Critical phenomena and analytic properties of an ellipse connected with singularities of its wave front

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Abstract

One finds explicit expressions of critical value of the distance and of all singularities of wave front of an ellipse and its complex extention. From asymptotic equalities for singular points it follows that sinuglarities of wave front of an ellipse are semicubical cusp points, and there exists only one value of the distance such that the singularities of corresponding complex extension of wave front are not semicubical cusp points.

Let us consider an ellipse, given in Cartesian coordinate system x, y by the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $0 < a < b$. Let $d > 0$. The wave front Π_d is formed by the points at the distant d in the forward direction from the points of the ellipse along each internal normal. In this paper we study also the extension of the ellipse in the complex domain, for which the real variable x takes values in the region $|x| > a$. In this case the variable y takes imaginary values, and the wave front Π_d admites the complex extension Π_d^*. The following geometrical picture is well-known in the days of Huygens. If the number d less or equal to some critical value $d_* > 0$, then the wave front Π_d is a smooth not self-intersecting curve, and if $d_* < a < a$, then the front Π_d intersects itself in two points and it acquires four singularities, which are semicubical cusp points such that the curve Π_d is given by equation $v^2 = u^3$ in some neighborhood of these singularities ([1]-[3]). However, in the literature there are no proofs of these statements, explicit description of all singularities and asymptotic behavior of curves Π_d and Π_d^* at their sufficiently small neighborhood. There are only the proofs for curves of general form (instead of an ellipse) ([4]). In present paper we prove all these statements and indicate explicit expressions for all these objects. From asymptotic equalities found here it follows that, if $\frac{a^2}{b} < d < \frac{b^2}{a}$, then the curve Π_d has the differentiable type of semicubical parabola at a small neighborhood of singular point (the statement 2 of Theorem 2). However, if $b < 1$, then there exists the unique number $d > \frac{b^2}{a}$, such that the curve Π_d^* is not equivalent to semicubical parabola in any neighborhood of singularity (the statement 3 of Theorem 2). It is in contrast with the conclusion, which follows from many publications concerning this problem (see, for example, [1]).

Theorem 1. If $0 < d \leq \frac{a^2}{b}$, then Π_d is the smooth not self-intersecting curve, and, if $\frac{a^2}{b} < d < a$, then Π_d intersects itself in two points $x = 0$, $y = \pm\sqrt{(a^2 - d^2)(b^2 - a^2)}/a$.

Proof. Let $P = (x, y)$ be a point of the ellipse such that the coordinate $y \geq 0$. Then the point $(x', y') \in \Pi_d$ at the distant d in the direction from the point P along internal normal going over P has the coordinates

$$x' = x - \frac{dbx}{\kappa}, \quad y' = y - \frac{da\sqrt{a^2 - x^2}}{\kappa}, \tag{1}$$

where

$$\kappa = \kappa(x) = \sqrt{a^4 + b^2x^2 - a^2x^2}. \tag{2}$$

From the equalities (1) and (2) it follows that, if $d \leq \frac{a^2}{b}$ and $x > 0$, then $x' > 0$, and, if $a > d > \frac{a^2}{b}$ and $x > 0$, then $x' = 0$ for the point with coordinates $x = \sqrt{(d^2b^2 - a^4)(b^2 - a^2)}$, $y = b^2a^{-1}\sqrt{(a^2 - d^2)(b^2 - a^2)}$. In addition we have $y' = a^{-1}\sqrt{(a^2 - d^2)(b^2 - a^2)}$. Since the point $(0, y')$ is at the same distance from two points (x, y) and $(-x, y)$ of the ellipse, then for $a > d > \frac{a^2}{b}$ the front Π_d intersects itself in this point, and one needs only to prove that for $d \leq \frac{a^2}{b}$ the front Π_d is the smooth not self-intersecting curve. We assume the contrary: there exists a point $(x', y') \in \Pi_d$ at the same distance from two points (x_1, y_1) and (x_2, y_2) of the ellipse. Let $x_1 < x_2$. Then by virtue of (1), (2) we have

$$1 = \frac{db}{\kappa(x_1)} + \frac{dbx_2}{x_1 - x_2}\left(\frac{1}{\kappa(x_1)} - \frac{1}{\kappa(x_2)}\right) = \frac{db}{\kappa(x_1)} + dbx_2\left(\frac{1}{\kappa(x)}\right)'_x(\xi), \tag{3}$$

where $\left(\frac{1}{\kappa(x)}\right)'_x(\xi)$ is the derivative of the function $\frac{1}{\kappa(x)}$ in x at the point ξ, and the number ξ satisfies the inequality

$$x_1 \leq \xi \leq x_2. \tag{4}$$

The derivative $\left(\frac{1}{\kappa(x)}\right)'_x(\xi) = -\frac{\xi(b^2-a^2)}{\kappa^3}$. Therefore, if numbers x_1 and x_2 have the same sign, then, according to (4) $dbx_2\left(\frac{1}{\kappa(x)}\right)'_x(\xi) < 0$, and since for $d \leq \frac{a^2}{b}$ the inequality $\frac{db}{\kappa(x_1)} \leq 1$ holds, then from two last inequalities it follows that the equality (3)is not valid. If the numbers x_1 and x_2 have different signs, then, connecting points (x_1, y_1) and (x_2, y_2) with the point (x', y') by segments, we obtain that one of these two segments intersects the axis $x = 0$. Let the segment with endpoints (x', y') and (x_2, y_2) intersects the axis $x = 0$ and $x_2 > 0$, $y_2 > 0$. Then $x' \leq 0$, but as indicated above, if $d \leq \frac{a^2}{b}$, then it is not valid. If $d \leq \frac{a^2}{b}$, then applying (1) and (2) we obtain that the curve Π_d is smooth. Theorem 1 is proved.

Theorem 2.

1) If $\frac{a^2}{b} < d < \frac{b^2}{a}$, then the front Π_d acquires four singularities $x' = \pm\hat{x}$, $y' = \pm\hat{y}$ (any combinations of signs are possible) which are semicubical cusp points such that

$$\hat{x} = \sqrt{\frac{(dba^4)^{\frac{2}{3}} - a^4}{b^2 - a^2}}\left(1 - \frac{db}{(dba^4)^{\frac{1}{3}}}\right) = x'(x_*),\ \hat{y} = \sqrt{\frac{a^2b^2 - (dba^4)^{\frac{2}{3}}}{b^2 - a^2}}\left(\frac{b}{a} - \frac{da}{(dba^4)^{\frac{1}{3}}}\right) = y'(x_*),$$

where $x_* = \sqrt{\frac{\kappa_*^2 - a^4}{b^2 - a^2}}$, $\kappa_* = (dba^4)^{\frac{1}{3}}$, and $x'(x_*)$ and $y'(x_*)$ are the values of functions $x'(x)$ and $y'(x)$ from (1) at $x = x_*$;

2) if $\frac{a^2}{b} < d < \frac{b^2}{a}$, a number $\varepsilon > 0$ is sufficiently small and $|x - x_*| \le \varepsilon$, then

$$x'(x) = \hat{x} + \frac{3dba^4(b^2 - a^2)x_*}{2\kappa_*^5}(x - x_*)^2 + O\left(|x - x_*|^3\right),$$

$$y'(x) = \hat{y} - \frac{3db^2a^3(b^2 - a^2)x_*^2}{2\kappa_*^5\sqrt{a^2 - x_*^2}}(x - x_*)^2 + O\left(|x - x_*|^3\right),$$

and there exists the linear change of variable $u = x' - \hat{x}$, $v = c_1(x' - \hat{x}) + c_2(y' - \hat{y})$ (c_1, c_2 are constants; $c_1 < 0, c_2 < 0$), such that in the coordinates u, v the curve has the form $u = k_1(x - x_*)^2 + O(|x - x_*|^3)$, $\quad v = k_2(x - x_*)^3 + O(|x - x_*|^4)$, where k_1 and k_2 are positive constants;

3) if $b < 1$, $d = \hat{d} = \frac{\sqrt{b}}{\sqrt{a^5}}\left(\frac{b^2 + ab - a^2 - a^3}{1 - b}\right)^{\frac{3}{2}}$, then in the coordinates u, v in the small neighborhood of singular point $(\hat{x}, \hat{y}) \in \Pi_d^*$ the curve Π_d^* has the form $u = k_1(x - x_*)^2 + O(|x - x_*|^3)$, $\quad v = O(|x - x_*|^4)$, where the constant $k_1 > 0$, and, if $d > \frac{b^2}{a}$, $d \neq \hat{d}$, then for the curve Π_d^* the statement 2) is valid.

Proof. At the singular points (x', y') of the curve Π_d the equalities $\frac{dx'}{dx}(x) = \frac{dy'}{dx}(x) = 0$ hold. By virtue of (1) and (2) these equalities are valid if and only if $\kappa(x) = \kappa(x_*) = \kappa_*$. This proves the statement 1). Assuming $c_1 = -x_*, c_2 = -a\sqrt{a^2 - x_*^2}$ and computing the second and third derivatives of functions $x'(x)$ and $y'(x)$ from (1) at the point $x = x_*$, we prove the statement 2) in which the constant $k_2 = \frac{dba^4(b^2 - a^2)x_*}{2\kappa_*^5}\left(1 + \frac{b(x_*^2 + a)}{a^2 - x_*^2}\right) > 0$. The statement 3) follows from the fact that for $d = \hat{d}$ the equality $x_* = \sqrt{\frac{a(a+b)}{1-b}}, k_2 = 0$ are valid. Theorem 2 is proved.

Conjecture. For $b < 1$, $d = d^*$ the singular point is equivalent to point $u = v = 0$ of the curve $v^2 = u^5$ up to diffeomorphism.

References

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