Nonlinear Fokker–Planck equations with fractional Laplacian and McKean–Vlasov SDEs with Lévy–Noise

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Abstract

This work is concerned with the existence of mild solutions to nonlinear Fokker–Planck equations with fractional Laplace operator $(-\Delta)^s$ for $s \in \left(\frac{1}{2}, 1\right)$. The uniqueness of Schwartz distributional solutions is also proved under suitable assumptions on diffusion and drift terms. As applications, weak existence and uniqueness of solutions to McKean–Vlasov equations with Lévy–Noise, as well as the Markov property for their laws are proved.

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1 Introduction

We consider here the nonlinear Fokker–Planck equation (NFPE)

$$\begin{aligned} &u_t + (-\Delta)^s \beta(u) + \operatorname{div}(Db(u)u) = 0, \quad \text{in } (0,\infty) \times \mathbb{R}^d, \\ &u(0,x) = u_0(x), \ x \in \mathbb{R}^d, \end{aligned} \tag{1.1}$$

where $\beta : \mathbb{R} \to \mathbb{R}$, $D : \mathbb{R}^d \to \mathbb{R}^d$, $d \geq 2$, and $b : \mathbb{R} \to \mathbb{R}$ are given functions to be made precise later on, while $(-\Delta)^s$, $0 < s < 1$, is the fractional Laplace

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operator defined as follows. Let $S' := S'(\mathbb{R}^d)$ be the dual of the Schwartz test function space $S := S(\mathbb{R}^d)$. Define

$$D_s := \{u \in S';\ \mathcal{F}(u) \in L^1_{\rm loc}(\mathbb{R}^d),\ |\xi|^{2s}\mathcal{F}(u) \in S'\}\ (\supset L^1(\mathbb{R}^d))$$

and

$$\mathcal{F}((-\Delta)^s u)(\xi) = |\xi|^{2s}\mathcal{F}(u)(\xi),\ \xi \in \mathbb{R}^d,\ u \in D_s, \tag{1.2}$$

where $\mathcal{F}$ stands for the Fourier transform in $\mathbb{R}^d$, that is,

$$\mathcal{F}(u)(\xi) = (2\pi)^{-d/2}\int_{\mathbb{R}^d} e^{ix\cdot\xi}u(x)dx,\ \xi \in \mathbb{R}^d,\ u \in L^1(\mathbb{R}^d). \tag{1.3}$$

($\mathcal{F}$ extends from S' to itself.)

NFPE (1.1) is used for modelling the dynamics of anomalous diffusion of particles in disordered media. The solution u may be viewed as the transition density corresponding to a distribution dependent stochastic differential equation with Lévy forcing term.

Hypotheses

(i) $\beta \in C^1(\mathbb{R}) \cap \mathrm{Lip}(\mathbb{R}),\ \beta'(r) > 0,\ \forall\, r \neq 0.$

(ii) $D \in L^\infty(\mathbb{R}^d;\mathbb{R}^d),\ \mathrm{div}\, D \in L^2_{\rm loc}(\mathbb{R}^d).$

(iii) $b \in C_b(\mathbb{R}).$

(iv) $(\mathrm{div}\, D)^- \in L^\infty,\ b \geq 0.$

Here, we shall study the existence of a *mild solution* to equation (1.1) (see Definition 1.1 below) and also the uniqueness of distributional solutions. As regards the existence, we shall follow the semigroup methods used in [6]–[9] in the special case $s = 1$. Namely, we shall represent (1.1) as an abstract differential equation in $L^1(\mathbb{R}^d)$ of the form

$$\begin{aligned} &\frac{du}{dt} + A(u) = 0, \quad t \geq 0,\\ &u(0) = u_0, \end{aligned} \tag{1.4}$$

where A is a suitable realization in $L^1(\mathbb{R}^d)$ of the operator

$$\begin{aligned} A_0(u) &= (-\Delta)^s\beta(u) + \mathrm{div}(Db(u)u),\ u \in D(A_0),\\ D(A_0) &= \left\{u \in L^1(\mathbb{R}^d); (-\Delta)^s\beta(u) + \mathrm{div}(Db(u)u) \in L^1(\mathbb{R}^d)\right\}, \end{aligned} \tag{1.5}$$

where div is taken in the sense of Schwartz distributions on $\mathbb{R}^d$.

Definition 1.1. A function $u \in C([0,\infty); L^1 := L^1(\mathbb{R}^d))$ is said to be a *mild solution* to (1.1) if, for each $0 < T < \infty$,

$$u(t) = \lim_{h\to 0} u_h(t) \text{ in } L^1(\mathbb{R}^d),\ t \in [0,T), \tag{1.6}$$

where

$$u_h(t) = u_h^j,\ \forall\, t \in [jh,(j+1)h),\ j = 0,1,...,N = \left[\tfrac{T}{h}\right], \tag{1.7}$$

$$u_h^{j+1} + hA_0(u_h^{j+1}) = u_h^j,\ j = 0,1,....,N_h, \tag{1.8}$$

$$u_h^j \in D(A_0),\ \forall\, j = 0,...,N_h;\ u_h^0 = u_0. \tag{1.9}$$

Of course, Definition 1.1 makes sense only if the range $R(I+hA_0)$ of the operator $I + hA_0$ is all of $L^1(\mathbb{R}^d)$. We note that, if u is a mild solution to (1.1), then it is also a *Schwartz distributional solution*, that is,

$$\begin{aligned}\int_0^\infty\!\!\int_{\mathbb{R}^d} &(u(t,x)\varphi_t(t,x) - (-\Delta)^s\varphi(t,x)\beta(u(t,x))\\ &+b(u(t,x))u(t,x)D(x)\cdot\nabla\varphi(t,x))dtdx\\ &+\int_{\mathbb{R}^d}\varphi(0,x)u_0(dx) = 0,\ \forall\,\varphi \in C_0^\infty([0,\infty)\times\mathbb{R}^d),\end{aligned} \tag{1.10}$$

where u_0 is a measure of finite variation on $\mathbb{R}^d$. The main existence result for equation (1.1) is given by Theorem 2.4 below, which amounts to saying that under Hypotheses (i)–(iv) there is a mild solution u represented as $u(t) = S(t)u_0$, $t \ge 0$, where $S(t)$ is a continuous semigroup of nonlinear contractions in L^1. In Section 3, the uniqueness of distributional solutions to (1.1), (1.10) respectively, in the class $(L^1\cap L^\infty)((0,T)\times\mathbb{R}^d)\cap L^\infty(0,T;L^2)$ will be proved for $s \in \left(\frac12,1\right)$ and $\beta'(r) > 0$, $\forall\, r \in \mathbb{R}$ and $\beta' \ge 0$ if $D \equiv 0$. In the special case of porous media equations with fractional Laplacian, that is, $D \equiv 0$, $\beta(u) \equiv |u|^{m-1}u$, $m > (d-2s)_+/d$, the existence of a strong solution was proved in [16], [17] (see also [15] for some earlier abstract results, which applies to this case as well).

Like in the present work, the results obtained in [16] are based on the Crandall & Liggett generation theorem of nonlinear contraction semigroups in $L^1(\mathbb{R}^d)$. However, the approach used in [16] cannot be adapted to cope with equation (1.1). In fact, the existence and uniqueness of a mild solution to (1.1) reduces to prove the m-accretivity in $L^1(\mathbb{R}^d)$ of the operator A_0, that is, $(I+\lambda A_0)^{-1}$ must be nonexpansive in $L^1(\mathbb{R}^d)$ for all $\lambda > 0$. If $D \equiv 0$

and $\beta(u) = |u|^{m-1}u$, $m > (d-2s)_+/d$, this follows as shown in [16] (see, e.g., Theorem 7.1) by regularity $u \in L^1(\mathbb{R}^d) \cap L^{m+1}(\mathbb{R}^d)$, $|u|^{m-1}u \in \dot{H}^s(\mathbb{R}^d)$ of solutions to the resolvent equation $u + \lambda(-\Delta)^s\beta(u) = f$ for $f \in L^1(\mathbb{R}^d)$. However, such a property might not be true in our case. For instance, if $s = 1$, this happens if $|b'(r)r + b(r)| \leq \alpha\beta'(r)$, $\forall r \in \mathbb{R}$, $b \geq 0$, $\beta' > 0$ on $\mathbb{R} \setminus \{0\}$, and D sufficiently regular ([9, Theorem 2.2]). To circumvent this situation, following [6] (see Section 2) we have constructed here an m-accretive restriction A of A_0 and derive so via the Crandall & Liggett theorem a semigroup of contractions $S(t)$ such that $u(t) = S(t)u_0$ is a mild solution to (1.1). In general, that is if $A \neq A_0$, this is not the unique mild solution to (1.1). However, as shown in Theorem 3.1 below, under Hypotheses (j) (resp. (j)$'$), (jj), (jjj) (see Section 3), for initial conditions in $L^1 \cap L^\infty$ it is the unique bounded, distributional solution to (1.1). For initial conditions in L^1, the uniqueness of mild solutions to (1.1) as happens for $s = 1$ ([9]) or for $D \equiv 0$, $s \in (0,1)$, as shown in [16], in the case of the present paper remains open. One may suspect, however, that one has in this case as for $s = 1$ (see [12], [14]) the existence of an entropy, resp. kinetic, solution to (1.1) for $u_0 \in L^1 \cap L^\infty$. But this remains to be done. Let us mention that there is a huge literature on the well-posedness of equation (1.1) for the case $s = 1$, in particular when $D \equiv 0$. We refer the reader e.g. to [3]–[9], [11], [12], [14], [15], [23] and the references therein.

In Section 4, we apply our results to the following McKean–Vlasov SDE on $\mathbb{R}^d$

$$\begin{aligned} dX_t &= D(X_t)b(u(t,X_t))dt + \left(\frac{\beta(u(t,X_{t-}))}{u(t,X_{t-})}\right)^{\frac{1}{2s}} dL_t, \\ \mathcal{L}_{X_t}(dx) &:= \mathbb{P} \circ X_t^{-1}(dx) = u(t,x)dx,\ t \in [0,T], \end{aligned} \tag{1.11}$$

where L is a d-dimensional isotropic $2s$-stable process with Lévy measure $dz/|z|^{d+2s}$ (see (4.7) below). We prove that provided $u(0,\cdot)$ is a probability density in L^∞, by our Theorem 2.4 and the superposition principle for non-local Kolmogorov operators (see [25, Theorem 1.5], which is an extension of the local case in [18] and [29]) it follows that (1.11) has a weak solution (see Theorem 4.1 below). Furthermore, we prove that our Theorem 3.1 implies that we have weak uniqueness for (1.11) among all solutions satisfying

$$\left((t,x) \mapsto \frac{d\mathcal{L}_{X_t}}{dx}(x)\right) \in L^\infty((0,T) \times \mathbb{R}^d),$$

(see Theorem 4.2). As a consequence, their laws form a nonlinear Markov process in the sense of McKean [22], thus realizing his vision formulated in that paper (see Remark 4.3). We stress that for the latter two results β is allowed to be degenerate, if $D \equiv 0$. We refer to Section 4 for details.

McKean–Vlasov SDEs for which (L_t) in (1.11) is replaced by a Wiener process (W_t) have been studied very intensively following the two fundamental papers [22], [31]. We refer to [19], [28] and the monograph [13] as well as the references therein. We stress that (1.11) is of *Nemytskii type*, i.e. distribution density dependent, also called *singular* McKean–Vlasov SDEs, so there is no weak continuity in the measure dependence of the coefficients, as usually assumed in the literature. This (also in case of Wiener noise) is a technically more difficult situation. Therefore, the literature on weak existence and uniqueness for (1.11) with Lévy noise is much smaller. In fact, since the diffusion coefficient is allowed to depend (nonlinearly) on the distribution density, except for [25], where weak existence (but not uniqueness) is proved for (1.11), if $D \equiv 0$ and $\beta(r) := |r|^{m-1}r$, $m > (d-2\sigma)_+/d$, we are not aware of any other paper adressing weak well-posedness in our case. If in (1.11) the Lévy process (L_t) is replaced by a Wiener process (W_t), we refer to [3], [4], [5], [6] for weak existence and to [7]–[9] for weak uniqueness, as well as the references therein.

Notation. $L^p(\mathbb{R}^d) = L^p$, $p \in [1,\infty]$ is the standard space of Lebesgue p-integrable functions on $\mathbb{R}^d$. We denote by L^p_{loc} the corresponding local space and by $|\cdot|_p$ the norm of L^p. The inner product in L^2 is denoted by $(\cdot,\cdot)_2$. Denote by $H^\sigma(\mathbb{R}^d) = H^\sigma$, $0 < \sigma < \infty$, the standard Sobolev spaces on $\mathbb{R}^d$ in L^2 and by $H^{-\sigma}$ its dual space. By $C_b(\mathbb{R})$ denote the space of continuous and bounded functions on $\mathbb{R}$ and by $C^1(\mathbb{R})$ the space of differentiable functions on $\mathbb{R}$. For any $T > 0$ and a Banach space $\mathcal{X}$, $C([0,T];\mathcal{X})$ is the space of $\mathcal{X}$-valued continuous functions on $[0,T]$ and by $L^p(0,T;\mathcal{X})$ the space of $\mathcal{X}$-valued L^p-Bochner integrable functions on $(0,T)$. We denote also by $C_0^\infty(\mathcal{O})$, $\mathcal{O} \subset \mathbb{R}^d$, the space of infinitely differentiable functions with compact support in $\mathcal{O}$ and by $\mathcal{D}'(\mathcal{O})$ its dual, that is, the space of Schwartz distributions on $\mathcal{O}$. By $C_0^\infty([0,\infty) \times \mathbb{R}^d)$ we denote the space of infinitely differentiable functions on $[0,\infty) \times \mathbb{R}^d$ with compact in $[0,\infty) \times \mathbb{R}^d$. By $S'(\mathbb{R}^d)$ we denote the space of tempered distributions on $\mathbb{R}^d$.

2 Existence of a mild solution

To begin with, let us construct the operator $A : D(A) \subset L^1 \to L^1$ mentioned in (1.4). To this purpose, we shall first prove the following lemmas.

Lemma 2.1. *Assume that* $\frac{1}{2} < s < 1$. *Let* $\lambda_0 > 0$ *be as defined in* (2.33) *below. Then, under Hypotheses* (i)–(iv) *there is a family of operators* $\{J_\lambda : L^1 \to L^1; \lambda > 0)\}$, *which for all* $\lambda \in (0, \lambda_0)$ *satisfies*

$$(I + \lambda A_0)(J_\lambda(f)) = f,\ \forall\, f \in L^1, \tag{2.1}$$

$$|J_\lambda(f_1) - J_\lambda(f_2)|_1 \le |f_1 - f_2|_1,\ \forall f_1, f_2 \in L^1, \tag{2.2}$$

$$J_{\lambda_2}(f) = J_{\lambda_1}\left(\frac{\lambda_1}{\lambda_2} f + \left(1 - \frac{\lambda_1}{\lambda_2}\right) J_{\lambda_2}(f)\right),\ \forall f \in L^1,\ \lambda_1, \lambda_2 > 0, \tag{2.3}$$

$$\int_{\mathbb{R}^d} J_\lambda(f) dx = \int_{\mathbb{R}^d} f\, dx,\ \forall f \in L^1, \tag{2.4}$$

$$J_\lambda(f) \ge 0, \quad a.e.\ on\ \mathbb{R}^d,\ \ if\ f \ge 0,\ \ a.e.\ on\ \mathbb{R}^d, \tag{2.5}$$

$$|J_\lambda(f)|_\infty \le (1 + ||D| + (\operatorname{div} D)^-|_\infty^{\frac{1}{2}})|f|_\infty,\ \forall\, f \in L^1 \cap L^\infty, \tag{2.6}$$

$$\beta(J_\lambda(f)) \in H^s \cap L^1 \cap L^\infty,\ \forall f \in L^1 \cap L^\infty. \tag{2.7}$$

Remark 2.2. By (2.6), (2.7) and for given $f \in L^1 \cap L^\infty$ changing β as in the proof of Theorem 3.1 below, we may drop the assumption $\beta \in \mathrm{Lip}(\mathbb{R})$ in Hypothesis (i).

Proof of Lemma 2.1. We shall first prove the existence of a solution $y = y_\lambda \in D(A_0)$ to the equation

$$y + \lambda A_0(y) = f \text{ in } S', \tag{2.8}$$

for $f \in L^1$. To this end, for $\varepsilon \in (0, 1]$ we consider the approximating equation

$$y + \lambda(\varepsilon I - \Delta)^s(\beta_\varepsilon(y)) + \lambda \operatorname{div}(D_\varepsilon b_\varepsilon(y)y) = f \text{ in } S', \tag{2.9}$$

where, for $r \in \mathbb{R}$, $\beta_\varepsilon(r) := \beta(r) + \varepsilon r$ and

$$D_\varepsilon := \eta_\varepsilon D,\ \eta_\varepsilon \in C_0^1(\mathbb{R}^d),\ 0 \le \eta_\varepsilon \le 1,\ |\nabla \eta_\varepsilon| \le 1,\ \eta_\varepsilon(x) = 1 \text{ if } |x| < \frac{1}{\varepsilon}.$$

Clearly, we have

$$\begin{gathered} |D_\varepsilon| \in L^2 \cap L^\infty,\ |D_\varepsilon| \le |D|,\ \lim_{\varepsilon \to \infty} D_\varepsilon(x) = D(x),\ \text{a.e. } x \in \mathbb{R}^d, \\ \operatorname{div} D_\varepsilon \in L^2,\ (\operatorname{div} D_\varepsilon)^- \le (\operatorname{div} D)^- + \mathbb{1}_{\left[|x| > \frac{1}{\varepsilon}\right]}|D|. \end{gathered} \tag{2.10}$$

As regards b_ε, it is of the form

$$b_\varepsilon(r) \equiv \frac{(b * \varphi_\varepsilon)(r)}{1+\varepsilon|r|},\ \forall\, r \in \mathbb{R},$$

where $\varphi_\varepsilon(r) = \frac{1}{\varepsilon}\ \varphi\left(\frac{r}{\varepsilon}\right)$ is a standard mollifier. We also set $b^*_\varepsilon(r) := b_\varepsilon(r)r$, $r \in \mathbb{R}$.

Now, let us assume that $f \in L^2$ and consider the approximating equation

$$F_{\varepsilon,\lambda}(y) = f \text{ in } S', \tag{2.11}$$

where $F_{\varepsilon,\lambda} : L^2 \to S'$ is defined by

$$F_{\varepsilon,\lambda}(y) := y + \lambda(\varepsilon I - \Delta)^s \beta_\varepsilon(y) + \lambda \operatorname{div}(D_\varepsilon b^*_\varepsilon(y)),\ y \in L^2,$$

where $(\varepsilon I - \Delta)^s : S \to S$ is defined as usual by Fourier transform and then it extends by duality to an operator $(\varepsilon I - \Delta)^s : S' \to S'$ (which is consistent with (1.2)).

We recall that the Bessel space of order $s \in \mathbb{R}$ is defined as

$$H^s := \{u \in S';\ (1+|\xi|^2)^{\frac{s}{2}}\mathcal{F}(u) \in L^2\}$$

and the Riesz space as

$$\dot{H}^s := \{u \in S';\ \mathcal{F}(u) \in L^1_{\rm loc} \text{ and } |\xi|^s\mathcal{F}(u) \in L^2\}$$

with respective norms

$$|u|^2_{H^s} := \int_{\mathbb{R}^d} (1+|\xi|^2)^s|\mathcal{F}(u)|^2(\xi)d\xi = \int_{\mathbb{R}^d} |(I-\Delta)^{\frac{s}{2}}u|^2 d\xi,$$

and

$$|u|^2_{\dot{H}^s} := \int_{\mathbb{R}^d} |\xi|^{2s}|\mathcal{F}(u)|^2(\xi)d\xi = \int_{\mathbb{R}^d} |(-\Delta)^{\frac{s}{2}}u|^2 d\xi.$$

H^s is a Hilbert space for all $s \in \mathbb{R}$, whereas $\dot{H}^s$ is only a Hilbert space if $s < \frac{d}{2}$ (see, e.g., [1, Proposition 1.34]).

Now, we shall show that (2.11) has a unique solution $y_\varepsilon \in L^2$. To this end, we rewrite (2.11) as

$$(\varepsilon I - \Delta)^{-s} F_{\varepsilon,\lambda}(y) = (\varepsilon I - \Delta)^{-s} f\ (\in H^{2s}),$$

i.e.,

$$(\varepsilon I-\Delta)^{-s}y+\lambda\beta_\varepsilon(y)+\lambda(\varepsilon I-\Delta)^{-s}\mathrm{div}(D_\varepsilon b_\varepsilon^*(y))=(\varepsilon I-\Delta)^{-s}f. \tag{2.12}$$

Clearly, since $D_\varepsilon b_\varepsilon^*(y)\in L^2$, hence $\mathrm{div}(D_\varepsilon b_\varepsilon^*(y))\in H^{-1}$, we have

$$(\varepsilon I-\Delta)^{-s}F_{\varepsilon,\lambda}(y)\in L^2,\ \forall y\in L^2,$$

because $s>\frac{1}{2}$. Now, it is easy to see that (2.12) has a unique solution, $y_\varepsilon\in L^2$, because, as the following chain of inequalities shows, $(\varepsilon I-\Delta)^{-s}F_{\varepsilon,\lambda}: L^2\to L^2$ is strictly monotone. By (2.12) we have, for $y_1,y_2\in L^2$,

$$\begin{aligned}
&((\varepsilon I-\Delta)^{-s}(F_{\varepsilon,\lambda}(y_2)-F_{\varepsilon,\lambda}(y_1)),y_2-y_1)_2\\
&\quad=((\varepsilon I-\Delta)^{-s}(y_2-y_1),y_2-y_1)_2+\lambda(\beta_\varepsilon(y_2)-\beta_\varepsilon(y_1),y_2-y_1)_2\\
&\quad-\lambda\ {}_{H^{-1}}\langle\mathrm{div}(D_\varepsilon(b_\varepsilon^*(y_2)-b_\varepsilon^*(y_1))),(\varepsilon I-\Delta)^{-s}(y_2-y_1)\rangle_{H^1}\\
&\quad\geq|y_2-y_1|_{H^{-s}}^2+\lambda\varepsilon|y_2-y_1|_2^2\\
&\qquad-\lambda c_1|D_\varepsilon(b_\varepsilon^*(y_2)-b_\varepsilon^*(y_1))|_2|(\varepsilon I-\Delta)^{\frac{1}{2}-s}(y_2-y_1)|_2\\
&\quad\geq|y_2-y_1|_{H^{-s}}^2+\lambda\varepsilon|y_2-y_1|_2^2-\lambda c|D|_\infty\mathrm{Lip}(b_\varepsilon^*)|y_2-y_1|_2|y_2-y_1|_{H^{1-2s}},
\end{aligned}\tag{2.13}$$

where $c\in(0,\infty)$ is independent of $\lambda,\varepsilon,y_1,y_2$ and $\mathrm{Lip}(b_\varepsilon^*)$ denotes the Lipschitz norm of b_ε^*. Since $-s<1-2s<0$, by interpolation we have for $\theta:=\frac{2s-1}{s}$ that

$$|y_2-y_1|_{H^{1-2s}}\leq|y_2-y_1|_2^{1-\theta}|y_2-y_1|_{H^{-s}}^\theta$$

(see [1, Proposition 1.52]). So, by Young's inequality we find that the left hand side of (2.13) dominates

$$\lambda(\varepsilon-\lambda c_\varepsilon)|y_2-y_1|_2^2+\frac{1}{2}|y_2-y_1|_{H^{-s}}^2$$

for some $c_\varepsilon\in(0,\infty)$ independent of λ,y_1 and y_2. Hence, for some $\lambda_\varepsilon\in(0,\infty)$, we conclude that $(\varepsilon I-\Delta)^{-s}F_{\varepsilon,\lambda}$ is strictly monotone on L^2 for $\lambda\in(0,\lambda_\varepsilon)$.

It follows from (2.12) that its solution y_ε belongs to H^{2s-1}, hence $b_\varepsilon^*(y_\varepsilon)\in H^{2s-1}$. Since $s>\frac{1}{2}$ and $D\in C^1(\mathbb{R}^d;\mathbb{R}^d)$, by simple bootstrapping (2.12) implies

$$y_\varepsilon\in H^1, \tag{2.14}$$

hence $\beta_\varepsilon(y_\varepsilon)\in H^1$. Furthermore, for $f\in L^2$ and $\lambda\in(0,\lambda_\varepsilon)$, y_ε is the unique solution of (2.9) in L^2.

Assume now that $\lambda \in (0, \lambda_\varepsilon)$ and $f \geq 0$, a.e. on $\mathbb{R}^d$. Then, we have

$$y_\varepsilon \geq 0, \quad \text{a.e. on } \mathbb{R}^d. \tag{2.15}$$

Here is the argument. For $\delta > 0$, consider the function

$$\eta_\delta(r) = \begin{cases} -1 & \text{for} \quad r \leq -\delta, \\ \dfrac{r}{\delta} & \qquad\quad r \in (-\delta, 0), \\ 0 & \text{for} \quad r \geq 0. \end{cases} \tag{2.16}$$

If we multiply equation (2.9), where $y = y_\varepsilon$, by $\eta_\delta(\beta_\varepsilon(y_\varepsilon))$ $(\in H^1)$ and integrate over $\mathbb{R}^d$, we get

$$\begin{aligned} &\int_{\mathbb{R}^d} y_\varepsilon \eta_\delta(\beta_\varepsilon(y_\varepsilon))dx + \lambda \int_{\mathbb{R}^d} (\varepsilon I - \Delta)^s (\beta_\varepsilon(y_\varepsilon))\eta_\delta(\beta_\varepsilon(y_\varepsilon))dx \\ &\quad = \int_{\mathbb{R}^d} f\eta_\delta(\beta_\varepsilon(y_\varepsilon))dx + \lambda \int_{\mathbb{R}^d} D_\varepsilon b^*_\varepsilon(y_\varepsilon)\eta'_\delta(\beta_\varepsilon(y_\varepsilon)) \cdot \nabla\beta_\varepsilon(y_\varepsilon)dx. \end{aligned} \tag{2.17}$$

By Lemma 5.2 in [16] we have (Stroock-Varopoulos inequality)

$$\int_{\mathbb{R}^d} (\varepsilon I - \Delta)^s u\Psi(u)dx \geq \int_{\mathbb{R}^d} |(\varepsilon I - \Delta)^{\frac{s}{2}} \widetilde{\Psi}(u)|^2 dx, \ u \in H^1(\mathbb{R}^d), \tag{2.18}$$

for any pair of functions $\Psi, \widetilde{\Psi} \in \mathrm{Lip}(\mathbb{R})$ such that $\Psi'(r) \equiv (\widetilde{\Psi}'(r))^2, \ r \in \mathbb{R}$. This yields

$$\int_{\mathbb{R}^d} (\varepsilon I - \Delta)^s \beta_\varepsilon(y_\varepsilon)\eta_\delta(\beta_\varepsilon(y_\varepsilon))dx \geq \int_{\mathbb{R}^d} |(\varepsilon I - \Delta)^{\frac{s}{2}} \widetilde{\Psi}(\beta_\varepsilon(y_\varepsilon))|^2 dx \geq 0, \tag{2.19}$$

where $\widetilde{\Psi}(r) = \int_0^r \sqrt{\eta'_\delta(s)}\, ds$. Taking into account that $y_\varepsilon \in H^1$ and that $|\beta_\varepsilon(y_\varepsilon)| \geq \varepsilon|y_\varepsilon|$, we have

$$\begin{aligned} &\left| \int_{\mathbb{R}^d} D_\varepsilon b_\varepsilon(y_\varepsilon) y_\varepsilon \eta'_\delta(\beta_\varepsilon(y_\varepsilon))\nabla\beta_\varepsilon(y_\varepsilon)dx \right| \\ &\qquad \leq \frac{1}{\delta}\, |b|_\infty \int_{\widetilde{E}^\delta_\varepsilon} |y_\varepsilon(x)|\, |\nabla\beta_\varepsilon(y_\varepsilon)|\, |D_\varepsilon(x)|dx \\ &\qquad \leq \frac{1}{\varepsilon}\, |b|_\infty \|D_\varepsilon\|_{L^2} \left(\int_{\widetilde{E}^\delta_\varepsilon} |\nabla\beta_\varepsilon(y_\varepsilon)|^2 dx \right)^{\frac{1}{2}} \to 0 \text{ as } \delta \to 0. \end{aligned} \tag{2.20}$$

Here $\widetilde{E}^\delta_\varepsilon = \{-\delta < \beta_\varepsilon(y_\varepsilon) \le 0\}$ and we used that $\nabla\beta_\varepsilon(y_\varepsilon) = 0$, a.e. on $\{\beta_\varepsilon(y_\varepsilon) = 0\}$.

Taking into account that $\operatorname{sign}\beta_\varepsilon(r) \equiv \operatorname{sign} r$, by (2.17)–(2.20) we get, for $\delta \to 0$, that $y^-_\varepsilon = 0$, a.e. on $\mathbb{R}^d$ and so (2.15) holds.

If $y_\varepsilon = y_\varepsilon(\lambda, f)$ is the solution to (2.9), we have for $f_1, f_2 \in L^1 \cap L^2$

$$\begin{aligned} &y_\varepsilon(\lambda, f_1) - y_\varepsilon(\lambda, f_2) + \lambda(\varepsilon I - \Delta)^s(\beta_\varepsilon(y_\varepsilon(\lambda, f_1)) - \beta_\varepsilon(y_\varepsilon(\lambda, f_2))) \\ &\qquad +\lambda \operatorname{div} D_\varepsilon(b^*_\varepsilon(y_\varepsilon(\lambda, f_1)) - b^*_\varepsilon(y_\varepsilon(\lambda, f_2))) = f_1 - f_2. \end{aligned} \tag{2.21}$$

Now, for $\delta > 0$ consider the function

$$\mathcal{X}_\delta(r) = \begin{cases} 1 & \text{for} \quad r \ge \delta, \\ \dfrac{r}{\delta} & \text{for } |r| < \delta, \\ -1 & \text{for} \quad r < -\delta. \end{cases}$$

If we multiply (2.21) by $\mathcal{X}_\delta(\beta_\varepsilon(y_\varepsilon(\lambda, f_1)) - \beta_\varepsilon(y_\varepsilon(\lambda, f_2)))$ $(\in H^1)$ and integrate over $\mathbb{R}^d$, we get

$$\begin{aligned} &\int_{\mathbb{R}^d} (y_\varepsilon(\lambda, f_1) - y_\varepsilon(\lambda, f_2))\mathcal{X}_\delta(\beta_\varepsilon(y_\varepsilon(\lambda, f_1)) - \beta_\varepsilon(y_\varepsilon(\lambda, f_2)))dx \\ &\quad \le \lambda\,\frac{1}{\delta}\int_{E^\delta_\varepsilon} |b^*_\varepsilon(y_\varepsilon(\lambda, f_1)) - b^*_\varepsilon(y_\varepsilon(\lambda, f_2))|\,|D_\varepsilon||\nabla(\beta_\varepsilon(y_\varepsilon(\lambda, f_1)) - \beta_\varepsilon(y_\varepsilon(\lambda, f_2)))|dx \\ &\quad +|f_1 - f_2|_1, \end{aligned}$$

because, by (2.18), we have

$$\int_{\mathbb{R}^d} (\varepsilon I - \Delta)^s(\beta_\varepsilon(y_\varepsilon, f_1) - \beta_\varepsilon(y_\varepsilon, f_2))\mathcal{X}_\delta(\beta_\varepsilon(y_\varepsilon, f_1) - \beta_\varepsilon(y_\varepsilon, f_2))dx \ge 0.$$

Set $E^\delta_\varepsilon = \{|\beta_\varepsilon(y_\varepsilon(\lambda, f_1)) - \beta_\varepsilon(y_\varepsilon(\lambda, f_2))| \le \delta\}$.

Since $|\beta_\varepsilon(r_1) - \beta_\varepsilon(r_2)| \ge \varepsilon|r_1 - r_2|$, $D_\varepsilon \in L^2(\mathbb{R}^d; \mathbb{R}^d)$ and $b^*_\varepsilon \in \operatorname{Lip}(\mathbb{R})$, we get that

$$\lim_{\delta\to 0} \frac{1}{\delta}\int_{E^\delta_\varepsilon} |b^*_\varepsilon(y_\varepsilon(\lambda, f_1)) - b^*_\varepsilon(y_\varepsilon(\lambda, f_2))||D_\varepsilon||\nabla(\beta_\varepsilon(y_\varepsilon(\lambda, f_1)) - \beta_\varepsilon(y_\varepsilon(\lambda, f_2)))|dx = 0,$$

because $y_\varepsilon(\lambda, f_i) \in H^1$, $i = 1, 2$, and $\nabla(\beta_\varepsilon(y_\varepsilon(\lambda, f_1)) - \beta_\varepsilon(y_\varepsilon(\lambda, f_2))) = 0$, a.e. on $\{\beta_\varepsilon(y_\varepsilon(\lambda, f_1)) - \beta_\varepsilon(y_\varepsilon(\lambda, f_2)) = 0\}$. This yields

$$|y_\varepsilon(\lambda, f_1) - y_\varepsilon(\lambda, f_2)|_1 \le |f_1 - f_2|_1, \ \forall \lambda \in (0, \lambda_\varepsilon). \tag{2.22}$$

In particular, it follows that

$$|y_\varepsilon(\lambda,f)|_1 \le |f|_1,\ \forall\, f\in L^1\cap L^2. \tag{2.23}$$

Now let us remove the restriction λ to be in $(0,\lambda_\varepsilon)$. To this end define the operator $A_\varepsilon : D_0(A_\varepsilon)\to L^1$ by

$$\begin{aligned} A_\varepsilon(y) &:= (\varepsilon I-\Delta)^s(\beta_\varepsilon(y)) + \operatorname{div}(D_\varepsilon b^*_\varepsilon(y)),\\ D_0(A_\varepsilon) &:= \{y\in L^1;\ (\varepsilon I-\Delta)^s\beta_\varepsilon(y)+\operatorname{div}(D_\varepsilon b^*_\varepsilon(y))\in L^1\}, \end{aligned}$$

and for $\lambda\in(0,\lambda_\varepsilon)$

$$J^\varepsilon_\lambda(f) := y_\varepsilon(\lambda,f),\ \ f\in L^1\cap L^2.$$

Then $J^\varepsilon_\lambda(L^1\cap L^2)\subset D_0(A_\varepsilon)\cap H^1$ and by (2.22) it extends by continuity to an operator $J^\varepsilon_\lambda : L^1\to L^1$. We note that the operator $(A_\varepsilon, D_0(A_\varepsilon))$ is closed as an operator on L^1. Hence (2.22) implies that

$$J^\varepsilon_\lambda(L^1)\subset D_0(A_\varepsilon) \tag{2.24}$$

and that $J^\varepsilon_\lambda(f)$ solves (2.9) for all $f\in L^1$.

Now define

$$D(A_\varepsilon) := J^\varepsilon_\lambda(L^1) \tag{2.25}$$

and restrict A_ε to $D(A_\varepsilon)$. It is easy to see that $D(A_\varepsilon)$ is independent of $\lambda\in(0,\lambda_\varepsilon)$.

Now let $0<\lambda_1<\lambda_\varepsilon$. Then, for $\lambda\ge\lambda_\varepsilon$, the equation

$$y+\lambda A_\varepsilon(y) = f\ (\in L^1),\ \ y\in D(A_\varepsilon), \tag{2.26}$$

can be rewritten as

$$y+\lambda_1 A_\varepsilon(y) = \left(1-\frac{\lambda_1}{\lambda}\right)y + \frac{\lambda_1}{\lambda}\,f.$$

Equivalently,

$$y = J^\varepsilon_{\lambda_1}\left(\left(1-\frac{\lambda_1}{\lambda}\right)y+\frac{\lambda_1}{\lambda}\,f\right). \tag{2.27}$$

Taking into account that, by (2.22), $|J^\varepsilon_{\lambda_1}(f_1)-J^\varepsilon_{\lambda_1}(f_2)|_1 \le |f_1-f_2|_1$, it follows that (2.27) has a unique solution $y_\varepsilon\in D(A_\varepsilon)$. Let $J^\varepsilon_\lambda(f) := y_\varepsilon$, $\lambda>0,\ f\in L^1$, denote this solution to (2.26). By (2.27) we see that (2.22), (2.23) extend to all $\lambda > 0$, $f \in L^1$.

Claim. Let $f \in L^1 \cap L^2$. Then

$$J^\varepsilon_\lambda(f) \in H^1 \text{ for all } \lambda > 0. \tag{2.28}$$

Proof. Fix $\lambda_1 \in [\lambda_\varepsilon/2, \lambda_\varepsilon)$ and set $\lambda := 2\lambda_1$. Define $T : L^1 \cap L^2 \to L^1 \cap H^1$ by

$$T(y) := J^\varepsilon_{\lambda_1}\left(\frac{1}{2}\, y + \frac{1}{2}\, f\right),\ y \in L^1 \cap L^2.$$

Then, as just proved, for any $f_0 \in L^1 \cap L^2$ fixed

$$\lim_{n\to\infty} T^n(f_0) = J^\varepsilon_\lambda(f) \text{ in } L^1. \tag{2.29}$$

It suffices to prove

$$J^\varepsilon_\lambda(f) \in L^2, \tag{2.30}$$

because then $J^\varepsilon_\lambda(f) = J^\varepsilon_{\lambda_1}(g)$ with $g := \frac{1}{2}\, J^\varepsilon_\lambda(f) + \frac{1}{2}\, f \in L^1 \cap L^2$, and so the claim follows by (2.14) which holds with $y_\varepsilon := J^\varepsilon_{\lambda_1}(g)$, because $\lambda_1 \in (0, \lambda_\varepsilon)$.

To prove (2.30) we note that we have, for $n \in \mathbb{N}$,

$$(I + \lambda_1 A_\varepsilon) T^n(f_0) = \frac{1}{2}\, T^{n-1}(f_0) + \frac{1}{2}\, f$$

with $T^n(f_0) \in H^1$. Hence, applying ${}_{H^{-1}}\langle \cdot, T^n(f_0)\rangle_{H^1}$ to this equation, we find

$$\begin{aligned}&|T^{(n)} f_0|_2^2 + \lambda_1\, {}_{H^{-1}}\langle (\varepsilon I - \Delta)^s \beta_\varepsilon(T^n(f_0)), T^n(f_0)\rangle_{H^1}\\ &= \lambda_1 \int_{\mathbb{R}^d} (D_\varepsilon b^*_\varepsilon(T^n(f_0))) \cdot \nabla(T^n(f_0)) d\xi + \left(\frac{1}{2} T^{n-1}(f_0) + \frac{1}{2} f, T^n(f_0)\right)_2.\end{aligned} \tag{2.31}$$

Setting

$$\psi_\varepsilon(r) := \int_0^r b^*_\varepsilon(\tau) d\tau,\ r \in \mathbb{R}, \tag{2.32}$$

by Hypotheses (iii), (iv) we have

$$0 \le \psi_\varepsilon(r) \le |b^*_\varepsilon|_\infty r,\ r \in \mathbb{R},$$

and hence the left hand side of (2.31) is equal to

$$-\lambda_1(\text{div } D_\varepsilon, \psi_\varepsilon(T^n(f_0)))_2 + \left(\frac{1}{2}\, (T^{n-1}(f_0) + f), T^n(f_0)\right)_2,$$

where we recall that $\operatorname{div} D_\varepsilon \in L^2$ by (2.10). By (2.19) we thus obtain

$$
\begin{aligned}
|T^{(n)}(f_0)|_2^2 \le\ & \lambda_1 |b_\varepsilon^*|_\infty |(\operatorname{div} D_\varepsilon)^-|_2 |T^n(f_0)|_2 + \frac{1}{2} |T^n(f_0)|_2^2 \\
& + \frac{1}{4}(|T^{n-1}(f_0)|_2^2 + |f|_2^2),
\end{aligned}
$$

therefore,

$$
|T^{(n)}(f_0)|_2^2 \le C_\varepsilon + \frac{1}{2}\, |T^{n-1}(f_0)|_2^2,
$$

where

$$
C_\varepsilon := \frac{\lambda_1}{1-\lambda_1}\, |b_\varepsilon^*|_\infty^2 |(\operatorname{div} D_\varepsilon)^-|_2^2 + \frac{1}{2}\, |f|_2^2.
$$

Iterating, we find

$$
|T^{(n)}(f_0)|_2^2 \le \sum_{k=1}^{n-2} 2^{-k} C_\varepsilon + \frac{1}{2^{n-1}}\, |T(f_0)|_2^2,\ n \in \mathbb{N}.
$$

Hence, by Fatou's lemma and (2.29),

$$
|J_\lambda^\varepsilon(f)|_2^2 \le \liminf_{n\to\infty} |T^n(f_0)|_2^2 \le C_\varepsilon < \infty,
$$

and (2.28) holds for $\lambda = 2\lambda_1$. Proceeding this way, we get (2.28) for all $\lambda > 0$. $\square$

Set

$$
\lambda_0 := \left(\big|(\text{div } D)^- + |D|\big|_\infty\right) + \left(\big|(\text{div } D)^- + |D|\big|_\infty^{\frac{1}{2}}\, |b|_\infty\right)^{-1}. \tag{2.33}
$$

Then, for $f \in L^1 \cap L^\infty$ and $y_\varepsilon := J_\lambda^\varepsilon(f)$, $\lambda > 0$, we have

$$
|y_\varepsilon|_\infty \le (1 + \big||D| + (\operatorname{div} D)^-\big|_\infty^{\frac{1}{2}})|f|_\infty,\ \forall \lambda \in (0, \lambda_0). \tag{2.34}
$$

Indeed, if we set $M_\varepsilon = |(\operatorname{div} D_\varepsilon)^-|_\infty^{\frac{1}{2}} |f|_\infty$, we get by (2.9) that

$$
\begin{aligned}
(y_\varepsilon - |f|_\infty - M_\varepsilon) + \lambda(\varepsilon I - \Delta)^s(\beta_\varepsilon(y_\varepsilon) - \beta_\varepsilon(|f|_\infty + M_\varepsilon)) + \varepsilon^s(\beta_\varepsilon(|f|_\infty + M_\varepsilon)) \\
+ \lambda \operatorname{div}(D_\varepsilon(b_\varepsilon^*(y_\varepsilon) - b_\varepsilon^*(|f|_\infty + M_\varepsilon))) \le 0.
\end{aligned}
$$

Here we used that

$$
1 \in \{u \in S';\ (\varepsilon + |\varepsilon|^2)^s \mathcal{F}(u) \in S'\}
$$

and that $(\varepsilon I-\Delta)^s 1=\varepsilon^s$, since $\mathcal{F}(1)=\delta_0$ (= Dirac measure in $0\in\mathbb{R}^d$). Then, taking the scalar product in L^2 with $\mathcal{X}_\delta((\beta_\varepsilon(y_\varepsilon)-\beta_\varepsilon(|f|_\infty+M_\varepsilon)^+)$, letting $\delta\to 0$ and using (2.18), we get by (2.10)

$$y_\varepsilon\le(1+||D|+(\operatorname{div}D)^-|_\infty^{\frac12})|f|_\infty,\ \text{a.e. in }\mathbb{R}^d,$$

and, similarly, for $-y_\varepsilon$ which yields (2.34) for $\lambda\in(0,\lambda_0)$.

In particular, it follows that

$$|J^\varepsilon_\lambda(f)|_1+|J^\varepsilon_\lambda(f)|_\infty\le c_1,\ \forall\varepsilon,\lambda>0,\tag{2.35}$$

where $c_1=c_1(|f|_1,|f|_\infty)$ is independent of ε and λ.

Now, fix $\lambda\in(0,\lambda_0)$ and $f\in L^1\cap L^\infty$. For $\varepsilon\in(0,1]$ set

$$y_\varepsilon:=J^\varepsilon_\lambda(f).$$

Then, since $\beta_\varepsilon(y_\varepsilon)\in H^1$, by (2.9) we get

$$\begin{aligned}(y_\varepsilon,\beta_\varepsilon(y_\varepsilon))_2&+\lambda\ {}_{H^{-1}}\langle(\varepsilon I-\Delta)^s\beta_\varepsilon(y_\varepsilon),\beta_\varepsilon(y_\varepsilon)\rangle_{H^1}\\ &=-\lambda(\operatorname{div}(D_\varepsilon b^*_\varepsilon(y_\varepsilon)),\beta_\varepsilon(y_\varepsilon))_2+(f,\beta_\varepsilon(y_\varepsilon))_2\\ &=\lambda\int_{\mathbb{R}^d}(D_\varepsilon b^*_\varepsilon(y_\varepsilon))\cdot\nabla\beta_\varepsilon(y_\varepsilon)dx+(f,\beta_\varepsilon(y_\varepsilon))_2.\end{aligned}\tag{2.36}$$

Setting

$$\widetilde{\psi}_\varepsilon(r):=\int_0^r b^*_\varepsilon(\tau)\beta'_\varepsilon(\tau)d\tau,\ r\in\mathbb{R},\tag{2.37}$$

by Hypotheses (iii), (iv) we have

$$0\le\widetilde{\psi}_\varepsilon(r)\le\frac12\ |b|_\infty(|\beta'|_\infty+1)r^2,\ \forall r\in\mathbb{R},$$

and hence, since $y_\varepsilon\in H^1$, the left hand side of (2.36) is equal to

$$-\lambda\int_{\mathbb{R}^d}\operatorname{div}D_\varepsilon\widetilde{\psi}_\varepsilon(y_\varepsilon)dx+(f,\beta_\varepsilon(y_\varepsilon))_2,$$

which, because $(y_\varepsilon,\beta_\varepsilon(y_\varepsilon))_2\ge 0$ and $H^1\subset H^s$, by (2.10) and Hypothesis (iv) implies that

$$\lambda|(\varepsilon I-\Delta)^{\frac s2}\beta_\varepsilon(y_\varepsilon)|_2^2\le\frac12\,(\lambda|b|_\infty(|\beta'|_\infty+1)|(\operatorname{div}D)^-+|D||_\infty)|y_\varepsilon|_2^2+\frac12|\beta_\varepsilon(y_\varepsilon)|_2^2+\frac12\,|f|_2^2.$$

Since $|\beta_\varepsilon(r)| \le (\mathrm{Lip}(\beta)+1)|r|,\ r \in \mathbb{R}$, by (2.23), (2.34) we obtain

$$\sup_{\varepsilon\in(0,1]} |(\varepsilon I-\Delta)^{\frac{s}{2}}\beta_\varepsilon(y_\varepsilon)|_2^2 \le C < \infty, \tag{2.38}$$

where

$$C := \lambda(|b|_\infty+1)(|(\mathrm{div}\,D)^- + |D|_\infty| + 2)^2(\mathrm{Lip}(\beta)+1)^2|f|_\infty|f|_1.$$

Since obviously for all $u \in H^s$ $(\subset \dot{H}^s)$, $\varepsilon \in (0,1]$,

$$|(-\Delta)^{\frac{s}{2}}u|_2^2 \le |(\varepsilon I-\Delta)^{\frac{s}{2}}u|_2^2 \le |(-\Delta)^{\frac{s}{2}}u|_2^2 + \varepsilon^s|u|_2^2 \le 2|(\varepsilon I-\Delta)^{\frac{s}{2}}u|_2^2, \tag{2.39}$$

and since $\beta_\varepsilon(y_\varepsilon) \in H^1 \subset H^s$ with $|\beta_\varepsilon(r)| \le (1+\mathrm{Lip}(\beta))|r|$, we conclude from (2.35) and (2.38) that (along a subsequence) as $\varepsilon \to 0$

$$\begin{array}{rcl} \beta_\varepsilon(y_\varepsilon) & \to & z \text{ weakly in } \dot{H}^s, \\ (\varepsilon I-\Delta)^s\beta_\varepsilon(y_\varepsilon) & \to & (-\Delta)^s z \text{ in } S', \\ y_\varepsilon & \to & y \text{ weakly in } L^2 \text{ and weakly}^* \text{ in } L^\infty, \end{array}$$

where the second statement follows, because

$$(\varepsilon I-\Delta)^s\varphi \to (-\Delta)^s\varphi \text{ in } L^2 \text{ for all } \varphi \in S.$$

By [1, Theorem 1.69], it follows that as $\varepsilon \to 0$

$$\beta_\varepsilon(y_\varepsilon) \to z \text{ in } L^2_{\mathrm{loc}}(\mathbb{R}^d),$$

so (selecting another subsequence, if necesary)

$$\beta(y_\varepsilon) \to z, \text{ a.e.}$$

Since β^{-1} (the inverse function of β) is continuous, it follows that as $\varepsilon \to 0$

$$y_\varepsilon \to \beta^{-1}(z) = y, \text{ a.e.},$$

so

$$z = \beta(y) \tag{2.40}$$

and

$$b^*_\varepsilon(y_\varepsilon) \to b^*(y) \text{ weakly in } L^2.$$

Recalling that y_ε solves (2.9), we can let $\varepsilon \to 0$ in (2.9) to find that

$$y + \lambda(-\Delta)^s \beta(y) + \lambda \operatorname{div}(Db^*(y)) = f \text{ in } S'. \tag{2.41}$$

Since $\beta \in \mathrm{Lip}(\mathbb{R})$, the operator $(A_0, D(A_0))$ defined in (1.5) is obviously closed as an operator on L^1. Again defining for y as in (2.41)

$$J_\lambda(f) := y \in D(A_0),$$

it follows by (2.22) and Fatou's lemma that for $f_1, f_2 \in L^1 \cap L^\infty$

$$|J_\lambda(f_1) - J_\lambda(f_2)|_1 \le |f_1 - f_2|_1. \tag{2.42}$$

Hence J_λ extends continuously to all of L^1, still satisfying (2.42) for all $f_1, f_2 \in L^1$. Then it follows by the closedness of $(A_0, D(A_0))$ on L^1 that $J_\lambda(f) \in D(A_0)$ and that it solves (2.41) for all $f \in L^1$.

Hence, Lemma 2.1 is proved except for (2.3) and (2.4). However, (2.3) is obvious, since by (2.1) it is equivalent to

$$(I + \lambda_1 A_0) J_{\lambda_2}(f) = \frac{\lambda_1}{\lambda_2} f + \left(1 - \frac{\lambda_1}{\lambda_2}\right) J_{\lambda_2} f,$$

which in turn is equivalent to

$$(I + \lambda_2 A_0) J_{\lambda_2}(f) = f.$$

Now let us prove (2.4). We may assume that $f \in L^1 \cap L^\infty$ and set $y := J_\lambda(f)$. Let $\mathcal{X}_n \in C_0^\infty(\mathbb{R}^d)$, $\mathcal{X}_n \uparrow 1$, as $n \to \infty$, with $\sup\limits_n |\nabla \mathcal{X}_n|_\infty < \infty$. Define

$$\varphi_n := (I + (-\Delta)^s)^{-1} \mathcal{X}_n = g_1^s * \mathcal{X}_n, \ n \in \mathbb{N},$$

where g_1^s is as in the Appendix. Then, clearly,

$$\begin{aligned} &\varphi_n \uparrow 1, \text{ as } n \to \infty, \\ &\sup_n (|\varphi_n|_\infty + |\nabla \varphi_n|_\infty) < \infty, \\ &\varphi_n \in L^1 \cap H^{2s}, \ n \in \mathbb{N}, \end{aligned} \tag{2.43}$$

where the last statement follows from (2.39). Furthermore,

$$(-\Delta)^s \varphi_n = \mathcal{X}_n - (I + (-\Delta)^s)^{-1} \mathcal{X}_n \in L^1 \cap L^\infty,$$

and, as $n \to \infty$,

$$(-\Delta)^s \varphi_n \to 0 \ dx - \text{a.e.},$$

hence, because $\beta(y) \in L^1 \cap L^\infty$,

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} (-\Delta)^s \varphi_n \, \beta(y) dx = 0. \tag{2.44}$$

Consequently, since $\beta(y) \in H^s$, $y \in D(A)$ with $A_0 y \in L^1$,

$$\begin{aligned}
&\int_{\mathbb{R}^d} A_0 y \, dx \\
&= \lim_{n\to\infty} \int_{\mathbb{R}^d} \varphi_n A_0 y \, dx \\
&= -\int_{\mathbb{R}^d} \beta(y) dx + \lim_{n\to\infty} {}_{H^{2s}}\langle \varphi_n, (I + (-\Delta)^s)\beta(y) + \mathrm{div}(Db^*(y))\rangle_{H^{-2s}} \\
&= -\int_{\mathbb{R}^d} \beta(y) dx + \lim_{n\to\infty} {}_{H^s}\langle \varphi_n, (I+(-\Delta)^s)\beta(y)\rangle_{H^{-s}} + \lim_{n\to\infty} \int_{\mathbb{R}^d} \nabla\varphi_n \cdot Db^*(y) dx,
\end{aligned}$$

which by (2.43) and (2.44) is equal to zero. Hence, integrating the equation

$$y + \lambda A_0 y = f$$

over $\mathbb{R}^d$, (2.4) follows, which concludes the proof of Lemma 2.1. $\square$

Now define

$$\begin{aligned} D(A) &:= J_\lambda(L^1) \ (\subset D(A_0)), \\ A(y) &:= A_0(y), \ y \in D(A). \end{aligned} \tag{2.45}$$

Again it is easy to see that $J_\lambda(L^1)$ is independent of $\lambda \in (0, \lambda_0)$ and that

$$J_\lambda = (I + \lambda A)^{-1}, \ \lambda \in (0, \lambda_0).$$

Therefore, we have

Lemma 2.3. *Under Hypotheses* (i)–(iv)*, the operator A defined by* (2.45) *is m-accretive in L^1 and $(I+\lambda A)^{-1} = J_\lambda$, $\lambda \in (0, \lambda_0)$. Moreover, if $\beta \in C^\infty(\mathbb{R})$, then $\overline{D(A)} = L^1$.*

Here, $\overline{D(A)}$ is the closure of $D(A)$ in L^1.

We note that, by (1.3), if $\beta \in C^\infty(\mathbb{R})$, it follows that

$$A_0(\varphi) = (-\Delta)^s\beta(\varphi) + \operatorname{div}(Db(\varphi)\varphi) \in L^1,\ \forall \varphi \in C_0^\infty(\mathbb{R}^d),$$

and so $\overline{D(A)} = L^1$, as claimed.

Then, by the Crandall & Liggett theorem (see, e.g., [2], p. 131), we have that the Cauchy problem (1.4), that is,

$$\begin{aligned} &\frac{du}{dt} + A(u) = 0, \quad t \geq 0,\\ &u(0) = u_0, \end{aligned}$$

has, for each $u_0 \in \overline{D(A)}$, a unique mild solution $u = u(t,u_0) \in C([0,\infty);L^1)$ and $S(t)u_0 = u(t,u_0)$ is a C_0-semigroup of contractions on L^1, that is,

$$\begin{gathered} |S(t)u_0 - S(t)\bar{u}_0|_1 \leq |u_0 - \bar{u}_0|_1,\ \forall u_0, \bar{u}_0 \in \overline{D(A)},\\ S(t+\tau)u_0 = S(t, S(\tau)u_0),\ \forall t,\tau > 0;\ u_0 \in \overline{D(A)},\\ \lim_{t\to 0} S(t)u_0 = u_0 \ \text{ in } L^1(\mathbb{R}^d). \end{gathered}$$

Moreover, by (2.5) and the exponential formula

$$S(t)u_0 = \lim_{n\to\infty}\left(I + \frac{t}{n}\,A\right)^{-n} u_0,\ \forall\, t \geq 0,$$

it follows that $S(t)u_0 \in L^\infty((0,T)\times\mathbb{R}^d)$, $T>0$, if $u_0 \in L^1 \cap L^\infty$.

Let us show now that $u = S(t)u_0$ is a Schwartz distributional solution, that is, (1.10) holds.

By (1.6)–(1.9), we have

$$\begin{aligned} &\int_0^\infty dt\left(\int_{\mathbb{R}^d}\varphi(t,x)(u_h(t,x) - u_h(t-h,x))\right)dx\\ &\qquad + \int_{\mathbb{R}^d}(\varphi(t,x)(-\Delta)^s\beta(u_h(t,x)) - \nabla_x\varphi(t,x)\cdot D(x)b^*(u_h((x)))dx) = 0,\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall \varphi \in C_0^\infty([0,\infty)\times\mathbb{R}^d). \end{aligned}$$

This yields

$$\begin{aligned} &\frac{1}{h}\int_0^\infty dt\left(\int_{\mathbb{R}^d}u_h(t,x)(\varphi(t+h,x) - \varphi(t,x))\right)dx\\ &\quad + \int_{\mathbb{R}^d}(\beta(u_h(t,x))(-\Delta)^s\varphi(t,x) - \nabla_x\varphi(t,x)\cdot D(x)b^*(u_h(t,x))dx)\\ &\quad + \frac{1}{h}\int_0^h dt\int_{\mathbb{R}^d}u_0(x)\varphi(t,x)dx = 0,\ \forall\varphi \in C_0^\infty([0,\infty)\times\mathbb{R}^d). \end{aligned}$$

Taking into account that, by (1.6) and (i)–(iii), $\beta(u_h) \to \beta(u)$, $b^*(u_h) \to b^*(u)$ in $C([0,T]; L^1)$ as $h \to 0$ for each $t > 0$, we get that (1.10) holds.

This together with Remark 2.2 implies the following existence result for equation (1.1).

Theorem 2.4. *Assume that Hypotheses* (i)–(iv) *hold. Then, there is a C_0-semigroup of contractions $S(t) : L^1 \to L^1$, $t \geq 0$, such that for each $u_0 \in \overline{D(A)}$, which is L^1 if $\beta \in C^\infty(\mathbb{R})$, $u(t, u_0) = S(t)u_0$ is a mild solution to* (1.1)*. Moreover, if $u_0 \geq 0$, a.e. in $\mathbb{R}^d$,*

$$u(t, u_0) \geq 0, \quad a.e. \ in \ \mathbb{R}^d, \ \forall\, t \geq 0, \tag{2.46}$$

and

$$\int_{\mathbb{R}^d} u(t, u_0)(x)dx = \int_{\mathbb{R}^d} u_0(x)dx, \quad \forall\, t \geq 0. \tag{2.47}$$

Moreover, u is a distributional solution to (1.1) *on $[0, \infty) \times \mathbb{R}^d$. Finally, if $u_0 \in L^1 \cap L^\infty$, then all above assertions remain true, if we drop the assumption $\beta \in \mathrm{Lip}(\mathbb{R})$ from Hypothesis* (i)*, and additionally we have that $u \in L^\infty((0,T) \times \mathbb{R}^d)$, $T > 0$.*

Remark 2.5. It should be emphasized that, in general, the mild solution u given by Theorem 2.4 is not unique because the operator A constructed in Lemma 2.3 is dependent of the approximating operator $A_\varepsilon y \equiv (\varepsilon I + (-\Delta)^s)\beta_\varepsilon(y) + \mathrm{div}(D_\varepsilon b_\varepsilon(y)y)$ and so $u = S(t)u_0$ may be viewed as a *viscosity-mild* solution to (1.1). However, as seen in the next section, this mild solution – which is also a distributional solution to (1.1) – is, under appropriate assumptions on β, D and b, unique in the class of solutions $u \in L^\infty((0,T) \times \mathbb{R}^d)$, $T > 0$.

3 The uniqueness of distributional solutions

In this section, we shall prove the uniqueness of distributional solutions to (1.1), where $s \in \left(\frac{1}{2}, 1\right)$, under the following Hypotheses:

(j) $\beta \in C^1(\mathbb{R}),\ \beta'(r) > 0,\ \forall\, r \in \mathbb{R},\ \beta(0) = 0.$

(jj) $D \in L^\infty(\mathbb{R}^d; \mathbb{R}^d)$.

(jjj) $b \in C^1(\mathbb{R})$.

Theorem 3.1. *Let $d \geq 1$, $s \in \left[\frac{1}{2}, 1\right]$, $T > 0$, and let $y_1, y_2 \in L^\infty((0,T)\times\mathbb{R}^d)$ be two distributional solutions to* (1.1) *on* $(0,T)\times\mathbb{R}^d$ *(in the sense of* (1.10)*) such that $y_1 - y_2 \in L^1((0,T)\times\mathbb{R}^d) \cap L^\infty(0,T;L^2)$ and*

$$\lim_{t\to 0} \operatorname*{ess\,sup}_{s\in(0,t)} |(y_1(s) - y_2(s), \varphi)_2| = 0, \ \forall \varphi \in C_0^\infty(\mathbb{R}^d). \tag{3.1}$$

Then $y_1 \equiv y_2$. If $D \equiv 0$, then Hypothesis (j) *can be relaxed to*

(j)$'$ $\beta \in C^1(\mathbb{R}),\ \beta'(r) \geq 0,\ \forall\, r \in \mathbb{R},\ \beta(0) = 0.$

Proof. (The idea of proof is borrowed from Theorem 3.2 in [9], but has to be adapted substantially.) Replacing, if necessary, the functions β and b by

$$\beta_N(r) = \begin{cases} \beta(r) & \text{if } |r| \leq N, \\ \beta'(N)(r-N) + \beta(N) & \text{if } r > N, \\ \beta'(-N)(r+N) + \beta(-N) & \text{if } r < -N, \end{cases}$$

and

$$b_N(r) = \begin{cases} b(r) & \text{if } |r| \leq N, \\ b'(N)(r-N) + b(N) & \text{if } r > N, \\ b'(-N)(r+N) + b(-N) & \text{if } r < -N, \end{cases}$$

where $N \geq \max\{|y_1|_\infty, |y_2|_\infty\}$, by (j) we may assume that

$$\beta', b' \in C_b(\mathbb{R}),\ \beta' > \alpha_2 \in (0,\infty) \tag{3.2}$$

and, therefore, we have

$$\begin{aligned} \alpha_1|\beta(r) - \beta(\bar r)| &\geq |b^*(r) - b^*(\bar r)|, \quad \forall\, r, \bar r \in \mathbb{R}, &(3.3)\\ |\beta(r) - \beta(\bar r)| &\geq \alpha_2|r - \bar r|, \quad \forall\, r, \bar r \in \mathbb{R}, &(3.4) \end{aligned}$$

where $b^*(r) = b(r)r$ and $\alpha_2 > 0$. We set

$$\begin{gathered} \Phi_\varepsilon(y) = (\varepsilon I + (-\Delta)^s)^{-1} y,\ \forall\, y \in L^2, \\ z = y_1 - y_2,\ w = \beta(y_1) - \beta(y_2),\ b^*(y_i) \equiv b(y_i)y_i,\ i = 1,2. \end{gathered} \tag{3.5}$$

It is well known that $\Phi_\varepsilon : L^p \to L^p$, $\forall p \in [1,\infty]$ and

$$\varepsilon|\Phi_\varepsilon(y)|_p \leq |y|_p, \quad \forall y \in L^p,\ \varepsilon > 0. \tag{3.6}$$

Moreover, $\Phi_\varepsilon(y) \in C_b(\mathbb{R}^d)$ if $y \in L^1 \cap L^\infty$. We have

$$z_t + (-\Delta)^s w + \operatorname{div} D(b^*(y_1) - b^*(y_2)) = 0 \text{ in } \mathcal{D}'((0,T) \times \mathbb{R}^d).$$

We set

$$z_\varepsilon = z * \theta_\varepsilon,\ w_\varepsilon = w * \theta_\varepsilon,\ \zeta_\varepsilon = (D(b^*(y_1) - b^*(y_2))) * \theta_\varepsilon,$$

where $\theta \in C_0^\infty(\mathbb{R}^d)$, $\theta_\varepsilon(x) \equiv \varepsilon^{-d}\theta\left(\frac{x}{\varepsilon}\right)$ is a standard mollifier. We note that $z_\varepsilon, w_\varepsilon, \zeta_\varepsilon, (-\Delta)^s w_\varepsilon, \operatorname{div} \zeta_\varepsilon \in L^2(0,T;L^2)$ and we have

$$(z_\varepsilon)_t + (-\Delta)^s w_\varepsilon + \operatorname{div} \zeta_\varepsilon = 0 \text{ in } \mathcal{D}'(0,T;L^2). \tag{3.7}$$

This yields $\Phi_\varepsilon(z_\varepsilon), \Phi_\varepsilon(w_\varepsilon), \operatorname{div} \Phi_\varepsilon(\zeta_\varepsilon) \in L^2(0,T;L^2)$ and

$$(\Phi_\varepsilon(z_\varepsilon))_t = -(-\Delta)^s \Phi_\varepsilon(w_\varepsilon) - \operatorname{div}\Phi_\varepsilon(\zeta_\varepsilon) = 0 \text{ in } \mathcal{D}'(0,T;L^2). \tag{3.8}$$

By (3.7), (3.8) it follows that $(z_\varepsilon)_t = \frac{d}{dt}\, z_\varepsilon$, $(\Phi_\varepsilon(z))_t = \frac{d}{dt}\, \Phi_\varepsilon(z_\varepsilon) \in L^2(0,T;L^2)$, where $\frac{d}{dt}$ is taken in the sense of L^2-valued vectorial distributions on $(0,T)$. This implies that $z_\varepsilon, \Phi_\varepsilon(z_\varepsilon) \in H^1(0,T;L^2)$ and both $[0,T] \ni t \mapsto z_\varepsilon(t) \in L^2$ and $[0,T] \ni t \to \Phi_\varepsilon(z_\varepsilon(t)) \in L^2$ are absolutely continuous. As a matter of fact, it follows by (3.6) and (3.8) that

$$\Phi_\varepsilon(z_\varepsilon), \Phi_\varepsilon(w_\varepsilon) \in L^2(0,T;C_b(\mathbb{R}^d) \cap L^2). \tag{3.9}$$

We set $h_\varepsilon(t) = (\Phi_\varepsilon(z_\varepsilon(t)), z_\varepsilon(t))_2$ and get, therefore,

$$\begin{aligned} h'_\varepsilon(t) &= 2(z_\varepsilon(t), (\Phi_\varepsilon(z_\varepsilon(t)))_t)_2 \\ &= 2(\varepsilon\Phi_\varepsilon(w_\varepsilon(t)) - w_\varepsilon(t) - \operatorname{div}\Phi_\varepsilon(\zeta_\varepsilon(t)), z_\varepsilon(t))_2 \\ &= 2\varepsilon(\Phi_\varepsilon(z_\varepsilon(t)), w_\varepsilon(t))_2 + 2(\nabla\Phi_\varepsilon(z_\varepsilon(t)), \zeta_\varepsilon(t))_2 \\ &\qquad -2(z_\varepsilon(t), w_\varepsilon(t))_2,\ \text{a.e. } t \in (0,T), \end{aligned} \tag{3.10}$$

where, $(\cdot,\cdot)_2$ is the scalar product in L^2. By (3.8)–(3.10) it follows that $t \to h_\varepsilon(t)$ has an absolutely continuous dt-version on $[0,T]$ which we shall consider from now on. Since, by (3.5), we have for $\alpha_3 := |\beta'|_\infty^{-1}$

$$(z_\varepsilon(t), w_\varepsilon(t))_2 \geq \alpha_3 |\,|w(t)| * \theta_\varepsilon|_2^2 + \gamma_\varepsilon(t), \tag{3.11}$$

where

$$\gamma_\varepsilon(t) := (z_\varepsilon(t), w_\varepsilon(t))_2 - (z(t), w(t))_2, \tag{3.12}$$

we get, therefore, by (3.3) and (3.9),

$$\begin{aligned} 0 \ \leq h_\varepsilon(t) \leq h_\varepsilon(0+) &+ 2\varepsilon \int_0^t (\Phi_\varepsilon(z_\varepsilon(s)), w_\varepsilon(s))_2 ds - 2\alpha_3 \int_0^t |w_\varepsilon(s)|_2^2 ds \\ +2\alpha_1 |D|_\infty \int_0^t |\nabla \Phi_\varepsilon(z_\varepsilon(s))|_2 |w_\varepsilon(s)|_2 ds &+ 2\int_0^t |\gamma_\varepsilon(s)| ds, \ \forall t \in [0,T]. \end{aligned} \tag{3.13}$$

Moreover, since $z \in L^\infty((0,T)\times \mathbb{R}^d)$ and by (3.6) we obtain

$$\varepsilon |\Phi_\varepsilon(z_\varepsilon(t))|_\infty \leq |z_\varepsilon(t)|_\infty \leq |z(t)|_\infty, \quad \text{a.e. } t \in (0,T). \tag{3.14}$$

Taking into account that $t \to \Phi_\varepsilon(z_\varepsilon(t))$ has an L^2 continuous version on $[0,T]$, there exists $f \in L^2$ such that

$$\lim_{t\to 0} \Phi_\varepsilon(z_\varepsilon(t)) = f \ \text{ in } L^2.$$

Furthermore, for every $\varphi \in C_0^\infty(\mathbb{R}^d)$, $s \in (0,T)$,

$$0 \leq h_\varepsilon(s) \leq |\Phi_\varepsilon(z_\varepsilon(s)) - f|_2 |z_\varepsilon(s)|_2 + |f-\varphi|_2 |z_\varepsilon(s)|_2 + |(\varphi * \theta_\varepsilon, z(s))_2|.$$

Hence, by (3.1),

$$\begin{aligned} 0 \leq h_\varepsilon(0+) \ &= \lim_{t\downarrow 0} h_\varepsilon(t) = \lim_{t\to 0} \ \operatorname*{ess\,sup}_{s\in(0,t)} h_\varepsilon(s) \\ &\leq \left(\lim_{t\to 0} |\Phi_\varepsilon(z_\varepsilon(t)) - f|_2 + |f-\varphi|_2\right) |z_\varepsilon|_{L^\infty(0,T;L^2)} \\ &+ \lim_{t\to 0} \ \operatorname*{ess\,sup}_{s\in(0,t)} |(\varphi*\theta_\varepsilon, z(s))_2| = |f-\varphi|_2 |z_\varepsilon|_{L^\infty(0,T;L^2)}. \end{aligned}$$

Since $C_0^\infty(\mathbb{R}^d)$ is dense in $L^2(\mathbb{R}^d)$, we find

$$h_\varepsilon(0+) = 0. \tag{3.15}$$

On the other hand, taking into account that, for a.e. $t \in (0,T)$,

$$\varepsilon \Phi_\varepsilon(z_\varepsilon(t)) + (-\Delta)^s \Phi_\varepsilon(z_\varepsilon(t)) = z_\varepsilon(t), \tag{3.16}$$

we get that

$$\begin{aligned} \varepsilon |\Phi_\varepsilon(z_\varepsilon(t))|_2^2 + |(-\Delta)^{\frac{s}{2}} \Phi_\varepsilon(z_\varepsilon(t))|_2^2 = (z_\varepsilon(t), \Phi_\varepsilon(z_\varepsilon(t)))_2 = h_\varepsilon(t), \\ \text{for a.e. } t \in (0,T), \end{aligned} \tag{3.17}$$

and

$$\varepsilon|(\Phi_\varepsilon(z_\varepsilon(t),w_\varepsilon(t)))_2| \le \varepsilon|\Phi_\varepsilon(z_\varepsilon(t))|_\infty|w_\varepsilon(t)|_1 \le |z(t)|_\infty|w(t)|_1,$$
$$\text{for a.e. } t\in(0,T).$$

We note that by (3.16) and Parseval's formula we have

$$|\nabla\Phi(z_\varepsilon(t))|_2^2 = \int_{\mathbb{R}^d} \frac{|\mathcal{F}(z_\varepsilon(t))(\xi)|^2|\xi|^2}{(\varepsilon+|\xi|^{2s})^2}\,d\xi,\ \forall t\in(0,T),$$

and

$$h_\varepsilon(t) = \int_{\mathbb{R}^d} \frac{|\mathcal{F}(z_\varepsilon(t))(\xi)|^2}{\varepsilon+|\xi|^{2s}}\,d\xi,\quad \forall t\in(0,T).$$

This yields

$$\begin{aligned}|\nabla\Phi_\varepsilon(z_\varepsilon(t))|_2^2 &\le R^{2(1-s)}\int_{[|\xi|\le R]} \frac{|\mathcal{F}(z_\varepsilon(t))(\xi)|^2}{\varepsilon+|\xi|^{2s}}\,d\xi\\ &\quad+\int_{[|\xi|\ge R]} |\mathcal{F}(z_\varepsilon(t))(\xi)|^2|\xi|^{2(1-2s)}d\xi\\ &\le R^{2(1-s)}h_\varepsilon(t)+R^{2(1-2s)}|z_\varepsilon(t)|_2^2,\ \forall t\in(0,T),\ R>0,\end{aligned}\tag{3.18}$$

because $2s\ge 1$.

We shall prove now that

$$\lim_{\varepsilon\to 0}\varepsilon(\Phi_\varepsilon(z_\varepsilon(t)),w_\varepsilon(t))_2 = 0,\ \text{a.e. } t\in(0,T).\tag{3.19}$$

Since by (3.14)

$$\varepsilon|(\Phi_\varepsilon(z_\varepsilon(t)),w_\varepsilon(t))_2| \le |z_\varepsilon(t)|_\infty|w_\varepsilon(t)|_1 \le |z(t)|_\infty|w(t)|_1,\tag{3.20}$$

it suffices to show that

$$\lim_{\varepsilon\to 0}\varepsilon|\Phi_\varepsilon(z_\varepsilon(t))|_\infty = 0,\quad \text{a.e. } t\in(0,T).\tag{3.21}$$

To prove (3.21) we proceed similarly as in the proof of [11, Lemma 1]. By (A.6) in the Appendix we have for a.e. $t\in(0,T)$

$$\varepsilon\Phi_\varepsilon(z_\varepsilon(t))(x) = \varepsilon^{\frac{d}{2s}}\int_{\mathbb{R}^d} g_1^s(\varepsilon^{\frac{1}{2s}}(x-\xi))z_\varepsilon(t)(\xi)d\xi \text{ for a.e. } x\in\mathbb{R}^d.$$

This yields for a.e. $x \in \mathbb{R}^d$

$$|\varepsilon\Phi_\varepsilon(z_\varepsilon(t))(x)| \le C_r\varepsilon^{d/2s}|z(t)|_1 + \varepsilon^{d/2s}|z(t)|_\infty \int_{[\varepsilon^{1/2s}|x-\xi|\le r]} g_1^s(\varepsilon^{1/2s}(x-\xi))d\xi,$$

where $C_r := \sup\{g_1^s(x); |x| \ge r\}$ ($< \infty$, since $g_1^s \in L^\infty(B_r(0)^C)$ by (A.7)). Therefore, for a.e. $x \in \mathbb{R}^d$,

$$|\varepsilon\Phi_\varepsilon(z_\varepsilon(t))(x)| \le C_r\varepsilon^{d/2s}|z(t)|_1 + |z(t)|_\infty \int_{[|\xi|\le r]} g_1^s(\xi)d\xi.$$

Since $g_1^s \in L^1$ by (A.4), letting first $\varepsilon \to 0$ and then $r \to 0$, (3.21) follows, as claimed.

By (3.20), (3.21) and the dominated convergence theorem, it follows that

$$\lim_{\varepsilon\to 0}\varepsilon\int_0^t (\Phi_\varepsilon(z_\varepsilon(s)), w_\varepsilon(s))_2 ds = 0,\ t \in [0,T]. \tag{3.22}$$

Next, by (3.13), (3.15) and (3.18), we have

$$\begin{aligned}
0 \le h_\varepsilon(t) &\le 2\varepsilon\int_0^t |(\Phi_\varepsilon(z_\varepsilon(s)), w_\varepsilon(s))_2| ds - 2\alpha_3\int_0^t |w_\varepsilon(s)|_2^2 ds \\
&\quad + 2\alpha_1|D|_\infty \int_0^t |\nabla\Phi_\varepsilon(z_\varepsilon(t))|_2 |w_\varepsilon(s)|_2 ds + 2\int_0^t |\gamma_\varepsilon(s)| ds \\
&\le \eta_\varepsilon(t) + 2\alpha_1|D|_\infty \int_0^t \left(R^{1-s}h_\varepsilon^{\frac{1}{2}}(t) + R^{1-2s}|z_\varepsilon(t)|_2\right)|w_\varepsilon(s)|_2 ds \\
&\quad - 2\alpha_3\int_0^t |w_\varepsilon(s)|_2^2 ds,\ \forall\, t \in [0,T],\ R > 0,
\end{aligned}$$

where

$$\eta_\varepsilon(t) := 2\varepsilon\int_0^t |(\Phi_\varepsilon(z_\varepsilon(s)), w_\varepsilon(s))_2| ds + 2\int_0^t |\gamma_\varepsilon(s)| ds.$$

This yields

$$\begin{aligned}
h_\varepsilon(t) &\le \eta_\varepsilon(t) + 2\alpha_1|D|_\infty \bigg(R^{2(1-s)}\lambda\int_0^t h_\varepsilon(s)ds + \int_0^t \Big(R^{1-2s}|z_\varepsilon(s)|_2^2 \\
&\quad + \Big(\frac{1}{4\lambda} + R^{1-2s}\Big)|w_\varepsilon(s)|_2^2\Big)ds\bigg) - 2\alpha_3\int_0^t |w_\varepsilon(s)|_2^2 ds,\ \forall \lambda > 0,\ R > 0.
\end{aligned}$$

Taking into account that, by (3.4),

$$|z(t)|_2 \leq \alpha_2^{-1}|w(t)|_2, \quad \forall t \in (0,T), \tag{3.23}$$

we have

$$|z_\varepsilon(t)|_2 \leq \alpha_2^{-1}|w_\varepsilon(t)|_2 + \nu_\varepsilon(t), \ \forall t \in (0,T),$$

where $\nu_\varepsilon \to 0$ in $L^2(0,T)$. Then, we get, for λ, $R > 0$, suitably chosen,

$$0 \leq h_\varepsilon(t) \leq \eta_\varepsilon(t) + C\int_0^t h_\varepsilon(s)ds, \quad \text{for } t \in [0,T], \tag{3.24}$$

where $C > 0$ is independent of ε and $\lim\limits_{\varepsilon\to 0} \eta_\varepsilon(t) = 0$ for all $t \in [0,T]$.

In particular, by (3.17), it follows that

$$0 \leq h_\varepsilon(t) \leq \eta_\varepsilon(t)\exp(Ct), \ \forall\, t \in [0,T]. \tag{3.25}$$

This implies that $h_\varepsilon(t) \to 0$ as $\varepsilon \to 0$ for every $t \in [0,T]$, hence by (3.17) the left hand side of (3.16) converges to zero in S'. Thus, $0 = \lim\limits_{\varepsilon\to 0} z_\varepsilon(t) = z(t)$ in S' for a.e. $t \in (0,T)$, which implies $y_1 \equiv y_2$. If $D \equiv 0$, we see by (3.13), (3.15) and (3.19) that $0 \leq h_\varepsilon(t) \leq \eta_\varepsilon(t)$, $\forall\, t \in (0,T)$, and so the conclusion follows without invoking that $\beta' > 0$, which was only used to have (3.4) and thus (3.23). □

Linearized uniqueness. In particular, the linearized uniqueness for equation (1.10) follows by Theorem 3.1. More precisely,

Theorem 3.2. *Under assumptions of Theorem* 3.1*, let* $T > 0$*,* $u \in L^\infty((0,T)\times\mathbb{R}^d)$ *and let* $y_1, y_2 \in L^\infty((0,T)\times\mathbb{R}^d)$ *with* $y_1 - y_2 \in L^1((0,T)\times\mathbb{R}^d) \cap L^\infty(0,T;L^2)$ *be two distributional solutions to the equation*

$$\begin{aligned} &y_t + (-\Delta)^s\left(\frac{\beta(u)}{u}y\right) + \operatorname{div}(yDb(u)) = 0 \ \textit{in } \mathcal{D}'((0,T)\times\mathbb{R}^d), \\ &y(0) = u_0, \end{aligned} \tag{3.26}$$

where u_0 *is a measure of finite variation on* $\mathbb{R}^d$ *and* $\frac{\beta(0)}{0} := 0$*. If* (3.1) *holds, then* $y_1 \equiv y_2$*.*

Proof. We note first that

$$\begin{aligned}&\frac{\beta(u)}{u},\ b(u)\in L^\infty((0,T)\times\mathbb{R}^d),\\&|Db(u)|_\infty\le C_1\left|\frac{\beta(u)}{u}\right|_\infty\le C_2.\end{aligned}$$

If $z=y_1-y_2,\ w=\frac{\beta(u)}{u}\,(y_1-y_2)$, we see that

$$\begin{aligned}&wz\ge\left|\tfrac{\beta(u)}{u}\right|_\infty+|w|^2,\quad \text{a.e. on } (0,T)\times\mathbb{R}^d,\\&|Db(u)z|\le C_2|w|,\qquad \text{a.e. on } (0,T)\times\mathbb{R}^d.\end{aligned}$$

Then, we have

$$z_t+(-\Delta)^s w+\operatorname{div}(Db(u)z)=0,$$

and so, arguing as in the proof of Theorem 3.1, we get that $y_1\equiv y_2$. The details are omitted. □

4 Applications to McKean–Vlasov equations with Lévy noise

4.1 Weak existence

To prove weak existence for (1.11), we use the recent results in [25] and Theorem 2.4.

Theorem 4.1. *Assume that Hypotheses* (i)–(iv) *from Section* 1 *hold and let* $u_0\in L^1\cap L^2$. *Assume that either* $u_0\in\overline{D(A)}$ *or that* $\beta\in C^\infty(\mathbb{R})$ (*see Theorem* 2.4) *and let* u *be the solution of* (1.1) *from Theorem* 2.4. *Then, there exists a stochastic basis* $\mathbb{B}:=(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\ge0},\mathbb{P})$ *and a* d*-dimensional isotropic* $2s$*-stable process* L *with Lévy measure* $\frac{dz}{|z|^{d+2s}}$ *as well as an* $(\mathcal{F}_t)$*-adapted càdlàg process* (X_t) *on* Ω *such that, for*

$$\mathcal{L}_{X_t}(x):=\frac{d(\mathbb{P}\circ X_t^{-1})}{dx}\,(x),\ t\ge0, \tag{4.1}$$

we have

$$\begin{aligned}dX_t&=D(X_t)b(\mathcal{L}_{X_t}(X_t))dx+\frac{\beta(\mathcal{L}_{X_t}(X_{t-}))}{\mathcal{L}_{X_t}(X_{t-})}\,dL_t,\\ \mathcal{L}_{X_0}&=u_0.\end{aligned} \tag{4.2}$$

Furthermore,

$$\mathcal{L}_{X_t} = u(t,\cdot),\ t \geq 0, \tag{4.3}$$

in particular, $((t,x) \mapsto \mathcal{L}_{X_t}(x)) \in L^\infty([0,T] \times \mathbb{R}^d)$ *for every* $T > 0$.

Proof. Let u be the mild (hence distributional) solution of (1.1) from Theorem 2.4.

By the well known formula that

$$(-\Delta)^s f(x) = -c_{d,s}\text{P.V.} - \int_{\mathbb{R}^d} (f(x+z) - f(x)) \frac{dz}{|z|^{d+2s}} \tag{4.4}$$

with $c_{d,s} \in (0,\infty)$ (see [26, Section 13]), and since, as an easy calculation shows,

$$\int_A \frac{\beta(u(t,x))}{u(t,x)} \frac{dz}{|z|^{d+2s}} = \int_{\mathbb{R}^d} \mathbb{1}_A \left(\left(\frac{\beta(u(t,x))}{u(t,x)} \right)^{\frac{1}{2s}} z \right) \frac{dz}{|z|^{d+2s}}, \qquad A \in B(\mathbb{R}^d \setminus \{0\}), \tag{4.5}$$

we have

$$\begin{aligned} &\frac{\beta(u(t,x))}{u(t,x)} (-\Delta)^s f(x) \\ &\quad = -c_{d,s}\text{P.V.} \int_{\mathbb{R}^d} \left(f\left(x + \left(\frac{\beta(u(t,x))}{u(t,x)} \right)^{\frac{1}{2s}} z \right) - f(x) \right) \frac{dz}{|z|^{d+2s}}. \end{aligned} \tag{4.6}$$

As is easily checked, Hypotheses (i)–(iv) imply that condition (1.18) in [25] holds. Furthermore, it follows by Theorem 2.4 that

$$\mu(dx) := u(t,x)(dx),\ t \geq 0,$$

solves the Fokker–Planck equation (1.10) with $u_0 := u_0(x)dx$. Hence, by [25, Theorem 1.5], (4.5), (4.6) and [20, Theorem 2.26, p. 157], there exists a stochastic basis $\mathbb{B}$ and $(X_t)_{t\geq 0}$ as in the assertion of the theorem, as well as a Poisson random measure N on $\mathbb{R}^d \times [0,\infty)$ with intensity $|z|^{-d-2s}dz\,dt$ on the stochastic basis $\mathbb{B}$ such that for

$$L_t := \int_0^t \int_{|z|\leq 1} z\widetilde{N}(dz\,ds) + \int_0^t \int_{|z|>1} z\,N(dz\,ds), \tag{4.7}$$

(4.1), (4.2) and (4.3) hold. Here,

$$\widetilde{N}(dz\,dt) := N(dz\,dt) - |z|^{-d-2s}dz\,dt. \qquad \square$$

4.2 Weak uniqueness

Theorem 4.2. *Assume that Hypotheses* (j)–(jjj), *resp.* (j)$'$, (jj), (jjj) *if* $D \equiv 0$, *from Section* 3 *hold and let* $T > 0$. *Let* (X_t) *and* $(\widetilde{X}_t)$ *be two càdlàg processes on two (possibly different) stochastic bases* $\mathbb{B}, \widetilde{\mathbb{B}}$ *that are weak solutions to* (4.2) *with (possibly different)* L *and* $\widetilde{L}$ *defined as in* (4.7). *Assume that*

$$((t,x) \mapsto \mathcal{L}_{X_t}(x)), \big((t,x) \mapsto \mathcal{L}_{\widetilde{X}_t}(x)\big) \in L^\infty((0,T) \times \mathbb{R}^d). \tag{4.8}$$

Then X *and* $\widetilde{X}$ *have the same laws, i.e.,*

$$\mathbb{P} \circ X^{-1} = \widetilde{\mathbb{P}} \circ \widetilde{X}^{-1}.$$

Proof. Clearly, by Dynkin's formula both

$$\mu_t(dx) := \mathcal{L}_{X_t}(x)dx \quad \text{and} \quad \widetilde{\mu}_t(dx) := \mathcal{L}_{\widetilde{X}_t}(x)dx$$

solve the Fokker–Planck equation (1.10) with the same initial condition $u_0(dx) := u_0(x)dx$, hence satisfy (3.1) with $y_1(t) := \mathcal{L}_{X_t}$ and $y_2(t) := \mathcal{L}_{\widetilde{X}_t}$. Hence, by Theorem 3.1,

$$\mathcal{L}_{X_t} = \mathcal{L}_{\widetilde{X}_t} \quad \text{for all } t \geq 0,$$

since $t \mapsto \mathcal{L}_{X_t}(x)dx$ and $t \mapsto \mathcal{L}_{\widetilde{X}_t}(x)dx$ are both narrowly continuous and are probability measures for all $t \geq 0$, so both are in $L^\infty(0,T;L^1 \cap L^\infty) \subset L^\infty(0,T;L^2)$.

Now, consider the linear Fokker–Planck equation

$$\begin{gathered} v_t + (-\Delta)^s \beta(\mathcal{L}_{X_t}) + \operatorname{div}(Db(\mathcal{L}_{X_t})v) = 0, \\ v(0,x) = u_0(x), \end{gathered} \tag{4.9}$$

again in the weak (distributional) sense analogous to (1.10). Then, by Theorem 3.2 we conclude that $\mathcal{L}_{X_t}$, $t \in [0,T]$, is the unique solution to (4.9) in $L^\infty(0,T;L^1 \cap L^\infty)$. Again by Dynkin's formula, both $\mathbb{P} \circ X^{-1}$ and $\widetilde{\mathbb{P}} \circ \widetilde{X}^{-1}$ solve the martingale problem with initial condition $u_0(dx) := u_0(x)dx$ for the linear Kolmogorov operator

$$K_{\mathcal{L}_{X_t}} := -\frac{\beta(\mathcal{L}_{X_t})}{\mathcal{L}_{X_t}} (-\Delta)^s + b(\mathcal{L}_{X_t}) D \cdot \nabla. \tag{4.10}$$

Since the above is true for all $u_0 \in L^1 \cap L^\infty$, and also holds when we consider (1.1) resp (1.10) with start in $s > 0$ instead of zero, it follows by exactly the same arguments as in the proof of Lemma 2.12 in [29] that

$$\mathbb{P} \circ X^{-1} = \widetilde{\mathbb{P}} \circ \widetilde{X}^{-1}. \qquad \square$$

Remark 4.3. Let for $s \in [0, \infty)$ and $\mathcal{Z} := \{\zeta \equiv \zeta(x)dx \mid \zeta \in L^1 \cap L^\infty,\ \zeta \geq 0, |\zeta|_1 = 1\}$

$$\mathbb{P}_{(s,\zeta)} := \mathbb{P} \circ X^{-1}(s, \zeta),$$

where $(X_t(s, \zeta))_{t \geq 0}$ on a stochastic basis $\mathbb{B}$ denotes the solution of (1.11) with initial condition ζ at s. Then, by Theorems 3.1, 3.2 and 4.2, exactly the same way as Corollary 4.6 in [24], one proves that $\mathbb{P}_{(s,\zeta)}$, $(s, \zeta) \in [0, \infty) \times \mathcal{Z}$, form a nonlinear Markov process in the sense of McKean (see [22]).

Remark 4.4. (4.3) in Theorem 4.1 says that our solution u of (1.1) from Theorem 2.4 is the law density of a càdlàg process solving (4.2) or resp. by Remark 4.3 above that it is the law density of a nonlinear Markov process. This realizes McKean's vision formulated in [22] for solutions to nonlinear parabolic PDE. So, our results show that it is also possibly for nonlocal PDE of type (1.1).

Remark 4.5. In a forthcoming paper [10], we achieve similar results as in this paper in the case where $(-\Delta)^s$ is replaced by $\psi(-\Delta)$, where ψ is a Bernstein function (see [26]).

Appendix: Representation and properties of the integral kernel of $(\varepsilon I + (-\Delta)^s)^{-1}$

Let $s \in (0, 1)$ and let $\mathcal{F} : S'(\mathbb{R}^d) \to S'(\mathbb{R}^d)$ be the Fourier transform, as defined in (1.3).

It is well known (see, e.g., [21, Chap. II, Sect. 4c]) that for $t > 0$ the integral kernel p_t^s of the operator $T_t^s := \exp(-t(-\Delta)^s)$ is related to the kernel p_t of $\exp(t\Delta)$ by the following subordination formula

$$p_t^s(x) = \int_0^\infty p_r(x) \eta_t^s(dr),\ x \in \mathbb{R}^d, \tag{A.1}$$

where $(\eta_t^s)_{t>0}$ is the one-sided stable semigroup of order $s \in (0, 1)$, which is defined through its Laplace transform by

$$\int_0^\infty e^{-\lambda r}\eta_t^s(dr) = e^{-t\lambda^s},\ \ \lambda > 0. \tag{A.2}$$

Furthermore, since $(\varepsilon I + (-\Delta)^s)^{-1}$ is the Laplace transform of the semigroup $T_t^{(s)}$, $t \geq 0$, it follows that

$$g_\varepsilon^s(x) = \int_0^\infty e^{-\varepsilon t}\int_0^\infty p_r(x)\eta_t^s(dr)dt,\ \ x \in \mathbb{R}^d, \tag{A.3}$$

is the integral kernel of $(\varepsilon I + (-\Delta)^s)^{-1}$. Obviously, since each η_t^s is a probability measure, we have

$$\varepsilon \int_{\mathbb{R}^d} g_\varepsilon^s(x)dx = 1. \tag{A.4}$$

Since

$$p_r(x) = \frac{1}{(4\pi r)^{d/2}}\ e^{-\frac{1}{4r}|x|^2} = (2\pi)^{-d}\int_{\mathbb{R}^d} e^{i\langle x,y\rangle}e^{-r|y|^2}dy,\ \ x \in \mathbb{R}^d, \tag{A.5}$$

plugging the first equality in (A.5) into (A.3), we obtain

$$g_\varepsilon^s(x) = \int_0^\infty e^{-\varepsilon t}\int_0^\infty \frac{1}{(4\pi r)^{d/2}}\ e^{-\frac{1}{4r}|x|^2}\eta_t^s(dr)dt.$$

It is well known and trivial to check from the definition that for $\gamma \in (0,\infty)$ the image measure of η_t^s under the map $r \mapsto \gamma r$ is equal to $\eta_{\gamma^s t}^s$. Hence, by an elementary computation we find

$$g_\varepsilon^s(x) = |x|^{-d+2s}\int_0^\infty\int_0^\infty e^{-\varepsilon|x|^{2s}t}\frac{1}{(4\pi r t^{1/s})^{d/2}}\ e^{-\frac{1}{4rt^{1/s}}}\,dt\,\eta_1^s(dr), \tag{A.6}$$

which, since $s < \frac{d}{2}$ and η_1^s is a probability measure, in turn implies

$$g_\varepsilon^s(x) \sim |x|^{-d+2s}\ \text{ as } |x| \to 0$$

and

$$g_\varepsilon^s \in L^\infty(\mathbb{R}^d \setminus B_R(0)),\ \forall R > 0. \tag{A.7}$$

Plugging the second equality in (A.5) into (A.3), it follows by (A.2) that

$$\begin{aligned} g_\varepsilon^s(x) &= (2\pi)^{-d}\int_0^\infty e^{-\varepsilon t}\int_{\mathbb{R}^d} e^{i\langle x,y\rangle}e^{-t|y|^{2s}}\,dy\,dt \\ &= (2\pi)^{-d/2}\int_0^\infty \mathcal{F}\left(e^{-t(\varepsilon+|\cdot|^{2s})}\right)(x)dt,\ \ x \in \mathbb{R}^d. \end{aligned} \tag{A.8}$$

Hence

$$g_\varepsilon^s = (2\pi)^{-d/2}\mathcal{F}\left(\frac{1}{\varepsilon + |\cdot|^{2s}}\right) \quad \text{in } S'(\mathbb{R}^d). \tag{A.9}$$

Finally, from (A.8) it follows that

$$g_\varepsilon^s(x) = \varepsilon^{\frac{d-2s}{2s}} g_1^s\left(\varepsilon^{\frac{1}{2s}}x\right), \ x \in \mathbb{R}^d. \tag{A.10}$$

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