ANOMALOUS AND TOTAL DISSIPATION DUE TO ADVECTION BY SOLUTIONS OF RANDOMLY FORCED NAVIER-STOKES EQUATIONS

MARTINA HOFMANOVÁ, UMBERTO PAPPALETTERA, RONGCHAN ZHU, AND XIANGCHAN ZHU

Abstract. We show the existence of a velocity field v, solution of (randomly) forced Navier-Stokes equations, which produces total dissipation of kinetic energy in finite time when advecting a passive scalar ρ. The total dissipation holds true uniformly in the viscosity parameter and the initial conditions ρ_0, in particular the dissipation is anomalous. Dissipation induced by single realizations of v is also discussed. Our results extend to the case when ρ is replaced by a solution to the two or three dimensional (deterministic) Navier-Stokes equations advected by v.

1. Introduction

Let $\mathbb{T}^d := (\mathbb{R}/\mathbb{Z})^d$ denote the d-dimensional torus, $d = 2, 3$, and let $v : \mathbb{T}^d \times \mathbb{R}_+ \to \mathbb{R}^d$ be a divergence-free velocity field given a priori. In this work we are interested in the anomalous dissipation induced by advection with respect to v in the passive scalar equation

$$\partial_t \rho + v \cdot \nabla \rho = \nu \Delta \rho, \tag{1.1}$$

in the unknown $\rho : \mathbb{T}^d \times \mathbb{R}_+ \to \mathbb{R}$. The viscosity (or diffusivity) parameter ν can freely vary in $(0, 1)$.

Our main goal is to find an advecting velocity field v that produces anomalous dissipation at time $t = 1$ in *every* weak solution ρ of (1.1), namely for every non-constant initial data $\rho_0 \in L^2$

$$\liminf_{\nu \to 0} \|\rho_1\|_{L^2} < \|\rho_0\|_{L^2}, \tag{1.2}$$

or equivalently, if energy equality holds

$$\limsup_{\nu \to 0} \nu \int_0^1 \|\nabla \rho_t\|^2_{L^2} dt > 0. \tag{1.3}$$

Physically speaking, the mechanism inducing enhanced dissipation (eventually anomalous) is the transfer of kinetic energy of ρ to small spatial scales by the convective term $v \cdot \nabla \rho$. However, this heuristic phenomenon is difficult to capture mathematically, and only a few rigorous results are known.

In the deterministic literature, one of the most studied examples is that of alternating shear flows, namely velocity fields v that are translation invariant along one time-dependent direction. These flows are known to induce enhanced dissipation [4, 11], and more recently [15] obtained a general criterion for anomalous dissipation based on inverse interpolation inequality. A more refined version of alternating shear flows, rearranging chessboard-like initial data so to develop small-scale spatial structures and induce anomalous dissipation, has been presented in [10], see also [6] for a result on anomalous dissipation for the forced Navier-Stokes equations. Other mechanisms have been shown to produce anomalous dissipation for particular initial data, see [7, 24, 23]. Let us

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On the other hand, in the stochastic literature it is now relatively well-understood that a white-in-time velocity field v can produce mixing and dissipation enhancement in (1.1), at least when v is concentrated only at very high Fourier modes and each Fourier mode is excited with very low intensity [18]. This is ultimately due to the presence of a Stratonovich-to-Itō corrector in the weak formulation of (1.1), that under suitable geometric conditions on the space structure of the noise acts as a large multiple of the Laplacian, and permits to prove smallness of the H^{-1} norm of the solution ρ. We point out that the mechanism producing dissipation enhancement is still the transfer of kinetic energy to high wavenumbers, since this *transport noise* is formally energy preserving; the algebraic reformulation of (1.1) with Stratonovich-to-Itō corrector is just a convenient framework where this transfer can be effectively quantified. Let us also mention that the transport noise can be engineered in such a way that the enhancement in dissipation is arbitrarily strong on short time intervals, see also [17]. Up to now, however, it was unknown whether the dissipation can be made anomalous, i.e. uniform in the vanishing viscosity limit $\nu \to 0$.

More importantly, it has been argued in the literature that a physically relevant velocity v shall not be white-in-time. This presents further difficulties as in particular no Stratonovich-to-Itō corrector appears in the weak formulation of (1.1) and the previous considerations can not be applied. But if v has a similar space structure as the aforementioned transport noise, it is reasonable to expect that the same transfer of energy to high wavenumbers and consequent dissipation enhancement occur, see for instance [25].

The main contribution of the present paper consists in capturing this phenomenon without the help of the Stratonovich-to-Itō corrector. The key idea is that we can define v to be a solution of a stochastic differential equation with a strong external (random) forcing, so that the force makes v behave similarly to a transport noise in the weak formulation of (1.1), notwithstanding the fact that v remains a function with positive decorrelation time. If the external forcing is sufficiently strong, we can even allow v to solve the forced Navier-Stokes equations, or other equations of fluid dynamics relevance. This is particularly interesting since, in all the anomalous dissipation examples we are aware of, the advecting velocity field is somewhat constructed *ad hoc* and it is not clear whether the dissipation mechanism is robust enough to be genuinely embedded into the Navier-Stokes dynamics[1].

Then, in order to quantify the induced decay of the H^{-1} norm of ρ, we add a small perturbation φ to the solution of (1.1) so to "reintroduce by hand" the temporal roughness in the equation that produced the Stratonovich-to-Itō corrector in the case of transport noise. The definition of the perturbation φ is inspired by homogenization methods as uses ideas from [13]. What we ultimately get is that the convective term "homogenizes" to a negative definite operator, which coincides with the same Stratonovich-to-Itō corrector of the case with transport noise, plus small reminders. Then, after a suitable tuning of the parameters defining the random forcing acting on v, we are able to prove enhanced, and eventually anomalous, dissipation in (1.1). Actually, what we are able to show is the following stronger result on *total* dissipation in finite time, uniformly in the initial condition and viscosity parameter.

Theorem 1.1. *There exist a countable family of Brownian motions $\{W^{k,\alpha}\}$, time-dependent velocity fields $\{\sigma_{k,\alpha}\}$, and a weak solution v of the Navier-Stokes equations with large friction and additive noise:*

$$\begin{cases} dv + (v\cdot\nabla)v\,dt + \nabla p_v\,dt = \Delta v\,dt - \varepsilon^{-1} v\,dt + \varepsilon^{-1}\sum_{k,\alpha}\sigma_{k,\alpha}dW^{k,\alpha}, \\ \operatorname{div} v = 0, \end{cases} \tag{1.4}$$

where $\varepsilon = \varepsilon(t)$ depends on t and is piecewise constant, such that

$$v \in C_w([0,1), L^2) \cap L^2_{loc}([0,1), H^1) \quad \textit{almost surely,}$$

[1]Of course one could first construct an advecting velocity v field producing anomalous dissipation and then define a forcing term $f = \partial_t v + (v\cdot\nabla)v + \nabla p_v - \Delta v$, so that v solves the forced Navier-Stokes equations. With "genuinely embedded into the Navier-Stokes dynamics" we mean that we define the forcing a priori, independently of v.

and every progressively measurable[2] *Leray-Hopf weak solution of* (1.1) *manifests total dissipation at time* $t = 1$*, i.e. for every zero-mean initial condition* $\rho_0 \in L^2$ *and* $\nu \in (0, 1)$ *it holds*

$$\lim_{t \uparrow 1} \|\rho_t\|_{L^2} = 0 \qquad \textit{almost surely.}$$

Analogously to the Navier-Stokes equations, a weak solution is called Leray-Hopf provided it satisfies the corresponding energy inequality, which is the key property in our proof. Note that Leray-Hopf solutions of (1.1) with the velocity v satisfying (1.4) exist and are unique due to [5].

The mechanism producing total (not just enhanced) dissipation is a forced transfer of the kinetic energy of ρ to higher and higher wavenumbers as time t approaches one, thanks to the large forcing terms at the right-hand-side of (1.4). As a side effect, the norm of v blows up as time $t \uparrow 1$, but we obtain a *universal* result in the sense that total dissipation holds true for every initial condition and every $\nu \in (0, 1)$. In particular, the dissipation is anomalous in the sense that (1.2) holds almost surely (by continuity extension), as well as (1.3) if ρ satisfies the energy equality[3].

As an intermediate auxiliary step, we prove the corresponding statement for the case of white-in-time velocity v given formally as

$$v = \sum_{k,\alpha} \sigma_{k,\alpha} \partial_t W^{k,\alpha}.$$

This result is also new and potentially interesting in its own right. The method of its proof then serves as a basis for the proof of Theorem 1.1.

Let us mention at this point that in principle one could take a generic realization $v = v(\omega)$ of the random velocity field v above and ask whether the *deterministic* vector field $v(\omega)$ induces total dissipation in (1.1). The problem in this case is that the full measure set of ω's for which $\lim_{t \uparrow 1} \|\rho_t\|_{L^2} = 0$ in Theorem 1.1 may in principle depend on the initial condition ρ_0 and the viscosity ν. As a consequence, we lose the universality of dissipation and only have the weaker result:

Corollary 1.2. *For every countable sets of initial conditions* $\mathscr{C} \subset L^2$ *with zero mean and viscosities* $\mathscr{V} \subset (0, 1)$ *there exists a deterministic vector field* $v = v(\mathscr{C}, \mathscr{V})$ *such that* $\lim_{t \uparrow 1} \|\rho_t\|_{L^2} = 0$ *for every Leray-Hopf weak solution* ρ *of* (1.1) *with initial condition* $\rho_0 \in \mathscr{C}$ *and viscosity* $\nu \in \mathscr{V}$*. More generally, given arbitrary probability measures* $\mathbb{P}_{\rho_0}$ *on* L^2 *with zero mean and* $\mathbb{P}_\nu$ *on* $(0, 1)$*, there exists* $v = v(\mathbb{P}_{\rho_0}, \mathbb{P}_\nu)$ *inducing total dissipation at time* $t = 1$ *in* (1.1) *for* $\mathbb{P}_{\rho_0} \otimes \mathbb{P}_\nu$ *almost every* (ρ_0, ν)*.*

If we desire to have better regularity of v, we can require it to solve an Ornstein-Uhlenbeck equation instead:

$$dv = -\varepsilon^{-1} v \, dt + \varepsilon^{-1} \sum_{k,\alpha} \sigma_{k,\alpha} dW^{k,\alpha}. \tag{1.5}$$

Theorem 1.3. *There exists* $v \in C^{1/2^-}_{loc}([0, 1), C^\infty)$ *almost surely, satisfying* (1.5) *above, such that the same total dissipation for solutions of* (1.1) *stated in Theorem 1.1 holds true.*

Our approach is robust enough to apply to nonlinear equations as well. For instance, let us consider the Navier-Stokes equations advected by v in dimension $d = 2$ or 3

$$\begin{cases} \partial_t u + (u \cdot \nabla) u + (v \cdot \nabla) u + \nabla p = \nu \Delta u \\ \operatorname{div} u = 0, \end{cases} \tag{1.6}$$

in the unknown $u : \mathbb{T}^d \times \mathbb{R}_+ \to \mathbb{R}^d$. The scalar pressure field $p : \mathbb{T}^d \times \mathbb{R}_+ \to \mathbb{R}$ is usually taken away from the analysis of (1.6) by applying the Leray projector Π. We have the following:

Theorem 1.4. *For the same velocity field* v *as in Theorem 1.1 or Theorem 1.3, every progressively measurable Leray-Hopf weak solution of* (1.6) *with zero-mean, divergence-free initial condition* $u_0 \in L^2$ *and* $\nu \in (0, 1)$ *satisfies* $\lim_{t \uparrow 1} \|u_t\|_{L^2} = 0$ *almost surely.*

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[2]By weak continuity of weak solutions to (1.1), progressive measurability is equivalent to require that ρ is *adapted*, namely for every $t \in [0, 1)$ the random variable ρ_t is measurable with respect to the sigma algebra $\mathcal{F}_t$, where $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ is a filtered probability space, satisfying the usual conditions, that supports the Brownian motions $\{W^{k,\alpha}\}$.

[3]For the supposed regularity of v, this is the case when $\rho_0 \in L^\infty$.

The previous theorem holds true both in dimension $d=2$ and $d=3$; however, there is a technical difference between these two cases. When $d=2$, by pathwise uniqueness of solutions to (1.6), given the velocity field v we can always find a Leray-Hopf weak solution u of (1.6) that is progressively measurable with respect to the filtration generated by the Brownian motions $\{W^{k,\alpha}\}$, and the analogue of Corollary 1.2 holds true. Moreover, energy equality gives the analogue of (1.3)

$$\limsup_{\nu\to 0} \nu \int_0^1 \|\nabla u_t\|_{L^2}^2 dt > 0 \quad \text{almost surely.} \tag{1.7}$$

In dimension $d=3$, the Navier-Stokes equations may have multiple Leray-Hopf weak solutions and thus it is unknown whether (1.6) has Leray-Hopf weak solutions adapted to $\{W^{k,\alpha}\}$ given a priori. In particular, a statement like that of Theorem 1.4 might be void. In this case, existence of progressively measurable Leray-Hopf weak solutions is recovered up to a possible change in the underlying probability space (this is usually referred to as probabilistically weak existence). More specifically, we have:

Theorem 1.5. *Let $d=3$. Then there exists a countable family of time-dependent velocity fields $\{\sigma_{k,\alpha}\}$ with the following property. For every countable sets of initial conditions $\mathscr{C}\subset L^2$ with null mean and divergence and viscosities $\mathscr{V}\subset(0,1)$ there exist a countable family of Brownian motions $\{W^{k,\alpha}\}$ and a vector field $v\in C_w([0,1),L^2)\cap L^2_{loc}([0,1),H^1)$ solution of (1.4) such that for every initial condition $u_0\in\mathscr{C}$ and viscosity $\nu\in\mathscr{V}$ there exists a progressively measurable Leray-Hopf solution u of (1.6) satisfying $\lim_{t\uparrow 1}\|u_t\|_{L^2}=0$ almost surely.*

Contrary to Corollary 1.2, if we consider fixed initial condition u_0 and viscosity ν and take a deterministic realization of v, in this case we can not prove total dissipation for every (deterministic) Leray-Hopf weak solution u. The reason is that we are not able to say whether a given Leray-Hopf weak solution u is the realization $u=\tilde u(\omega)$ of a progressively measurable Leray-Hopf weak solution $\tilde u$. Nonetheless, we have total dissipation at time $t=1$ for at least one Leray-Hopf weak solution:

Corollary 1.6. *Let $d=3$. Then for every zero-mean initial condition $u_0\in L^2$ with null divergence and viscosity $\nu\in(0,1)$ there exist a deterministic vector field $v=v(u_0,\nu)$ and a Leray-Hopf weak solution u of (1.6) satisfying $\lim_{t\uparrow 1}\|u_t\|_{L^2}=0$.*

Finally, let us mention that for u solution of (1.6) in dimension $d=3$ we are not able to deduce (1.7), since we do not know whether energy equality holds and we might only have an energy inequality. Additional details will be given in section 2 and section 3.

1.1. **Structure of the noise.** Let us focus on the case $d=3$ only. The case $d=2$ is readily covered with a similar construction and we omit it for the sake of brevity.

We construct the noise in (1.4) taking inspiration from [22] and [20]. Let $\mathbb{Z}^3_0 := \mathbb{Z}^3\setminus\{\mathbf{0}\}$ and let $\{\Lambda,-\Lambda\}$ be a partition of $\mathbb{Z}^3_0$. Let us introduce a family $\{B^{k,\alpha}\}_{k\in\mathbb{Z}^3_0,\alpha\in\{1,2\}}$ of i.i.d. standard Brownian motions on a filtered probability space $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},\mathbb{P})$ satisfying the usual conditions. Define the complex-valued Brownian motions

$$W^{k,\alpha} := \begin{cases} B^{k,\alpha}+iB^{-k,\alpha}, & \text{if } k\in\Lambda,\\ B^{k,\alpha}-iB^{-k,\alpha}, & \text{if } k\in-\Lambda,\end{cases}$$

whose quadratic covariation is given by $[W^{k,\alpha},W^{l,\beta}]_t = 2t\delta_{k,-l}\delta_{\alpha,\beta}$. For every $k\in\Lambda$ let $\{a_{k,1},a_{k,2}\}$ be orthonormal basis of $k^\perp$ such that $\{a_{k,1},a_{k,2},k/|k|\}$ is right-handed. Let $a_{k,\alpha}=a_{-k,\alpha}$ for every $k\in\mathbb{Z}^3_0$. The coefficients $\{\sigma_{k,\alpha}\}_{k\in\mathbb{Z}^3_0,\alpha\in\{1,2\}}$ are given by

$$\sigma_{k,\alpha}(x,t) := \theta_k(t)a_{k,\alpha}e_k(x), \qquad e_k(x) := e^{2\pi i k\cdot x},$$

for some suitable intensity coefficients $\theta_k:\mathbb{R}_+\to\mathbb{R}$. Notice that $\{a_{k,\alpha}e_k\}_{k,\alpha}$ is a complete orthonormal system of the space H of zero-mean, divergence-free, square integrable velocity fields on the torus; in particular $\sigma_{k,\alpha}(\cdot,t)$ is H valued for every k,α and t. For $s\in\mathbb{R}$, denote H^s the H-based Sobolev space of velocity fields on the torus with zero average and null divergence in the sense of distributions.

We introduce the dependence of θ_k on time in order to switch the noise on at higher and higher Fourier modes as time approaches $t=1$. To rigorously describe this, let $\{\tau_q\}_{q\in\mathbb{N}}$ be a sequence of times such that $\tau_0 = 1$ and $\tau_q \to 0$ monotonically as $q \to \infty$. On the time interval $(1-\tau_q, 1-\tau_{q+1}]$ we choose the coefficients $\theta_k(t) = \theta_k^q$ as:

$$\theta_k^q := \mathbf{1}_{\{N_q \le |k| \le 2N_q\}},$$

where N_q is a sequence such that $N_q \to \infty$ sufficiently fast as $q \to \infty$. We also define

$$\sum_{k\in\mathbb{Z}_0^2} (\theta_k^q)^2 =: \kappa_q \sim N_q^3 \to \infty \qquad \text{as } q \to \infty.$$

1.2. **Reintroducing roughness at small time scales.** Let us describe the heuristics behind the proof of Theorem 1.1. For the sake of simplicity, here we focus on the scalar case but we shall see that same arguments apply to the Navier-Stokes case. Fix $q \in \mathbb{N}$ and consider the time interval $(1-\tau_q, 1-\tau_{q+1}]$. Suppose on this time interval $\varepsilon(t) \equiv \varepsilon_q$ and we can decompose a solution v of (1.4) as

$$v = \varepsilon_q^{-1/2} w + r,$$

where w is the "noisy" part of v, satisfying

$$dw = -\varepsilon_q^{-1} w dt + \varepsilon_q^{-1/2} \sum_{k,\alpha} \sigma_{k,\alpha} dW^{k,\alpha},$$

and is renormalized such that w is on average of order one, whereas r is a remainder. Then, if $\varepsilon_q \ll 1$ the dynamics (1.1) has a potentially large term $\varepsilon_q^{-1/2} w\cdot\nabla\rho$, which however can be cancelled out considering the dynamics of the process $\rho - \varepsilon_q^{1/2} w\cdot\nabla\rho$ instead:

$$d\left(\rho - \varepsilon_q^{1/2} w\cdot\nabla\rho\right) = -r\cdot\nabla\rho\, dt + \nu\Delta\rho\, dt + w\cdot\nabla(w\cdot\nabla\rho)\, dt - \sum_{k,\alpha} \sigma_{k,\alpha}\cdot\nabla\rho\, dW^{k,\alpha} + O(\varepsilon_q^{1/2}).$$

Here we have the term $w\cdot\nabla(w\cdot\nabla\rho)$ which oscillates extremely fast with respect to time, and together with the quadratic interaction in w this produces the extra dissipation. To see this, consider

$$\frac{\varepsilon_q}{2} d\left(w\cdot\nabla(w\cdot\nabla\rho)\right) = -w\cdot\nabla(w\cdot\nabla\rho)\, dt + \sum_{k,\alpha} \sigma_{k,\alpha}\cdot\nabla(\sigma_{-k,\alpha}\cdot\nabla\rho)\, dt + O(\varepsilon_q^{1/2}),$$

which compensates exactly the fast oscillations $w\cdot\nabla(w\cdot\nabla\rho)$ in the dynamics of $\rho - \varepsilon_q^{1/2} w\cdot\nabla\rho$ and simultaneously makes appear the Stratonovich-to-Itō corrector $\sum_{k,\alpha} \sigma_{k,\alpha}\cdot\nabla(\sigma_{-k,\alpha}\cdot\nabla\rho)$.

If the parameters $\sigma_{k,\alpha}$ are tuned properly, this permits to prove smallness of a negative Sobolev norm of the process $\rho - \varepsilon_q^{1/2} w\cdot\nabla\rho + \frac{\varepsilon_q}{2} w\cdot\nabla(w\cdot\nabla\rho) + V \sim \rho$, from which we deduce dissipation. Here V is another auxiliary process, of average size proportional to ε_q, which helps to control the term $-r\cdot\nabla\rho$ in the dynamics of ρ. For more details, we refer the reader to section 3, where the rigorous argument is described.

Remark 1.7. The same arguments apply a fortiori to v solution of (1.5). Indeed, in this case one has $r = 0$ in the decomposition $v = \varepsilon_q^{-1/2} w + r$.

1.3. **Organization of the paper.** Hereafter, in order to keep the discussion as concise as possible, we prefer to restrict ourselves to the vectorial case (1.6), which is technically more demanding than the scalar case (1.1) due to the more complex form of the Stratonovich-to-Itō corrector. Also, in view of Remark 1.7 we will only consider v solution of (1.4), being the case when v solves (1.5) easier. Major differences in the proofs will be detailed when necessary.

We point out that the presence of the nonlinearity $(u\cdot\nabla)u$ in (1.6) is not the main source of difficulties, nor it is the ultimate reason why anomalous dissipation occurs. This may sound unsatisfactory since this implies we are not capturing the energy cascade to small scales that naturally occurs in solutions to the three-dimensional Navier-Stokes equations; yet we believe our results are interesting, since we have identified a new mechanism producing scalar anomalous dissipation that is: *i*) strong enough to produce total dissipation in finite time; *ii*) universal, in the sense that a single velocity field v works for every initial condition and value of the viscosity; and *iii*) robust enough to be extended to the nonlinear case as the Navier-Stokes equations (and likely to many other models of fluid dynamics).

The paper is structured as follows.

In section 2 we tackle the preliminary problem of white-in-time velocity v, see Proposition 2.1. This is useful to our analysis since it allows us to isolate the difficulties coming from the Leray projection Π in the expression of the Stratonovich-to-Itō corrector (subsection 2.1) and the main strategy behind the proof of total dissipation (subsection 2.2). In subsection 2.3, we also show that in dimension $d = 3$ there exist weak solutions of (1.6) perturbed by transport noise that are not Leray-Hopf and do not dissipate their kinetic energy at time $t = 1$. This is reasonable to expect, since non Leray-Hopf weak solutions do not need to satisfy the energy inequality, see (2.2).

Next, in section 3 we consider the more realistic case of v solution of the forced Navier-Stokes equations. The main content of this section consists in making rigorous the heuristic arguments presented in subsection 1.2, in the (more difficult) case of solutions to (1.6). As already pointed out in Remark 1.7, the proof of Theorem 1.3 when v solves (1.5) descends easily from the Navier-Stokes case (1.4).

2. Total dissipation by transport noise

In this section we shall focus on the case of transport noise:

$$v(x,t) = \sum_{k,\alpha} \sigma_{k,\alpha}(x,t)\partial_t W^{k,\alpha},$$

where the coefficients $\sigma_{k,\alpha}$ are as in subsection 1.1 and the white noise $\partial_t W^{k,\alpha}$ is meant in the Stratonovich sense. This is a good starting point, propaedeutic to the study of (1.6) advected by a more realistic velocity field solution v solution of (1.4). With this choice of v, equation (1.6) reads as

$$\begin{cases} du + (u\cdot\nabla)u\,dt + \sum_{k,\alpha}(\sigma_{k,\alpha}\cdot\nabla)u\circ dW^{k,\alpha} + \nabla p\,dt = \nu\Delta u\,dt \\ \operatorname{div} u = 0. \end{cases} \tag{2.1}$$

The symbol $\circ\, dW^{k,\alpha}$ in the equation above is a notational shorthand for the Stratonovich interpretation of the stochastic integral. This is a sensible modelling choice since Stratonovich integration satisfies the chain rule and thus the advection term $\sum_{k,\alpha}(\sigma_{k,\alpha}\cdot\nabla)u\circ dW^{k,\alpha}$ is formally energy preserving.

The goal of this section is to prove the following:

Proposition 2.1. *We can choose parameters τ_q, N_q and $\kappa_q \sim N_q^3$, $q\in\mathbb{N}$, such that the following holds true. Every probabilistically weak, progressively measurable Leray-Hopf weak solution of (2.1) with zero-mean, divergence-free initial condition $u_0 \in L^2$ and viscosity $\nu\in(0,1)$ satisfies $\lim_{t\uparrow 1}\|u_t\|_{L^2} = 0$ almost surely.*

A similar statement holds true in the passive scalar case. Before moving on, let us clarify what we mean by probabilistically weak Leray-Hopf weak solution of (2.1) in this case.

Definition 2.2. *A probabilistically weak, progressively measurable Leray-Hopf weak solution of (2.1) is defined as a probability space $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 1},\mathbb{P})$ supporting a family of i.i.d. Brownian motions $\{W^{k,\alpha}\}_{k,\alpha}$ and a progressively measurable stochastic process $u:\Omega\to C_w([0,1),H)\cap L^2([0,1),H^1)$ almost surely such that, for every divergence-free test function $f\in C_c^\infty(\mathbb{T}^3\times[0,1),\mathbb{R}^3)$ it holds almost surely for every $0\leq s<r<1$*

$$\langle u_r, f_r\rangle - \langle u_s, f_s\rangle = \int_s^r \langle u_t, \partial_t f_t + (u_t\cdot\nabla)f_t + \nu\Delta f_t\rangle dt + \sum_{k,\alpha}\int_s^r \langle u_t, (\sigma_{k,\alpha}(\cdot,t)\cdot\nabla)f_t\rangle\circ dW_t^{k,\alpha},$$

and for almost every $\omega\in\Omega$ there exists a full Lebesgue measure set $\mathcal{T}\subset[0,1)$ such that $0\in\mathcal{T}$ and for every $r\in\mathcal{T}$, $r<t<1$ the following energy inequality holds almost surely

$$\|u_t\|_{L^2}^2 + 2\nu\int_r^t \|\nabla u_s\|_{L^2}^2 ds \leq \|u_r\|_{L^2}^2. \tag{2.2}$$

We can rewrite the Stratonovich integral in Definition 2.2 in the equivalent Itō form as follows: for every divergence-free test function $f \in C_c^\infty(\mathbb{T}^3 \times [0,1), \mathbb{R}^3)$

$$\langle u_r, f_r \rangle - \langle u_s, f_s \rangle = \int_s^r \langle u_t, \partial_t f_t + (u_t \cdot \nabla) f_t + \nu \Delta f_t + S(f_t) \rangle dt$$
$$+ \sum_{k,\alpha} \int_s^r \langle u_t, (\sigma_{k,\alpha}(\cdot, t) \cdot \nabla) f_t \rangle dW_t^{k,\alpha},$$

where S is the so-called Stratonovich-to-Itō (or simply Stratonovich) corrector, described in details in the next subsection. Existence of solutions according to Definition 2.2 has been shown in [19].

Finally, let us comment briefly on the energy inequality (2.2). It ultimately is the reason why the transfer of energy to high wavenumbers increases the rate of dissipation in solutions of the Navier-Stokes equations, since a larger time integral of $\|\nabla u_s\|_{L^2}^2$ necessitates a smaller $\|u_t\|_{L^2}^2$ to ensure the validity of (2.2). However, the energy inequality can not be deduced from the equation (2.1) itself, and must be postulated a priori: this is the difference between weak solutions and Leray-Hopf weak solutions of (2.1). In particular, weak solutions of (2.1) that do not satisfy the energy inequality do not need to dissipate their energy, and explicit examples have been constructed recently by convex integration techniques [8], [9], [26]. In subsection 2.3 we give an example following the construction of [26].

In the passive scalar case, the weak formulation of the equation is obtained *mutatis mutandis*, but by linearity and pathwise uniqueness [16, Theorem 5.2] the stronger energy equality holds true: one can take $\mathcal{T} = [0,1)$ and has almost surely for every $t > r$

$$\|\rho_t\|_{L^2}^2 + 2\nu \int_r^t \|\nabla \rho_s\|_{L^2}^2 ds = \|\rho_r\|_{L^2}^2.$$

This can be obtained for instance taking a space mollification ρ^ϵ of the solution, using a commutator estimate à la Di Perna-Lions [14, Lemma II.1] to deduce energy equality for ρ^ϵ up to in infinitesimal error, and then passing to the limit $\epsilon \to 0$. The passage to the limit is justified by local-in-time smoothness of the coefficients $\{\sigma_{k,\alpha}\}$.

2.1. **Stratonovich corrector.** Equation (2.1) can be rewritten without the pressure term by using the Leray projector $\Pi = Id + \nabla(-\Delta)^{-1}\mathrm{div}$ onto the divergence-free velocity fields:

$$du + \Pi[(u \cdot \nabla) u]\, dt + \sum_{k,\alpha} \Pi[(\sigma_{k,\alpha} \cdot \nabla) u] \circ dW^{k,\alpha} = \nu \Delta u\, dt. \tag{2.3}$$

The advantage of (2.3) compared to (2.1) is that the Stratonovich-to-Itō corrector S can be readily computed,[4] see also equation (1.6) in [20]. We have

$$du + \Pi[(u \cdot \nabla) u]\, dt + \sum_{k,\alpha} \Pi[(\sigma_{k,\alpha} \cdot \nabla) u]\, dW^{k,\alpha} = \nu \Delta u\, dt + S(u)\, dt,$$

with

$$S(u) = \sum_{k,\alpha} \Pi[(\sigma_{k,\alpha} \cdot \nabla) \Pi[(\sigma_{-k,\alpha} \cdot \nabla) u]]. \tag{2.4}$$

On the time interval $(1-\tau_q, 1-\tau_{q+1}]$, we can rewrite $S = S_q$ (cf. (2.6) and (2.7) and choose C_ν properly therein) as

$$S_q(u) = \frac{2}{3} \kappa_q \Delta u - S_q^\perp(u),$$
$$S_q^\perp(u) := \sum_{k,\alpha} (\theta_k^q)^2\, \Pi[(a_{k,\alpha} e_k \cdot \nabla) \Pi^\perp (a_{k,\alpha} e_{-k} \cdot \nabla) u],$$

where $\Pi^\perp = -\nabla(-\Delta)^{-1}\mathrm{div}$ is the orthogonal complement of the Leray projector Π.

Decompose u as $u = \sum_{\ell,\beta} u_{\ell,\beta} a_{\ell,\beta} e_\ell$. By [20, Corollary 5.3] it holds

$$S_q^\perp(u) = -4\pi^2 \sum_{\ell,\beta} u_{\ell,\beta} |\ell|^2 \Pi \left[\sum_k (\theta_k^q)^2 \sin^2(\sphericalangle_{k,\ell}) (a_{\ell,\beta} \cdot (k-\ell)) \frac{k-\ell}{|k-\ell|^2} e_\ell \right],$$

[4]The problem with formulation (2.1) is related to the difficulty of computing the quadratic covariation between the Brownian motions $W^{k,\alpha}$ and the pressure p.

where $<_{k,\ell}$ denotes the angle between the $\mathbb{Z}^3$ vectors k and ℓ.

In order to better describe the behaviour of $S_q^\perp$, split u into low and high Fourier modes

$$u = u^L + u^H, \quad u^L := \Pi_L u,$$

where Π_L denotes the Fourier projector onto modes $|\ell| \le N_q^{1-\delta}$ for some small $\delta > 0$. It holds

$$\begin{aligned} S_q^\perp(u^L) &= -4\pi^2 \sum_{\ell,\beta} u^L_{\ell,\beta} |\ell|^2 \Pi \left[\sum_k (\theta^q_k)^2 \sin^2(<_{k,\ell})(a_{\ell,\beta}\cdot k)\frac{k}{|k|^2} e_\ell \right] \\ &\quad - 4\pi^2 \sum_{\ell,\beta} u^L_{\ell,\beta} |\ell|^2 \Pi \left[\sum_k (\theta^q_k)^2 \sin^2(<_{k,\ell}) \left((a_{\ell,\beta}\cdot (k-\ell))\frac{k-\ell}{|k-\ell|^2} - (a_{\ell,\beta}\cdot k)\frac{k}{|k|^2} \right) e_\ell \right]. \end{aligned}$$

Next we show that the second line on the right-hand-side in the expression above is negligible when compared to the first line, at least when taking the H^{-1} norm thereof. Recall also that Π is a bounded operator from H^{-1} to H^{-1} and acts diagonally on the Fourier elements e_ℓ. By [20, Lemma 5.5] using $u^L_{\ell,\beta} = 0$ for $|\ell| > N_q^{1-\delta}$ and $\theta^q_k = 0$ for $|k| < N_q$ it holds

$$\begin{aligned} &\left\| \sum_{\ell,\beta} u^L_{\ell,\beta} |\ell|^2 \Pi \left[\sum_k (\theta^q_k)^2 \sin^2(<_{k,\ell}) \left((a_{\ell,\beta}\cdot (k-\ell))\frac{k-\ell}{|k-\ell|^2} - (a_{\ell,\beta}\cdot k)\frac{k}{|k|^2} \right) e_\ell \right] \right\|_{H^{-1}} \\ &\quad \le \left(\sum_{\ell,\beta} |u^L_{\ell,\beta}|^2 |\ell|^2 \left(\sum_k (\theta^q_k)^2 \left| (a_{\ell,\beta}\cdot (k-\ell))\frac{k-\ell}{|k-\ell|^2} - (a_{\ell,\beta}\cdot k)\frac{k}{|k|^2} \right| \right)^2 \right)^{1/2} \\ &\quad \lesssim \left(\sum_{\ell,\beta} |u^L_{\ell,\beta}|^2 |\ell|^2 \left(\sum_k (\theta^q_k)^2 \frac{|\ell|}{N_q} \right)^2 \right)^{1/2} \lesssim \|u^L\|_{H^1} \kappa_q N_q^{-\delta}. \end{aligned}$$

On the other hand, one can check that the convergences in [20, Proposition 5.4, Lemma 5.6] hold true with $\gamma = 0$ and are uniform in $|\ell| \le N_q^{1-\delta}$, namely we have

$$\left\| \sum_{\ell,\beta} u^L_{\ell,\beta} |\ell|^2 \Pi \left[\sum_k (\theta^q_k)^2 \left(\sin^2(<_{k,\ell})(a_{\ell,\beta}\cdot k)\frac{k}{|k|^2} - \frac{4}{15} a_{\ell,\beta} \right) e_\ell \right] \right\|_{H^{-1}} \lesssim \|u^L\|_{H^1} \kappa_q N_q^{-1}.$$

Therefore, since

$$-4\pi^2 \sum_{\ell,\beta} u^L_{\ell,\beta} |\ell|^2 \sum_k (\theta^q_k)^2 a_{\ell,\beta} e_\ell = \kappa_q \Delta u^L,$$

we have for some universal constant C

$$\left\| \frac{2}{5}\kappa_q \Delta u^L - S_q(u^L) \right\|_{H^{-1}} = \left\| S_q^\perp(u^L) - \frac{4}{15}\kappa_q \Delta u^L \right\|_{H^{-1}} \le C \kappa_q N_q^{-\delta} \|u^L\|_{H^1}. \tag{2.5}$$

A similar computation shows the continuity of $S_q : \Pi_L H^{s+2} \to H^s$ for other values of $s \in \mathbb{R}$.

Remark 2.3. In the passive scalar case, there is no need to reformulate the equation as in (2.3) since no Leray projection is needed. As a consequence, the Stratonovich-to-Itō corrector takes the simpler form

$$S_q(\rho) = \sum_{k,\alpha} (\theta^q_k)^2 a_{k,\alpha} e_k \cdot \nabla (a_{k,\alpha} e_{-k} \cdot \nabla \rho) = \frac{2}{3}\kappa_q \Delta \rho.$$

2.2. **Dissipation.** In principle, we are interested in the case where the initial condition $u_0 \in L^2$ is deterministic. However, without any additional difficulty we can assume u_0 random and independent of the Brownian motions $\{W^{k,\alpha}\}$, and that $u_0 \in L^2$ holds almost surely. Indeed, this setting is equivalent to requiring $\|u_0\|_{L^2} \le M$ almost surely, for some deterministic constant $1 \le M < \infty$. To see this, introduce $\Omega_M := \{M-1 < \|u_0\|_{L^2} \le M\}$ for $M \in \mathbb{N}$, $M \ge 1$ and write $u_0 = \sum_M u_0 \mathbf{1}_{\Omega_M}$. Denoting u^M a Leray-Hopf weak solution of (2.1) with initial condition $u_0 \mathbf{1}_{\Omega_M}$, then $u := \sum_M u^M$ solves (2.1) with initial condition u_0. Viceversa, every Leray-Hopf weak solution to (2.1) can be decomposed as above.

Let $q \in \mathbb{N}$ be given. A key step to the proof of total dissipation is the following uniform estimate on the H^{-1} norm of u^L, the low-modes projection of u. This implies smallness of the H^{-1} norm of u since u^H is high-modes by construction, and thus the standard estimate $\|u^H\|_{H^{-1}} \lesssim N_q^{\delta-1}\|u^H\|_{L^2} \leq N_q^{\delta-1}\|u\|_{L^2}$ holds true (recall that by definition of H the process u has zero space average for every t almost surely).

Lemma 2.4. *For every $q \in \mathbb{N}$ there exists a choice of the parameters τ_q, N_q and $\kappa_q \sim N_q^3$ such that the following holds true. Let u be any Leray-Hopf weak solution of (2.1) with initial condition u_0, $\|u_0\|_{L^2} \leq M$. Then it holds for every $\beta > 3/2$ and $\delta \in (0, 1 \wedge (\beta - 3/2))$*

$$\mathbb{E}\left[\sup_{t\in[1-\tau_q/2,1-\tau_{q+1}]} \|u^L\|^2_{H^{-1}}\right] \lesssim_{\beta,\delta} \frac{M^{2\frac{\beta+1}{\beta}}}{(\nu+\kappa_q)^{\frac{1-\delta}{\beta}}}.$$

Proof. Let us decompose $u = u^L + u^H$ and write down the mild formulation for u^L with $t \in (1-\tau_q, 1-\tau_{q+1}]$

$$\begin{aligned} u^L_t &= P(t-(1-\tau_q))u^L_{1-\tau_q} - \int_{1-\tau_q}^t P(t-s)\Pi_L \Pi \operatorname{div}(u_s \otimes u_s) ds \\ &- \sum_{k,\alpha} \int_{1-\tau_q}^t P(t-s)\Pi_L\Pi \operatorname{div}(u_s \otimes \sigma_{k,\alpha}) dW^{k,\alpha}_s \\ &+ \int_{1-\tau_q}^t P(t-s)\left(S_q(u^L_s) - \frac{2}{5}\kappa_q \Delta u^L_s\right) ds. \end{aligned}$$

In the expression above we have conveniently denoted $P = e^{(\nu+2\kappa_q/5)\Delta}$ the semigroup generated by the operator $(\nu + 2\kappa_q/5)\Delta$, and we have used that $\Pi_L S_q = S_q \Pi_L$ since S_q acts diagonally on the Fourier elements e_ℓ.

Since u is a Leray-Hopf solution, the energy inequality guarantees $\|u^L_t\|_{L^2} \leq \|u_t\|_{L^2} \leq M$ almost surely for every t. Thus using $\|P(t)\|_{H^{-1}\to H^{-1}} \lesssim \frac{1}{(\nu+2\kappa_q/5)t}$ we have

$$\mathbb{E}\left[\sup_{t\in[1-\tau_q/2,1-\tau_{q+1}]} \|P(t-(1-\tau_q))u^L_{1-\tau_q}\|^2_{H^{-1}}\right] \lesssim \frac{M^2}{(\nu+\kappa_q)^2\tau_q^2},$$

and by boundedness of $\Pi_L\Pi$ and the embedding $L^1 \subset H^{-\beta+\delta}$

$$\begin{aligned} &\mathbb{E}\left[\sup_{t\in[1-\tau_q/2,1-\tau_{q+1}]} \int_{1-\tau_q}^t \|P(t-s)\Pi_L\Pi \operatorname{div}(u_s\otimes u_s)\|^2_{H^{-\beta}} ds\right] \\ &\lesssim \mathbb{E}\left[\sup_{t\in[1-\tau_q/2,1-\tau_{q+1}]} \int_{1-\tau_q}^t \frac{\|u_s\otimes u_s\|^2_{H^{-\beta+\delta}}}{(\nu+\kappa_q)^{1-\delta}(t-s)^{1-\delta}} ds\right] \\ &\lesssim \mathbb{E}\left[\sup_{t\in[1-\tau_q/2,1-\tau_{q+1}]} \int_{1-\tau_q}^t \frac{\|u_s\otimes u_s\|^2_{L^1}}{(\nu+\kappa_q)^{1-\delta}(t-s)^{1-\delta}} ds\right] \lesssim \frac{M^4}{(\nu+\kappa_q)^{1-\delta}}. \end{aligned}$$

In principle, the parameter δ in the equation above does not need to coincide with the parameter δ defining the low modes projector Π_L; however, we have chosen to use the same value for both quantities, so to avoid the introduction of an additional parameter and keep notation lighter.

To control the stochastic integral, we want to apply [18, Lemma 2.5], in particular equation (2.3) therein. Notice that, with respect to that lemma, here we have in addition the projector $\Pi_L\Pi$; but it is easy to check that the very same proof applies also in this case. Therefore

$$\mathbb{E}\left[\sup_{t\in[1-\tau_q/2,1-\tau_{q+1}]} \left\|\sum_{k,\alpha}\int_{1-\tau_q}^t P(t-s)\Pi_L\Pi \operatorname{div}(u_s\otimes\sigma_{k,\alpha}) dW^{k,\alpha}_s\right\|^2_{H^{-\beta}}\right] \lesssim \frac{M^2}{(\nu+\kappa_q)^{1-\delta}}.$$

Finally, for the last term we have by (2.5) and using that P and S_q commute

$$\left\|\int_{1-\tau_q}^t P(t-s)\left(S_q(u^L_s)-\frac{2}{5}\kappa_q\Delta u^L_s\right)ds\right\|_{H^{-1}} \lesssim \kappa_q N_q^{-\delta}\int_{1-\tau_q}^t \|P(t-s)u^L_s\|_{H^1}ds$$

$$\lesssim \kappa_q N_q^{-\delta}\int_{1-\tau_q}^t \|P(t-s)u^L_s\|_{H^{2-2\varepsilon}}ds \lesssim \kappa_q N_q^{-\delta}\int_{1-\tau_q}^t (\nu+\kappa_q)^{-1+\varepsilon}(t-s)^{-1+\varepsilon}\|u^L_s\|_{L^2}ds$$

$$\lesssim M\kappa_q^{\varepsilon}N_q^{-\delta},$$

where $\varepsilon>0$ is small enough. Accordingly,

$$\mathbb{E}\left[\sup_{t\in[1-\tau_q/2,1-\tau_{q+1}]}\left\|\int_{1-\tau_q}^t P(t-s)\left(S_q(u^L_s)-\frac{2}{5}\kappa_q\Delta u^L_s\right)ds\right\|^2_{H^{-1}}\right] \lesssim M^2\kappa_q^{2\varepsilon}N_q^{-2\delta}.$$

Putting all together and assuming for every $q\in\mathbb{N}$

$$\kappa_q^{2\varepsilon}N_q^{-2\delta}\lesssim\frac{1}{(\nu+\kappa_q)^{1-\delta}},\qquad (\nu+\kappa_q)^{1+\delta}\tau_q^2\gtrsim 1, \tag{2.6}$$

by interpolation we arrive to

$$\mathbb{E}\left[\sup_{t\in[1-\tau_q/2,1-\tau_{q+1}]}\|u^L\|^2_{H^{-1}}\right]\le M^{2\frac{\beta-1}{\beta}}\,\mathbb{E}\left[\sup_{t\in[1-\tau_q/2,1-\tau_{q+1}]}\|u^L\|^2_{H^{-\beta}}\right]^{1/\beta}\lesssim\frac{M^{2\frac{\beta+1}{\beta}}}{(\nu+\kappa_q)^{\frac{1-\delta}{\beta}}}.$$

□

With this lemma at hand we are ready to prove our dissipation result in the case of transport noise.

Proof of Proposition 2.1. Let (2.6) holds true. Without loss of generality we can assume $\|u_0\|_{L^2}\le M$ almost surely, for some deterministic constant $M\in(0,\infty)$. For simplicity we denote

$$\tilde c_q^2 := C_{\beta,\delta}\frac{M^{2/\beta}}{(\nu+\kappa_q)^{\frac{1-\delta}{\beta}}},$$

where $C_{\beta,\delta}$ is the implicit constant in the previous lemma and depends only on β and δ. The same lemma implies, by Markov inequality:

$$\|u_t\|^2_{H^{-1}}\le\|u^L_t\|^2_{H^{-1}}+\|u^H_t\|^2_{H^{-1}}\le(\tilde c_q+N_q^{2(\delta-1)})M^2=:c_qM^2$$

for every $t\in[1-\tau_q/2,1-\tau_{q+1}]$, with probability at least equal to $1-\tilde c_q$.

Recall the energy inequality (2.2) satisfied by Leray-Hopf weak solutions of (2.1): there exists a full Lebesgue measure set of times $\mathcal{T}=\mathcal{T}(\omega)$ such that for every $r\in\mathcal{T}$, $t>r$ it holds

$$\|u_t\|^2_{L^2}+2\nu\int_r^t\|u_s\|^2_{H^1}ds\le\|u_r\|^2_{L^2}.$$

Formally, in order to prove smallness of $\|u_{1-\tau_{q+1}}\|_{L^2}$ we would like to apply the following inequality with $c_q\ll 1$:

$$-\frac{d}{dt}\left(\frac{1}{\|u_t\|^2_{L^2}}\right)=\frac{1}{\|u_t\|^4_{L^2}}\frac{d}{dt}\|u_t\|^2_{L^2}\le-2\nu\frac{\|u_t\|^2_{H^1}}{\|u_t\|^4_{L^2}}\le-\frac{2\nu}{\|u_t\|^2_{H^{-1}}}\le-\frac{2\nu}{c_qM^2}, \tag{2.7}$$

where the second to last inequality comes from interpolation $\|u_t\|^4_{L^2}\le\|u_t\|^2_{H^1}\|u_t\|^2_{H^{-1}}$. However, the kinetic energy $t\mapsto\|u_t\|^2_{L^2}$ may have jump discontinuities and even vanish at some time $t\in[0,1)$. In order to rigorously make sense of the previous line, let us consider instead the function

$$E(t):=\begin{cases}\|u_t\|^2_{L^2}, & \text{if } t\in\mathcal{T},\\ \liminf\limits_{s\in\mathcal{T},\,s\to t}\|u_s\|^2_{L^2}, & \text{if } t\in\mathcal{T}^c.\end{cases}$$

The function E is non-increasing by definition of $\mathcal{T}$, therefore of class BV almost surely. In particular, $\liminf_{s\in\mathcal{T},s\to t}\|u_s\|^2_{L^2}$ always equals the right limit $\lim_{s\in\mathcal{T},s\downarrow t}\|u_s\|^2_{L^2}$ and E is right continuous. Moreover, also the map $t\mapsto\|u_t\|^2_{L^2}$ is of class BV almost surely since it coincides with E on the full Lebesgue measure set $\mathcal{T}$. In particular, it holds as Radon measures

$$d\|u\|^2_{L^2}((s,t])=dE((s,t])=E(t)-E(s).$$

Finally, $E(t) \geq \|u_t\|^2_{L^2}$ for every $t \in [0,1)$ since

$$\liminf_{s\in\mathcal{T},\, s\to t} \|u_s\|^2_{L^2} \geq \liminf_{s\to t} \|u_s\|^2_{L^2} \geq \|u_t\|^2_{L^2}$$

by lower semicontinuity of the L^2 norm (recall that we require Leray-Hopf weak solutions to be of class $C_w([0,1),H)$ almost surely).

Let us consider the Lebesgue decomposition of the measure dE:

$$dE(t) = e_{Leb}(t)dt + dE_{Can}(t) + \sum_{t\in\mathcal{D}} (E(t^+) - E(t^-))\delta_t,$$

where $e_{Leb}(t)$ is the density with respect to the Lebesgue measure of the absolutely continuous part of dE, dE_{Can} in the Cantor part of dE, $\mathcal{D}$ is the (at most countable) discontinuity set of E and δ_t denotes the delta Dirac measure at time t.

By energy inequality (2.2) we have for almost every $t \in [0,1)$

$$e_{Leb}(t) \leq -2\nu\|u_t\|^2_{H^1},$$

and the Cantor and atomic parts of dE are non-positive measures:

$$dE_{Can}(t) \leq 0, \qquad \sum_{t\in\mathcal{D}} (E(t^+) - E(t^-))\delta_t \leq 0.$$

Recall that we want to show that $\|u_{1-\tau_{q+1}}\|^2_{L^2}$ is small. Assume $E(1-\tau_{q+1}) > 0$ (otherwise there is nothing to prove); by Vol'pert formula (i.e. the chain rule for BV functions, see for instance [1]) we have with probability no less than $1 - \tilde{c}_q$

$$\begin{aligned}
-\frac{1}{E(1-\tau_{q+1})} &\leq \frac{1}{E(1-\tau_q/2)} - \frac{1}{E(1-\tau_{q+1})} \\
&= \int_{1-\tau_q/2}^{1-\tau_{q+1}} \frac{e_{Leb}(t)dt}{E(t)^2} + \int_{1-\tau_q/2}^{1-\tau_{q+1}} \frac{dE_{Can}(t)}{E(t)^2} \\
&\quad + \sum_{t\in\mathcal{D}} \mathbf{1}_{\{1-\tau_q/2<t\leq 1-\tau_{q+1}\}} \left(-\frac{1}{E(t^+)} + \frac{1}{E(t^-)} \right) \\
&\leq -2\nu \int_{1-\tau_q/2}^{1-\tau_{q+1}} \frac{\|u_t\|^2_{H^1} dt}{\|u_t\|^4_{L^2}} \leq -2\nu \int_{1-\tau_q/2}^{1-\tau_{q+1}} \frac{dt}{\|u_t\|^2_{H^{-1}}} \leq -\frac{\nu(\tau_q - 2\tau_{q+1})}{c_q M^2}.
\end{aligned}$$

In the last line we have used $\|u_t\|^2_{L^2} = E(t)$ for every $t \in \mathcal{T}$ and $\|u_t\|^2_{H^{-1}} \leq c_q M^2$ for every $t \in [1-\tau_q/2, 1-\tau_{q+1}]$ with probability at least $1 - \tilde{c}_q$.

Therefore, as long as $q \in \mathbb{N}$ is such that $E(1-\tau_{q+1}) > 0$ and assuming $\tau_q - 2\tau_{q+1} \geq \tau_q/2$, the previous formula implies

$$\mathbb{P}(A_q) \geq 1 - \tilde{c}_q, \quad A_q := \left\{ E(1-\tau_{q+1}) \leq M^2 \frac{2c_q}{\nu\tau_q} \right\}.$$

Let us now fix parameters $\beta = 12/5$, $\delta = 4/5$ and

$$\tau_q := 4^{-q}, \quad N_q := \tau_q^{-10}.$$

Since $\sum_q \tilde{c}_q < \infty$ by our choice of parameters, by Borel-Cantelli Lemma almost every $\omega \in \Omega$ belongs to the set A_q for every q larger than a certain $q_\star = q_\star(\omega)$, and thus

$$\sup_{t\geq 1-\tau_{q+1}} \|u_t\|^2_{L^2} \leq E(1-\tau_{q+1}) \to 0$$

almost surely as $q \to \infty$, since for every fixed value of M and ν

$$\lim_{q\to\infty} M^2 \frac{c_q}{\nu\tau_q} = 0.$$

This obviously implies $\lim_{t\uparrow 1} \|u_t\|_{L^2} = 0$ almost surely, and the proof is complete. □

Remark 2.5. As a consequence, Leray-Hopf weak solutions of (2.1) can be extended with $u_1 = 0$ to continuous functions at time $t = 1$ with respect to the strong topology on H.

Actually, from the proof of the previous proposition one can deduce the following refinement. Recall that the largest integer q such that $E(1-\tau_{q+1}) > M^2 \frac{c_q}{\nu\tau_q}$ is almost surely finite by Borel-Cantelli Lemma. In particular, we have the following almost sure "rate" of dissipation:

$$\limsup_{q\to\infty} \left(\frac{\nu\tau_q}{M^2 c_q}\right)^{1-\delta} E(1-\tau_{q+1}) \le \lim_{q\to\infty} \left(\frac{M^2 c_q}{\nu\tau_q}\right)^{\delta} = 0,$$

where $\delta \in (0,1)$, implying

$$\limsup_{q\to\infty} \left(\frac{\nu\tau_q}{M^2 c_q}\right)^{1-\delta} \sup_{t\ge 1-\tau_{q+1}} \|u_t\|_{L^2}^2 = 0. \tag{2.8}$$

2.3. **Non dissipating solutions.** The key property that allowed us to prove total dissipation at time $t=1$ in Proposition 2.1 was the energy inequality (2.2) satisfied by Leray-Hopf weak solutions of (2.1). More than that, (2.8) shows that in this case some sort of *enhanced dissipation* holds even before time $t=1$, although for $t<1$ it is neither total nor anomalous. In this subsection we show that weak solutions not satisfying the energy inequality may not dissipate energy close to time $t=1$.

Proposition 2.6. *Let $d=3$. For every $\nu>0$ and zero-mean, divergence-free $u_0 \in L^2$ almost surely there exists a progressively measurable weak solution u to (2.1) on the time interval $[0,1)$, with continuous trajectories in H^{-1} and initial condition $u|_{t=0}=u_0$, such that almost surely*

$$\int_r^1 \|u_t\|_{L^2}^2 dt = \infty, \quad \forall r \in (0,1). \tag{2.9}$$

In particular, (2.8) can not hold true for the constructed solution.

The proof is based on a modification of the convex integration scheme of [26]. Let Z be the unique weak solution on $[0,1)$ of the Stokes system

$$\begin{cases} dZ + \sum_{k,\alpha} (\sigma_{k,\alpha}\cdot\nabla) Z \circ dW^{k,\alpha} + \nabla p_Z dt = \nu \Delta Z dt, \\ \operatorname{div} Z = 0, \\ Z|_{t=0} = u_0. \end{cases}$$

It is sufficient to prove the following:

Lemma 2.7. *For every $K>0$ and $q\in\mathbb{N}$ there exists a progressively measurable weak solution u^q to (2.1) on the time interval $[1-\tau_q, 1-\tau_{q+1}]$ with continuous trajectory in H^{-1} and such that $u^q_{1-\tau_q} = Z_{1-\tau_q}$ and $u^q_{1-\tau_{q+1}} = Z_{1-\tau_{q+1}}$ almost surely, and with probability at least $1-(q+1)^{-2}$ it holds*

$$\int_{1-\tau_q}^{1-\tau_{q+1}} \|u^q_t\|_{L^2}^2 dt \ge K.$$

Proof. Without loss of generality we can assume $\|u_0\|_{L^2} \le M$ almost surely. The idea is to construct a solution u^q as the sum of the solution Z to the Stokes system and a perturbation v, under the additional constraint $v_{1-\tau_q} = v_{1-\tau_{q+1}} = 0$. In order to do so, we modify the convex integration scheme in [26, Proposition 4.2] using two sided cutoffs χ such that

$$\chi(t) = \begin{cases} 0, & \text{if } -\infty < t \le 1-\tau_q(1-2^{-n-1}), \\ 1, & \text{if } 1-\tau_q(1-2^{-n}) \le t \le 1-\tau_{q+1}(1+2^{-n}), \\ 0, & \text{if } 1-\tau_{q+1}(1+2^{-n-1}) \le t < \infty, \end{cases}$$

and monotone in between of these intervals. Notice that imposing the terminal value $v_{1-\tau_{q+1}} = 0$ does not compromise adaptedness, since the time $1-\tau_{q+1}$ is deterministic. Then the estimates in [26, Section 4.2] remain the same, with the only differences that the the iterative estimates can now depend on $\nu>0$ and time derivative of χ is now controlled with $|\chi'| \lesssim \tau_{q+1}^{-1} 2^n$ and therefore may depend on q; however this gives no additional problem since here ν, q are fixed (notice however that the constants $C_v, C_R, ...$ in [26, Proposition 4.2] may depend on ν and τ_{q+1}). The lower bound on the kinetic energy of u^q comes from (here u^q_n denotes the solution of the Navier-Stokes-Reynolds system obtained as n-th iteration of the convex integration scheme of [26, Proposition 4.2])

$$\int_{1-\tau_q(1-2^{-n+2})\wedge \mathfrak{t}}^{\mathfrak{t}} \left| \|u^q_{n+1}(t) - Z(t)\|^2_{L^2} - \|u^q_n(t) - Z(t)\|^2_{L^2} - 3\gamma_{n+1} \right| dt \leq C_e \delta_{n+1},$$

where δ_n is a given parameter going to zero sufficiently fast as $n \to \infty$, $\mathfrak{t} \leq 1 - \tau_{q+1}$ is a suitable stopping time with $\mathbb{P}\{\mathfrak{t} > 1 - \tau_q/2\} \geq 1 - (q+1)^{-2}$, and C_e may depend on ν and τ_{q+1}. Then, choosing n_0 sufficiently large (possibly depending on ν and q) and $\gamma_{n_0} \geq C\tau_q^{-1}(K + M^2)$, we get with probability no less than $1 - (q+1)^{-2}$:

$$\begin{aligned}
2\int_{1-\tau_q}^{1-\tau_{q+1}} \|u^q_t\|^2_{L^2} dt &\geq \int_{1-\tau_q(1-2^{-n+2})\wedge \mathfrak{t}}^{\mathfrak{t}} \|u^q_t - Z(t)\|^2_{L^2} dt - 2M^2 \\
&\geq \int_{1-\tau_q(1-2^{-n+2})\wedge \mathfrak{t}}^{\mathfrak{t}} \|u^q_{n_0}(t) - Z(t)\|^2_{L^2} dt - C_e \sum_{n=n_0}^{\infty} \delta_{n+1} - 3\sum_{n=n_0}^{\infty} \gamma_{n+1} - 2M^2 \\
&\geq \int_{1-\tau_q(1-2^{-n+2})\wedge \mathfrak{t}}^{\mathfrak{t}} 3\gamma_{n_0} dt - C_e \sum_n \delta_{n+1} - 3\sum_{n\neq n_0} \gamma_n - 2M^2 \geq K.
\end{aligned}$$

□

Proof of Proposition 2.6. It follows from the previous lemma by gluing solutions u^q on different time intervals and Borel-Cantelli Lemma. Gluing is possible because $Z_{1-\tau_{q+1}} \in L^2$ almost surely, and the glued process solves (2.1) and enjoys continuity in H^{-1}. Indeed for every divergence-free test function $f \in C^\infty_c(\mathbb{T}^3 \times [0,1), \mathbb{R}^3)$ and $0 < s < 1 - \tau_q < ... < 1 - \tau_{q+k} < t < 1$, since u is a solution on every time interval $[1 - \tau_{q+i-1}, 1 - \tau_{q+i}]$ (the endpoints s and t being similar) it holds

$$\begin{aligned}
\langle u^-_{1-\tau_{q+i}}, f_{1-\tau_{q+i}} \rangle - \langle u^+_{1-\tau_{q+i-1}}, f_{1-\tau_{q+i-1}} \rangle &= \int_{1-\tau_{q+i-1}}^{1-\tau_{q+i}} \langle u_r, \partial_t f_r + (u_r \cdot \nabla) f_r + \nu \Delta f_r \rangle dr \\
&+ \sum_{k,\alpha} \int_{1-\tau_{q+i-1}}^{1-\tau_{q+i}} \langle u_r, (\sigma_{k,\alpha} \cdot \nabla) f_r \rangle \circ dW^{k,\alpha},
\end{aligned}$$

where $u^-_{1-\tau_{q+i}}$ denotes the left limit of u at time $1 - \tau_{q+i}$ and $u^+_{1-\tau_{q+i-1}}$ denotes the right limit of u at time $1 - \tau_{q+i-1}$. By continuity in H^{-1}

$$\langle u^-_{1-\tau_{q+i}}, f_{1-\tau_{q+i}} \rangle - \langle u^+_{1-\tau_{q+i-1}}, f_{1-\tau_{q+i-1}} \rangle = \langle u_{1-\tau_{q+i}}, f_{1-\tau_{q+i}} \rangle - \langle u_{1-\tau_{q+i-1}}, f_{1-\tau_{q+i-1}} \rangle,$$

and therefore

$$\begin{aligned}
\langle u_t, f_t \rangle - \langle u_s, f_s \rangle &= \langle u_t, f_t \rangle - \langle u_{1-\tau_{q+k}}, f_{1-\tau_{q+k}} \rangle + ... + \langle u_{1-\tau_q}, f_{1-\tau_q} \rangle - \langle u_s, f_s \rangle \\
&= \int_s^t \langle u_r, \partial_t f_r + (u_r \cdot \nabla) f_r + \nu \Delta f_r \rangle dr + \sum_{k,\alpha} \int_s^t \langle u_r, (\sigma_{k,\alpha} \cdot \nabla) f_r \rangle \circ dW^{k,\alpha}.
\end{aligned}$$

□

3. Total dissipation by solution of randomly forced Navier-Stokes equations

In this section we consider a relatively more regular (in time) approximation of transport noise. Let the coefficients $\{\sigma_{k,\alpha}\}_{k,\alpha}$ be as in the previous section, and consider the Navier-Stokes equations with large friction and additive noise (1.4):

$$\begin{cases} dv + (v \cdot \nabla)v\, dt + \nabla p_v\, dt = \Delta v\, dt - \varepsilon^{-1} v\, dt + \varepsilon^{-1} \sum_{k,\alpha} \sigma_{k,\alpha} dW^{k,\alpha}, \\ \operatorname{div} v = 0, \end{cases}$$

where $\varepsilon = \varepsilon(t)$ depends on t and is constantly equal to $\varepsilon_q \in (0,1)$ on the intervals of the form $(1 - \tau_q, 1 - \tau_{q+1}]$. We shall assume $\varepsilon_q \to 0$ sufficiently fast as $q \to \infty$.

Definition 3.1. *Given a probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 1}, \mathbb{P})$ supporting a family of i.i.d. Brownian motions $\{W^{k,\alpha}\}_{k,\alpha}$, a weak solution of (1.4) is defined as a progressively measurable stochastic* *processes $v:\Omega\to C_w([0,1),H)\cap L^2_{loc}([0,1),H^1)$ almost surely such that, for every divergence-free test function $f\in C_c^\infty(\mathbb{T}^3\times[0,1),\mathbb{R}^3)$ it holds almost surely for every $0\le s<r<1$*

$$\langle v_r,f_r\rangle-\langle v_s,f_s\rangle=\int_s^r\langle v_t,\partial_t f_t+(v_t\cdot\nabla)f_t+\Delta f_t\rangle dt$$
$$-\int_s^r\varepsilon_t^{-1}\langle v_t,f_t\rangle dt+\sum_{k,\alpha}\int_s^r\varepsilon_t^{-1}\langle\sigma_{k,\alpha}(\cdot,t),f_t\rangle dW_t^{k,\alpha}.$$

Our main results are about total dissipation for progressively measurable (Leray-Hopf) weak solutions. For the sake of completeness, here we specify our notion of solutions.

Definition 3.2. *Given a probability space $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\ge1},\mathbb{P})$ supporting a family of i.i.d. Brownian motions $\{W^{k,\alpha}\}_{k,\alpha}$ and a weak solution v of (1.4), a progressively measurable weak solution of (1.1) is defined as a progressively measurable stochastic process $\rho:\Omega\to C_w([0,1),H)\cap L^2([0,1),H^1)$ almost surely such that, for every test function $f\in C_c^\infty(\mathbb{T}^3\times[0,1),\mathbb{R}^3)$ it holds almost surely for every $0\le s<r<1$*

$$\langle\rho_r,f_r\rangle-\langle\rho_s,f_s\rangle=\int_s^r\langle\rho_t,\partial_t f_t+\nu\Delta f_t\rangle dt+\int_s^r\langle\rho_t,(v_t\cdot\nabla)f_t\rangle dt,$$

and for almost every $\omega\in\Omega$ there exists a full Lebesgue measure set $\mathcal{T}\subset[0,1)$ such that $0\in\mathcal{T}$ and for every $r\in\mathcal{T}$, $t>r$ the following energy inequality holds almost surely

$$\|\rho_t\|_{L^2}^2+2\nu\int_r^t\|\nabla\rho_s\|_{L^2}^2ds\le\|\rho_r\|_{L^2}^2.$$

Definition 3.3. *Given a probability space $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\ge1},\mathbb{P})$ supporting a family of i.i.d. Brownian motions $\{W^{k,\alpha}\}_{k,\alpha}$ and a weak solution v of (1.4), a progressively measurable Leray-Hopf weak solution of (1.6) is defined as a progressively measurable stochastic process $u:\Omega\to C_w([0,1),H)\cap L^2([0,1),H^1)$ almost surely such that, for every divergence-free test function $f\in C_c^\infty(\mathbb{T}^3\times[0,1),\mathbb{R}^3)$ it holds almost surely for every $0\le s<r<1$*

$$\langle u_r,f_r\rangle-\langle u_s,f_s\rangle=\int_s^r\langle u_t,\partial_t f_t+(u_t\cdot\nabla)f_t+\nu\Delta f_t\rangle dt+\int_s^r\langle u_t,(v_t\cdot\nabla)f_t\rangle dt,$$

and for almost every $\omega\in\Omega$ there exists a full Lebesgue measure set $\mathcal{T}\subset[0,1)$ such that $0\in\mathcal{T}$ and for every $r\in\mathcal{T}$, $t>r$ the following energy inequality holds almost surely

$$\|u_t\|_{L^2}^2+2\nu\int_r^t\|\nabla u_s\|_{L^2}^2ds\le\|u_r\|_{L^2}^2.$$

Let us comment briefly on the previous notions of solutions Definition 3.2 and Definition 3.3.

Suppose $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\ge1},\mathbb{P})$, $\{W^{k,\alpha}\}_{k,\alpha}$ and v are given. For the passive scalar case Definition 3.2, since we have assumed $v\in C_w([0,1),H)\cap L^2_{loc}([0,1),H^1)$ almost surely and $\rho_0\in L^2$ there exists a unique weak solution ρ, which satisfies the energy inequality. This can be shown following the lines of [14, Corollary II.1]. By uniqueness, the restriction $\rho|_{[0,t]}$ of ρ to a time interval $[0,t]$, $t<1$ coincides, up to time t, with the solution obtained from the same initial condition and advecting velocity $v|_{[0,t]}$. In particular, ρ is necessarily adapted to the filtration $\{\mathcal{F}_t\}_{t\ge1}$ and therefore progressively measurable. The same applies to the Navier-Stokes case Definition 3.3 in dimension $d=2$. However, in dimension $d=3$ we can not prove uniqueness of solutions, and the restriction $u|_{[0,t]}$ of a solution may depend on the values of v after time t. For instance, it could be that the converging subsequence obtained by compactness depends on the whole trajectory of v. As a consequence, there is no guarantee that u is progressively measurable, and in order to regain adaptedness we may need to change the underlying probability space, see for instance [19].

The proof of Theorem 1.1, Theorem 1.4 and Theorem 1.5 is based on a generator approach inspired by [13], which roughly speaking permits us to mimic the proof of Lemma 2.4 even though technically speaking there is no Stratonovich corrector in the equations (1.1) and (1.6), since now v has positive decorrelation time and therefore ρ and u are processes with finite variation.

Similar arguments can be applied without additional difficulties to the Ornstein-Ulhenbeck approximation (1.5), but we shall omit details for the sake of brevity.

Let us consider (1.4) and let us split v on the time interval $[1-\tau_q,1-\tau_{q+1}]$ as

$$v=\varepsilon_q^{-1/2}w+r,$$

where $w(1-\tau_q)=0$, $r(1-\tau_q)=v(1-\tau_q)\in H$ almost surely, and w,r are divergence-free and evolve according to

$$dw=-\varepsilon_q^{-1}wdt+\varepsilon_q^{-1/2}Q_q^{1/2}dW,$$

$$dr=-\varepsilon_q^{-1}rdt+A(\varepsilon_q^{-1/2}w+r)dt+b(\varepsilon_q^{-1/2}w+r,\varepsilon_q^{-1/2}w+r)dt.$$

In the lines above we have denoted for simplicity

$$Q_q:=\sum_{k,\alpha}(\theta_k^q)^2(a_{k,\alpha}e_k\otimes a_{k,\alpha}e_{-k}),\qquad W:=\sum_{k,\alpha}a_{k,\alpha}e_kW^{k,\alpha},$$

so that $Tr(Q_q)=\kappa_q$ and

$$Q_q^{1/2}dW=\sum_{k,\alpha}\theta_k^q a_{k,\alpha}e_kdW^{k,\alpha}=\sum_{k,\alpha}\sigma_{k,\alpha}dW^{k,\alpha},$$

and the operators A and b are defined as

$$A:=\Delta,\qquad b(v_1,v_2):=-\Pi[(v_1\cdot\nabla)v_2].$$

Next we are going to collect some energy-type a priori estimates on the processes v, w, and r. Then our total dissipation results hold true as soon as v is a weak solution to (1.4) that can be decomposed as $v=\varepsilon_q^{-1/2}w+r$ on each interval of the form $[1-\tau_q,1-\tau_{q+1}]$, these energy estimates hold true, and ρ (resp. u) is a progressively measurable weak solution to (1.1) (resp. Leray-Hopf weak solution to (1.6)).

We point out that it is easy to exhibit at least one process v satisfying this property, for instance by taking the limit of the Galerkin approximations on $(1-\tau_q,1-\tau_{q+1}]$

$$v^n=\varepsilon_q^{-1/2}w^n+r^n,\qquad n\in\mathbb{N},$$

to produce a (Leray-Hopf) weak solution v to (1.4) on $[1-\tau_q,1-\tau_{q+1}]$, living in a probability space $(\Omega^q,\mathcal{F}^q,\{\mathcal{F}_t^q\}_{t\geq0},\mathbb{P}^q)$ supporting the Brownian motions $\{W^{k,\alpha}\}_{k,\alpha}$ with $N_q\leq|k|\leq 2N_q$, and then using continuity of v e.g. in H^{-1} to glue together solutions on different time intervals, see also [21].

It is worth mentioning that since each Brownian motion $W^{k,\alpha}$ has non-zero intensity θ_k for at most one time interval of the form $(1-\tau_q,1-\tau_{q+1}]$, the probability space $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq0},\mathbb{P})$ supporting the whole family $\{W^{k,\alpha}\}_{k,\alpha}$ can be just taken as the product of the probability spaces $(\Omega^q,\mathcal{F}^q,\{\mathcal{F}_t^q\}_{t\geq0},\mathbb{P}^q)$.

Notice that, given $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq0},\mathbb{P})$, $\{W^{k,\alpha}\}_{k,\alpha}$ and v as above, there always exists a progressively measurable weak solution ρ to (1.1) and (in dimension $d=2$ only) u to (1.6). This is a consequence of probabilistically weak existence and pathwise uniqueness, by Yamada-Watanabe Theorem.

However, when $d=3$ we are not able to construct Leray-Hopf weak solutions to (1.6) that are adapted to the filtration $\{\mathcal{F}_t\}_{t\geq0}$, and thus we need to define the probability space $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq0},\mathbb{P})$ and Brownian motions $\{W^{k,\alpha}\}_{k,\alpha}$ taking into account the adaptedness of u, too. This can be done considering simultaneously the Galerkin approximations of v and u, working at fixed divergence-free initial condition $u_0\in L^2$ and viscosity $\nu\in(0,1)$. More generally, one can fix countable families $\mathscr{C}\subset L^2$ with null divergence and $\mathscr{V}\subset(0,1)$ and consider simultaneously the Galerkin approximations of v and $u^{u_0,\nu}$, where $u^{u_0,\nu}$ solves (1.6) with initial condition $u_0\in\mathscr{C}$ and viscosity $\nu\in\mathscr{V}$. This marks the difference between the statements of Theorem 1.1, Theorem 1.4 and the statement of Theorem 1.5.

3.0.1. *Estimates on v.* The basic a priori estimate on v_t we get from (1.4), which in particular holds true on the Galerkin approximations v^n for every $t\in(1-\tau_q,1-\tau_{q+1}]$, is the following

$$\mathbb{E}\|v_t^n\|_{L^2}^2+2\int_{1-\tau_q}^t\mathbb{E}\|v_s^n\|_{H^1}^2ds+2\varepsilon_q^{-1}\int_{1-\tau_q}^t\mathbb{E}\|v_s^n\|_{L^2}^2ds$$
$$\leq\mathbb{E}\|\Pi_nv_{1-\tau_q}\|_{L^2}^2+\varepsilon_q^{-2}\kappa_q(\tau_q-\tau_{q+1}),$$

where in the right-hand-side Π_n is the Fourier projector on modes $|k|\leq n$, and $v_{1-\tau_q}$ is considered as a given initial condtion (we suppose to have already defined the solution v for times $t\leq1-\tau_q$). It is obtained by applying the Itō formula to $\|v_t^n\|_{L^2}^2$ and taking expectations.

Therefore we deduce the following energy estimate on v:

$$\begin{aligned}\sup_{t\in(1-\tau_q,1-\tau_{q+1}]} \mathbb{E}\|v_t\|_{L^2}^2 &\leq \mathbb{E}\|v_{1-\tau_q}\|_{L^2}^2 + \varepsilon_q^{-2}\kappa_q(\tau_q-\tau_{q+1}) \\ &\leq \sup_{t\in(1-\tau_{q-1},1-\tau_q]} \mathbb{E}\|v_t\|_{L^2}^2 + \varepsilon_q^{-2}\kappa_q(\tau_q-\tau_{q+1}),\end{aligned}$$

and iterating for $q, q-1, q-2, \dots, 1$ we obtain (define $v_t \equiv 0$ for times $t \leq 0$)

$$\sup_{t\in(1-\tau_q,1-\tau_{q+1}]} \mathbb{E}\|v_t\|_{L^2}^2 \leq \sum_{k\leq q} \varepsilon_k^{-2}\kappa_k(\tau_k-\tau_{k+1}) \lesssim \varepsilon_q^{-2}\kappa_q. \tag{3.1}$$

Here we are assuming $\varepsilon_q^{-2}\kappa_q \gg \varepsilon_{q-1}^{-2}\kappa_{q-1}$. Once we have the estimate above for the L^2 norm of v at every fixed time, we can deduce as well

$$\int_{1-\tau_q}^{1-\tau_{q+1}} \mathbb{E}\|v_s\|_{H^1}^2 ds \lesssim \varepsilon_q^{-2}\kappa_q, \tag{3.2}$$

and

$$\int_{1-\tau_q}^{1-\tau_{q+1}} \mathbb{E}\|v_s\|_{L^2}^2 ds \lesssim \varepsilon_q^{-1}\kappa_q. \tag{3.3}$$

Applying the Itō Formula to $\|v_t^n\|_{L^2}^4$ we get with similar arguments

$$\int_{1-\tau_q}^{1-\tau_{q+1}} \mathbb{E}\|v_s\|_{L^2}^4 ds \lesssim \varepsilon_q^{-2}\kappa_q^2. \tag{3.4}$$

3.0.2. *Estimates on w.* We have the explicit expression for the stochastic convolution

$$\begin{aligned}w_t &= \varepsilon_q^{-1/2}\int_{1-\tau_q}^t e^{-\varepsilon_q^{-1}(t-s)} Q_q^{1/2} dW_s \\ &= \sum_{k,\alpha} \theta_k^q a_{k,\alpha} e^{2\pi i k\cdot x} \varepsilon_q^{-1/2} \int_{1-\tau_q}^t e^{-\varepsilon_q^{-1}(t-s)} dW_s^{k,\alpha}.\end{aligned}$$

The estimates we need on w are the following: for every $t \in (1-\tau_q, 1-\tau_{q+1}]$ and $\theta \geq 0$,

$$\mathbb{E}\left\|\varepsilon_q^{-1/2}\int_{1-\tau_q}^t e^{-\varepsilon_q^{-1}(t-s)} Q_q^{1/2} dW_s\right\|_{H^\theta}^2 \lesssim \kappa_q N_q^{2\theta}, \tag{3.5}$$

which can be proved on the Galerkin approximations w^n by Itō isometry. By Gaussianity we also have

$$\mathbb{E}\left\|\varepsilon_q^{-1/2}\int_{1-\tau_q}^t e^{-\varepsilon_q^{-1}(t-s)} Q_q^{1/2} dW_s\right\|_{H^\theta}^4 \lesssim \kappa_q^2 N_q^{4\theta}. \tag{3.6}$$

If we want to put the supremum over time inside the expectation (this will be needed in the proof of Theorem 1.5), we can invoke [3, Lemma 3.1]. As a result we get a similar estimate as (3.6), up to a logarithmic factor in ε_q^{-1}

$$\mathbb{E}\sup_{t\in[1-\tau_q,1-\tau_{q+1}]}\left\|\varepsilon_q^{-1/2}\int_{1-\tau_q}^t e^{-\varepsilon_q^{-1}(t-s)} Q_q^{1/2} dW_s\right\|_{H^\theta}^4 \lesssim \log^2(1+\varepsilon_q^{-1})\kappa_q^2 N_q^{4\theta}. \tag{3.7}$$

3.0.3. *Estimates on r.* We are left with the a priori estimates on r. Let $C_\varepsilon = -Id + \varepsilon A$ and rewrite

$$\begin{aligned}dr &= -\varepsilon^{-1} r dt + A(\varepsilon^{-1/2} w + r)dt + b(\varepsilon^{-1/2}w + r, \varepsilon^{-1/2}w + r)dt \\ &= \varepsilon^{-1} C_\varepsilon r dt + \varepsilon^{-1/2} A w dt + b(v, \varepsilon^{-1/2} w + r)dt.\end{aligned}$$

Testing the equation against the solution itself, we have the following estimate for the Galerkin truncations r^n, for every $t \in (1-\tau_q, 1-\tau_{q+1}]$

$$\begin{aligned}
&\|r^n_t\|^2_{L^2} + 2\int_{1-\tau_q}^t \|r^n_s\|^2_{H^1} ds + 2\varepsilon_q^{-1}\int_{1-\tau_q}^t \|r^n_s\|^2_{L^2} ds \\
&\qquad \le \|v_{1-\tau_q}\|^2_{L^2} + 2\varepsilon_q^{-1/2}\int_{1-\tau_q}^t \langle A w_s, r^n_s\rangle ds + 2\varepsilon_q^{-1}\int_{1-\tau_q}^t \langle b(\varepsilon_q^{1/2} v^n_s, w^n_s), r^n_s\rangle ds.
\end{aligned}$$

By Young's inequality, there exist unimportant constants $c<1$ and $C<\infty$ such that

$$\begin{aligned}
\|r^n_t\|^2_{L^2} &+ 2\int_{1-\tau_q}^t \|r^n_s\|^2_{H^1} ds + 2\varepsilon_q^{-1}\int_{1-\tau_q}^t \|r^n_s\|^2_{L^2} ds \\
&\le \|v_{1-\tau_q}\|^2_{L^2} + c\varepsilon_q^{-1}\int_{1-\tau_q}^t \|r^n_s\|^2_{L^2} ds + C\int_{1-\tau_q}^t \|w^n_s\|^2_{H^2} ds \\
&\quad + C\varepsilon_q^{-1}\int_{1-\tau_q}^t \|\varepsilon_q^{1/2} v^n_s\|_{L^2}\|w^n_s\|_{H^3}\|r^n_s\|_{L^2} ds \\
&\le \|v_{1-\tau_q}\|^2_{L^2} + c\varepsilon_q^{-1}\int_{1-\tau_q}^t \|r^n_s\|^2_{L^2} ds + C\int_{1-\tau_q}^t \|w^n_s\|^2_{H^2} ds \\
&\quad + C\varepsilon_q^{-1}\int_{1-\tau_q}^t \|\varepsilon_q^{1/2} v^n_s\|^4_{L^2} ds + C\varepsilon_q^{-1}\int_{1-\tau_q}^t \|w^n_s\|^4_{H^3} ds.
\end{aligned}$$

Taking expectations, using (3.4), (3.6) and passing to the limit $n\to\infty$, we deduce the following preliminary estimate on r:

$$\int_{1-\tau_q}^{1-\tau_{q+1}} \mathbb{E}\|r_s\|^2_{L^2} ds \lesssim \kappa_q^2 N_q^{12}, \tag{3.8}$$

which is not very good (recall that $\kappa_q^2 N_q^{12}\to\infty$ relatively fast as $q\to\infty$) but is auxiliary to isolate the leading order terms in the dynamics of r. Indeed, rewrite

$$\begin{aligned}
dr &= \varepsilon^{-1} C_\varepsilon r dt + \varepsilon^{-1/2} A w dt + b(v, \varepsilon^{-1/2} w + r) dt \\
&= \varepsilon^{-1} C_\varepsilon r dt + \varepsilon^{-1/2} A w dt + \varepsilon^{-1} b(w,w) dt + \varepsilon^{-1/2} b(r,w) dt + \varepsilon^{-1/2} b(\varepsilon^{1/2} v, r) dt.
\end{aligned}$$

Taking into account $r_{1-\tau_q} = v_{1-\tau_q}$, the mild formulation of the previous equation takes the form

$$\begin{aligned}
r_t &= e^{\varepsilon^{-1}C_\varepsilon(t-1+\tau_q)} v_{1-\tau_q} + \varepsilon^{-1/2}\int_{1-\tau_q}^t e^{\varepsilon^{-1}C_\varepsilon(t-s)} A w_s ds + \varepsilon^{-1}\int_{1-\tau_q}^t e^{\varepsilon^{-1}C_\varepsilon(t-s)} b(w_s,w_s) ds \\
&\quad + \varepsilon^{-1/2}\int_{1-\tau_q}^t e^{\varepsilon^{-1}C_\varepsilon(t-s)} b(r_s,w_s) ds + \varepsilon^{-1/2}\int_{1-\tau_q}^t e^{\varepsilon^{-1}C_\varepsilon(t-s)} b(\varepsilon^{1/2} v_s, r_s) ds.
\end{aligned}$$

Let $\theta_0 > 5/2$. After taking expectation and time integral on $[1-\tau_q, 1-\tau_{q+1}]$, we can separately estimate in $H^{-\theta_0}$ each term on the right-hand-side of the equation above as follows. First,

$$\int_{1-\tau_q}^{1-\tau_{q+1}} \mathbb{E}\left\| e^{\varepsilon^{-1}C_\varepsilon(t-1+\tau_q)} v_{1-\tau_q}\right\|_{H^{-\theta_0}} dt \lesssim \varepsilon_q \mathbb{E}\|v_{1-\tau_q}\|_{L^2} \lesssim \varepsilon_q \varepsilon_{q-1}^{-1}\kappa_{q-1}^{1/2} \lesssim \varepsilon_q^{1/2},$$

where we have used (3.1) and assuming for every $q\ge 1$

$$\varepsilon_{q-1}^{-1}\kappa_{q-1}^{1/2} \lesssim \varepsilon_q^{-1/2}. \tag{3.9}$$

Moreover, by Young's convolution inequality and previous estimates (3.3), (3.5) on v, w and (3.8) on r we have

$$\begin{aligned}
&\varepsilon_q^{-1/2}\int_{1-\tau_q}^{1-\tau_{q+1}} \mathbb{E}\left\|\int_{1-\tau_q}^t e^{\varepsilon^{-1}C_\varepsilon(t-s)} A w_s\, ds\right\|_{H^{-\theta_0}} dt \\
&\lesssim \varepsilon_q^{-1/2}\int_{1-\tau_q}^{1-\tau_{q+1}}\int_{1-\tau_q}^t e^{-\varepsilon^{-1}(t-s)}\mathbb{E}\|w_s\|_{L^2}\, ds dt \lesssim \varepsilon_q^{1/2}\kappa_q^{1/2},
\end{aligned}$$

and similarly

$$\varepsilon_q^{-1/2}\int_{1-\tau_q}^{1-\tau_{q+1}}\mathbb{E}\left\|\int_{1-\tau_q}^{t}e^{\varepsilon^{-1}C_\varepsilon(t-s)}b(r_s,w_s)\,ds\right\|_{H^{-\theta_0}}dt\lesssim\varepsilon_q^{1/2}\kappa_q^{3/2}N_q^6,$$

$$\varepsilon_q^{-1/2}\int_{1-\tau_q}^{1-\tau_{q+1}}\mathbb{E}\left\|\int_{1-\tau_q}^{t}e^{\varepsilon^{-1}C_\varepsilon(t-s)}b(\varepsilon_q^{1/2}v_s,r_s)\,ds\right\|_{H^{-\theta_0}}dt\lesssim\varepsilon_q^{1/2}\kappa_q^{3/2}N_q^6.$$

In the second and last inequality we have used that the operator $b:H\times H\to H^{-\theta_0}$ is bounded, therefore

$$\mathbb{E}\,\|b(r_s,w_s)\|_{H^{-\theta_0}}\lesssim\mathbb{E}\|r_s\|_{L^2}\|w_s\|_{L^2}\le\left(\mathbb{E}\|r_s\|_{L^2}^2\right)^{1/2}\left(\mathbb{E}\|w_s\|_{L^2}^2\right)^{1/2},$$

and similarly for the term $b(\varepsilon_q^{1/2}v_s,r_s)$. Putting all together, we obtain

$$\int_{1-\tau_q}^{1-\tau_{q+1}}\mathbb{E}\|r_s-\tilde r_s\|_{H^{-\theta_0}}ds\lesssim\varepsilon_q^{1/2}\kappa_q^{3/2}N_q^6, \tag{3.10}$$

where we have defined

$$\tilde r_t=\varepsilon_q^{-1}\int_{1-\tau_q}^{t}e^{\varepsilon^{-1}C_\varepsilon(t-s)}b(w_s,w_s)ds.$$

An estimate similar to (3.8) holds for $\tilde r$, indeed

$$\begin{aligned}\int_{1-\tau_q}^{1-\tau_{q+1}}\mathbb{E}\|\tilde r_s\|_{L^2}^2ds&\lesssim\int_{1-\tau_q}^{1-\tau_{q+1}}\int_{1-\tau_q}^{s}\varepsilon_q^{-1}e^{-\varepsilon^{-1}(s-r)}\mathbb{E}\|b(w_r,w_r)\|_{L^2}^2drds\\&\lesssim\int_{1-\tau_q}^{1-\tau_{q+1}}\int_{1-\tau_q}^{s}\varepsilon_q^{-1}e^{-\varepsilon^{-1}(s-r)}\mathbb{E}\|w_r\|_{H^3}^4drds\lesssim\kappa_q^2N_q^{12},\end{aligned}$$

and more generally, using $b:H^{3+\theta}\times H^{3+\theta}\to H^\theta$ continuously for every $\theta\ge0$,

$$\int_{1-\tau_q}^{1-\tau_{q+1}}\mathbb{E}\|\tilde r_s\|_{H^\theta}^2ds\lesssim\kappa_q^2N_q^{12+4\theta}. \tag{3.11}$$

By interpolation we also get for every $\theta\in(0,\theta_0)$ and p such that $\frac{p\theta}{\theta_0}+\frac{p(1-\theta/\theta_0)}{2}\le1$

$$\begin{aligned}\int_{1-\tau_q}^{1-\tau_{q+1}}\mathbb{E}\|r_s-\tilde r_s\|_{H^{-\theta}}^pds&\lesssim\int_{1-\tau_q}^{1-\tau_{q+1}}\mathbb{E}\|r_s-\tilde r_s\|_{H^{-\theta_0}}^{p\theta/\theta_0}\|r_s-\tilde r_s\|_{L^2}^{p(1-\theta/\theta_0)}ds\\&\lesssim\left(\varepsilon_q^{1/2}\kappa_q^{3/2}N_q^6\right)^{p\theta/\theta_0}\left(\kappa_q^2N_q^{12}\right)^{p(1-\theta/\theta_0)/2}.\end{aligned}$$

In particular, for $\theta=1/2$ and $p=4/3$

$$\left(\int_{1-\tau_q}^{1-\tau_{q+1}}\mathbb{E}\|r_s-\tilde r_s\|_{H^{-1/2}}^{4/3}ds\right)^{3/4}\lesssim\left(\varepsilon_q^{1/2}\kappa_q^{3/2}N_q^6\right)^{1/2\theta_0}\left(\kappa_q^2N_q^{12}\right)^{(1-1/2\theta_0)/2}\lesssim\varepsilon_q^{1/12}\kappa_q^2N_q^6. \tag{3.12}$$

3.1. **Dissipation.** As we have seen in section 2, when v is a white-in-time noise we can prove anomalous dissipation for u thanks to the presence of a Stratonovich corrector in the Itō formulation of (2.1). The goal of this subsection is to "find" the hidden Stratonovich corrector in the dynamics of u solution of (1.6). We will focus on the Navier-Stokes case (1.6) only, and in particular on Theorem 1.5; but the same arguments work with minor modifications in all the other cases.

Technically speaking, there isn't any Stratonovich corrector in (1.6) since u has finite variation; but morally speaking, the solution v of (1.4) excites the small scales of u just as well as the white-in-time transport noise. Therefore, it is reasonable to expect that the same dissipation mechanism, induced by transfer of energy to high wavenumbers, can happen in this case, too.

Let us work on a fixed time interval $(1-\tau_q,1-\tau_{q+1}]$. Let us consider the process

$$U:=u+\varepsilon_q^{1/2}b(w,u)+\frac{\varepsilon_q}{2}b(w,b(w,u))+V,$$

where the two correctors $\varepsilon_q^{1/2} b(w,u)$ and $\frac{\varepsilon_q}{2} b(w, b(w,u))$ are motivated by the heuristic arguments presented in subsection 1.2 and serve to reintroduce the time roughness producing the Stratonovich corrector in (1.6), and the auxiliary process V is defined as

$$V := \frac{\varepsilon_q}{2} b(b(w,w),u) + \varepsilon_q b((-C_\varepsilon)^{-1}\tilde{r}, u),$$

and is needed to compensate for the term $b(r,u)dt$ appearing in the dynamics of u. Indeed, by Itō formula we have

$$\begin{aligned}\frac{\varepsilon_q}{2} d\left(b(b(w,w),u)\right) = &-b(b(w,w),u)\,dt + \sum_{k,\alpha} (\theta_k^q)^2 b(b(a_{k,\alpha}e_k, a_{k,\alpha}e_{-k}),u)\,dt \\ &+ \frac{\varepsilon_q^{1/2}}{2} b(b(Q_q^{1/2}dW_t, w),u) + \frac{\varepsilon_q^{1/2}}{2} b(b(w, Q_q^{1/2}dW_t),u) \\ &+ \frac{\varepsilon_q^{1/2}}{2} b(b(w,w), b(w,u))\,dt + \frac{\varepsilon_q}{2} b(b(w,w), \nu A u + b(u,u) + b(r,u))\,dt\end{aligned}$$

and

$$\begin{aligned}\varepsilon_q d\left(b((-C_\varepsilon)^{-1}\tilde{r},u)\right) = &-b(\tilde{r},u)\,dt + b((-C_\varepsilon)^{-1} b(w,w),u)\,dt \\ &+ \varepsilon_q b((-C_\varepsilon)^{-1}\tilde{r}, \nu A u + b(u,u) + b(r,u))\,dt \\ &+ \varepsilon_q^{1/2} b((-C_\varepsilon)^{-1}\tilde{r}, b(w,u))\,dt.\end{aligned}$$

Notice that the Itō corrector $\sum_{k,\alpha} (\theta_k^q)^2 b(b(a_{k,\alpha}e_k, a_{k,\alpha}e_{-k}),u)$ in the dynamics of $\frac{\varepsilon_q}{2} b(b(w,w),u)$ equals zero since $b(a_{k,\alpha}e_k, a_{k,\alpha}e_{-k}) = 0$ for every $k \in \mathbb{Z}_0^3$ and $\alpha \in \{1,2\}$. Moreover, since it holds $(-C_\varepsilon)^{-1} - Id = \varepsilon_q A(-C_\varepsilon)^{-1}$ (this can be checked multiplying both expression by $-C_\varepsilon$) we also have

$$b((-C_\varepsilon)^{-1} b(w,w),u) - b(b(w,w),u) = \varepsilon_q b(A(-C_\varepsilon)^{-1} b(w,w),u).$$

Therefore, the process U evolves according to

(3.13)

$$\begin{aligned}dU = &\nu A u\,dt + b(u,u)\,dt + b(r-\tilde{r},u)\,dt + \sum_{k,\alpha} (\theta_k^q)^2 b(a_{k,\alpha}e_k, b(a_{k,\alpha}e_{-k},u))\,dt + b(Q_q^{1/2}dW_t, u) \\ &+ \varepsilon_q^{1/2} b(w, \nu A u + b(u,u) + b(r,u))\,dt + \frac{\varepsilon_q^{1/2}}{2} b(w, b(w, b(w,u))\,dt \\ &+ \frac{\varepsilon_q}{2} b(w, b(w, \nu A u + b(u,u) + b(r,u)))\,dt \\ &+ \frac{\varepsilon_q^{1/2}}{2} b(Q_q^{1/2}dW_t, b(w,u)) + \frac{\varepsilon_q^{1/2}}{2} b(w, b(Q_q^{1/2}dW_t, u)) \\ &+ \frac{\varepsilon_q^{1/2}}{2} b(b(Q_q^{1/2}dW_t, w),u) + \frac{\varepsilon_q^{1/2}}{2} b(b(w, Q_q^{1/2}dW_t),u) \\ &+ \frac{\varepsilon_q^{1/2}}{2} b(b(w,w), b(w,u))\,dt + \frac{\varepsilon_q}{2} b(b(w,w), \nu A u + b(u,u) + b(r,u))\,dt \\ &+ \varepsilon_q b((-C_\varepsilon)^{-1}\tilde{r}, \nu A u + b(u,u) + b(r,u))\,dt \\ &+ \varepsilon_q^{1/2} b((-C_\varepsilon)^{-1}\tilde{r}, b(w,u))\,dt + \varepsilon_q b(A(-C_\varepsilon)^{-1} b(w,w),u)\,dt.\end{aligned}$$

The term $\sum_{k,\alpha} (\theta_k^q)^2 b(a_{k,\alpha}e_k, b(a_{k,\alpha}e_{-k},u))$ comes from the second derivative of $\frac{\varepsilon_q}{2} b(w, b(w,u))$ with respect to w, and recalling (2.4) it coincides with the Stratonovich-to-Itō corrector applied to u:

$$\sum_{k,\alpha} (\theta_k^q)^2 b(a_{k,\alpha}e_k, b(a_{k,\alpha}e_{-k},u)) = S_q(u).$$

Thus, we can rewrite

$$S_q(u) = \frac{2}{5}\kappa_q \Delta U + \left(S_q(U) - \frac{2}{5}\kappa_q \Delta U\right) - \varepsilon_q^{1/2} S_q(b(w,u) - \frac{\varepsilon_q}{2} S_q(b(w, b(w,u))) - S_q(V), \tag{3.14}$$

and the term $\frac{2}{5}\kappa_q \Delta U$ gives us enough dissipation to control a negative Sobolev norm in the mild formulation of U. More precisely, let Π_L be the Fourier projector onto modes $|k| \leq N_q^{1-\delta}$, for some $\delta \in (0,1)$, and denote $U^L = \Pi_L U$. We have

Lemma 3.4. *Let u be a Leray-Hopf solution to (1.6) with zero-mean, divergence-free initial condition u_0 satisfying $\|u_0\|_{L^2} \leq M$ for some deterministic $1 \leq M < \infty$, and let U^L be defined as above. Then for every $q \in \mathbb{N}$ there exists a choice of the parameters τ_q, N_q, ε_q and $\kappa_q \sim N_q^3$ such that for every $\delta \in (1/6, 1)$*

$$\mathbb{E}\left[\sup_{t\in[1-\tau_q/2,1-\tau_{q+1}]} \|U^L\|_{H^{-4}}\right] \lesssim_\delta \frac{M^2}{(\nu+\kappa_q)^{1-\delta}} + \frac{\varepsilon_q^{1/12}\kappa_q^2 N_q^6 M}{\nu^{1/4}}.$$

Proof. Let P be the semigroup generated by $\nu A + 2\kappa_q \Delta/5 = (\nu + 2\kappa_q/5)\Delta$, and consider the mild formulation of (3.13) for times $t \in [1-\tau_q/2, 1-\tau_{q+1}]$, taking also (3.14) into account.

First, at time $t = 1-\tau_q$ we have $U^L = u^L$ by definition of w, therefore

$$\mathbb{E}\left[\sup_{t\in[1-\tau_q/2,1-\tau_{q+1}]} \|P(t-(1-\tau_q))U^L_{1-\tau_q}\|_{H^{-4}}\right] \lesssim \frac{M}{(\nu+\kappa_q)\tau_q}.$$

We shall assume hereafter condition (2.6) on τ_q and κ_q as in Lemma 2.4, and moreover $(\nu+\kappa_q)^{-\delta} \leq \tau_q$. In addition, we will take ε_q satisfying (3.9) and small with respect to the other parameters, so to control easily all the terms multiplied by powers of ε_q. For these terms, we do not need the action of the semigroup P to prove smallness, and we will use the simple estimate

$$\mathbb{E}\left[\sup_{t\in[1-\tau_q/2,1-\tau_{q+1}]} \left\|\int_{1-\tau_q}^t P(t-s)\Pi_L \dots ds\right\|_{H^{-4}}\right] \lesssim \int_{1-\tau_q}^{1-\tau_{q+1}} \mathbb{E}\,\|\dots\|_{H^{-4}}\, ds.$$

Having said this, let us control the terms appearing in (3.14), which permits us to isolate the strong dissipation term $\frac{2}{5}\kappa_q \Delta U^L$. By (3.5), (3.8), using $\|S_q(u)\|_{H^{-4}} \lesssim \kappa_q \|u\|_{H^{-2}}$ and the Sobolev embedding we have for $\theta_0 > 5/2$

$$\begin{aligned}
&\varepsilon_q^{1/2}\mathbb{E}\left[\int_{1-\tau_q}^{1-\tau_{q+1}} \|S_q(b(w_s,u_s))\|_{H^{-4}}\, ds\right] \lesssim \varepsilon_q^{1/2}\kappa_q \mathbb{E}\left[\int_{1-\tau_q}^{1-\tau_{q+1}} \|w_s \otimes u_s\|_{H^{-1}} ds\right] \\
&\lesssim \varepsilon_q^{1/2}\kappa_q \int_{1-\tau_q}^{1-\tau_{q+1}} \mathbb{E}\|w_s\|_{L^\infty}\|u_s\|_{L^2} ds \lesssim \varepsilon_q^{1/2}\kappa_q^{3/2} N_q^{\theta_0-1} M,
\end{aligned}$$

$$\begin{aligned}
&\varepsilon_q \mathbb{E}\left[\int_{1-\tau_q}^{1-\tau_{q+1}} \|S_q(b(w_s,b(w_s,u_s)))\|_{H^{-4}}\, ds\right] \lesssim \varepsilon_q\kappa_q \mathbb{E}\left[\int_{1-\tau_q}^{1-\tau_{q+1}} \|w_s \otimes b(w_s,u_s)\|_{H^{-1}} ds\right] \\
&\lesssim \varepsilon_q\kappa_q \int_{1-\tau_q}^{1-\tau_{q+1}} \mathbb{E}\|w_s\|^2_{H^{\theta_0}}\|u_s\|_{L^2} ds \lesssim \varepsilon_q\kappa_q^2 N_q^{2\theta_0} M,
\end{aligned}$$

and similarly by (3.11)

$$\varepsilon_q \mathbb{E}\left[\int_{1-\tau_q}^{1-\tau_{q+1}} \|S_q(b(b(w_s,w_s),u_s)))\|_{H^{-4}}\, ds\right] \lesssim \varepsilon_q\kappa_q^2 N_q^{2\theta_0} M,$$

$$\varepsilon_q \mathbb{E}\left[\int_{1-\tau_q}^{1-\tau_{q+1}} \left\|S_q(b((-C_\varepsilon)^{-1}\tilde{r}_s,u_s))\right\|_{H^{-4}}\, ds\right] \lesssim \varepsilon_q\kappa_q N_q^{6+2\theta_0} M.$$

Moreover, by (3.6) and (3.11) the correctors giving the difference $U-u$ are small, thus

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[1-\tau_q/2,1-\tau_{q+1}]} \left\|\int_{1-\tau_q}^t P(t-s)\left(S_q(U^L) - \frac{2}{5}\kappa_q\Delta U^L\right) ds\right\|_{H^{-4}}\right] \\
&\lesssim \kappa_q N_q^{-\delta}\mathbb{E}\left[\sup_{t\in[1-\tau_q/2,1-\tau_{q+1}]} \int_{1-\tau_q}^t \left\|P(t-s)U^L\right\|_{H^{-2}} ds\right] \\
&\lesssim \kappa_q^\varepsilon N_q^{-\delta} M\left(1 + \varepsilon_q^{1/2}\kappa_q^{1/2}N_q^{\theta_0} + \varepsilon_q\kappa_q N_q^{6+2\theta_0}\right),
\end{aligned}$$

for arbitrary $\varepsilon > 0$ small, coming from the action of the semigroup.

Similarly we have

$$\mathbb{E}\left[\sup_{t\in[1-\tau_q/2,1-\tau_{q+1}]}\left\|\int_{1-\tau_q}^t P(t-s)\Pi_L \nu A(u_s-U_s)ds\right\|_{H^{-4}}\right] \lesssim M\left(\varepsilon_q^{1/2}\kappa_q^{1/2}N_q^{\theta_0}+\varepsilon_q\kappa_q N_q^{6+2\theta_0}\right)$$

Let us now move to the other terms in (3.13). We use (3.6), (3.8) and have

$$\begin{aligned}\varepsilon_q^{1/2}\mathbb{E}&\left[\int_{1-\tau_q}^{1-\tau_{q+1}}\|b(w_s,\nu Au_s+b(u_s,u_s)+b(r_s,u_s))\|_{H^{-4}}\,ds\right]\\&\lesssim \varepsilon_q^{1/2}\mathbb{E}\left[\int_{1-\tau_q}^{1-\tau_{q+1}}(\|w_s\|_{H^{\theta_0+1}}\|u_s\|_{L^2}+\|w_s\|_{H^{\theta_0}}(\|u_s\|_{L^2}^2+\|u_s\|_{L^2}\|r_s\|_{L^2})ds\right]\\&\lesssim \varepsilon_q^{1/2}\kappa_q^{3/2}N_q^{6+\theta_0}M^2,\end{aligned}$$

$$\begin{aligned}\varepsilon_q\mathbb{E}&\left[\int_{1-\tau_q}^{1-\tau_{q+1}}\|b(w_s,b(w_s,\nu Au_s+b(u_s,u_s)+b(r_s,u_s)))\|_{H^{-4}}\,ds\right]\\&\lesssim \varepsilon_q\mathbb{E}\left[\int_{1-\tau_q}^{1-\tau_{q+1}}(\|w_s\|_{H^{\theta_0+1}}\|w_s\|_{H^{\theta_0+2}}\|u_s\|_{L^2}+\|w_s\|_{H^{\theta_0}}\|w_s\|_{H^{\theta_0+1}}(\|u_s\|_{L^2}^2+\|u_s\|_{L^2}\|r_s\|_{L^2}))ds\right]\\&\lesssim \varepsilon_q\kappa_q^2N_q^{2\theta_0+7}M^2,\end{aligned}$$

and similarly by (3.11)

$$\varepsilon_q\mathbb{E}\left[\int_{1-\tau_q}^{1-\tau_{q+1}}\|b(b(w_s,w_s),\nu Au_s+b(u_s,u_s)+b(r_s,u_s))\|_{H^{-4}}\,ds\right]\lesssim \varepsilon_q\kappa_q^2N_q^{2\theta_0+8}M^2,$$

$$\varepsilon_q\mathbb{E}\left[\int_{1-\tau_q}^{1-\tau_{q+1}}\left\|b((-C_\varepsilon)^{-1}\tilde{r}_s,\nu Au_s+b(u_s,u_s)+b(r_s,u_s))\right\|_{H^{-4}}\,ds\right]\lesssim \varepsilon_q\kappa_q^2N_q^{14+2\theta_0}M^2.$$

In addition,

$$\begin{aligned}\varepsilon_q^{1/2}\mathbb{E}&\left[\int_{1-\tau_q}^{1-\tau_{q+1}}\|b(w_s,b(w_s,b(w_s,u_s))\|_{H^{-4}}\,ds\right]\\&\lesssim \varepsilon_q^{1/2}\mathbb{E}\left[\int_{1-\tau_q}^{1-\tau_{q+1}}\|w_s\|_{H^{\theta_0+1}}\|w_s\|_{H^{\theta_0}}\|w_s\|_{H^{\theta_0-1}}\|u_s\|_{L^2}ds\right]\\&\lesssim \varepsilon_q^{1/2}\kappa_q^{3/2}N_q^{3\theta_0}M,\end{aligned}$$

and

$$\varepsilon_q^{1/2}\mathbb{E}\left[\int_{1-\tau_q}^{1-\tau_{q+1}}\|b(b(w_s,w_s),b(w_s,u_s))\|_{H^{-4}}\,ds\right]\lesssim \varepsilon_q^{1/2}\kappa_q^{3/2}N_q^{3\theta_0}M,$$

$$\varepsilon_q^{1/2}\mathbb{E}\left[\int_{1-\tau_q}^{1-\tau_{q+1}}\left\|b((-C_\varepsilon)^{-1}\tilde{r}_s,b(w_s,u_s))\right\|_{H^{-4}}\,ds\right]\lesssim \varepsilon_q^{1/2}\kappa_q^{3/2}N_q^{6+3\theta_0}M,$$

$$\varepsilon_q\mathbb{E}\left[\int_{1-\tau_q}^{1-\tau_{q+1}}\left\|b(A(-C_\varepsilon)^{-1}b(w_s,w_s),u_s)\right\|_{H^{-4}}\,ds\right]\lesssim \varepsilon_q\kappa_qN_q^{2\theta_0}M.$$

All these quantities are small assuming

$$\varepsilon_q\kappa_q^3N_q^{30}\le 1.$$

For the terms involving stochastic integrals, we have by maximal inequality for stochastic convolution [12, Theorem 1.1]

$$\begin{aligned}&\varepsilon_q\mathbb{E}\left[\sup_{t\in[1-\tau_q/2,1-\tau_{q+1}]}\left\|\int_{1-\tau_q}^t P(t-s)\Pi_L b(Q_q^{1/2}dW_s,b(w_s,u_s))\right\|_{H^{-4}}^2\right]\\&\quad\lesssim \varepsilon_q\sum_{k,\alpha}\int_{1-\tau_q}^{1-\tau_{q+1}}\mathbb{E}\left[\|b(\theta_k^q a_{k,\alpha}e_k,b(w_s,u_s))\|_{H^{-4}}^2\right]ds\lesssim \varepsilon_q\kappa_q^2N_q^{4\theta_0}M^2,\end{aligned}$$

and in the same fashion

$$\varepsilon_q\mathbb{E}\left[\sup_{t\in[1-\tau_q/2,1-\tau_{q+1}]}\left\|\int_{1-\tau_q}^t P(t-s)\Pi_L b(w_s,b(Q_q^{1/2}dW_s,u_s))\right\|_{H^{-4}}^2\right]\lesssim \varepsilon_q\kappa_q^2N_q^{4\theta_0}M^2,$$

$$\varepsilon_q\mathbb{E}\left[\sup_{t\in[1-\tau_q/2,1-\tau_{q+1}]}\left\|\int_{1-\tau_q}^t P(t-s)\Pi_L b(b(Q_q^{1/2}dW_s,w_s),u_s)\right\|_{H^{-4}}^2\right]\lesssim \varepsilon_q\kappa_q^2N_q^{4\theta_0}M^2,$$

$$\varepsilon_q\mathbb{E}\left[\sup_{t\in[1-\tau_q/2,1-\tau_{q+1}]}\left\|\int_{1-\tau_q}^t P(t-s)\Pi_L b(b(w_s,Q_q^{1/2}dW_s),u_s)\right\|_{H^{-4}}^2\right]\lesssim \varepsilon_q\kappa_q^2N_q^{4\theta_0}M^2,$$

which are small under the same assumptions on ε_q.

As for the other terms, we have

$$\begin{aligned}&\mathbb{E}\left[\sup_{t\in[1-\tau_q/2,1-\tau_{q+1}]}\left\|\int_{1-\tau_q}^t P(t-s)\Pi_L b(u_s,u_s)ds\right\|_{H^{-4}}\right]\\&\quad\lesssim \mathbb{E}\left[\sup_{t\in[1-\tau_q/2,1-\tau_{q+1}]}\int_{1-\tau_q}^t\frac{\|u_s\|_{L^2}^2}{(\nu+\kappa_q)^{1-\delta}(t-s)^{1-\delta}}ds\right]\lesssim\frac{M^2}{(\nu+\kappa_q)^{1-\delta}},\end{aligned}$$

and recalling (3.12)

$$\begin{aligned}&\mathbb{E}\left[\sup_{t\in[1-\tau_q/2,1-\tau_{q+1}]}\left\|\int_{1-\tau_q}^t P(t-s)\Pi_L b(r_s-\tilde r_s,u_s)ds\right\|_{H^{-4}}\right]\\&\lesssim \mathbb{E}\left[\int_{1-\tau_q}^{1-\tau_{q+1}}\|u_s\|_{H^{1/2}}\|r_s-\tilde r_s\|_{H^{-1/2}}ds\right]\\&\lesssim\left(\int_{1-\tau_q}^{1-\tau_{q+1}}\mathbb{E}\|u_s\|_{H^{1/2}}^4ds\right)^{1/4}\left(\int_{1-\tau_q}^{1-\tau_{q+1}}\mathbb{E}\|r_s-\tilde r_s\|_{H^{-1/2}}^{4/3}ds\right)^{3/4}\lesssim\frac{\varepsilon_q^{1/12}\kappa_q^2N_q^6M}{\nu^{1/4}}.\end{aligned}$$

Here we additionally need to ask

$$\varepsilon_q\kappa_q^{24}N_q^{72}\le 1. \tag{3.15}$$

The last Itō integral was already controlled in Lemma 2.4 by [18, Lemma 2.5] as

$$\mathbb{E}\left[\sup_{t\in[1-\tau_q/2,1-\tau_{q+1}]}\left\|\int_{1-\tau_q}^t P(t-s)\Pi_L b(Q_q^{1/2}dW_s,u_s)\right\|_{H^{-4}}^2\right]\lesssim\frac{M^2}{(\nu+\kappa_q)^{1-\delta}}.$$

Putting all together, we get for our choice of parameters (2.6) and (3.15)

$$\mathbb{E}\left[\sup_{t\in[1-\tau_q/2,1-\tau_{q+1}]}\|U_t^L\|_{H^{-4}}\right]\lesssim\frac{M^2}{(\nu+\kappa_q)^{1-\delta}}+\frac{\varepsilon_q^{1/12}\kappa_q^2N_q^6M}{\nu^{1/4}}.$$

□

We are finally ready to give the proof of our main result.

Proof of Theorem 1.5. The proof is similar to that of Proposition 2.1 in the previous section.

First of all, by Lemma 3.4, (3.7) and condition (3.15) it holds

$$\mathbb{E}\left[\sup_{t\in[1-\tau_q/2,1-\tau_{q+1}]}\left\|u_t^L\right\|_{H^{-4}}\right]\lesssim \mathbb{E}\left[\sup_{t\in[1-\tau_q/2,1-\tau_{q+1}]}\left\|U_t^L\right\|_{H^{-4}}\right]+o(M\varepsilon_q^{1/4})$$
$$\leq C_\delta\left(\frac{M^2}{(\nu+\kappa_q)^{1-\delta}}+\frac{\varepsilon_q^{1/12}\kappa_q^2N_q^6M}{\nu^{1/4}}\right)=: M\tilde{c}_q^2.$$

and therefore with probability at least $1-\tilde{c}_q$ we have for every $t\in[1-\tau_q/2,1-\tau_{q+1}]$

$$\|u_t\|_{H^{-1}}^2\leq M^{3/2}\|u_t\|_{H^{-4}}^{1/2}\leq\left(\tilde{c}_q^2+N_q^{8(\delta-1)}\right)^{1/4}M^2=: c_qM^2.$$

We take δ, τ_q, κ_q and N_q as in Proposition 2.1, and ε_q satisfying (3.9) and (3.15), for instance $\varepsilon_q=2^{-4^q}\kappa_q^{-36}N_q^{-72}$. From now on the proof goes exactly as that of Proposition 2.1, and we omit it. □

As already mentioned, the proofs of Theorem 1.1 and Theorem 1.3 descend easily by the same arguments presented above. Finally, let us give the:

Proof of Corollary 1.2. Let $\mathbb{P}_{\rho_0}$ and $\mathbb{P}_\nu$ be as in the statement of the corollary. Without loss of generality we may assume $\|\rho_0\|_{L^2}\leq M$ for $\mathbb{P}_{\rho_0}$ almost every ρ_0 and, possibly replacing $\mathbb{P}_\nu$ with an equivalent measure, $\nu^{-1/4}$ integrable with respect to $\mathbb{P}_\nu$.

Let us denote $\tilde{\mathbb{P}}:=\mathbb{P}_{\rho_0}\otimes\mathbb{P}_\nu\otimes\mathbb{P}$, with expectation $\tilde{\mathbb{E}}$. For every triple $\tilde{\omega}=(\rho_0,\nu,\omega)$ there exists a unique solution $\rho=\rho(\tilde{\omega})$ of (1.1) which satisfies, by the same computations of Lemma 3.4:

$$\tilde{\mathbb{E}}\left[\sup_{t\in[1-\tau_q/2,1-\tau_{q+1}]}\|\rho^L\|_{H^{-4}}\right]\lesssim\frac{M^2}{\kappa_q^{1/5}}+\varepsilon_q^{1/12}\kappa_q^2N_q^6M.$$

This implies, arguing as in the proof of Proposition 2.1

$$\tilde{\mathbb{P}}(A_q)\geq 1-\tilde{c}_q,\quad A_q=\left\{E(1-\tau_{q+1})\leq M^2\frac{2c_q}{\nu\tau_q}\right\}$$

for every $q\in\mathbb{N}$ and suitable c_q, $\tilde{c}_q$, decreasing fast enough so that Borel-Cantelli Lemma gives

$$\lim_{t\uparrow 1}\|\rho_t\|_{L^2}=0\quad\tilde{\mathbb{P}}-\text{almost surely.}$$

In particular, by Fubini Theorem we have a full $\mathbb{P}$-probablity set $\Omega_0=\Omega_0(\mathbb{P}_{\rho_0},\mathbb{P}_\nu)\subset\Omega$ such that for every $\omega\in\Omega_0$ it holds $\lim_{t\uparrow 1}\|\rho_t\|_{L^2}=0$ for $\mathbb{P}_{\rho_0}\otimes\mathbb{P}_\nu$ almost every (ρ_0,ν). In particular, total dissipation for almost every initial condition and viscosity occurs for a generic realization $v=v(\omega)$, $\omega\in\Omega_0$. It is interesting to observe that $\mathbb{P}(\Omega_0)=1$. □

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(M. Hofmanová) Fakultät für Mathematik, Universität Bielefeld, D-33501 Bielefeld, Germany
E-mail address: hofmanova(at)math.uni-bielefeld.de

(U. Pappalettera) Fakultät für Mathematik, Universität Bielefeld, D-33501 Bielefeld, Germany
E-mail address: upappale(at)math.uni-bielefeld.de

(R. Zhu) Department of Mathematics, Beijing Institute of Technology, Beijing 100081, China
E-mail address: zhurongchan(at)126.com

(X. Zhu) Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China
E-mail address: zhuxiangchan(at)126.com