GAUSSIAN FLUCTUATIONS FOR INTERACTING PARTICLE SYSTEMS WITH SINGULAR KERNELS

ZHENFU WANG, XIANLIANG ZHAO, AND RONGCHAN ZHU

ABSTRACT. We consider the asymptotic behaviour of the fluctuations for the empirical measures of interacting particle systems with singular kernels. We prove that the sequence of fluctuation processes converges in distribution to a generalized Ornstein-Uhlenbeck process.

Our result considerably extends classical results to singular kernels, including the Biot-Savart law. The result applies to the point vortex model approximating the 2D incompressible Navier–Stokes equation and the 2D Euler equation. We also obtain Gaussianity and optimal regularity of the limiting Ornstein-Uhlenbeck process. The method relies on the martingale approach and the Donsker-Varadhan variational formula, which transfers the uniform estimate to some exponential integrals. Estimation of those exponential integrals follows by cancellations and combinatorics techniques and is of the type of large deviation principle.

CONTENTS

1. INTRODUCTION

In this article, we consider interacting particle systems characterized by the following system of SDEs on the torus $\mathbb{T}^d$, $d \geqslant 2$,

$$\mathrm{d}X_i = \frac{1}{N} \sum_{j \neq i} K(X_i - X_j)\mathrm{d}t + F(X_i)\mathrm{d}t + \sqrt{2\sigma_N}\mathrm{d}B_t^i, \quad i = 1, ..., N, \tag{1.1}$$

with random initial data $\{X_i(0)\}_{i=1}^N$. The collection $\{B^i\}_{i=1}^N$ consists of N independent d dimensional Brownian motions on a stochastic basis, i.e. $(\Omega, \mathcal{F}, \mathbb{P})$ with a normal filtration $(\mathcal{F}_t)$, induced by the Laplacian operator on the torus, independent of $\{X_i(0)\}_{i=1}^N$. The coefficient $\sigma_N \geqslant 0$ is a non-negative scalar for simplicity. In this model, $X^N(t) := (X_1(t), ..., X_N(t)) \in (\mathbb{T}^d)^N$ represents the positions of particles, which are interacting through the kernel K and confined by the exterior force F.

Many particle systems written in the canonical form (1.1) or its variant are now quite ubiquitous. Such systems are usually formulated by first-principle agent based models which are conceptually

simple. For instance, in physics those particles X_i can represent ions and electrons in plasma physics [Dob79], or molecules in a fluid [JO04] or even large scale galaxies [Jea15] in some cosmological models; in biological sciences, they typically model the collective behavior of animals or micro-organisms (for instance flocking, swarming and chemotaxis and other aggregation phenomena [CCH14]); in economics or social sciences particles are usually individual "agents" or "players" for instance in opinion dynamics [FJ90] or in the study of mean-field games [LL07, HMC$^+$06]. Motivation even extends to the analysis of large biological [BFT15] or artificial [MMN18] neural networks in neuroscience or in machine learning.

Under mild assumptions, it is well-known (see for instance [MJ, BH77, Dob79, Osa86, Szn91, FHM14, JW18, Ser20b, Jab14, BJW20] and Section 1.3 for more details) that the empirical measure $\mu_N(t) := \frac{1}{N} \sum_{i=1}^{N} \delta_{X_i(t)}$ of the particle system (1.1) converges to the solution $\bar{\rho}(t)$ of the following deterministic mean-field PDE

$$\partial_t \bar{\rho} = \sigma \Delta \bar{\rho} - \operatorname{div}([F + K * \bar{\rho}]\bar{\rho}), \tag{1.2}$$

as $N \to \infty$, where $\sigma = \lim_{N \to \infty} \sigma_N$. This is equivalent to the *propagation of chaos*, i.e. the $k-$th marginal $\rho_{N,k}$ of the particle system (1.1) will converge to the tensor product of the limit law $\bar{\rho}^{\otimes k}$ as N goes to infinity, given for instance the i.i.d. initial data. This law of large numbers type result implies that the continuum model (1.2) is a suitable approximation to the particle system (1.1) when N is large, i.e. $\mu_N \approx \bar{\rho} + o(1)$.

Inspired in particular by quantitative estimates of propagation of chaos by Jabin and Wang [JW18], which is $\|\rho_{N,k} - \bar{\rho}^{\otimes k}\|_{L^\infty([0,T],L^1)} \leqslant C_T/\sqrt{N}$, we aim to study the central limit theorem of (1.1), which provides a better continuum approximation to (1.1). More precisely, we study the limit of fluctuation measures around the mean-field law, which are defined by

$$\eta^N := \sqrt{N}(\mu_N - \bar{\rho}) = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} (\delta_{X_i} - \bar{\rho}). \tag{1.3}$$

In this article, we establish that the fluctuation measure η^N converges in distribution as $N \to \infty$ to an infinite-dimensional continuous Gaussian process η for a large class of particle systems (1.1). This implies that there exists a continuum model η such that

$$\mu_N \overset{\mathrm{d}}{\approx} \bar{\rho} + \frac{1}{\sqrt{N}}\eta + o(\frac{1}{\sqrt{N}}),$$

where $\overset{\mathrm{d}}{\approx}$ means that the approximation holds in distribution.

1.1. **Assumptions.** To state our main results we first give the framework in this article. Recall that the relative entropy $H(\mu|\nu)$ between probability measures μ and ν on a Polish space E is defined by

$$H(\mu|\nu) := \begin{cases} \int_E \frac{\mathrm{d}\mu}{\mathrm{d}\nu} \log \frac{\mathrm{d}\mu}{\mathrm{d}\nu} \mathrm{d}\nu & \text{if } \mu \ll \nu, \\ \infty & \text{otherwise,} \end{cases}$$

where $\frac{\mathrm{d}\mu}{\mathrm{d}\nu}$ is the Radon-Nikodym derivative of μ with respect to ν. Note that throughout this article, all the relative entropy is of the classical form. We will not normalize it as what have been done for instance in [JW18].

Our assumptions are listed as follows.

(A1)-CLT for initial values. There exists η_0, which belongs to the space of tempered distributions $\mathcal{S}'(\mathbb{T}^d)$, such that the sequence $\{\eta_0^N\}_{N \geqslant 1}$ converges in distribution to η_0 in $\mathcal{S}'(\mathbb{T}^d)$. Here η_0 will be the initial data for our expected limit SPDE (1.5) below.

(A2)-Regularity of the kernel. The kernel $K : \mathbb{T}^d \to \mathbb{R}^d$, $d \geqslant 2$, satisfies one of the following conditions

(1) K is bounded.
(2) For each $x \in \mathbb{T}^d$, $K(x) = -K(-x)$ and $|x|K(x) \in L^\infty$.

(A3)-Uniform relative entropy bound. Let $X^N(t) = (X_1(t), ..., X_N(t))$ be a solution to the particle system (1.1), and let $\rho_N(t)$ represent the joint distribution of $X^N(t)$. It holds that

$$\sup_{t \in [0,T]} \sup_N H(\rho_N(t)|\bar{\rho}_N(t)) < \infty,$$

where $\bar{\rho}_N(t)$ denotes the tensor product $\bar{\rho}_t^{\otimes N}$. We may use $H_t(\rho_N|\bar{\rho}_N)$ to represent $H(\rho_N(t)|\bar{\rho}_N(t))$ for simplicity.

The global well-posedness of the limit equation (1.5) will be obtained by two different approaches, depending on the diffusion coefficient σ is positive or zero. Hence we distinguish the extra assumptions into the following two cases.

For the case when $\sigma > 0$, in addition to the assumptions **(A1)-(A3)**, we need the following extra assumption:

(A4)-The case with non-vanishing diffusion.

(1) $\sigma > 0$ and $|\sigma_N - \sigma| = \mathcal{O}\left(\frac{1}{N}\right)$.
(2) There exists some $\beta > d/2$ such that $\bar{\rho} \in C([0,T], C^\beta(\mathbb{T}^d))$ and $F \in C^\beta(\mathbb{T}^d)$, where $\bar{\rho}$ solves equation (1.2) in the weak sense.

On the other hand, for the case with vanishing diffusion, besides **(A1)-(A3)**, we require that

(A5)-The case with vanishing diffusion. The diffusion coefficients and the mean-field equation (1.2) satisfy that,

(1) $\sigma = 0$ and $|\sigma_N - \sigma| = \mathcal{O}\left(\frac{1}{N}\right)$.
(2) $\bar{\rho} \in C^1([0,T], C^{\beta+2}(\mathbb{T}^d))$ and $F \in C^{\beta+1}(\mathbb{T}^d)$ with $\beta > d/2$ where $\bar{\rho}$ solves (1.2).
(3) $\mathrm{div}K \in L^1$.

We make several remarks on our assumptions. Firstly, when $\{X_i(0)\}_{i \in \mathbb{N}}$ are i.i.d. with a common probability density function μ, which is the usual setting to study the fluctuations, one can easily check that **(A1)** holds true. Indeed, for each $\varphi \in C^\infty(\mathbb{T}^d)$, we have

$$\langle \eta_0^N, \varphi \rangle = \frac{1}{\sqrt{N}} \sum_{i=1}^N \left[\varphi(X_i(0)) - \langle \varphi, \mu \rangle \right] \xrightarrow{N \to \infty} \mathcal{N}\left(0, \langle \varphi^2, \mu \rangle - \langle \varphi, \mu \rangle^2\right),$$

where $\mathcal{N}(0, a)$ denotes the centered Gaussian distribution on $\mathbb{R}$ with variance a. Hereafter we use the bracket $\langle \cdot, \cdot \rangle$ as a shorthand notation for integration. We also state a central limit theorem under an assumption on $H(\rho_N(0)|\bar{\rho}_N(0))$ in Section 5.2, where $\{X_i(0)\}_{i \in \mathbb{N}}$ can be neither independent nor identically distributed.

Assumption **(A2)** on interaction kernels allows our framework to cover smooth kernels and some singular kernels, in particular the Biot-Savart kernel related to the vorticity formulation of 2D Navier-Stokes/Euler equation on the torus. See Theorem 1.7 and Section 5 for more details.

Assumption **(A3)** seems quite nontrivial and demanding, but fortunately it has been established by Jabin and Wang in [JW18] for a quite large family of interacting kernels, including all the kernels satisfying **(A2)**. Indeed, once we have that the relative entropy between the joint distribution ρ_N of the interacting particle system (1.1) and the tensorized law $\bar{\rho}^{\otimes N}$ of the mean-field PDE (1.2) is uniformly bounded with respect to N, then easily the particle system (1.1) converges to the mean-field equation (1.2) with a rate $\frac{C_T}{\sqrt{N}}$, in the total variation norm or the Wasserstein metric. More precisely, since all particles in (1.1) are indistinguishable, the joint distribution ρ_N is thus assumed to be symmetric/exchangeable, so is any k-marginal distribution $\rho_{N,k}$ of ρ_N, which is defined as

$$\rho_{N,k}(t, x_1, ..., x_k) := \int_{\mathbb{T}^{d(N-k)}} \rho_N(t, x_1, ..., x_N) \mathrm{d}x_{k+1}...\mathrm{d}x_N.$$

Then by the sub-additivity of relative entropy, in particular $H(\rho_{N,k}|\bar{\rho}^{\otimes k}) \leqslant \frac{k}{N} H(\rho_N|\bar{\rho}^{\otimes N})$ and the classical Csiszár–Kullback–Pinsker inequality [Vil08, (22.25)], it follows that for fixed $k \in \mathbb{N}$,

$$W_1\left(\rho_{N,k}(t), \bar{\rho}^{\otimes k}(t)\right) \lesssim \|\rho_{N,k}(t) - \bar{\rho}^{\otimes k}(t)\|_{TV} \leqslant \sqrt{2H_t(\rho_{N,k}|\bar{\rho}^{\otimes k})} \lesssim \sqrt{\frac{k}{N}} \longrightarrow 0, \qquad (1.4)$$

where $W_1(\cdot, \cdot)$ denotes the 1-Wasserstein distance, $\|\cdot\|_{TV}$ denotes the total variation norm and the first inequality is guaranteed by [Vil08, Theorem 6.15] since now $\mathbb{T}^d$ is compact.

1.2. **Main results.** Under the assumptions **(A1)-(A3)** and either **(A4)** or **(A5)**, depending on $\sigma > 0$ or $\sigma = 0$, we establish that as $N \to \infty$, the sequence of the fluctuation measures (η^N) converges in distribution to the centered Gaussian process η solving the following stochastic PDE (SPDE)

$$\partial_t \eta = \sigma \Delta \eta - \nabla \cdot (\bar{\rho} K * \eta) - \nabla \cdot (\eta K * \bar{\rho}) - \nabla \cdot (F\eta) - \sqrt{2\sigma} \nabla \cdot \left(\sqrt{\bar{\rho}} \xi\right), \qquad \eta(0) = \eta_0, \qquad (1.5)$$

where η_0 is given in Assumption **(A1)** and ξ is vector-valued space-time white noise on $\mathbb{R}^+ \times \mathbb{T}^d$, i.e. a family of centered Gaussian random variables $\{\xi(h) : h \in L^2(\mathbb{R}^+ \times \mathbb{T}^d; \mathbb{R}^d)\}$ such that $\mathbb{E}[|\xi(h)|^2] = \|h\|^2_{L^2(\mathbb{R}^+ \times \mathbb{T}^d; \mathbb{R}^d)}$, and $\bar{\rho}$ solves the mean-field equation (1.2). But when $\sigma = 0$, the SPDE (1.5) becomes a deterministic PDE.

As a first step, we need a proper notion of solutions to the SPDE (1.5). When $\sigma > 0$, it turns out to be the martingale solutions defined as below.

Definition 1.1. *We call η a martingale solution to the SPDE (1.5) on some stochastic basis $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ if*

(1) *η is a continuous $(\mathcal{F}_t)$-adapted process with values in $H^{-\alpha-2}$ and $\eta \in L^2([0,T], H^{-\alpha})$ for every $\alpha > d/2$, $\mathbb{P}$-almost surely.*

(2) *For each $\varphi \in C^\infty(\mathbb{T}^d)$ and $t \in [0,T]$, it holds that*

$$\langle \eta_t, \varphi \rangle - \langle \eta_0, \varphi \rangle = \int_0^t \langle \sigma \Delta \varphi, \eta \rangle \mathrm{d}s + \int_0^t \langle \nabla \varphi, \bar{\rho} K * \eta + \eta K * \bar{\rho} + F\eta \rangle \mathrm{d}s + \mathcal{M}_t(\varphi),$$

where $\mathcal{M}$ is a continuous $(\mathcal{F}_t)$-adapted centered Gaussian process with values in $H^{-\alpha-1}$ for every $\alpha > d/2$ and its covariance given by

$$\mathbb{E}[\mathcal{M}_t(\varphi_1)\mathcal{M}_s(\varphi_2)] = 2\sigma \int_0^{s \wedge t} \langle \nabla \varphi_1 \cdot \nabla \varphi_2, \bar{\rho}_r \rangle \mathrm{d}r,$$

for each $\varphi_1, \varphi_2 \in C^\infty(\mathbb{T}^d)$ and $s, t \in [0,T]$.

Remark 1.2.   (1) *The stochastic basis in Definition 1.1 might be different from the stochastic basis where the particle system (1.1) lives.*

(2) *By Lemma A.2 and Lemma A.4 given in Appendix A, $\bar{\rho} K * \eta$, $\eta K * \bar{\rho}$, and $F\eta$ are all well-defined under Assumption **(A4)**.*

(3) *The noise $\mathcal{M}$ is equivalent to be characterized as: for each $\varphi \in C^\infty$, $\mathcal{M}(\varphi)$ is a continuous $(\mathcal{F}_t)$-adapted martingale with quadratic variation given by*

$$\mathbb{E}\left[|\mathcal{M}_t(\varphi)|^2\right] = 2\sigma \int_0^t \langle |\nabla \varphi|^2, \bar{\rho}_r \rangle \mathrm{d}r.$$

Similarly, when $\sigma = 0$, the equation (1.5) actually becomes a deterministic PDE. We define solutions to this first order PDE as follows.

Definition 1.3. *Given that $\sigma = 0$, we call η a solution to the PDE (1.5) with random initial data η_0, if*

(1) *$\eta \in L^2([0,T], H^{-\alpha}) \cap C([0,T], H^{-\alpha-2})$ for every $\alpha > d/2$ almost surely.*

(2) *For each $\varphi \in C^\infty(\mathbb{T}^d)$ and $t \in [0,T]$, it holds that*

$$\langle \eta_t, \varphi \rangle = \langle \eta_0, \varphi \rangle + \int_0^t \langle \nabla \varphi, \bar{\rho} K * \eta + \eta K * \bar{\rho} + F\eta \rangle \mathrm{d}s.$$

Our first main result gives the convergence of fluctuation measures when the diffusion coefficient σ is positive (which may be generalized to the case with a non-degenerate coefficient matrix though).

Theorem 1.4. *Under the assumptions* **(A1)-(A4)**, *the sequence η^N defined in* (1.3) *converges in distribution to η in the space $L^2([0,T], H^{-\alpha}) \cap C([0,T], H^{-\alpha-2})$ for every $\alpha > d/2$, where η is the unique martingale solution to the SPDE* (1.5).

The proof of Theorem 1.4 will be given in Section 3.2.

It is worth emphasizing that the condition $\alpha > d/2$ is optimal due to the optimal regularity of η established in Proposition 3.18. Since the driven noise of equation (1.5) is very rough, so are the solutions. In Section 3.3, we rewrite the equation (1.5) in the mild form and study the regularity of the stochastic part by Kolmogorov's theorem. Using the Schauder estimate, we obtain in Proposition 3.18 the optimal regularity of η given by $C([0,T], C^{-\alpha})$ $\mathbb{P}$-a.s. for every $\alpha > d/2$.

Comparing to the previous result by Fernandez and Méléard [FM97], Theorem 1.4 requires less regularity of the kernel but more regularity of the solution to the mean-field equation, which eventually would lead to a more restrictive condition on the initial value $\bar{\rho}(0)$. The extra assumption on the mean-field equation is indeed fairly reasonable, moreover, by regularity analysis, the condition $\beta > d/2$ in **(A4)** is optimal on the scale of Hölder spaces.

Furthermore, η is characterized as a solution to the following generalized Ornstein-Uhlenbeck process in the weak formulation:

$$\langle \eta_t, \varphi \rangle = \langle \eta_0, Q_{0,t}\varphi \rangle + \sqrt{2\sigma} \int_0^t \int_{\mathbb{T}^d} (\nabla Q_{s,t}\varphi) \sqrt{\bar{\rho}_s}\, \xi(\mathrm{d}x, \mathrm{d}s), \tag{1.6}$$

for each $\varphi \in C^\infty$. Here the time evolution operators $\{Q_{s,t}\}_{0 \leqslant s \leqslant t \leqslant T}$ is defined for each $t \in [0,T]$ and $\varphi \in C^\infty$,

$$Q_{\cdot,t}\varphi := f(\cdot), \tag{1.7}$$

with

(1) $f \in L^2([0,t], H^{\beta+2}) \cap C([0,t], H^{\beta+1})$ with $\partial_t f \in L^2([0,t], H^\beta)$ for $\beta > d/2$.
(2) f is the unique solution with terminal value φ to the following backward equation

$$f_s = \varphi + \sigma \int_s^t \Delta f_r \mathrm{d}r + \int_s^t \Big[K * \bar{\rho}_r \cdot \nabla f_r + K(-\cdot) * (\nabla f_r \bar{\rho}_r) + F \nabla f_r \Big] \mathrm{d}r, \quad s \in [0,t],$$

where $K(-\cdot) * g(x) := \int K(y-x)g(y)\mathrm{d}y$ and we use this convention throughout the article. For the definition of $\{Q_{s,t}\}$ we refer to Section 3.4 for more details. The formulation (1.6) gives rise to the Gaussianity of the limit of fluctuation measures. We state the result as follows and we give the proof in Section 3.4.

Proposition 1.5. *Under the assumptions* **(A1)-(A4)**, *for the η obtained in Theorem 1.4, assume in addition that $\bar{\rho} \in C([0,T], C^{\beta+1}(\mathbb{T}^d))$, $F \in C^{\beta+1}(\mathbb{T}^d)$ with $\beta > d/2$, and η_0 in* **(A1)** *is characterized by*

$$\langle \eta_0, \varphi \rangle \sim \mathcal{N}(0, \langle \varphi^2, \bar{\rho}_0 \rangle - \langle \varphi, \bar{\rho}_0 \rangle^2), \quad \varphi \in C^\infty(\mathbb{T}^d).$$

Then it holds for each test function $\varphi \in C^\infty$ and $t \in [0,T]$ that

$$\langle \eta_t, \varphi \rangle \sim \mathcal{N}\Big(0, \langle |Q_{0,t}\varphi|^2, \bar{\rho}_0 \rangle - \langle Q_{0,t}\varphi, \bar{\rho}_0 \rangle^2 + 2\sigma \int_0^t \langle |\nabla Q_{s,t}\varphi|^2, \bar{\rho}_s \rangle \mathrm{d}s \Big).$$

We now focus on the fluctuation problem for the case with vanishing diffusion. In contrast to the non-degenerate case with $\sigma > 0$, due to the vanishing diffusion, the limit equation becomes a deterministic PDE but with random initial data. We then analyze the limit equation with the method of characteristics, and obtain the following result in Section 4.

Theorem 1.6. *Under the assumptions* **(A1)-(A3)** *and* **(A5)***, assume further that η_0 in* **(A1)** *is characterized by*

$$\langle \eta_0, \varphi \rangle \sim \mathcal{N}(0, \langle \varphi^2, \bar{\rho}_0 \rangle - \langle \varphi, \bar{\rho}_0 \rangle^2), \quad \varphi \in C^\infty(\mathbb{T}^d).$$

Let η be the unique solution to (1.5) *with $\sigma = 0$ on the same stochastic basis with the particle system* (1.1)*. Then the sequence η^N defined in* (1.3) *converges in probability to η in the space $L^2([0,T], H^{-\alpha}) \cap C([0,T], H^{-\alpha-2})$ for every $\alpha > d/2$. Furthermore, η satisfies*

$$\langle \eta_t, \varphi \rangle = \langle \eta_0, Q_{0,t}\varphi \rangle \sim \mathcal{N}\left(0, \langle |Q_{0,t}\varphi|^2, \bar{\rho}_0 \rangle - \langle Q_{0,t}\varphi, \bar{\rho}_0 \rangle^2 \right),$$

for each test function φ and $t \in [0,T]$. Here the time evolution operator $\{Q_{0,t}\}_{0 \leqslant t \leqslant T}$ is given by (1.7) *with $\sigma = 0$.*

Our main results validate that the relative entropy bound $\sup_{t \in [0,T]} \sup_N H(\rho_N | \bar{\rho}_N) \lesssim 1$ which has been established by Jabin and Wang in [JW18] is actually optimal. But the convergence rate for the marginal distributions $\|\rho_{N,k} - \bar{\rho}^{\otimes k}\|_{L^\infty([0,T],L^1)} \lesssim C_T/\sqrt{N}$ is less optimal possibly due to the naive application of the CKP inequality as in (1.4). Notice that very recently Lacker [Lac21] obtains a sharp estimate for marginal distributions, which is $\|\rho_{N,k} - \bar{\rho}^{\otimes k}\|_{L^\infty([0,T],L^1)} \lesssim C_T/N$ by local relative entropy analysis of the BBGKY hierarchy, but under stronger assumptions $H(\rho_{N,k}(0)|\bar{\rho}_0^{\otimes k}) \lesssim k^2/N^2$ as well. Even though it is well-known that the convergence of empirical measures and the k-marginal distributions are equivalent in the qualitative sense for instance in [Szn91], their quantitative behaviors can be quite complicated when it comes to the order of N. See some related discussions in [Lac21, MM13, HM14, MMW15].

As an guiding example for our main results in Theorem 1.4 and Theorem 1.6, we consider the famous vortex model for approximating the 2D Navier-Stokes equation in the vorticity formulation when $\sigma > 0$ and also the 2D Euler equation when $\sigma = 0$. More precisely, given a sequence of i.i.d. initial random variables $\{X_i(0)\}_{i \in \mathbb{N}}$ with a common probability density function $\bar{\rho}_0$ on $\mathbb{T}^2$, and consider the particle system

$$\mathrm{d}X_i = \frac{1}{N} \sum_{j \neq i} K(X_i - X_j)\mathrm{d}t + \sqrt{2\sigma}\mathrm{d}B_t^i, \quad i = 1, 2, \cdots, N, \tag{1.8}$$

with the Biot–Savart law $K : \mathbb{T}^2 \to \mathbb{R}^2$ defined by

$$K = \nabla^\perp G = (-\partial_2 G, \partial_1 G) \tag{1.9}$$

where G is the Green function of the Laplacian on the torus $\mathbb{T}^2$ with mean 0. Note in particular that

$$K(x) = \frac{1}{2\pi} \frac{x^\perp}{|x|^2} + K_0(x),$$

where $x^\perp = (x_1, x_2)^\perp = (-x_2, x_1) \in \mathbb{R}^2$ and K_0 is a smooth correction to periodize K on the torus $\mathbb{T}^2$. Obviously the Biot-Savart kernel K satisfies our assumption **(A2)**.

One major corollary of our main results Theorem 1.4 and Theorem 1.6 is the following result.

Theorem 1.7. *If $\bar{\rho}_0 \in C^3(\mathbb{T}^2)$ when $\sigma > 0$ and $\bar{\rho}_0 \in C^4(\mathbb{T}^2)$ when $\sigma = 0$, and $\inf \bar{\rho}_0 > 0$ for both cases, then the sequence of fluctuation measures $\{\eta^N\}_{N \in \mathbb{N}}$ associated with* (1.8) *converges in distribution to η in the space $L^2([0,T], H^{-\alpha}) \cap C([0,T], H^{-\alpha-2})$ for every $\alpha > 1$. Here η is a generalized Ornstein-Uhlenbeck process solving the equation* (1.5) *with K given by* (1.9) *and $F = 0$. Moreover, $\langle \eta, \varphi \rangle$ is a centered continuous Gaussian process with covariance*

$$\langle |Q_{0,t}\varphi|^2, \bar{\rho}_0 \rangle - \langle Q_{0,t}\varphi, \bar{\rho}_0 \rangle^2 + 2\sigma \int_0^t \langle |\nabla Q_{s,t}\varphi|^2, \bar{\rho}_s \rangle \mathrm{d}s,$$

where $\{Q_{s,t}\}$ is introduced in (1.7) *with $F = 0$ and $\bar{\rho}$ is the solution to the vorticity formulation of 2D imcompressible Navier-Stokes equation when $\sigma > 0$ and 2D Eurler equation when $\sigma = 0$.*

The point vortex approximation towards 2D Navier-Stokes/Euler equation arouses lots of interests since 1980s. The well-posedness of the point vortex model (1.8) was established in [Osa85, MP12, Tak85, FM07]. The main part is to show that $X_i(t) \neq X_j(t)$ for all $t \in [0, T]$ and $i \neq j$ almost surely, thus the singularity of the kernel will not be visited almost surely. The routine method for instance in [Tak85] is based on estimating the quantity $\sum_{i \neq j} G(|X_i - X_j|)$, where G is the Green function. Using the fact $\nabla G \cdot \nabla^\perp G = 0$ and by regularization in the intermediate step, it can be shown that $\sum_{i \neq j} G(|X_i - X_j|)$ is finite almost surely for all $t \in [0, T]$. In [MP12] by Marchioro and Pulvirenti and [FM07] by Fontbona and Martinez, the well-posedness of point vortex model with more general circulations/intensities was established by estimating the displacements of particles. Osada in [Osa85] obtained the same result by an analytic approach, which depends on Gaussian upper and lower bounds for the fundamental solution and the result from [Kan67].

Osada [Osa86] firstly obtained a propogation of chaos result for (1.8) with bounded initial distribution and large viscosity. More recently, Fournier, Hauray, and Mischler [FHM14] obtained entropic propagation of chaos by the compactness argument and their result applies to all viscosity, as long as it is positive, and all initial distributions with finite k-moment ($k > 0$) and finite Boltzmann entropy. A quantitative estimate of propagation of chaos has been established by Jabin and Wang in [JW18] by evolving the relative entropy between the joint distribution of (1.8) and the tensorized law at the limit. Note in particular that [JW18] provided the uniform relative entropy bound as in **(A3)** for all the kernels satisfying **(A2)**, including the Biot-Savart law.

To the authors' knowledge, Theorem 1.7 is the first result on the fluctuation problems for the 2D Navier-Stokes/Euler equation.

1.3. **Related literatures.** Mean field limit and propagation of chaos for the 1st order system given in our canonical form (1.1) have been extensively studied over the last decade. The basic idea of deriving some effective PDE describing the large scale behaviour of interacting particle systems dates back to Maxwell and Boltzmann. But in our setting, the very first mathematical investigation can be traced back to McKean in [MJ]. See also the classical mean field limit from Newton dynamics towards Vlasov Kinetic PDEs in [Dob79, BH77, JH15, Laz16] and the review [Jab14]. Recently much progress has been made in the mean field limit for systems as (1.1) with singular interaction kernels, including those results focusing on the vortex model [Osa86, FHM14], Dyson Brownian motions [BO19, SYY20, LLX20] and very recently quantitative convergence results on general singular kernels for example as in [JW18, BJW20] and [Ser20b, Due16, Ros20, NRS21]. See also the references therein for more complete development on the mean field limit.

However, the study of central limit theory for the system (1.1), in particular for those with singular interactions, is quite limited, due to the lack of proper mathematical tools. The fluctuation problem around a limiting PDE was popularized for the Boltzmann equation in 1970-1980s for instance in [McK75, Tan82, Tan83, Uch83], but those results focus more on the jump-type particle systems. We also refer to [BGSRS20] for the recent breakthrough on the deviation of the hard sphere dynamics from the kinetic Boltzmann equation. For the fluctuations of interacting diffusions, which is the focus of our article, to the best of the authors' knowledge, one of the earliest results is due to Itô [Itô83] , where he showed that for the system of 1D independent and identically distributed Brownian motions, the limit of the corresponding fluctuations is a Gaussian process. In the literature, there are mainly two type results for the fluctuations of interacting processes, either in the path space or in the time marginals. This article focuses on the later one for bounded kernels and some singular ones. When studying fluctuations in the path space, one treats processes $\{X_i\}$ as random variables valued in some functional space, for instance the fluctuation measures may be defined as $\sqrt{N}\left(\frac{1}{N}\sum_{i=1}^{N}\delta_{X_i} - \mathcal{L}(X)\right)$, with the process $X \in C([0, T], \mathbb{R}^d)$ solves the nonlinear stochastic differential equation

$$X(t) = X(0) + \int_0^t \int_{\mathbb{R}^d} K(X(s) - x) \mathrm{d}\mu_s(x) + \sqrt{2\sigma} B_t, \quad \text{with } \mu_s = \mathcal{L}(X(s)).$$

To the best of our knowledge, the fluctuation in path space type result was firstly obtained by Tanaka and Hitsuda [TH81] and by Tanaka [Tan84] for interacting diffusions. They proved that the fluctuation measures on the path space converges to a Gaussian random field when the interacting kernels are bounded and Lipschitz continuous on $\mathbb{R}^1$, with respect to differentiable test functions on the path space. Later Sznitmann [Szn84] removed the differentiability condition on test functions and generalized the result to $\mathbb{R}^d$, using Girsanov's formula and the method of U-statistics.

The article by Fernandez and Méléard [FM97] is probably the one closest to our article when it comes to the basic setting, where they studied interacting diffusions with regular enough coefficients, using the so-called Hilbertian approach. Their result cannot cover kernels which are only bounded or even singular. The systems they consider are on the whole space and allow multiplicative independent noises. It is worth emphasizing that the Hilbertian approach introduced in [FM97] has been amplified to study various interacting models, see [JM98, Che17, CF16, LS16] etc. The Hilbertian approach is based on the martingale method (as used in this article and many other stochastic problems), coupling method, and analysis in negative weighted Sobolev spaces. The coupling method, which is based on directly comparing the $N-$ particle system (1.1) and $N-$copies of the limit McKean-Vlasov equation, is also widely used in classical propagation of chaos result [Szn91], but usually requires strong assumptions on the interacting kernels and diffusion coefficients. In contrast, our new method enables us to obtain uniform estimates and hence convergence results through directly comparing the Liouville equation and the limit mean-field equation.

We also quickly review some related central limit theory result for general interacting particle systems. In a classical work [BH77] by Braun and Hepp, the authors established the stability of characteristic flow in the phase space $\mathbb{R}^{2d}$ with respect to the initial measure and thus established the mean field limit for Newton dynamics with regular interactions towards the Vlasov kinetic PDE. Furthermore, the authors proved that the limiting behavior of normalized fluctuations around the mean-field characteristics is a Gaussian process and a precise SDE governing this limit was also presented. See its recent generalization by Lancellotti [Lan09]. Budhiraja and Wu [BW16] studied some general interacting systems with possible common factors, which do not necessarily have the exchangability property as usual. Their result is in the flavor of fluctuation in the path space and its proof follows the strategy by Sznitmann [Szn84], i.e. using Girsanov transform and U-statistics. Furthermore, Kurtz and Xiong [KX04] studied some interacting particle system from filtering problems. Those SDE's are driven by common noise. The fluctuation result is similar to the one driven by independent noises, but the limiting fluctuation is not Gaussian in general.

For particle systems in the stationary state with possible singular interaction kernels, there are also many results in the flavor of central limit theory. We only refer to a few results and readers can find more in reference therein. Fluctuations for point vortices charged by canonical Gibbs ensembles with the limits given by the so-called energy-enstrophy Gaussian random distributions has been studied by Bodineau and Guionnet [BG99] and recently by Grotto and Romito [GR20]. Those results can be regarded as stationary counterparts of our main theorem in the 2D Euler setting. See also a recent generalization [GR21] for more singular point vortex model leading to generalized 2D Euler equation but also in the stationary setting. Moreover, Leblé and Serfaty [LS18] and Serfaty [Ser20a] considered the fluctuation of Coulomb gas on dimension 2 and 3, where the joint distribution of N-particle is given by the following Gibbs measure

$$\mathrm{d}\mathbb{P}_{N,\beta} = \frac{1}{Z_{N,\beta}} e^{-\frac{\beta}{2} H_N(X_N)},$$

where $Z_{N,\beta}$ is the partition function, β is the temperature, and H_N is the energy including interacting and confining potentials. Now the fluctuation measure, defined as $\sum_{i=1}^N \delta_{x_i} - N\mu_0$, where μ_0 is the equilibrium measure, is shown to converge to a Gaussian free field by using the Laplace transform and many delicate analysis. In a similar context as above, the large deviation principle for the empirical measure charged by a Gibbs distribution with possible singular Hamiltonian has been obtained by Liu and Wu in [LW20], even though its dynamical counterpart is still missing and believed to be

challenging. Moreover, see for instance [PAT07] on some study the fluctuations of eigenvalues of random matrices and a particular case when the eigenvalues are given by Dyson Brownian motions investigated in Theorem 4.3.20 [AGZ10] where the fluctuations of moments, when properly normalized, converge to Gaussian processes.

1.4. **Methodology and difficulties.** The main result (Theorem 1.4) follows by the martingale approach, which has also been used to study the fluctuation problem of interacting diffusions with regular kernels as in [Mél96, FM97]. The proof consists of three steps: tightness, identifying the limits of converging subsequences, and well-posedness of the SPDE (1.5). By Itô's formula, we have

$$d\langle \eta_t^N, \varphi \rangle = \langle \sigma \Delta \varphi, \eta_t^N \rangle dt + \mathcal{K}_t^N(\varphi) dt + \langle \nabla \varphi, F \eta_t^N \rangle dt + \frac{\sqrt{2\sigma_N}}{\sqrt{N}} \sum_{i=1}^{N} \nabla \varphi(X_i) dB_t^i$$
$$+ \sqrt{N}(\sigma_N - \sigma)\langle \Delta \varphi, \mu_N(t) \rangle dt, \tag{1.10}$$

$\mathbb{P}$-a.s. for each $\varphi \in C^\infty(\mathbb{T}^d)$. Here the interacting term $\mathcal{K}_t^N : C^\infty(\mathbb{T}^d) \to \mathbb{R}$ is defined by

$$\mathcal{K}_t^N(\varphi) = \sqrt{N}\langle \nabla \varphi, K * \mu_N(t)\mu_N(t) \rangle - \sqrt{N}\langle \nabla \varphi, \bar{\rho}K * \bar{\rho} \rangle. \tag{1.11}$$

To show the tightness of η^N, we need to derive some uniform estimates for η^N in (1.10). However, due to the singularity of kernels K in Assumption **(A2)**, it seems challenging to directly obtain uniform estimates for terms involving η^N in the negative Sobolev spaces. In fact, the optimal regularity for the limit η obtained in Section 3.3 is in $C_T C^{-\alpha}$ with $\alpha > d/2$. It is natural to consider the energy estimate for η^N in $H^{-\alpha}$ using (1.10). For the purpose of illustration, let us assume that $\sigma_N = \sigma \equiv 0$, and the exterior force $F = 0$ as well, so we can rewrite (1.10) as the following form

$$\partial_t \eta^N + \mathrm{div}(\mu_N K * \eta^N) + \mathrm{div}(\eta^N K * \bar{\rho}) = 0.$$

To control nonlinear terms appearing in the time evolution $\frac{d}{dt}\langle \eta^N, \eta^N \rangle_{H^{-\alpha}}$, such as $\langle \nabla \eta^N, K * \mu_N \eta^N \rangle_{H^{-\alpha}}$, we need $K \in C^\beta$ with $\beta > d/2$ by multiplicative inequality in Appendix A, which is much more demanding than the assumptions we made on our kernels K.

We overcome this difficulty caused by the singularity of interaction kernels by using the Donsker-Varadhan variational formula [DE11, Proposition 4.5.1] (see (2.1) below) and two large deviation type estimates, one is from [JW18, Theorem 4] and the other is our contribution (see Lemma 2.3). More precisely, now the uniform estimate of fluctuation measures can be controlled by two terms, one is the relative entropy $H(\rho_N|\bar{\rho}_N)$ and the other is some exponential integrals with a tensorized reference measure $\bar{\rho}_N = \bar{\rho}^{\otimes N}$ (see (2.2)). On one hand, the uniform bound on $H(\rho_N|\bar{\rho}_N)$, as summarized in Assumption **(A3)**, has already been established by Jabin and Wang in [JW18] for a large family of interaction kernels, in particular including those specified in **(A2)**. On the other hand, exploiting cancellation properties from the interaction terms for instance $\mathcal{K}_t^N$ would enable us to obtain a uniform bound of the exponential integrals (see Lemma 2.1 and Lemma 2.3 for details). This large deviation type estimate enables the authors of [JW18] to conclude quantitative estimates of propagation of chaos and also serves a major technical contribution in our proof. See Remark 2.5 for further comments about the exponential integrals in terms of U-statistics.

Recall the decomposition (1.10), we also need to estimate the martingale part and show its convergence as well. We shall find a pathwise realization $\mathcal{M}^N$ of the martingale part (see Appendix B) and then establish its tightness. The tightness of laws of fluctuation measures then follows by applying Arzela-Ascoli theorem.

When characterizing the limit of a converging subsequence, the difficulty still comes from the singularity of kernels. For the illustration example ($\sigma_N \equiv 0$ and $F = 0$), we notice that it has the following representation

$$\partial_t \eta^N + \mathrm{div}(\eta^N K * \bar{\rho}) + \mathrm{div}(\bar{\rho} K * \bar{\eta}^N) + \frac{1}{\sqrt{N}} \mathrm{div}(\eta^N K * \eta^N) = 0, \tag{1.12}$$

More precisely, the convergence of the interaction term $\mathcal{K}_t^N(\varphi)$ cannot be directly deduced from convergence of μ_N and η^N. We notice that the interaction term can be splitted into two terms. One term is a continuous function of η^N, or more precisely $\bar{\rho}K * \eta^N + \eta^N K * \bar{\rho}$, which definitely converges as N goes to infinity. The other term is of the form $\frac{1}{\sqrt{N}}\eta^N K * \eta^N$, which is not easy to handle directly since the formal limit $K * \eta\eta$ is not well-defined (see Lemma A.2) in the classical sense due to the singularity of K. Instead, we obtain a uniform bound of this singular term by using the variational formula trick again (see Lemma 2.9 below). The remaining part for identifying limits is classical.

The last step to Theorem 1.4 is the uniqueness of martingale solutions to the SPDE (1.5), which follows by pathwise uniqueness (see Lemma 3.11) and Yamada-Watanabe theorem. Proposition 1.5 is obtained by solving the dual backward equation of (1.5) without noises, which gives the Gaussianity of the limit process of fluctuation measures.

For the case with vanishing diffusion (which includes the purely deterministic dynamics with $\sigma_N \equiv 0$), the only difference is on the well-posedness of the limit equation (1.5), which is a first order PDE. The well-posedness follows from the method of characteristics. Since now the limit equation is deterministic, by a useful lemma in [GK96] by Gyöngy and Krylov (see Lemma 4.3 below) we obatin the convergence in probability of the fluctuation measures.

1.5. **Notations.** Throughout the paper, we use the notation $a \lesssim b$ if there exists a universal constant $C > 0$ such that $a \leqslant Cb$. We shall use $\{e_k\}_{k \in \mathbb{Z}^d}$ to represent the Fourier basis on $\mathbb{T}^d$ or $e_k(x) = e^{\sqrt{-1}k \cdot x}$. For simplicity, we define $\langle k \rangle := \sqrt{1 + |k|^2}$.

We will mostly work on Sobolev spaces, Besov spaces, and the space of k-differentiable functions. The norm of Sobolev space $H^\alpha(\mathbb{T}^d)$, $\alpha \in \mathbb{R}$, is defined by

$$\|f\|_{H^\alpha}^2 := \sum_{k \in \mathbb{Z}^d} \langle k \rangle^{2\alpha} |\langle f, e_k \rangle|^2,$$

with the inner product $\langle \cdot, \cdot \rangle_{H^\alpha}$. Moreover, we also use the bracket $\langle \cdot, \cdot \rangle$ to denote integrals when the space and underlying measure are clear from the context. The precise definition and some basic properties of Besov spaces on torus $B_{p,q}^\alpha(\mathbb{T}^d)$ with $\alpha \in \mathbb{R}$ and $1 \leqslant p, q \leqslant \infty$, will be given in the Appendix A for completeness. We remark that $B_{2,2}^\alpha(\mathbb{T}^d)$ coincides with Sobolev space $H^\alpha(\mathbb{T}^d)$. We say $f \in C^\alpha(\mathbb{T}^d)$, $\alpha \in \mathbb{N}$, if f is α-times differentiable. For $\alpha \in \mathbb{R} \setminus \mathbb{N}$, the $C^\alpha(\mathbb{T}^d)$ is given by $C^\alpha(\mathbb{T}^d) = B_{\infty,\infty}^\alpha(\mathbb{T}^d)$. We will often write $\| \cdot \|_{C^\alpha}$ instead of $\| \cdot \|_{B_{\infty,\infty}^\alpha}$. In the case $\alpha \in \mathbb{R}^+ \setminus \mathbb{N}$, $C^\alpha(\mathbb{T}^d)$ coincides with the usual Hölder space. We use $C^\infty(\mathbb{T}^d)$ to denote the space of infinitely differentiable functions on $\mathbb{T}^d$, $\mathcal{S}(\mathbb{R}^d)$ to denote the class of Schwartz functions on $\mathbb{R}^d$ and $\mathcal{S}'(\mathbb{T}^d)$ to denote the space of tempered distributions. Given a Banach space E with a norm $\| \cdot \|_E$ and $T > 0$, we write $C_T E = C([0, T]; E)$ for the space of continuous functions from $[0, T]$ to E, equipped with the supremum norm $\|f\|_{C_T E} = \sup_{t \in [0,T]} \|f(t)\|_E$. For $p \in [1, \infty]$ we write $L_T^p E = L^p([0, T]; E)$ for the space of L^p-integrable functions from $[0, T]$ to E, equipped with the usual L^p-norm.

For simplicity, we may omit the underlying space $\mathbb{T}^d$ without causing confusions.

1.6. **Structure of the paper.** This paper is organized as follows. Section 2 is devoted to obtaining three main estimates which are based on the variational formula and the large deviation type, including uniform estimates on terms related to η^N, $\mathcal{K}^N$, and a singular term derived from $\mathcal{K}^N(\varphi)$. The critical part is to establish some uniform in N estimate of some partition functions. The proof of Theorem 1.4 and Proposition 1.5 is completed in Section 3. First, in Section 3.1, we obtain tightness of the laws of $\{\eta^N\}$ in the space $C([0, T], H^{-\alpha})$ for every $\alpha > d/2 + 2$, meanwhile we prove tightness of laws for the pathwise realizations $\{\mathcal{M}^N\}$ of the martingale part in (1.10). Then we identify the limits of converging (in distribution) subsequences of $\{\eta^N\}$ as a martingale solution to the SPDE (1.5) and finish the proof of Theorem 1.4 in Section 3.2. The optimal regularity of solutions to the SPDE (1.5) is shown in Section 3.3. Lastly, we prove Proposition 1.5 in Section 3.4.

Section 4 is concerned with the case with vanishing diffusion, where we give the proof of Theorem 1.6. Section 5 focuses on some examples which fulfill assumptions **(A1)-(A5)**, including the point

vortex model approximating the vorticity formulations of the 2D Navier-Stokes/Euler equation on the torus.

Finally in Appendix A, we collect the notations and lemmas about Besov spaces used throughout the paper for completeness. In Appendix B we give the proof of Lemma 3.1, which shows existence of pathwise realizations.

2. Large Deviation Type Estimates

This section collects uniform estimates on $\mu_N - \bar{\rho}$, the interaction term $\mathcal{K}^N$, and a singular term derived from $\mathcal{K}^N(\varphi)$, where $\mathcal{K}^N$ is defined in (1.11). These estimates shall play crucial roles in obtaining tightness and identifying the limit in Section 3. Indeed, proving the uniform estimates is the main difficulty and technical contribution of this article. Surprisingly, this type estimate, which has been shown to be very useful for many purposes, can be actually obtained through a simple unified idea. The quantity we want to bound can be put in the integral form $\int \Phi \rho_N$, where Φ is a nonnegative function on $\mathbb{T}^{dN}$. Applying the famous variational formula from [DE11, Proposition 4.5.1], that is

$$\log \int_{\mathbb{T}^{dN}} \bar{\rho}_N e^{\Phi} \mathrm{d}X^N = \sup_{\nu \in \mathcal{P}(\mathbb{T}^{dN}), H(\nu|\bar{\rho}_N) < \infty} \left\{ \int_{\mathbb{T}^{dN}} \Phi \mathrm{d}\nu - H(\nu|\bar{\rho}_N) \right\}, \quad \forall \Phi \geqslant 0, \qquad (2.1)$$

with $X^N := (x_1, .., x_N)$, $\mathcal{P}(\mathbb{T}^{dN})$ the probability measures on $\mathbb{T}^{dN}$, one can easily control $\int \Phi \rho_N$ as follows

$$\int_{\mathbb{T}^{dN}} \Phi \rho_N \mathrm{d}X^N \leqslant \frac{1}{\kappa N} \left(H(\rho_N|\bar{\rho}_N) + \log \int_{\mathbb{T}^{dN}} \bar{\rho}_N e^{\kappa N \Phi} \mathrm{d}X^N \right), \qquad (2.2)$$

for any $\kappa > 0$, simply noticing that ρ_N plays the role of ν and replacing Φ with $\kappa N \Phi$. See also a direct proof of this inequality (2.2) in [JW18, Lemma 1] . As we will see in Lemma 3.3, the extra factor $\frac{1}{N}$ is essential to obtain uniform estimate for fluctuations $\eta^N = \sqrt{N}(\mu_N - \bar{\rho})$, but it comes with a cost that we have to bound the exponential integral $\int \bar{\rho}_N \exp(\kappa N \Phi)$ uniformly in N. Controlling such exponential integrals will be achieved in Section 2.1, then the uniform estimates will be stated and proved in Section 2.2.

2.1. **Large deviation type estimates.** As we mentioned before, the major difficulty of our main estimates is to bound some exponential integrals, which can be understood as some proper partition functions. To prove Lemma 2.6 below, the following result from Jabin and Wang [JW18] is crucial, and we adapt it a bit below for convenience.

Lemma 2.1 (Jabin and Wang [JW18, Theorem 4]). *For any probability measure $\bar{\rho}$ on $\mathbb{T}^d$, and any $\phi(x,y) \in L^{\infty}(\mathbb{T}^{2d})$ with*

$$\gamma := C\|\phi\|_{L^{\infty}}^2 < 1,$$

where C is a universal constant. Assume that ϕ satisfies the following cancellations

$$\int_{\mathbb{T}^d} \phi(x,y)\bar{\rho}(x)\mathrm{d}x = 0 \quad \forall y, \qquad \int_{\mathbb{T}^d} \phi(x,y)\bar{\rho}(y)\mathrm{d}y = 0 \quad \forall x.$$

Then

$$\sup_{N \geqslant 2} \int_{\mathbb{T}^{dN}} \bar{\rho}_N \exp\left(N\langle \phi, \mu_N \otimes \mu_N \rangle\right) \mathrm{d}X^N \leqslant \frac{2}{1-\gamma} < \infty,$$

where $\mu_N = \frac{1}{N}\sum_{i=1}^{N} \delta_{x_i}$, $X^N := (x_1, .., x_N) \in \mathbb{T}^{dN}$, and $\bar{\rho}_N = \bar{\rho}^{\otimes N}$.

Here we abuse the notation $\bar{\rho}$ since applications below are for the solution $\bar{\rho}$ to (1.2). In this section we also abuse the notations μ_N and X^N, but we shall always point out the dependence on time when we mention the empirical measure and vector associated to the particle system (1.1).

Remark 2.2. *The proof of the above lemma in* [JW18] *relies on the observation that* $e^A \leqslant e^A + e^{-A}$ *and it is then sufficient to control the series*

$$\sum_{k=0}^{\infty} \frac{1}{(2k)!} \int_{\mathbb{T}^{dN}} \bar{\rho}_N A^{2k} \mathrm{d}X^N.$$

Then of course under the same assumptions, we also have

$$\int_{\mathbb{T}^{dN}} \bar{\rho}_N \exp\left(N|\langle \phi, \mu_N \otimes \mu_N \rangle|\right) \mathrm{d}X^N \leqslant \frac{2}{1-\gamma} < \infty.$$

Here adding $|\cdot|$ *in the exponential part will be convenient for proving Lemma* 2.9.

We also need the following novel large deviation type estimate on the uniform in N control of a partition function, and use it to obtain the uniform estimate of the interaction term. The proof below is inspired by [JW18, Theorem 4], using combinatorics techniques and some cancellation properties of functions.

Lemma 2.3. *For any probability measure* $\bar{\rho}$ *on* $\mathbb{T}^d$. *Assume further that functions* $\phi(x, y) \in L^{\infty}(\mathbb{T}^{2d})$ *with* $\|\phi\|_{L^{\infty}}$ *is small enough, and that*

$$\int_{\mathbb{T}^{2d}} \bar{\rho}(x)\bar{\rho}(y)\phi(x, y)\mathrm{d}x\mathrm{d}y = 0. \tag{2.3}$$

Then

$$\int_{\mathbb{T}^{dN}} \bar{\rho}_N \exp\left(N|\langle \phi, \mu_N \otimes \mu_N \rangle|^2\right) \mathrm{d}X^N \leqslant 1 + \frac{\alpha_0}{1-\alpha_0} + \frac{\beta_0}{1-\beta_0},$$

where

$$\alpha_0 := e^9 \|\phi\|_{L^{\infty}}^2 < 1, \quad \beta_0 := 4e\|\phi\|_{L^{\infty}}^2 < 1.$$

Proof. We start with the Taylor expansion:

$$\int_{\mathbb{T}^{dN}} \bar{\rho}_N \exp\left(N|\langle \phi, \mu_N \otimes \mu_N \rangle|^2\right) \mathrm{d}X^N = \sum_{m=0}^{\infty} \frac{1}{m!} \int_{\mathbb{T}^{dN}} \bar{\rho}_N \left(N|\langle \phi, \mu_N \otimes \mu_N \rangle|^2\right)^m \mathrm{d}X^N.$$

For the m-th term, we use $\mu_N = \frac{1}{N}\sum_{i=1}^{N} \delta_{x_i}$ to expand it as

$$\frac{1}{m!} \int_{\mathbb{T}^{dN}} \bar{\rho}_N \left(N|\langle \phi, \mu_N \otimes \mu_N \rangle|^2\right)^m \mathrm{d}X^N = \frac{1}{m!} N^m \int_{\mathbb{T}^{dN}} \bar{\rho}_N \left(\frac{1}{N^2}\sum_{i,j=1}^{N} \phi(x_i, x_j)\right)^{2m} \mathrm{d}X^N$$

$$= \frac{1}{m!} N^{-3m} \sum_{i_1,\dots,i_{2m},j_1,\dots,j_{2m}=1}^{N} \int_{\mathbb{T}^{dN}} \bar{\rho}_N \prod_{\nu=1}^{2m} \phi(x_{i_\nu}, x_{j_\nu}) \mathrm{d}X^N. \tag{2.4}$$

We shall divide the rest proof into two different cases: $4m > N$ and $4m \leqslant N$.

In the case of $4m > N$, the $m-$th term, i.e. (2.4) can be bounded trivially

$$\frac{1}{m!} N^{-3m} \sum_{i_1,\dots,i_{2m},j_1,\dots,j_{2m}=1}^{N} \int_{\mathbb{T}^{dN}} \bar{\rho}_N \prod_{\nu=1}^{2m} \phi(x_{i_\nu}, x_{j_\nu}) \mathrm{d}X^N$$

$$\leqslant \frac{1}{m!} N^{-3m} N^{4m} \|\phi\|_{L^{\infty}}^{2m} \leqslant m^{-\frac{1}{2}} 4^m e^m \|\phi\|_{L^{\infty}}^{2m}.$$

Here we used the following Stirling's formula with $x = m$

$$x! = c_x \sqrt{2\pi x} \left(\frac{x}{e}\right)^x, \tag{2.5}$$

where $1 < c_x < \frac{11}{10}$ and $c_x \to 1$ as $x \to \infty$.

In the case of $4 \leqslant 4m \leqslant N$, we estimate the m-th term via counting how many choices of multi-indices $(i_1, \ldots, i_{2m}, j_1, \ldots, j_{2m})$ that lead to a non-vanishing integral. If there exists a couple (i_q, j_q) such that

$$i_q \neq j_q \text{ and } i_q, j_q \notin \{i_\nu, j_\nu\} \text{ for any } \nu \neq q,$$

then variables x_{i_q} and x_{j_q} enter exactly once in the integration. For simplicity, let $(x_{i_q}, x_{j_q}) = (x_1, x_2)$, then by Fubini and the cancellation rule (2.3) of ϕ,

$$\int_{\mathbb{T}^{dN}} \bar{\rho}_N \prod_{\nu=1}^{2m} \phi(x_{i_\nu}, x_{j_\nu}) \mathrm{d}X^N$$

$$= \int_{\mathbb{T}^{d(N-2)}} \left(\int_{\mathbb{T}^{2d}} \bar{\rho}(x_1)\bar{\rho}(x_2)\phi(x_1, x_2)\mathrm{d}x_1\mathrm{d}x_2 \right) \cdot \left(\prod_{\nu=2}^{2m} \phi(x_{i_\nu}, x_{j_\nu}) \right) \left(\prod_{i \neq 1,2} \bar{\rho}(x_i) \right) \mathrm{d}x_3 \ldots \mathrm{d}x_N$$

$$= 0.$$

In this case, we introduce auxiliary notations:

- l denotes the number of x_{i_ν} or x_{j_ν} which appears exactly once in the integral.
- p denotes the number of x_{i_ν} or x_{j_ν} which appears at least twice in the integral.

A crucial observation is that for multi-indices $(i_1, \ldots, i_{2m}, j_1, \ldots, j_{2m})$ which lead to a non-vanishing integral, these l variables enter in different couples. This gives $0 \leqslant l \leqslant 2m$. We summarize the following relations among $\{l, p, m, N\}$ as

$$4 \leqslant 4m \leqslant N; \quad 0 \leqslant l \leqslant 2m; \quad 1 \leqslant p \leqslant (4m-l)/2$$

For a fixed (l, p), notice that there are $\binom{N}{l}\binom{N-l}{p}$ choices of variables. For each choice of variables, there exists $\binom{2m}{l}2^l$ choices of place to arrange the l unique variables. Lastly, for each arrangement, there are at most $l!p^{4m-l}$ plans where $l!$ is for the l unique variables while p^{4m-l} is for the other p variables.

In conclusion, we have when $4 \leqslant 4m \leqslant N$,

$$\frac{1}{m!} N^{-3m} \sum_{i_1, \ldots, i_{2m}, j_1, \ldots, j_{2m}=1}^{N} \int_{\mathbb{T}^{dN}} \bar{\rho}_N \prod_{\nu=1}^{2m} \phi(x_{i_\nu}, x_{j_\nu}) \mathrm{d}X^N$$

$$\leqslant \frac{1}{m!} N^{-3m} \|\phi\|_{L^\infty}^{2m} \sum_{l=0}^{2m} \sum_{p=1}^{2m-l/2} \binom{N}{l}\binom{N-l}{p}\binom{2m}{l} 2^l \, l! \, p^{4m-l}$$

$$= \|\phi\|_{L^\infty}^{2m} \sum_{l=0}^{2m} \sum_{p=1}^{2m-l/2} \frac{N! N^{-3m}}{(N-p-l)!} \frac{\binom{2m}{l} 2^l \, p^{4m-l}}{m! p!}. \tag{2.6}$$

Applying Stirling's formula (2.5) with $x = m, p$ gives

$$\sum_{l=0}^{2m} \sum_{p=1}^{2m-l/2} \frac{N! N^{-3m}}{(N-p-l)!} \frac{\binom{2m}{l} 2^l p^{4m-l}}{m! p!} \leqslant \sum_{l=0}^{2m} \sum_{p=1}^{2m-l/2} N^{p+l-3m} 2^l e^{p+m} \frac{\binom{2m}{l} p^{4m-l-p}}{m^m}. \tag{2.7}$$

Furthermore, observe that

$$\frac{\binom{2m}{l} p^m}{m^m} \leqslant \frac{2^{2m} \left(2m - \frac{l}{2}\right)^m}{m^m} \leqslant 2^{3m},$$

where we used $2^{2m} = \sum_{l=0}^{2m} \binom{2m}{l}$ and $p \leqslant 2m - \frac{l}{2}$. Taking this estimate into (2.7) yields

$$\sum_{l=0}^{2m} \sum_{p=1}^{2m-l/2} N^{p+l-3m} 2^l e^{p+m} \frac{\binom{2m}{l} p^{4m-l-p}}{m^m} \leqslant \sum_{l=0}^{2m} \sum_{p=1}^{2m-l/2} N^{p+l-3m} 2^{l+3m} e^{p+m} p^{3m-p-l}.$$

Since $p + l - 3m \leqslant 0$ and $p < N$, the above inequality is bounded by e^{9m}. Combining this with (2.6), we find for every $m \in \left[1, \left\lfloor \frac{N}{4} \right\rfloor\right]$,

$$\frac{1}{m!} N^{-3m} \sum_{i_1, \ldots, i_{4m}=1}^{N} \int_{\mathbb{T}^{dN}} \bar{\rho}_N \prod_{\nu=1}^{2m} \phi(x_{i_\nu}, x_{j_\nu}) \mathrm{d}X^N \leqslant \|\phi\|_{L^\infty}^{2m} e^{9m}.$$

Combining the two cases $4 \leqslant 4m \leqslant N$ and $4m > N$, it follows that

$$\int_{\mathbb{T}^{dN}} \bar{\rho}_N \exp\left(N|\langle\phi, \mu_N \otimes \mu_N\rangle|^2\right) \mathrm{d}X^N \leqslant 1 + \sum_{m=1}^{\left\lfloor \frac{N}{4} \right\rfloor} \|\phi\|_{L^\infty}^{2m} e^{9m} + \sum_{\left\lfloor \frac{N}{4} \right\rfloor+1}^{\infty} m^{-\frac{1}{2}} 4^m e^m \|\phi\|_{L^\infty}^{2m}.$$

Recall that

$$\alpha_0 = e^9 \|\phi\|_{L^\infty}^2 < 1, \quad \beta_0 = 4e\|\phi\|_{L^\infty}^2 < 1.$$

The proof is thus completed by noticing that

$$\sum_{m=1}^{\left\lfloor \frac{N}{4} \right\rfloor} \|\phi\|_{L^\infty}^{2m} e^{9m} \leqslant \sum_{m=1}^{\infty} \alpha_0^m = \frac{\alpha_0}{1 - \alpha_0}$$

and

$$\sum_{\left\lfloor \frac{N}{4} \right\rfloor+1}^{\infty} m^{-\frac{1}{2}} 4^m e^m \|\phi\|_{L^\infty}^{2m} \leqslant \sum_{m=1}^{\infty} \beta_0^m = \frac{\beta_0}{1 - \beta_0}.$$

$\square$

Remark 2.4. *Lemma 2.3 and Lemma 2.1 can be generalized in several aspects. Firstly, the space $\mathbb{T}^d$ could be replaced by any measurable spaces. Also, when ϕ is vector-valued, the result still holds with a slight modification in the proof as follows*

$$\prod_{\nu=1}^{2m} \phi(x_{i_\nu}, x_{j_\nu}) \xrightarrow{replaced\ by} \prod_{\nu=1}^{m} \phi(x_{i_\nu}, x_{j_\nu}) \cdot \phi(x_{k_\nu}, x_{l_\nu}).$$

Indeed, given ϕ a vector-valued function, the modification only comes from the expanding

$$|\langle\phi, \mu_N \otimes \mu_N\rangle|^2 = \left| \frac{1}{N^2} \sum_{i,j=1}^{N} \phi(x_i, x_j) \right|^2 = \frac{1}{N^4} \sum_{i,j,k,l=1}^{N} \phi(x_i, x_j) \cdot \phi(x_k, x_l).$$

Remark 2.5. *In Lemma 2.1 and Lemma 2.3, we proved that two exponential integrals which are in the form of $\mathbb{E}e^{NU_N}$ are uniformly bounded with respect to N. In the first case as in [JW18],*

$$U_N^1 = \frac{1}{N^2} \sum_{i,j=1}^{N} \phi(X_i, X_j),$$

while in the second case, $U_N^2 = (U_N^1)^2$, or U_N^2 may be expressed as

$$U_N^2 = \frac{1}{N^4} \sum_{i_1, i_2, i_3, i_4=1}^{N} \phi(X_{i_1}, X_{i_2}) \phi(X_{i_3}, X_{i_4}).$$

Those U_N in both cases are U-statistics, which are symmetric functions of N i.i.d random variables. The degree of U_N is said to be k if U_N is a symmetric version of an interaction function between k variables. So U_N in Lemma 2.1 is of degree 2 and U_N in Lemma 2.3 is of degree 4. Here the cancellation properties actually imply the first order degeneracy of U-statistics, which together with the boundedness condition gives the weak convergence of the law of NU_N^i, $i = 1, 2$. We refer to [Lee19] for more details.

2.2. **Uniform estimates.** Now we are in the position to state and prove the uniform estimates.

The first estimate concerns on the convergence from $\mu_N(t)$ to $\bar{\rho}_t$ in $H^{-\alpha}$, for $\alpha > d/2$.

Lemma 2.6. *For each $\alpha > d/2$, there exists a constant C_α such that, for all $t \in [0, T]$,*

$$\mathbb{E}\|\mu_N(t) - \bar{\rho}_t\|_{H^{-\alpha}}^2 \leqslant \frac{C_\alpha}{N}(H_t(\rho_N|\bar{\rho}_N) + 1),$$

where we recall $\mu_N(t) = \frac{1}{N}\sum_{i=1}^N \delta_{X_i(t)}$ and the expectation is taken according to the joint distribution $\rho_N(t, \cdot)$ of the particle system (1.1) *and $\bar{\rho}_N(t, \cdot) = \bar{\rho}(t)^{\otimes N}$.*

This lemma has a direct consequence. Recall that the fluctuation measure $\eta^N(t) = \sqrt{N}(\mu_N(t) - \bar{\rho}_t)$. Under Assumption **(A3)**, i.e. $\sup_{t\in[0,T]}\sup_N H_t(\rho_N|\bar{\rho}_N) \lesssim 1$, one can then immediately obtain

$$\sup_{t\in[0,T]}\sup_N \mathbb{E}\|\eta^N(t)\|_{H^{-\alpha}}^2 \lesssim 1, \quad \text{for } \alpha > d/2.$$

Proof. Since the Dirac measure belongs to $H^{-\alpha}(\mathbb{T}^d)$ for every $\alpha > d/2$, it follows that $\mu_N(t) - \bar{\rho}_t \in H^{-\alpha}(\mathbb{T}^d)$. Then by (2.2) , we find for any $\kappa > 0$,

$$\begin{aligned}
\mathbb{E}\|\mu_N(t) - \bar{\rho}_t\|_{H^{-\alpha}}^2 &= \int_{\mathbb{T}^{dN}} \|\mu_N - \bar{\rho}_t\|_{H^{-\alpha}}^2 \rho_N(t, X^N)\mathrm{d}X^N \\
&\leqslant \frac{1}{\kappa N}\left(H_t(\rho_N|\bar{\rho}_N) + \log\int_{\mathbb{T}^{dN}}\exp\left(\kappa N\|\mu_N - \bar{\rho}_t\|_{H^{-\alpha}}^2\right)\bar{\rho}_N\mathrm{d}X^N\right).
\end{aligned} \tag{2.8}$$

Recalling $\{e_k\}_{k\in\mathbb{Z}^d}$ is the Fourier basis, and

$$\|\mu_N - \bar{\rho}_t\|_{H^{-\alpha}}^2 = \sum_{k\in\mathbb{Z}^d}\langle k\rangle^{-2\alpha}|\langle e_k, \mu_N - \bar{\rho}_t\rangle|^2.$$

Since the exponential function is convex, using Jensen's inequality gives that

$$\begin{aligned}
\int_{\mathbb{T}^{dN}}\exp\left(\kappa N\|\mu_N - \bar{\rho}_t\|_{H^{-\alpha}}^2\right)\bar{\rho}_N\mathrm{d}X^N &\leqslant \frac{1}{C}\sum_{k\in\mathbb{Z}^d}\langle k\rangle^{-2\alpha}\int_{\mathbb{T}^{dN}}\exp\left(\kappa NC|\langle e_k, \mu_N - \bar{\rho}_t\rangle|^2\right)\bar{\rho}_N\mathrm{d}X^N \\
&\leqslant \sup_{k\in\mathbb{Z}^d}\int_{\mathbb{T}^{dN}}\exp\left(\kappa NC|\langle e_k, \mu_N - \bar{\rho}_t\rangle|^2\right)\bar{\rho}_N\mathrm{d}X^N, \tag{2.9}
\end{aligned}$$

where the constant $C = \sum_{k\in\mathbb{Z}^d}\langle k\rangle^{-2\alpha}$ depends only on α and is finite since $\alpha > d/2$.

We define

$$\phi_1(t, k, x, y) := [e_k(x) - \langle e_k, \bar{\rho}_t\rangle][e_{-k}(y) - \langle e_{-k}, \bar{\rho}_t\rangle],$$

therefore

$$\int_{\mathbb{T}^{dN}}\exp\left(\kappa NC|\langle e_k, \mu_N - \bar{\rho}_t\rangle|^2\right)\bar{\rho}_N\mathrm{d}X^N = \int_{\mathbb{T}^{dN}}\exp\left(\kappa NC\langle\phi_1(t, k, \cdot, \cdot), \mu_N \otimes \mu_N\rangle\right)\bar{\rho}_N\mathrm{d}X^N.$$

Since $\bar{\rho}$ is a probability measure, $\|\phi_1\|_{L^\infty}$ is bounded uniformly in t and k. One can also easily check that

$$\int_{\mathbb{T}^d}\phi_1(t, k, x, y)\bar{\rho}_t(x)\mathrm{d}x = 0 \quad \forall y, \qquad \int_{\mathbb{T}^d}\phi_1(t, k, x, y)\bar{\rho}_t(y)\mathrm{d}y = 0 \quad \forall x.$$

Then by Lemma 2.1 with κ (depending on α) small enough, we deduce that

$$\begin{aligned}
&\sup_N\sup_{k\in\mathbb{Z}^d}\int_{\mathbb{T}^{dN}}\exp\left(\kappa NC|\langle e_k, \mu_N - \bar{\rho}_t\rangle|^2\right)\bar{\rho}_N\mathrm{d}X^N \\
&= \sup_N\sup_{k\in\mathbb{Z}^d}\int_{\mathbb{T}^{dN}}\bar{\rho}_N\exp(\kappa NC\langle\phi_1(t, k, \cdot, \cdot), \mu_N \otimes \mu_N\rangle)\mathrm{d}X^N \\
&= \sup_N\sup_{k\in\mathbb{Z}^d}\int_{\mathbb{T}^{dN}}\bar{\rho}_N\exp(\kappa NC\langle\mathrm{Re}\,\phi_1(t, k, \cdot, \cdot), \mu_N \otimes \mu_N\rangle)\mathrm{d}X^N < \infty, \tag{2.10}
\end{aligned}$$

where the equalities follows by

$$|\langle e_k, \mu_N - \bar{\rho}_t \rangle|^2 = \langle \phi_1(t, k, \cdot, \cdot), \mu_N \otimes \mu_N \rangle \in \mathbb{R}.$$

Combining (2.8)-(2.10) yields

$$\mathbb{E}\|\mu_N(t) - \bar{\rho}_t\|_{H^{-\alpha}}^2 \leqslant \frac{1}{\kappa N}(H_t(\rho_N|\bar{\rho}_N) + C_\alpha),$$

where C_α is a constant depending only on α. We thus arrive at the result. $\qquad\square$

In particular, Lemma 2.6 gives the tightness of laws of $\{\eta^N(0)\}$ on $H^{-\alpha}$ under the condition that $H(\rho_N(0)|\bar{\rho}_N(0))$ is finite, which together with Assumption **(A1)** yields the convergence of $\{\eta^N(0)\}$ in the negative Sobolev spaces.

Corollary 2.7. *For every $\alpha > d/2$, η_0^N converges in distribution to η_0 given by **(A1)** in $H^{-\alpha}$.*

The next lemma concerns on the interaction part in the decomposition (1.10).

Lemma 2.8. *If the kernel K satisfies Assumption **(A2)**, then for each $\alpha > d/2 + 2$, there exists a constant C_α such that, for all $t \in [0, T]$,*

$$\mathbb{E}\|\nabla \cdot [K * \mu_N(t)\mu_N(t) - \bar{\rho}_t K * \bar{\rho}_t]\|_{H^{-\alpha}}^2 \leqslant \frac{C_\alpha}{N}(H_t(\rho_N|\bar{\rho}_N) + 1),$$

where the expectation is taken according to the joint distribution $\rho_N(t, \cdot)$ of the particle system (1.1).

Proof. The proof is similar to Lemma 2.6. First, by (2.2) we find for any $\kappa > 0$,

$$\mathbb{E}\|\nabla \cdot [K * \mu_N(t)\mu_N(t) - \bar{\rho}_t K * \bar{\rho}_t]\|_{H^{-\alpha}}^2$$
$$\leqslant \frac{1}{\kappa N}\left(H_t(\rho_N|\bar{\rho}_N) + \log \int_{\mathbb{T}^{dN}} \bar{\rho}_N \exp\left(\kappa N\|\nabla \cdot [K * \mu_N\mu_N - \bar{\rho}_t K * \bar{\rho}_t]\|_{H^{-\alpha}}^2\right) \mathrm{d}X^N\right). \qquad (2.11)$$

Next, we find that

$$\|\nabla \cdot [K * \mu_N\mu_N - \bar{\rho}_t K * \bar{\rho}_t]\|_{H^{-\alpha}}^2 = \sum_{k \in \mathbb{Z}^d} \langle k \rangle^{-2\alpha} |\langle \nabla e_k, K * \mu_N\mu_N - \bar{\rho}_t K * \bar{\rho}_t \rangle|^2$$
$$\leqslant \sum_{k \in \mathbb{Z}^d \backslash \{0\}} \langle k \rangle^{-2\alpha} |k|^2 |\langle e_k, K * \mu_N\mu_N - \bar{\rho}_t K * \bar{\rho}_t \rangle|^2.$$

For the case $|x|K(x) \in L^\infty$ and $K(x) = -K(-x)$, we do a symmetrization trick. That is, for any $\varphi \in C^\infty(\mathbb{T}^d)$ and a probability measure μ,

$$\int_{\mathbb{T}^d} \varphi(x) K * \mu(x)\mu(\mathrm{d}x) = \int_{\mathbb{T}^{2d}} \varphi(x) K(x - y)\mu^{\otimes 2}(\mathrm{d}x\mathrm{d}y)$$
$$= \frac{1}{2} \int_{\mathbb{T}^{2d}} (\varphi(x) - \varphi(y)) \cdot K(x - y)\mu^{\otimes 2}(\mathrm{d}x\mathrm{d}y).$$

We define that

$$\mathbb{K}_\varphi(x, y) := \frac{1}{2} K(x - y)[\varphi(x) - \varphi(y)], \quad \forall \varphi \in C^\infty(\mathbb{T}^d).$$

Thus in this case, $\|\mathbb{K}_\varphi\|_{L^\infty} \lesssim \|\nabla\varphi\|_{L^\infty}\||x|K\|_{L^\infty}$. Consequently, since

$$\langle e_k, K * \mu_N\mu_N - \bar{\rho}_t K * \bar{\rho}_t \rangle = \langle \mathbb{K}_{e_k}(\cdot, \cdot), \mu_N^{\otimes 2} - \bar{\rho}_t^{\otimes 2} \rangle,$$

and $\|\mathbb{K}_{e_k}\| \lesssim |k|$, one proceeds as

$$\|\nabla \cdot [K * \mu_N\mu_N - \bar{\rho}_t K * \bar{\rho}_t]\|_{H^{-\alpha}}^2 = \sum_{k \in \mathbb{Z}^d \backslash \{0\}} \langle k \rangle^{-2\alpha} |k|^2 |\langle \mathbb{K}_{e_k}, \mu_N \otimes \mu_N - \bar{\rho}_t \otimes \bar{\rho}_t \rangle|^2$$
$$= \sum_{k \in \mathbb{Z}^d \backslash \{0\}} \langle k \rangle^{-2\alpha} |k|^4 |\langle \frac{\mathbb{K}_{e_k}}{|k|}, \mu_N \otimes \mu_N - \bar{\rho}_t \otimes \bar{\rho}_t \rangle|^2,$$

where now $\frac{\mathbb{K}_{e_k}}{|k|}$ is bounded.

For the case that $K \in L^\infty$ but K is not necessarily anti-symmetric, we directly write

$$\langle e_k, K * \mu_N \mu_N - \bar{\rho}_t K * \bar{\rho}_t \rangle = \langle e_k(x)K(x-y), \mu_N^{\otimes 2} - \bar{\rho}_t^{\otimes 2} \rangle.$$

To sum it up, we define $\phi_2 : [0,T] \times \{\mathbb{Z}^d \setminus \{0\}\} \times \mathbb{T}^{2d} \to \mathbb{R}^d$ by

$$\phi_2(t,k,x,y) := \begin{cases} K(x-y)e_k(x) - \langle e_k, \bar{\rho}_t K * \bar{\rho}_t \rangle, & \text{if } K \in L^\infty \\ \frac{\mathbb{K}_{e_k}(x,y)}{|k|} - \langle \frac{\mathbb{K}_{e_k}}{|k|}, \bar{\rho}_t \otimes \bar{\rho}_t \rangle, & \text{if } |x|K(x) \in L^\infty, K(x) = -K(-x). \end{cases}$$

Using Jensen's inequality, for both cases we have

$$\int_{\mathbb{T}^{dN}} \bar{\rho}_N \exp\left(\kappa N \|\nabla \cdot [K * \mu_N \mu_N - \bar{\rho}_t K * \bar{\rho}_t]\|_{H^{-\alpha}}^2\right) \mathrm{d}X^N$$

$$= \int_{\mathbb{T}^{dN}} \bar{\rho}_N \exp\left(\kappa N \sum_{k \in \mathbb{Z}^d \setminus \{0\}} \langle k \rangle^{-2\alpha} |k|^2 |\langle e_k, K * \mu_N \mu_N - \bar{\rho}_t K * \bar{\rho}_t \rangle|^2 \right) \mathrm{d}X^N$$

$$\leqslant \int_{\mathbb{T}^{dN}} \bar{\rho}_N \exp\left(\kappa N \sum_{k \in \mathbb{Z}^d \setminus \{0\}} \langle k \rangle^{-2\alpha+4} |\langle \phi_2(t,k,\cdot,\cdot), \mu_N \otimes \mu_N \rangle|^2 \right) \mathrm{d}X^N$$

$$\leqslant \frac{1}{C} \sum_{k \in \mathbb{Z}^d \setminus \{0\}} \langle k \rangle^{-2\alpha+4} \int_{\mathbb{T}^{dN}} \bar{\rho}_N \exp\left(\kappa N C |\langle \phi_2(t,k,\cdot,\cdot), \mu_N \otimes \mu_N \rangle|^2 \right) \mathrm{d}X^N$$

$$\leqslant \sup_{k \in \mathbb{Z}^d \setminus \{0\}} \int_{\mathbb{T}^{dN}} \bar{\rho}_N \exp\left(\kappa N C |\langle \phi_2(t,k,\cdot,\cdot), \mu_N \otimes \mu_N \rangle|^2 \right) \mathrm{d}X^N, \tag{2.12}$$

where the constant $C := \sum_{k \in \mathbb{Z}^d \setminus 0} \langle k \rangle^{-2\alpha+4}$ depends only on α, and is finite since $\alpha > d/2 + 2$.

Furthermore, since ϕ_2 is complex-valued, we find

$$\sup_N \sup_{k \in \mathbb{Z}^d \setminus \{0\}} \int_{\mathbb{T}^{dN}} \bar{\rho}_N \exp\left(\kappa N C |\langle \phi_2(t,k,\cdot,\cdot), \mu_N \otimes \mu_N \rangle|^2 \right) \mathrm{d}X^N$$

$$\leqslant \frac{1}{2} \sup_N \sup_{k \in \mathbb{Z}^d \setminus \{0\}} \int_{\mathbb{T}^{dN}} \bar{\rho}_N \exp\left(2\kappa N C |\langle \mathrm{Re}\, \phi_2(t,k,\cdot,\cdot), \mu_N \otimes \mu_N \rangle|^2 \right) \mathrm{d}X^N$$

$$+ \frac{1}{2} \sup_N \sup_{k \in \mathbb{Z}^d \setminus \{0\}} \int_{\mathbb{T}^{dN}} \bar{\rho}_N \exp\left(2\kappa N C |\langle \mathrm{Im}\, \phi_2(t,k,\cdot,\cdot), \mu_N \otimes \mu_N \rangle|^2 \right) \mathrm{d}X^N, \tag{2.13}$$

where the inequality follows by Jensen's inequality and the fact that

$$|\langle \phi_2(t,k,\cdot,\cdot), \mu_N \otimes \mu_N \rangle|^2 = |\langle \mathrm{Re}\, \phi_2(t,k,\cdot,\cdot), \mu_N \otimes \mu_N \rangle|^2 + |\langle \mathrm{Im}\, \phi_2(t,k,\cdot,\cdot), \mu_N \otimes \mu_N \rangle|^2.$$

One can easily find that $\|\phi_2\|_{L^\infty}$ is bounded uniformly in (t,k), and satisfies the cancellation

$$\int_{\mathbb{T}^{2d}} \phi_2(t,k,x,y) \bar{\rho}_t(x) \bar{\rho}_t(y) \mathrm{d}x \mathrm{d}y = 0,$$

and so do the real and imaginary part of ϕ_2.

Choosing κ (depending on α) sufficiently small, then we are able to apply Lemma 2.3 to obtain that

$$\sup_N \sup_{k \in \mathbb{Z}^d \setminus 0} \int_{\mathbb{T}^{dN}} \bar{\rho}_N \exp\left(\kappa N C |\langle \phi_2(t,k,\cdot,\cdot), \mu_N \otimes \mu_N \rangle|^2 \right) \mathrm{d}X^N \leqslant C_\alpha,$$

where the universal constant C_α only depends on α. The proof is then completed by combining this with (2.11) and (2.12). $\square$

The last estimate in this section plays a crucial role in identifying the limit in Section 3.2.

Lemma 2.9. *If the kernel K satisfies assumption* **(A2)***, then for each $\varphi \in C^1$, there exists a universal constant C such that, for all $t \in [0,T]$,*

$$\mathbb{E}|\langle \varphi K * (\mu_N(t) - \bar{\rho}_t), \mu_N(t) - \bar{\rho}_t \rangle| \leqslant \frac{C}{N}(H_t(\rho_N|\bar{\rho}_N) + 1),$$

where the expectation is taken according to the joint distribution $\rho_N(t, \cdot)$ of the particle system (1.1).

Proof. We first write the quantity in the following form

$$\mathbb{E}|\langle \varphi K * (\mu_N(t) - \bar{\rho}_t), \mu_N(t) - \bar{\rho}_t \rangle| = \mathbb{E}|\Phi(t, X_t^N)| = \int_{\mathbb{T}^{dN}} |\Phi(t, X^N)| \rho_N \mathrm{d}X^N, \qquad (2.14)$$

where Φ is defined by

$$\Phi(t, X^N) = \langle \varphi K * (\mu_N - \bar{\rho}_t), \mu_N - \bar{\rho}_t \rangle.$$

For the case $K \in L^\infty$, we find

$$\Phi(t, X^N) = \langle \phi_3(t, \cdot, \cdot), \mu_N \otimes \mu_N \rangle,$$

with ϕ_3 defined by

$$\phi_3(t, x, y) := K(x - y)\varphi(x) - \varphi(x)K * \bar{\rho}_t(x) - \langle K(\cdot - y)\varphi, \bar{\rho}_t \rangle + \langle \varphi K * \bar{\rho}_t, \bar{\rho}_t \rangle.$$

For the case $|x|K(x) \in L^\infty$ and $K(x) = -K(-x)$, we do a symmetrization for Φ as in the proof of Lemma 2.8, i.e.

$$\Phi(t, X^N) = \langle \mathbb{K}_\varphi, (\mu_N - \bar{\rho}_t) \otimes \mu_N - \bar{\rho}_t \rangle = \langle \phi_3(t, \cdot, \cdot), \mu_N \otimes \mu_N \rangle,$$

with ϕ_3 defined by

$$\phi(t, x, y) := \mathbb{K}_\varphi(x, y) - \langle \mathbb{K}_\varphi(x, \cdot), \bar{\rho}_t \rangle - \langle \mathbb{K}_\varphi(\cdot, y), \bar{\rho}_t \rangle + \langle \mathbb{K}_\varphi, \bar{\rho}_t \otimes \bar{\rho}_t \rangle.$$

By (2.2), it holds for any $\kappa > 0$ that

$$\int_{\mathbb{T}^{dN}} |\Phi(t, X^N)| \rho_N \mathrm{d}X^N \leqslant \frac{1}{\kappa N} \left(H(\rho_N|\bar{\rho}_N) + \log \int_{\mathbb{T}^{dN}} \bar{\rho}_N e^{\kappa N|\Phi|} \mathrm{d}X^N \right). \qquad (2.15)$$

On the other hand, one can easily check the following cancellations

$$\int_{\mathbb{T}^d} \phi_3(t, x, y)\bar{\rho}_t(x)\mathrm{d}x = 0 \quad \forall y, \qquad \int_{\mathbb{T}^d} \phi_3(t, x, y)\bar{\rho}_t(y)\mathrm{d}y = 0 \quad \forall x.$$

Since in both cases, ϕ_3 is bounded uniformly in t, we can choose κ such that $\sqrt{\kappa}\|\phi_3\|_{L^\infty}$ sufficiently small. Letting $\bar{\rho}_t$ and $\sqrt{\kappa}\phi_3$ play the roles of $\bar{\rho}$ and ϕ in Remark 2.2 respectively, we deduce that

$$\int_{\mathbb{T}^{dN}} \bar{\rho}_N e^{\kappa N|\Phi|} \mathrm{d}X^N \leqslant C,$$

where C is a constant depending only on φ. Combining this with (2.14) and (2.15), we thus arrive at the result. $\qquad\qquad\square$

3. THE SPDE LIMIT

The aim of this section is to analyze fluctuation behavior of the empirical measure μ^N for the non-degenerate case, i.e. $\sigma > 0$. It will be shown that $\eta^N = \sqrt{N}(\mu_N - \bar{\rho})$ converges in distribution to the unique solution η to the linear SPDE (1.5). We shall start with proving that the sequence of $(\eta^N)_{N \geqslant 1}$ is tight. Then each tight limit of the subsequence from $(\eta^N)_{N \geqslant 1}$ will be identified as a martingale solution to the equation (1.5). The next step is to show pathwise uniqueness of (1.5), which allows us to conclude the proof of Theorem 1.4. In Section 3.3, we prove the optimal regularity of solutions to the limit SPDE (1.5). Finally, the proof of Proposition 1.5, which gives the Gaussianity of the unique limit of fluctuation measures, is given in Section 3.4.

3.1. **Tightness.** Before proving tightness, we introduce pathwise realization of the martingale part in the decomposition (1.10). Recall that the martingale part is given by

$$\frac{\sqrt{2\sigma_N}}{\sqrt{N}} \sum_{i=1}^{N} \int_0^t \nabla\varphi(X_i) \mathrm{d}B_s^i,$$

for each $\varphi \in C^\infty(\mathbb{T}^d)$. Formally, one could define a random operator $\mathcal{M}_t^N : \Omega \times H^\alpha \to \mathbb{R}$, $\alpha > d/2 + 1$, for each $t \in [0, T]$ through

$$\mathcal{M}_t^N(\varphi) = \frac{\sqrt{2\sigma_N}}{\sqrt{N}} \sum_{i=1}^{N} \int_0^t \nabla\varphi(X_i) \mathrm{d}B_s^i, \quad \mathbb{P}-a.s. \tag{3.1}$$

However, the measurability of $\mathcal{M}_t^N : \Omega \to H^{-\alpha}$ is nontrivial due to the fact that the above stochastic integral is defined as a $\mathbb{P}$-equivalence class for each φ. Finding a measurable map $\mathcal{M}_t^N$ from Ω to $H^{-\alpha}$ requires a pathwise meaning of the map $\mathcal{M}_t^N(\varphi)$, such that $\mathcal{M}_t^N(\varphi)$ is continuous with respect to φ for almost every $\omega \in \Omega$.

Pathwise realization has been studied for different function spaces, for instance in [Itô83, Theorem 3.1], [FGGT05], [MW17, Lemma 9], etc. Adapting the idea of investigating stochastic currents in [FGGT05] by Flandoli, Gubinelli, Giaquinta, and Tortorelli, a pathwise realization $\mathcal{M}^N$ with values in Hilbert spaces can be obtained with a relatively simple proof, which is postponed into Appendix B. We state the result as follows.

Lemma 3.1. *For each N, there exists a progressively measurable process $\mathcal{M}^N$ with values in $H^{-\alpha}$, for any $\alpha > d/2 + 1$, such that (3.1) holds almost surely for all $t \in [0, T]$ and $\varphi \in C^\infty$.*

In the following, we are going to prove tightness of $(\eta^N, \mathcal{M}^N)_{N \geqslant 1}$. To start, recall the following tightness criterion given by Arzela-Ascoli theorem [Kel17, Theorem 7.17]. Suppose that $(u^N)_{N \geqslant 1}$ is a class of random variables in $C([0, T], E)$ with a given Polish space E. The sequence of $(u^N)_{N \geqslant 1}$ is tight in $C([0, T], E)$ if and only if the following conditions hold:

(1) For each $\epsilon > 0$ and each $t \in [0, T]$, there is a compact set $A \subset E$ (possibly depending on t) such that

$$\sup_N \mathbb{P}(u_t^N \in A) > 1 - \epsilon.$$

(2) For each $\epsilon > 0$,

$$\lim_{h \to 0} \sup_N \mathbb{P}(\sup_{s,t \in [0,T]} \sup_{|t-s| \leqslant h} \|u_t^N - u_s^N\|_E > \epsilon) = 0.$$

Since the embedding $H^{-\alpha'} \hookrightarrow H^{-\alpha}$ is compact if $\alpha' < \alpha$ (see [Tri06, Proposition 4.6]), using Chebyshev's inequality, one can get the following sufficient conditions for tightness in $C([0, T], H^{-\alpha})$

(i) For each $t \in [0, T]$, there exists some $\alpha' < \alpha$ such that

$$\sup_N \mathbb{E}\|u_t^N\|_{H^{-\alpha'}} < \infty. \tag{3.2}$$

(ii) There exists $\theta > 0$ such that

$$\sup_N \mathbb{E}\|u_t^N\|_{C^\theta([0,T], H^{-\alpha})} = \sup_N \mathbb{E}\left(\sup_{0 \leqslant s < t \leqslant T} \frac{\|u_t^N - u_s^N\|_{H^{-\alpha}}}{(t-s)^\theta}\right) < \infty. \tag{3.3}$$

Therefore to obtain tightness of $\{\eta^N, \mathcal{M}^N\}_{N \in \mathbb{N}}$ it suffices to justify (i) and (ii) with $\mathcal{M}^N$ and η^N playing the role of u^N.

The following lemma gives tightness of the martingale part.

Lemma 3.2. *For every $\alpha > d/2 + 1$, the sequence of $(\mathcal{M}^N)_{N \geqslant 1}$ is tight in the space $C([0, T], H^{-\alpha})$.*

Proof. By the above tightness criterion, it is indeed sufficient to prove that: for each $\alpha > d/2 + 1$ and $\theta' \in (0, \frac{1}{2})$, it holds that

$$\sup_N \mathbb{E}(\|\mathcal{M}^N\|^2_{C^{\theta'}([0,T],H^{-\alpha})}) < \infty.$$

First, for the Fourier basis $\{e_k\}_{k \in \mathbb{Z}^d}$ and $t \in [0, T]$, we find

$$\mathcal{M}^N_t(e_k) = \frac{\sqrt{2\sigma_N}}{\sqrt{N}} \sum_{i=1}^N \int_0^t \nabla e_k(X_i) \cdot dB^i_s = \sqrt{-1} \frac{\sqrt{2\sigma_N}}{\sqrt{N}} \sum_{i=1}^N \int_0^t e_k(X_i) k \cdot dB^i_s. \tag{3.4}$$

For any $\theta > 1$ and $0 \leqslant s < t \leqslant T$, we deduce from Hölder's inequality that

$$\sup_N \mathbb{E}(\|\mathcal{M}^N_t - \mathcal{M}^N_s\|^{2\theta}_{H^{-\alpha}}) = \sup_N \mathbb{E}\left[\left(\sum_{k \in \mathbb{Z}^d} \langle k \rangle^{-2\alpha} |\mathcal{M}^N_t(e_k) - \mathcal{M}^N_s(e_k)|^2\right)^{\theta}\right]$$

$$\leqslant \sup_N \mathbb{E}\left(\sum_{k \in \mathbb{Z}^d} \langle k \rangle^{-\alpha_1 \theta} |\mathcal{M}^N_t(e_k) - \mathcal{M}^N_s(e_k)|^{2\theta}\right)\left(\sum_{k \in \mathbb{Z}^d} \langle k \rangle^{-\alpha_2 \frac{\theta}{\theta-1}}\right)^{\theta-1},$$

where $\alpha_1 + \alpha_2 = 2\alpha$ and $\alpha_1, \alpha_2 > 0$. Further choosing $\alpha_1 = \alpha + 1 - \frac{d}{2} + \frac{d}{\theta}$ and $\alpha_2 = \alpha - 1 + \frac{d}{2} - \frac{d}{\theta}$, then we have $(\alpha_1 - 2)\theta > d$ and $\alpha_2 \frac{\theta}{\theta-1} > d$, due to $\theta > 1$ and the condition $\alpha > d/2 + 1$. Hence the summation $\sum_{k \in \mathbb{Z}^d} \langle k \rangle^{-\alpha_2 \frac{\theta}{\theta-1}}$ is finite. Moreover, using the equality (3.4) gives

$$\sup_N \mathbb{E}(\|\mathcal{M}^N_t - \mathcal{M}^N_s\|^{2\theta}_{H^{-\alpha}}) \lesssim_{\alpha_2,\theta} \sup_N \sum_{k \in \mathbb{Z}^d} \langle k \rangle^{-\alpha_1 \theta} \mathbb{E}(|\mathcal{M}^N_t(e_k) - \mathcal{M}^N_s(e_k)|^{2\theta})$$

$$\lesssim_{\alpha_2,\theta} \sum_{k \in \mathbb{Z}^d} \langle k \rangle^{-\alpha_1 \theta + 2\theta} \sup_N \mathbb{E}\left|\frac{\sqrt{2\sigma_N}}{\sqrt{N}} \sum_{i=1}^N \int_s^t e_k(X_i) dB^i_r\right|^{2\theta}$$

$$\lesssim_{\alpha_2,\theta} \sum_{k \in \mathbb{Z}^d} \langle k \rangle^{-\alpha_1 \theta + 2\theta} \sup_N \mathbb{E}\left(\int_s^t \frac{2\sigma_N}{N} \sum_{i=1}^N |e_k(X_i)|^2 dr\right)^{\theta}$$

$$\lesssim_{\alpha_2,\theta} (t-s)^\theta \sum_{k \in \mathbb{Z}^d} \langle k \rangle^{-\alpha_1 \theta + 2\theta} \lesssim_{\alpha_1,\alpha_2,\theta} (t-s)^\theta, \tag{3.5}$$

where the third inequality follows by the Burkholder-Davis-Gundy's inequality. Therefore, (3.5) allows us to apply the Kolmogorov continuity theorem [BFH18, Theorem 2.3.11], and we find

$$\sup_N \mathbb{E}(\|\mathcal{M}^N\|^{2\theta}_{C^{\theta'}([0,T],H^{-\alpha})}) < \infty, \tag{3.6}$$

for any $0 < \theta' < \frac{\theta-1}{2\theta}$, $\theta > 1$, and $\alpha > d/2 + 1$. The result follows by arbitrary $\theta > 1$. □

Next, we need the tightness of the fluctuation measures.

Lemma 3.3. *Under the assumptions* **(A2)-(A4)**, *for every* $\alpha > d/2 + 2$, *the sequence of* $(\eta^N)_{N \geqslant 1}$ *is tight in the space* $C([0, T], H^{-\alpha})$.

Proof. First, by Assumption **(A3)** and Lemma 2.6, one can easily deduce (3.2) with η^N playing the role of u^N for any $\alpha > d/2 + 2$. Indeed, taking $\mu_N(\cdot) - \bar{\rho} = \frac{1}{\sqrt{N}} \eta^N$ into Lemma 2.6 immediately gives

$$\sup_{t \in [0,T]} \sup_N \mathbb{E}\|\eta^N_t\|^2_{H^{-\alpha+2}} \lesssim \sup_{t \in [0,T]} \sup_N H(\rho_N | \bar{\rho}_N)(t) + 1. \tag{3.7}$$

Then **(A3)** implies that the right hand side of (3.7) is finite. Thus (3.2) follows by $\alpha - 2$ playing the role of α'.

As for (3.3), it suffices to prove the case $\alpha - 2 \in (d/2, \beta)$, where β is given in Assumption **(A5)**. Recall the decomposition (1.10), $\|\eta_t^N - \eta_s^N\|_{H^{-\alpha}}, 0 \leqslant s < t < T$, is controlled via the following relation

$$\|\eta_t^N - \eta_s^N\|_{H^{-\alpha}}^2 \lesssim \sum_{i=1}^5 J_{s,t}^i, \tag{3.8}$$

where $J_{s,t}^i$, $i = 1, \ldots, 5$, are defined by

$$J_{s,t}^1 := \left\| \sigma \int_s^t \Delta \eta_r^N \mathrm{d}r \right\|_{H^{-\alpha}}^2, \qquad J_{s,t}^2 := \left\| \int_s^t \mathcal{K}_r^N \mathrm{d}r \right\|_{H^{-\alpha}}^2,$$

$$J_{s,t}^3 := \left\| \int_s^t \nabla \cdot (F \eta_r^N) \mathrm{d}r \right\|_{H^{-\alpha}}^2, \quad J_{s,t}^4 := \left\| \int_s^t \sqrt{N}(\sigma_N - \sigma) \Delta \mu_N(r) \mathrm{d}r \right\|_{H^{-\alpha}}^2,$$

$$J_{s,t}^5 := \|\mathcal{M}_t^N - \mathcal{M}_s^N\|_{H^{-\alpha}}^2.$$

For $J_{s,t}^1$, applying Hölder's inequality gives

$$\sup_N \mathbb{E} \left(\sup_{0 \leqslant s < t \leqslant T} \frac{J_{s,t}^1}{t-s} \right) \lesssim \sup_N \mathbb{E} \int_0^T \|\Delta \eta_t^N\|_{H^{-\alpha}}^2 \mathrm{d}t \lesssim \sup_N \sup_{t \in [0,T]} \mathbb{E}\|\eta_t^N\|_{H^{-\alpha+2}}^2 < \infty, \tag{3.9}$$

where we used (3.7) in the last step.

For $J_{s,t}^2$, similarly, applying Hölder's inequality gives

$$\sup_N \mathbb{E} \left(\sup_{0 \leqslant s < t \leqslant T} \frac{J_{s,t}^2}{t-s} \right) \lesssim \sup_N \mathbb{E} \left(\int_0^T \|\mathcal{K}_t^N\|_{H^{-\alpha}}^2 \mathrm{d}t \right)$$

$$\lesssim \sup_N \sup_{t \in [0,T]} \mathbb{E}\|\mathcal{K}_t^N\|_{H^{-\alpha}}^2.$$

Recall that $\mathcal{K}_t^N = \sqrt{N} \nabla \cdot [K * \mu_N(t)\mu_N(t) - \bar{\rho}_t K * \bar{\rho}_t]$, and thus Lemma 2.8 and the assumptions **(A2)-(A3)** deduces that

$$\sup_N \mathbb{E} \left(\sup_{0 \leqslant s < t \leqslant T} \frac{J_{s,t}^2}{t-s} \right) \lesssim \sup_N \sup_{t \in [0,T]} H(\rho_N | \bar{\rho}_N) + 1 < \infty. \tag{3.10}$$

Similarly, we obtain

$$\sup_N \mathbb{E} \left(\sup_{0 \leqslant s < t \leqslant T} \frac{J_{s,t}^3}{t-s} \right) \lesssim \sup_N \sup_{t \in [0,T]} \mathbb{E}\|F \eta_t^N\|_{H^{-\alpha+2}}^2,$$

and

$$\sup_N \mathbb{E} \left(\sup_{0 \leqslant s < t \leqslant T} \frac{J_{s,t}^4}{t-s} \right) \lesssim \sup_N \sup_{t \in [0,T]} N|\sigma_N - \sigma|^2 \mathbb{E}\|\mu_N(t)\|_{H^{-\alpha+2}}^2.$$

Furthermore, Lemma A.1 together with Lemma A.2 shows that

$$\|F\eta\|_{H^{-\alpha+2}} \lesssim \|F\|_{C^\beta} \|\eta^N\|_{H^{-\alpha+2}}.$$

Hence using (3.7) and Assumption **(A4)** gives

$$\sup_N \mathbb{E} \left(\sup_{0 \leqslant s < t \leqslant T} \frac{J_{s,t}^3}{t-s} \right) \lesssim \sup_N \sup_{t \in [0,T]} \|F\|_{C^\beta}^2 \mathbb{E}\|\eta_t^N\|_{H^{-\alpha+2}}^2 < \infty. \tag{3.11}$$

On the other hand, Assumption **(A4)**, $\mu_N = N^{-1/2}\eta^N + \bar{\rho}$ and (3.7) imply that

$$\sup_N \mathbb{E} \left(\sup_{0 \leqslant s < t \leqslant T} \frac{J_{s,t}^4}{t-s} \right) \to 0. \tag{3.12}$$

For $J_{s,t}^5$, we deduce from (3.6) that for any $\theta \in \left(0, \frac{1}{2}\right)$,

$$\sup_N \mathbb{E}\left(\sup_{0 \leqslant s < t \leqslant T} \frac{J_{s,t}^5}{(t-s)^{2\theta}}\right) = \sup_N \mathbb{E}(\|\mathcal{M}^N\|_{C^\theta([0,T],H^{-\alpha})}^2) < \infty. \tag{3.13}$$

We are in a position to conclude (3.3) with η^N playing the role of u^N for any $\alpha > d/2 + 2$, and tightness of the sequence $(\eta^N)_{N \geqslant 1}$ follows. Indeed, combining (3.8)-(3.13) yields that

$$\sup_N \mathbb{E}(\|\eta^N\|_{C^\theta([0,T],H^{-\alpha})}^2) = \sup_N \mathbb{E}\left(\sup_{0 \leqslant s < t \leqslant T} \frac{\|\eta_t^N - \eta_s^N\|_{H^{-\alpha}}^2}{(t-s)^{2\theta}}\right)$$

$$\lesssim_T \sup_N \sum_{i=1}^5 \mathbb{E}\left(\sup_{0 \leqslant s < t \leqslant T} \frac{J_{s,t}^i}{(t-s)^{2\theta}}\right) < \infty,$$

for any $\theta \in \left(0, \frac{1}{2}\right)$. The result then follows. $\square$

Remark 3.4. *Careful readers may find that it suffices to assume $|\sigma_N - \sigma| = o(\frac{1}{\sqrt{N}})$ in order to obtain the tightness of (η^N). But we still adopt the assumption that $|\sigma_N - \sigma| = \mathcal{O}\left(\frac{1}{N}\right)$ in Assumption **(A4)** and Assumption **(A5)** since this is one of the assumptions used in [JW18] to obtain the uniform bound for $H(\rho_N|\bar{\rho}^{\otimes N})$, i.e. our Assumption **(A3)**.*

Define the topological space $\mathcal{X}$:

$$\mathcal{X} := \left\{\bigcap_{k \in \mathbb{N}} \left[C([0,T], H^{-\frac{d}{2}-2-\frac{1}{k}}) \cap L^2([0,T], H^{-\frac{d}{2}-\frac{1}{k}})\right]\right\} \times \left\{\bigcap_{k \in \mathbb{N}} C([0,T], H^{-\frac{d}{2}-1-\frac{1}{k}})\right\}.$$

The space $Y := \cap_{k \in \mathbb{N}} Y_k$ with $C([0,T], H^{-\frac{d}{2}-2-\frac{1}{k}}) \cap L^2([0,T], H^{-\frac{d}{2}-\frac{1}{k}})$ or $C([0,T], H^{-\frac{d}{2}-1-\frac{1}{k}})$ playing the role of Y_k is endowed with the metric $d_Y(f,g) = \sum_{k=1}^\infty 2^{-k}(1 \wedge \|f-g\|_{Y_k})$. Thus the convergence in Y is equivalent to the convergence in Y_k for every $k \in \mathbb{N}$. Moreover, $\mathcal{X}$ is a Polish space.

We then deduce the following result by the Skorokhod theorem.

Theorem 3.5. *There exists a subsequence of $(\eta^N, \mathcal{M}^N)_{N \geqslant 1}$, still denoted by $(\eta^N, \mathcal{M}^N)$ for simplicity, and a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ with $\mathcal{X}$-valued random variables $(\tilde{\eta}^N, \tilde{\mathcal{M}}^N)_{N \geqslant 1}$ and $(\tilde{\eta}, \tilde{M})$ such that*

(1) For each $N \in \mathbb{N}$, the law of $(\tilde{\eta}^N, \tilde{\mathcal{M}}^N)$ coincides with the law of $(\eta^N, \mathcal{M}^N)$.
(2) The sequence of $\mathcal{X}$-valued random variables $(\tilde{\eta}^N, \tilde{\mathcal{M}}^N)_{N \geqslant 1}$ converges to $(\tilde{\eta}, \tilde{M})$ in $\mathcal{X}$ $\tilde{\mathbb{P}}$-a.s.

Proof. By the Skorokhod theorem, the result follows by justifying the fact that the joint law of $(\eta^N, \mathcal{M}^N)_{N \geqslant 1}$ is tight on $\mathcal{X}$.

We start with proving the set A defined below is relatively compact in the space $C([0,T], H^{-\alpha-2}) \cap L^2([0,T], H^{-\alpha})$ for each $\alpha > d/2$,

$$A := \left\{u \in K; \int_0^T \|u(t)\|_{H^{-\frac{2\alpha+d}{4}}}^2 \, dt \leqslant M, \right\},$$

where K is relatively compact in $C([0,T], H^{-\alpha-2})$. Suppose a sequence $\{u_n\} \subset A$, then there is a subsequence $\{u_{n_m}\}$ converging in $C([0,T], H^{-\alpha-2})$. On the other hand, by the Sobolev interpolation theorem [BCD11, Proposition 1.52], we find for $n, n' \in \mathbb{N}$ and $-\alpha - 2 < -\alpha < -\frac{2\alpha+d}{4}$

$$\int_0^T \|u_n(t) - u_{n'}(t)\|_{H^{-\alpha}}^2 dt$$

$$\leqslant \int_0^T \|u_n(t) - u_{n'}(t)\|_{H^{-\frac{2\alpha+d}{4}}}^{2\theta} \|u_n(t) - u_{n'}(t)\|_{H^{-\alpha-2}}^{2(1-\theta)} dt$$

$$\leqslant \left(\int_0^T \|u_n(t) - u_{n'}(t)\|_{H^{-\frac{2\alpha+d}{4}}}^2 dt\right)^\theta \left(\int_0^T \|u_n(t) - u_{n'}(t)\|_{H^{-\alpha-2}}^2 dt\right)^{1-\theta}.$$

$$\leqslant \left(\int_0^T \|u_n(t) - u_{n'}(t)\|^2_{H^{-\frac{2\alpha+d}{4}}} \mathrm{d}t\right)^\theta \left(T \sup_{t\in[0,T]} \|u_n(t) - u_{n'}(t)\|^2_{H^{-\alpha-2}}\right)^{1-\theta},$$

where the interpolation constant $\theta \in (0,1)$ depends on α and d. This implies the convergence of the subsequence $\{u_{n_m}\}$ in $L^2([0,T], H^{-\alpha})$ for each $\alpha > d/2$, and A is thus relatively compact in $C([0,T], H^{-\alpha-2}) \cap L^2([0,T], H^{-\alpha})$. For each $\epsilon > 0$, by (3.7) and Lemma 3.3, one can find M sufficiently large and a compact set K in $C([0,T], H^{-\alpha-2})$ such that

$$\mathbb{P}(\eta^N \notin A) \leqslant P(\eta^N \notin K) + \mathbb{P}\Big(\int_0^T \|\eta^N(t)\|^2_{H^{-\frac{2\alpha+d}{4}}} \mathrm{d}t > M\Big)$$

$$\leqslant P(\eta^N \notin K) + \frac{T}{M} \sup_{t\in[0,T]} \sup_N \mathbb{E}\|\eta^N(t)\|^2_{H^{-\frac{2\alpha+d}{4}}} < \epsilon,$$

where the second line follows by Chebyshev's inequality. Therefore the sequence of laws of $(\eta^N)_{N\geqslant 1}$ is tight on $C([0,T], H^{-\alpha-2}) \cap L^2([0,T], H^{-\alpha})$ for every $\alpha > d/2$.

Furthermore, recall that Lemma 3.2 gives that the sequence of laws of $(\mathcal{M}^N)_{N\geqslant 1}$ is tight on $C([0,T], H^{-\alpha-1})$ for every $\alpha > d/2$. For each $\epsilon > 0$ and $k \in \mathbb{N}$, choose compact sets A_k^ϵ and B_k^ϵ in $C([0,T], H^{-\frac{d}{2}-\frac{1}{k}-2}) \cap L^2([0,T], H^{-\frac{d}{2}-\frac{1}{k}})$ and $C([0,T], H^{-\frac{d}{2}-\frac{1}{k}-1})$, respectively, such that

$$\mathbb{P}(\eta^N \notin A_k^\epsilon) < \epsilon 2^{-k}, \quad \mathbb{P}(\mathcal{M}^N \notin B_k^\epsilon) < \epsilon 2^{-k}, \quad \forall N \in \mathbb{N}.$$

Thus the set $A^\epsilon \times B^\epsilon$ in $\mathcal{X}$ defined by

$$A^\epsilon \times B^\epsilon := \left(\bigcap_{k\in\mathbb{N}} A_k^\epsilon\right) \times \left(\bigcap_{k\in\mathbb{N}} B_k^\epsilon\right)$$

is relatively compact and satisfies

$$\mathbb{P}\Big((\eta^N, \mathcal{M}^N) \notin A^\epsilon \times B^\epsilon\Big) \leqslant \sum_{k\in\mathbb{N}} \mathbb{P}(\eta^N \notin A_k^\epsilon) + \mathbb{P}(\mathcal{M}^N \notin B_k^\epsilon) < 2\epsilon, \quad \forall N \in \mathbb{N},$$

which shows the tightness of $(\eta^N, \mathcal{M}^N)_{N\geqslant 1}$ in $\mathcal{X}$. $\qquad\square$

Corollary 3.6. *For every $\alpha > d/2$, it holds that*

$$\tilde{\mathbb{E}} \int_0^T \|\tilde{\eta}_t^N - \tilde{\eta}_t\|_{H^{-\alpha}} \mathrm{d}t \xrightarrow{N\to\infty} 0. \tag{3.14}$$

Proof. Notice that

$$\tilde{\mathbb{E}} \int_0^T \|\tilde{\eta}_t\|^2_{H^{-\alpha}} \mathrm{d}t \leqslant \sup_N \tilde{\mathbb{E}} \int_0^T \|\tilde{\eta}_t^N\|^2_{H^{-\alpha}} \mathrm{d}t \leqslant T \sup_{t\in[0,T]} \sup_N \tilde{\mathbb{E}}\|\tilde{\eta}_t^N\|^2_{H^{-\alpha}} < \infty, \quad \forall \alpha > \frac{d}{2},$$

which provides the uniform (in $[0,T] \times \Omega$) integrability of $\|\tilde{\eta}_t^N - \tilde{\eta}_t\|_{H^{-\alpha}}$, thus the convergence of $\|\tilde{\eta}_t^N - \tilde{\eta}_t\|_{H^{-\alpha}} \, \mathrm{d}t \times \mathrm{d}\tilde{\mathbb{P}}$-a.e. leads to (3.14). $\qquad\square$

For each N, let $(\tilde{\mathcal{F}}_t^N)_{t\geqslant 0}$ and $(\tilde{\mathcal{F}}_t)_{t\geqslant 0}$ be the normal filtration generated by $(\tilde{\eta}^N, \tilde{\mathcal{M}}^N)$ and $(\tilde{\eta}, \tilde{\mathcal{M}})$, respectively. Then we have

$$\tilde{\mathcal{M}}_t^N = \tilde{\eta}_t^N - \tilde{\eta}_0^N - \sigma \int_0^t \Delta\tilde{\eta}_s^N \mathrm{d}s + \int_0^t \tilde{\mathcal{K}}_s^N \mathrm{d}s + \int_0^t \nabla(F\tilde{\eta}_s^N)\mathrm{d}s - \tilde{R}_t^N, \tag{3.15}$$

where $\tilde{\mathcal{K}}^N$ and $\tilde{R}^N$ are defined with $\tilde{\mu}_N := \bar{\rho} + \frac{1}{\sqrt{N}}\tilde{\eta}^N$ and

$$\tilde{\mathcal{K}}_t^N := \sqrt{N}\nabla \cdot \Big(K * \tilde{\mu}_N(t)\tilde{\mu}_N(t) - K * \bar{\rho}_t\bar{\rho}_t\Big), \quad \tilde{R}_t^N := \sqrt{N}(\sigma_N - \sigma)\int_0^t \Delta\tilde{\mu}_N(s)\mathrm{d}s.$$

Here $\tilde{\mathcal{K}}^N$ is well-defined since $\tilde{\mu}_N$ is linear combination of Dirac measure and $K * \tilde{\mu}_N \tilde{\mu}_N$ is understood as

$$\langle K * \tilde{\mu}_N \tilde{\mu}_N, \varphi \rangle = \int_{\mathbb{T}^d \times \mathbb{T}^d} K(x-y) \varphi(x) \tilde{\mu}_N(\mathrm{d}x) \tilde{\mu}_N(\mathrm{d}y),$$

for $\varphi \in C^1$.

3.2. **Characterization of the limit.** In this section, we conclude that the original sequence $(\eta^N)_{N \geqslant 1}$ converges in distribution to the equation (1.5). Recall that the sequence $(\tilde{\eta}^N)_{N \geqslant 1}$ converges in $C([0,T], H^{-\alpha-2}) \cap L^2([0,T], H^{-\alpha})$ $\tilde{\mathbb{P}}$-a.s. for $\alpha > d/2$ and shares the same distribution with a subsequence of $(\eta^N)_{N \geqslant 1}$. Hence it is sufficient to justify two facts. One is that each limit $\tilde{\eta}$ is a martingale solution to (1.5). The other is that the law of the solution to (1.5) is unique, which would follow by pathwise uniqueness and the Yamada-Watanabe theorem.

Throughout this section, we always assume **(A1)-(A4)**.

Identifying the limit of the interacting term $\tilde{\mathcal{K}}^N$ is one of the main difficulties in this article, it derseves to be treated separately from other terms in the decomposition (1.10). The following lemma identifies the limit of the interacting term $\tilde{\mathcal{K}}^N$. The idea of the proof is to split the interacting term into some regular part and a term in the form of a function of $\mu_N - \bar{\rho}$, which can be controlled in Lemma 2.9 by the techniques developed in Section 2.

Lemma 3.7. *For each $\varphi \in C^\infty(\mathbb{T}^d)$, it holds that*

$$\tilde{\mathbb{E}} \left(\sup_{t \in [0,T]} \left| \int_0^t \tilde{\mathcal{K}}_s^N(\varphi) - \langle \bar{\rho}_s K * \tilde{\eta}_s + \tilde{\eta}_s K * \bar{\rho}_s, \nabla\varphi \rangle \mathrm{d}s \right| \right) \xrightarrow{N \to \infty} 0.$$

Proof. Direct computations give the following identity

$$\sqrt{N} \left(\tilde{\mu}_N K * \tilde{\mu}_N - \bar{\rho} K * \bar{\rho} \right) = \bar{\rho} K * \eta^N + \eta^N K * \bar{\rho} + \frac{1}{\sqrt{N}} \eta^N K * \eta^N.$$

Consequently, for each $\varphi \in C^\infty$,

$$\sup_{t \in [0,T]} \left| \int_0^t \tilde{\mathcal{K}}_s^N(\varphi) - \langle \bar{\rho}_s K * \tilde{\eta}_s + \tilde{\eta}_s K * \bar{\rho}_s, \nabla\varphi \rangle \mathrm{d}s \right| \leqslant J_1^N(\varphi) + J_2^N(\varphi), \tag{3.16}$$

where

$$J_1^N(\varphi) := \sqrt{N} \int_0^T |\langle \nabla\varphi K * (\tilde{\mu}_N(t) - \bar{\rho}_t), \tilde{\mu}_N(t) - \bar{\rho}_t \rangle| \mathrm{d}t,$$

$$J_2^N(\varphi) := \int_0^T |\langle \bar{\rho}_t K * \tilde{\eta}_t^N + \tilde{\eta}_t^N K * \bar{\rho}_t, \nabla\varphi \rangle - \langle \bar{\rho}_t K * \tilde{\eta}_t + \tilde{\eta}_t K * \bar{\rho}_t, \nabla\varphi \rangle| \mathrm{d}t.$$

On one hand, we deduce from Lemma 2.9 that

$$\tilde{\mathbb{E}} J_1^N(\varphi) \leqslant T \sqrt{N} \sup_{t \in [0,T]} \tilde{\mathbb{E}} |\langle \nabla\varphi K * (\tilde{\mu}_N(t) - \bar{\rho}_t), \tilde{\mu}_N(t) - \bar{\rho}_t \rangle|$$

$$= T \sqrt{N} \sup_{t \in [0,T]} \mathbb{E} |\langle \nabla\varphi K * (\mu_N(t) - \bar{\rho}_t), \mu_N(t) - \bar{\rho}_t \rangle|$$

$$\lesssim N^{-\frac{1}{2}} \sup_N \sup_{t \in [0,T]} (H(\rho_N | \bar{\rho}_N) + 1) \xrightarrow{N \to \infty} 0,$$

where the limit follows by Assumption **(A3)**. On the other hand, we find

$$\tilde{\mathbb{E}} J_2^N(\varphi) \leqslant \tilde{\mathbb{E}} \int_0^T |\langle \bar{\rho}_t K * (\tilde{\eta}_t^N - \tilde{\eta}_t), \nabla\varphi \rangle| + |\langle (\tilde{\eta}_t^N - \tilde{\eta}_t) K * \bar{\rho}_t, \nabla\varphi \rangle| \mathrm{d}t. \tag{3.17}$$

For each $t \in [0,T]$, it holds for every $\alpha \in (d/2, \beta)$ that

$$|\langle \bar{\rho}_t K * (\tilde{\eta}_t^N - \tilde{\eta}_t), \nabla\varphi \rangle| = |\langle K(-\cdot) * (\bar{\rho}_t \nabla\varphi), \tilde{\eta}_t^N - \tilde{\eta}_t \rangle| \leqslant \|\tilde{\eta}_t^N - \tilde{\eta}_t\|_{H^{-\alpha}} \|K(-\cdot) * (\bar{\rho}_t \nabla\varphi)\|_{H^\alpha},$$

$$|\langle(\tilde{\eta}_t^N - \tilde{\eta}_t)K * \bar{\rho}_t, \nabla\varphi\rangle| \leqslant \|\tilde{\eta}_t^N - \tilde{\eta}_t\|_{H^{-\alpha}}\|\nabla\varphi \cdot K * \bar{\rho}_t\|_{H^\alpha},$$

where

$$K(-\cdot) * g(x) := \int K(y - x)g(y)\mathrm{d}y. \tag{3.18}$$

Applying Lemma A.4 with $p = p_1 = q = 2$ and Lemma A.3 yields that

$$|\langle\bar{\rho}_t K * (\tilde{\eta}_t^N - \tilde{\eta}_t), \nabla\varphi\rangle| \lesssim \|\tilde{\eta}_t^N - \tilde{\eta}_t\|_{H^{-\alpha}}\|K\|_{L^1}(\|\bar{\rho}_t\|_{H^\alpha}\|\nabla\varphi\|_{L^\infty} + \|\bar{\rho}_t\|_{L^\infty}\|\nabla\varphi\|_{H^\alpha}),$$

$$|\langle(\tilde{\eta}_t^N - \tilde{\eta}_t)K * \bar{\rho}_t, \nabla\varphi\rangle| \lesssim \|\tilde{\eta}_t^N - \tilde{\eta}_t\|_{H^{-\alpha}}\|K\|_{L^1}(\|\bar{\rho}_t\|_{H^\alpha}\|\nabla\varphi\|_{L^\infty} + \|\bar{\rho}_t\|_{L^\infty}\|\nabla\varphi\|_{H^\alpha}).$$

Here the fact $K \in L^1$ follows by Assumption **(A2)**. Then taking these two estimates into (3.17), applying Sobolev embedding $H^\alpha \hookrightarrow L^\infty$ with $\alpha > d/2$, we thus arrive at

$$\tilde{\mathbb{E}}J_2^N(\varphi) \lesssim_\varphi \|K\|_{L^1}\sup_{t\in[0,T]}\|\bar{\rho}_t\|_{H^\alpha}\mathbb{E}\int_0^T\|\tilde{\eta}_t^N - \tilde{\eta}_t\|_{H^{-\alpha}}\mathrm{d}t \xrightarrow{N\to\infty} 0,$$

where the limit follows by (3.14). Using inequality (3.16) and $\mathbb{E}J_1^N(\varphi) \to 0$, the proof is completed. □

Remark 3.8. *One may easily find that in the "identifying the limit" part, we only need to assume that the relative entropy grow slower than the order $\sqrt{N}$, i.e. $H(\rho_N|\bar{\rho}_N) = o(\sqrt{N})$ as $N \to \infty$. However, in the tightness part we need a stronger assumption, namely our Assumption **(A3)**. As a separate question, it would be interesting to show whether or not there exists some symmetric probability measure $\rho_N \in \mathcal{P}_{Sym}(S^N)$ such that $H(\rho_N|\bar{\rho}^{\otimes N}) = N^\theta$ with $\theta \in (0,1)$, where $\bar{\rho}$ is a given probability measure on the Polish space S.*

Now we are in the position to conclude that $\tilde{\eta}$ solves (1.5).

Theorem 3.9. *The limit $\tilde{\eta}$ is a martingale solution to* (1.5) *in the sense of Definition 1.1.*

Proof. We deduce from (3.15) that

$$\tilde{\mathcal{M}}_t^N(\varphi) = \langle\tilde{\eta}_t^N, \varphi\rangle - \langle\tilde{\eta}_0^N, \varphi\rangle - \sigma\int_0^t\langle\Delta\varphi, \tilde{\eta}_s^N\rangle\mathrm{d}s - \int_0^t\tilde{\mathcal{K}}_s^N(\varphi)\mathrm{d}s - \int_0^t\langle\nabla\varphi, F\tilde{\eta}_s^N\rangle\mathrm{d}s$$
$$- \sqrt{N}(\sigma_N - \sigma)\int_0^t\langle\Delta\varphi, \tilde{\mu}_N(s)\rangle\mathrm{d}s,$$

for each $\varphi \in C^\infty(\mathbb{T}^d)$ and $t \in [0,T]$. By Lemma 3.7, $\sigma_N - \sigma = \mathcal{O}\left(\frac{1}{N}\right)$, and the fact that $\tilde{\eta}^N$ converges to $\tilde{\eta}$ in $C([0,T], H^{-\alpha-2}) \cap L^2([0,T], H^{-\alpha})$ for every $\alpha > d/2$ $\tilde{\mathbb{P}}$-a.s., one can take limit of every term above on both sides and have

$$\tilde{\mathcal{M}}_t(\varphi) = \langle\tilde{\eta}_t, \varphi\rangle - \langle\tilde{\eta}_0, \varphi\rangle - \sigma\int_0^t\langle\Delta\varphi, \tilde{\eta}_s\rangle\mathrm{d}s - \int_0^t\langle\nabla\varphi, \bar{\rho}_s K * \tilde{\eta}_s + \tilde{\eta}_s K * \bar{\rho}_s + F\tilde{\eta}_s\rangle\mathrm{d}s, \quad \tilde{\mathbb{P}} - a.s.$$

To indentify $\tilde{\eta}$ is a martingale solution, we need to justify properties of $\tilde{\mathcal{M}}$. Since $\tilde{\mathcal{M}}^N$ are centered Gaussian process and by Theorem 3.5, the limit $\tilde{\mathcal{M}}$ is a centered Gaussian process with values in $H^{-\alpha-1}$ for every $\alpha > d/2$.

As for the covariance functions, on one hand, applying Burkholder-Davis-Gundy's inequality, we have for each $1 < \theta \leqslant 2$

$$\sup_N\tilde{\mathbb{E}}[\sup_{t\in[0,T]}|\tilde{\mathcal{M}}_t^N(\varphi)|^{2\theta}] = \sup_N\mathbb{E}[\sup_{t\in[0,T]}|\mathcal{M}_t^N(\varphi)|^{2\theta}]$$
$$\lesssim \sup_N\mathbb{E}\left(\int_0^T\sum_{i=1}^N\frac{\sigma_N}{N}|\nabla\varphi(X_i)|^2\,\mathrm{d}t\right)^\theta = \sup_N\mathbb{E}\left(\int_0^T\sigma_N\langle|\nabla\varphi|^2, \mu_N(t)\rangle\mathrm{d}t\right)^\theta$$
$$\lesssim_\varphi \sup_N\sigma_N^\theta T^\theta\|\nabla\varphi\|_{L^\infty}^{2\theta} < \infty.$$

This implies uniform integrability of $|\tilde{\mathcal{M}}_t^N(\varphi)|^2$ for each $t \in [0,T]$.

On the other hand, we have that $|\tilde{\mathcal{M}}_t^N(\varphi_1)\tilde{\mathcal{M}}_s^N(\varphi_2)|$ converges to $|\tilde{\mathcal{M}}_t(\varphi_1)\tilde{\mathcal{M}}_s(\varphi_2)|$ $\tilde{\mathbb{P}}$-a.s. for $s, t \in [0, T]$ and $\varphi_1, \varphi_2 \in C^\infty$. Thus by the uniform integrability of $|\tilde{\mathcal{M}}_t^N(\varphi)|^2$ and

$$|\tilde{\mathcal{M}}_t^N(\varphi_1)\tilde{\mathcal{M}}_s^N(\varphi_2)| \leqslant |\tilde{\mathcal{M}}_t^N(\varphi_1)|^2 + |\tilde{\mathcal{M}}_s^N(\varphi_2)|^2,$$

we arrive at

$$\tilde{\mathbb{E}}[\tilde{\mathcal{M}}_t(\varphi_1)\tilde{\mathcal{M}}_s(\varphi_2)] = \lim_{N\to\infty} \tilde{\mathbb{E}}[\tilde{\mathcal{M}}_t^N(\varphi_1)\tilde{\mathcal{M}}_s^N(\varphi_2)] = \lim_{N\to\infty} \mathbb{E}[\mathcal{M}_t^N(\varphi_1)\mathcal{M}_s^N(\varphi_2)].$$

Furthermore, using (3.1) and Ito's isometry we obtain that

$$\begin{aligned}
\mathbb{E}[\mathcal{M}_t^N(\varphi_1)\mathcal{M}_s^N(\varphi_2)] =& \frac{2\sigma_N}{N}\mathbb{E}\left[\left(\sum_{i=1}^N \int_0^t \nabla\varphi_1(X_i)\mathrm{d}B_r^i\right)\left(\sum_{i=1}^N \int_0^s \nabla\varphi_2(X_i)\mathrm{d}B_r^i\right)\right]\\
=& \frac{2\sigma_N}{N}\mathbb{E}\left[\sum_{i=1}^N \int_0^{s\wedge t} \nabla\varphi_1(X_i)\nabla\varphi_2(X_i)\mathrm{d}r\right]\\
=& 2\sigma_N \int_0^{s\wedge t} \mathbb{E}\langle\nabla\varphi_1 \cdot \nabla\varphi_2, \mu_N(r)\rangle\mathrm{d}r.
\end{aligned}$$

The proof is thus completed since

$$2\sigma_N \int_0^{s\wedge t} \mathbb{E}\langle\nabla\varphi_1 \cdot \nabla\varphi_2, \mu_N(r)\rangle\mathrm{d}r \xrightarrow{N\to\infty} 2\sigma \int_0^{s\wedge t} \langle\nabla\varphi_1 \cdot \nabla\varphi_2, \bar{\rho}_r\rangle\mathrm{d}r.$$

$\square$

The rest of this subsection is devoted to obtain the well-posedness of the SPDE (1.5), and finish the proof of Theorem 1.4.

Let us first introduce an equivalent definition of martingale solutions to (1.5), which is used in the proof of Theorem 1.4. For notations' simplicity, we omit the tildes in the following.

Definition 3.10. *We call* $(\eta, \mathcal{M})$ *a probabilistically weak solution to* (1.5) *on stochastic basis* $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ *with initial data* η_0 *if*

(1) η *is a continuous* $(\mathcal{F}_t)$-*adapted process with values in* $H^{-\alpha-2}$ *and* $\eta \in L^2([0,T], H^{-\alpha})$ *for every* $\alpha > d/2$, $\mathbb{P}$-*a.s.*

(2) $\mathcal{M}$ *is a continuous* $(\mathcal{F}_t)$-*adapted centered Gaussian process with values in* $H^{-\alpha-1}$ *for every* $\alpha > d/2$, *with covariance given by*

$$\mathbb{E}[\mathcal{M}_t(\varphi_1)\mathcal{M}_s(\varphi_2)] = 2\sigma \int_0^{s\wedge t} \langle\nabla\varphi_1 \cdot \nabla\varphi_2, \bar{\rho}_r\rangle\mathrm{d}r, \tag{3.19}$$

for each $\varphi_1, \varphi_2 \in C^\infty$ *and* $s, t \in [0, T]$.

(3) For each $\varphi \in C^\infty(\mathbb{T}^d)$ *and* $t \in [0, T]$, *it holds that*

$$\begin{aligned}
\mathcal{M}_t(\varphi) =& \langle\eta_t, \varphi\rangle - \langle\eta_0, \varphi\rangle - \int_0^t \langle\sigma\Delta\varphi, \eta\rangle\mathrm{d}s - \int_0^t \langle\nabla\varphi, \bar{\rho}K * \eta\rangle\mathrm{d}s - \int_0^t \langle\nabla\varphi, \eta K * \bar{\rho}\rangle\mathrm{d}s\\
& - \int_0^t \langle\nabla\varphi, F\eta\rangle\mathrm{d}s.
\end{aligned}$$

Furthermore, given a centered Gaussian process $\mathcal{M}$ *on stochastic basis* $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ *with covariance characterized by* (3.19), *we call* η *is a probabilistically strong solution to* (1.2) *if* $(\eta, \mathcal{M})$ *is a probabilistically weak solution and* η *is adapted to the normal filtration generated by* $\mathcal{M}$.

Uniqueness in law of the solutions to (1.5) usually follows by the Yamada-Watanabe theorem, which requires existence of probabilistically weak solutions and pathwise uniqueness. Since the martingale solutions and the probabilistically weak solutions are equivalent, Theorem 3.9 means that there exists a stochastic basis $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ such that $(\eta, \mathcal{M})$ is a probabilistically weak solution to (1.5), it thus suffices to prove the pathwise uniqueness.

We now briefly explain the concept of pathwise uniqueness of probabilistically weak solutions introduced before. Equation (1.5) can be viewed as a system, for which the information of the initial data and the nosie is given (i.e. the distribution of $(\mathcal{M}, \eta_0)$ is fixed), and $(\mathcal{M}, \eta_0)$ can be seen as the input and η is the output. Pathwise uniqueness means that if on some fixed stochastic basis there exist two outputs η and $\tilde{\eta}$ with given η_0 and $\mathcal{M}$, then η coincides with $\tilde{\eta}$ $\mathbb{P}$-a.s..

Notice that the covariance function of $\mathcal{M}$ and Assumption **(A1)** have determined the distribution of $(\mathcal{M}, \eta_0)$. Since equation (1.5) is linear and is driven by additive noise, pathwise uniqueness of solutions to the equation (1.5) follows from uniqueness of solutions to the following PDE

$$\partial_t u = \sigma \Delta u - \nabla \cdot (\bar{\rho} K * u) - \nabla \cdot (u K * \bar{\rho}) - \nabla \cdot (F u), \quad u_0 = 0. \tag{3.20}$$

Lemma 3.11. *Under the assumptions* **(A2)** *and* **(A4)** *with parameter β, for each $\alpha \in (d/2, \beta)$, $u \equiv 0$ is the only solution with zero initial value to* (3.20) *in the sense that*

(1) $u \in L^2([0,T], H^{-\alpha}) \cap C([0,T], H^{-\alpha-2})$.
(2) For each $\varphi \in C^\infty$ and $t \in [0,T]$,

$$\langle u_t, \varphi \rangle = \int_0^t \langle \sigma u_s, \Delta \varphi \rangle \mathrm{d}s + \int_0^t \langle \bar{\rho}_s K * u_s + u_s K * \bar{\rho}_s + F u_s, \nabla \varphi \rangle \mathrm{d}s.$$

Proof. Testing u with the Fourier basis $\{e_k\}_{k \in \mathbb{Z}^d}$, then we find for every $t \in [0,T]$ and $k \in \mathbb{Z}^d$,

$$\partial_t |\langle u_t, e_k \rangle|^2 = -2\sigma |k|^2 \langle u_t, e_k \rangle \langle u_t, e_{-k} \rangle + \langle u_t, e_{-k} \rangle [J_t^1(k) + J_t^2(k)] + \langle u_t, e_k \rangle [J_t^1(-k) + J_t^2(-k)]$$
$$+ \sqrt{-1} k \langle u_t, e_{-k} \rangle \langle F u_t, e_k \rangle - \sqrt{-1} k \langle u_t, e_k \rangle \langle F u_t, e_{-k} \rangle, \tag{3.21}$$

where $J_t^1(k)$ and $J_t^2(k)$, for each $k \in \mathbb{Z}^d$, are defined by

$$J_t^1(k) := \sqrt{-1} k \int_{\mathbb{T}^d} K * u_t(x) e_k(x) \bar{\rho}_t(x) \mathrm{d}x,$$

$$J_t^2(k) := \sqrt{-1} k \int_{\mathbb{T}^d} K * \bar{\rho}_t(x) e_k(x) u_t(x) \mathrm{d}x.$$

Integrating (3.21) over time, summing up over k with weight $\langle k \rangle^{-2\alpha-2}$, and applying Young's inequality yields that there exists a constant C_ϵ for each $\epsilon > 0$ such that

$$\sum_{k \in \mathbb{Z}^d} \langle k \rangle^{-2\alpha-2} |\langle u_t, e_k \rangle|^2 + 2\sigma \sum_{k \in \mathbb{Z}^d} \langle k \rangle^{-2\alpha} \int_0^t |\langle u_t, e_k \rangle|^2 \mathrm{d}s$$

$$\leqslant C_\epsilon \sum_{k \in \mathbb{Z}^d} \langle k \rangle^{-2\alpha-2} \int_0^t |\langle u_s, e_k \rangle|^2 \mathrm{d}s + \epsilon \sum_{k \in \mathbb{Z}^d} \langle k \rangle^{-2\alpha-2} \int_0^t |J_s^1(-k) + J_s^2(-k)|^2 \mathrm{d}s$$

$$+ \epsilon \sum_{k \in \mathbb{Z}^d} \langle k \rangle^{-2\alpha-2} |k|^2 \int_0^t |\langle F u_s, e_k \rangle|^2 \mathrm{d}s. \tag{3.22}$$

To make (3.22) suitable for applying Gronwall's lemma, we first find estimates related to $J_t^1(k)$ and $J_t^2(k)$,

$$\sum_{k \in \mathbb{Z}^d} \langle k \rangle^{-2\alpha-2} |J^1(k)|^2 = \sum_{k \in \mathbb{Z}^d} \langle k \rangle^{-2\alpha-2} |k|^2 \langle K * u\bar{\rho}, e_k \rangle \langle K * u\bar{\rho}, e_{-k} \rangle \leqslant \|K * u\bar{\rho}\|_{H^{-\alpha}}^2,$$

and

$$\sum_{k \in \mathbb{Z}^d} \langle k \rangle^{-2\alpha-2} |J^2(k)|^2 = \sum_{k \in \mathbb{Z}^d} \langle k \rangle^{-2\alpha-2} |k|^2 \langle K * \bar{\rho} u, e_k \rangle \langle K * \bar{\rho} u, e_{-k} \rangle \leqslant \|K * \bar{\rho} u\|_{H^{-\alpha}}^2.$$

Then applying Lemmas A.1 and A.2 gives that

$$\|K * u\bar{\rho}\|_{H^{-\alpha}} \leqslant C_\alpha \|K * u\|_{H^{-\alpha}} \|\bar{\rho}\|_{C^\beta}, \quad \|K * \bar{\rho} u\|_{H^{-\alpha}} \leqslant C_\alpha \|u\|_{H^{-\alpha}} \|K * \bar{\rho}\|_{C^\beta}.$$

Furthermore, by Lemma A.4, we deduce

$$\sum_{k\in\mathbb{Z}^d}\langle k\rangle^{-2\alpha-2}|J^1(k)|^2 + \sum_{k\in\mathbb{Z}^d}\langle k\rangle^{-2\alpha-2}|J^2(k)|^2 \leqslant C_\alpha\|u\|_{H^{-\alpha}}^2\|K\|_{L^1}^2\|\bar\rho\|_{C^\beta}^2. \tag{3.23}$$

Similarly, we obtain

$$\sum_{k\in\mathbb{Z}^d}\langle k\rangle^{-2\alpha-2}|k|^2|\langle Fu,e_k\rangle|^2 \leqslant \|Fu\|_{H^{-\alpha}}^2 \leqslant C_\alpha\|F\|_{C^\beta}^2\|u\|_{H^{-\alpha}}^2. \tag{3.24}$$

Since $u\in L^2([0,T],H^{-\alpha})$, we obtain $\partial_t u\in L^2([0,T],H^{-\alpha-2})$, which by Lions-Magenes Lemma implies $u\in C([0,T],H^{-\alpha-1})$. Combining (3.22)-(3.24) leads to

$$\|u_t\|_{H^{-\alpha-1}}^2 + 2\sigma\int_0^t\|u_s\|_{H^{-\alpha}}^2\mathrm{d}s$$

$$\leqslant C_\epsilon\int_0^t\|u_s\|_{H^{-\alpha-1}}^2\mathrm{d}s + \epsilon C_\alpha\int_0^t\left(\|K\|_{L^1}^2\|\bar\rho_s\|_{C^\beta}^2 + \|F\|_{C^\beta}^2\right)\|u_s\|_{H^{-\alpha}}^2\mathrm{d}s. \tag{3.25}$$

Choosing ϵ such that

$$\epsilon C_\alpha\|K\|_{L^1}^2\left(\sup_{s\in[0,t]}\|\bar\rho_s\|_{C^\beta}^2 + \|F\|_{C^\beta}^2\right)<2\sigma,$$

then using Gronwall's inequality gives

$$\|u_t\|_{H^{-\alpha-1}}^2 + \int_0^t\|u_s\|_{H^{-\alpha}}^2\mathrm{d}s = 0.$$

This completes the proof.   □

Proof of Theorem 1.4 . We have proved that the sequence of laws of $\{\eta^N\}_{N\in\mathbb{N}}$ is tight and every tight limit is a martingale solution to (1.5) (Theorem 3.9). As a result, existence of martingale solutions (equivalently probabilistically weak solutions) follows. On the other hand, Lemma 3.11 together with Corollary 2.7 implies pathwise uniqueness of probabilistically weak solutions. Then applying the general Yamada-Watanabe theorem [Kur14, Theorem 1.5] gives that the law of martingale solutions starting from the same initial distribution is unique, and every probabilistically weak solution is a probabilistically strong solution. Therefore η^N converges in distribution to the unique (in distribution) martingale solution η.   □

Remark 3.12. *From the proof of Theorem 1.4, we also obtain the well-posedness of probabilistically strong solutions to the SPDE* (1.5).

3.3. **Optimal regularity.** In this subsection we improve the regularity of η by using the mild formulation and the smooth effect of the heat kernel.

Recall that $\mathcal{M}$ is a centered Gaussian process with covariance given by

$$\mathbb{E}[\mathcal{M}_t(\varphi_1)\mathcal{M}_s(\varphi_2)] = 2\sigma\int_0^{s\wedge t}\langle\nabla\varphi_1\cdot\nabla\varphi_2,\bar\rho_r\rangle\mathrm{d}r,$$

for $\varphi_1,\varphi_2\in C^\infty(\mathbb{T}^d)$. Therefore, the distribution of $\mathcal{M}$ is uniquely determined, and one can regard $\mathcal{M}$ as $\nabla\cdot\int_0^\cdot\sqrt{\bar\rho}\xi(\mathrm{d}s,\mathrm{d}x)$ with $\xi=(\xi^i)_{i=1}^d$ being vector valued space-time white noise on $\mathbb{R}^+\times\mathbb{T}^d$. In fact, for every $\varphi\in C^\infty$,

$$\mathcal{M}_t(\varphi) \overset{d}{=} -\sqrt{2\sigma}\int_0^t\int_{\mathbb{T}^d}\nabla\varphi(x)\sqrt{\bar\rho_s(x)}\xi(\mathrm{d}s,\mathrm{d}x), \tag{3.26}$$

where $\overset{d}{=}$ means equal in distribution and we omit the inner product in $\mathbb{R}^d$ between ξ and $\nabla\varphi$. We start with investigating the regularity of a stochastic integral, which will be the stochastic term in the mild form of equation (1.5). Define a stochastic process Z as

$$Z_t := \int_0^t\int_{\mathbb{T}^d}\nabla\Gamma_{t-s}(\cdot-y)\sqrt{\bar\rho_s(y)}\xi(\mathrm{d}s,\mathrm{d}y), \tag{3.27}$$

where Γ is the heat kernel of $\sigma\Delta$ on $\mathbb{T}^d$.

Recall that $\{\chi_n\}_{n\geqslant -1}$ is the Littlewood-Paley partition functions and $\chi_n(\cdot) = \chi_0(2^{-n}\cdot)$ for $n \geqslant 0$ (see Appendix A). Denote ψ_n be the inverse Fourier transform of χ_n for every n, we then have the following result.

Lemma 3.13. *For each $\kappa > 0$ and every $n \geqslant -1$, it holds for all $s, t \in [0, T]$ and $x \in \mathbb{T}^d$ that*

$$\mathbb{E}\left|\langle Z_t, \psi_n(\cdot - x)\rangle\right|^2 \lesssim 2^{dn},$$

$$\mathbb{E}\left|\langle Z_t, \psi_n(\cdot - x)\rangle - \langle Z_s, \psi_n(\cdot - x)\rangle\right|^2 \lesssim 2^{dn+2\kappa n}(t - s)^\kappa,$$

where the proportional constants depend on $\|\bar{\rho}\|_{C_T L^\infty}$.

Proof. For simplicity we set $\sigma = 1$ in the proof. First, we use Fourier transform to represent $\langle Z_t, \psi_n(\cdot - x)\rangle$ as follows,

$$\langle Z_t, \psi_n(\cdot - x)\rangle = \int_0^t \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \nabla\Gamma_{t-r}(y - z)\sqrt{\bar{\rho}_r(z)}\psi_n(y - x)\mathrm{d}y\xi(\mathrm{d}r, \mathrm{d}z)$$

$$= \int_0^t \int_{\mathbb{T}^d} \langle\nabla\Gamma_{t-r}(\cdot - z), \psi_n(\cdot - x)\rangle \sqrt{\bar{\rho}_r(z)}\xi(\mathrm{d}r, \mathrm{d}z)$$

$$= \int_0^t \int_{\mathbb{T}^d} \sum_{k\in\mathbb{Z}^d} G_{t-r}(k)e_{-k}(z)\sqrt{\bar{\rho}_r(z)}\xi(\mathrm{d}r, \mathrm{d}z), \tag{3.28}$$

where $G_t(k)$ is defined by

$$G_t(k) := \int_{\mathbb{T}^d} \langle\nabla\Gamma_t(\cdot - z'), \psi_n(\cdot - x)\rangle e_k(z')\mathrm{d}z'$$

and we used $\langle\nabla\Gamma_{t-r}(\cdot - z), \psi_n(\cdot - x)\rangle \in L^2(\mathbb{T}^d)$ and the sum in (3.28) converges in $L^2(\mathbb{T}^d)$. Furthermore, noticing that $G_t(-k)$ is the complex conjugate of $G_t(k)$, we thus have

$$\mathbb{E}\left|\langle Z_t, \psi_n(\cdot - x)\rangle\right|^2 = \int_0^t \int_{\mathbb{T}^d} \left|\sum_{k\in\mathbb{Z}^d} G_{t-r}(k)e_{-k}(z)\right|^2 \bar{\rho}_r(z)\mathrm{d}z\mathrm{d}r$$

$$\lesssim \|\bar{\rho}\|_{C_T L^\infty} \sum_{k_1\in\mathbb{Z}^d} \sum_{k_2\in\mathbb{Z}^d} \int_0^t \int_{\mathbb{T}^d} G_{t-r}(k_1)G_{t-r}(-k_2)e_{-k_1}(z)e_{k_2}(z)\mathrm{d}z\mathrm{d}r$$

$$\lesssim \|\bar{\rho}\|_{C_T L^\infty} \sum_{k\in\mathbb{Z}^d} \int_0^t G_{t-r}(k)G_{t-r}(-k)\mathrm{d}r, \tag{3.29}$$

where the last inequality follows by $\int_{\mathbb{T}^d} e_{k_2-k_1}(z)\mathrm{d}z = C_d\delta_{k_2=k_1}$, C_d is the volume of $\mathbb{T}^d$.

For each $k \in \mathbb{Z}^d$ and $t \in [0, T]$, we find that

$$G_t(k) = \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \nabla\Gamma_t(y - z')\psi_n(y - x)e_k(z')\mathrm{d}y\mathrm{d}z'$$

$$= \langle\nabla\Gamma_t, e_{-k}\rangle \langle\psi_n, e_k\rangle e_k(x)$$

$$= -\sqrt{-1}ke^{-t|k|^2}\chi_n(-k)e_k(x). \tag{3.30}$$

Here we used the facts that Γ is the heat kernel on $\mathbb{T}^d$ and $\langle\psi_n, e_{-k}\rangle = \chi_n(k)$.

Combining (3.29) and (3.30) yields that

$$\mathbb{E}\left|\langle Z_t, \psi_n(\cdot - x)\rangle\right|^2 \lesssim \|\bar{\rho}\|_{C_T L^\infty} \sum_{k\in\mathbb{Z}^d} \int_0^t |k|^2 e^{-2|k|^2(t-r)}\chi_n(-k)\chi_n(k)\mathrm{d}r.$$

Notice that

$$\int_0^t |k|^2 e^{-2|k|^2(t-r)}\mathrm{d}r = \frac{1}{2}\left(1 - e^{-2|k|^2 t}\right),$$

which implies that for $n \geqslant 0$

$$\mathbb{E}\left|\langle Z_t, \psi_n(\cdot - x)\rangle\right|^2 \lesssim_{\bar{\rho}} \sum_{k \in \mathbb{Z}^d} \chi_n(-k)\chi_n(k)\left(1 - e^{-2|k|^2 t}\right)$$

$$\lesssim_{\bar{\rho}} 2^{dn} \int \chi_0(-k')\chi_0(k')\left(1 - e^{-2^{2n+1}|k'|^2 t}\right)\mathrm{d}k' \lesssim_{\bar{\rho}} 2^{dn}. \tag{3.31}$$

Here we used $\chi_n(\cdot) = \chi_0(2^{-n}\cdot)$ and the fact that χ_0 is of compact support. The case $n = -1$ is similar.

Next, we deduce by (3.28) that

$$\langle Z_t, \psi_n(\cdot - x)\rangle - \langle Z_s, \psi_n(\cdot - x)\rangle$$

$$= \int_0^t \int_{\mathbb{T}^d} \sum_{k \in \mathbb{Z}^d} G_{t-r}(k)e_{-k}(z)\sqrt{\bar{\rho}_r(z)}\xi(\mathrm{d}r, \mathrm{d}z) - \int_0^s \int_{\mathbb{T}^d} \sum_{k \in \mathbb{Z}^d} G_{s-r}(k)e_{-k}(z)\sqrt{\bar{\rho}_r(z)}\xi(\mathrm{d}r, \mathrm{d}z)$$

$$= \int_s^t \int_{\mathbb{T}^d} \sum_{k \in \mathbb{Z}^d} G_{t-r}(k)e_{-k}(z)\sqrt{\bar{\rho}_r(z)}\xi(\mathrm{d}r, \mathrm{d}z) + \int_0^s \int_{\mathbb{T}^d} \sum_{k \in \mathbb{Z}^d} \left[G_{t-r}(k) - G_{s-r}(k)\right]e_{-k}(z)\sqrt{\bar{\rho}_r(z)}\xi(\mathrm{d}r, \mathrm{d}z)$$

$$:= J_1^n + J_2^n.$$

Moreover, we have

$$\mathbb{E}\left|\langle Z_t, \psi_n(\cdot - x)\rangle - \langle Z_s, \psi_n(\cdot - x)\rangle\right|^2 \leqslant 2\mathbb{E}|J_1^n|^2 + 2\mathbb{E}|J_2^n|^2. \tag{3.32}$$

Again, it suffices to check the cases with $n \geqslant 0$. Similar as in (3.31), we have

$$\mathbb{E}|J_1^n|^2 \lesssim_{\bar{\rho}} \sum_{k \in \mathbb{Z}^d} \int_s^t |k|^2 e^{-2|k|^2(t-r)}\chi_n(k)\chi_n(-k)\mathrm{d}r$$

$$\lesssim_{\bar{\rho}} \sum_{k \in \mathbb{Z}^d} \left(1 - e^{-2|k|^2(t-s)}\right)\chi_n(k)\chi_n(-k)$$

$$\lesssim_{\bar{\rho}} 2^{dn} \int \left(1 - e^{-2^{2n+1}|k'|^2(t-s)}\right)\chi_0(k')\chi_0(-k')\mathrm{d}k' \lesssim_{\bar{\rho}} 2^{dn}(1 - e^{-C2^{2n}(t-s)}),$$

where $C > 0$ is a universal constant determinded by the support of χ_0. Notice that for each $\kappa > 0$, it holds that $1 - e^{-a} \lesssim a^\kappa$ for $a \geqslant 0$. Therefore, for each $\kappa > 0$, let $C2^{2n}(t-s)$ in the above inequality play the role of a, we arrive at

$$\mathbb{E}|J_1^n|^2 \lesssim_{\bar{\rho}} 2^{dn+2\kappa n}(t-s)^\kappa. \tag{3.33}$$

Similarly, one can study J_2^n and find

$$\mathbb{E}|J_2^n|^2 \lesssim_{\bar{\rho}} \sum_{k \in \mathbb{Z}^d} \int_0^s |k|^2 \chi_n(k)\chi_n(-k)\left(e^{-|k|^2(s-r)} - e^{-|k|^2(t-r)}\right)^2 \mathrm{d}r$$

$$\lesssim_{\bar{\rho}} \sum_{k \in \mathbb{Z}^d} \left(1 - e^{-2|k|^2 s}\right)\chi_n(k)\chi_n(-k)\left(1 - e^{-|k|^2(t-s)}\right)^2$$

$$\lesssim_{\bar{\rho}} 2^{dn} \int_{k' \in \mathbb{R}^d} \chi_0(k')\chi_0(-k')\left(1 - e^{-2^{2n}|k'|^2(t-s)}\right)^2 \mathrm{d}k' \lesssim_{\bar{\rho}} 2^{dn+2\kappa n}(t-s)^\kappa,$$

for each $\kappa > 0$. This together with (3.32) and (3.33) leads to

$$\mathbb{E}\left|\langle Z_t, \psi_n(\cdot - x)\rangle - \langle Z_s, \psi_n(\cdot - x)\rangle\right|^2 \lesssim_{\bar{\rho}} 2^{dn+2\kappa n}(t-s)^\kappa.$$

The proof is thus completed. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We now apply the above result to study regularity of the process Z.

Lemma 3.14. *Suppose that $\bar{\rho} \in C([0,T], L^\infty)$. It holds that $Z \in C([0,T], C^{-\alpha})$ $\mathbb{P}$-a.s. for every $\alpha > d/2$. Moreover, for all $p > 2$,*

$$\mathbb{E} \sup_{t \in [0,T]} \|Z_t\|_{C^{-\alpha}}^p < \infty.$$

Proof. Since Z is a centered Gaussian process, Lemma 3.13 together with the hypercontractivity property [Nua06, Theorem 1.4.1] implies that

$$\mathbb{E} |\langle Z_t, \psi_n(\cdot - x)\rangle|^p \lesssim \left(\mathbb{E} |\langle Z_t, \psi_n(\cdot - x)\rangle|^2\right)^{\frac{p}{2}} \lesssim 2^{\frac{dnp}{2}},$$

$$\mathbb{E} |\langle Z_t, \psi_n(\cdot - x)\rangle - \langle Z_s, \psi_n(\cdot - x)\rangle|^p \lesssim \left(\mathbb{E} |\langle Z_t, \psi_n(\cdot - x)\rangle - \langle Z_s, \psi_n(\cdot - x)\rangle|^2\right)^{\frac{p}{2}} \lesssim 2^{(\frac{d}{2}+\kappa)np}(t - s)^{\frac{\kappa p}{2}},$$

for each $\kappa > 0$, $p > 2$, and every $n \geqslant -1$. This allows us to apply the Kolmogorov criterion [MW17, Lemma 10] to conclude that $Z \in C([0,T], B_{p,p}^{-\alpha})$ $\mathbb{P}$-a.s., for each $p > 2$ and every $\alpha > d/2 + 2/p$. Moreover,

$$\mathbb{E} \sup_{t \in [0,T]} \|Z_t\|_{B_{p,p}^{-\alpha}}^p < \infty.$$

The result follows by the embedding $B_{p,p}^{-\alpha} \hookrightarrow B_{\infty,\infty}^{-\beta}$ for $\beta > \alpha + d/p$ (see Lemma A.1). $\qquad\square$

Next we rewrite the unique solution η to (1.5), which has been obtained in Section 3.2, in the mild form.

Proposition 3.15. *Under the assumptions* **(A1)-(A4)***, the unique solution η to* (1.5) *satisfies*

$$\eta_t = \Gamma_t * \eta_0 - \int_0^t \nabla\Gamma_{t-s} * (\bar{\rho}K * \eta + \eta K * \bar{\rho} + F\eta)\mathrm{d}s - \sqrt{2\sigma}\tilde{Z}_t, \quad \mathbb{P} - a.s,$$

where $\tilde{Z}$ has the same distribution as Z.

Proof. We start with the following statement: for every function φ of class $C^1([0,t], C^\infty(\mathbb{T}^d))$ and $t \in [0,T]$, it holds that,

$$\langle\eta_t, \varphi(t)\rangle - \langle\eta_0, \varphi(0)\rangle = \int_0^t \langle\eta_s, \partial_s\varphi + \sigma\Delta\varphi\rangle \,\mathrm{d}s + \int_0^t \langle\bar{\rho}_s K * \eta_s + \eta_s K * \bar{\rho}_s + F\eta_s, \nabla\varphi\rangle\mathrm{d}s$$
$$+ \sqrt{2\sigma} \int_0^t \int_{\mathbb{T}^d} \nabla\varphi(x)\sqrt{\bar{\rho}_s(x)}\xi(\mathrm{d}s, \mathrm{d}x). \tag{3.34}$$

It is straightforward to check the statement for finite linear combinations of functions φ of the form $\varphi(s,x) = \varphi_1(s)\varphi_2(x)$, where $\varphi_1 \in C^\infty([0,t])$ and $\varphi_2 \in C^\infty(\mathbb{T}^d)$. Then one can uniformly approximate functions in $C^1([0,t], C^\infty(\mathbb{T}^d))$ with such combinations and find (3.34).

For every $\varphi_0 \in C^\infty(\mathbb{T}^d)$ and $0 \leqslant s \leqslant t$, define $\varphi(s) := \Gamma_{t-s} * \varphi_0$, then $\partial_s\varphi(s) = -\sigma\Delta\varphi(s)$ and $\varphi(t) = \varphi_0$. By (3.34), we find

$$\langle\eta_t, \varphi_0\rangle - \langle\eta_0, \Gamma_t * \varphi_0\rangle = \int_0^t \langle\bar{\rho}_s K * \eta_s + \eta_s K * \bar{\rho}_s + F\eta_s, \nabla\Gamma_{t-s} * \varphi_0\rangle\mathrm{d}s$$
$$+ \sqrt{2\sigma} \int_0^t \int_{\mathbb{T}^d} (\nabla\Gamma_{t-s} * \varphi_0)(x)\sqrt{\bar{\rho}_s(z)}\xi(\mathrm{d}s, \mathrm{d}x)$$
$$= -\int_0^t \langle\nabla\Gamma_{t-s} * (\bar{\rho}K * \eta + \eta K * \bar{\rho} + F\eta), \varphi_0\rangle \,\mathrm{d}s$$
$$- \sqrt{2\sigma} \int_{\mathbb{T}^d} \varphi_0(x)\left(\int_0^t \int_{\mathbb{T}^d} \nabla\Gamma_{t-s}(x - y)\sqrt{\bar{\rho}(y)}\xi(\mathrm{d}s, \mathrm{d}y)\right)\mathrm{d}x,$$

where we used symmetry of Γ at the last inequality. The result then follows by arbitrary $\varphi_0 \in C^\infty(\mathbb{T}^d)$ and the definition of Z. $\qquad\square$

This result gives rise to the definition of mild solutions to (1.5) on a stochastic basis $(\Omega, \mathcal{F}_t, \mathcal{F}, \mathbb{P})$. We set Z given by (3.27) with ξ being vector-valued space-time white noise on $(\Omega, \mathcal{F}_t, \mathcal{F}, \mathbb{P})$.

Definition 3.16. *Assume that* $K \in L^1$, $\bar{\rho} \in C([0, T], C^\beta)$, *and* $F \in C^\beta$ *for some* $\beta > d/2$. *We call* $\eta \in C([0, T], \mathcal{S}'(\mathbb{T}^d)) \cap L^2([0, T], B_{p,q}^{-\alpha})$ *with* $\alpha < \beta$, $p, q \in [1, \infty]$ *is a mild solution to* (1.5) *with initial condition* η_0 *if for every* $\varphi \in C^\infty$

$$\langle \eta_t, \varphi \rangle = \langle \Gamma_t * \eta_0, \varphi \rangle - \int_0^t \langle \nabla\Gamma_{t-s} * (\bar{\rho}K * \eta + \eta K * \bar{\rho} + F\eta), \varphi \rangle \, \mathrm{d}s - \sqrt{2\sigma} \langle Z_t, \varphi \rangle.$$

Remark 3.17. *By Proposition 3.15, we know, under the assumptions* **(A1)-(A4)**, *the solutions to* (1.5) *obtained from Theorem 3.9 have the same law as the mild solutions.*

To make sense of $\bar{\rho}K * \eta$, $\eta K * \bar{\rho}$, *and* $F\eta$ *in the definition of mild solutions, we need the condition* $\eta_t \in B_{p,q}^{-\alpha}$ *for a.e.* $t \in [0, T]$ *with* $\alpha < \beta$ *and* $p, q \in [1, \infty]$, *where* β *is from* **(A4)**.

The following result based on the smoothing effect of heat kernel (see Lemma A.5) gives the optimal regularity of η.

Proposition 3.18. *Suppose that Assumption* **(A4)** *holds with parameter* $\beta > d/2$ *and* η *is a mild solution to* (1.5), *and assume* $\eta_0 \in L^r(\Omega, B_{p,q}^{-\alpha})$ *for some* $r > 2$, $\alpha \in (d/2, \beta)$, *and* $p, q \in [1, \infty]$. *Then* $\eta \in C([0, T], B_{p,q}^{-\alpha})$ *almost surely. Moreover,*

$$\mathbb{E} \sup_{t \in [0,T]} \|\eta_t\|_{B_{p,q}^{-\alpha}}^r < \infty.$$

Proof. Firstly, applying Lemma A.5, we have

$$\|\eta_t\|_{B_{p,q}^{-\alpha}} \lesssim \|\eta_0\|_{B_{p,q}^{-\alpha}} + \int_0^t (t-s)^{-\frac{1}{2}} \left[\|\bar{\rho}K * \eta\|_{B_{p,q}^{-\alpha}} + \|\eta K * \bar{\rho}\|_{B_{p,q}^{-\alpha}} + \|F\eta\|_{B_{p,q}^{-\alpha}} \right] \mathrm{d}s$$
$$+ \|Z_t\|_{B_{p,q}^{-\alpha}}.$$

To further estimate the right hand side of the above inequality, by $\alpha < \beta$, applying Lemmas A.1-A.4 gives that

$$\|\bar{\rho}K * \eta\|_{B_{p,q}^{-\alpha}} + \|\eta K * \bar{\rho}\|_{B_{p,q}^{-\alpha}} \lesssim \|\bar{\rho}\|_{C^\beta} \|K\|_{L^1} \|\eta\|_{B_{p,q}^{-\alpha}}, \quad \|F\eta\|_{B_{p,q}^{-\alpha}} \lesssim \|F\|_{C^\beta} \|\eta\|_{B_{p,q}^{-\alpha}}.$$

Hence,

$$\|\eta_t\|_{B_{p,q}^{-\alpha}} \lesssim \|\eta_0\|_{B_{p,q}^{-\alpha}} + \int_0^t (t-s)^{-\frac{1}{2}} \|\eta_s\|_{B_{p,q}^{-\alpha}} \mathrm{d}s + \|Z_t\|_{B_{p,q}^{-\alpha}}. \tag{3.35}$$

By Hölder's inequality, we find

$$\int_0^t (t-s)^{-\frac{1}{2}} \|\eta_s\|_{B_{p,q}^{-\alpha}} \mathrm{d}s \lesssim \left(\int_0^t \|\eta_s\|_{B_{p,q}^{-\alpha}}^r \mathrm{d}s \right)^{\frac{1}{r}} \left(\int_0^t (t-s)^{-\frac{r}{2(r-1)}} \mathrm{d}s \right)^{\frac{r-1}{r}}$$
$$\lesssim \left(\int_0^t \|\eta_s\|_{B_{p,q}^{-\alpha}}^r \mathrm{d}s \right)^{\frac{1}{r}} t^{\frac{1}{2} - \frac{1}{r}}, \tag{3.36}$$

where in the last step we used $r > 2$ to have $\frac{r}{2(r-1)} < 1$. Furthermore, applying Gronwall's inequality to (3.35) yields that

$$\mathbb{E} \sup_{t \in [0,T]} \|\eta_t\|_{B_{p,q}^{-\alpha}}^r \lesssim \mathbb{E}\|\eta_0\|_{B_{p,q}^{-\alpha}}^r + \mathbb{E}\|Z\|_{C_T B_{p,q}^{-\alpha}}^r + 1.$$

By the assumption on η_0 and Lemma 3.14, the right hand side of the above inequality is thus finite.

Using (3.35), the continuity of η on $[0, T]$ follows by the continuity of Z and continuity of Γ_t from Lemma A.5. The proof is thus completed. $\qquad\square$

Remark 3.19. *By* [Hai14] *we know the space-time white noise* $\xi \in C_{t,x}^{-\frac{d}{2}-1-\varepsilon}$ $\mathbb{P}$-*a.s. for every* $\varepsilon > 0$, *where* $C_{t,x}^{-\frac{d}{2}-1-\varepsilon}$ *is endowed with suitable parabolic time space scaling. Hence by Schauder estimates* $\eta \in C([0, T], C^{-\alpha})$ *for* $\alpha > d/2$ *gives the best regularity by taking* $p, q = \infty$ *in Proposition 3.18.*

Remark 3.20. *By the optimal regularity of η and Lemma A.2, $K * \bar{\rho}$ needs to stay in C^β with $\beta > d/2$ so that the term $K * \bar{\rho}\eta$ appearing in SPDE (1.5) is well-defined. Applying Lemma A.4 and noticing $K \in L^1$ for K satisfying **(A2)**, the assumption about $\bar{\rho}$ in **(A4)** is thus a sufficient condition for $K * \bar{\rho} \in C^\beta$. Moreover, $\beta > d/2$ is optimal in general. With appropriate modifications of the proof in Section 3.2 and by the convolution inequality in Lemma A.4, we could also weaken the condition of $\bar{\rho}$ to $\bar{\rho} \in C^{\beta - \beta_1}$ with $\beta > d/2$ and $\beta_1 \in (0, \frac{d}{2})$, at the cost of stronger condition on the interacting kernel: $K \in C^{\beta_1}$.*

3.4. **Gaussianity.** This section is devoted to the proof of Proposition 1.5. As mentioned in the introduction, we need a class of time evolution operators $\{Q_{s,t}\}$ in order to rewrite η as the generalized Ornstein-Uhlenbeck process (1.6), which would be given by the following result.

Lemma 3.21. *Assume that $\bar{\rho} \in C([0,T], C^{\beta+1}(\mathbb{T}^d))$ and $F \in C^{\beta+1}(\mathbb{T}^d)$ with $\beta > d/2$, for each $\varphi \in C^\infty(\mathbb{T}^d)$ and $t \in [0,T]$, there exists a unique solution $f \in L^2([0,t], H^{\beta+2}) \cap C([0,t], H^{\beta+1})$ with $\partial_s f \in L^2([0,t], H^\beta)$ to the following backward equation:*

$$f_s = \varphi + \sigma \int_s^t \Delta f_r \mathrm{d}r + \int_s^t \Big[K * \bar{\rho}_r \cdot \nabla f_r + K(-\cdot) * (\nabla f_r \bar{\rho}_r) + F \cdot \nabla f_r \Big] \mathrm{d}r, \quad s \in [0,t], \qquad (3.37)$$

where $K(-\cdot) * g$ is given in (3.18).

Proof. Similar to (3.25), we obtain the following a priori energy estimate for any $\epsilon > 0$

$$\|f_s\|_{H^{\beta+1}}^2 + 2\sigma \int_s^t \|f_r\|_{H^{\beta+2}}^2 \mathrm{d}r$$

$$\leqslant \|\varphi\|_{H^{\beta+1}}^2 + C_\epsilon \int_s^t \|f_r\|_{H^{\beta+1}}^2 \mathrm{d}r + \epsilon(\|K\|_{L^1}^2 \|\bar{\rho}\|_{C_T C^{\beta+1}}^2 + \|F\|_{C^{\beta+1}}^2) \int_s^t \|f_r\|_{H^{\beta+2}}^2 \mathrm{d}r.$$

Choosing $\epsilon > 0$ sufficiently small, $f \in L^2([0,t], H^{\beta+2})$ follows from the Gronwall's inequality. Furthermore, by Lemma A.2 and Lemma A.4, we find

$$\|K * \bar{\rho} \cdot \nabla f\|_{H^\beta} \lesssim \|K * \bar{\rho}\|_{C^\beta} \|f\|_{H^{\beta+1}} \lesssim \|K\|_{L^1} \|\bar{\rho}\|_{C^\beta} \|f\|_{H^{\beta+1}}, \quad \|F \cdot \nabla f\|_{H^\beta} \lesssim \|F\|_{C^\beta} \|f\|_{H^{\beta+1}},$$

$$\|K(-\cdot) * (\bar{\rho}\nabla f)\|_{H^\beta} \lesssim \|K\|_{L^1} \|\bar{\rho}\nabla f\|_{H^\beta} \lesssim \|K\|_{L^1} \|\bar{\rho}\|_{C^\beta} \|f\|_{H^{\beta+1}}.$$

Hence we deduce from equation (3.37) that $\partial_t f \in L^2([0,t], H^\beta)$, which combined with $f \in L^2([0,t], H^{\beta+2})$ implies that $f \in C([0,t], H^{\beta+1})$ by Lions-Magenes Lemma. When $\varphi = 0$, the above energy estimate implies that $f = 0$. This fact together with linearity of equation implies the uniqueness of solutions. On the other hand, one can obtain the existence of solutions to (3.37) by classical Galerkin method (cf. [Eva98, Chapter 7]). $\square$

Define the space $\mathcal{X}_t^\beta$ and time evolution operators $\{Q_{\cdot,t}\}_{0 \leqslant t \leqslant T} : C^\infty(\mathbb{T}^d) \to \mathcal{X}_t^\beta$ as

$$\mathcal{X}_t^\beta := \big\{ f \in L^2([0,t], H^{\beta+2}) \cap C([0,t], H^{\beta+1}); \partial_s f \in L^2([0,t], H^\beta) \big\},$$

$$Q_{s,t}\varphi := f(s).$$

where f is the unique solution to (3.37) with terminal value φ at time t and is given by Lemma 3.21.

Now we are in the position to justify the Gaussianity of the unique (in distribution) limit of fluctuation measures.

Proof of Proposition 1.5. Recall that $\eta \in L^2([0,T], H^{-\alpha})$ for any $\alpha > d/2$. Then for each test function $f \in \mathcal{X}_t^\beta$ with $\beta > d/2$, by Lemma A.2 and Lemma A.4, we have

$$\|\langle \eta_s, \partial_s f \rangle\|_{L_T^1} \lesssim \|\eta\|_{L_T^2 H^{-\beta}} \|\partial_s f\|_{L_T^2 H^\beta}, \quad \|\langle \eta_s, \Delta f_s \rangle\|_{L_T^1} \lesssim \|\eta\|_{L_T^2 H^{-\beta}} \|f\|_{L_T^2 H^{\beta+2}},$$

$$\|\langle \bar{\rho}_s K * \eta_s + \eta_s K * \bar{\rho}_s + F\eta_s, \nabla f_s \rangle\|_{L_T^1} \lesssim \Big(\|K\|_{L^1} \|\bar{\rho}\|_{C_T C^\beta} + \|F\|_{C^\beta} \Big) \|\eta\|_{L_T^2 H^{-\beta+\epsilon}} \|f\|_{L_T^2 H^{\beta+1}} \lesssim 1,$$

where $\epsilon > 0$ is sufficiently small, so that the weak formulation (3.34) extends to all the test fucntions $f \in \mathcal{X}_t^\beta$ with $\beta > d/2$. For each $\varphi \in C^\infty$, choosing $Q_{\cdot,t}\varphi$ as the test function in (3.34), we find

$$
\begin{aligned}
\langle \eta_t, \varphi \rangle - \langle \eta_0, Q_{0,t}\varphi \rangle &= \int_0^t \langle \eta_s, \partial_s Q_{s,t}\varphi + \sigma\Delta Q_{s,t}\varphi \rangle \, \mathrm{d}s + \int_0^t \langle \bar{\rho}_s K * \eta_s + \eta_s K * \bar{\rho}_s + F\eta_s, \nabla Q_{s,t}\varphi \rangle \mathrm{d}s \\
&\quad + \sqrt{2\sigma} \int_0^t \int_{\mathbb{T}^d} \nabla Q_{s,t}\varphi(x)\sqrt{\bar{\rho}_s(x)}\xi(\mathrm{d}s, \mathrm{d}x) \\
&= \sqrt{2\sigma} \int_0^t \int_{\mathbb{T}^d} \nabla Q_{s,t}\varphi(x)\sqrt{\bar{\rho}_s(x)}\xi(\mathrm{d}s, \mathrm{d}x),
\end{aligned}
$$

where we used Lemma 3.21 with $Q_{s,t}\varphi = f(s)$. The result follows by the assumption on η_0 and the fact that the stochastic integral is a centered Gaussian process with quadratic variation

$$
2\sigma \int_0^t \langle |\nabla Q_{s,t}\varphi|^2, \bar{\rho}_s \rangle \, \mathrm{d}s.
$$

$\square$

4. The Vanishing Diffusion Case

In this section, we study particle systems with vanishing diffusion, i.e. $\sigma = 0$. We denote the fluctuation measures by $\eta^N := \sqrt{N}(\mu^N - \bar{\rho})$ as well. Instead of the SPDE limit (1.5) in the case when $\sigma > 0$, now the fluctuation measures converge to a deterministic first order nonlocal PDE with random initial value η_0, which reads

$$
\partial_t \eta = -\nabla \cdot (\bar{\rho}K * \eta) - \nabla \cdot (\eta K * \bar{\rho}) - \nabla \cdot (F\eta). \tag{4.1}
$$

With the same proof as in Section 3.1 and Section 3.2, we can deduce that under the assumptions **(A1)-(A3), (A5)**, we have

(1) The sequence of laws of $(\eta^N)_{N \geqslant 1}$ is tight in the space $L^2([0,T], H^{-\alpha}) \cap C([0,T], H^{-\alpha-2})$, for every $\alpha > d/2$.

(2) Any limit η of converging (in distribution) subsequence of $(\eta^N)_{N \geqslant 1}$ is an analytic weak solution to (4.1) in the sense that

$$
\langle \eta_t, \varphi(t) \rangle = \langle \eta_0, \varphi(0) \rangle + \int_0^t \int_{\mathbb{T}^d} \eta_s[\partial_s\varphi + K(-\cdot) * (\bar{\rho}\nabla\varphi) + K * \bar{\rho} \cdot \nabla\varphi + F\nabla\varphi]\mathrm{d}x\mathrm{d}s, \tag{4.2}
$$

for every $\varphi \in C^1([0,t], C^{\beta+1})$ with $\beta > d/2$, $\mathbb{P}$-a.s. and $K(-\cdot) * g$ is given in (3.18).

However, for the case with vanishing diffusion, we cannot deduce uniqueness of the solutions to (4.1) by the proof of Lemma 3.11, due to the lack of the energy inequality (3.25). The following uniqueness result follows by the method of characteristics. We also recall the definition of flow from [Kun97, Chapter 4][1], which is used in the following proof. $\phi_{s,t}$ be a continuous map from $\mathbb{T}^d$ into itself for any $s, t \in [0,T]$ is called a flow if it satisfies the following property

(1) $\phi_{s,u} = \phi_{t,u} \circ \phi_{s,t}$ holds for all s, t, u, where $\circ$ denotes the composition of maps;
(2) $\phi_{s,s} = \mathrm{Id}$;
(3) $\phi_{s,t} : \mathbb{T}^d \to \mathbb{T}^d$ is an onto homeomorphism for all s, t.

We refer to [Kun97, Chapter 4] for the relation between flows and ODEs. In general the solutions to ODEs with regular coefficients could generate a flow.

Proposition 4.1. *Under the assumptions* **(A2)** *and* **(A5)**, $\eta \equiv 0$ *is the only solution with zero initial value to* (4.1) *in the space* $L^2([0,T], H^{-\alpha}) \cap C([0,T], H^{-\alpha-2})$ *for* $\alpha \in (d/2, \beta)$, *the parameter* β *is from* **(A5)**.

---

[1]Although the framework in [Kun97] is for the flow on $\mathbb{R}^d$, it also holds for the periodic case since the functions on $\mathbb{T}^d$ could be viewed as periodic functions on $\mathbb{R}^d$ and the framework in [Kun97, Chapter 4] has been extended to Riemannian manifold.

Proof. We first claim that a similar result to Lemma 3.21 holds for $\sigma = 0$. That is, there exists a unique solution $\varphi \in C^1([0, t], C^{\beta'})$ with $\beta' \in (\alpha+1, \beta+1)$ for any $\varphi(t, x) = \psi(x) \in C^\infty$ to the following backward equation

$$\partial_s \varphi + K(-\cdot) * (\bar{\rho}\nabla\varphi) + K * \bar{\rho} \cdot \nabla\varphi + F \cdot \nabla\varphi = 0, \quad \forall s \in [0, t]. \tag{4.3}$$

Suppose that the claim holds. Then for every $\psi \in C^\infty$ and $t \in [0, T]$, we use (4.2) with the test function given by the solution φ to (4.3) and $\eta_0 = 0$. Then we conclude that $\int_{\mathbb{T}^d} \eta(t, x)\psi(x)\mathrm{d}x = 0$, which implies the result. It thus suffices to justify the claim.

In the following we verify the claim by considering the backward flow $(\phi_{t,s})_{0 \leqslant s \leqslant t \leqslant T}$ generated by

$$\phi_{t,s} = x + \int_s^t (K * \bar{\rho}_r(\phi_{t,r}) + F(\phi_{t,r}))\mathrm{d}r, \quad x \in \mathbb{T}^d, s \in [0, t]. \tag{4.4}$$

Define the forward flow $\phi_{s,t} := \phi_{t,s}^{-1}$, $0 \leqslant s \leqslant t \leqslant T$. Since $F \in C^{\beta+1}$ and $K * \bar{\rho} \in C^1([0, T], C^{\beta+2})$ by Assumption **(A5)** and Lemma A.4, the existence of the flow $(\phi_{t,s})_{t,s \in [0,T]}$ in $C^{\beta'}$ for any $\beta' < \beta+1$ follows from [Kun97, Theorem 4.6.5]. For fixed $t \in [0, T]$, denote $\phi_s := \phi_{t,s}$ and $\phi_{-s} := \phi_{s,t}$ for $s \in [0, t]$, then $\phi_{-s} \circ \phi_s = \mathrm{Id}$.

The next step is the one-to-one correspondence between the solutions to (4.3) and the solutions to the following equation

$$\begin{aligned} g_s(x) =&\psi(x) - \int_s^t \left[K(-\cdot) * (g_r \circ \phi_{-r}\nabla\bar{\rho}) \right] \circ \phi_r(x)\mathrm{d}r - \int_s^t \left[\mathrm{div}K(-\cdot) * (g_r \circ \phi_{-r}\bar{\rho}) \right] \circ \phi_r(x)\mathrm{d}r \\ :=&\psi(x) + \Phi_s(g), \quad s \in [0, t]. \end{aligned} \tag{4.5}$$

Here the notation $K(-\cdot) * f$ is given in (3.18). We also write $g(s, x) := g_s(x)$. We will prove that $g(s, x) := \varphi(s, \phi_s)$ satisfies (4.5). Indeed, suppose that $\varphi \in C^1([0, t], C^{\beta'})$ with $\beta' > \alpha+1$ solves (4.3), then by the chain rule, we have

$$\begin{aligned} \partial_s\{\varphi(s, \phi_s(x))\} =&\partial_s\varphi(s, \phi_s(x)) + \nabla\varphi(s, \phi_s(x)) \cdot \partial_s\phi_s(x) = -K(-\cdot) * (\bar{\rho}\nabla\varphi)(\phi_s(x)) \\ =&K(-\cdot) * (\varphi\nabla\bar{\rho})(\phi_s(x)) + \mathrm{div}\, K(-\cdot) * (\varphi\bar{\rho})(\phi_s(x)) \\ =&\left[K(-\cdot) * \{(\varphi \circ \phi_s \circ \phi_{-s})\nabla\bar{\rho}\} \right] \circ \phi_s(x) + \left[\mathrm{div}K(-\cdot) * \{(\varphi \circ \phi_s \circ \phi_{-s})\bar{\rho}\} \right] \circ \phi_s(x), \end{aligned}$$

where we used integration by parts formula in the third step. Therefore $\varphi(s, \phi_s)$ satisfies (4.5).

Conversely, if $g \in C^1([0, t], C^{\beta'})$ is a solution to equation (4.5), let $\varphi(s, x) := g(s, \phi_{-s}(x))$, then $g(s, x) = \varphi(s, \phi_s(x))$. Similarly, we have

$$\begin{aligned} \partial_s g(s, x) =&\partial_s\{\varphi(s, \phi_s(x))\} = \partial_s\varphi(s, \phi_s(x)) + \nabla\varphi(s, \phi_s(x)) \cdot \partial_s\phi_s(x), \\ \partial_s g(s, x) =&K(-\cdot) * (\varphi\nabla\bar{\rho})(\phi_s(x)) + \mathrm{div}\, K(-\cdot) * (\varphi\bar{\rho})(\phi_s(x)) = -K(-\cdot) * (\bar{\rho}\nabla\varphi)(\phi_s(x)), \end{aligned}$$

where the first line follows by the chain rule, while the second line follows by integration by parts. Substituting (4.4) into the first line, we obtain that φ is a solution to (4.3). Hence justifying the claim is turned into obtaining the well-posedness of backward equation (4.5) in $C^1([0, t], C^{\beta'})$.

Notice that $\Phi_{\cdot} : C([s, t], C^{\beta'}) \to C([s, t], C^{\beta'})$ satisfies

$$\begin{aligned} \sup_{r \in [s,t]} \|\Phi_r(g)\|_{C^{\beta'}} \leqslant& \int_s^t \left\| \left[K(-\cdot) * (g_r \circ \phi_{-r}\nabla\bar{\rho}) \right] \circ \phi_r \right\|_{C^{\beta'}} + \left\| \left[\mathrm{div}K(-\cdot) * (g_r \circ \phi_{-r}\bar{\rho}) \right] \circ \phi_r \right\|_{C^{\beta'}} \mathrm{d}r \\ \lesssim& \int_s^t \|K\|_{L^1} \|g_r\|_{C^{\beta'}} \left(1 + \|\phi_{-r}\|_{C^{\beta'}}^{\beta'} \right) \|\bar{\rho}\|_{C^{\beta'+1}} (1 + \|\phi_r\|_{C^{\beta'}}^{\beta'})\mathrm{d}r \\ &+ \int_s^t \|\mathrm{div}K\|_{L^1} \|g_r\|_{C^{\beta'}} \left(1 + \|\phi_{-r}\|_{C^{\beta'}}^{\beta'} \right) \|\bar{\rho}\|_{C^{\beta'}} \left(1 + \|\phi_r\|_{C^{\beta'}}^{\beta'} \right)\mathrm{d}r, \end{aligned} \tag{4.6}$$

where we used Lemma A.2, Lemma A.4, and the fact that $\|f_1 \circ f_2\|_{C^\alpha} \lesssim \|f_1\|_{C^\alpha}(1 + \|f_2\|_{C^\alpha}^\alpha)$ when $\alpha \geqslant 1$. Recalling Assumption **(A5)** and $\phi \in C([0, t], C^{\beta'})$, we find

$$\sup_{r \in [s,t]} \|\Phi_r(g)\|_{C^{\beta'}} \lesssim (t - s) \sup_{r \in [s,t]} \|g_r\|_{C^{\beta'}}.$$

Choosing s close to t enough, by the linearity of Φ, we find $g \mapsto \psi + \Phi_\cdot(g)$ is a contraction mapping on $C([s, t], C^{\beta'})$, hence it has a unique fixed point solving (4.5) on $[s, t]$. Applying this argument a finite number of times, we remove the constraint on s. Therefore, there exists a unique solution $g \in C([0, t], \beta')$ to the ODE (4.5). Finally, we deduce $g \in C^1([0, t], \beta')$ from $\partial_s \Phi_s \in C([0, t], C^{\beta'})$, which follows by the calculation in (4.6). Now we obtain the global well-posedness of (4.5) in $C^1([0, t], \beta')$, which by the one-to-one correspondence between φ and g concludes the result. □

Now we are able to conclude the result for vanishing diffusion cases similar to Theorem 1.4.

Theorem 4.2. *Under the assumptions* **(A1)-(A3)**, **(A5)**, *let η be the unique solution to* (4.1) *on the same stochastic basis with the particle system* (1.1), *the sequence* $(\eta^N)_{N \geqslant 1}$ *converges in probability to η in $L^2([0, T], H^{-\alpha}) \cap C([0, T], H^{-\alpha-2})$, for every $\alpha > d/2$.*

Proof. By the facts that $(\eta^N)_{N \geqslant 1}$ are tight, the tight limits of converging subsequences solve (4.1), and there exists a unique analytic weak solution to the equation (4.1), which is ensured by Proposition 4.1, it follows immediately that the sequence $(\eta^N)_{N \geqslant 1}$ converges in distribution to the unique solution η .

Similar to $(\eta^N)_{N \geqslant 1}$, one can first obtain tightness of laws of $(\eta^l, \eta^m)_{l,m \in \mathbb{N}}$. Without loss of generality, we assume $(\eta^l, \eta^m)_{l,m \in \mathbb{N}}$ to be two converging subsequences. Then using Skorohod theorem and identifying the limit we deduce that (η^l) and (η^m) converge in distribution to η and η', which both solve random PDE (4.1) with the same initial value η_0. Furthermore, Proposition 4.1 leads to $\eta = \eta'$ $\mathbb{P}$-a.s.. Therefore, we can deduce by Lemma 4.3 below that $(\eta^N)_{N \geqslant 1}$ converges in probability to the unique solution η. □

Lemma 4.3 (Gyöngy and Krylov [GK96]). *Let $(Z^N)_{N \geqslant 1}$ be a sequence of random elements in a Polish space E equipped with the Borel σ-algebra. Then $(Z^N)_{N \geqslant 1}$ converges in probability to an E-valued random element if and only if for every pair of subsequences (Z^l) and (Z^m) there exists a subsequence $u^k := (Z^{l(k)}, Z^{m(k)})$ converging in distribution to a random element u supported on the diagonal $\{(x, y) \in E \times E : x = y\}$.*

Proof of Theorem 1.6. The proof is similar to Proposition 1.5. In addition to the convergence obtained in Theorem 4.2, we need to check the Gaussianity of the unique solution η to (4.1) with Gaussian initial value η_0.

Define the time evolution operators $\{Q_{s,t}\}_{0 \leqslant s \leqslant t \leqslant T}$ by $Q_{s,t}\varphi := f(s)$, where $f \in C^1([0, t], C^{\beta'})$, $\beta' \in (d/2 + 1, \beta + 1)$, is the unique solution to (4.3) with terminal value φ at time t. Then for each $\varphi \in C^\infty$ and $t \in [0, T]$, let $Q_{\cdot,t}\varphi$ play the role of test function in (4.2). We find

$$\langle \eta_t, \varphi \rangle = \langle \eta_0, Q_{0,t}\varphi \rangle.$$

Finally, the result follows by the assumption on η_0. □

5. Applications

In this section we finish the proof of Theorem 1.7 and then give a similar result for the particle system (1.1) with C^1 kernels. At last, we prove a central limit theorem for the initial values, which gives a sufficient condition to **(A1)**.

5.1. **Examples.** Let us start with proving Theorem 1.7, which concerns on the most important example of this article: the Biot-Savart law.

Proof of Theorem 1.7. By our main results Theorem 1.4, Proposition 1.5, and Theorem 1.6, it suffices to check the assumptions **(A1)-(A5)**.

   (A1) is automatically satisfied since the point vortex model (1.8) is of i.i.d initial data, and one can easily check that the Biot-Savart law satisfies the second case of Assumption **(A2)**. Moreover, by [JW18, Theorem 2], the following condition (5.1) yields **(A3)**,

$$\bar{\rho} \in C([0,T], C^3) \quad \text{and} \quad \inf \bar{\rho} > 0. \tag{5.1}$$

Now we check (5.1). On one hand, when $\sigma > 0$ we deduce $\bar{\rho} \in C([0,T], C^3)$ under the assumption that $\bar{\rho}_0 \in C^3$ by [BA94, Theorem A]. The fact that $\bar{\rho} \in C([0,T], C^3)$ for the case $\sigma = 0$ follows by [MP12, Theorem 2.4.1]. On the other hand, $\bar{\rho} \in C([0,T], C^3)$ and Lemma A.4 implies that $K * \bar{\rho}$ is bounded and Lipschitz continuous, which yields the global well-posedness for the Cauchy problem to the following SDE:

$$\varphi_t = x + \int_0^t K * \bar{\rho}_s(\varphi_s)\mathrm{d}s + \sqrt{2\sigma}B_t. \tag{5.2}$$

where B is a d-dimensional Brownian motion. Let $\{\varphi_t\}_{t \in [0,T]}$ be the unique solution to (5.2), and notice that $\bar{\rho}$ is the density of the time marginal law of solution to (5.2) with initial value $\bar{\rho}_0$-distributed.

   Since K is divergence free and σ is a constant, the flow $\{\varphi_t\}_{t \in [-T,T]}$ (recall that $\varphi_{-t} := \varphi_t^{-1}$) is measure preserving (see [Kun97, Lemma 4.3.1]). Then for any bounded measurable function f, we have

$$\langle f, \bar{\rho}_t \rangle = \mathbb{E} \int_{\mathbb{T}^d} f(\varphi_t(x))\bar{\rho}_0(x)\mathrm{d}x = \mathbb{E} \int_{\mathbb{T}^d} f(x)\bar{\rho}_0(\varphi_{-t}(x))\mathrm{d}x,$$

which implies $\inf \bar{\rho} > 0$ since $\inf \bar{\rho}_0 > 0$. Thus we obtain (5.1) and thus obviously Assumption **(A4)** for the 2D Navier-Stokes equations. Lastly, for the 2D Euler equations, we deduce Assumption **(A5)** from [MP12, Theorem 2.4.1]. □

   For general cases, our main result Theorem 1.4 only requires bounded kernels. However, in order to check Assumption **(A3)** using [JW18] , the extra condtion $\mathrm{div} K \in L^\infty$ is necessary. Thus we consider the system (1.1) with C^1 kernels below, and give a complete result with the only assumption on the initial data and the confined potential F. Nevertheless, the following result considerably relaxes the condition on kernels in the classical work by Fernandez and Méléard [FM97], where the kernel K they considered should be regular enough, for instance in $C^{2+d/2}$.

Example 5.1 (C^1 kernels). *Consider the particle system (1.1) on $\mathbb{T}^d$ and a sequence of independent initial random variables $\{X_i(0)\}_{i \in \mathbb{N}}$ with identical probability density $\bar{\rho}_0$. Assume that $\sigma > 0$, $K \in C^1$, $F, \rho_0 \in C^\beta$ for some $\beta > 2 \vee d/2$, and $\inf \bar{\rho}_0 > 0$. Then the assumptions **(A1)-(A4)** hold. In particular, Theorem 1.4 and Proposition 1.5 hold in this case.*

Proof. The assumptions **(A1)-(A2)** follow immediately. For simplicity, we set $\sigma = 1$ and prove the required regularity for $\bar{\rho}$. Consider the McKean-Vlasov equation:

$$\mathrm{d}X_t = K * \bar{\rho}_t(X_t)\mathrm{d}t + F(X_t)\mathrm{d}t + \sqrt{2}\mathrm{d}B_t, \quad \bar{\rho}_t = \mathcal{L}(X_t), \tag{5.3}$$

where B is a d-dimensional Brownian motion. Since K and F are bounded and Lipschitz continuous, one can obtain the well-posedness of (5.3), c.f. [CG19, Theorem 3.3]. Furthermore, applying Itô's formula and superposition principle (see [BRS21, Tre16]) we have the one-to-one correspondance between the solutions to the Mckean-Vlasov equation (5.3) and the solutions to the mean field equation (1.2), which implies the global well-posedness of the mean field equation in the space $C([0,T], \mathcal{P}(\mathbb{R}^d))$. Recall that the mild form of the mean-field equation (1.2) is stated as

$$\bar{\rho}_t = \Gamma_t * \bar{\rho}_0 + \int_0^t \nabla\Gamma_{t-s} * (K * \bar{\rho}_s\bar{\rho}_s + F\bar{\rho}_s)\mathrm{d}s,$$

where Γ is the heat kernel of Δ. Since K is bounded and Lipschitz continuous, $K * \rho_t$ is bounded and Lipschitz continuous as well, hence belongs to the Besov space $B^1_{\infty,\infty}$. Next we consider the following linearized equation

$$\rho_t = \Gamma_t * \bar{\rho}_0 + \int_0^t \nabla\Gamma_{t-s} * (K * \bar{\rho}_s\rho_s + F\rho_s)\mathrm{d}s, \tag{5.4}$$

where $\bar{\rho}$ is the unique solution to the mean field equation in $C([0,T], \mathcal{P}(\mathbb{R}^d))$. We are going to exploit the regularity of the kernel to improve the regularity of $\bar{\rho}$. Notice first that $\Gamma. * \bar{\rho}_0 \in C([0,T], C^\beta)$ and we use Lemma A.5 to have

$$\left\| \int_0^t \nabla\Gamma_{t-s} * (K * \bar{\rho}_s\rho_s + F\rho_s)\mathrm{d}s \right\|_{B^1_{\infty,\infty}}^p \lesssim \left(\int_0^t (t-s)^{-\frac{1}{2}} \|K * \bar{\rho}_s\rho + F\rho\|_{B^1_{\infty,\infty}} \mathrm{d}s \right)^p.$$

Furthermore, let $p > 2$, then Lemma A.2 and Hölder's inequality yield that

$$\left\| \int_0^t \nabla\Gamma_{t-s} * (K * \bar{\rho}_s\rho_s + F\rho_s)\mathrm{d}s \right\|_{B^1_{\infty,\infty}}^p \lesssim \left(\int_0^t (t-s)^{-\frac{1}{2}} \|K * \bar{\rho}_s + F\|_{B^1_{\infty,\infty}} \|\rho\|_{B^1_{\infty,\infty}} \mathrm{d}s \right)^p$$

$$\lesssim_{K,\bar{\rho},F} \left(\int_0^t (t-s)^{-\frac{p}{2(p-1)}} \mathrm{d}s \right)^{\frac{p-1}{p}} \int_0^t \|\rho_s\|_{B^1_{\infty,\infty}}^p \mathrm{d}s$$

$$\lesssim_{K,\bar{\rho},F} t^{\frac{p-2}{2p}} \int_0^t \|\rho_s\|_{B^1_{\infty,\infty}}^p \mathrm{d}s.$$

The constant omitted here depends on $\|F\|_{C^\beta}$ and $\sup_{t\in[0,T]} \|K * \bar{\rho}_s\|_{B^1_{\infty,\infty}}$. Thus we have

$$\|\rho_t\|_{B^1_{\infty,\infty}}^p \lesssim \|\bar{\rho}_0\|_{B^1_{\infty,\infty}}^p + \int_0^t \|\rho_s\|_{B^1_{\infty,\infty}}^p \mathrm{d}s,$$

for any ρ satisfies the linearized equation (5.4). Since the solution $\bar{\rho}$ to the mean field equation also satisfies (5.4), we find $\bar{\rho} \in C([0,T], B^1_{\infty,\infty})$ by Gronwall's inequality.

Recall that we first obtained probability measure-valued solution to (1.2). As $\bar{\rho} \in C([0,T], B^1_{\infty,\infty})$, the coefficient $K * \bar{\rho}$ has better regularity, which provides the possibility to improve the regularity of $\bar{\rho}$. In fact, by $K \in B^1_{\infty,\infty}$, Lemma A.2 and Lemma A.4, we deduce $K * \bar{\rho}_t \in B^{\alpha+1-\epsilon}_{\infty,\infty}$ whenever $\bar{\rho} \in B^\alpha_{\infty,\infty}$, for sufficiently small $\epsilon > 0$. Therefore, $\bar{\rho}$ helps improving the regularity of the coefficient to the linearized equation (5.4). As a result, we could repeat the above estimates with $B^1_{\infty,\infty}$-norm replaced by $B^{2-\epsilon}_{\infty,\infty}$-norm for some $\epsilon > 0$ and conclude $\bar{\rho} \in C([0,T], B^{2-\epsilon}_{\infty,\infty})$. We iterate this procedure again and we get $\bar{\rho} \in C([0,T], B^\beta_{\infty,\infty})$ for $\beta > 2 \vee d/2$, which implies Assumption (**A4**).

As to Assumption (**A3**), by [JW18, Theorem 2] and $\bar{\rho} \in C([0,T], B^\beta_{\infty,\infty})$, it is sufficient to check $\inf \bar{\rho} > 0$. Similar to the proof of Theorem 1.7, we need the auxiliary SDE

$$\varphi_t = x + \int_0^t K * \bar{\rho}_s(\varphi_s)\mathrm{d}s + \int_0^t F(\varphi_s)\mathrm{d}s + \sqrt{2}B_t.$$

Then for any nonnegative measurable function f on $\mathbb{T}^d$, we have

$$\langle f, \bar{\rho}_t \rangle = \mathbb{E} \int_{\mathbb{T}^d} f(\varphi_t(x))\bar{\rho}_0(x)\mathrm{d}x = \mathbb{E} \int_{\mathbb{T}^d} f(x)\bar{\rho}_0(\varphi_{-t}(x))|\det \partial\varphi_{-s}(x)|\mathrm{d}x,$$

$$\geqslant \inf \bar{\rho}_0 \int_{\mathbb{T}^d} f(x)e^{-\int_0^s \|\mathrm{div}(K*\bar{\rho}_r+F)\|_{L^\infty} dr}\mathrm{d}x$$

$$\geqslant \inf \bar{\rho}_0 e^{-T(\|K\|_{C^1}+\|F\|_{C^1})} \int_{\mathbb{T}^d} f(x)\mathrm{d}x,$$

where the first inequality follows by the representation of the determinant for the Jacobian matrix $\det \partial\varphi_{-s}$, see [Kun97, Lemma 4.3.1]. This implies (**A3**). $\qquad\square$

5.2. **A sufficient condition for (A1).** Motivated by the main Assumption **(A3)** in this article, we give a sufficient condition for central limit theorem for random variables in terms of relative entropy. These random variables may be neither independent nor identical distributed. This result could be applied to check **(A1)**.

In the following, we let $\{X_i^N\}_{1 \leqslant i \leqslant N}$ be a class of random variables with values in $\mathbb{R}^d$ or $\mathbb{T}^d$, here $\{X_i^N\}_{1 \leqslant i \leqslant N}$ plays the role of $\{X_i(0)\}_{1 \leqslant i \leqslant N}$ in **(A1)**. More general, the laws of $\{X_i^N\}_{1 \leqslant i \leqslant N}$ are allowed to be different and these random variables might take values in the whole space $\mathbb{R}^d$ so that one can compare our result with the general central limit theorems in the literature. We abuse the notation ρ_N to denote the joint distribution of $\{X_i^N\}_{1 \leqslant i \leqslant N}$, and let $\bar{\rho}$ denote a probability measure on $\mathbb{R}^d$ (or $\mathbb{T}^d$). For simplicity, we omit the torus case in the following discussion.

For fixed $\varphi \in \mathcal{S}(\mathbb{R}^d)$, define Y_φ^N by

$$Y_\varphi^N := \frac{\sum_{i=1}^N \varphi(X_i^N) - N \langle \varphi, \bar{\rho} \rangle}{\sqrt{N}}.$$

Thus $\frac{1}{\sqrt{N}} \sum_{i=1}^N (\delta_{X_i^N} - \bar{\rho})$ converges in distribution to the Gaussian variable η_0 in $\mathcal{S}'(\mathbb{R}^d)$ which satisfies

$$\langle \eta_0, \varphi \rangle \overset{d}{\sim} \mathcal{N}(0, \langle \varphi^2, \bar{\rho} \rangle - \langle \varphi, \bar{\rho} \rangle^2), \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^d),$$

if and only if the law of Y_φ^N converges weakly to $\mathcal{N}(0, \langle \varphi^2, \bar{\rho} \rangle - \langle \varphi, \bar{\rho} \rangle^2)$, which is denoted by G_φ.

Recall that the bounded Lipschitz distance $d_{bL}(\mu, \nu)$ between two probability measures μ, ν on $\mathbb{R}^d$ is defined as

$$d_{bL}(\mu, \nu) = \sup \left\{ \int_{\mathbb{R}^d} g(y)\mu(\mathrm{d}y) - \int_{\mathbb{R}^d} g(y)\nu(\mathrm{d}y); \quad \|g\|_{L^\infty} + \|g\|_{\mathrm{Lip}} \leqslant 1 \right\}, \tag{5.5}$$

where $\|g\|_{\mathrm{Lip}}$ denotes the Lipschitz constant of g. The bounded Lipschitz distance metrizes the weak convergence (see [Vil08]). We are going to control the bounded Lipschitz distance between the law of Y_φ^N and G_φ, denoted by $d_{bL}(\mathcal{L}(Y_\varphi^N), G_\varphi)$, by the relative entropy $H(\rho_N | \bar{\rho}_N)$ and the bounded Lipschitz distance between G_φ and S_φ^N, which is defined as

$$S_\varphi^N := \frac{\sum_{i=1}^N \varphi(Z_i) - N \langle \varphi, \bar{\rho} \rangle}{\sqrt{N}},$$

where $\{Z_i\}_{i \in \mathbb{N}}$ is a class of i.i.d random variables with distribution $\bar{\rho}$.

Proposition 5.2. *There is $\lambda > 0$ such that for every $\kappa > 0$,*

$$d_{bL}(\mathcal{L}(Y_\varphi^N), G_\varphi) \leqslant \frac{1}{\kappa} H(\rho_N | \bar{\rho}_N) + \frac{\kappa}{2\lambda} + d_{bL}(\mathcal{L}(S_\varphi^N), G_\varphi).$$

By taking $\kappa = \sqrt{2\lambda H(\rho_N | \bar{\rho}_N)}$, then

$$d_{bL}(\mathcal{L}(Y_\varphi^N), G_\varphi) \leqslant \sqrt{\frac{2}{\lambda}} \sqrt{H(\rho_N | \bar{\rho}_N)} + d_{bL}(\mathcal{L}(S_\varphi^N), G_\varphi).$$

Assume further

$$H(\rho_N | \bar{\rho}_N) \xrightarrow{N \to \infty} 0,$$

then $\mathcal{L}(Y_\varphi^N)$ converges to G_φ weakly, as $N \to \infty$.

Proof. From (5.5) we know

$$d_{bL}(\mathcal{L}(Y_\varphi^N), G_\varphi) = \sup \left\{ \int_{\mathbb{R}^d} g(y)\mathrm{d}\mathcal{L}(Y_\varphi^N) - \int_{\mathbb{R}^d} g(y)G_\varphi(y)\mathrm{d}y; \quad \|g\|_{L^\infty} + \|g\|_{\mathrm{Lip}} \leqslant 1 \right\}. \tag{5.6}$$

First, we find

$$\int_{\mathbb{R}^d} g(y)\mathrm{d}\mathcal{L}(Y_\varphi^N) = \mathbb{E}g(Y_\varphi^N) = \mathbb{E}g\left(\frac{\sum_{i=1}^N \varphi(X_i^N) - N \langle \varphi, \bar{\rho} \rangle}{\sqrt{N}} \right)$$

$$= \int_{\mathbb{R}^{dN}} g\Big(\frac{\sum_{i=1}^{N} \varphi(x_i) - N \langle \varphi, \bar{\rho} \rangle}{\sqrt{N}} \Big) \rho_N \mathrm{d}x^N$$

$$:= \int_{\mathbb{R}^{dN}} g \circ \Phi_N(x^N) \rho_N \mathrm{d}x^N,$$

with $x^N = (x_1, \ldots, x_N)$. Then applying the Donsker-Varadhan variational formula (2.1) gives

$$\int_{\mathbb{R}^d} g(y) \mathrm{d}\mathcal{L}(Y_\varphi^N) \leqslant \frac{1}{\kappa} \left(H(\rho_N | \bar{\rho}_N) + \log \int_{\mathbb{R}^{dN}} e^{\kappa g \circ \Phi_N(x^N)} \bar{\rho}_N \mathrm{d}x^N \right), \tag{5.7}$$

for every $\kappa > 0$. Since $\{Z_i\}_{i \in \mathbb{N}}$ is a class of i.i.d random variables with distribution $\bar{\rho}$, we have

$$\int_{\mathbb{R}^{dN}} e^{\kappa g \circ \Phi_N(x^N)} \bar{\rho}_N \mathrm{d}x^N = \mathbb{E} \exp\Big[\kappa g \Big(\frac{\sum_{i=1}^{N} \varphi(Z_i) - N \langle \varphi, \bar{\rho} \rangle}{\sqrt{N}} \Big) \Big] = \int_{\mathbb{R}} e^{\kappa g(y)} \mathrm{d}\mathcal{L}(S_\varphi^N). \tag{5.8}$$

Combining (5.6)-(5.8), we arrive at

$$d_{bL}(\mathcal{L}(Y_\varphi^N), G_\varphi) \leqslant \frac{1}{\kappa} H(\rho_N | \bar{\rho}_N) + d_{bL}(\mathcal{L}(S_\varphi^N, G_\varphi))$$

$$+ \sup \left\{ \frac{1}{\kappa} \log \int_{\mathbb{R}} e^{\kappa g(y)} \mathrm{d}\mathcal{L}(S_\varphi^N) - \int_{\mathbb{R}} g(y) \mathrm{d}\mathcal{L}(S_\varphi^N); \quad \|g\|_{L^\infty} + \|g\|_{\mathrm{Lip}} \leqslant 1 \right\}, \tag{5.9}$$

for every $\kappa > 0$.

To handle the last term on the right hand side of (5.9), we need uniform Gaussian concentration of $\{\mathcal{L}(S_\varphi^N)\}_{N \in \mathbb{N}}$. That is, there is $a > 0$ such that

$$\sup_N \int_{\mathbb{R}} e^{ay^2} \mathrm{d}\mathcal{L}(S_\varphi^N) < \infty. \tag{5.10}$$

Indeed, recall (5.8) and the definition of Φ_N, for (5.10) it is sufficient to show that

$$\sup_N \int_{\mathbb{R}^{dN}} \exp\left(aN \Big| \Big\langle \varphi, \frac{1}{N} \sum_{i=1}^{N} \delta_{x_i} - \bar{\rho} \Big\rangle \Big|^2 \right) \bar{\rho}_N \mathrm{d}x^N$$

$$= \sup_N \int_{\mathbb{R}^{dN}} \exp\left(aN \Big| \Big\langle \varphi - \langle \varphi, \bar{\rho} \rangle, \frac{1}{N} \sum_{i=1}^{N} \delta_{x_i} \Big\rangle \Big|^2 \right) \bar{\rho}_N \mathrm{d}x^N < \infty.$$

This follows by Lemma 2.1 with sufficient small a, which depends only on $\|\varphi\|_{L^\infty}$. Hence (5.10) holds.

Therefore, this uniform Gaussian concentration (5.10) allows us to apply [Vil08, Theorem 22.10], which yields that $\{\mathcal{L}(S_\varphi^N)\}_{N \in \mathbb{N}}$ satisfies the dual formulation of Talagrand-1 inequality uniformly. More precisely, there is $\lambda > 0$ such that for any $g \in C_b(\mathbb{R}^d)$, $N \in \mathbb{N}$, and $\kappa \geqslant 0$,

$$\int_{\mathbb{R}} \exp\Big[\kappa \inf_{z \in \mathbb{R}} \Big(g(z) + |y - z| \Big) \Big] \mathrm{d}\mathcal{L}(S_\varphi^N) \leqslant \exp\Big(\frac{\kappa^2}{2\lambda} + \kappa \int_{\mathbb{R}} g(y) \mathrm{d}\mathcal{L}(S_\varphi^N) \Big).$$

Recall (5.9), we are only interested in functions with Lipschitz constant smaller than 1, for which functions it holds that $\inf_{y \in \mathbb{R}} \Big(g(z) + |y - z| \Big) = g(y)$. Thus we have

$$\frac{1}{\kappa} \log \int_{\mathbb{R}} e^{\kappa g(y)} \mathrm{d}\mathcal{L}(S_\varphi^N) - \int_{\mathbb{R}} g(y) \mathrm{d}\mathcal{L}(S_\varphi^N) \leqslant \frac{\kappa}{2\lambda}, \quad \forall \kappa > 0.$$

Taking the above dual formulation of Talagrand-1 inequality into (5.9) gives

$$d_{bL}(\mathcal{L}(Y_\varphi^N), G_\varphi) \leqslant \frac{1}{\kappa} H(\rho_N | \bar{\rho}_N) + d_{bL}(\mathcal{L}(S_\varphi^N, G_\varphi)) + \frac{\kappa}{2\lambda},$$

the result then follows by the canonical central limit theorem. $\square$

Remark 5.3. *Similar as the central limit theorems in general, the convergence still holds if just a fixed number of elements are changed. This gives rise to the more reasonable condition*

$$\lim_{m\to\infty} \lim_{N\to\infty} H(\rho_{N-\tau_{N,m}}|\bar{\rho}_{N-m}) = 0,$$

where $\tau_{N,m}$ is a subset of $\{1,...,N\}$ with m elements and $\rho_{N-\tau_{N,m}}$ be the joint distribution of $\{X_i^N\}_{i\notin\tau_{N,m}}$. Indeed, define $Y_\varphi^{\tau_{N,m}}$ as

$$Y_\varphi^{\tau_{N,m}} := \frac{\sqrt{N}}{\sqrt{N-m}}\Big(Y_\varphi^N - \frac{\sum_{i\in\tau_{N,m}}\varphi(X_i^N) - m\langle\varphi,\bar{\rho}\rangle}{\sqrt{N}}\Big).$$

It is easy to check that $d_{bL}(\mathcal{L}(Y_\varphi^N),\mathcal{L}(Y_\varphi^{\tau_{N,m}})) \lesssim m/\sqrt{N}$, which together with Proposition 5.2 applied to $\{X_i^N\}_{i\notin\tau_{N,m}}$ yields the convergence of $\mathcal{L}(Y_\varphi^N)$ to the Gaussian distribution G_φ.

APPENDIX A. BESOV SPACES

In this section we collect useful results related to Besov spaces. Recall that Besov spaces on the torus $B_{p,q}^\alpha(\mathbb{T}^d)$ (c.f [Tri06], [MW17]), with $\alpha \in \mathbb{R}$ and $1 \leqslant p, q \leqslant \infty$, are defined as the completion of C^∞ with respect to the norm

$$\|f\|_{B_{p,q}^\alpha} := \Big(\sum_{n\geqslant -1}\Big(2^{n\alpha q}\|\mathscr{F}^{-1}(\chi_n\mathscr{F}(f))\|_{L^p(\mathbb{T}^d)}^q\Big)\Big)^{\frac{1}{q}},$$

where $\mathscr{F}$ represents Fourier transform on $\mathbb{R}^d$ and $\{\chi_n\}_{n\geqslant -1} : \mathbb{R}^d \to [0,1]$ are compact supported smooth functions satisfying

$$\text{supp}\chi_{-1} \subseteq B(0,\frac{4}{3}); \quad \text{supp}\chi_0 \subseteq B(0,\frac{8}{3})\setminus B(0,\frac{4}{3}), \quad \chi_n(\cdot) = \chi_0(2^{-n}\cdot) \text{ for } n\geqslant 0; \quad \sum_{n\geqslant -1}\chi_n = 1.$$

Here $B(0,R)$ denotes the ball of center 0 and radius R.

We collect the following results which are frequently used in this article.

Lemma A.1 ([Tri06, Proposition4.6]). *Let $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}$ and $p_1, p_2, q_1, q_2 \in [1,\infty]$. Then the embedding*

$$B_{p_1,q_2}^\alpha \hookrightarrow B_{p_2,q_2}^\beta$$

is compact if and only if,

$$\alpha - \beta > d\Big(\frac{1}{p_1} - \frac{1}{p_2}\Big)_+.$$

Lemma A.2. *(i) Let $\alpha, \beta \in \mathbb{R}$ and $p, p_1, p_2, q \in [1,\infty]$ be such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. The bilinear map $(u,v) \mapsto uv$ extends to a continuous map from $B_{p_1,q}^\alpha \times B_{p_2,q}^\beta$ to $B_{p,q}^{\alpha\wedge\beta}$ if $\alpha + \beta > 0$ (cf. [MW17, Corollary 2]).*

(ii) (Duality.) Let $\alpha \in (0,1)$, $p, q \in [1,\infty]$, p' and q' be their conjugate exponents, respectively. Then the mapping $(u,v) \mapsto \langle u,v\rangle = \int uv\mathrm{d}x$ extends to a continuous bilinear form on $B_{p,q}^\alpha \times B_{p',q'}^{-\alpha}$, and one has $|\langle u,v\rangle| \lesssim \|u\|_{B_{p,q}^\alpha}\|v\|_{B_{p',q'}^{-\alpha}}$ (cf. [MW17, Proposition 7]).

Lemma A.3 ([BCD11, Corollary 2.86]). *For any positive real number α and any $p, q \in [1,\infty]$, it holds that*

$$\|fg\|_{B_{p,q}^\alpha} \lesssim \|f\|_{L^\infty}\|g\|_{B_{p,q}^\alpha} + \|f\|_{B_{p,q}^\alpha}\|g\|_{L^\infty},$$

with the proportional constant independent of f, g.

Lemma A.4 ([KS21, Theorem 2.1 and 2.2]). *Let $\alpha, \beta \in \mathbb{R}, q, q_1, q_2 \in (0,\infty]$ and $p, p_1, p_2 \in [1,\infty]$ be such that*

$$1 + \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \quad \frac{1}{q} \leqslant \frac{1}{q_1} + \frac{1}{q_2}.$$

(1) If $f \in B_{p_1,q}^\alpha$ and $g \in L^{p_2}$, then $f * g \in B_{p,q}^\alpha$ and

$$\|f * g\|_{B_{p,q}^\alpha} \lesssim \|f\|_{B_{p_1,q}^\alpha} \cdot \|g\|_{L^{p_2}},$$

with the proportional constant independent of f, g.

(2) If $f \in B_{p_1,q_1}^\alpha$ and $g \in B_{p_2,q_2}^\beta$, then $f * g \in B_{p,q}^{\alpha+\beta}$ and

$$\|f * g\|_{B_{p,q}^{\alpha+\beta}} \lesssim \|f\|_{B_{p_1,q_1}^\alpha} \cdot \|g\|_{B_{p_2,q_2}^\beta}.$$

Recall the result about smoothing effect of the heat kernel Γ.

Lemma A.5 ([MW17, Propositions 3.11, 3.12]). *Let $u \in B_{p,q}^\alpha$ for some $\alpha \in \mathbb{R}$, $1 \leqslant p, q \leqslant \infty$. Then for every $\kappa \geqslant 0$*

$$\|\Gamma_t * u\|_{B_{p,q}^{\alpha+2\kappa}} \lesssim t^{-\kappa} \|u\|_{B_{p,q}^\alpha},$$

and

$$\|\Gamma_t * u - u\|_{B_{p,q}^\alpha} \lesssim t^{\kappa/2} \|u\|_{B_{p,q}^{\alpha+\kappa}}.$$

Appendix B. Proof of Lemma 3.1

In this section we give the proof of Lemma 3.1. First recall the following result from [FGGT05].

Lemma B.1. *Let $\varphi \to S(\varphi)$ be a linear continuous mapping from a separable Banach space E to $L^0(\Omega)$ (random variables with convergence in probability). Assume that there exists a random variable $C(\omega)$ such that for any given $\varphi \in E$ we have*

$$|S(\varphi)(\omega)| \leqslant C(\omega)\|\varphi\|_E \quad \text{for} \quad \mathbb{P} - a.s. \quad \omega \in \Omega.$$

Then there exists a pathwise realization $\mathcal{S}$ of $S(\varphi)$ from $(\Omega, \mathcal{F}, \mathbb{P})$ to the dual space of E in the sense that

$$[S(\varphi)](\omega) = [\mathcal{S}(\omega)](\varphi), \quad \mathbb{P} - a.s,$$

for every $\varphi \in E$.

Proof of Lemma 3.1. The proof consists of two step. The first step is to find a pathwise realization for each $t \in [0, T]$. The second step is justifying the pathwise realization forms a progressively measurable process. We denote $\frac{\sqrt{2\sigma_N}}{\sqrt{N}}$ by C_N below for simplicity.

We first apply [FGGT05, Lemma 8] to obtain the following equality, for $\varphi \in C^\infty$,

$$C_N \sum_{i=1}^N \int_0^t \nabla\varphi(X_i)\mathrm{d}B_s^i = C_N \sum_{i=1}^N \sum_{k_i \in \mathbb{Z}^d} \langle \nabla\varphi, e_{-k_i} \rangle \int_0^t e_{k_i}(X_i)\mathrm{d}B_s^i, \quad \mathbb{P} - a.s.$$

Furthermore, using Hölder's inequality we have

$$\left| C_N \sum_{i=1}^N \int_0^t \nabla\varphi(X_i)\mathrm{d}B_s^i \right| \leqslant C_N \sum_{i=1}^N \left(\sum_{k_i \in \mathbb{Z}^d} \langle k_i \rangle^{2\alpha-2} |\langle \nabla\varphi, e_{-k_i} \rangle|^2 \right)^{\frac{1}{2}}$$

$$\times \left(\sum_{k_i \in \mathbb{Z}^d} \langle k_i \rangle^{-2\alpha+2} \left| \int_0^t e_{k_i}(X_i)\mathrm{d}B_s^i \right|^2 \right)^{\frac{1}{2}}$$

$$\leqslant \|\varphi\|_{H^\alpha} C_N \sum_{i=1}^N \left(\sum_{k_i \in \mathbb{Z}^d} \langle k_i \rangle^{-2\alpha+2} \left| \int_0^t e_{k_i}(X_i)\mathrm{d}B_s^i \right|^2 \right)^{\frac{1}{2}}.$$

To apply Lemma B.1 with $E = H^\alpha(\mathbb{T}^d)$ for $\alpha > d/2 + 1$, it is sufficient to find

$$\mathbb{E}\left(\sum_{i=1}^{N}\left(\sum_{k_i \in \mathbb{Z}^d}\langle k_i\rangle^{-2\alpha+2}\left|\int_0^t e_{k_i}(X_i)\mathrm{d}B_s^i\right|^2\right)^{\frac{1}{2}}\right)^2$$

$$\lesssim_N \mathbb{E}\left(\sum_{i=1}^{N}\sum_{k_i \in \mathbb{Z}^d}\langle k_i\rangle^{-2\alpha+2}\left|\int_0^t e_{k_i}(X_i)\mathrm{d}B_s^i\right|^2\right)$$

$$\lesssim_N t\sum_{i=1}^{N}\sum_{k_i \in \mathbb{Z}^d}\langle k_i\rangle^{-2\alpha+2} < \infty.$$

Therefore, we thus obtain a pathwise realization of $C_N\sum_{i=1}^{N}\int_0^t \nabla\varphi(X_i)\mathrm{d}B_s^i$ for each $t \in [0, T]$, denoted by $\mathcal{M}_t^N$.

Define $\mathcal{M}^N := (\mathcal{M}_t^N)_{t \in [0,T]}$. Since the stochastic integrals are t-continuous, the equality

$$\mathcal{M}_t^N(\varphi) = C_N\sum_{i=1}^{N}\int_0^t \nabla\varphi(X_i)\mathrm{d}B_s^i$$

holds almost surely for all $t \in [0, T]$ and $\varphi \in C^\infty(\mathbb{T}^d)$. To justify measurability of $\mathcal{M}^N$. Notice that for each $\varphi \in C^\infty$, $\langle\mathcal{M}^N, \varphi\rangle = \mathcal{M}_t^N(\varphi)$ is a continuous adapted process. Hence for each $t \in [0, T]$, $\langle\mathcal{M}^N, \varphi\rangle : \Omega \times [0, t] \to \mathbb{R}$ is $\mathcal{F}_t \times \mathcal{B}([0, t])$-measurable. Since C^∞ is dense in the separable Hilbert space H^α, using Pettis measurability theorem and Lemma B.1 we thus find $\mathcal{M}^N : \Omega \times [0, T] \to H^{-\alpha}$ is progressively measurable. $\square$

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

REFERENCES

[AGZ10]   Greg W Anderson, Alice Guionnet, and Ofer Zeitouni. *An introduction to random matrices*. Number 118. Cambridge university press, 2010.

[BA94]   Matania Ben-Artzi. Global solutions of two-dimensional navier-stokes and euler equations. *Archive for rational mechanics and analysis*, 128(4):329–358, 1994.

[BCD11]   Hajer Bahouri, Jean-Yves Chemin, and Raphaël Danchin. *Fourier analysis and nonlinear partial differential equations*, volume 343. Springer Science & Business Media, 2011.

[BFH18]   Dominic Breit, Eduard Feireisl, and Martina Hofmanová. *Stochastically forced compressible fluid flows*, volume 3. Walter de Gruyter GmbH & Co KG, 2018.

[BFT15]   Mireille Bossy, Olivier Faugeras, and Denis Talay. Clarification and complement to "mean-field description and propagation of chaos in networks of hodgkin–huxley and fitzhugh–nagumo neurons". *The Journal of Mathematical Neuroscience (JMN)*, 5(1):1–23, 2015.

[BG99]   Thierry Bodineau and Alice Guionnet. About the stationary states of vortex systems. In *Annales de l'Institut Henri Poincare (B) Probability and Statistics*, volume 35, pages 205–237. Elsevier, 1999.

[BGSRS20] Thierry Bodineau, Isabelle Gallagher, Laure Saint-Raymond, and Sergio Simonella. Statistical dynamics of a hard sphere gas: fluctuating boltzmann equation and large deviations. *arXiv preprint arXiv:2008.10403*, 2020.

[BH77]   Werner Braun and K Hepp. The vlasov dynamics and its fluctuations in the 1/n limit of interacting classical particles. *Communications in mathematical physics*, 56(2):101–113, 1977.

[BJW20]   Didier Bresch, Pierre-Emmanuel Jabin, and Zhenfu Wang. Mean-field limit and quantitative estimates with singular attractive kernels. *arXiv preprint arXiv:2011.08022*, 2020.

[BO19]   Robert J Berman and Magnus Onnheim. Propagation of chaos for a class of first order models with singular mean field interactions. *SIAM Journal on Mathematical Analysis*, 51(1):159–196, 2019.

[BRS21]   Vladimir I Bogachev, Michael Röckner, and Stanislav V Shaposhnikov. On the ambrosio–figalli–trevisan superposition principle for probability solutions to fokker–planck–kolmogorov equations. *Journal of Dynamics and Differential Equations*, 33(2):715–739, 2021.

[BW16]   Amarjit Budhiraja and Ruoyu Wu. Some fluctuation results for weakly interacting multi-type particle systems. *Stochastic Processes and their Applications*, 126(8):2253–2296, 2016.

[CCH14]   José Antonio Carrillo, Young-Pil Choi, and Maxime Hauray. The derivation of swarming models: mean-field limit and wasserstein distances. In *Collective dynamics from bacteria to crowds*, pages 1–46. Springer, 2014.

[CF16]   Zhen-Qing Chen and Wai-Tong Louis Fan. Fluctuation limit for interacting diffusions with partial annihilations through membranes. *Journal of Statistical Physics*, 164(4):890–936, 2016.

[CG19]   Michele Coghi and Benjamin Gess. Stochastic nonlinear fokker–planck equations. *Nonlinear Analysis*, 187:259–278, 2019.

[Che17]   Julien Chevallier. Fluctuations for mean-field interacting age-dependent hawkes processes. *Electronic Journal of Probability*, 22, 2017.

[DE11]   Paul Dupuis and Richard S Ellis. *A weak convergence approach to the theory of large deviations*, volume 902. John Wiley & Sons, 2011.

[Dob79]   Roland L'vovich Dobrushin. Vlasov equations. *Functional Analysis and Its Applications*, 13(2):115–123, 1979.

[Due16]   Mitia Duerinckx. Mean-field limits for some riesz interaction gradient flows. *SIAM Journal on Mathematical Analysis*, 48(3):2269–2300, 2016.

[Eva98]   Lawrence C Evans. Partial differential equations. *Graduate studies in mathematics*, 19(2), 1998.

[FGGT05] Franco Flandoli, Massimiliano Gubinelli, Mariano Giaquinta, and Vincenzo M Tortorelli. Stochastic currents. *Stochastic processes and their applications*, 115(9):1583–1601, 2005.

[FHM14]   Nicolas Fournier, Maxime Hauray, and Stéphane Mischler. Propagation of chaos for the 2d viscous vortex model. *Journal of the European Mathematical Society*, 16(7):1423–1466, 2014.

[FJ90]   Noah E Friedkin and Eugene C Johnsen. Social influence and opinions. *Journal of Mathematical Sociology*, 15(3-4):193–206, 1990.

[FM97]   Begoña Fernandez and Sylvie Méléard. A hilbertian approach for fluctuations on the mckean-vlasov model. *Stochastic processes and their applications*, 71(1):33–53, 1997.

[FM07]   Joaquin Fontbona and Miguel Martinez. Paths clustering and an existence result for stochastic vortex systems. *Journal of Statistical Physics*, 128(3):699–719, 2007.

[GK96]   István Gyöngy and Nicolai Krylov. Existence of strong solutions for itô's stochastic equations via approximations. *Probability theory and related fields*, 105(2):143–158, 1996.

[GR20]   Francesco Grotto and Marco Romito. A central limit theorem for gibbsian invariant measures of 2d euler equations. *Communications in Mathematical Physics*, 376(3):2197–2228, 2020.

[GR21]   Carina Geldhauser and Marco Romito. Limit theorems and fluctuations for point vortices of generalized euler equations. *Journal of Statistical Physics*, 182(3):1–27, 2021.

[Hai14]   Martin Hairer. A theory of regularity structures. *Inventiones mathematicae*, 198(2):269–504, 2014.

[HM14]   Maxime Hauray and Stéphane Mischler. On kac's chaos and related problems. *Journal of Functional Analysis*, 266(10):6055–6157, 2014.

[HMC$^+$06] Minyi Huang, Roland P Malhamé, Peter E Caines, et al. Large population stochastic dynamic games: closed-loop mckean-vlasov systems and the nash certainty equivalence principle. *Communications in Information & Systems*, 6(3):221–252, 2006.

[Itô83]   Kiyosi Itô. Distribution-valued processes arising from independent brownian motions. *Mathematische Zeitschrift*, 182(1):17–33, 1983.

[Jab14]   Pierre-Emmanuel Jabin. A review of the mean field limits for vlasov equations. *Kinetic & Related Models*, 7(4):661, 2014.

[Jea15]   J.H. Jeans. On the theory of star-streaming and the structure of the universe. *Monthly Notices of the Royal Astronomical Society*, 76:70–84, 1915.

[JH15]   Pierre-Emmanuel Jabin and Maxime Hauray. Particles approximations of vlasov equations with singular forces: Propagation of chaos. In *Annales scientifiques de l'École Normale Supérieure*, 2015.

[JM98]   Benjamin Jourdain and Sylvie Méléard. Propagation of chaos and fluctuations for a moderate model with smooth initial data. In *Annales de l'Institut Henri Poincare (B) Probability and Statistics*, volume 34, pages 727–766. Elsevier, 1998.

[JO04]   Pierre-Emmanuel Jabin and Felix Otto. Identification of the dilute regime in particle sedimentation. *Communications in mathematical physics*, 250(2):415–432, 2004.

[JW18]   Pierre-Emmanuel Jabin and Zhenfu Wang. Quantitative estimates of propagation of chaos for stochastic systems with $w^{-1,\infty}$ kernels. *Inventiones mathematicae*, 214(1):523–591, 2018.

[Kan67]   Mamoru Kanda. Regular points and green functions in markov processes. *Journal of the Mathematical Society of Japan*, 19(1):46–69, 1967.

[Kel17]   John L Kelley. *General topology*. Courier Dover Publications, 2017.

[KS21]   Franziska Kuhn and Rene L. Schilling. Convolution inequalities for besov and triebel lizorkin spaces, and applications to convolution semigroups. 2021.

[Kun97]   Hiroshi Kunita. *Stochastic flows and stochastic differential equations*, volume 24. Cambridge university press, 1997.

[Kur14]   Thomas Kurtz. Weak and strong solutions of general stochastic models. *Electronic Communications in Probability*, 19, 2014.

[KX04]   Thomas G Kurtz and Jie Xiong. A stochastic evolution equation arising from the fluctuations of a class of interacting particle systems. *Communications in Mathematical Sciences*, 2(3):325–358, 2004.

[Lac21]   Daniel Lacker. Hierarchies, entropy, and quantitative propagation of chaos for mean field diffusions. *arXiv preprint arXiv:2105.02983*, 2021.

[Lan09]   Carlo Lancellotti. On the fluctuations about the vlasov limit for n-particle systems with mean-field interactions. *Journal of Statistical Physics*, 136(4):643–665, 2009.

[Laz16]   Dustin Lazarovici. The vlasov-poisson dynamics as the mean field limit of extended charges. *Communications in Mathematical Physics*, 347(1):271–289, 2016.

[Lee19]   A J Lee. *U-statistics: Theory and Practice*. Routledge, 2019.

[LL07]   Jean-Michel Lasry and Pierre-Louis Lions. Mean field games. *Japanese journal of mathematics*, 2(1):229–260, 2007.

[LLX20]   Songzi Li, Xiang-Dong Li, and Yong-Xiao Xie. On the law of large numbers for the empirical measure process of generalized dyson brownian motion. *Journal of Statistical Physics*, 181(4):1277–1305, 2020.

[LS16]   Eric Lucon and Wilhelm Stannat. Transition from gaussian to non-gaussian fluctuations for mean-field diffusions in spatial interaction. *The Annals of Applied Probability*, 26(6):3840–3909, 2016.

[LS18]   Thomas Leblé and Sylvia Serfaty. Fluctuations of two dimensional coulomb gases. *Geometric and Functional Analysis*, 28(2):443–508, 2018.

[LW20]   Wei Liu and Liming Wu. Large deviations for empirical measures of mean-field gibbs measures. *Stochastic Processes and their Applications*, 130(2):503–520, 2020.

[McK75]   HP McKean. Fluctuations in the kinetic theory of gases. *Communications on pure and applied mathematics*, 28(4):435–455, 1975.

[Mél96]   Sylvie Méléard. Asymptotic behaviour of some interacting particle systems; mckean-vlasov and boltzmann models. In *Probabilistic models for nonlinear partial differential equations*, pages 42–95. Springer, 1996.

[MJ]   HP McKean Jr. Propagation of chaos for a class of non-linear parabolic equations. 1967 stochastic differential equations (lecture series in differential equations, session 7, catholic univ., 1967) pp. 41–57 air force office sci. *Res., Arlington, Va.*

[MM13]   Stéphane Mischler and Clément Mouhot. Kac's program in kinetic theory. *Inventiones mathematicae*, 193(1):1–147, 2013.

[MMN18]   Song Mei, Andrea Montanari, and Phan-Minh Nguyen. A mean field view of the landscape of two-layer neural networks. *Proceedings of the National Academy of Sciences*, 115(33):E7665–E7671, 2018.

[MMW15]   Stéphane Mischler, Clément Mouhot, and Bernt Wennberg. A new approach to quantitative propagation of chaos for drift, diffusion and jump processes. *Probability Theory and Related Fields*, 161(1):1–59, 2015.

[MP12]   Carlo Marchioro and Mario Pulvirenti. *Mathematical theory of incompressible nonviscous fluids*, volume 96. Springer Science & Business Media, 2012.

[MW17]   Jean-Christophe Mourrat and Hendrik Weber. Global well-posedness of the dynamic phi4 model in the plane. *The Annals of Probability*, 45(4):2398–2476, 2017.

[NRS21]   Quoc Hung Nguyen, Matthew Rosenzweig, and Sylvia Serfaty. Mean-field limits of riesz-type singular flows with possible multiplicative transport noise. *arXiv preprint arXiv:2107.02592*, 2021.

[Nua06]   David Nualart. *The Malliavin calculus and related topics*, volume 1995. Springer, 2006.

[Osa85]   Hirofumi Osada. A stochastic differential equation arising from the vortex problem. *Proceedings of the Japan Academy, Series A, Mathematical Sciences*, 61(10):333–336, 1985.

[Osa86]   Hirofumi Osada. Propagation of chaos for the two dimensional navier-stokes equation. *Proceedings of the Japan Academy, Series A, Mathematical Sciences*, 62(1):8–11, 1986.

[PAT07]   Víctor Pérez-Abreu and Constantin Tudor. Functional limit theorems for trace processes in a dyson brownian motion. *Communications on Stochastic Analysis*, 1(3):5, 2007.

[Ros20]   Matthew Rosenzweig. Mean-field convergence of point vortices without regularity. *arXiv preprint arXiv:2004.04140*, 2020.

[Ser20a]   Sylvia Serfaty. Gaussian fluctuations and free energy expansion for 2d and 3d coulomb gases at any temperature. *arXiv preprint arXiv:2003.11704*, 2020.

[Ser20b]   Sylvia Serfaty. Mean field limit for coulomb-type flows. *Duke Mathematical Journal*, 169(15):2887–2935, 2020.

[SYY20]   Jian Song, Jianfeng Yao, and Wangjun Yuan. High-dimensional limits of eigenvalue distributions for general wishart process. *The Annals of Applied Probability*, 30(4):1642–1668, 2020.

[Szn84]   Alain-Sol Sznitman. Nonlinear reflecting diffusion process, and the propagation of chaos and fluctuations associated. *Journal of functional analysis*, 56(3):311–336, 1984.

[Szn91]   Alain-Sol Sznitman. Topics in propagation of chaos. In *Ecole d'été de probabilités de Saint-Flour XIX—1989*, pages 165–251. Springer, 1991.

[Tak85]   Satoshi Takanobu. On the existence and uniqueness of sde describing an n-particle system interacting via a singular potential. *Proceedings of the Japan Academy, Series A, Mathematical Sciences*, 61(9):287–290, 1985.

[Tan82]   Hiroshi Tanaka. Fluctuation theory for kac's one dimensional model of maxwellian molecules. *The Indian Journal of Statistics, Series A*, pages 23–46, 1982.

[Tan83]   Hiroshi Tanaka. Some probabilistic problems in the spatially homogeneous boltzmann equation. In *Theory and application of random fields*, pages 258–267. Springer, 1983.

[Tan84]   Hiroshi Tanaka. Limit theorems for certain diffusion processes with interaction. In *North-Holland Mathematical Library*, volume 32, pages 469–488. Elsevier, 1984.

[TH81]   Hiroshi Tanaka and Masuyuki Hitsuda. Central limit theorem for a simple diffusion model of interacting particles. *Hiroshima Mathematical Journal*, 11(2):415–423, 1981.

[Tre16]   Dario Trevisan. Well-posedness of multidimensional diffusion processes with weakly differentiable coefficients. *Electronic Journal of Probability*, 21:1–41, 2016.

[Tri06]   Hans Triebel. Theory of function spaces. iii,. *Monographs in Mathematics*, 2006.

[Uch83]   Kohei Uchiyama. Fluctuations of markovian systems in kac's caricature of a maxwellian gas. *Journal of the Mathematical Society of Japan*, 35(3):477–499, 1983.

[Vil08]   Cédric Villani. *Optimal transport: old and new*, volume 338. Springer Science & Business Media, 2008.

(Z. Wang) Beijing International Center for Mathematical Research, Peking University, Beijing 100871, China

Email address: zwang@bicmr.pku.edu.cn

(X. Zhao) Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China; Fakultät für Mathematik, Universität Bielefeld, D-33501 Bielefeld, Germany

Email address: xzhao@math.uni-bielefeld.de

(R. Zhu) Department of Mathematics, Beijing Institute of Technology, Beijing 100081, China

Email address: zhurongchan@126.com