-W(s)}\varphi)\rangle dr \\ &+ \int_s^t\langle L(r)v(r) - L(s)v(s),(-i)e^{-W(s)}\varphi\rangle dr \\ =&: \langle i\Delta(e^{W(s)}v(s)),\varphi\rangle(t-s) + K_{11} + K_{12},\end{aligned} \tag{2.9}$$

where $L(r)v(r) := (b(r)\cdot\nabla + c(r))v(r)$, and $L(s)v(s)$ is defined similarly.

By the integration by parts formula,

$$\begin{aligned}K_{11} =& \int_s^t\langle e^{-ir\Delta}v(r) - e^{-is\Delta}v(s),(-i)e^{-is\Delta}\Delta(e^{-W(s)}\varphi)\rangle dr \\ &+ \int_s^t\langle v(r),(-i)(1-e^{i(r-s)\Delta})\Delta(e^{-W(s)}\varphi)\rangle dr \\ \leq& \int_s^t \|e^{-ir\Delta}u(r) - e^{-is\Delta}v(s)\|_{L^2}\|\Delta(e^{-W(s)}\varphi)\|_{L^2}dr\end{aligned}$$

$$+\int_s^t \|v(r)\|_{L^2}\|(1-e^{i(r-s)\Delta})\Delta(e^{-W(s)}\varphi)\|_{L^2}dr. \tag{2.10}$$

We claim that, there exists $\zeta>0$ such that,

$$\|e^{-ir\Delta}v(r)-e^{-is\Delta}v(s)\|_{L^2}\leq C(T)(r-s)^{\zeta}, \tag{2.11}$$

where $C(T)$ depends on $\|v\|_{C([T_0,T];H^1)}$.

To this end, by equation (1.25),

$$\|e^{-ir\Delta}v(r)-e^{-is\Delta}v(s)\|_{L^2}\leq \left\|\int_s^r e^{-is'\Delta}\left(f(v(s'))+b(s')\cdot\nabla v(s')+c(s')v(s')\right)ds'\right\|_{L^2}. \tag{2.12}$$

We use the Strichartz estimate (see, e.g., [13, 63]) with the Strichartz pair $(p+1,q)$, $q=\frac{4(p+1)}{d(p-1)}$. Note that, $\frac{2}{q}=d(\frac12-\frac{1}{p+1})$, so $(p+1,q)$ is admissible. We then get

$$\left\|\int_s^r e^{-is'\Delta}f(v(s'))ds'\right\|_{L^2}\leq C\|f(v)\|_{L^{q'}(s,r;L^{\frac{p+1}{p}})}\leq C(r-s)^{1-\frac{d(p-1)}{4}}\|v\|^p_{L^q(s,r;L^{p+1})}, \tag{2.13}$$

which, via Sobolev's embedding, yields that

$$\left\|\int_s^r e^{-is'\Delta}f(v(s'))ds'\right\|_{L^2}\leq C(r-s)^{1-\frac{d(p-1)}{4}+\frac{p}{q}}\|v\|^p_{C([s,r];H^1)}. \tag{2.14}$$

Moreover, we have

$$\left\|\int_s^r e^{-is'\Delta}\left(b(s')\cdot\nabla v(s')+c(s')v(s')\right)ds'\right\|_{L^2}\leq C(r-s)\|v\|_{C([s,r];H^1)}. \tag{2.15}$$

Hence, plugging (2.14) and (2.15) into (2.12) we obtain (2.11), as claimed.

Thus, using (2.11) and the estimate that for any multi-index υ,

$$\|(1-e^{i(r-s)\Delta})\partial^{\upsilon}(e^{-W(s)}\varphi)\|_{L^2}\leq C(r-s)\|e^{-W(s)}\varphi\|_{H^{2+|\upsilon|}}\leq C(T,\upsilon)(r-s), \tag{2.16}$$

we derive from (2.10) that

$$K_{11}\leq C(T)\int_s^t (r-s)^{\zeta}+(r-s)dr=o(t-s). \tag{2.17}$$

Similarly, we compute

$$\begin{aligned}K_{12}=&\int_s^t\langle e^{-ir\Delta}v(r)-e^{-is\Delta}v(s),(-i)e^{-is\Delta}L^*(s)(e^{-W(s)}\varphi)\rangle dr\\&+\int_s^t\langle v(r),(-i)(1-e^{i(r-s)\Delta})L^*(s)(e^{-W(s)}\varphi)\rangle dr\\&+\int_s^t\langle (L(r)-L(s))v(r),(-i)e^{-W(s)}\varphi\rangle dr,\end{aligned}$$

where $L^*(s)$ is the adjoint operator of $L(s)$. Since by the expressions (1.26) and (1.27),

$$\begin{aligned}\|(L(r)-L(s))v(r)\|_{L^2}\leq& C(1+\|\nabla W\|_{C([s,r];L^\infty)})\|v\|_{C([s,r];H^1)}\sum_{l=1}^N\left|\int_s^r g_l(s')dB_l(s')\right|\\ \leq& C(T)(r-s)^{\alpha},\end{aligned} \tag{2.18}$$

using (2.11) and (2.16) we get

$$K_{12}\leq C(T)\int_s^t (r-s)^{\zeta}+(r-s)+(r-s)^{\alpha}dr=o(t-s). \tag{2.19}$$

Thus, plugging (2.17) and (2.19) into (2.9) we conclude that

$$K_1 = \langle i\Delta(e^{W(s)}v(s)), \varphi\rangle(t-s) + o(t-s). \tag{2.20}$$

Regarding the second term K_2 in (2.8), we see that

$$K_2 = \langle if(e^{W(s)}v(s)), \varphi\rangle(t-s) + \int_s^t \langle f(v(r)) - f(v(s)), (-i)e^{-W(s)}\varphi\rangle dr. \tag{2.21}$$

Since

$$|f(v(r)) - f(v(s))| \leq C(|v(r)|^{p-1} + |v(s)|^{p-1})|v(r) - v(s)|, \tag{2.22}$$

Sobolev's embedding $H^1 \hookrightarrow L^{p+1}$ yields that

$$\begin{aligned}\left|\langle f(v(r)) - f(v(s)), (-i)e^{-W(s)}\varphi\rangle\right| \leq& \|e^{-W}\varphi\|_{C([s,t];L^{p+1})}\|f(v(r)) - f(v(s))\|_{L^{\frac{p+1}{p}}} \\ \leq& \|e^{-W}\varphi\|_{C([s,t];L^{p+1})}\left(\|v(r)\|^{p-1}_{L^{p+1}} + \|v(s)\|^{p-1}_{L^{p+1}}\right)\|v(r) - v(s)\|_{L^{p+1}} \\ \leq& C\|e^{-W}\varphi\|_{C([s,t];L^{p+1})}\|v\|^{p-1}_{C([s,t];H^1)}\|v(r) - v(s)\|_{H^1},\end{aligned}$$

which yields that

$$\begin{aligned}&\left|\int_s^t \langle f(v(r)) - f(v(s)), (-i)e^{-W(s)}\varphi\rangle dr\right| \\ \leq& C\|e^{-W}\varphi\|_{C([T_0,T];L^{p+1})}\|v\|^{p-1}_{C([T_0,T];H^1)} \sup_{s\leq r\leq t}\|v(r) - v(s)\|_{H^1}(t-s) = o(t-s),\end{aligned} \tag{2.23}$$

where in the last step we used the fact that $\sup_{s\leq r\leq t}\|v(r) - v(s)\|_{H^1} = o(1)$ as $t \to s$, due to the continuity of v in H^1. Thus, we obtain

$$K_2 = \langle if(e^{W(s)}v(s)), \varphi\rangle(t-s) + o(t-s). \tag{2.24}$$

Therefore, plugging (2.20) and (2.24) into (2.8) we obtain (2.7), as claimed.

(iii) *Estimate of* $\langle(\delta e^W)_{st}\delta u_{st}, \varphi\rangle$. By the integration by parts formula and Hölder's inequality,

$$\begin{aligned}\langle(\delta e^W)_{st}\delta v_{st}, \varphi\rangle =& \int_s^t \langle v(r), (-i)e^{-W(r)}\Delta\left(e^{W(r)}\overline{(\delta e^W)_{st}}\varphi\right)\rangle + \langle f(v(r)), (-i)\overline{(\delta e^W)_{st}}\varphi\rangle dr \\ \leq& \int_s^t \|v(r)\|_{L^2}\|\Delta\left(e^{W(r)}\overline{(\delta e^W)_{st}}\varphi\right)\|_{L^2} + \|v(r)\|^p_{L^{\rho p}}\|(\delta e^W)_{st}\|_{L^\infty}\|\varphi\|_{L^{\rho'}}dr,\end{aligned} \tag{2.25}$$

where $\rho \in (1,\infty)$ is taken such that $2 \leq \rho p \leq 2 + \frac{4}{d-2}$ if $d \geq 3$, $2 \leq \rho p < \infty$ if $d = 1, 2$, $\frac{1}{\rho} + \frac{1}{\rho'} = 1$. Since for any multi-index υ,

$$\|\partial^\upsilon_x(\delta e^W)_{st}\|_{L^\infty} \leq C(T,\alpha)(t-s)^\alpha,$$

using Sobolev's embedding $H^1 \hookrightarrow L^{\rho p}$, we obtain

$$\begin{aligned}\langle(\delta e^W)_{st}\delta v_{st}, \varphi\rangle \leq& C\int_s^t \|v(r)\|_{L^2} + \|v(r)\|^p_{H^1}dr(t-s)^\alpha \\ \leq& C(1 + \|v\|^p_{C([s,T];H^1)})(t-s)^\alpha = o(t-s).\end{aligned} \tag{2.26}$$

Now, plugging (2.6), (2.7) and (2.26) into (2.1) and using $X = e^W v$ we obtain

$$\langle\delta X_{st}, \varphi\rangle = \langle i\Delta X(s) + if(X(s)) - \mu X(s), \varphi\rangle(t-s) + \sum_{k=1}^N \langle i\phi_k g_k(s)X(s), \varphi\rangle\delta B_{k,st}$$

$$+\sum_{j,k=1}^{N}\langle -\phi_j\phi_k g_j(s)g_k(s)X(s)+i\phi_k g'_{kj}(s)X(s),\varphi\rangle\mathbb{B}_{jk,st}+o(t-s). \tag{2.27}$$

In particular, this yields that for any $\varphi\in C_c^\infty$,

$$\langle X,\varphi\rangle\in C^\alpha([T_0,T],\mathbb{R}). \tag{2.28}$$

Let $Y:=(Y_k)$ with $Y_k:=\langle i\phi_k g_k X,\varphi\rangle$. We claim that

$$Y\in\mathcal{D}_B^{2\alpha}([T_0,T];\mathbb{R}^N), \tag{2.29}$$

with the Gubinelli derivative

$$Y'_{kj}=\langle -\phi_j\phi_k g_j g_k X+i\phi_k g'_{kj}X,\varphi\rangle\in C^\alpha([T_0,T];\mathbb{R}). \tag{2.30}$$

To this end, using (2.28) and the fact that $g_k\in C^\alpha([T_0,T];\mathbb{R})$, we have

$$Y_k=g_k\langle X,-i\phi_k\varphi\rangle\in C^\alpha([T_0,T];\mathbb{R}).$$

Moreover, note that

$$\delta Y_{k,st}=\delta g_{k,st}\langle X(s),-i\phi_k\varphi\rangle+g_k(s)\langle\delta X_{st},-i\phi_k\varphi\rangle+\delta g_{k,st}\langle\delta X_{st},-i\phi_k\varphi\rangle. \tag{2.31}$$

Since $(g_k)\in\mathcal{D}_B^{2\alpha}([T_0,T];\mathbb{R}^N)$, we have

$$\delta g_{k,st}=\sum_{j=1}^{N}g'_{kj}(s)\delta B_{j,st}+O((t-s)^{2\alpha}). \tag{2.32}$$

It also follows from (2.27) that

$$\langle\delta X_{st},-i\phi_k\varphi\rangle=-\sum_{j=1}^{N}\langle\phi_j g_j(s)X(s),\phi_k\varphi\rangle\delta B_{j,st}+O((t-s)^{2\alpha}). \tag{2.33}$$

Plugging (2.32) and (2.33) into (2.31) we obtain

$$\delta Y_{k,st}=\sum_{j=1}^{N}\langle -\phi_j\phi_k g_j(s)g_k(s)X(s)+i\phi_k g'_{kj}(s)X(s),\varphi\rangle\delta B_{j,st}+O((t-s)^{2\alpha}), \tag{2.34}$$

which yields (2.29) and (2.30), as claimed.

Thus, we conclude from (2.28), (2.29) and (2.30) that $\langle X,\varphi\rangle\in\mathcal{D}_B^{2\alpha}([T_0,T];\mathbb{R})$ with the Gubinelli derivative $\langle i\phi_k g_k X,\varphi\rangle$, $\alpha\in(\frac13,\frac12)$, $X:=e^W v$ satisfies equation (1.1) in the sense of Definition 1.2 and (1.19) follows from (2.34). Therefore, the proof is complete. □

3. Geometrical decomposition

This section mainly treats the geometrical decomposition of solutions to equation

$$\begin{cases} i\partial_t u+\Delta u+|u|^{p-1}u+b_*\cdot\nabla u+c_*u=0,\\ u(T)=R(T),\end{cases} \tag{3.1}$$

where $R=\sum_{k=1}^K R_k$, $\{R_k\}$ are given by (1.12) and $T>0$ is sufficiently large. We mainly focus on the critical case, i.e., $p=1+\frac4d$, as the subcritical case is easier and can be proved similarly.

For convenience, we set $\mathcal{P}_k:=(\alpha_k,\theta_k,w_k)\in\mathbb{X}:=\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}$, $1\le k\le K$, and $\mathcal{P}:=(\mathcal{P}_1,\cdots,\mathcal{P}_K)\in\mathbb{X}^K$. For simplicity of exposition, we will omit the dependence on $\omega(\in\Omega)$.

3.1. **Critical case.**

Proposition 3.1. *(Geometrical decomposition) Assume that u solves* (3.1) *with* $p = 1 + \frac{4}{d}$, $d \geq 1$. *For any T sufficiently large, there exist* $0 \leq T^* < T$ *and unique modulation parameters* $\mathcal{P} \in C^1([T^*, T]; \mathbb{X}^K)$*, such that u admits the geometrical decomposition*

$$u(t,x) = \sum_{k=1}^{K} \widetilde{R}_k(t,x) + \varepsilon(t,x) \ \left(=: \widetilde{R}(t,x) + \varepsilon(t,x)\right), \tag{3.2}$$

with the modulation parameters $\mathcal{P}_k := (\alpha_k, \theta_k, w_k) \in \mathbb{X}$ *and*

$$\widetilde{R}_k(t,x) := Q_{w_k(t)}\,(x - v_k t - \alpha_k(t))\, e^{i\left(\frac{1}{2}v_k\cdot x - \frac{1}{4}|v_k|^2 t + (w_k^0)^{-2}t + \theta_k(t)\right)}, \tag{3.3}$$

satisfying

$$\varepsilon(T) = 0, \ \ \mathcal{P}_k(T) = (x_k^0, \theta_k^0, w_k^0). \tag{3.4}$$

Moreover, the following orthogonality conditions hold on $[T^*, T]$*: for every* $1 \leq k \leq K$*,*

$$\begin{aligned} &\mathrm{Re}\int \nabla\widetilde{R}_k(t)\overline{\varepsilon}(t)dx = 0, \ \ \mathrm{Im}\int \widetilde{R}_k(t)\overline{\varepsilon}(t)dx = 0, \\ &\mathrm{Re}\int \left(\Lambda_k\widetilde{R}_k(t) - \frac{i}{2}v_k\cdot y_k(t)\widetilde{R}_k(t)\right)\bar{\varepsilon}(t)dx = 0, \end{aligned} \tag{3.5}$$

where

$$\Lambda_k := \frac{2}{p-1}I_d + y_k\cdot\nabla, \ \ \textit{with } y_k(t) = x - v_k t - \alpha_k(t). \tag{3.6}$$

Remark 3.2. (i). *The orthogonality conditions in* (3.5) *correspond to the coercivity of linearized operators around the ground state in Lemma 6.2. The only one remaining unstable direction* $\mathrm{Re}\langle\widetilde{R}_k, \varepsilon\rangle$ *will be controlled by the almost conservation of the local mass in Corollary 4.2 below.*

(ii). *We note that, the frequency* w_k^0 *in the phase of* $\widetilde{R}_k$ *is fixed, but the frequency parameter* $w_k(t)$ *in* $Q_{w_k(t)}$ *varies with time. In Proposition 3.4 below we are also able to fix the frequency* w_k^0 *in* $Q_{w_k^0}$ *in the subcritical case. This is possible because the linearized operators in the subcritical case have one less unstable direction than those in the critical case.*

The proof of Proposition 3.1 is based on the implicit function theorem. See, e.g., [47]. For the reader's convenience, we present the proof in the subsection 6.3 of the Appendix in the fashion close to that of [12].

In the sequel, we set $B_{*,l}(t) := \int_t^\infty g_l(s)dB_l(s)$, $B_*(t) = \sup_{t\leq s<\infty}\sum_{l=1}^N |B_{*,l}(s)|$. Since $g_l \in L^2(\mathbb{R}^+)$, we have

$$\lim_{t\to\infty} B_*(t) = 0, \ \ \mathbb{P}-a.s.. \tag{3.7}$$

In particular, for $\mathbb{P}$-a.e. $\omega \in \Omega$, we may take a large (random) time $T_* = T_*(\omega) > 0$ such that

$$\sup_{t\geq T_*} B_*(t) \leq 1. \tag{3.8}$$

We also consider $T^* \geq T_*(\omega)$ sufficiently large such that for any $t \in [T^*, T]$,

$$\sup_{T^*\leq t\leq T} \|\varepsilon(t)\|_{H^1} < 1, \tag{3.9}$$

and

$$B_*(t) + |w_k(t) - w_k^0| + |\alpha_k(t) - x_k^0| \leq \frac{1}{10}\min\{1, w_k^0, x_k^0\}, \tag{3.10}$$

where $1 \leq k \leq K$. Hence, B_*, w_k, w_k^{-1} and $|\alpha_k|$ are bounded by a deterministic constant on $[T^*, T]$.

Let us mention that, under the conditions (3.9) and (3.10), the constants in the estimates of the modulations equations and functionals below are uniformly bounded, which are important in the bootstrap arguments when constructing multi-solitons.

As we shall see in Proposition 5.2 later, there exists $T_0 > 0$ large enough such that the conditions (3.9) and (3.10) hold on $[T_0, T]$.

Next, the dynamic of geometric parameters are controlled by the modulation equation below.

Proposition 3.3. *(Control of modulation equations) Define the modulation equations by*

$$Mod_k(t) := |\dot{w}_k(t)| + |\dot{\alpha}_k(t)| + |\dot{\theta}_k(t) - (w_k^{-2}(t) - (w_k^0)^{-2})|, \tag{3.11}$$

where $1 \leq k \leq K$, and set $Mod := \sum_{k=1}^K Mod_k$. For any T sufficiently large, and for T^* close to T such that Proposition 3.1 and the a prior bounds (3.9) *and* (3.10) *hold. Then there exist universal deterministic constants $C, \delta_1, \delta_2 > 0$, depending on w_k^0, x_k^0, v_k and δ_0, such that*

$$Mod(t) \leq C(\|\varepsilon(t)\|_{H^1} + B_*(t)\phi(\delta_1 t) + e^{-\delta_2 t}), \quad \forall t \in [T^*, T]. \tag{3.12}$$

Proof. For simplicity we set the phase function

$$\Phi_k(t,x) := \frac{1}{2} v_k \cdot x - \frac{1}{4}|v_k|^2 t + (w_k^0)^{-2} t + \theta_k(t), \tag{3.13}$$

where $1 \leq k \leq K$. Using the explicit formula (3.3) we compute

$$\begin{aligned} i\partial_t \widetilde{R}_k(t,x) = & \left(\frac{|v_k|^2}{4} - (w_k^0)^{-2} - \dot{\theta}_k(t)\right) \widetilde{R}_k(t,x) - i(\dot{\alpha}_k(t) + v_k) \cdot \nabla Q_{w_k(t)}(x - v_k t - \alpha_k) e^{i\Phi_k(t,x)} \\ & - i\frac{\dot{w}_k(t)}{w_k(t)} \Lambda_k Q_{w_k(t)}(x - v_k t - \alpha_k) e^{i\Phi_k(t,x)}, \end{aligned} \tag{3.14}$$

and

$$\nabla \widetilde{R}_k(t,x) - \frac{i}{2} v_k \widetilde{R}_k(t,x) = \nabla Q_{w_k(t)}(x - v_k t - \alpha_k) e^{i\Phi_k(t,x)}, \tag{3.15}$$

$$\Delta \widetilde{R}_k(t,x) = \left(\Delta Q_{w_k(t)} + i v_k \cdot \nabla Q_{w_k(t)} - \frac{|v_k|^2}{4} Q_{w_k(t)}\right)(x - v_k t - \alpha_k) e^{i\Phi_k(t,x)}. \tag{3.16}$$

Then, it follows from (1.9), (3.14) and (3.16) that

$$\begin{aligned} & i\partial_t \widetilde{R}_k(t,x) + \Delta \widetilde{R}_k(t,x) + |\widetilde{R}_k(t,x)|^{p-1} \widetilde{R}_k(t,x) \\ = & \left(-i\frac{\dot{w}_k(t)}{w_k(t)} \Lambda_k Q_{w_k(t)} - i\dot{\alpha}_k(t) \nabla Q_{w_k(t)} - \left(\dot{\theta}_k(t) - (w_k^{-2}(t) - (w_k^0)^{-2})\right) Q_{w_k(t)}\right)(x - v_k t - \alpha_k) e^{i\Phi_k(t,x)}. \end{aligned} \tag{3.17}$$

Moreover, set

$$H_1 := -\sum_{j \neq k} \left(i\frac{\dot{w}_j}{w_j} \Lambda_j Q_{w_j} + i\dot{\alpha}_j \nabla Q_{w_j} + \left(\dot{\theta}_j - (w_j^{-2} - (w_j^0)^{-2})\right) Q_{w_j}\right)(x - v_k t - \alpha_k) e^{i\Phi_j}, \tag{3.18}$$

$$H_2 := |\widetilde{R}|^{p-1} \widetilde{R} - \sum_{k=1}^K |\widetilde{R}_k|^{p-1} \widetilde{R}_k, \tag{3.19}$$

$$H_3 := |\widetilde{R} + \varepsilon|^{p-1}(\widetilde{R} + \varepsilon) - |\widetilde{R}|^{p-1} \widetilde{R}. \tag{3.20}$$

It then follows from equation (3.1), (3.2) and (3.14) that

$$i\partial_t \varepsilon(t,x) + \Delta \varepsilon(t,x)$$

$$- \left(i\frac{\dot{w}_k(t)}{w_k(t)}\Lambda_k Q_{w_k(t)} + i\dot{\alpha}_k(t)\nabla Q_{w_k(t)} + \left(\dot{\theta}_k(t) - (w_k^{-2}(t) - (w_k^0)^{-2})\right) Q_{w_k(t)}\right)(x - v_k t - \alpha_k)e^{i\Phi_k(t,x)}$$
$$= - H_1(t,x) - H_2(t,x) - H_3(t,x) - b_*(t,x)\cdot(\nabla\widetilde{R}(t,x) + \nabla\varepsilon(t,x)) - c_*(\widetilde{R}(t,x) + \varepsilon(t,x)). \quad (3.21)$$

We are now in position to derive the estimates of modulation equations.

(i) Estimate of $\dot{\alpha}_k$. Taking the inner product of (3.21) with $\nabla\widetilde{R}_k - \frac{i}{2}v_k\widetilde{R}_k$, then taking the imaginary part and using (3.15) we get

$$\mathrm{Re}\langle \partial_t\varepsilon, \nabla\widetilde{R}_k - \frac{i}{2}v_k\widetilde{R}_k\rangle + \mathrm{Im}\langle \Delta\varepsilon, \nabla\widetilde{R}_k - \frac{i}{2}v_k\widetilde{R}_k\rangle - \frac{\dot{w}_k}{w_k}\mathrm{Re}\langle \Lambda Q_{w_k}, \nabla Q_{w_k}\rangle - \dot{\alpha}_k\|\nabla Q_{\omega_k}\|_{L^2}^2$$
$$- \left(\dot{\theta}_k - (w_k^{-2} - (w_k^0)^{-2})\right)\mathrm{Im}\langle Q_{w_k}, \nabla Q_{w_k}\rangle$$
$$= - \mathrm{Im}\langle H_1 + H_2 + H_3, \nabla\widetilde{R}_k - \frac{i}{2}v_k\widetilde{R}_k\rangle - \mathrm{Im}\langle b_*\cdot(\nabla\widetilde{R} + \nabla\varepsilon) + c_*(\widetilde{R} + \varepsilon), \nabla\widetilde{R}_k - \frac{i}{2}v_k\widetilde{R}_k\rangle. \quad (3.22)$$

We first treat the L.H.S. of (3.22). It follows from the orthogonality conditions (3.5) that,

$$\mathrm{Re}\langle \partial_t\varepsilon, \nabla\widetilde{R}_k - \frac{i}{2}v_k\widetilde{R}_k\rangle = \frac{d}{dt}\mathrm{Re}\langle \varepsilon, \nabla\widetilde{R}_k - \frac{i}{2}v_k\widetilde{R}_k\rangle - \mathrm{Re}\langle \varepsilon, \partial_t\left(\nabla\widetilde{R}_k - \frac{i}{2}v_k\widetilde{R}_k\right)\rangle$$
$$= - \mathrm{Re}\langle \varepsilon, \partial_t\left(\nabla\widetilde{R}_k - \frac{i}{2}v_k\widetilde{R}_k\right)\rangle.$$

Then by using (3.9), (3.10) and (3.15), there exists $C > 0$ depending on w_k^0, x_k^0, v_k, such that

$$|\mathrm{Re}\langle \partial_t\varepsilon, \nabla\widetilde{R}_k - \frac{i}{2}v_k\widetilde{R}_k\rangle| = |\mathrm{Re}\langle \varepsilon, \partial_t\left(\nabla Q_{w_k}(x - v_k t - \alpha_k)e^{i\Phi_k}\right)\rangle| \le C(Mod_k + 1)\|\varepsilon\|_{L^2}, \quad (3.23)$$

and similarly we have

$$|\mathrm{Im}\langle \Delta\varepsilon, \nabla\widetilde{R}_k - \frac{i}{2}v_k\widetilde{R}_k\rangle| = |\mathrm{Im}\langle \varepsilon, \Delta(\nabla Q_{w_k}(x - v_k t - \alpha_k)e^{i\Phi_k})\rangle| \le C\|\varepsilon\|_{L^2}. \quad (3.24)$$

Moreover, by the radial symmetry of Q_{w_k},

$$\langle \Lambda Q_{w_k}, \nabla Q_{w_k}\rangle = 0, \ \ \langle Q_{w_k}, \nabla Q_{w_k}\rangle = 0. \quad (3.25)$$

Regarding the R.H.S. of (3.22), we claim that there exist deterministic positive constants C, δ_1, δ_2 depending on w_k^0, x_k^0, v_k and δ_0 which is defined as in (1.5), such that

$$|\text{R.H.S. of } (3.22)| \le C(\|\varepsilon(t)\|_{H^1} + B_*(t)\phi(\delta_1 t) + (Mod(t) + 1)e^{-\delta_2 t}). \quad (3.26)$$

In order to prove (3.26), we use (3.9), (3.10) and Lemma 6.3 to derive that

$$|\langle H_1(t), \nabla\widetilde{R}_k(t) - \frac{i}{2}v_k\widetilde{R}_k(t)\rangle| + |\langle H_2(t), \nabla\widetilde{R}_k(t) - \frac{i}{2}v_k\widetilde{R}_k(t)\rangle| \le C(Mod(t) + 1)e^{-\delta_2 t}. \quad (3.27)$$

Moreover, since $p \le 1 + \frac{4}{d}$, we may take $\rho \ge 1$ close to 1 such that $\rho p \le \frac{2d}{d-2}$. Taking into account

$$|H_3| \le C(|\widetilde{R}|^{p-1} + |\varepsilon|^{p-1})|\varepsilon|, \quad (3.28)$$

Gagliardo-Nirenberg's inequality and (3.15), we get

$$|\langle H_3, \widetilde{R}_k\rangle| \le \int (|\widetilde{R}|^{p-1} + |\varepsilon|^{p-1})|\varepsilon||\nabla\widetilde{R}_k - \frac{i}{2}v_k\widetilde{R}_k|dx$$
$$\le \sum_{k=1}^{K} \|\widetilde{R}^{p-1}(\nabla\widetilde{R}_k - \frac{i}{2}v_k\widetilde{R}_k)\|_{L^2}\|\varepsilon\|_{L^2} + \|\nabla\widetilde{R}_k - \frac{i}{2}v_k\widetilde{R}_k\|_{L^{\rho'}}\|\varepsilon\|_{L^{\rho p}}^p$$
$$\le C\left(\|\varepsilon\|_{L^2} + \|\varepsilon\|_{H^1}^p\right) \le C\|\varepsilon\|_{H^1}. \quad (3.29)$$

Concerning the lower order perturbations, applying Lemma 6.3 again we have

$$\mathrm{Re}\langle b_* \cdot \nabla\widetilde{R} + c_*\widetilde{R}, \nabla\widetilde{R}_k - \frac{i}{2}v_k\widetilde{R}_k\rangle = \mathrm{Re}\langle b_* \cdot \nabla\widetilde{R}_k + c_*\widetilde{R}_k, \nabla\widetilde{R}_k - \frac{i}{2}v_k\widetilde{R}_k\rangle + O(e^{-\delta_2 t}), \tag{3.30}$$

where the implicit constant is independent of ω, due to (3.10). Note that, by (1.30), (3.15) and the change of variables,

$$\begin{aligned}
&|\mathrm{Re}\langle b_* \cdot \nabla\widetilde{R}_k, \widetilde{R}_k\rangle| = 2\Big|\sum_{l=1}^{N} B_{*,l}\mathrm{Im}\langle\nabla\phi_l \cdot \nabla\widetilde{R}_k, \nabla\widetilde{R}_k - \frac{i}{2}v_k\widetilde{R}_k\rangle\Big| \\
\leq& CB_*\sum_{l=1}^{N}\left(|v_k|\int|\nabla\phi_l(y+v_kt+\alpha_k)||Q_{w_k}\nabla Q_{w_k}(y)|dy + \int|\nabla\phi_l(y+v_kt+\alpha_k)||\nabla Q_{w_k}(y)|^2dy\right).
\end{aligned} \tag{3.31}$$

Since by (3.10), $|\alpha_k| \leq 2|x_k^0|$, and for $|y| \leq \frac{|v_k|t}{2w_k}$ and t large enough such that $t \geq \frac{8|x_k^0|}{|w_k|}$, $|y + v_kt + \alpha_k| \geq \frac{1}{2}|v_k|t - |\alpha_k| \geq \frac{1}{4}|v_k|t$. Then, by the exponential decay of Q, the lower bound $\inf_t w_k > 0$, $\min_{1\leq k\leq K}|v_k| > 0$, the decay conditions in Assumption $(A1)$ and the decay function ϕ given by (1.18), the first integration on the R.H.S. above can be bounded by

$$\begin{aligned}
&CB^*\int_{|y|\leq\frac{|v_k|t}{2}}|\nabla\phi_l(y+v_kt+\alpha_k)||Q_{w_k}\nabla Q_{w_k}(y)|dy + C\int_{|y|\geq\frac{|v_k|t}{2}}|Q_{w_k}\nabla Q_{w_k}(y)|dy \\
\leq& CB^*\left(\phi(\frac{1}{4}|v_k|t)\int|\nabla Q_{w_k}Q_{w_k}(y)|dy + e^{-\delta_2 t}\right) \\
\leq& CB^*\left(\phi(\frac{1}{4}|v_k|t) + e^{-\delta_2 t}\right),
\end{aligned} \tag{3.32}$$

for some positive constants C, δ_2 depending on w_k^0, x_k^0, v_k and δ_0. Similarly, we have

$$\int|\nabla\phi_l(y+v_kt+\alpha_k)||\nabla Q_{w_k}(y)|^2dy \leq C\left(\phi(\frac{1}{4}|v_k|t) + e^{-\delta_2 t}\right). \tag{3.33}$$

Hence, plugging (3.32) and (3.33) into (3.31) we obtain that

$$|\mathrm{Re}\langle b_*(t) \cdot \nabla\widetilde{R}_k(t), \nabla\widetilde{R}_k(t) - \frac{i}{2}v_k\widetilde{R}_k(t)\rangle| \leq CB^*(t)\left(\phi(\delta_1 t) + e^{-\delta_2 t}\right), \tag{3.34}$$

where $C, \delta_1, \delta_2 > 0$ are universal deterministic constants depending on w_k^0, x_k^0, v_k and δ_0.

Moreover, by (1.31) and analogous arguments,

$$\begin{aligned}
&|\mathrm{Re}\langle c_*\widetilde{R}_k, \nabla\widetilde{R}_k - \frac{i}{2}v_k\widetilde{R}_k\rangle| \\
=&\Big|\mathrm{Re}\langle\sum_{j=1}^{d}\left(\sum_{l=1}^{N}B_{*,l}\partial_j\phi_l\right)^2\widetilde{R}_k - i\sum_{l=1}^{N}\Delta\phi_l B_{*,l}\widetilde{R}_k, \nabla\widetilde{R}_k - \frac{i}{2}v_k\widetilde{R}_k\rangle\Big| \\
\leq& C\sum_{j=1}^{d}\sum_{l=1}^{N}(B_{*,l} + B_{*,l}^2)\int(|\partial_j\phi_l(y+v_kt+\alpha_k)|^2 + |\Delta\phi_l(y+v_kt+\alpha_k)|)|Q_{w_k}\nabla Q_{w_k}(y)|dy \\
\leq& C\sum_{j=1}^{d}\sum_{l=1}^{N}B_{*,l}\left(\int_{|y|\leq\frac{|v_k|t}{2}}(|\partial_j\phi_l(y+v_kt+\alpha_k)|^2 + |\Delta\phi_l(y+v_kt+\alpha_k)|)|Q_{w_k}\nabla Q_{w_k}(y)|dy + e^{-\delta_2 t}\right) \\
\leq& CB_*\left(\phi(\delta_1 t) + e^{-\delta_2 t}\right).
\end{aligned} \tag{3.35}$$

Hence, plugging (3.34) and (3.35) into (3.30) we obtain

$$|\mathrm{Re}\langle b_*(t)\cdot\nabla\widetilde{R}(t)+c_*(t)\widetilde{R}(t),\nabla\widetilde{R}_k(t)-\frac{i}{2}v_k\widetilde{R}_k(t)\rangle|\leq CB_*(t)\left(\phi(\delta_1 t)+e^{-\delta_2 t}\right). \tag{3.36}$$

Using Hölder's inequality and $\|\widetilde{R}_k\|_{H^1}\leq C$ we also have

$$|\mathrm{Re}\langle b_*(t)\cdot\nabla\varepsilon(t)+c_*(t)\varepsilon(t),\nabla\widetilde{R}_k(t)-\frac{i}{2}v_k\widetilde{R}_k(t)\rangle|\leq CB_*(t)\|\varepsilon(t)\|_{H^1}. \tag{3.37}$$

Here, the constants in (3.36) and (3.37) are independent of ω.

Thus, combining (3.27), (3.29), (3.36) and (3.37) we prove (3.26), as claimed.

Therefore, we conclude from (3.23)-(3.26) and the lower bound, via (3.10),

$$\|\nabla Q_{w_k}\|_{L^2}=w_k^{-1}\|\nabla Q\|_{L^2}\geq\frac{1}{2}(w_k^0)^{-1}\|\nabla Q\|_{L^2}$$

that

$$\frac{1}{2}(w_k^0)^{-1}\|\nabla Q\|_{L^2}|\dot{\alpha}_k(t)|\leq C\left((\|\varepsilon(t)\|_{L^2}+e^{-\delta_2 t})Mod(t)+\|\varepsilon(t)\|_{H^1}+B_*(t)\phi(\delta_1 t)+e^{-\delta_2 t}\right), \tag{3.38}$$

where $C,\delta_1,\delta_2>0$ are universal deterministic constants depending on w_k^0, x_k^0, v_k and δ_0.

(ii) Estimate of $\dot{\theta}_k$. Taking the inner product of (3.21) with $\widetilde{R}_k$ and taking the real part we get

$$\begin{aligned}&-\mathrm{Im}\langle\partial_t\varepsilon,\widetilde{R}_k\rangle+\mathrm{Re}\langle\Delta\varepsilon,\widetilde{R}_k\rangle-\left(\dot{\theta}_k-(w_k^{-2}-(w_k^0)^{-2})\right)\|Q_{w_k}\|_{L^2}^2\\ =&-\mathrm{Re}\langle H_1+H_2+H_3,\widetilde{R}_k\rangle-\mathrm{Re}\langle b_*\cdot\nabla\widetilde{R}+c_*\widetilde{R},\widetilde{R}_k\rangle-\mathrm{Re}\langle b_*\cdot\nabla\varepsilon+c_*\varepsilon,\widetilde{R}_k\rangle.\end{aligned} \tag{3.39}$$

Similarly to (3.26), we have

$$|\text{R.H.S. of (3.39)}|\leq C\left(\|\varepsilon(t)\|_{H^1}+B_*(t)\phi(\delta_1 t)+(Mod(t)+1)e^{-\delta_2 t}\right), \tag{3.40}$$

where C,δ_1,δ_2 are universal deterministic constants.

For the L.H.S. of (3.39), we note that, by (3.5), (3.9) and (3.10),

$$|\mathrm{Im}\langle\partial_t\varepsilon,\widetilde{R}_k\rangle|=|\frac{d}{dt}\mathrm{Im}\langle\varepsilon,\widetilde{R}_k\rangle-\mathrm{Im}\langle\varepsilon,\partial_t\widetilde{R}_k\rangle|=|\mathrm{Im}\langle\varepsilon,\partial_t\widetilde{R}_k\rangle|\leq C(Mod_k+1)\|\varepsilon\|_{L^2}, \tag{3.41}$$

and

$$|\mathrm{Re}\langle\Delta\varepsilon,\widetilde{R}_k\rangle|=|\mathrm{Re}\langle\varepsilon,\Delta\widetilde{R}_k\rangle|\leq C\|\varepsilon\|_{L^2}. \tag{3.42}$$

Therefore, we conclude from (3.40)-(3.42) and the identity

$$\|\widetilde{R}_k\|_{L^2}^2=\|Q_{w_k}\|_{L^2}^2=\|Q\|_{L^2}^2$$

that for some deterministic constants $C,\delta_1,\delta_2>0$,

$$\begin{aligned}&\|Q\|_{L^2}^2|\dot{\theta}_k(t)-(w_k^{-2}(t)-w_{k,0}^{-2})|\\ \leq&C\left((\|\varepsilon(t)\|_{L^2}+e^{-\delta_2 t})Mod(t)+\|\varepsilon(t)\|_{H^1}+B_*(t)\phi(\delta_1 t)+e^{-\delta_2 t}\right).\end{aligned} \tag{3.43}$$

(iii) Estimate of $\dot{w}_k$. Taking the inner product of (3.21) with $\Lambda_k\widetilde{R}_k-\frac{i}{2}v_k\cdot y_k\widetilde{R}_k$, y_k as in (3.6), then taking the imaginary part and using the identity

$$\Lambda_k\widetilde{R}_k(t,x)-\frac{i}{2}v_k\cdot(x-v_kt-\alpha_k)\widetilde{R}_k(t,x)=\Lambda_kQ_{w_k(t)}(x-v_kt-\alpha_k)e^{i\Phi_k(t,x)}, \tag{3.44}$$

we derive that

$$\mathrm{Re}\langle\partial_t\varepsilon,\Lambda_k\widetilde{R}_k-\frac{i}{2}v_k\cdot y_k\widetilde{R}_k\rangle+\mathrm{Im}\langle\Delta\varepsilon,\Lambda_k\widetilde{R}_k-\frac{i}{2}v_k\cdot y_k\widetilde{R}_k\rangle-\frac{\dot{w}_k}{w_k}\|\Lambda Q_{w_k}\|_{L^2}^2$$

$$
\begin{aligned}
&- \dot{\alpha}_k \mathrm{Re}\langle \nabla Q_{w_k}, \Lambda Q_{w_k}\rangle - \left(\dot{\theta}_k + ((w_k^0)^{-2} - w_k^{-2})\right) \mathrm{Im}\langle Q_{w_k}, \Lambda Q_{w_k}\rangle \\
=& - \mathrm{Im}\langle H_1 + H_2 + H_3, \Lambda_k \widetilde{R}_k - \frac{i}{2} v_k \cdot y_k \widetilde{R}_k\rangle \\
&- \mathrm{Im}\langle b_* \cdot (\nabla \widetilde{R} + \nabla \varepsilon) + c_* (\widetilde{R} + \varepsilon), \Lambda_k \widetilde{R}_k - \frac{i}{2} v_k \cdot y_k \widetilde{R}_k\rangle. \qquad (3.45)
\end{aligned}
$$

Again the R.H.S. of (3.45) contributes the orders as in (3.26) and (3.40).

For the L.H.S. of (3.45), by (3.5), (3.9), (3.10), (3.44) and the exponential decay (1.5),

$$
|\mathrm{Re}\langle \partial_t \varepsilon, \Lambda_k \widetilde{R}_k - \frac{i}{2} v_k \cdot y_k \widetilde{R}_k\rangle| = |\mathrm{Re}\langle \varepsilon, \partial_t (\Lambda_k \widetilde{R}_k - \frac{i}{2} v_k \cdot y_k \widetilde{R}_k)\rangle| \leq C(Mod_k + 1)\|\varepsilon\|_{L^2}, \qquad (3.46)
$$

and

$$
|\mathrm{Im}\langle \Delta \varepsilon, \Lambda_k \widetilde{R}_k - \frac{i}{2} v_k \cdot y_k \widetilde{R}_k\rangle| = |\mathrm{Im}\langle \varepsilon, \Delta (\Lambda_k \widetilde{R}_k - \frac{i}{2} v_k \cdot y_k \widetilde{R}_k)\rangle| \leq C\|\varepsilon\|_{L^2}.
$$

Moreover, we have

$$
\mathrm{Re}\langle \nabla Q_{w_k}, \Lambda Q_{w_k}\rangle = 0, \quad \mathrm{Im}\langle Q_{w_k}, \Lambda Q_{w_k}\rangle = 0. \qquad (3.47)
$$

Thus, takeing w_k close to w_k^0 such that $\|\Lambda Q_{w_k}\|_{L^2} \geq \frac{1}{2}\|\Lambda Q_{w_k^0}\|_{L^2}$ we obtain that

$$
\frac{1}{2}\|\Lambda Q_{w_k^0}\|_{L^2}^2 |\dot{w}_k(t)| \leq C\left((\|\varepsilon(t)\|_{L^2} + e^{-\delta_2 t}) Mod(t) + \|\varepsilon(t)\|_{H^1} + B_*(t)\phi(\delta_1 t) + e^{-\delta_2 t}\right). \qquad (3.48)
$$

Therefore, combining (3.38), (3.43) and (3.48) together we conclude that

$$
Mod(t) \leq C\left((\|\varepsilon(t)\|_{L^2} + e^{-\delta_2 t}) Mod(t) + \|\varepsilon(t)\|_{H^1} + B_*(t)\phi(\delta_1 t) + e^{-\delta_2 t}\right). \qquad (3.49)
$$

where $C, \delta_1, \delta_2 > 0$ are deterministic constants. Hence, for t close to T and large enough such that $C(\|\varepsilon(t)\|_{L^2} + e^{-\delta_2 t}) \leq 1/2$ we obtain

$$
Mod(t) \leq C\left(\|\varepsilon(t)\|_{H^1} + B_*(t)\phi(\delta_1 t) + e^{-\delta_2 t}\right). \qquad (3.50)
$$

The proof of Proposition 3.3 is complete. □

3.2. **Subcritical case.** In the subcritical case, we only need to control three unstale directions, corresponding to the coercivity of the linearized operator. Two of them will be controlled by the following geometrical decomposition and the remaining one $\mathrm{Re}\langle \widetilde{R}_k, \varepsilon\rangle$ can be controlled by the almost conservation of the local mass in Section 4 below.

Proposition 3.4. *(Geometrical decomposition) Assume that u solves* (3.1) *with* $1 < p < 1 + \frac{4}{d}$*. For any T sufficiently large, there exist $0 \leq T^* < T$ and unique modulation parameters $\mathcal{P}_k := (\alpha_k, \theta_k) \in C^1([T^*, T]; \mathbb{R}^d \times \mathbb{R})$, $1 \leq k \leq K$, such that u admits the geometrical decomposition*

$$
u(t,x) = \sum_{k=1}^{K} \widetilde{R}_k(t,x) + \varepsilon(t,x) \ \left(=: \widetilde{R}(t,x) + \varepsilon(t,x)\right), \qquad (3.51)
$$

where for every $1 \leq k \leq K$,

$$
\widetilde{R}_k(t,x) := Q_{w_k^0}(x - v_k t - \alpha_k(t))\, e^{i(\frac{1}{2} v_k \cdot x - \frac{1}{4}|v_k|^2 t + (w_k^0)^{-2} t + \theta_k(t))}, \qquad (3.52)
$$

the modulation parameters satisfy

$$
\varepsilon(T) = 0, \quad \mathcal{P}_k(T) = (x_k^0, \theta_k^0), \qquad (3.53)
$$

and the following orthogonality conditions hold on $[T^*, T]$*:*

$$\mathrm{Re}\int \nabla\widetilde{R}_k(t)\overline{\varepsilon}(t)dx = 0, \quad \mathrm{Im}\int \widetilde{R}_k(t)\overline{\varepsilon}(t)dx = 0. \tag{3.54}$$

Remark 3.5. (i). *We note that, in* (3.52) $Q_{w_k^0}$ *is indexed by a fixed parameter* w_k^0*, which is different from the previous soliton profile* (3.3) *in the critical case and from* [46] *in the subcritical case, where the parameter* w_k *depends on time.*

(ii). *The proof of Proposition 3.4 is quite similar to that of Proposition 3.1. Actually, the corresponding Jacobian matrices* $(\frac{\partial F^k}{\partial \widetilde{\mathcal{P}}_j})$ *can be obtained from those in the proof of Lemma 6.4 by removing* f_3^k *and* $\widetilde{w}_k$*. Hence, by* (6.23)*, the Jacobian matrix* $\frac{\partial F}{\partial \widetilde{\mathcal{P}}}$ *is still uniformly non-degenerate, the arguments there are applicable in the subcritical case.*

As in the previous critical case, for $\mathbb{P}$-a.e. $\omega \in \Omega$, we take a random time $T_*(\omega) > 0$ large enough such that (3.8)-(3.10) hold on $[T^*, T]$, and so B_*, $\|\varepsilon\|_{H^1}$ and $|\alpha_k|$ bounded by a deterministic constant on $[T^*, T]$.

Using the decomposition (3.51) and orthogonality condition (3.54), we can use similar arguments as in the proof of Proposition 3.3 to derive the control of modulation equations.

Proposition 3.6. *(Control of modulation equations) For any* T *sufficiently large, and for* T^* *close to* T *such that Proposition 3.4 and the a prior bounds* (3.9) *to* (3.10) *hold. Then there exist universal deterministic constants* $C, \delta_1, \delta_2 > 0$*, depending on* w_k^0*,* x_k^0*,* v_k *and* δ_0*, such that,*

$$\sum_{k=1}^{K}(|\dot{\alpha}_k(t)| + |\dot{\theta}_k(t)|) \le C(\|\varepsilon(t)\|_{H^1} + B_*(t)\phi(\delta_1 t) + e^{-\delta_2 t}), \quad t \in [T^*, T]. \tag{3.55}$$

Proof. The arguments follow the lines as in the proof of Proposition 3.3. Using the explicit formula (3.52) we compute

$$i\partial_t\widetilde{R}_k(t,x) = \left(\frac{|v_k|^2}{4} - (w_k^0)^{-2} - \dot{\theta}_k(t)\right)\widetilde{R}_k(t,x) - i(\dot{\alpha}_k(t) + v_k)\cdot\nabla Q_{w_k^0}(x - v_k t - \alpha_k)e^{i\Phi_k(t,x)}, \tag{3.56}$$

where Φ_k is as in (3.13).

Then, by (1.9), $\widetilde{R}_k$ satisfies the equation

$$\begin{aligned} &i\partial_t\widetilde{R}_k(t,x) + \Delta\widetilde{R}_k(t,x) + |\widetilde{R}_k(t,x)|^{p-1}\widetilde{R}_k(t,x) \\ &= -\, i\dot{\alpha}_k(t)\nabla Q_{w_k^0}(x - v_k t - \alpha_k)e^{i\Phi_k(t,x)} - \dot{\theta}_k(t)\widetilde{R}_k(t,x) \end{aligned} \tag{3.57}$$

It then follows from equations (1.29) and (3.57) that

$$\begin{aligned} &i\partial_t\varepsilon + \Delta\varepsilon - i\dot{\alpha}_k\nabla Q_{w_k^0}(x - v_k t - \alpha_k)e^{i\Phi_k} - \dot{\theta}_k\widetilde{R}_k \\ &= -\, H_1 - H_2 - H_3 - b_*\cdot(\nabla\widetilde{R} + \nabla\varepsilon) - c_*(\widetilde{R} + \varepsilon), \end{aligned} \tag{3.58}$$

where

$$H_1 := -\sum_{j\neq k}\left(i\dot{\alpha}_j\nabla Q_{w_j^0} + \dot{\theta}_j Q_{w_j^0}\right)(x - v_k t - \alpha_k)e^{i\Phi_j}, \tag{3.59}$$

$$H_2 := |\widetilde{R}|^{p-1}\widetilde{R} - \sum_{k=1}^{K}|\widetilde{R}_k|^{p-1}\widetilde{R}_k, \tag{3.60}$$

$$H_3 := |\widetilde{R} + \varepsilon|^{p-1}(\widetilde{R} + \varepsilon) - |\widetilde{R}|^{p-1}\widetilde{R}. \tag{3.61}$$

Now, taking the inner product of (3.58) with $\widetilde{R}_k$ and then taking the real part we can control the dynamic of $\dot{\theta}_k$

$$\|Q_{w_k^0}\|_{L^2}^2|\dot{\theta}_k(t)| \leq C(\|\varepsilon(t)\|_{H^1} + B_*(t)\phi(\delta_1 t) + (Mod(t)+1)e^{-\delta_2 t}). \tag{3.62}$$

Moreover, taking the inner product of (3.58) with $\nabla\widetilde{R}_k - \frac{i}{2}v_k\widetilde{R}_k$ and then taking the imaginary part we get the estimate of $\dot{\alpha}_k$

$$\|\nabla Q_{w_k^0}\|_{L^2}^2|\dot{\alpha}_k(t)| \leq C(\|\varepsilon(t)\|_{H^1} + B_*(t)\phi(\delta_1 t) + (Mod(t)+1)e^{-\delta_2 t}). \tag{3.63}$$

Here C, δ_1, δ_2 are deterministic positive constants depending on w_k^0, x_k^0, v_k and δ_0. Therefore, summing over k and taking t close to T we obtain (3.55) and finish the proof. □

4. Local quantities and Lyapunov type functional

In this section we control several important functionals, including the local mass, local momentum, energy and the Lyapunov type functional, for the subcritical and critical cases where $1 < p \leq 1 + \frac{4}{d}$ simultaneously.

Note that, these functionals depend on Brownian paths and the energy is no longer conserved in the stochastic case. Below we perform the path-by-path analysis in order to obtain the sharp estimates. As in Section 3, for $\mathbb{P}$-a.e. $\omega \in \Omega$, we take a random time $T_*(\omega) > 0$ large enough such that (3.8)-(3.10) hold on $[T^*(\omega), T]$, and so $B_*(\omega)$, $\|\varepsilon(\omega)\|_{H^1}$, $|\alpha_k(\omega)|$, $|w_k(\omega)|$ and $|w_k^{-1}(\omega)|$ are bounded by a deterministic constant on $[T^*(\omega), T]$. For simplicity, the dependence on ω is omitted.

4.1. **Local mass and local momentum.**

Let us start with the analysis of the local mass. Because equation (1.29) is invariant under the orthogonal transform, we may take an orthonormal basis $\{\mathbf{e}_j\}_{j=1}^d$ of $\mathbb{R}^d$ as in [46], such that $(v_j - v_k)\cdot \mathbf{e}_1 \neq 0$ for any $j \neq k$. Let $v_{k,1} := v_k \cdot \mathbf{e}_1, 1 \leq k \leq K$. Without loss of generality, we may assume that $v_{1,1} < v_{2,1} < \cdots < v_{K,1}$. Following [16] (see also [46]), we set $A_0 := \frac{1}{4}\min_{2\leq k\leq K}\{v_{k,1} - v_{k-1,1}\}$ and $\sigma_k := \frac{1}{2}(v_{k-1,1} + v_{k,1})$, $2 \leq k \leq K$. Let $\psi(x)$ be a smooth nondecreasing function on $\mathbb{R}$ such that $0 \leq \psi \leq 1$, $\psi(x) = 0$ for $x \leq -A_0$, $\psi(x) = 1$ for $x > A_0$, and there exists $C > 0$ such that

$$(\psi'(x))^2 \leq C\psi(x), \quad (\psi''(x))^2 \leq C\psi'(x), \ \ x \in \mathbb{R}^d. \tag{4.1}$$

The localization functions are defined by

$$\begin{aligned} \varphi_1(t,x) = 1 - \psi\left(\frac{x_1 - \sigma_2 t}{t}\right), \quad \varphi_K(t,x) = \psi\left(\frac{x_1 - \sigma_K t}{t}\right), \\ \varphi_k(t,x) = \psi\left(\frac{x_1 - \sigma_k t}{t}\right) - \psi\left(\frac{x_1 - \sigma_{k+1} t}{t}\right), \quad 2 \leq k \leq K-1. \end{aligned} \tag{4.2}$$

We have the partition of unity $\sum_{k=1}^K \varphi_k(t,x) = 1$. Moreover, for every $1 \leq k \leq K$,

$$|\varphi_k'(t,x)| + |\varphi_k'''(t,x)| + |\partial_t\varphi_k(t,x)| \leq \frac{C}{t}. \tag{4.3}$$

For $1 \leq k \leq K$, define the local mass and local momentum by

$$I_k(t) := \int |u(t,x)|^2\varphi_k(t,x)dx, \quad M_k(t) := \mathrm{Im}\int \nabla u(t,x)\bar{u}(t,x)\varphi_k(t,x)dx. \tag{4.4}$$

Though the local mass and local momentum are no longer conserved, the explicit estimates in Proposition 4.1 below show that both local quantities are almost conserved.

Proposition 4.1. *(Control of local mass and local momentum) We have that for any $t \in [T^*, T]$,*

$$\left|\frac{d}{dt}I_k(t)\right| \leq \frac{C}{t}(\|\varepsilon(t)\|_{H^1}^2 + e^{-\delta_2 t}), \tag{4.5}$$

and

$$\left|\frac{d}{dt}M_k(t)\right| \leq \frac{C}{t}(\|\varepsilon(t)\|_{H^1}^2 + e^{-\delta_2 t}) + CB_*(t)(\|\varepsilon(t)\|_{H^1}^2 + \phi(\delta_1 t) + e^{-\delta_2 t}), \tag{4.6}$$

where C, δ_1, δ_2 are deterministic positive constants depending on w_k^0, x_k^0, v_k and δ_0.

Proof. Using the integration-by-parts formula we compute

$$\frac{d}{dt}I_k = \mathrm{Im}\int(2\bar{u}\partial_{x_1}u + b_*|u|^2)\cdot\nabla\varphi_k dx + \int|u|^2\partial_t\varphi_k dx. \tag{4.7}$$

Note that, the supports of φ_k' and $\partial_t\varphi_k$ are contained in the regime

$$\Omega_1 = [(-A_0+\sigma_2)t, (A_0+\sigma_2)t]\times\mathbb{R}^{d-1}, \quad \Omega_K = [(-A_0+\sigma_K)t, (A_0+\sigma_K)t]\times\mathbb{R}^{d-1},$$

$$\Omega_k = [(-A_0+\sigma_k)t, (A_0+\sigma_k)t]\times\mathbb{R}^{d-1}\cup[(-A_0+\sigma_{k+1})t, (A_0+\sigma_{k+1})t]\times\mathbb{R}^{d-1}, 2\leq k\leq K-1.$$

Taking into account (4.3) we obtain that for some $C > 0$

$$\left|\frac{d}{dt}I_k(t)\right| \leq \frac{C}{t}\int_{\Omega_k}|u(t)|^2 + |\nabla u(t)|^2 dx, \tag{4.8}$$

where we also used (1.30), (3.10) to bounded the coefficient b_*. Note that, for $x\in\Omega_k$ and t large enough so that $t \geq 4A_0^{-1}\max_{1\leq k\leq K}\{1, |x_k^0|\}$,

$$|x - v_l t - \alpha_l| \geq |x_1 - v_{l,1}t| - |\alpha_l| \geq A_0 t - |\alpha_l| \geq \frac{1}{2}A_0 t, \quad 1\leq l\leq K.$$

Using the exponential decay of the ground state we thus obtain

$$\left|\frac{d}{dt}I_k(t)\right| \leq \frac{C}{t}(\|\varepsilon(t)\|_{H^1(\Omega_k)}^2 + \|\widetilde{R}(t)\|_{H^1(\Omega_k)}^2) \leq \frac{C}{t}(\|\varepsilon(t)\|_{H^1}^2 + e^{-\delta_2 t}), \tag{4.9}$$

for $C, \delta_2 > 0$ depending on w_k^0, x_k^0, v_k and δ_0, which yields (4.5).

Concerning the local momentum, straightforward computations show that

$$\begin{aligned}\frac{d}{dt}\mathrm{Im}\int\partial_{x_1}u\bar{u}\varphi_k dx =& 2\int|\partial_{x_1}u|^2\varphi_k' dx - \frac{1}{2}\int|u|^2\varphi_k''' dx - \frac{p-1}{p+1}\int|u|^{p+1}\varphi_k' dx + \mathrm{Im}\int\partial_{x_1}u\bar{u}\partial_t\varphi_k dx\\ &- 2\mathrm{Re}\langle\partial_1 u\varphi_k, b_*\cdot\nabla u + c_*u\rangle - \mathrm{Re}\langle u\partial_1\varphi_k, b_*\cdot\nabla u + c_*u\rangle,\end{aligned} \tag{4.10}$$

and for $2\leq j\leq d$,

$$\begin{aligned}\frac{d}{dt}\mathrm{Im}\int\partial_{x_j}u\bar{u}\varphi_k dx =& 2\mathrm{Re}\int\partial_{x_1}u\partial_{x_j}\bar{u}\varphi_k' dx + \int\partial_{x_j}u\bar{u}\partial_t\varphi_k dx\\ &- 2\mathrm{Re}\langle\partial_j u\varphi_k, b_*\cdot\nabla u + c_*u\rangle.\end{aligned} \tag{4.11}$$

The first line in (4.10) and (4.11) can be bounded similarly as above by, up to a universal constant,

$$\frac{1}{t}\left(\|\varepsilon(t)\|_{H^1}^2 + e^{-\delta_2 t}\right). \tag{4.12}$$

Regarding the remaining inner products involving lower order perturbations, as in the proof of (3.32), the key fact is that, since Q_{w_k} is well localized, x is essentially localized around $|v_k|t$, i.e., $|x|\sim|v_k|t$. Hence, taking into account the decay conditions in Assumption $(A1)$ we get

$$|\mathrm{Re}\langle\partial_1\widetilde{R}(t)\varphi_k(t), b_*(t)\cdot\nabla\varepsilon(t)\rangle + \mathrm{Re}\langle\partial_1\varepsilon(t)\varphi_k(t), b_*(t)\cdot\nabla\widetilde{R}(t)\rangle + \mathrm{Re}\langle\partial_1\widetilde{R}(t)\varphi_k(t), b_*(t)\cdot\nabla\widetilde{R}(t)\rangle|$$

$$
\begin{aligned}
&\leq CB_*(t)(1 + \|\nabla\varepsilon(t)\|_{L^2})(\phi(\delta_1 t) + e^{-\delta_2 t}) \\
&\leq CB_*(t)(\phi(\delta_1 t) + e^{-\delta_2 t}). \qquad (4.13)
\end{aligned}
$$

Taking into account

$$
|\mathrm{Re}\langle \partial_1 \varepsilon \varphi_k, b_* \nabla \varepsilon\rangle| \leq CB_* \|\nabla\varepsilon\|_{L^2}^2, \qquad (4.14)
$$

we thus obtain

$$
|\mathrm{Re}\langle \partial_1 u(t)\varphi_k(t), b_*(t)\cdot\nabla u(t)\rangle| \leq CB_*(t)(\phi(\delta_1 t) + \|\nabla\varepsilon(t)\|_{L^2}^2 + e^{-\delta_2 t}). \qquad (4.15)
$$

Using analogous arguments we obtain

$$
\begin{aligned}
&|\langle \partial_1 u(t)\varphi_k(t), b_*(t)\cdot\nabla u(t) + c_*(t)u(t)\rangle| + |\langle u(t)\partial_1\varphi_k(t), b_*(t)\cdot\nabla u(t) + c_*(t)u(t)\rangle| \\
&\quad + |\langle \partial_j u(t)\varphi_k(t), b_*(t)\cdot\nabla u(t) + c_*(t)u(t)\rangle| \\
&\leq CB_*(t)(\phi(\delta_1 t) + \|\varepsilon(t)\|_{H^1}^2 + e^{-\delta_2 t}) \qquad (4.16)
\end{aligned}
$$

Therefore, combining (4.12) and (4.16) we obtain (4.6). The proof is complete. □

One important outcome of the almost conservation of local mass is the following control of the unstable direction $\mathrm{Re}\langle \widetilde{R}_k, \varepsilon\rangle$ in both the critical and subcritical settings.

Corollary 4.2. *(Control of unstable direction) We have that for any $t \in [T^*, T]$,*

$$
\left|\mathrm{Re}\int \widetilde{R}_k(t)\bar{\varepsilon}(t)dx\right| \leq C\left(\int_t^\infty \frac{1}{s}\|\varepsilon(s)\|_{H^1}^2 ds + \|\varepsilon(t)\|_{L^2}^2 + e^{-\delta_2 t}\right), \qquad (4.17)
$$

where C, δ_2 are deterministic positive constants depending on w_k^0, x_k^0, v_k and δ_0.

Proof. Using the decomposition (3.2) and (3.51), respectively, in the critical and subcritical case, we expand

$$
I_k = \int |\widetilde{R}|^2\varphi_k dx + 2\mathrm{Re}\int \widetilde{R}\bar{\varepsilon}\varphi_k dx + \int |\varepsilon|^2\varphi_k dx. \qquad (4.18)
$$

Note that, by the decoupling Lemma 6.3,

$$
\int |\widetilde{R}|^2\varphi_k dx = \int |\widetilde{R}_k|^2\varphi_k dx + \sum_{j\neq k}\int |\widetilde{R}_j|^2\varphi_k dx + O(e^{-\delta_2 t}). \qquad (4.19)
$$

Since α_k is uniformly bounded, $|\alpha_k| \leq 2|x_k^0|$, and on the support of φ_k, $|x - v_j t| \geq A_0 t$, $j \neq k$, we infer that for t large enough

$$
|x - v_j t - \alpha_j| \geq A_0 t - |\alpha_j| \geq \frac{1}{2}A_0 t, \quad j \neq k, \qquad (4.20)
$$

which yields that

$$
\int |\widetilde{R}_j|^2\varphi_k dx \leq C \int_{|x-v_jt-\alpha_j|\geq \frac{1}{2}A_0 t} Q_{w_j}^2(x - v_j t - \alpha_j)dx \leq Ce^{-\delta_2 t}, \quad j \neq k. \qquad (4.21)
$$

Moreover, since on the support of $1 - \varphi_k$, $|x - v_k t| \geq A_0 t$, and so for t very large it holds that $|x - v_k t - \alpha_k| \geq \frac{1}{2}A_0 t$, we have

$$
\left|\int |\widetilde{R}_k|^2\varphi_k dx - \int |\widetilde{R}_k|^2 dx\right| \leq Ce^{-\delta_2 t}. \qquad (4.22)
$$

Thus, we derive from (4.19), (4.21) and (4.22) that

$$|\int|\widetilde{R}|^2\varphi_k dx-\|\widetilde{R}_k\|_{L^2}^2|\leq Ce^{-\delta_2 t}. \tag{4.23}$$

Similarly,

$$|\mathrm{Re}\int\widetilde{R}\bar{\varepsilon}\varphi_k dx-\mathrm{Re}\int\widetilde{R}_k\bar{\varepsilon}dx|\leq Ce^{-\delta_2 t}\|\varepsilon\|_{L^2}. \tag{4.24}$$

Thus, we conclude from (4.18), (4.23) and (4.24) that

$$I_k(t)=\|\widetilde{R}_k\|_{L^2}^2+2\mathrm{Re}\int\widetilde{R}_k\bar{\varepsilon}dx+\int|\varepsilon|^2\varphi_k dx+O\left(e^{-\delta_2 t}\right),\quad as\ t\to T. \tag{4.25}$$

In particular, letting $t=T$ and using $\varepsilon(T)=0$ we get

$$|I_k(T)-\|\widetilde{R}_k(T)\|_{L^2}^2|\leq Ce^{-\delta_2 T}. \tag{4.26}$$

Note that, in both the critical and subcritical cases,

$$\|\widetilde{R}_k(t)\|_{L^2}=\|\widetilde{R}_k(T)\|_{L^2}. \tag{4.27}$$

In fact, via the scaling invariance, one has $\|\widetilde{R}_k(t)\|_{L^2}=\|\widetilde{R}_k(T)\|_{L^2}=\|Q\|_{L^2}$ in the critical case. While in the subcritical case, since $w_k\equiv w_k^0$, one has $\|\widetilde{R}_k(t)\|_{L^2}=\|\widetilde{R}_k(T)\|_{L^2}=(w_k^0)^{\frac{d}{2}-\frac{2}{p-1}}\|Q\|_{L^2}$.

Therefore, plugging (4.26) and (4.27) into (4.25) we then obtain

$$\mathrm{Re}\int\widetilde{R}_k(t)\bar{\varepsilon}(t)dx=\frac{1}{2}(I_k(t)-I_k(T))-\frac{1}{2}\int|\varepsilon(t)|^2\varphi_k dx+O\left(e^{-\delta_2 t}\right),\quad as\ t\to T, \tag{4.28}$$

which, via Proposition 4.1, yields that

$$\begin{aligned}\left|\mathrm{Re}\int\widetilde{R}_k(t)\bar{\varepsilon}(t)dx\right|&\leq\frac{1}{2}\int_t^T\left|\frac{dI_k}{ds}\right|ds+\frac{1}{2}\int|\varepsilon(t)|^2dx+C\left(e^{-\delta_2 t}\right)\\&\leq C\int_t^T\frac{1}{s}(\|\varepsilon\|_{H^1}^2+e^{-\delta_2 s})ds+C\left(\|\varepsilon(t)\|_{L^2}^2+e^{-\delta_2 t}\right),\end{aligned} \tag{4.29}$$

thereby proving (4.17) by letting T tend to infinity. □

4.2. **Energy.** Proposition 4.3 below is concerned with the control of energy defined by

$$E(u):=\frac{1}{2}\|\nabla u\|_{L^2}^2-\frac{1}{p+1}\|u\|_{L^{p+1}}^{p+1}, \tag{4.30}$$

where u is the solution to equation (3.1).

Again the energy is no longer conserved due to the presence of lower order perturbations (or noise). The variation control of the energy is estimated in the following proposition.

Proposition 4.3. *(Control of energy) There exist deterministic constants $C,\delta_1,\delta_2>0$ depending on w_k^0, x_k^0, v_k and δ_0, such that*

$$\left|\frac{d}{dt}E(u(t))\right|\leq CB_*(t)(\phi(\delta_1 t)+\|\varepsilon(t)\|_{H^1}^2+e^{-\delta_2 t}),\quad\forall t\in[T^*,T]. \tag{4.31}$$

Proof. Using (3.1) and the integration-by-parts formula we compute

$$\frac{d}{dt}E(u)=-\mathrm{Im}\int(\overline{b_*}\cdot\nabla\bar{u}+\overline{c_*u})(\Delta u+|u|^{p-1}u)dx$$

$$
\begin{aligned}
=&2\sum_{l=1}^N B_{*,l}\mathrm{Re}\int \nabla^2\phi_l(\nabla u,\nabla\overline{u})dx-\frac{1}{2}\sum_{l=1}^N B_{*,l}\int\Delta^2\phi_l|u|^2dx\\
&-\frac{p-1}{p+1}\sum_{l=1}^N B_{*,l}\int\Delta\phi_l|u|^{p+1}dx-\mathrm{Im}\int\nabla\sum_{j=1}^d\left(\sum_{l=1}^N\partial_j\phi_lB_{*,l}\right)^2\cdot\nabla u\overline{u}dx.
\end{aligned}
\tag{4.32}
$$

Then, using (3.2) and estimating similarly as in the proof of (3.32), we obtain

$$
\left|\frac{d}{dt}E(u(t))\right|\leq CB_*(t)(\phi(\delta_1 t)+\|\varepsilon(t)\|_{H^1}^2+\|\varepsilon(t)\|_{H^1}^{p+1}+e^{-\delta_2 t}),
\tag{4.33}
$$

where $C,\delta_1,\delta_2>0$ depending on w_k^0, x_k^0, v_k and δ_0, and thus (4.31) follows. □

4.3. **Lyapunov type functional.** The key ingredient to control the size of remainder is the following Lyapunov type functional

$$
G(t):=2E(u(t))+\sum_{k=1}^K\left\{\left((w_k^0)^{-2}+\frac{|v_k|^2}{4}\right)I_k(t)-v_k\cdot M_k(t)\right\}.
\tag{4.34}
$$

Recall that, in the subcritical case, we have $w_k(t)\equiv w_k^0$ in the geometrical decomposition (3.51).

The main estimate for $G(t)$ is formulated in Proposition 4.4 below.

Proposition 4.4. *(Expansion of Lyapunov type functional) Let* $1<p\leq 1+\frac{4}{d}$, $d\geq 1$. *Then, for any* $t\in[T^*,T]$ *we have*

$$
\begin{aligned}
G(t)=&\sum_{k=1}^K(2E(Q_{w_k^0})+(w_k^0)^{-2}\|Q_{w_k^0}\|_{L^2}^2)+H(\varepsilon(t))+\mathcal{O}(|w_k(t)-w_k^0|\,\|\varepsilon(t)\|_{H^1}^2+e^{-\delta_2 t})\\
&+o(\|\varepsilon(t)\|_{H^1}^2)+\mathcal{O}\left(\sum_{k=1}^K\left|(w_k^0-w_k(t))\mathrm{Re}\int\widetilde{R}_k(t)\bar{\varepsilon}(t)dx\right|\right),\quad as\ t\to T,
\end{aligned}
\tag{4.35}
$$

where $H(\varepsilon)$ *contains the quadratic terms of* ε, *i.e.,*

$$
\begin{aligned}
H(\varepsilon)=&\int|\nabla\varepsilon|^2dx-\sum_{k=1}^K\int\frac{p+1}{2}|\widetilde{R}_k|^{p-1}|\varepsilon|^2+(p-1)|\widetilde{R}_k|^{p-3}[\mathrm{Re}(\widetilde{R}_k\bar{\varepsilon})^2]dx\\
&+\sum_{k=1}^K\left\{\left(w_k^{-2}+\frac{|v_k|^2}{4}\right)\int|\varepsilon|^2\varphi_kdx-v_k\cdot\mathrm{Im}\int\nabla\varepsilon\bar{\varepsilon}\varphi_kdx\right\},
\end{aligned}
\tag{4.36}
$$

and the implicit constant and δ *are independent of* ω.

Remark 4.5. (*i*). *It should be mentioned that, the main part in* (4.35) *is independent of time. This fact is obvious in the subcritical case because the parameter* $\omega_k\equiv w_k^0$ *is independent of time. While, in the critical case it relies on the scaling invariance and Pohozaev identity (see* (4.52) *below).*

(*ii*). *Another important property is the coercivity of the quadratic term* $H(\varepsilon)$, *i.e., for some* $C>0$,

$$
H(\varepsilon)\geq C\|\varepsilon\|_{H^1}^2-\frac{1}{C}\sum_{k=1}^K\left(\mathrm{Re}\int\widetilde{R}_k\bar{\varepsilon}dx\right)^2.
\tag{4.37}
$$

The coercivity in particular enables us to control the remainder ε *in the geometrical decomposition. In the subcritical case,* (4.37) *follows from the orthogonality conditions* (3.54) *and an immediate adaptation to all dimensions of the proof given for the one-dimensional case in* [47, Appendix B].

In the critical case, (4.37) *follows from the orthogonality conditions* (3.5) *and* [16, Proposition 3.17].

Proof of Proposition 4.4. First, using (3.2) (or (3.51)) and Lemma 6.3 we expand the kinetic energy

$$\begin{aligned}\int|\nabla u|^2dx&=\int|\nabla\widetilde{R}|^2dx+\int|\nabla\varepsilon|^2dx-2\mathrm{Re}\int\Delta\widetilde{R}\bar{\varepsilon}dx\\&=\sum_{k=1}^K\left(\int|\nabla\widetilde{R}_k|^2dx+\int|\nabla\varepsilon|^2dx-2\mathrm{Re}\int\Delta\widetilde{R}_k\bar{\varepsilon}dx\right)+O(e^{-\delta_2t})\\&=\sum_{k=1}^K\left(\int|\nabla Q_{w_k}|^2dx+\frac{|v_k|^2}{4}\int|Q_{w_k}|^2dx+\int|\nabla\varepsilon|^2dx-2\mathrm{Re}\int\Delta\widetilde{R}_k\bar{\varepsilon}dx\right)+O(e^{-\delta_2t}).\end{aligned}\tag{4.38}$$

Moreover, for the potential energy we expand

$$\begin{aligned}\int|u|^{p+1}dx=&\int|\widetilde{R}|^{p+1}dx+(p+1)\mathrm{Re}\int|\widetilde{R}|^{p-1}\widetilde{R}\bar{\varepsilon}dx\\&+\frac{p+1}{2}\int\frac{p+1}{2}|\widetilde{R}|^{p-1}|\varepsilon|^2+(p-1)|\widetilde{R}|^{p-3}[\mathrm{Re}(\widetilde{R}\bar{\varepsilon})^2]dx+O(Er),\end{aligned}\tag{4.39}$$

where the error term

$$Er:=\int\sum_{z^*,\overline{z}^*\in\{z,\bar{z}\}}\int_0^1r\int_0^1\left(\partial_{z^*\overline{z}^*}g(\widetilde{R}+sr\varepsilon)-\partial_{z^*\overline{z}^*}g(\widetilde{R})\right)\varepsilon^2drdsdx,\tag{4.40}$$

and $g:=|u|^{p+1}$. We note that, since $1<p\leq1+\frac{4}{d}$, we may take $\rho\in(1,\infty)$ such that $\frac{1}{\rho}=(\frac{1}{2}-\frac{1}{d})(p-1)$ if $d\geq3$, and $\rho=\frac{p-1}{8}$ if $d=1,2$. Then, $2\leq\rho(p-1)\leq2+\frac{4}{d-2}$ and $2\leq2\rho'\leq2+\frac{4}{d-2}$ if $d\geq3$, $2\leq2\rho'<\infty$ and $2\leq\rho(p-1)<\infty$ if $d=1,2$. By Sobolev's embedding $H^1\hookrightarrow L^{2\rho'}$,

$$\begin{aligned}|Er|\leq&C\|\varepsilon\|_{2\rho'}^2\sum_{z^*,\overline{z}^*\in\{z,\bar{z}\}}\left\|\int_0^1r\int_0^1\left(\partial_{z^*\overline{z}^*}g(\widetilde{R}+sr\varepsilon)-\partial_{z^*\overline{z}^*}g(\widetilde{R})\right)drds\right\|_{L^\rho}\\\leq&C\|\varepsilon\|_{H^1}^2\sum_{z^*,\overline{z}^*\in\{z,\bar{z}\}}\left\|\int_0^1r\int_0^1\left(\partial_{z^*\overline{z}^*}g(\widetilde{R}+sr\varepsilon)-\partial_{z^*\overline{z}^*}g(\widetilde{R})\right)drds\right\|_{L^\rho}.\end{aligned}\tag{4.41}$$

Moreover, since $\|\varepsilon(t)\|_{H^1}\to0$ as $t\to T$, we infer that for any sequence $\{t_n\}$, $t_n\to T$, there exists a subsequence (still denoted by $\{n\}$) such that $\varepsilon(t_n)\to0$, dx-a.e.. By Sobolev's embedding $H^1\hookrightarrow L^{\rho(p-1)}$,

$$|\partial_{z^*\overline{z}^*}g(\widetilde{R}(t_n)+sr\varepsilon(t_n))-\partial_{z^*\overline{z}^*}g(\widetilde{R}(t_n))|\leq C(|\widetilde{R}(t_n)|^{p-1}+|\varepsilon(t_n)|^{p-1})\in L^\rho.\tag{4.42}$$

Hence, by the dominated convergence theorem,

$$\left\|\int_0^1r\int_0^1\left(\partial_{z^*\overline{z}^*}g(\widetilde{R}(t_n)+sr\varepsilon(t_n))-\partial_{z^*\overline{z}^*}g(\widetilde{R}(t_n))\right)drds\right\|_{L^\rho}\to0,\ \ as\ t_n\to T.\tag{4.43}$$

Since $\{t_n\}$ is any arbitrary sequence converging to T, we obtain that the above convergence is valid for any $t\to T$, and thus

$$|Er|=o(\|\varepsilon\|_{H^1}^2),\quad as\ t\to T.\tag{4.44}$$

Moreover, we claim that

$$\frac{2}{p+1}\int|\widetilde{R}|^{p+1}dx+2\mathrm{Re}\int|\widetilde{R}|^{p-1}\widetilde{R}\bar{\varepsilon}dx+\int\frac{p+1}{2}|\widetilde{R}|^{p-1}|\varepsilon|^2+(p-1)|\widetilde{R}|^{p-3}[\mathrm{Re}(\widetilde{R}\bar{\varepsilon})^2]dx$$
$$=\frac{2}{p+1}\sum_{k=1}^{K}\int|\widetilde{R}_k|^{p+1}dx+2\sum_{k=1}^{K}\mathrm{Re}\int|\widetilde{R}_k|^{p-1}\widetilde{R}_k\bar{\varepsilon}dx$$
$$+\sum_{k=1}^{K}\int\frac{p+1}{2}|\widetilde{R}_k|^{p-1}|\varepsilon|^2+(p-1)|\widetilde{R}_k|^{p-3}[\mathrm{Re}(\widetilde{R}_k\bar{\varepsilon})^2]dx+O(e^{-\delta_2 t}). \tag{4.45}$$

The proof of (4.45) is postponed to Subsection 6.4 of the Appendix. Thus, plugging (4.44) and (4.45) into (4.39) we then obtain

$$\frac{2}{p+1}\int|u|^{p+1}dx=\frac{2}{p+1}\sum_{k=1}^{K}\int|\widetilde{R}_k|^{p+1}dx+2\sum_{k=1}^{K}\mathrm{Re}\int|\widetilde{R}_k|^{p-1}\widetilde{R}_k\bar{\varepsilon}dx$$
$$+\sum_{k=1}^{K}\int\frac{p+1}{2}|\widetilde{R}_k|^{p-1}|\varepsilon|^2+(p-1)|\widetilde{R}_k|^{p-3}[\mathrm{Re}(\widetilde{R}_k\bar{\varepsilon})^2]dx+o(\|\varepsilon\|_{H^1}^2)+O(e^{-\delta_2 t}). \tag{4.46}$$

Thus, combining (4.38) and (4.46) together we obtain

$$2E(u)=\sum_{k=1}^{K}\left(2E(Q_{w_k})+\frac{|v_k|^2}{4}\|Q_{w_k}\|_{L^2}^2\right)-\sum_{k=1}^{K}2\mathrm{Re}\int(\Delta\widetilde{R}_k+|\widetilde{R}_k|^{p-1}\widetilde{R}_k)\bar{\varepsilon}dx$$
$$+\int|\nabla\varepsilon|^2dx-\sum_{k=1}^{K}\int\frac{p+1}{2}|\widetilde{R}_k|^{p-1}|\varepsilon|^2+(p-1)|\widetilde{R}_k|^{p-3}[\mathrm{Re}(\widetilde{R}_k\bar{\varepsilon})^2]dx+o(\|\varepsilon\|_{H^1}^2)+O(e^{-\delta t}). \tag{4.47}$$

We also see from (4.25) that

$$\left((w_k^0)^{-2}+\frac{|v_k|^2}{4}\right)I_k=\left((w_k^0)^{-2}+\frac{|v_k|^2}{4}\right)\|Q_{w_k}\|_{L^2}^2+\left(2(w_k^0)^{-2}+\frac{|v_k|^2}{2}\right)\mathrm{Re}\int\widetilde{R}_k\bar{\varepsilon}dx$$
$$+\left(w_k^{-2}+\frac{|v_k|^2}{4}\right)\int|\varepsilon|^2\varphi_k dx+((w_k^0)^{-2}-w_k^{-2})\int|\varepsilon|^2\varphi_k dx+O(e^{-\delta_2 t}). \tag{4.48}$$

Regarding the local momentum, we expand

$$M_k=\mathrm{Im}\int\nabla\widetilde{R}_k\bar{\widetilde{R}}_k dx+2\mathrm{Im}\int\nabla\widetilde{R}_k\bar{\varepsilon}dx+\mathrm{Im}\int\nabla\varepsilon\bar{\varepsilon}\varphi_k dx+O(e^{-\delta_2 t})$$
$$=\frac{v_k}{2}\int|Q_{w_k}|^2dx+2\mathrm{Im}\int\nabla\widetilde{R}_k\bar{\varepsilon}dx+\mathrm{Im}\int\nabla\varepsilon\bar{\varepsilon}\varphi_k dx+O(e^{-\delta_2 t}), \tag{4.49}$$

which yields that

$$v_k\cdot M_k=\frac{|v_k|^2}{2}\|Q_{w_k}\|_{L^2}^2+2v_k\cdot\mathrm{Im}\int\nabla\widetilde{R}_k\bar{\varepsilon}dx+v_k\cdot\mathrm{Im}\int\nabla\varepsilon\bar{\varepsilon}\varphi_k dx+O(e^{-\delta_2 t}). \tag{4.50}$$

Therefore, collecting (4.47), (4.48) and (4.49) altogether we conclude that

$$G(t)=\sum_{k=1}^{K}(2E(Q_{w_k(t)})+(w_k^0)^{-2}\|Q_{w_k(t)}\|_{L^2}^2)$$

$$
\begin{aligned}
&-2\sum_{k=1}^{K}\mathrm{Re}\int(\Delta\widetilde{R}_k(t)-(w_k^0)^{-2}\widetilde{R}_k(t)+|\widetilde{R}_k(t)|^{p-1}\widetilde{R}_k(t))\bar{\varepsilon}(t)dx+\sum_{k=1}^{K}\frac{|v_k|^2}{2}\mathrm{Re}\int\widetilde{R}_k(t)\bar{\varepsilon}(t)dx \\
&-2\sum_{k=1}^{K}v_k\cdot\mathrm{Im}\int\nabla\widetilde{R}_k(t)\bar{\varepsilon}(t)dx+H(\varepsilon(t))+\sum_{k=1}^{K}((w_k^0)^{-2}-w_k^{-2}(t))\int|\varepsilon(t)|^2\varphi_k(t)dx \\
&+o(\|\varepsilon(t)\|_{H^1}^2)+O(e^{-\delta_2 t}).
\end{aligned} \tag{4.51}
$$

Now, let us estimate the R.H.S. of (4.51). For the first term, we claim that

$$
2E(Q_{w_k(t)})+(w_k^0)^{-2}\|Q_{w_k(t)}\|_{L^2}^2=2E(Q_{w_k^0})+(w_k^0)^{-2}\|Q_{w_k^0}\|_{L^2}^2. \tag{4.52}
$$

Note that, the R.H.S. above only depends on w_k^0 which is independent of time. Hence, the identity (4.52) shows that the modulation parameter $w_k(t)$ (depending on time in the critical case) indeed does not affect the main part of the Lyapunov type functional.

This identity is obvious in the subcritical case as $w_k(t)\equiv w_k^0$. Concerning the critical case, the scaling invariance in the L^2-critical case yields that

$$
2E(Q_{w_k})+(w_k^0)^{-2}\|Q_{w_k}\|_{L^2}^2-2E(Q_{w_k^0})-(w_k^0)^{-2}\|Q_{w_k^0}\|_{L^2}^2=2(w_k^{-2}-(w_k^0)^{-2})E(Q). \tag{4.53}
$$

Then, by the key Pohozaev identity

$$
(d-2)\|\nabla Q\|_{L^2}^2+d\|Q\|_{L^2}^2=\frac{2d}{p+1}\|Q\|_{L^{p+1}}^{p+1}, \tag{4.54}
$$

we obtain

$$
E(Q)=0, \tag{4.55}
$$

which along with (4.53) yields (4.52), as claimed.

For the linear terms of ε on the R.H.S. of (4.51), by (1.9), (3.15) and (3.16),

$$
\Delta\widetilde{R}_k-(w_k^0)^{-2}\widetilde{R}_k+|\widetilde{R}_k|^p\widetilde{R}_k=\left(w_k^{-2}-(w_k^0)^{-2}\right)\widetilde{R}_k+iv_k\cdot\nabla\widetilde{R}_k+\frac{1}{4}|v_k|^2\widetilde{R}_k, \tag{4.56}
$$

which yields the identity

$$
\begin{aligned}
&-2\mathrm{Re}\int(\Delta\widetilde{R}_k-(w_k^0)^{-2}\widetilde{R}_k+|\widetilde{R}_k|^p\widetilde{R}_k)\bar{\varepsilon}dx+\frac{|v_k|^2}{2}\mathrm{Re}\int\widetilde{R}_k\bar{\varepsilon}dx-2v_k\cdot\mathrm{Im}\int\nabla\widetilde{R}_k\bar{\varepsilon}dx \\
=&2((w_k^0)^{-2}-w_k^{-2})\mathrm{Re}\int\widetilde{R}_k\bar{\varepsilon}dx.
\end{aligned} \tag{4.57}
$$

Therefore, plugging (4.52) and (4.57) into (4.51) we obtain (4.35) and finish the proof. □

As a consequence, we have the crucial coercivity type control of the remainder.

Proposition 4.6. *(Coercivity type control of remainder) Let $1<p\leq 1+\frac{4}{d}$, $d\geq 1$. Then, there exist deterministic constants $C,\delta_1,\delta_2>0$ depending on w_k^0, x_k^0, v_k and δ_0, such that for $t\in[T^*,T]$,*

$$
\begin{aligned}
\|\varepsilon(t)\|_{H^1}^2\leq &C\left(\int_t^\infty\frac{1}{s}\|\varepsilon(s)\|_{H^1}^2ds+\left(\int_t^\infty\frac{1}{s}\|\varepsilon(s)\|_{H^1}ds\right)^2\right) \\
&+C\left(\int_t^\infty B_*(s)(\|\varepsilon\|_{H^1}^2+\phi(\delta_1 s))ds+e^{-\delta_2 t}\right).
\end{aligned} \tag{4.58}
$$

Proof. Since $\varepsilon(T)=0$, by (4.35),

$$G(T)=\sum_{k=1}^{K}(2E(Q_{w_k^0})+(w_k^0)^{-2}\|Q_{w_k^0}\|_{L^2}^2)+O(e^{-\delta_2 T}), \tag{4.59}$$

which along with Proposition 4.4 yields that

$$\begin{aligned} H(\varepsilon(t)) =& G(t)-G(T)+O\left(\sum_{k=1}^{K}\left|(w_k(t)-w_k^0)\mathrm{Re}\int \widetilde{R}_k(t)\bar{\varepsilon}(t)dx\right|\right) \\ &+O(|w_k(t)-w_k^0|\,\|\varepsilon(t)\|_{H^1}^2+e^{-\delta_2 t})+o(\|\varepsilon(t)\|_{H^1}^2). \end{aligned} \tag{4.60}$$

Taking into account the coercivity (4.37) we then come to, for t close to T,

$$\begin{aligned} \|\varepsilon(t)\|_{H^1}^2 \le& C\left(|G(t)-G(T)|+\sum_{k=1}^{K}\left(\mathrm{Re}\int \widetilde{R}_k(t)\bar{\varepsilon}(t)dx\right)^2+\sum_{k=1}^{K}\left|(w_k(t)-w_k^0)\mathrm{Re}\int \widetilde{R}_k(t)\bar{\varepsilon}(t)dx\right|\right) \\ &+C\left(|w_k(t)-w_k^0|\,\|\varepsilon(t)\|_{H^1}^2+e^{-\delta_2 t}\right)+o(\|\varepsilon(t)\|_{H^1}^2). \end{aligned} \tag{4.61}$$

Note that, by definition (4.34) and Propositions 4.1 and 4.3,

$$\begin{aligned} |G(t)-G(T)| \le& C|E(t)-E(T)|+C\sum_{k=1}^{K}(|I_k(t)-I_k(T)|+|M_k(t)-M_k(T)|) \\ \le& C\int_t^\infty \frac{1}{s}(\|\varepsilon(s)\|_{H^1}^2+e^{-\delta_2 s})ds+C\int_t^\infty B_*(s)(\|\varepsilon(s)\|_{H^1}^2+\phi(\delta_1 s)+e^{-\delta_2 s})ds, \end{aligned} \tag{4.62}$$

where $C>0$. Moreover, by Corollary 4.2,

$$\left|\mathrm{Re}\int \widetilde{R}_k(t)\bar{\varepsilon}(t)dx\right| \le C\left(\int_t^\infty \frac{1}{s}\|\varepsilon(s)\|_{H^1}^2ds+\|\varepsilon(t)\|_{H^1}^2+e^{-\delta_2 t}\right). \tag{4.63}$$

Therefore, combing (4.60), (4.61), (4.62), (4.63) and letting T^* close to T such that $|w_k-w_k^0|$ is small enough we obtain (4.58) and thus finish the proof. □

5. Proof of main results

This section is devoted to the proof of main results. As in Section 4, we shall perform the path-by-path analysis for $\mathbb{P}$-a.e. $\omega\in\Omega$. The crucial ingredients of the proof are the uniform estimates of the remainder and geometrical parameters.

5.1. **Uniform estimates.**

Take any increasing sequence $\{T_n\}$ such that $\lim_{n\to\infty}T_n=+\infty$ and consider the approximating solutions u_n satisfying the equation on $[T_0,T_n]$ (for the definition of T_0 see Theorem 5.1 below)

$$\begin{cases} i\partial_t u_n+\Delta u_n+|u_n|^{\frac{4}{d}}u_n+(b_*\cdot\nabla+c_*)u_n=0, \\ u_n(T_n)=\sum_{k=1}^{K}R_k(T_n)\ (=: R(T_n)). \end{cases} \tag{5.1}$$

The uniform estimates of the remainder and geometrical parameters are contained in Theorem 5.1 below.

Theorem 5.1. *(Uniform estimates) Let $\delta_1, \delta_2 > 0$ be as in Propositions 3.3, 3.6 and 4.6. Let $\widetilde{\delta} \in (0, \delta_1 \wedge \delta_2)$ in* Case (I)*, and $\widetilde{\delta} = 1$ in* Case (II)*. Then for υ_* sufficiently large, there exists $T_0 > 0$ such that for n large enough, u_n admits the geometrical decomposition* (3.2) *and* (3.51) *on* $[T_0, T_n]$ *in the critical and subcritical cases, respectively, and u_n obeys the following estimate:*

$$\|\varepsilon_n(t)\|_{H^1}^2 \leq \phi(\widetilde{\delta} t), \quad t \in [T_0, T_n], \tag{5.2}$$

where ϕ is the decay function given by (1.18)*.*

Moreover, let $\mathcal{P}_{n,k} = (\alpha_{n,k}, \theta_{n,k}, w_{n,k}) \in \mathbb{X}$, $1 \leq k \leq K$, be the corresponding modulation parameters. Then, there exists $C, \widetilde{\delta} > 0$ such that for n large enough,

$$\sum_{k=1}^{K} (|w_{n,k}(t) - w_k^0| + |\alpha_{n,k}(t) - x_k^0| + |\theta_{n,k}(t) - \theta_k^0|) \leq C \int_t^\infty s\phi^{\frac{1}{2}}(\widetilde{\delta} s) ds, \quad \forall t \in [T_0, T_n]. \tag{5.3}$$

The proof of Theorem 5.1 relies crucially on the following bootstrap estimate.

Proposition 5.2. *(Bootstrap estimate) Let $\widetilde{\delta}$ be as in Theorem 5.1. Then for υ_* sufficiently large, there exists $T_0 > 0$ such that, for n large enough, the following holds.*

For any $t^* \in (T_0, T_n]$ such that u_n admits the decomposition (3.2) *(resp.* (3.51)*) in the critical case (resp. the subcritical case) and obeys the following estimates on* $[t^*, T_n]$*:*

$$\begin{aligned} &\|\varepsilon_n(t)\|_{H^1}^2 \leq \phi(\widetilde{\delta} t), \\ &\sum_{k=1}^{K} (|w_{n,k}(t) - w_k^0| + |\alpha_{n,k}(t) - x_k^0|) \leq t\phi^{\frac{1}{2}}(\widetilde{\delta} t), \end{aligned} \tag{5.4}$$

there exists $t_* \in (T_0, t^*)$ such that the decomposition and the following improved estimate hold on $[t_*, T_n]$*:*

$$\begin{aligned} &\|\varepsilon_n(t)\|_{H^1}^2 \leq \frac{1}{2}\phi(\widetilde{\delta} t), \\ &\sum_{k=1}^{K} (|w_{n,k}(t) - w_k^0| + |\alpha_{n,k}(t) - x_k^0|) \leq \frac{1}{2} t\phi^{\frac{1}{2}}(\widetilde{\delta} t). \end{aligned} \tag{5.5}$$

Proof. By (5.4) and the continuity of solutions in H^1 and of the Jacobian matrices in the proof of geometrical decomposition, we may take $t_* (< t^*)$ close to t^*, such that the geometrical decompositions (3.2), (3.51) and the following estimate holds on $[t_*, T_n]$,

$$\begin{aligned} &\|\varepsilon_n(t)\|_{H^1}^2 \leq 2\phi(\widetilde{\delta} t), \\ &\sum_{k=1}^{K} (|w_{n,k}(t) - w_k^0| + |\alpha_{n,k}(t) - x_k^0|) \leq 2t\phi^{\frac{1}{2}}(\widetilde{\delta} t). \end{aligned} \tag{5.6}$$

Thus by taking $\upsilon_* > 2$ in (1.18) and using (5.6), there exists $T_0 > 0$ such that (3.9) and (3.10) hold for all $t \in [T_0, T_n]$. Moreover, the estimates in the previous subsection are all valid for $t \in [T_0, T_n]$, with some uniform positive constant C independent of n. So, we can apply Propositions 3.3, 3.6 and (5.6) to obtain that for any $t \in [t_*, T_n]$,

$$\sum_{k=1}^{K} (|\dot{w}_{n,k}(t)| + |\dot{\alpha}_{n,k}(t)|) \leq C(\|\varepsilon_n(t)\|_{H^1} + B_*(t)\phi(\widetilde{\delta} t) + e^{-\widetilde{\delta} t}) \leq C\phi^{\frac{1}{2}}(\widetilde{\delta} t). \tag{5.7}$$

Below we consider Case (I) and Case (II) separately. In Case (I), by integrating (5.7), for any $t \in [t_*, T_n]$, we get

$$\sum_{k=1}^{K}(|w_{n,k}(t) - w_k^0| + |\alpha_{n,k}(t) - x_k^0|) \leq C \int_t^\infty \phi^{\frac{1}{2}}(\widetilde{\delta} s)ds \leq \frac{C}{\widetilde{\delta} t} t e^{-\frac{\widetilde{\delta}}{2}t},$$

and by (1.15), (4.58) and (5.6),

$$\begin{aligned}
\|\varepsilon_n(t)\|_{H^1}^2 \leq & C\left(\int_t^\infty \frac{1}{s} e^{-\widetilde{\delta} s} ds + \left(\int_t^\infty \frac{1}{s} e^{-\widetilde{\delta} s} ds\right)^2 + \int_t^\infty B_* e^{-\widetilde{\delta} s} ds + e^{-\delta_2 t}\right) \\
\leq & C\left(\frac{1}{\widetilde{\delta} t} + \frac{1}{(\widetilde{\delta} t)^2} + \frac{1}{\widetilde{\delta}} B_*(t) + e^{-(\delta_2 - \widetilde{\delta})t}\right) e^{-\widetilde{\delta} t}. \qquad (5.8)
\end{aligned}$$

Taking T_0 large enough such that

$$C\left(\frac{1}{\widetilde{\delta} T_0} + \frac{1}{(\widetilde{\delta} T_0)^2} + \frac{1}{\widetilde{\delta}} B_*(T_0) + e^{-(\delta_2 - \widetilde{\delta})T_0}\right) \leq \frac{1}{2},$$

we get that for any $t \in [t_*, T_n]$,

$$\|\varepsilon_n(t)\|_{H^1}^2 \leq \frac{1}{2} e^{-\widetilde{\delta} t},$$

$$\sum_{k=1}^{K}(|w_{n,k}(t) - w_k^0| + |\alpha_{n,k}(t) - x_k^0|) \leq \frac{1}{2} t e^{-\frac{\widetilde{\delta}}{2}t}.$$

This verifies estimate (5.5) in Case (I).

Concerning Case (II), by integrating (5.7), for any $t \in [t_*, T_n]$, we get

$$\sum_{k=1}^{K}(|w_{n,k}(t) - w_k^0| + |\alpha_{n,k}(t) - x_k^0|) \leq C \int_t^\infty \phi^{\frac{1}{2}}(\widetilde{\delta} s)ds \leq \frac{C}{\upsilon_* - 2} t^{1-\frac{\upsilon_*}{2}},$$

and by using (1.17), (4.58) and (5.6) we infer that

$$\begin{aligned}
\|\varepsilon_n(t)\|_{H^1}^2 \leq & C\left(\int_t^\infty s^{-\upsilon_* - 1} ds + \left(\int_t^\infty s^{-\upsilon_* - 1} ds\right)^2 + \int_t^\infty B_*(s) s^{-\upsilon_*} ds + e^{-\delta_2 t}\right) \\
\leq & C\left(\frac{1}{\upsilon_*} + \frac{t^{-\upsilon_*}}{\upsilon_*^2} + \frac{t B_*(t)}{\upsilon_* - 1} + \frac{t^{\upsilon_*} e^{-\delta_2 t}}{\delta_2}\right) t^{-\upsilon_*}. \qquad (5.9)
\end{aligned}$$

Using the theorem on time change for continuous martingales and the Levy Hölder continuity of Brownian motions we derive from (1.16) that $\mathbb{P}$-a.s. for t large enough,

$$|B_{*,l}(t)| \leq 2\left(\int_t^\infty g_l^2 ds \log\left(\int_t^\infty g_l^2 ds\right)^{-1}\right)^{\frac{1}{2}} \leq \frac{2\sqrt{c^*}}{t}, \quad 1 \leq l \leq N, \qquad (5.10)$$

which yields that $\mathbb{P}$-a.s. for t large enough $tB_*(t) \leq 2\sqrt{c^*}$.

Thus, we may take υ_* and T_0 large enough such that,

$$C\left(\frac{1}{\upsilon_* - 2} + \frac{T_0^{-\upsilon_*}}{\upsilon_*^2} + \frac{T_0 B_*(T_0)}{\upsilon_* - 1} + \frac{T_0^{\upsilon_*} e^{-\delta_2 T_0}}{\delta_2}\right) \leq \frac{1}{2},$$

which in particular yields that for any $t \in [t_*, T_n]$,

$$\|\varepsilon_n(t)\|_{H^1}^2 \leq \frac{1}{2} t^{-\upsilon_*},$$

$$\sum_{k=1}^{K}(|w_{n,k}(t)-w_k^0|+|\alpha_{n,k}(t)-x_k^0|)\leq \frac{1}{2}t^{-\frac{\upsilon_*}{2}}.$$

Therefore, estimate (5.5) in Case (II) is verified. The proof is complete. □

Proof of Theorem 5.1. By using the bootstrap estimate in Proposition 5.2 and standard continuity arguments (see, e.g., [46, Page 7], [59, Page 43] for relevant arguments), we get that for any $t\in[T_0,T_n]$,

$$\begin{aligned}&\|\varepsilon_n(t)\|_{H^1}^2\leq \phi(\widetilde{\delta}t),\\ &\sum_{k=1}^{K}(|w_{n,k}(t)-w_k^0|+|\alpha_{n,k}(t)-x_k^0|)\leq t\phi^{\frac{1}{2}}(\widetilde{\delta}t).\end{aligned} \tag{5.11}$$

Therefore, estimate (5.2) is verified. Moreover, since (3.9) and (3.10) hold for all $t\in[T_0,T_n]$, it follows from (3.12), (3.55) and (5.11) that

$$\sum_{k=1}^{K}|\dot{\theta}_{n,k}(t)|\leq C(|w_{n,k}(t)-w_k^0|+\|\varepsilon_n(t)\|_{H^1}+B_*(t)\phi(\widetilde{\delta}t)+e^{-\widetilde{\delta}t})\leq Ct\phi^{\frac{1}{2}}(\widetilde{\delta}t), \tag{5.12}$$

where the last step is due to the fact that $\int_t^\infty e^{-\frac{1}{2}\widetilde{\delta}s}ds\leq 2\widetilde{\delta}^{-1}e^{-\frac{1}{2}\widetilde{\delta}t}$ and $\int_t^\infty s^{-\frac{\upsilon_*}{2}}ds=\frac{2}{\upsilon_*-2}t^{-\frac{\upsilon_*}{2}+1}\leq 2t\phi^{\frac{1}{2}}(\widetilde{\delta}t)$ if $\upsilon_*\geq 3$.

Then, integrating (5.12) over $[t,T_n]$ for any $t\in[T_0,T_n]$ and combing with (5.11), we get that

$$\sum_{k=1}^{K}(|w_{n,k}(t)-w_k^0|+|\alpha_{n,k}(t)-x_k^0|+|\theta_{n,k}(t)-\theta_k^0|)\leq C\int_t^\infty s\phi^{\frac{1}{2}}(\widetilde{\delta}s)ds,$$

which verifies (5.3) and completes the proof. □

5.2. **Proof of main results.** We are now in position to prove Theorems 1.6 and 1.3.

Proof of Theorem 1.6. By Theorem 5.1, $\{u_n(T_0)\}$ is uniformly bounded in H^1. This yields that up to a subsequence (still denoted by $\{n\}$), for some $u_0\in H^1$,

$$u_n(T_0)\rightharpoonup u_0 \ \ weakly\ in\ H^1,\ as\ n\to\infty. \tag{5.13}$$

We claim that the convergence is strong in L^2, i.e.,

$$u_n(T_0)\to u_0\ \ in\ L^2,\ as\ n\to\infty. \tag{5.14}$$

For this purpose, it suffices to prove that $\{u_n(T_0)\}$ is uniformly integrable, i.e., for any $\epsilon>0$, there exists $A_\epsilon>0$ such that for all n large,

$$\int_{|x|\geq A_\epsilon}|u_n(T_0)|^2dx\leq\epsilon. \tag{5.15}$$

In order to prove (5.15), we first fix $T_1>T_0$ such that

$$\|\varepsilon(T_1)\|_{H^1}^2\leq\phi(\widetilde{\delta}T_1)\leq\frac{1}{6}\epsilon. \tag{5.16}$$

By (5.3), we may take $A_0=A_0(v_k,T_1,x_k^0,1\leq k\leq K)$ large enough such that for $|x|\geq A_0$ and for $1\leq k\leq K$,

$$|x-v_kT_1-\alpha_{n,k}(T_1)|\geq|x|-|v_k|T_1-\sup_{n\geq1,t\geq T_0}|\alpha_{n,k}(t)|\geq A_0-\frac{1}{2}A_0\geq\frac{1}{2}A_0. \tag{5.17}$$

Hence, by the exponential decay of ground state, for A_0 possibly larger,

$$\sup_{n\geq 1}\int_{|x|\geq A_0}|\widetilde{R}_n(T_1)|^2dx \leq C\int_{|x|\geq \frac{1}{2}A_0} e^{-\delta_2|x|}dx \leq Ce^{-\frac{\delta_2}{4}A_0} \leq \frac{\epsilon}{6}. \tag{5.18}$$

Then, it follows from (5.16) and (5.18) that for all n large enough,

$$\int_{|x|\geq A_0}|u_n(T_1)|^2dx \leq 2\int_{|x|\geq A_0}|\widetilde{R}_n(T_1)|^2dx + 2\|\varepsilon_n(T_1)\|_{H^1}^2 \leq \frac{2\epsilon}{3}. \tag{5.19}$$

Moreover, let χ be a smooth cut off function on $\mathbb{R}$ such that $0 \leq \chi(x) \leq 1$, $\chi(x) = 0$ for $|x| \leq \frac{1}{2}$; $\chi(x) = 1$ for $|x| \geq 1$ and $|\chi'| \leq 2$. Let $\chi_{A_\epsilon}(x) := \chi\left(\frac{|x|}{A_\epsilon}\right)$, where $A_\epsilon = \max\{\frac{3\widetilde{C}(T_1-T_0)}{\epsilon}, 2A_0\}$ and $\widetilde{C}$ is the constant in (5.20) below. By the integration-by-parts formula,

$$\left|\frac{d}{dt}\int \chi_{A_\epsilon}|u_n(t)|^2dx\right| = \left|\mathrm{Im}(2\overline{u_n}\nabla u_n + b_*|u_n|^2)\cdot\nabla\chi_{A_\epsilon}\right| \leq \frac{\widetilde{C}}{A_\epsilon} \leq \frac{\epsilon}{3(T_1-T_0)}, \tag{5.20}$$

Thus, we derive from (5.19) and (5.20) that, for n large enough,

$$\begin{aligned}\int_{|x|\geq A_\epsilon}|u_n(T_0)|^2dx &\leq \int_{\mathbb{R}^d}|u_n(T_1)|^2\chi_{A_\epsilon}dx + \int_{T_0}^{T_1}\left|\frac{d}{dt}\int_{\mathbb{R}^d}|u_n(t)|^2\chi_{A_\epsilon}dx\right|dt\\ &\leq \int_{|x|\geq A_0}|u_n(T_1)|^2dx + \frac{\epsilon}{3} \leq \epsilon,\end{aligned} \tag{5.21}$$

which yields (5.15), and thus proves (5.14), as claimed.

Now, for n large enough, since u_n solves the equation (5.1) on $[T_0, T_n]$ with $\lim_{n\to\infty} T_n = +\infty$ and obey the uniform estimates in $C([T_0,T];H^1)$ for any $T_0 < T < \infty$, using the asymptotic (5.14) and comparison arguments (see, e.g., [41, 60, 64]) we infer that, there exists a unique L^2-solution u to (1.29) on $[T_0,\infty)$ such that

$$\lim_{n\to\infty}\|u_n(t)-u(t)\|_{L^2} = 0, \quad \forall t\in[T_0,\infty). \tag{5.22}$$

Moreover, since $u_0 \in H^1$, the preservation of H^1-regularity also yields $u(t) \in H^1$ for $t \in [T_0,\infty)$.

Furthermore, by (5.3) and straightforward computations, if $R := \sum_{k=1}^K R_k$ with R_k given by (1.12),

$$\begin{aligned}\|\widetilde{R}_n(t)-R(t)\|_{H^1} \leq& C\sum_{k=1}^K(|w_{n,k}(t)-\omega_k^0| + |\alpha_{n,k}(t)-x_k^0| + |\theta_{n,k}(t)-\theta_k^0|)\\ \leq& C\int_t^\infty s\phi^{\frac{1}{2}}(\widetilde{\delta}s)ds.\end{aligned} \tag{5.23}$$

Taking into account estimate (5.2) we then obtain

$$\begin{aligned}\|u_n(t)-R(t)\|_{H^1} \leq& \|\varepsilon_n(t)\|_{H^1} + \|\widetilde{R}_n(t)-R(t)\|_{H^1}\\ \leq& C\left(\phi^{\frac{1}{2}}(\widetilde{\delta}t) + \int_t^\infty s\phi^{\frac{1}{2}}(\widetilde{\delta}s)ds\right)\\ \leq& C\int_t^\infty s\phi^{\frac{1}{2}}(\widetilde{\delta}s)ds,\end{aligned} \tag{5.24}$$

where the last step is due to the explicit expression (1.18) of the decay function ϕ.

In particular, this yields that $u_n(t)-R(t)$ is uniformly bounded in H^1 for every $t\in[T_0,\infty)$, which along with (5.22) implies that, up to a subsequence (still denoted by $\{n\}$ which may depend on t),

$$u_n(t)-R(t) \rightharpoonup u(t)-R(t), \quad \textit{weakly in } H^1, \textit{ as } n\to\infty. \tag{5.25}$$

This yields that

$$\|u(t) - R(t)\|_{H^1} \leq \liminf_n \|u_n(t) - R(t)\|_{H^1} \leq C \int_t^\infty s\phi^{\frac{1}{2}}(\widetilde{\delta} s)ds. \tag{5.26}$$

Therefore, the proof of Theorem 1.6 is complete. □

Proof of Theorem 1.3. Theorem 1.3 now follows from Theorem 1.6 and Theorem 1.5 via the Doss-Sussman type transforms.

More precisely, by Theorem 1.6, there exists a unique solution u to (1.29) on $[T_0, \infty)$ with $T_0 > 0$ sufficiently large, and the asymptotic behavior (1.32) holds. This yields that

$$v := e^{-W(\infty)}u \tag{5.27}$$

is a unique solution to equation (1.25) on $[T_0, \infty)$. Thus, applying Theorem 1.5 we obtain that

$$X := e^W v = e^{W_*}u \tag{5.28}$$

solves equation (1.1) on $[T_0, \infty)$ in the sense of Definition 1.2. The asymptotic behavior (1.20) thus follows from (1.32).

Regarding (1.22), it suffices to prove that

$$\|X(t) - e^{-W_*(t)}X(t)\|_{H^1} \leq C \sum_{l=1}^N \left(\int_t^\infty g_l^2 ds \log \left(\int_t^\infty g_l^2 ds \right)^{-1} \right)^{\frac{1}{2}} =: CL(t). \tag{5.29}$$

To this end, we see that

$$\begin{aligned} \|X(t) - e^{-W_*(t)}X(t)\|_{H^1} \leq & \|(1 - e^{-W_*(t)})X(t)\|_{L^2} + \|\nabla(1 - e^{-W_*(t)})X(t)\|_{L^2} + \|(1 - e^{-W_*(t)})\nabla X(t)\|_{L^2} \\ =: & J_1(t) + J_2(t) + J_3(t). \end{aligned} \tag{5.30}$$

Using the inequality $|1 - e^{-x}| \leq e|x|$ for $|x| \leq 1$, the fact that for t large enough,

$$\|W_*(t)\|_{W^{1,\infty}} \leq C \sum_{l=1}^N \left| \int_t^\infty g_l(s) dB_l(s) \right| \leq C \sum_{l=1}^N \left(\int_t^\infty g_l^2(s) ds \log \left(\int_t^\infty g_l^2(s) ds \right)^{-1} \right)^{\frac{1}{2}} \tag{5.31}$$

and the mass conservation law of $X(t)$ we derive

$$J_1(t) + J_2(t) \leq C\|W_*(t)\|_{W^{1,\infty}} \|X(t)\|_{L^2} \leq C\|X(T_0)\|_{L^2} L(t). \tag{5.32}$$

Regarding the third term J_3, we claim that the following uniform boundedness holds, i.e., for some $C > 0$,

$$\|\nabla X\|_{C([T_0,\infty);L^2)} \leq C. \tag{5.33}$$

Actually, it follows from (5.28) that

$$\nabla X(t) = e^{W_*(t)} \nabla u(t) + \nabla W_*(t) X(t).$$

By (1.20) and the expression of the solitary wave (1.7),

$$\|\nabla u(t)\|_{L^2} \leq \|u(t) - \sum_{k=1}^K R_k(t)\|_{H^1} + \sum_{k=1}^K \|R_k(t)\|_{H^1} \leq C \left(\int_t^\infty s\phi^{\frac{1}{2}}(\delta s) ds + \sum_{k=1}^K \|Q_{w_k^0}\|_{H^1} \right) \leq C,$$

where C is a universal positive constant, independent of t. Taking into account $|e^{W_*(t)}| = 1$, $\|X(t)\|_{L^2} = \|X(T_0)\|_{L^2}$ and (5.31) we then obtain (5.33), as claimed.

We then infer from (5.31) and (5.33) that

$$J_3(t) \leq C\|W_*(t)\|_{L^\infty} \|\nabla X(t)\|_{L^2} \leq CL(t). \tag{5.34}$$

Thus, plugging (5.32) and (5.34) into (5.30) we prove (5.29), and so (1.22) follows.

At last, in the L^2-subcritical case, using the fixed point arguments as in [5, 6], based on the Strichartz and local smoothing estimates, we may extend the solution u to a larger time interval $[\sigma^*, \infty)$, where $\sigma^* \in [0, T_0)$ is a non-negative random variable. Because of the subcriticality of the nonlinearity, σ^* depends on the H^1-norm of the solution, and thus $\sigma^* = 0$ if the following uniform H^1-bound holds

$$\sup_{t\in(\sigma^*,T_0]} \|u(t)\|_{H^1} < \infty. \tag{5.35}$$

In order to prove (5.35), we derive from the evolution formula (4.32) of energy that, for any $t \in (\sigma^*, T_0]$,

$$E(u(t)) \leq E(u(T_0)) + C\int_t^{T_0} \|u(s)\|_{H^1}^2 + \|u(s)\|_{L^{p+1}}^{p+1} ds. \tag{5.36}$$

which, via the interpolation estimate (see [6, Lemma 3.5]), for some $\rho > 2$,

$$\|u\|_{L^{p+1}}^{p+1} \leq C_\varepsilon \|u\|_{L^2}^{\rho} + \varepsilon\|u\|_{H^1}^2, \tag{5.37}$$

and the conservation law of mass, yields that

$$E(u(t)) \leq C\left(1 + \int_t^{T_0} \|u(s)\|_{H^1}^2 ds\right), \tag{5.38}$$

where $C > 0$ is independent of t. Taking into account the definition of energy (4.30) and using (5.37) and the conservation law of mass again we thus arrive at

$$\begin{aligned}\frac{1}{2}\|u(t)\|_{H^1}^2 =& E(u(t)) + \frac{1}{p+1}\|u(t)\|_{L^{p+1}}^{p+1} \\ \leq& C\left(1 + \int_t^{T_0} \|u(s)\|_{H^1}^2 ds\right) + \frac{\varepsilon}{p+1}\|u\|_{H^1}^2,\end{aligned} \tag{5.39}$$

which yields (5.35) by taking $\varepsilon < \frac{1}{4}(p+1)$ and applying Gronwall's inequality.

Therefore, it follows that $\sigma^* = 0$, u and so X can be extended to the whole time regime $[0, \infty)$ in the subcritical case. The proof of Theorem 1.3 is complete. □

6. Appendix

6.1. **Linearized operator.** Let $L = (L_+, L_-)$ be the linearized operator around the ground state state defined by

$$L_+ := -\Delta + I - (1+p)Q^p, \quad L_- := -\Delta + I - Q^p. \tag{6.1}$$

For any complex valued H^1 function, set $f := f_1 + if_2$ in terms of the real and imaginary parts and

$$(Lf, f) := \int f_1 L_+ f_1 dx + \int f_2 L_- f_2 dx. \tag{6.2}$$

The crucial coercivity property of linearized operators in the subcritical and critical case are summarized below.

Lemma 6.1. *([61], see also [47, Lemma 2.2]) Let $1 < p < 1 + \frac{4}{d}$. Then, there exists $C > 0$ such that*

$$(Lf, f) \geq C\|f\|_{H^1}^2 - \frac{1}{C}\left((\int Qf_1 dx)^2 + (\int Qf_2 dx)^2 + (\int \nabla Qf_1 dx)^2\right), \tag{6.3}$$

where $f = f_1 + if_2$.

Lemma 6.2. *(*[16, Proposition 3.17]*) Let* $p = 1 + \frac{4}{d}$*. Then, there exists* $C > 0$ *such that*

$$(Lf, f) \geq C\|f\|_{H^1}^2 - \frac{1}{C}\left((\int Qf_1 dx)^2 + (\int Qf_2 dx)^2 + (\int \nabla Q f_1 dx)^2 + (\int x \cdot \nabla Q f_1 dx)^2 \right), \quad (6.4)$$

where $f = f_1 + if_2$*.*

6.2. **Decoupling lemma.**

Lemma 6.3. *(Decoupling lemma) Let* δ_0 *be as in* (1.5)*. For every* $1 \leq k \leq K$*, let*

$$G_{i,k}(t,x) = w_k^{-\frac{2}{p-1}} g_i(\frac{x - v_k t - \alpha_k}{w_k}), \quad i = 1, 2, \quad (6.5)$$

where $1 \leq p \leq 1 + \frac{4}{d}$*,* $g_i \in C_b^2$ *decays exponentially fast at infinity, i.e., for some* $C_1 > 0$*,*

$$|g_i(y)| \leq C_1 e^{-\delta_0 |y|}, \quad y \in \mathbb{R}^d, \ i = 1, 2, \quad (6.6)$$

the parameters $w_k > 0$*,* $v_k, \alpha_k \in \mathbb{R}^d$*, satisfying that*

$$w_k^{-1} + w_k + |v_k| + |\alpha_k| \leq C_2. \quad (6.7)$$

Then, if $v_j \neq v_k$*,* $j \neq k$*, we have that for any* $p_1, p_2 > 0$*,*

$$\int |G_{1,j}(t)|^{p_1} |G_{2,k}(t)|^{p_2} dx \leq Ce^{-\delta |v_j - v_k| t}, \quad (6.8)$$

where $C, \delta_2 > 0$ *depend on* δ_0, C_i, p_i*,* $i = 1, 2$*.*

Proof. We use (6.5) and the change of variables to compute

$$\begin{aligned}\int |G_{1,j}(t)|^{p_1} |G_{2,k}(t)|^{p_2} dx =& w_j^{d-\frac{2p_1}{p-1}} w_k^{-\frac{2p_2}{p-1}} \int |g_1|^{p_1}(y) |g_2|^{p_2} \left(\frac{w_j y + (v_j - v_k)t + (\alpha_j - \alpha_k)}{w_k} \right) dy \\ =& w_j^{d-\frac{2p_1}{p-1}} w_k^{-\frac{2p_2}{p-1}} \left(\int_\Omega + \int_{\Omega^c} \right) |g_1|^{p_1}(y) |g_2|^{p_2} \left(\frac{w_j y + (v_j - v_k)t + (\alpha_j - \alpha_k)}{w_k} \right) dy \\ =&: I_1 + I_2. \end{aligned} \quad (6.9)$$

where $\Omega := \{y \in \mathbb{R}^d : |y| \leq \frac{1}{2w_j}|v_j - v_k|t\}$ and $\Omega^c = \mathbb{R}^d \setminus \Omega$. On one hand, by (6.7), for t large enough,

$$\left| w_j y + (v_j - v_k)t + (\alpha_j - \alpha_k) \right| \geq \frac{1}{2}|v_j - v_k|t - |\alpha_j - \alpha_k| \geq \frac{1}{4}|v_j - v_k|t, \quad y \in \Omega,$$

which along with the exponential decay (6.6) and $w_j, w_k \geq C_2^{-1} > 0$ yields that

$$I_1 \leq Ce^{-\frac{\delta_1 p_2}{4w_k}|v_j - v_k|t} \int g_1^{p_1}(y) dy \leq Ce^{-\delta' |v_j - v_k| t}, \quad (6.10)$$

where $C, \delta' > 0$ depend on δ_0, C_i, p_i, $i = 1, 2$.

On the other hand, using (6.6) again we infer that

$$|g_1(y)| \leq C_1 e^{-\frac{\delta_0}{2w_j}|v_j - v_k|t}, \quad y \in \Omega^c,$$

and thus

$$I_2 \leq Ce^{-\frac{\delta_0 p_1}{2w_j}|v_j - v_k|t} \int_{\Omega^c} g_2^{p_2} \left(\frac{w_j y + (v_j - v_k)t + (\alpha_j - \alpha_k)}{w_k} \right) dy \leq Ce^{-\delta'' |v_j - v_k| t}, \quad (6.11)$$

where where $C, \delta_2 > 0$ depend on δ_0, C_i, p_i, $i = 1, 2$.

Therefore, plugging (6.10) and (6.11) into (6.9) we obtain (6.8) and finish the proof. □

6.3. **Proof of Proposition 3.1.** Below we present the proof of the geometrical decomposition in Proposition 3.1 in a fashion close to that of [12]. Given any $L > 0$, $w_k^0 \in \mathbb{R}^+$, $x_k^0, v_k \in \mathbb{R}^d$, $\theta_k^0 \in \mathbb{R}$, $1 \le k \le K$, set

$$R_L(x) := \sum_{k=1}^K R_{k,L}(x) = \sum_{k=1}^K (w_k^0)^{-\frac{2}{p-1}} Q\left(\frac{x - v_k L - x_k^0}{w_k^0}\right) e^{i\left(\frac{1}{2}v_k\cdot x - \frac{1}{4}|v_k|^2 L + (w_k^0)^{-2}L + \theta_k^0\right)}. \tag{6.12}$$

Note that, if $L = t$, then $R_{k,L} = R_k$ with R_k given by (1.12).

Lemma 6.4. *There exists a universal small constant $\delta_* > 0$ such that the following holds. For any $0 < r, L^{-1} < \delta_*$ and for any $u \in H^{-1}(\mathbb{R}^d)$ satisfying $\|u - R_L\|_{H^{-1}} \le r$, there exist unique C^1 functions $\mathcal{P}(u) = (\widetilde{\alpha}, \widetilde{\theta}, \widetilde{w}) : H^{-1} \to \mathbb{X}^K$ such that u admits the decomposition*

$$u = \sum_{k=1}^K (\widetilde{w}_k w_k^0)^{-\frac{2}{p-1}} Q\left(\frac{x - v_k L - x_k^0 - \widetilde{\alpha}_k}{\widetilde{w}_k w_k^0}\right) e^{i\left(\frac{1}{2}v_k\cdot x - \frac{1}{4}|v_k|^2 L + (w_k^0)^{-2}L + \theta_k^0 + \widetilde{\theta}_k\right)} + \varepsilon_L \ \ \left(=: \sum_{k=1}^K \widetilde{R}_{k,L} + \varepsilon_L\right), \tag{6.13}$$

and the following orthogonality conditions hold: for $1 \le k \le K$,

$$\begin{aligned} &\mathrm{Re}\ {}_{H^1}\langle \nabla \widetilde{R}_{k,L}, \varepsilon_L\rangle_{H^{-1}} = 0, \ \ \mathrm{Im}\ {}_{H^1}\langle \widetilde{R}_{k,L}, \varepsilon_L\rangle_{H^{-1}} = 0, \\ &\mathrm{Re}\ {}_{H^1}\langle \frac{d}{2}\widetilde{R}_{k,L} + y_k\cdot\nabla\widetilde{R}_{k,L} - \frac{i}{2}v_k\cdot y_k\widetilde{R}_{k,L}, \varepsilon_L\rangle_{H^{-1}} = 0, \end{aligned} \tag{6.14}$$

where $y_k := x - v_k L - x_k^0 - \widetilde{\alpha}_k$. Moreover, there exists a universal constant $C > 0$ such that,

$$\|\varepsilon_L\|_{H^{-1}} + \sum_{k=1}^K (|\widetilde{\alpha}_k| + |\widetilde{\theta}_k| + |\widetilde{w}_k - 1|) \le C\|u - R_L\|_{H^{-1}}. \tag{6.15}$$

Proof. The proof proceeds in four steps.

Step 1. Set $\widetilde{\mathcal{P}}_{0,k} := (0,0,1) \in \mathbb{X}$ and $\widetilde{\mathcal{P}}_0 = (\widetilde{\mathcal{P}}_{0,1}, \cdots, \widetilde{\mathcal{P}}_{0,K}) \in \mathbb{X}^K$. Similarly, let $\widetilde{\mathcal{P}}_k := (\widetilde{\alpha}_k, \widetilde{\theta}_k, \widetilde{w}_k) \in \mathbb{X}$, $\widetilde{\mathcal{P}} := (\widetilde{\mathcal{P}}_1, \cdots, \widetilde{\mathcal{P}}_K) \in \mathbb{X}^K$. Let

$$\alpha_k := \widetilde{\alpha}_k + x_k^0, \quad \theta_k := \widetilde{\theta}_k + \theta_k^0, \quad w_k := \widetilde{w}_k w_k^0. \tag{6.16}$$

For any $u_0 \in H^1$, let $B_\delta(u_0, \widetilde{\mathcal{P}}_0)$ denote the closed ball centered at $(u_0, \widetilde{\mathcal{P}}_0)$ of radius δ, i.e.,

$$B_\delta(u_0, \widetilde{\mathcal{P}}_0) := \{(u, \widetilde{\mathcal{P}}) \in H^{-1} \times \mathbb{X}^K : \ \|u - u_0\|_{H^{-1}} \le \delta, \ |\widetilde{\mathcal{P}} - \widetilde{\mathcal{P}}_0| \le \delta\}, \tag{6.17}$$

where δ is a small constant to be chosen later, and

$$|\widetilde{\mathcal{P}} - \widetilde{\mathcal{P}}_0| := \sum_{k=1}^K |\widetilde{\mathcal{P}}_k - \widetilde{\mathcal{P}}_{0,k}| = \sum_{k=1}^K (|\widetilde{\alpha}_k| + |\widetilde{\theta}_k| + |\widetilde{w}_k - 1|). \tag{6.18}$$

For $1 \le k \le K$, let

$$\begin{aligned} &f_{1,j}^k(u, \widetilde{\mathcal{P}}) := \mathrm{Re}\ {}_{H^1}\langle \partial_j \widetilde{R}_{k,L}, \varepsilon_L\rangle_{H^{-1}}, \ \ 1 \le j \le d, \\ &f_2^k(u, \widetilde{\mathcal{P}}) := \mathrm{Im}\ {}_{H^1}\langle \widetilde{R}_{k,L}, \varepsilon_L, \rangle_{H^{-1}}, \\ &f_3^k(u, \widetilde{\mathcal{P}}) := \mathrm{Re}\ {}_{H^1}\langle \frac{2}{p-1}\widetilde{R}_{k,L} + y_k\cdot\nabla\widetilde{R}_{k,L} - \frac{i}{2}v_k\cdot y_k\widetilde{R}_{k,L}, \varepsilon_L\rangle_{H^{-1}}, \end{aligned}$$

where y_k is as in (6.14). Let $F^k := (f^k_{1,1}, \cdots, f^k_{1,d}, f^k_2, f^k_3)$ and $\frac{\partial F^k}{\partial \widetilde{\mathcal{P}}_j}$ denote the Jacobian matrix

$$\frac{\partial F^k}{\partial \widetilde{\mathcal{P}}_j} := \begin{pmatrix} \frac{\partial f^k_{1,1}}{\partial \widetilde{\alpha}_{j,1}} & \cdots \frac{\partial f^k_{1,1}}{\partial \widetilde{\alpha}_{j,d}}, & \frac{\partial f^k_{1,1}}{\partial \widetilde{\theta}_j}, & \frac{\partial f^k_{1,1}}{\partial \widetilde{w}_j} \\ \vdots & & & \vdots \\ \frac{\partial f^k_3}{\partial \widetilde{\alpha}_{j,1}} & \cdots \frac{\partial f^k_3}{\partial \widetilde{\alpha}_{j,d}}, & \frac{\partial f^k_3}{\partial \widetilde{\theta}_j}, & \frac{\partial f^k_3}{\partial \widetilde{w}_j} \end{pmatrix}, \ \ 1 \le j, k \le K, \tag{6.19}$$

where $\widetilde{\alpha}_j := (\widetilde{\alpha}_{j,l}, 1 \le l \le d) \in \mathbb{R}^d$. Similarly, let $F := (F^1, \cdots, F^K)$ and $\frac{\partial F}{\partial \widetilde{\mathcal{P}}} := (\frac{\partial F^k}{\partial \widetilde{\mathcal{P}}_j})_{1 \le j,k \le K}$.

Note that, by the definition (6.12) of R_L, $F^k(R_L, \widetilde{\mathcal{P}}_0) = 0$, $1 \le k \le K$. Moreover, for any $(u, \widetilde{\mathcal{P}}) \in B_\delta(R_L, \widetilde{\mathcal{P}}_0)$, we have that, if $\widetilde{R}_L := \sum_{k=1}^K \widetilde{R}_{k,L}$,

$$\|\varepsilon_L\|_{H^{-1}} \le \|u - R_L\|_{H^{-1}} + \|R_L - \widetilde{R}_L\|_{H^{-1}}. \tag{6.20}$$

By the explicit expressions of R_L and $\widetilde{R}_L$ in (6.12) and (6.13), respectively,

$$\|R_L - \widetilde{R}_L\|_{H^{-1}} \le \|R_L - \widetilde{R}_L\|_{L^2} \le C \sum_{k=1}^K (|\widetilde{\alpha}_k| + |\widetilde{\theta}_k| + |\widetilde{w}_k - 1|) \le C|\widetilde{\mathcal{P}} - \widetilde{\mathcal{P}}_0|, \tag{6.21}$$

where $C > 0$. Thus, we get that for a universal constant $\widetilde{C} > 0$,

$$\|\varepsilon_L\|_{H^{-1}} \le \widetilde{C}(\|u - R_L\|_{H^{-1}} + |\widetilde{\mathcal{P}} - \widetilde{\mathcal{P}}_0|) \le 2\widetilde{C}\delta, \ \ \forall (u, \widetilde{\mathcal{P}}) \in B_\delta(R_L, \widetilde{\mathcal{P}}_0). \tag{6.22}$$

Step 2. We claim that, there exist small constants $\delta_*, c_1, c_2 > 0$ such that for any $0 < \delta, L^{-1} \le \delta_*$,

$$0 < c_1 \le \left| \det \frac{\partial F}{\partial \widetilde{\mathcal{P}}}(u, \widetilde{\mathcal{P}}) \right| \le c_2 < \infty, \ \ \forall (u, \widetilde{\mathcal{P}}) \in B_\delta(R_L, \widetilde{\mathcal{P}}_0). \tag{6.23}$$

To this end, we compute that for $1 \le j, k \le d$,

$$\begin{aligned} &\partial_{\widetilde{\alpha}_{k,j}} f^k_{1,j} = -w_k^{-2} \|\partial_j Q_{w_k}\|^2_{L^2} + O(\|\varepsilon_L\|_{H^{-1}}), \ \ \partial_{\widetilde{\theta}_k} f^k_{1,j} = -\frac{v_{k,j}}{2} \|Q_{w_k}\|^2_{L^2} + O(\|\varepsilon_L\|_{H^{-1}}), \\ &\partial_{\widetilde{\theta}_k} f^k_2 = \|Q_{w_k}\|^2_{L^2} + O(\|\varepsilon_L\|_{H^{-1}}), \ \ \partial_{\widetilde{w}_k} f^k_3 = w_k^{-1} \|\Lambda Q_{w_k}\|^2_{L^2} + O(\|\varepsilon_L\|_{H^{-1}}). \end{aligned} \tag{6.24}$$

Moreover, by the exponential decay of Q, we infer that, there exists $\delta > 0$ such that the other terms in the Jacobian matrices are of the order $O(\|\varepsilon\|_{H^{-1}} + e^{-\delta L})$. This yields that

$$\left| \det \left(\frac{\partial F}{\partial \widetilde{\mathcal{P}}} \right) \right| = \prod_{k=1}^K \left((w_k^0)^{-2d} \widetilde{w}_k^{-2d-1} \|Q_{w_k}\|^2_{L^2} \|\Lambda Q_{w_k}\|^2_{L^2} \prod_{j=1}^d \|\partial_j Q_{w_k}\|^2_{L^2} \right) + O\left(\|\varepsilon\|_{H^{-1}} + e^{-\delta L} \right). \tag{6.25}$$

Taking into account $|\widetilde{\mathcal{P}} - \widetilde{\mathcal{P}}_0| \le \delta$ we obtain (6.23), as claimed.

Step 3. In this step, we claim that there exists a universal constant $C_* (\ge 1)$ such that, for any $0 < \delta, L^{-1} \le \delta_*$ and any $(u_1, \widetilde{\mathcal{P}}(u_1)), (u_2, \widetilde{\mathcal{P}}(u_2)) \in B_\delta(R_L, \widetilde{\mathcal{P}}_0)$, if $F(u_1, \widetilde{\mathcal{P}}(u_1)) = F(u_2, \widetilde{\mathcal{P}}(u_2)) = 0$, then

$$|\widetilde{\mathcal{P}}(u_1) - \widetilde{\mathcal{P}}(u_2)| \le C_* \|u_1 - u_2\|_{H^{-1}}. \tag{6.26}$$

To this end, we infer that

$$F(u_1, \widetilde{\mathcal{P}}(u_1)) - F(u_1, \widetilde{\mathcal{P}}(u_2)) = F(u_2, \widetilde{\mathcal{P}}(u_2)) - F(u_1, \widetilde{\mathcal{P}}(u_2)). \tag{6.27}$$

By the differential mean value theorem,

$$\left(\frac{\partial F}{\partial \widetilde{\mathcal{P}}}(u_1, \widetilde{\mathcal{P}}_r) \right) (\widetilde{\mathcal{P}}(u_1) - \widetilde{\mathcal{P}}(u_2))^t = (F(u_2, \widetilde{\mathcal{P}}(u_2)) - F(u_1, \widetilde{\mathcal{P}}(u_2)))^t, \tag{6.28}$$

where $\widetilde{\mathcal{P}}_r = r\widetilde{\mathcal{P}}(u_1) + (1-r)\widetilde{\mathcal{P}}(u_2)$ for some $0 < r < 1$, and the superscript t means the transpose of matrices. Since the Jacobian matrix $\frac{\partial F}{\partial \widetilde{\mathcal{P}}}(u_1, \widetilde{\mathcal{P}}_r)$ is invertible by (6.23), this leads to

$$(\widetilde{\mathcal{P}}(u_1) - \widetilde{\mathcal{P}}(u_2))^t = \left(\frac{\partial F}{\partial \widetilde{\mathcal{P}}}(u_1, \widetilde{\mathcal{P}}_r)\right)^{-1} (F(u_2, \widetilde{\mathcal{P}}(u_2)) - F(v_1, \widetilde{\mathcal{P}}(u_2)))^t. \tag{6.29}$$

Note that, by (6.24), there exists a universal constant $C > 0$ such that

$$\left\| \left(\frac{\partial F}{\partial \widetilde{\mathcal{P}}}(u_1, \widetilde{\mathcal{P}}_r)\right)^{-1} \right\| \leq C, \tag{6.30}$$

where $\|\cdot\|$ denotes the Hilbert-Schmidt norm of matrices. Moreover, by (1.5),

$$|F(u_2, \widetilde{\mathcal{P}}(u_2)) - F(u_1, \widetilde{\mathcal{P}}(u_2))| \leq C\|u_2 - u_1\|_{H^{-1}}. \tag{6.31}$$

Thus, we infer from (6.29), (6.30) and (6.31) that (6.26) holds, as claimed.

Step 4. Let δ_*, C_* be the universal constants as in Step 1 and Step 2, respectively, and set

$$B := \{v \in B_{\frac{\delta_*}{C_*}}(R_L) : \exists \widetilde{\mathcal{P}} \in B_{\delta_*}(\widetilde{\mathcal{P}}_0),\ \textit{such that } F(v, \widetilde{\mathcal{P}}) = 0\}. \tag{6.32}$$

Since $B_{\frac{\delta_*}{C_*}}(R_L)$ is connected and $R_L \in B$, in order to prove that

$$B = B_{\frac{\delta_*}{C_*}}(R_L). \tag{6.33}$$

we only need to show that B is both open and closed in $B_{\frac{\delta_*}{C_*}}(R_L)$.

To this end, For any $u \in B$, by definition there exists $\widetilde{\mathcal{P}}(u) \in B_{\delta_*}(\widetilde{\mathcal{P}}_0)$ such that $F(u, \widetilde{\mathcal{P}}(u)) = 0$. Taking into account the non-degeneracy of the Jacobian matrix at $(u, \widetilde{\mathcal{P}}(u))$ due to (6.23), we can apply the implicit function theorem to get a small open neighborhood $\mathcal{U}(u)$ of u in $B_{\frac{\delta_*}{C_*}}(R_L)$ such that $\mathcal{U}(u) \subseteq B$. This yield that B is open in $B_{\frac{\delta_*}{C_*}}(R_L)$.

Moreover, for any sequence $\{u_n\} \subseteq B$ such that $u_n \to u_*$ in H^{-1} for some $u_* \in B_{\frac{\delta_*}{C_*}}(R_L)$, by definition there exist modulation parameters $\widetilde{\mathcal{P}}(u_n) \in B_{\delta_*}(\widetilde{\mathcal{P}}_0)$ such that $F(u_n, \widetilde{\mathcal{P}}(u_n)) = 0$, $n \geq 1$. In particular, $\{\widetilde{\mathcal{P}}(v_n)\} \subseteq \mathbb{X}^K$ is uniformly bounded and so, along a subsequence (still denoted by $\{n\}$), $\widetilde{\mathcal{P}}(v_n) \to \widetilde{\mathcal{P}}_*$ $(\in B_{\delta_*}(\widetilde{\mathcal{P}}_0))$ for some $\widetilde{\mathcal{P}}_* \in \mathbb{X}^K$.

Then, let $\widetilde{R}_{k,L,\widetilde{\mathcal{P}}(u_n)}$ and $\widetilde{R}_{k,L,\widetilde{\mathcal{P}}_*}$ be the k-th soliton profiles corresponding to $\widetilde{\mathcal{P}}(u_n)$ and $\widetilde{\mathcal{P}}_*$, respectively. By the above convergence of u_n and $\widetilde{\mathcal{P}}(u_n)$ we infer that $u_n - \sum_{k=1}^K \widetilde{R}_{k,L,\widetilde{\mathcal{P}}(u_n)} \to u_* - \sum_{k=1}^K \widetilde{R}_{k,L,\widetilde{\mathcal{P}}_*}$ in H^{-1}. Taking $n \to \infty$ and using the fact that $F(u_n, \widetilde{\mathcal{P}}(u_n)) = 0$ we obtain $F(u_*, \widetilde{\mathcal{P}}_*) = 0$, and so $u_* \in B$. Hence, B is also closed in $B_{\frac{\delta_*}{C_*}}(R_L)$.

Therefore, (6.33) is verified. The geometrical decomposition (6.13) and the orthogonality conditions in (6.14) hold. Moreover, estimate (6.15) follows from (6.22) and (6.26) by taking $u_1 = u$ and $u_2 = R_L$. The proof of Lemma 6.4 is complete. □

Proof of Proposition 3.1. Since $u(T) = R(T)$, by the local wellposedness theory, there exists T^* close to T, such that $u(t) \in C^1([T^*, T]; H^{-1}) \bigcap C([T^*, T]; H^1)$ and $\|u(t) - R(T)\|_{H^1} \in B_\delta(u(T))$ for all $t \in [T^*, T]$, where $\delta > 0$ is as in Lemma 6.4.

Hence, applying Lemma 6.4 to $\{u(t)\}$ we obtain that for T large enough, there exist unique C^1 functions $(\alpha_k, \theta_k, \omega_k) \in C^1([T^*, T]; \mathbb{X}^K)$, $1 \leq k \leq K$, such that for any $t \in [T^*, T]$, $u(t)$ admits the decomposition (6.13) and the orthogonality conditions in (6.14) hold with t replacing T.

Then, taking into account $u(t) \in H^1$ and (6.13), the remainder $\varepsilon(t)$ is indeed in the space H^1. Thus, the parings between H^{-1} and H^1 in (6.14) are exactly the L^2 inner products, which yields the orthogonality conditions in (3.5) for any $[T^*, T]$. Therefore, the proof is complete. □

6.4. **Proof of** (4.45)**.** We set $\widetilde{S}_k := \sum_{j=k}^{K} \widetilde{R}_j$, $1 \leq k \leq K$. Then,

$$\widetilde{S}_k = \widetilde{R}_k + \widetilde{S}_{k+1}, \quad 1 \leq k \leq K-1. \tag{6.34}$$

Lemma 6.5. *Let* $0 < q < \infty$*, we have*

$$\Big| |\widetilde{S}_k|^q - |\widetilde{R}_k|^q \Big| \leq C h(\widetilde{S}_{k+1}), \tag{6.35}$$

where $C > 0$, $h(\widetilde{S}_{k+1}) = |\widetilde{S}_{k+1}|^q$ *if* $0 < q < 1$*, and* $h(\widetilde{S}_{k+1}) = |\widetilde{S}_{k+1}|$ *if* $1 \leq q < \infty$.

Proof. The case where $0 < q < 1$ follows from the inequality

$$(a+b)^q \leq a^q + b^q, \quad a, b \geq 0,$$

while the case $1 \leq q < \infty$ follows from the inequality

$$||\widetilde{S}_k|^q - |\widetilde{R}_k|^q| \leq C(|\widetilde{S}_{k+1}|^{q-1} + |\widetilde{R}_k|^{q-1})|\widetilde{S}_{k+1}|$$

and the uniform boundedness of $\widetilde{R}_j$, $1 \leq j \leq K$. □

Lemma 6.6. *There exist constants* $C, \delta_2 > 0$ *depending on* w_k^0*,* x_k^0*,* v_k *and* δ_0 *such that*

$$\Big| \int |\widetilde{S}_k|^{p+1} - |\widetilde{R}_k|^{p+1} - |\widetilde{S}_{k+1}|^{p+1} dx \Big| \leq C e^{-\delta_2 t}. \tag{6.36}$$

Proof. Using the expansion

$$|\widetilde{S}_k|^2 = |\widetilde{R}_k|^2 + |\widetilde{S}_{k+1}|^2 + 2\mathrm{Re}(\widetilde{R}_k \widetilde{S}_{k+1}),$$

and Lemmas 6.3 and 6.5 we have

$$\begin{aligned}
&\Big| \int |\widetilde{S}_k|^{p+1} - |\widetilde{R}_k|^{p+1} - |\widetilde{S}_{k+1}|^{p+1} dx \Big| \\
\leq & \int \Big| |\widetilde{S}_k|^{p-1} - |\widetilde{R}_k|^{p-1} \Big| |\widetilde{R}_k|^2 + \Big| |\widetilde{S}_k|^{p-1} - |\widetilde{S}_{k+1}|^{p-1} \Big| |\widetilde{S}_{k+1}|^2 + 2|\widetilde{S}_k|^{p-1} |\widetilde{R}_k \widetilde{S}_{k+1}| dx \\
\leq & C \int h(\widetilde{S}_{k+1}) |\widetilde{R}_k|^2 + h(\widetilde{R}_k) |\widetilde{S}_{k+1}|^2 + 2|\widetilde{S}_k|^{p-1} |\widetilde{R}_k \widetilde{S}_{k+1}| dx \\
\leq & C e^{-\delta_2 t},
\end{aligned}$$

which yields (6.36). □

Lemma 6.7. *There exist constants* $C, \delta_2 > 0$ *depending on* w_k^0*,* x_k^0*,* v_k *and* δ_0 *such that*

$$\Big| \int (|\widetilde{S}_k|^{p-1} \widetilde{S}_k - |\widetilde{R}_k|^{p-1} \widetilde{R}_k - |\widetilde{S}_{k+1}|^{p-1} \widetilde{S}_{k+1}) \bar{\varepsilon} dx \Big| \leq C e^{-\delta_2 t} \|\varepsilon\|_{L^2}. \tag{6.37}$$

Proof. By the expansion (6.34), Lemmas 6.3 and 6.5 and Hölder's inequality,

$$\begin{aligned}
&\Big| \int (|\widetilde{S}_k|^{p-1} \widetilde{S}_k - |\widetilde{R}_k|^{p-1} \widetilde{R}_k - |\widetilde{S}_{k+1}|^{p-1} \widetilde{S}_{k+1}) \bar{\varepsilon} dx \Big| \\
\leq & \int \Big(\Big| |\widetilde{S}_k|^{p-1} - |\widetilde{R}_k|^{p-1} \Big| |\widetilde{R}_k| + \Big| |\widetilde{S}_k|^{p-1} - |\widetilde{S}_{k+1}|^{p-1} \Big| |\widetilde{S}_{k+1}| \Big) |\varepsilon| dx \\
\leq & C(\|h(\widetilde{S}_{k+1}) \widetilde{R}_k\|_{L^2} + \|h(\widetilde{R}_k) \widetilde{S}_{k+1}\|_{L^2}) \|\varepsilon\|_{L^2} \\
\leq & C e^{-\delta t} \|\varepsilon\|_{L^2},
\end{aligned}$$

which yields (6.37). □

Lemma 6.8. *There exist constants $C, \delta_2 > 0$ depending on w_k^0, x_k^0, v_k and δ_0 such that*

$$|\int (|\widetilde{S}_k|^{p-1} - |\widetilde{R}_k|^{p-1} - |\widetilde{S}_{k+1}|^{p-1})|\varepsilon|^2 dx| \leq Ce^{-\delta_2 t}\|\varepsilon\|_{L^2}^2. \tag{6.38}$$

Proof. Let $\Omega_k := \{x : |x - v_k t| \leq \frac{1}{2}\min_{j\neq k}|v_k - v_j|t\}$. By Lemma 6.5,

$$\begin{aligned}
&\left|\int (|\widetilde{S}_k|^{p-1} - |\widetilde{R}_k|^{p-1} - |\widetilde{S}_{k+1}|^{p-1})|\varepsilon|^2 dx\right| \\
\leq & 2\int_{\Omega_k} (h(\widetilde{S}_{k+1}) + |\widetilde{S}_{k+1}|^{p-1})|\varepsilon|^2 dx + 2\int_{\Omega_k^c} (h(\widetilde{R}_k) + |\widetilde{R}_k|^{p-1})|\varepsilon|^2 dx \\
\leq & C\|h(\widetilde{S}_{k+1}) + |\widetilde{S}_{k+1}|^{p-1}\|_{L^\infty(\Omega_k)}\|\varepsilon\|_{L^2}^2 + C\|h(\widetilde{R}_k) + |\widetilde{R}_k|^{p-1}\|_{L^\infty(\Omega_k^c)}\|\varepsilon\|_{L^2}^2.
\end{aligned} \tag{6.39}$$

Note that, for $x \in \Omega_k$, for any $j \neq k$,

$$|x - v_j t - \alpha_j| \geq |v_j - v_k|t - |x - v_k t| - |\alpha_j| \geq \frac{1}{4}|v_j - v_k|t,$$

and thus by the exponential decay of Q,

$$\|h(\widetilde{S}_{k+1}) + |\widetilde{S}_{k+1}|^{p-1}\|_{L^\infty(\Omega_k^c)} \leq Ce^{-\delta_2 t}. \tag{6.40}$$

Similarly, for $x \in \Omega_k^c$, there exists $c > 0$ such that for t large enough,

$$|x - v_k t - \alpha_k| \geq \frac{1}{2}\min_{j\neq k}\{|v_j - v_k|t\} - |\alpha_k| \geq ct,$$

and thus

$$\|h(\widetilde{R}_k) + |\widetilde{R}_k|^{p-1}\|_{L^\infty(\Omega_k^c)} \leq Ce^{-\delta_2 t}. \tag{6.41}$$

Therefore, plugging (6.40) and (6.41) into (6.39) we obtain (6.38) and finish the proof. □

Lemma 6.9. *There exist constants $C, \delta_2 > 0$ depending on w_k^0, x_k^0, v_k and δ_0 such that*

$$|\int (|\widetilde{S}_k|^{p-3}\widetilde{S}_k^2 - |\widetilde{R}_k|^{p-3}\widetilde{R}_k^2 - |\widetilde{S}_{k+1}|^{p-3}\widetilde{S}_{k+1}^2)\overline{\varepsilon}^2 dx| \leq Ce^{-\delta_2 t}. \tag{6.42}$$

Proof. Since

$$|\widetilde{S}_k|^{p-1}\frac{\widetilde{S}_k^2}{|\widetilde{S}_k^2|} = |\widetilde{R}_k|^{p-1}\frac{\widetilde{S}_k^2}{|\widetilde{S}_k^2|} + |\widetilde{S}_{k+1}|^{p-1}\frac{\widetilde{S}_k^2}{|\widetilde{S}_k^2|} + O(||\widetilde{S}_k|^{p-1} - |\widetilde{R}_k|^{p-1} - |\widetilde{S}_{k+1}|^{p-1}|),$$

we have

$$\begin{aligned}
&\left|\int (|\widetilde{S}_k|^{p-3}\widetilde{S}_k^2 - |\widetilde{R}_k|^{p-3}\widetilde{R}_k^2 - |\widetilde{S}_{k+1}|^{p-3}\widetilde{S}_{k+1}^2)\overline{\varepsilon}^2 dx\right| \\
\leq & \int \left||\widetilde{S}_k|^{p-1}\frac{\widetilde{S}_k^2}{|\widetilde{S}_k|^2} - |\widetilde{R}_k|^{p-1}\frac{\widetilde{R}_k^2}{|\widetilde{R}_k|^2} - |\widetilde{S}_{k+1}|^{p-1}\frac{\widetilde{S}_{k+1}^2}{|\widetilde{S}_{k+1}|^2}\right||\varepsilon|^2 dx \\
\leq & \int |\widetilde{R}_k|^{p-1}\left|\frac{\widetilde{S}_k^2}{|\widetilde{S}_k|^2} - \frac{\widetilde{R}_k^2}{|\widetilde{R}_k|^2}\right||\varepsilon|^2 dx + \int |\widetilde{S}_{k+1}|^{p-1}\left|\frac{\widetilde{S}_k^2}{|\widetilde{S}_k|^2} - \frac{\widetilde{S}_{k+1}^2}{|\widetilde{S}_{k+1}|^2}\right||\varepsilon|^2 dx \\
& + O\left(\int \left||\widetilde{S}_k|^{p-1} - |\widetilde{S}_k|^{p-1} - |\widetilde{R}_k|^{p-1}\right||\varepsilon|^2 dx\right) \\
= & \int |\widetilde{R}_k|^{p-1}\left|\frac{\widetilde{S}_k^2}{|\widetilde{S}_k|^2} - \frac{\widetilde{R}_k^2}{|\widetilde{R}_k|^2}\right||\varepsilon|^2 dx + \int |\widetilde{S}_{k+1}|^{p-1}\left|\frac{\widetilde{S}_k^2}{|\widetilde{S}_k|^2} - \frac{\widetilde{S}_{k+1}^2}{|\widetilde{S}_{k+1}|^2}\right||\varepsilon|^2 dx + O(e^{-\delta_2 t})
\end{aligned}$$

$$=: J_1 + J_2 + O(e^{-\delta_2 t}). \tag{6.43}$$

where the last step is due to Lemma 6.8.

Below we estimate J_1 and J_2 separately. For this purpose, let us set $d_* := \min_{k\leq j\neq l\leq K}\{|v_j t + \alpha_j - v_l t - \alpha_l|\}$. Similarly, let $w_* := \min_{k\leq j\leq K} w_j$, $w^* := \max_{k\leq j\leq K} w_j$. For every $k \leq j \leq K$, set

$$\Omega_j := \Big\{x \in \mathbb{R}^d : |x - v_j t - \alpha_j| \leq \varepsilon d_*\Big\},$$

where ε is a small constant to be specified below.

(i) Estimate of J_1. We decompose

$$J_1 = \int_{\Omega_k^c} |\widetilde{R}_k|^{p-1}\Big|\frac{\widetilde{S}_k^2}{|\widetilde{S}_k|^2} - \frac{\widetilde{R}_k^2}{|\widetilde{R}_k|^2}\Big||\varepsilon|^2 dx + \int_{\Omega_k} |\widetilde{R}_k|^{p-1}\Big|\frac{\widetilde{S}_k^2}{|\widetilde{S}_k|^2} - \frac{\widetilde{R}_k^2}{|\widetilde{R}_k|^2}\Big||\varepsilon|^2 dx := J_{11} + J_{12}. \tag{6.44}$$

Note that, for $x \in \Omega_k^c$, since

$$|x - v_k t - \alpha_k| \geq \varepsilon d_* > \frac{c}{2} t \tag{6.45}$$

for t large enough, where $c > 0$, by (1.5), there exist $C, \delta_2 > 0$ such that

$$J_{11} \leq C\|R_k\|^{p-1}_{L^\infty(\Omega_k^c)}\|\varepsilon\|^2_{L^2} \leq Ce^{-\delta_2 t}\|\varepsilon\|^2_{L^2}. \tag{6.46}$$

Concerning the first term J_{12} in (6.44), since $Q(x) \sim e^{-\delta_0|x|}$ (see [10]), we infer that

$$|\widetilde{R}_k(t,x)| \geq Ce^{-\delta_0\frac{\varepsilon d_*}{w_k}} \geq Ce^{-\delta_0\frac{\varepsilon d_*}{w_*}}, \quad x \in \Omega_k. \tag{6.47}$$

On the other hand, for $x \in \Omega_k$ and any $j \neq k$,

$$|x - v_j t - \alpha_j| \geq |(v_k - v_j)t + \alpha_k - \alpha_j| - |x - v_k t - \alpha_k| \geq (1-\varepsilon)d_*,$$

which yields that

$$|\widetilde{S}_{k+1}(t,x)| \leq C\sum_{j=k+1}^{K} e^{-\delta_0\frac{(1-\varepsilon)d_*}{w_j}} \leq Ce^{-\delta_0\frac{(1-\varepsilon)d_*}{w^*}}, \quad x \in \Omega_k. \tag{6.48}$$

Hence, we obtain from (6.47) and (6.48) that, for ε small enough such that

$$\varepsilon < \frac{w_*}{w^* + w_*},$$

there exist $C, \delta_2 > 0$ such that

$$\Big|\frac{\widetilde{S}_{k+1}(t,x)}{\widetilde{R}_k(t,x)}\Big| \leq Ce^{-\delta_0 d_*(\frac{(1-\varepsilon)}{w^*} - \frac{\varepsilon}{w_*})} \leq Ce^{-\delta_2 t}, \quad x \in \Omega_k. \tag{6.49}$$

Taking into account

$$\begin{aligned}\frac{\widetilde{S}_k^2}{|\widetilde{S}_k^2|} - \frac{\widetilde{R}_k^2}{|\widetilde{R}_k^2|} =& \frac{\widetilde{S}_k^2|\widetilde{R}_k|^2 - |\widetilde{S}_k|^2\widetilde{R}_k^2 + 2\widetilde{R}_k\widetilde{S}_{k+1}|\widetilde{R}_k|^2 - 2\mathrm{Re}(\widetilde{R}_k\widetilde{S}_{k+1})\widetilde{R}_k^2}{|\widetilde{R}_k + \widetilde{S}_{k+1}|^2|\widetilde{R}_k|^2}\\ \leq& \frac{|\widetilde{R}_k^{-1}\widetilde{S}_{k+1}|^2 + |\widetilde{R}_k^{-1}\widetilde{S}_{k+1}|}{|1 + \widetilde{R}_k^{-1}\widetilde{S}_{k+1}|^2}\end{aligned}$$

we thus lead to

$$\Big|\frac{\widetilde{S}_k^2}{|\widetilde{S}_k^2|} - \frac{\widetilde{R}_k^2}{|\widetilde{R}_k^2|}\Big| \leq Ce^{-\delta_2 t}, \quad x \in \Omega_k, \tag{6.50}$$

which yields that

$$J_{12} \le Ce^{-\delta_2 t}\|\varepsilon\|_{L^2}^2. \tag{6.51}$$

Thus, plugging (6.46) and (6.51) into (6.44) we obtain

$$J_2 \le Ce^{-\delta_2 t}. \tag{6.52}$$

(ii) Estimate of J_2. Set

$$\Omega = \bigcup_{j=k+1}^{K} \Omega_j = \bigcup_{j=k+1}^{K} \left\{ x \in \mathbb{R}^d : |x - v_j t - \alpha_j| \le \varepsilon d_* \right\} \tag{6.53}$$

and decompose

$$\begin{aligned} J_2 = & \int_{\Omega} |\widetilde{S}_{k+1}|^{p-1} \left| \frac{\widetilde{S}_k^2}{|\widetilde{S}_k|^2} - \frac{\widetilde{S}_{k+1}^2}{|\widetilde{S}_{k+1}|^2} \right| |\varepsilon|^2 dx + \int_{\Omega^c} |\widetilde{S}_{k+1}|^{p-1} \left| \frac{\widetilde{S}_k^2}{|\widetilde{S}_k|^2} - \frac{\widetilde{S}_{k+1}^2}{|\widetilde{S}_{k+1}|^2} \right| |\varepsilon|^2 dx \\ = & J_{21} + J_{22}. \end{aligned} \tag{6.54}$$

Note that, for every $k+1 \le j \le K$, since $Q(x) \sim e^{-\delta_0 |x|}$,

$$|\widetilde{R}_j(t,x)| \ge Ce^{-\delta_0 \frac{\varepsilon d_*}{w_j}} \ge Ce^{-\delta_0 \frac{\varepsilon d_*}{w_*}}, \quad x \in \Omega_j. \tag{6.55}$$

Moreover, for $x \in \Omega/\Omega_j$, there exists $j' \ne j$ such that $x \in \Omega_{j'}$ and so

$$|x - v_j t - \alpha_j| \ge |v_{j'} t + \alpha_{j'} - v_j t - \alpha_j| - |x - v_{j'} t - \alpha_{j'}| \ge (1-\varepsilon) d_*.$$

This yields that

$$|\widetilde{R}_j(t,x)| \le Ce^{-\delta_0 \frac{(1-\varepsilon)d_*}{w_j}} \le Ce^{-\delta_0 \frac{(1-\varepsilon)d_*}{w^*}}, \quad x \in \Omega/\Omega_j.$$

Hence, for ε very small such that

$$\varepsilon < \frac{w_*}{w_* + w^*},$$

we obtain that

$$|\widetilde{S}_{k+1}| \ge Ce^{-\delta_0 \frac{\varepsilon d_*}{w_*}} - C' e^{-\delta_0 \frac{(1-\varepsilon)d_*}{w^*}} \ge \frac{1}{2} Ce^{-\delta_0 \frac{\varepsilon d_*}{w_*}}, \quad x \in \Omega, \ k+1 \le j \le K, \tag{6.56}$$

which yields that there exist $C, \delta > 0$ such that

$$|\widetilde{S}_{k+1}| \ge Ce^{-\delta_0 \frac{\varepsilon d_*}{w_*}}, \quad x \in \Omega. \tag{6.57}$$

Moreover, for any $x \in \Omega$, there exists $k+1 \le j \le K$ such that $x \in \Omega_j$ and so

$$|x - v_k t - \alpha_k| \ge |(v_j - v_k)t - (\alpha_j - \alpha_k)| - |x - v_j t - \alpha_j| \ge (1-\varepsilon) d_*, \tag{6.58}$$

which yields that

$$|\widetilde{R}_k(t,x)| \le e^{-\delta_0 \frac{(1-\varepsilon)d_*}{w_k}} \le e^{-\delta_0 \frac{(1-\varepsilon)d_*}{w^*}}, \quad x \in \Omega. \tag{6.59}$$

Thus, we infer that for ε possibly smaller such that

$$\varepsilon < \frac{w_*}{w_* + w^*},$$

then for $x \in \Omega$,

$$\left| \frac{\widetilde{R}_k(t,x)}{\widetilde{S}_{k+1}(t,x)} \right| \le Ce^{-\delta_0 d_* (\frac{(1-\varepsilon)}{w^*} - \frac{\varepsilon}{w_*})} \le Ce^{-\delta_2 t}, \quad x \in \Omega. \tag{6.60}$$

Then, similar to (6.50), we have

$$\left|\frac{\widetilde{S}_k^2}{|\widetilde{S}_k^2|} - \frac{\widetilde{S}_{k+1}^2}{|\widetilde{S}_{k+1}^2|}\right| \leq C\frac{|\widetilde{S}_{k+1}^{-1}\widetilde{R}_k|^2 + |\widetilde{S}_{k+1}^{-1}\widetilde{R}_k|}{|1+\widetilde{S}_{k+1}^{-1}\widetilde{R}_k|^2} \leq Ce^{-\delta_2 t}, \quad x\in\Omega, \tag{6.61}$$

which yields that

$$J_{21} \leq Ce^{-\delta_2 t}\|\varepsilon\|_{L^2}^2. \tag{6.62}$$

Concerning J_{22}, we see that for $x\in\Omega^c$, for $k+1\leq j\leq K$,

$$|x - v_j t - \alpha_j| \geq \varepsilon d_*,$$

and so

$$|\widetilde{R}_j(t,x)| \leq Ce^{-\delta_0\frac{\varepsilon d_*}{w_j}}, \quad x\in\Omega^c. \tag{6.63}$$

This yields that there exist $C,\delta>0$ such that

$$|\widetilde{S}_{k+1}| \leq C\sum_{j=k+1}^{K}|\widetilde{R}_j| \leq Ce^{-\delta_2 t}, \quad x\in\Omega^c, \tag{6.64}$$

and thus

$$J_{22} \leq Ce^{-\delta_2 t}\|\varepsilon\|_{L^2}^2. \tag{6.65}$$

Thus, we obtain from (6.54), (6.62) and (6.65) that

$$J_2 \leq Ce^{-\delta_2 t}. \tag{6.66}$$

Therefore, plugging (6.52) and (6.66) into (6.43) we prove (6.42) and thus finish the proof. □

Now, estimate (4.45) follows from Lemmas 6.6, 6.7, 6.8 and 6.9.

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