Absolute Continuity and Regularity of Infinitesimally Invariant Measures under Minimal Conditions: the Elliptic Case

by

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Abstract

Let $A = (a^{ij})$ be a matrix valued Borel mapping on a domain $\Omega \subset \mathbb{R}^n$, let $b = (b^i)$ be a vector field on Ω, and let $L_{A,b}\varphi = a^{ij} D_i D_j \varphi + b^i D_i \varphi$. We study Borel measures μ on Ω that satisfy the elliptic equation $L^*_{A,b}\mu = 0$ in the weak sense: $\int L_{A,b}\varphi \, d\mu = 0$ for any $\varphi \in C_0^\infty(\Omega)$. We prove that μ has a density under mild conditions. If A is locally uniformly nondegenerate, $A \in H^{1,p}_{\mathrm{loc}}$ and $b \in L^p_{\mathrm{loc}}$ for some $p > n$, then this density belongs to $H^{1,p}_{\mathrm{loc}}$.

Contents

Preface

Before I begin with the real scientific considerations, I want to make some general remarks about my task and my way of implementing it. It was up to me to form the elliptic case of [9] to a self-contained paper, which should be learnable and understandable for some mathematician taking his doctor's degree, i. e. for someone, who is at least one year in advance with respect to me. The latter gave pointer to me about what I could assume as standard facts and what I had to work out explicitely. Therefore, I assume standard facts from a Functional Analysis course and also from a Measure Theory course, in terms of books let us say "half of [3]" plus "half of [5]"; additionally some elementary knowledge of an introductory PDE course is assumed.
In the original paper the authors often put their proofs together by citing. Furthermore, for drawing a full picture of the arguments, it was sometimes even better to resign on given quotes (e.g. [26]) in favour of other literature, which could happen to be found accidentally (e. g. [13]). What has come out is more a book than a paper, still not being perfectly self-containing. Cuts were decided to be made concerning Hölder space theory (see 2.1 and 2.5), while the needed L^p-theory is completely given; complete in the sense of the above conditions. Proofs found in literature were not only copied but revised and very often polished for the reader! I myself tried to avoid "clearly" and "easy to see" phrases, if a handful exceptions are forgiven.
The way of ordering sections as presented is mainly due to the following motivations:

- leading the reader towards the main theorems 2.28 and 3.54:
  the settings of these theorems in mind, I do not always formulate the assertions within the preparation parts as general as possible. For example, we mostly consider balls as domains, so that we do not have to bother about regularity of boundary portions – let them be smooth.

- motivate the reader to go on:
  In fact, the first main theorem uses about a hundred pages of preparations! If I put them all in front of it, probably no one would ever arrive. Therefore the tension bow looks like $\wedge\wedge$, the highlights being found on the top of the pyramids, but each of it understood not before arriving downstairs again (where the last downhill shall symbolize the appendix).

- putting wild computations and may be standard results into an appendix to avoid overloading the main part of the paper.

I want to thank Dr. Wilhelm Stannat for his admirable know-how and patience with me; I was inspired for example at the crucial point Proposition 2.6. Not for nothing I labeled it "wilhelm" in LaTeX. Moreover I want to thank Prof. Röckner, who has not been losing his energy before, while and after teaching his ten courses in a row. He motivates the students by carrying us with him and really respecting our work.

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Chapter 1

INTRODUCTION AND NOTATIONS

1.1 Introduction

The work in hand treats of the main results of the elliptic case of [9], which appear below as Theorem 2.28 and Theorem 3.54 in our numerical order. In opposite to the original presentation, where their proofs are built up with the help of quotations of other literature (sometimes not the best possible for understanding), this paper contains a complete deduction of the arguments that come in. It moreover tries to be as self-contained as possible in a sensible manner, assuming an ordinary PDE and Functional Analysis Course plus some knowledge in Measure Theory.

Before we point out the content of this work in detail, as in [9] we first want to give a brief introduction into its mathematical context. For further applications and former contributions to the part of mathematical research in demand we refer to the introduction of [9]. The references concerning the elliptic case given there, are also included into the bibliography of this work.

A fundamental problem in probability theory is the study of equilibrium distributions (i. e., invariant measures) of diffusion processes governed by the stochastic differential equation on $\mathbb{R}^n$

$$d\xi_t = \sigma(\xi_t)\, dw_t + b(\xi_t)\, dt \tag{1.1}$$

under minimal conditions on the coefficients σ and b. The main issues are existence and uniqueness of such invariant measures, but also their regularity properties. In order to avoid a priori conditions that ensure the existence of a solution to (1.1) or a corresponding semigroup describing its transition probabilities, the purely analytic reformulation of the notion of an invariant measure in terms of the underlying generator

$$L = \frac{1}{2}(\sigma\sigma^t) = a^{ij} D_i D_j + b^i D_i$$

is used, i. e., we consider solutions μ to the equation

$$L^*\mu = 0.$$

To be more specific, let Ω be an open set in $\mathbb{R}^n$, let $A = (a^{ij})$ be a Borel mapping on Ω with values in the space of nonnegative matrices on $\mathbb{R}^n$ and let $b = (b^i) : \Omega \longrightarrow \mathbb{R}^n$ be a Borel vector field. Let us set

$$L_{A,b}\,\varphi = a^{ij} D_i D_j \varphi + b^i D_i \varphi, \quad \forall \varphi \in C_0^\infty(\Omega), \tag{1.2}$$

where the standard summation rule for repeated indices is used. Suppose that μ is a locally finite Borel measure on Ω, such that $a^{ij}, |b| \in L^1_{\text{loc}}(\mu)$ and

$$L^*_{A,b}\mu = 0 \tag{1.3}$$

in the following sense:

$$\int_\Omega L_{A,b}\,\varphi\, d\mu = 0, \quad \forall \varphi \in C_0^\infty(\Omega). \tag{1.4}$$

The first main result, Theorem 2.28, states that if μ is nonnegative, then the measure $(\det A)^{\frac{1}{n}}\mu$ has a density in $L^{\frac{n}{n-1}}_{\text{loc}}(\Omega, dx)$ and if A is Hölder continuous and nondegenerate, then the same is true for μ itself even if μ only is a signed measure. This extends a result from [21], where b was assumed to be locally bounded. The proof is based on the following four main ingredients:

(I1) C^2-solvability of the special Dirichlet problem (2.13),

(I2) an estimate of the solution's $H^{2,p}$-norm (see Prop. 2.6),

(I3) Sobolev embedding (see Appendix D) and

(I4) Proposition 2.31.

Section 2.1 treats (I1) by following [19, Chapters 2-6] straightway towards (2.13). The needed propositions mostly are standard results but not necessarily topics of ordinary Functional Analysis or PDE courses. In order to keep this paper to a reasonable size, as an exception, the principle of self-containedness is injured in this section by citing Propositions 2.5 and 2.1 (global Schauder estimate). Instead, the analogous result (Proposition 2.23) to the latter in L^p is proved in complete detail in Section 2.2. It is fundamental for the desired estimate (I2). For 2.23, we first follow [13, p. 37-47], who in turn exploit potential operator techniques from [19] and make use of the Calderón-Zygmund Decomposition Lemma 2.12 to show L^p-continuity of the special operator T defined in (2.15). This operator leads us to the interior estimate 2.18. For the original references of some of the used results we refer to the bibliography of [13]. In order to extend 2.18 to the boundary, we turn back to [19] and use smoothness assumptions on our domain Ω to transform problems from Ω to B_R^+ with the help of C^∞-diffeomorphisms. Although [19, Proposition 9.17] asserts and proves (even a global version of) the required estimate (I2), we again leave [19] in favour of a more advantageous proof of W. Stannat, who does not need [19, Section 9.1

and Lemma 9.16] for proving the uniqueness assertion of [19, Proposition 9.15], which in turn is used in the proof of [19, Proposition 9.17]. For (I3), $p < n$, we give the proof of [19], for $p > n$ we follow the more general version of [3], since we need the Hölder continuity in, e. g., Theorem 3.52. Since the Sobolev Embeddings play an important role in the proofs of the main theorems and since also advanced students of mathematics may not have seen their proofs yet, they are put into Appendix D for completeness.
In Section 2.3 the first main theorem (Theorem 2.28) plus some corollaries are given, although we are not prepared for (I4), which is used in the proof of the main theorem. Since the necessary elaboration of the geometric and analytic part of Krylov's paper [22] is far away from what we have been doing up to this point, it was postponed to Section 2.4 in order to avoid mental confusion.

Chapter 3 prepares the reader for the second main theorem, Theorem 3.54. There, it is proved that if the a^{ij} belong to the Sobolev class $H^{1,p}_{\mathrm{loc}}(\Omega)$, $\det A \geq c > 0$, and $|b| \in L^p_{\mathrm{loc}}(\Omega)$, where $p > n$, then μ has a density from $H^{1,p}_{\mathrm{loc}}$. The main idea of the proof consists of reducing the assumptions of the theorem by a finite iteration process (bootstrapping) to the case, where μ is assumed to be an element of $L^r_{\mathrm{loc}}(\Omega, \mathrm{dx})$ for some $r \in \left(\frac{pn}{p-n}, \infty\right)$ instead of $r \in (p', \infty)$ only. Each of its iteration steps is essentially based on the results of Chapter 2, several applications of the Hölder inequality and an a priori estimate for weak $H^{1,p}$-solutions of the elliptic Dirichlet problem $L_A u = f$, which is deduced in Section 3.1. For that, we first prove the existence of a weak solution of Problem (3.3). This might be a standard result, but note that neither [13, Section I.1.2], where n was assumed to be larger or equal to 3, nor [19, Section 8.2] (cf. Assumption (8.8) and boundedness of b, d) are general enough for our application in Corollary 3.56. Therefore, we mix their considerations. In the next subsection, we introduce Morrey-, Campanato- and BMO-spaces, which are subspaces of L^p and which are useful for regularity considerations of weak solutions. Moreover, we investigate their interdependencies. In Subsection 3.1.3 we present further parts of the L^2-theory of weak solutions mainly as in [13, Chapters 1 and 8] and give the proof of Friedrich's Theorem. Then we follow the methods of S. Campanato (cf. [12]), which means investigating regularity of weak solutions in smooth domains with the help of Morrey- and Campanato spaces. Using also BMO spaces and Stampacchia interpolation, we shall arrive at the desired L^p-estimate. For additional references corresponding to this proceeding, see the bibliographies of [12], [13] and [18]. Neither the refurbishments of these methods in [13] or [18], nor the original [12] itself present the complete chain of proofs. [13] is rigorous up to the local result, Proposition 3.42, but then only states the global version, Proposition 3.28, and somehow misleads the reader by posing the continuous case behind. Note that for proving the global version, one has to carry equations from semispheres back to Ω, which transforms constant coefficients into non-constant (, but still continuous) ones. Thus, although we actually use [12] and [13] only to prove the desired L^p-estimate in the constant coefficient case, while we use [18] for generalizing to continuous ones, some elements of Campanato's considerations of the continuous case are

necessary. In the proof of Proposition 3.48, [12] conceals a technically hard part of the proof by reducing to two cases without loss of generality. With the help of fragments in [18, Section 3.4], we give the complete proof. The chapter closes with the proof (see description above) of the second main theorem, the Regularity Theorem 3.54, which improves a result from [10], where the a^{ij} were assumed to be infinitely differentiable.
The appendices contain proofs of several interpolation inequalities, completions of proofs of the main part of the paper and, as mentioned before, a presentation of the Sobolev Embeddings. Due to the aim of the given work, these appendices complete the chain of necessary arguments for the main theorems.

Chapter 2

EXISTENCE RESULT

2.1 Starting point: Dirichlet problem

Throughout this section $Lu = f$ shall denote the equation

$$Lu(x) = a^{ij}(x)D_{ij}u + b^i(x)D_iu + c(x)u = f(x), \quad a^{ij} = a^{ji}, \tag{2.1}$$

where the coefficients and f are defined in a bounded domain $\Omega \subset \mathbb{R}^n$. For technical reasons we further assume that $\sup\limits_{\Omega} \frac{|b_i(x)|}{|\lambda(x)|} < \infty$, where $\lambda(x)$ denotes the minimal eigenvalue, if L is assumed to be elliptic.

Since the underlying work of my diploma thesis consists of results in L^p-spaces and for reasons of avoiding a boundless presentation, we only want to touch the theory of Hölder-spaces slightly and appeal the reader to believe Propositions 2.1 and 2.5, which will occur within this section.

Proposition 2.1 (global Schauder Estimate) *Let Ω be a $C^{2,\alpha}$-domain in $\mathbb{R}^n$ and let $u \in C^{2,\alpha}(\bar{\Omega})$ be a solution of $Lu = f$ in Ω, where $f \in C^{0,\alpha}(\bar{\Omega})$ and the coefficients of L satisfy, for positive constants λ, Λ,*

$$a^{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2 \quad \forall x \in \Omega, \xi \in \mathbb{R}^n,$$

and

$$|a^{ij}|_{0,\alpha;\Omega}, |b^i|_{0,\alpha;\Omega}, |c|_{0,\alpha;\Omega} \leq \Lambda.$$

Let $\varphi \in C^{2,\alpha}(\bar{\Omega})$ and suppose $u = \varphi$ on $\partial\Omega$. Then

$$|u|_{2,\alpha;\Omega} \leq C(|u|_{0;\Omega} + |\varphi|_{2,\alpha;\Omega} + |f|_{0,\alpha;\Omega}), \tag{2.2}$$

where $C = C(n, \alpha, \lambda, \Lambda, \Omega)$.

For the definition of $C^{k,\alpha}$-domains, see Definition 2.20, or better do not bother now.

Proof. see [19, Theorem 6.6] □

As we will see later, this proposition is fundamental for the proof of existence of a $C^{2,\alpha}(\bar{B})$-solution of the Dirichlet problem

$$Lu = f \quad \text{in} \quad B, \qquad u = \varphi \quad \text{on} \quad \partial B, \tag{2.3}$$

where $B := B_R(x_0)$ for some $x_0 \in \mathbb{R}^n$ and $\varphi \in C^{2,\alpha}(\bar{B})$. Before that we need another a priori estimate:

Proposition 2.2 (cf. [19, Theorem 3.7]) *Let $Lu \geq f$ in Ω, where L is elliptic, $c \leq 0$ and $u \in C^0(\bar{\Omega}) \cap C^2(\Omega)$. Then*

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u^+ + C \sup_{\Omega} \frac{|f^-|}{\lambda},$$

where C is a constant only depending on diamΩ and $\beta = \sup \frac{|b|}{\lambda}$. In particular, if $Lu = f$ in Ω,

$$\sup_{\Omega} |u| \leq \sup_{\partial\Omega} |u| + C \sup_{\Omega} \frac{|f|}{\lambda}. \tag{2.4}$$

*Proof.*Let Ω lie in the slab $0 < x_1 < d$ and set $L_o := a^{ij}D_iD_j + b^iD_i$. For $\alpha \geq \beta + 1$ we have

$$\begin{aligned} L_o e^{\alpha x_1} &= (\alpha^2 a^{11} + \alpha b^1)e^{\alpha x_1} \geq \lambda(\alpha^2 - \alpha\beta)e^{\alpha x_1} \\ &\geq \lambda\alpha e^{\alpha x_1} \geq \lambda (> 0), \end{aligned}$$

where $\lambda(x)$ denotes the minimal eigenvalue of $(a^{ij}(x))$.
Let $v = \sup_{\partial\Omega} u^+ + (e^{\alpha d} - e^{\alpha x_1}) \sup_{\Omega} \frac{|f^-|}{\lambda}$. Then, since

$$\begin{aligned} Lv &= L_0 v + cv \\ &= -\sup_{\Omega} \frac{|f^-|}{\lambda} L_0 e^{\alpha x_1} + c \sup_{\partial\Omega} u^+ + c(e^{\alpha d} - e^{\alpha x_1}) \sup_{\Omega} \frac{|f^-|}{\lambda} \\ &\leq -\lambda \sup_{\Omega} \frac{|f^-|}{\lambda} \qquad (c \leq 0!), \end{aligned}$$

we have

$$L(v-u) \leq -\lambda \sup_{\Omega} \frac{|f^-|}{\lambda} - f = -\lambda \left(\sup_{\Omega} \frac{|f^-|}{\lambda} + \frac{f}{\lambda} \right) \leq 0 \quad \text{in} \quad \Omega$$

and

$$v - u \geq \sup_{\partial\Omega} u^+ - u^+ + u^- \geq 0 \quad \text{on} \quad \partial\Omega$$

by the weak maximum principle. Hence, for $C = e^{\alpha d} - 1$ and $\alpha \geq \beta + 1$, we obtain the desired result,

$$\sup_{\Omega} u \leq \sup_{\Omega} v \leq \sup_{\partial\Omega} u^+ + C \sup_{\Omega} \frac{|f^-|}{\lambda}.$$

(2.4) follows by replacing u by $-u$, since $(-u)^+ = u^-, |u| = u^+ + u^-$. □

We now reduce Problem (2.3) to the Dirichlet problem for Poisson's equation

$$\triangle v = f \quad \text{in} \quad B \quad , \quad v = \varphi \quad \text{on} \quad \partial B \tag{2.5}$$

with the help of

Proposition 2.3 (cf. [19, Theorem 6.8]) *Let Ω be a $C^{2,\alpha}$-domain in $\mathbb{R}^n$, and let the operator L be strictly elliptic in Ω with coefficients in $C^{0,\alpha}(\bar{\Omega})$ and with $c \leq 0$. Then, if the Dirichlet problem for Poisson's equation, $\triangle v = f$ in Ω, $v = \varphi$ on $\partial\Omega$, has a $C^{2,\alpha}(\bar{\Omega})$-solution for any $f \in C^{0,\alpha}(\bar{\Omega})$ and any $\varphi \in C^{2,\alpha}(\bar{\Omega})$, the problem (2.3) also has a (unique) $C^{2,\alpha}(\bar{\Omega})$-solution for any such f and φ.*

Proof. By hypothesis we may assume that the coefficients of L satisfy the conditions

$$\begin{aligned} \lambda|\xi|^2 &\leq a^{ij}\xi_i\xi_j \quad \forall x \in \Omega, \xi \in \mathbb{R}^n, \\ |a^{ij}|_{0,\alpha;\bar{\Omega}}, |b^i|&_{0,\alpha;\bar{\Omega}}, |c|_{0,\alpha;\bar{\Omega}} \leq \Lambda, \end{aligned} \tag{2.6}$$

with positive constants λ, Λ. It suffices to restrict consideration to zero boundary values, since problem (2.3) is equivalent to $L\tilde{u} = f - L\varphi =: f'$ in Ω, $\tilde{u} = 0$ on $\partial\Omega$ (after solving this define $u := \tilde{u} + \varphi$).
We consider the family of equations,

$$L_t u := tLu + (1-t)\triangle u = f, \quad 0 \leq t \leq 1. \tag{2.7}$$

We note that $L_0 = \triangle$, $L_1 = L$, and that the coefficients of L_t satisfy (2.6) with

$$\lambda_t = \min(1, \lambda), \qquad \Lambda_t = \max(1, \Lambda).$$

The operator L_t may be considered a bounded linear operator from the Banach space $\mathcal{B}_1 := \{u \in C^{2,\alpha}(\bar{\Omega}) | u = 0 \text{ on } \partial\Omega\}$ into the Banach space $\mathcal{B}_2 := C^{0,\alpha}(\bar{\Omega})$, since

$$\begin{aligned} |\triangle u|_{0,\alpha;\bar{\Omega}} &\leq \sum_{|\alpha|=2} |D^\alpha u|_{0,\alpha;\bar{\Omega}} \leq |u|_{2,\alpha;\bar{\Omega}}, \\ |Lu|_{0,\alpha;\bar{\Omega}} &\leq 3\Lambda |u|_{C^{2,\alpha}(\bar{\Omega})}. \end{aligned}$$

The solvability of the Dirichlet problem, $L_t u = f$ in Ω, $u = 0$ on $\partial\Omega$, for arbitrary $f \in C^{0,\alpha}(\bar{\Omega})$ is then equivalent to the invertibility of the mapping L_t (injectivity is given by the boundary condition $u = 0$ on $\partial\Omega$ and the weak maximum principle). Let $u \in \mathcal{B}_1$. By virtue of Proposition 2.2, we have the bound

$$|u|_0 \leq C \sup_\Omega |L_t u| \leq C |L_t u|_{0,\alpha},$$

where C only depends on λ, Λ and the diameter of Ω. Hence from (2.2) we have

$$|u|_{2,\alpha} \leq C_1(|u|_0 + |L_t u|_{0,\alpha}) \leq C_2 |L_t u|_{0,\alpha},$$

that is,

$$\|u\|_{\mathcal{B}_1} \leq C_2 \|L_t u\|_{\mathcal{B}_2},$$

the constant C_2 being independent of t. Since by hypothesis, L_0 maps B_1 onto B_2, the method of continuity, Proposition 2.4 below, is applicable and the theorem follows. $\square$

Proposition 2.4 (method of continuity, cf. [19, Theorem 5.2]) *Let $L_0, L_1 : B_1 \longrightarrow B_2$ be bounded linear operators between Banach-spaces B_1, B_2. We set*

$$L_t := tL_1 + (1-t)L_0 \quad \text{for} \quad 0 \le t \le 1.$$

Assume, there exists a constant c independent of t such that

$$\|u\|_{B_1} \le c\|L_t u\|_{B_2} \quad \text{for any} \quad u \in B_1. \tag{2.8}$$

Then, if L_0 is surjective, so is L_1.

Proof. Assume, L_τ is surjective for some $\tau \in [0,1]$. By (2.8), L_τ is injective and therefore bijective. Hence, there exists $L_\tau^{-1} : B_2 \longrightarrow B_1$. For $t \in [0,1]$ and $f \in B_2$, the equation $L_t u = f$ is equivalent to the equation

$$L_\tau u = f + (L_\tau - L_t)u = f + (t-\tau)(L_0 u - L_1 u),$$

which in turn is equivalent to the equation

$$u = L_\tau^{-1} f + (t-\tau)L_\tau^{-1}(L_0 - L_1)u =: \Lambda u.$$

So, solving $L_t u = f$ means finding a fixpoint of the operator $\Lambda : B_1 \longrightarrow B_2$. Banach's fixpoint theorem is applicable, if there exists $q < 1$ such that

$$\|\Lambda u - \Lambda v\|_{B_2} \le q\|u - v\|_{B_1}.$$

Now

$$\|\Lambda u - \Lambda v\| \le \|L_\tau^{-1}\|(\|L_0\| + \|L_1\|)|t-\tau|\|u-v\|.$$

From (2.8) we have $\|L_\tau^{-1}\| \le c$. Thus we have to choose $|t-\tau| \le \frac{1}{2}(c(\|L_0\| + \|L_1\|))^{-1} =: \eta$ in order to obtain the fixpoint. I. e., if $L_\tau u = f$ is solvable, then $L_t u = f$ is for any t with $|t-\tau| \le \eta$. Since L_0 is surjective by assumption, it follows that L_t is surjective for $0 \le t \le \eta$. Applying the above assertion on $t = \eta$, we obtain surjectivity for $\eta \le t \le 2\eta$. In this manner we obtain iteratively that all $L_t, t \in [0,1]$ are surjective, in particular L_1. □

We stop our top-down method (i. e., reducing (2.3) to (2.5)) here and do some bottom-up in order to meet in the middle.

It is well known (e. g. [19, Theorem 2.6]) that

$$h(x) := \begin{cases} \frac{R^2 - |x|^2}{n\omega_n R} \int_{\partial B} \frac{\varphi(y)}{|x-y|^n}\, dS(y) & \text{, for} \quad x \in B \\ \varphi(x) & \text{, for} \quad x \in \partial B \end{cases} \tag{2.9}$$

belongs to $C^2(B) \cap C^0(\bar{B})$ and satisfies

$$\triangle h = 0 \quad \text{in} \quad B, \quad h = \varphi \quad \text{on} \quad \partial B \tag{2.10}$$

Moreover, consider the Newton potential

$$Nf(x) := w(x) := \int_{\mathbb{R}^n} \Gamma(x-y)f(y)\, dy, \quad x \in \mathbb{R}^n$$

of $f \in L^1(\mathbb{R}^n)$, where

$$\Gamma(\xi) = \Gamma(|\xi|) = \begin{cases} \frac{|\xi|^{2-n}}{n(2-n)\omega_n} & , \text{for} \quad n > 2 \\ \frac{\log|\xi|}{2\pi} & , \text{for} \quad n = 2 \end{cases} \quad , \xi \neq 0,$$

is the fundamental solution of the Laplace equation $\triangle h = 0$ and ω_n denotes the volume of the unit ball in $\mathbb{R}^n$. For later use:

$$\begin{aligned} D_i\Gamma(\xi) &= \frac{1}{n\omega_n}\xi_i|\xi|^{-n} \\ D_{ij}\Gamma(\xi) &= \frac{1}{n\omega_n}(|\xi|^2\delta_{ij} - n\xi_i\xi_j)|\xi|^{-n-2} \\ |D_{ijk}\Gamma(\xi)| &\leq C|\xi|^{-n-1} \end{aligned} \tag{2.11}$$

For $f \in C_0^\infty(\mathbb{R}^n)$ we compute

$$\begin{aligned} \triangle w(x) &= \int_{\mathbb{R}^n} \Gamma(y)\triangle f(x-y)\,dy \\ &= \sum_{i=1}^{n} -\lim_{\epsilon\downarrow 0}\int_{|y|\geq\epsilon} D_i\Gamma(y)D_if(x-y)\,dy \quad \text{(int. by parts, Lebesgue)} \\ &= \lim_{\epsilon\downarrow 0}\sum_{i=1}^{n}\int_{|y|=\epsilon} D_i\Gamma(y)f(x-y)\frac{y_i}{|y|}\,dS(y) \quad \text{(divergence thm.)} \\ &= \lim_{\epsilon\downarrow 0} ⨍_{|y|=\epsilon} f(x-y)\,dS(y) \\ &= f(x). \end{aligned} \tag{2.12}$$

We have arrived at the meeting point, because of our computation, problem (2.5) is now equivalent to problem (2.10) (solved with (2.9)!) by setting $v := h + w$ after considering $\triangle h = 0$ in B, $h = \varphi - w$ on ∂B.

Furthermore, the resulting solution v of (2.5) belongs to $C^{2,\alpha}(\bar{B})$ by the following proposition, which shall be the second of the propositions the reader is suggested to believe.

Proposition 2.5 *Let B be a ball in $\mathbb{R}^n$ and let $v \in C^2(B)\cap C^0(\bar{B})$ be a solution of $\triangle v = f$ in B, $v = 0$ on ∂B , where $f \in C^{0,\alpha}(\bar{B})$. Then $v \in C^{2,\alpha}(\bar{B})$.*

Proof. see [19, Theorem 4.13] □

Using Proposition 2.3 we are now able to find a $C^{2,\alpha}(\bar{B})$-solution u of

$$Lu = f \quad \text{in} \quad B \quad , \quad u = 0 \quad \text{on} \quad \partial B, \tag{2.13}$$

if $a^{ij} \in C^{0,\alpha}(B), f \in C_0^\infty(B)$.

2.2 An L^p-estimate

In the proof of the main theorem of this chapter we shall need an estimate in $H^{2,p}(B)$ for the solution u of (2.13), which is stated now:

Proposition 2.6 *Let $\Omega \subset \mathbb{R}^n$ be open, $p \geq 2$ and*

$$L := a^{ij}(x) D_i D_j, \tag{2.14}$$

where $a_{ij} \in C(\bar{\Omega})$, $a_{ij} = a_{ji}$ and (a_{ij}) is nondegenerate. Then for any $x_0 \in \Omega$ there exists $r_0 := r(x_0) > 0$ such that

$$\|u\|_{2,p;B_r(x_0)} \leq C(r_0, (a_{ij})) \|Lu\|_{p;B_r(x_0)}$$

for any $r \leq r(x_0), u \in (H^{2,p} \cap H_0^{1,p})(B_r(x_0))$.

To prove this appears to be of proper effort, but we will progress step by step and start as in [13, p. 37-47].

Definition 2.7 *For $f \in L^1(\Omega)$ and $t \geq 0$, we set*

$$A_t(f) := \{x \in \Omega | \, |f(x)| > t\}.$$

The function

$$\lambda_f(t) := |A_t(f)|$$

is called the distribution function of f. Let $p \geq 1$. A measurable function f is said to belong to the weak L^p-space $L^p_w(\Omega)$, if

$$\|f\|_{L^p_w(\Omega)} := \inf\{A | \lambda_f(t) \leq t^{-p} A^p \quad \forall t > 0\} < \infty.$$

Let us get more familiar with the new notions.

Lemma 2.8 **(i)** *If $f \in L^p(\Omega)$ and $1 \leq p < \infty$, then*

$$\int_\Omega |f|^p \, dx = p \int_0^\infty t^{p-1} |A_t(f)| \, dt.$$

(ii) *$L^p(\Omega) \subsetneq L^p_w(\Omega) \subset L^q(\Omega) \quad \forall 1 \leq q < p$, and $\| \cdot \|_{L^p_w(\Omega)}$ is not a norm on $L^p_w(\Omega)$.*

(iii) *$L^\infty_w(\Omega) = L^\infty(\Omega)$*

Proof.(i):

$$\begin{aligned} \int_\Omega |f|^p \, dx &= \int_\Omega \int_0^{|f(x)|} p t^{p-1} \, dt \, dx \\ &= \int_\Omega \int_0^\infty p t^{p-1} 1_{A_t(f)} \, dt \, dx \end{aligned}$$

Now $E := \{(x,t) \in \Omega \times (0,\infty) | |f(x)| > t\} \in \mathcal{B}(\Omega) \otimes \mathcal{B}((0,\infty))$. Indeed, consider $F(x,t) := (|f(x)|, t)$, then F is $\mathcal{B}(\Omega) \otimes \mathcal{B}((0,\infty))$-measurable, since each component is and $Id_{|(0,\infty)}$ is continuous. Therefore $E = F^{-1}(\{(x,y) \in \mathbb{R}^2 | x > y\}) \in \mathcal{B}(\Omega) \otimes \mathcal{B}((0,\infty))$. By applying Fubini, (i) is proven.

(ii): For $f \in L^p(\Omega)$, $t^p|A_tf| \leq \int_{A_tf} |f(x)|^p\,dx \leq \int_\Omega |f(x)|^p\,dx$. Therefore, $\|f\|_{L^p_w} \leq \|f\|_{L^p}$.Next, if $f \in L^p_w(\Omega)$, then by (i),

$$\begin{aligned}\int_\Omega |f|^q\,dx &= q\int_0^\infty t^{q-1}|A_tf|\,dt \\ &\leq q\int_0^1 t^{q-1}|A_tf|\,dt + q\int_1^\infty t^{q-1}|A_tf|\,dt \\ &\leq q|\Omega| + \|f\|^p_{L^p_w} q\int_1^\infty t^{q-1-p}\,dt < \infty.\end{aligned}$$

Let $\Omega = (0,1)$, $f(x) := x^p, g(x) := 1 - x^p, x \in \Omega$. Then

$$\|f+g\|_{L^p_w(\Omega)} = 1.$$

Moreover,

$$\begin{aligned} dx(\{f > t\}) &= dx(\{x > t^{\frac{1}{p}}\}) = 1 - t^{\frac{1}{p}} \\ dx(\{g > t\}) &= dx(\{x < (1-t)^{\frac{1}{p}}\}) = (1-t)^{\frac{1}{p}}\end{aligned}$$

By Analysis 1 methods we obtain

$$\begin{aligned}\|f\|_{L^p_w(\Omega)} &= \sup_{t>0} dx(\{f > t\})t^p = \sup_{0<t<1}(1 - t^{\frac{1}{p}})t^p = \frac{p^{2p^2}}{(1+p^2)^{1+p^2}} \\ \|g\|_{L^p_w(\Omega)} &= \sup_{t>0} dx(\{g > t\})t^p = \sup_{0<t<1}(1-t)^{\frac{1}{p}}t^p = \frac{p^{2p}}{(1+p^2)^{1+p}},\end{aligned}$$

and therefore

$$\begin{aligned}\|f\|_{L^p_w(\Omega)} + \|g\|_{L^p_w(\Omega)} &= \frac{p^{2p^2}}{(1+p^2)^{1+p^2}} + \frac{p^{2p}}{(1+p^2)^{1+p}} \\ &= \frac{p^{2p^2}(1+p^2)^p + p^{2p}(1+p^2)^{p^2}}{(1+p^2)^{1+p+p^2}} \\ &< \frac{(1+p^2)^{p^2+p} + (1+p^2)^{p+p^2}}{(1+p^2)^{1+p+p^2}} \\ &= \frac{2}{1+p^2} \leq 1,\end{aligned}$$

which shows that the triangle inequality is not valid. Now define $f(x) := x^{-\frac{1}{p}} \notin L^p(\Omega)$.Then

$$dx(\{f > t\}) = dx(\{x < t^{-p}\}) = t^{-p},$$

thus $dx(\{f > t\})t^p = 1$ and $f \in L^p_w(\Omega) \setminus L^p(\Omega)$.
(iii):$\|f\|_\infty = \inf\{t|\lambda_f(t) = 0\}$and therefore, $\lambda_f(t) = 0$ for $t \geq \|f\|_\infty$. Moreover, $\lambda_f(t) > 0$ for $t < \|f\|_\infty$. Thus, the definition implies (iii). □

Definition 2.9 *A linear map $T : L^p(\Omega) \longrightarrow L^q(\Omega)$ is said to be of strong type (p,q), if there exists $C > 0$ such that*

$$\|Tf\|_{L^q(\Omega)} \le C\|f\|_{L^p(\Omega)} \quad \forall f \in L^p(\Omega).$$

T is said to be of weak type (p,q), if

$$\|Tf\|_{L^q_w(\Omega)} \le C\|f\|_{L^p(\Omega)} \quad \forall f \in L^p(\Omega).$$

We have recourse to the Newton potential, and for fixed $1 \le i,j \le n$ we define the following linear operator $T : C_0^\infty(\mathbb{R}^n) \longrightarrow C^\infty(\mathbb{R}^n)$ by

$$Tf := D_{ij}(Nf) \tag{2.15}$$

We are going to examine T, since it leads us to an interior $H^{2,p}$-estimate for solutions $u \in H^{2,p}_{\text{loc}}(\Omega)$ of (2.1).

Lemma 2.10 *T is a bounded linear operator from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$ and $\|T\|_{(2,2)} \le 1$. In particular, T is of strong type $(2,2)$.*

Proof. First we assume $f \in C_0^\infty(\mathbb{R}^n)$. By (2.12) and the divergence theorem for any $B_R = B_R(0)$,

$$\begin{aligned}
\int_{B_R} f^2\,dx &= \int_{B_R} (\triangle w)^2\,dx = \sum_{i,j}\int_{B_R} D_{ii}wD_{jj}w\,dx \\
&= \sum_{i,j}\left(-\int_{B_R} D_iwD_iD_{jj}w\,dx + \int_{\partial B_R} D_iwD_{jj}w\frac{x_i}{R}\,dS\right) \\
&= \sum_{i,j}\left(\int_{B_R} D_{ji}wD_{ij}w\,dx - \int_{\partial B_R} D_iwD_{ij}w\frac{x_j}{R}\,dS\,+\right. \\
&\qquad \left.\int_{\partial B_R} D_iwD_{jj}w\frac{x_i}{R}\,dS\right) \qquad (2.16) \\
&= \int_{B_R}\sum_{i,j}(D_{ij}w)^2\,dx + \int_{\partial B_R}\sum_{i,j} D_iw\left(D_{jj}w\frac{x_i}{R} - D_{ij}w\frac{x_j}{R}\right)dS
\end{aligned}$$

Now we suppose that $\text{supp}f \subset B_{R_0}$. Then for $R > 2R_0$ and $x \in \partial B_R$,

$$\begin{aligned}
|D_iw(x)| &\le \int_{B_{R_0}} |D_i\Gamma(x-y)||f(y)|\,dy \le \int_{B_{R_0}} c\frac{|x_i - y_i|}{|x-y|^n}\,dy \le \frac{c}{R^{n-1}} \\
|D^2w(x)| &\le \int_{B_{R_0}} |D^2\Gamma(x-y)||f(y)|\,dy \le \frac{c}{R^n}.
\end{aligned}$$

Letting $R \longrightarrow \infty$ in (2.16), we obtain

$$\int_{\mathbb{R}^n}\sum_{i,j}(D_{ij}w)^2\,dx = \int_{\mathbb{R}^n} f^2\,dx. \tag{2.17}$$

This equality implies that

$$\|Tf\|_2 \le \|f\|_2 \quad \forall f \in C_0^\infty(\mathbb{R}^n).$$

Since $C_0^\infty(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$, T can be extended uniquely to a bounded linear operator on $L^2(\mathbb{R}^n)$, i. e., T is of strong type $(2,2)$. □

To show that T also is of weak type $(1,1)$, we need two lemmas:

Lemma 2.11 *For the fundamental solution* Γ, *the following estimate holds:*

$$J := \sup_{y\neq 0, 1\leq i,j\leq n} \int_{|x|\geq 2|y|} |D_{ij}\Gamma(x-y) - D_{ij}\Gamma(x)|\, dx < \infty,$$

where J *only depends on* n.

Proof. By the mean-value-theorem

$$\int_{|x|\geq 2|y|} |D_{ij}\Gamma(x-y) - D_{ij}\Gamma(x)|\, dx \leq \int_{|x|\geq 2|y|} \sum_{k=1}^{n} |D_{ijk}\Gamma(x-\lambda y)||y_k|\, dx,$$

where $0<\lambda<1$. Using (2.11) and noticing that $|x-\lambda y| \geq \frac{|x|}{2}$ for $|x|\geq 2|y|$, we obtain

$$\begin{aligned}\int_{|x|\geq 2|y|} |D_{ij}\Gamma(x-y) - Dij\Gamma(x)|\, dx &\leq \int_{|x|\geq 2|y|} \frac{c|y|}{|x-\lambda y|^{n+1}}\, dx \\ &\leq c|y| \int_{|x|\geq 2|y|} |x|^{-n-1}\, dx = c|y|(2|y|)^{n+(-n-1)} \leq c,\end{aligned}$$

where c changes during computation. □

We introduce the following notation:

$$⨍_M u\, dx := \frac{1}{|M|}\int_M u\, dx$$

for any Borel set $M \neq \emptyset$ and measurable function u.

Lemma 2.12 (Calderón-Zygmund decomposition) *For* $f \in L^1(\mathbb{R}^n)$, $f \geq 0$ *and fixed* $\alpha > 0$, *there exist two sets* F *and* $\tilde{\Omega}$, *such that*

(i) $\mathbb{R}^n = F \cup \tilde{\Omega}, \quad F \cap \tilde{\Omega} = \emptyset$

(ii) $f(x) \leq \alpha$ *a. e.* $x \in F$

(iii) $\tilde{\Omega} = \bigcup_{k=1}^{\infty} Q_k$, *where* (Q_k) *are nonoverlapping (disjoint interior) cubes with their sides parallel to the coordinate axes, such that*

$$\alpha < ⨍_{Q_k} f\, dx \leq 2^n \alpha, \quad k = 1, 2, \ldots . \tag{2.18}$$

Proof. Since $f \in L^1(\mathbb{R}^n)$, we can decompose $\mathbb{R}^n$ into congruent cubes, such that for any cube Q',

$$⨍_{Q'} f\, dx \leq \alpha.$$

We devide each Q' into 2^n equal cubes Q''. There are only two possibilities:

1. $⨍_{Q''} f\,dx \le \alpha$
2. $⨍_{Q''} f\,dx > \alpha$

We take those Q'' satisfying Case 2 into our family of (Q_k) as stated in the lemma. For such Q'', (2.18) is valid, since

$$\alpha < ⨍_{Q''} f\,dx \le \frac{1}{2^{-n}|Q'|}\int_{Q'} f\,dx \le 2^n\alpha$$

We will further divide those cubes satisfying Case 1. This process will continue until Case 2 appears. Let $\tilde{\Omega}$ be the union of all such cubes satisfying Case 2, and let $F = \mathbb{R}^n \backslash \tilde{\Omega}$. By our construction process it is obvious that (i) and (iii) hold. Now we prove (ii). For any $x \in F$, there exist cubes $(\tilde{Q}_l)$, such that $x \in \tilde{Q}_l$, $|\tilde{Q}_l| \longrightarrow 0$ as $l \longrightarrow \infty$. Furthermore, for each $\tilde{Q}_l$, Case 1 is valid. If we can show that

$$f(x) = \lim_{k\to\infty} \frac{1}{|\tilde{Q}_{l_k}|}\int_{\tilde{Q}_{l_k}} f\,dy \quad \text{for a. e. } x \in F,$$

then (ii) follows. This is done in Proposition 2.13. □

Proposition 2.13 (Lebesgue differentiation theorem, cf. [18]) *Suppose that $f \in L^1(\mathbb{R}^n)$, then for a. e. $x \in \mathbb{R}^n$ we have*

$$f(x) = \lim_{l\to 0} ⨍_{Q_{x,l}} f(y)\,dy,$$

where $Q_{x,r}$ denotes the cube in $\mathbb{R}^n$ with center x and side length $2r$.

Proof. Define $T_l f(x) := \frac{1}{|Q_{x,l}|}\int_{Q_{x,l}} f(y)\,dy$. For $g \in C_0(\mathbb{R}^n)$,

$$\begin{aligned} |T_l g(x) - g(x)| &= \left|\frac{1}{|Q_{x,l}|}\int_{Q_{x,l}} g(y)\,dy - g(x)\right| \\ &\le \frac{1}{|Q_{x,l}|}\int_{Q_{x,l}} |g(y) - g(x)|\,dy \longrightarrow 0, \end{aligned}$$

as $l \longrightarrow 0$, for any $x \in \mathbb{R}^n$. For $f \in L^1(\mathbb{R}^n)$ and $\epsilon > 0$ there exists $g \in C_0(\mathbb{R}^n)$ such that $\|f - g\|_{L^1(\mathbb{R}^n)} \le \frac{\epsilon}{3}$. Hence,

$$\begin{aligned} \|T_l g - T_l f\|_{L^1(\mathbb{R}^n)} &\le \int_{\mathbb{R}^n} ⨍_{Q_{x,l}} |g(y) - f(y)|\,dy\,dx \\ &= \frac{1}{|Q_{x,l}|}\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} 1_{Q_{x,l}}(y)\,dx\right)|f(y) - g(y)|\,dy \\ &= \|g - f\|_{L^1(\mathbb{R}^n)}, \end{aligned}$$

since the term in brackets equals $|Q_{x,l}|$. Choose a sequence $(g_m) \subset C_0(\mathbb{R}^n)$ such that $g_m \longrightarrow g$ in $L^1(\mathbb{R}^n)$;

$$\|T_l f - f\|_{L^1(\mathbb{R}^n)} \le \|T_l(f - g_m)\| + \|T_l g_m - g_m\| + \|f - g_m\|$$

$$\leq 2\|f - g_m\| + \|T_l g_m - g_m\| < \epsilon$$

for large m, l. Therefore $T_l f \longrightarrow f$ in $L^1(\mathbb{R}^n)$ (and consequently a. e. along some subsequence (l_k), which already completes the proof of Lemma 2.12).
Now, it suffices to show that

$$Sf(x) := \limsup_{l \longrightarrow 0} \fint_{Q(x,l)} f(y)\, dy - \liminf_{l \longrightarrow 0} \fint_{Q(x,l)} f(y)\, dy = 0 \quad \text{for a. e.} \quad x \in \mathbb{R}^n.$$

Note that $Sg = 0$ a. e., if g is continuous. Now $\forall \epsilon > 0 \exists g \in C_0^\infty(\mathbb{R}^n)$, such that $\|f - g\|_1 < \epsilon$. Suppose that $Sf(x) > s$ for some $s > 0$, then, since $f = g + (f - g) =: g + h$, this implies $Sh > s$. On the other hand $Sh \leq 2M_0 h$, where M_0 denotes the centered Hardy-Littlewood maximal function (cf. A.18). Therefore, $\{Sf > s\} \subset \left\{M_0 h > \frac{s}{2}\right\}$ and by the Hardy-Littlewood-Theorem (cf. A.12 and A.9),

$$\mathrm{dx}(\{Sf > s\}) \leq \mathrm{dx}\left(\left\{Mh > \frac{s}{2}\right\}\right) \leq \frac{2c(n)\epsilon}{s},$$

which tends to zero as ϵ does. □

Note that in the setting of Lemma 2.12, by (2.18),

$$|\tilde{\Omega}| = \sum_{k=1}^{\infty} |Q_k| < \sum_{k=1}^{\infty} \frac{1}{\alpha} \int_{Q_k} f(x)\, dx \leq \frac{1}{\alpha} \|f\|_{1;\mathbb{R}^n}.$$

Lemma 2.14 *T is of weak type* $(1,1)$.

Proof. First we assume that $f \in C_0^\infty(\mathbb{R}^n)$. Applying our decomposition lemma to $|f|$, we get two sets F and $\tilde{\Omega}$, such that

- $\mathbb{R}^n = F \cup \tilde{\Omega}, \quad F \cap \tilde{\Omega} = \emptyset$
- $|f| \leq \alpha$, for a. e. $x \in F$
- $\tilde{\Omega} = \bigcup_{k=1}^{\infty} Q_k, \quad |\tilde{\Omega}| \leq \alpha^{-1} \|f\|_{L^1}$
- $\fint_{Q_k} |f|\, dx \leq 2^n \alpha$

Define

$$g(x) := \begin{cases} f(x) & , \quad x \in F \\ \fint_{Q_k} f(y)\, dy, & x \in Q_k \end{cases} \tag{2.19}$$

and $b(x) := f(x) - g(x)$. For g and b we have

$$|g(x)| \leq 2^n \alpha, \quad \text{a. e. } x \in \mathbb{R}^n \tag{2.20}$$

$$\|g\|_{1;\mathbb{R}^n} \leq \|f\|_{1;\mathbb{R}^n}, \|b\|_{1;\mathbb{R}^n} \leq 2\|f\|_{1;\mathbb{R}^n}, \tag{2.21}$$

since

$$\int_{\mathbb{R}^n} |g|\, dx = \int_{\tilde{\Omega}} |g|\, dx + \int_F |f|\, dx$$

and

$$\begin{aligned}\int_{\tilde\Omega} |g|\,dx &= \int_{\bigcup Q_k} |g|\,dx = \sum_k \int_{Q_k} |g|\,dx \\ &\leq \sum_k \int_{Q_k} \fint_{Q_k} |f(y)|\,dy\,dx = \int_{\tilde\Omega} |f(y)|\,dy,\end{aligned}$$

$$\begin{aligned}\int_{\mathbb{R}^n} |b|\,dx &= \sum_k \int_{Q_k} |f(x) - \fint_{Q_k} f(y)\,dy|\,dx \\ &\leq \sum_k \int_{Q_k} |f|\,dx + \int_{Q_k}\fint_{Q_k} |f(y)|\,dy\,dx \\ &\leq 2\|f\|_{1;\mathbb{R}^n}\end{aligned}$$

Moreover,

$$b = 0 \quad \text{on} \quad F, \quad \fint_{Q_k} b\,dx = 0 \quad (k = 1, 2, \ldots) \tag{2.22}$$

Since $f \in C_0^\infty(\mathbb{R}^n)$, both g and b belong to $L^2(\mathbb{R}^n)$. Clearly $Tf = Tg + Tb$. For any $x > 0$

$$\begin{aligned}\lambda_{Tf}(\alpha) &= |\{|Tf| > \alpha\}| = |\{|Tg + Tb| > \alpha\}| \\ &\leq \lambda_{Tg}(\frac{\alpha}{2}) + \lambda_{Tb}(\frac{\alpha}{2})\end{aligned} \tag{2.23}$$

By Lemma 2.10, T is of strong type $(2,2)$. Using also (2.20), (2.21), we have

$$\lambda_{Tg}(\frac{\alpha}{2}) \leq \frac{4}{\alpha^2}\|Tg\|^2_{L^2_w} \leq \frac{4}{\alpha^2}\|g\|_2^2 \leq \frac{4}{\alpha^2}\|g\|_{L^\infty}\|g\|_{L^1} \leq \frac{2^{n+2}}{\alpha}\|f\|_{L^1}. \tag{2.24}$$

Next, we estimate $\lambda_{Tb}(\frac{\alpha}{2})$. Let Q_k^* be a cube with the same center as Q_k and its side length $2\sqrt{n}$ times that of Q_k. Set

$$\tilde\Omega^* := \bigcup_{k=1}^\infty Q_k^*, \quad F^* := \mathbb{R}^n \backslash \tilde\Omega^*. \tag{2.25}$$

Obviously,

$$|\tilde\Omega^*| \leq (2\sqrt{n})^n |\tilde\Omega| \leq \frac{(2\sqrt{n})^n}{\alpha}\|f\|_{L^1}. \tag{2.26}$$

Set $b_k := b1_{Q_k}$, then by (2.22), $\int_{Q_k} b_k\,dx = 0$. We now prove that there exists a sequence $(b_k^l)_l \subset C_0^\infty(Q_k)$, such that

$$\|b_k^l - b_k\|_{2;Q_k} \longrightarrow 0 \quad \text{as} \quad l \longrightarrow \infty, \quad \int_{Q_k} b_k^l\,dx = 0 \quad : \tag{2.27}$$

Since $b_k \in L^2(Q_k)$, there exists $\tilde b_k^l \in C_0^\infty(Q_k), l \in \mathbb{N}$, such that $\tilde b_k^l \longrightarrow b_k$ in $L^2(Q_k)$ as $l \longrightarrow \infty$. Necessarily, $\alpha_l := \int_{Q_k} \tilde b_k^l\,dx \longrightarrow 0$, since

$$\left|\int_{Q_k} \tilde b_k^l\,dx\right| \leq \left|\int_{Q_k} \tilde b_k^l - b_k\,dx\right| + \left|\int_{Q_k} b_k\,dx\right|$$

$$\leq \quad \|1\|_{L^2(Q_k)}\|\tilde{b}^l_k - b_k\|_{L^2(Q_k)} \longrightarrow 0.$$

Take $\chi \in C^\infty_0(Q_k)$, such that $\int_{Q_k} \chi\, dx = 1$ and define $b^l_k := \tilde{b}^l_k - \alpha_l \chi (\in C^\infty_0(Q_k))$. Then $\int_{Q_k} b^l_k\, dx = 0 \quad \forall l \in \mathbb{N}$ and
$\|b^l_k - b_k\|_{L^2(Q_k)} \leq \|\tilde{b}^l_k - b_k\|_{L^2(Q_k)} + \alpha_l \|\chi\|_{L^2(Q_k)} \longrightarrow 0$ as $l \longrightarrow \infty$.

By definition of T and (2.27), we have for $x \in (Q^*_k)^c$,

$$\begin{aligned} Tb^l_k(x) &= D_{ij} \int_{Q_k} \Gamma(x-y) b^l_k(y)\, dy \\ &= \int_{Q_k} [D_{ij}\Gamma(x-y) - D_{ij}\Gamma(x-x^k)] b^l_k(y)\, dy, \end{aligned}$$

where x^k is the center of Q_k. Integrating over $(Q^*_k)^c$, applying also Lemma 2.11, we obtain for each l,

$$\begin{aligned} &\int_{(Q^*_k)^c} |Tb^l_k(x)|\, dx \\ &\leq \int_{(Q^*_k)^c} \int_{Q_k} |D_{ij}\Gamma(x-y) - D_{ij}\Gamma(x-x^k)||b^l_k(y)|\, dy\, dx \\ &\leq \sup_{Q_k \setminus \{x^k\}} \int_{(Q^*_k)^c} |D_{ij}\Gamma(x-y) - D_{ij}\Gamma(x-x^k)|\, dx \|b^l_k\|_{1;Q_k} \\ &= \sup_{Q_k \setminus \{x^k\}} \int_{(Q^*_k)^c} |D_{ij}\Gamma((x-x^k) - (y-x^k)) - D_{ij}\Gamma(x-x^k)|\, dx \|b^l_k\|_{1;Q_k} \\ &\leq J \|b^l_k\|_{1;Q_k}, \end{aligned}$$

since $|x - x^k| \geq 2|y - x^k|$, because $y \in Q_k, x \in (Q^*_k)^c$. As $l \longrightarrow \infty$, b^l_k and Tb^l_k converge in $L^2(\mathbb{R}^n)$ to b_k and Tb_k respectively. Since Tb^l_k converges even pointwisely a. e. for some suitable subsequence (l_m), by the above inequality and Fatou's lemma we have that

$$\begin{aligned} \int_{(Q^*_k)^c} |Tb_k|\, dx &\leq \liminf_{m \longrightarrow \infty} \int_{(Q^*_k)^c} |Tb^{l_m}_k|\, dx \\ &\leq J \liminf_{m \longrightarrow \infty} \int_{Q_k} |b^{l_m}_k|\, dx \\ &= J \int_{Q_k} |b_k|\, dx \quad (|Q_k| < \infty) \\ &= J \int_{Q_k} |b|\, dx. \end{aligned}$$

Therefore, since $\tilde{b}_l := \sum_{k=1}^l b_k \longrightarrow b1_{\tilde{\Omega}}$ in $L^2(\mathbb{R}^n)$ as $l \longrightarrow \infty$, for a suitable subsequence (l_k) we get

$$\begin{aligned} \int_{F^*} |Tb|\, dx &= \int_{F^*} \lim_k |T\tilde{b}_{l_k}|\, dx \leq \liminf_k \int_{F^*} |Tb_1 + \cdots + Tb_{l_k}|\, dx \\ &\leq \sum_{k=1}^{\infty} \int_{F^*} |Tb_k|\, dx \leq \sum_{k=1}^{\infty} \int_{(Q^*_k)^c} |Tb_k|\, dx \end{aligned}$$

$$\leq \quad J\|b\|_{L^1} \leq 2J\|f\|_{L^1},$$

which implies that $|\{|Tb| > \frac{\alpha}{2}\}| \cap |F^*| \leq \frac{4}{\alpha} J\|f\|_{L^1}$. By (2.26),

$$\lambda_{Tb}\left(\frac{\alpha}{2}\right) \leq |\tilde{\Omega}^*| + \left|\left\{|Tb| > \frac{\alpha}{2}\right\}\right| \leq [4J + (2\sqrt{n})^n]\frac{\|f\|_{L^1}}{\alpha}. \tag{2.28}$$

Substituting (2.24), (2.28) into (2.23), we get

$$\lambda_{Tf}(\alpha) \leq (2^{n+2} + 4J + (2\sqrt{n})^n)\frac{\|f\|_{L^1}}{\alpha} \quad \forall f \in C_0^\infty(\mathbb{R}^n). \tag{2.29}$$

Since $C_0^\infty(\mathbb{R}^n)$ is dense in $L^1(\mathbb{R}^n)$, the above inequality also holds for any $f \in L^1(\mathbb{R}^n)$. □

Proposition 2.15 *For $1 < p < \infty$, T is of strong type (p,p).*

*Proof.*By Lemma 2.10 and 2.14, T is both of strong type $(2,2)$ and of weak type $(1,1)$. From the Marcinkiewicz interpolation (see A.1) it follows that T is of strong type (p,p) for any $1 < p \leq 2$. For $2 < p < \infty$ and any $f, h \in C_0^\infty(\mathbb{R}^n)$,

$$\begin{aligned} \int Tf(x)h(x)\,dx &= \int h(x)D_{ij}\left[\int \Gamma(x-y)f(y)\,dy\right]dx \\ &= \int D_{ij}h(x)\int \Gamma(x-y)f(y)\,dy\,dx \\ &= \int f(y)\int \Gamma(x-y)D_{ij}h(x)\,dx\,dy \\ &= \int f(y)D_{ij}\left(\int \Gamma(x-y)h(x)\,dx\right)dy \\ &\leq \|f\|_{L^p}\|Th\|_{L^{p'}} \quad \text{(Hölder)} \quad . \end{aligned}$$

Since $1 < p' < 2$, the case $1 < p \leq 2$ implies that

$$\int Tf(x)h(x)\,dx \leq C\|f\|_{L^p}\|h\|_{L^{p'}} \quad \forall f, h \in C_0^\infty(\mathbb{R}^n)$$

Therefore $\|Tf\|_{L^p} \leq C\|f\|_{L^p} \quad \forall f \in C_0^\infty(\mathbb{R}^n)$, which implies that T is of strong type (p,p) □

We employ our examination of T to obtain the following

Proposition 2.16 *Let $\Omega \subset \mathbb{R}^n$ be a domain. Suppose that $u \in H_0^{2,p}(\Omega)$ satisfies $\triangle u = f$. Then for $1 < p < \infty$, there exists a constant C, only depending on n, p, such that*

$$\|D^2u\|_{L^p(\Omega)} \leq C\|f\|_{L^p(\Omega)}.$$

*Proof.*Assume without loss of generality that $u \in C_0^\infty(\Omega)$. Applying (2.12) we obtain $\triangle(Nu) = u$ and compute

$$u(x) \quad = \quad \triangle\int \Gamma(x-y)u(y)\,dy = \triangle\int \Gamma(y)u(x-y)\,dy$$

$$= \int \Gamma(y)\triangle u(x-y)\,dy = \int \Gamma(x-y)f(y)\,dy$$

With Proposition 2.15 it follows that

$$\begin{aligned}\|D^2u\|_{L^p(\Omega)} &= \sum_{i,j}\left\|D_{ij}\int \Gamma(x-y)f(y)\,dy\right\|_{L^p(\Omega)} \\ &= \sum_{i,j}\|Tf\|_{L^p(\Omega)} \le C\|f\|_{L^p(\Omega)}.\end{aligned}$$

□

We now get the curve to the promised interior $H^{2,p}$-estimate. Let us redefine our setting first.

It is the boundary value problem

$$Lu = f \quad \text{in} \quad \Omega \tag{2.30}$$
$$u = 0 \quad \text{on} \quad \partial\Omega, \tag{2.31}$$

where L is as in (2.1) and the coefficients satisfy

$$a^{ij}\xi_i\xi_j \ge \lambda|\xi|^2 \quad \forall x\in\Omega, \xi\in\mathbb{R}^n \quad \text{, where} \quad \lambda > 0 \tag{2.32}$$
$$\sum_{i,j}\|a^{ij}\|_{L^\infty} + \sum_i \|b^i\|_{L^\infty} + \|c\|_{L^\infty} \le \Lambda \tag{2.33}$$
$$a^{ij} \in C(\bar{\Omega}), 1 \le i,j \le n \tag{2.34}$$

Lemma 2.17 *Let the assumptions (2.32)-(2.34) be in force. Then there exists $R_0 > 0$, only depending on $n, p, \Lambda, \lambda, a^{ij}$, such that for any $0 < R < R_0$ and any $u \in H_0^{2,p}(B_R), 1 < p < \infty$, satisfying (2.30) almost everywhere, the following estimate holds:*

$$\|D^2u\|_{L^p(B_R)} \le C\left(\frac{1}{\lambda}\|f\|_{L^p(B_R)} + \frac{1}{R^2}\|u\|_{L^p(B_R)}\right), \tag{2.35}$$

where C only depends on n, p, Λ, λ.

Proof. Assume without loss of generality that $\lambda = 1$. Let $B_R = B_R(x_0)$. We freeze the coefficients at x_0 and rewrite (2.30) as

$$a^{ij}(x_0)D_{ij}u = \tilde{f}, \tag{2.36}$$

where $\tilde{f} = f + [a^{ij}(x_0) - a^{ij}(x)]D_{ij}u - b^iD_iu - cu$.
For the constant coefficients elliptic equation (2.36), the result of Proposition 2.16 holds, since we can reduce it to $\triangle\bar{u} = \tilde{f}$, as can be seen in the following computation, where $A := (a^{ij}(x_0))$:

$$D_i\sum_j (D_ju)\circ A^{\frac{1}{2}}(x)A^{\frac{1}{2}}_{ij} = \sum_k\sum_j (D_kD_ju)\circ A^{\frac{1}{2}}(x)A^{\frac{1}{2}}_{ij}A^{\frac{1}{2}}_{ik}$$

$$\begin{aligned}
\Rightarrow \qquad \triangle(\underbrace{u \circ A^{\frac{1}{2}}}_{\bar{u}})(x) &= \sum_i D_i \left(\sum_j (D_j u) \circ A^{\frac{1}{2}}(x) A^{\frac{1}{2}}_{ij} \right) \\
&= \sum_{i,j,k} (D_{kj} u) \circ A^{\frac{1}{2}}(x) A^{\frac{1}{2}}_{ij} A^{\frac{1}{2}}_{ki} \\
&= \sum_{j,k} A_{jk} (D_{jk} u) \circ A^{\frac{1}{2}}(x) \\
&= (L_A u)(A^{\frac{1}{2}}(x)) = (\tilde{f} \cdot A^{\frac{1}{2}})(x) =: \bar{f}(x).
\end{aligned}$$

Therefore, by integral transformation

$$\|D^2 u\|_{L^p(B_R)} \leq C \|\tilde{f}\|_{L^p(B_R)}, \tag{2.37}$$

where C only depends on n, p, Λ. Define

$$w(R) := \sup_{\substack{|x-y| \leq R, \\ 1 \leq i,j \leq n}} |a^{ij}(y) - a^{ij}(x)|. \tag{2.38}$$

By (2.34), $w(R) \longrightarrow 0$ as $R \longrightarrow 0$. Thus (2.37) implies that

$$\|D^2 u\|_{L^p} \leq C(\|f\|_{L^p} + w(R)\|D^2 u\|_{L^p} + \|u\|_{H^{1,p}}).$$

We take $R_0 \leq 1$ such that $Cw(R) \leq \frac{1}{2}$ for $0 < R < R_0$. Then

$$\begin{aligned}
\|D^2 u\|_p &\leq C(\|f\|_p + \|u\|_{1,p}) \quad \text{for} \quad 0 < R \leq R_0 \\
&\leq C \left(\|f\|_p + \|u\|_p + \sum_{i=1}^n (C'\epsilon \|D^2 u\|_p + \frac{C''}{\epsilon} \|u\|_p) \right) \\
&\qquad \text{(Sobolev interpolation, see A.3, and Poincaré, see A.7)} \\
&\leq C \left(\|f\|_p + \left(\frac{nC''}{\epsilon} + 1 \right) \|u\|_p + nC'\epsilon \|D^2 u\|_p \right).
\end{aligned}$$

We now choose ϵ such that $C'Cn\epsilon = \frac{1}{2}$ and obtain

$$\begin{aligned}
\|D^2 u\|_p &\leq C'''(\|f\|_p + \|u\|_p) \\
&\leq C'''(\|f\|_p + R^{-2}\|u\|_p)
\end{aligned}$$

□

Proposition 2.18 *Let the assumptions (2.32)-(2.34) be in force. Suppose that* $u \in H^{2,p}_{loc}(\Omega) \cap L^p(\Omega)$ *satisfies (2.30) almost everywhere. Then for any domain* $\Omega' \subset\subset \Omega$,

$$\|u\|_{2,p;\Omega'} \leq C \left(\frac{1}{\lambda} \|f\|_{p;\Omega} + \|u\|_{p;\Omega} \right),$$

where $C = C(n, p, \Lambda, \lambda, \Omega', \Omega, a^{ij})$.

Proof. Assume without loss of generality that $\lambda = 1$. Let R_0 be the constant in Lemma 2.17 and $\bar{R}_0 = \min\left(R_0, \frac{1}{2}\mathrm{dist}(\Omega', \partial\Omega)\right)$. For any $x_0 \in \Omega'$ and $\frac{\bar{R}_0}{2} \leq \rho < R \leq \bar{R}_0$, we let $0 \leq \zeta \leq 1$ be a cutoff function on $B_R(x_0)$ such that $\zeta \in C_0^\infty(B_R(x_0)), \zeta \equiv 1$ on $B_\rho(x_0)$ and

$$|D^k\zeta| \leq \frac{C}{(R-\rho)^k} \quad (k = 1, 2) \tag{2.39}$$

The function $v := \zeta u$ satisfies the equation

$$a^{ij}D_{ij}v + b^iD_iv + cv = \tilde{f},$$

where $\tilde{f} = \zeta f + [a^{ij}D_{ij}\zeta + b^iD_i\zeta]u - 2a^{ij}D_i\zeta D_ju$. By Lemma 2.17,

$$\begin{aligned}
\|D^2v\|_{p;B_R(x_0)} &\leq C\left(\|\tilde{f}\|_{p;B_R(x_0)} + \frac{1}{R^2}\|v\|_{p;B_R(x_0)}\right) \\
&\leq C\left[\|f\|_{p;B_R(x_0)} + \frac{1}{R-\rho}\|Du\|_{p;B_R(x_0)} + \frac{1}{(R-\rho)^2}\|u\|_{p;B_R(x_0)}\right],
\end{aligned}$$

since ($R_0 \leq 1 \Rightarrow R - \rho \leq 1$). Denote $B_R = B_R(x_0)$. Using Proposition A.5, we obtain

$$\begin{aligned}
\|D^2u\|_{p;B_\rho} &= \|D^2v\|_{p;B_\rho} \leq \|D^2v\|_{p;B_R} \\
&\leq C\left[\|f\|_p + \frac{1}{R-\rho}(\tilde{\epsilon}\|D^2u\|_p + C_{\tilde{\epsilon}}\|u\|_p) + \frac{1}{(R-\rho)^2}\|u\|_p\right] \\
&\leq C\left[\|f\|_p + \frac{\tilde{\epsilon}}{R-\rho}\|D^2u\|_p + \left(\frac{C_{\tilde{\epsilon}}}{(R-\rho)} + \frac{1}{(R-\rho)^2}\right)\|u\|_p\right] \\
&\leq C\left[\epsilon\|D^2u\|_p + \|f\|_p + \frac{C_\epsilon}{(R-\rho)^2}\|u\|_p\right].
\end{aligned}$$

By Lemma 2.19 (see below), for sufficient small ϵ it follows that

$$\|D^2u\|_{p;B_\rho} \leq C\left[\|f\|_{p;B_R} + \frac{1}{(R-\rho)^2}\|u\|_{p;B_R}\right]$$

for $\frac{\bar{R}_0}{2} \leq \rho < R \leq \bar{R}_0$. We take $R = \bar{R}_0$, $\rho = \frac{\bar{R}_0}{2}$ and then cover Ω' with finitely many balls of radius $\frac{\bar{R}_0}{2}$ to obtain the proposition. □

Lemma 2.19 *Let φ be a bounded nonnegative function defined on the interval $[T_0, T_1]$, where $T_1 > T_0 \geq 0$. Suppose that for any $T_0 \leq t < s \leq T_1$, φ satisfies*

$$\varphi(t) \leq \theta\varphi(s) + \frac{A}{(s-t)^\alpha} + B, \tag{2.40}$$

where θ, A, B, α are nonnegative constants, $\theta < 1$. Then

$$\varphi(\rho) \leq C\left[\frac{A}{(R-\rho)^\alpha} + B\right] \quad \forall T_0 \leq \rho < R \leq T_1, \tag{2.41}$$

where $C = C(\alpha, \theta)$.

Proof. Let $t_0 := \rho, t_{i+1} := t_i + (1-\tau)\tau^i(R-\rho), i \in \{0,1,2,\ldots\}$, where $0 < \tau < 1$ is to be determined. From (2.40),

$$\varphi(t_i) \le \theta\varphi(t_{i+1}) + \frac{A}{[(1-\tau)\tau^i(R-\rho)]^\alpha} + B, i \in \{0,1,2,\ldots\}.$$

By induction,

$$\varphi(t_0) \le \theta^k\varphi(t_k) + \left[\frac{A}{(1-\tau)^\alpha(R-\rho)^\alpha} + B\right]\sum_{i=0}^{k-1}\theta^i\tau^{-i\alpha} \quad \left(B \le \frac{B}{\tau^{i\alpha}}\forall i\right).$$

We choose τ such that $\theta\tau^{-\alpha} < 1$. By letting $k \to \infty$, we get (2.41). $\square$

In order to extend Proposition 2.18 to the boundary $\partial\Omega$, we must get to know coordinate transformation and boundary portions.

Definition 2.20 (cf. [15], [3]) *A bounded domain Ω in $\mathbb{R}^n$ and its boundary are of class $C^{k,\alpha}, 0 \le \alpha \le 1, k \in \{0,1,2,\ldots\}$, if at each point $x_0 \in \partial\Omega$ there exists $r = r(x_0) > 0$ and a $C^{k,\alpha}$- function $g : \mathbb{R}^{n-1} \longrightarrow \mathbb{R}$ such that – upon relabeling and reorienting the coordinate axes if necessary – we have*

$$\begin{aligned} \Omega \cap B_r(x_0) &= \{x \in B_r(x_0) | x_n > g(x')\} \\ \partial\Omega \cap B_r(x_0) &= \{x \in B_r(x_0) | x_n = g(x')\}, \end{aligned}$$

where we set $x = (x', x_n)$ for $x \in \mathbb{R}^n$. Likewise, $\partial\Omega$ is C^∞, if $\partial\Omega$ is C^k for any $k \in \mathbb{N}$.
A domain Ω will be said to have a boundary portion $T \subset \partial\Omega$ of class $C^{k,\alpha}$, if at each point $x_0 \in T$ there is a ball $B_{r(x_0)}(x_0)$, in which the conditions of the first part of the definition are satisfied and such that $B_{r(x_0)}(x_0) \cap \partial\Omega \subset T$.

Remark 2.21 *We want to transform the given definition of regular domains into the setting of* [19, Definition 6.2], *because we are going to leave* [13] *in favour of* [19] *for gaining global estimates. Let $\partial\Omega$ be of class $C^{k,\alpha}$. Define*

$$\psi(x) := (x', x_n - g(x')) \quad \textit{for} \quad x \in B_r(x_0)$$

and write $y = \psi(x)$. Similarly, set

$$\varphi(y) := (y', y_n + g(y')) \quad \textit{for} \quad y \in \psi(B_r(x_0))$$

and write $x = \varphi(y)$. Then $\varphi = \psi^{-1}$ and the mapping $x \longmapsto \psi(x) = y$ "straightens out $\partial\Omega$" near x_0. Observe that

$$D\psi = \begin{pmatrix} 1 & & & \\ & & & \\ & & 1 & \\ -D_1 g & \ldots & -D_{n-1}g & 1 \end{pmatrix}, D\varphi = \begin{pmatrix} 1 & & & \\ & & & \\ & & 1 & \\ D_1 g & \ldots & D_{n-1}g & 1 \end{pmatrix},$$

$|\det D\psi| = |\det D\varphi| = 1$. *Therefore,*

$$\psi : B_r(x_0) \longrightarrow \psi(B_r(x_0)) =: D \in C^{k,\alpha}(B_r(x_0)),$$

$$\begin{aligned}\psi^{-1}: D &\longrightarrow B_r(x_0) \in C^{k,\alpha}(D)\\ \psi(B_r(x_0)\cap\Omega)\subset \mathbb{R}^n_+ \quad &, \quad \psi(B_r(x_0)\cap\partial\Omega)\subset\partial\mathbb{R}^n_+.\end{aligned}$$

By restricting ψ to $B_{r-\epsilon_{x_0}}$, we may assume without loss of generality that $\psi \in C^{k,\alpha}(\overline{B_r(x_0)}), \psi^{-1} \in C^{k,\alpha}(\bar{D})$, respectively.

Letting $\Omega^+ := \Omega \cap \mathbb{R}^n_+ = \{x \in \Omega | x_n > 0\}, (\partial\Omega)^+ := (\partial\Omega)\cap\mathbb{R}^n_+ = \{x \in \partial\Omega | x_n > 0\}$, we have the following lemma. We will say that $u = 0$ near $(\partial\Omega)^+$, if for any $x \in (\partial\Omega)^+$ there is a neighbourhood V of x, such that $u_{|V\cap\Omega} = 0$.

Lemma 2.22 (cf. [19, Lemma 9.12]) *Let $u \in H^{1,1}_0(\Omega^+), f \in L^p(\Omega^+), 1 < p < \infty$, satisfy $\triangle u = f$ weakly in Ω^+ with $u = 0$ near $(\partial\Omega)^+$. Then $u \in H^{2,p}(\Omega^+)\cap H^{1,p}_0(\Omega^+)$ and*

$$\|D^2u\|_{p;\Omega^+} \le C\|f\|_{p;\Omega^+}, \tag{2.42}$$

where $C = C(n,p)$.

Proof. We extend u and f to all of $\mathbb{R}^n_+$ by setting $u = f = 0$ in $\mathbb{R}^n_+\setminus\Omega$ and then to all of $\mathbb{R}^n$ by odd reflection, that is, by setting

$$u(x',x_n) := -u(x',-x_n), f(x',x_n) := -f(x',-x_n) \quad \text{for} \quad x_n < 0,$$

where $x' := (x_1,\dots,x_{n-1})$. It follows that the extended functions satisfy $\triangle u = f$ weakly in $\mathbb{R}^n$. To show this we take an arbitrary test function $\varphi \in C^\infty_0(\mathbb{R}^n)$, and for $\epsilon > 0$ let η be an even function in $C^\infty(\mathbb{R})$ such that $\eta(t) = 0$ for $|t| \le \epsilon$, $\eta(t) = 1$ for $|t| \ge 2\epsilon$ and $|\eta'| \le \frac{2}{\epsilon}$. Then

$$\begin{aligned}-\int \eta(x_n) f\varphi\, dx &= -\int \eta\varphi\triangle u\, dx = \int DuD(\eta\varphi)\, dx\\ &= \int \eta DuD\varphi\, dx + \int \varphi\eta' D_n u\, dx.\end{aligned}$$

Now

$$\begin{aligned}&\left|\int \varphi\eta' D_n u\, dx\right|\\ &= \left|\int_{\{0<x_n<2\epsilon\}} \varphi(x',x_n)\eta' D_n u\, dx + \int_{\{0>x_n>2\epsilon\}} \varphi(x',x_n)\eta' D_n u\, dx\right|\\ &= \left|\int_{0<x_n<2\epsilon} (\varphi(x',x_n) - \varphi(x',-x_n))\eta' D_n u\right|\end{aligned}$$

(since $\int\limits_{\{0>x_n>2\epsilon\}} \varphi(x',x_n)\eta' D_n u\, dx = -\int\limits_{\{0<x_n<2\epsilon\}} \varphi(x',-x_n)\eta' D_n u(x',x_n)\, dx$)

$$\le 8\max|D\varphi| \int_{0<x_n<2\epsilon} |D_n u|\, dx \longrightarrow 0 \quad \text{as} \quad \epsilon \longrightarrow 0$$

by the mean-value-theorem. Consequently, letting $\epsilon \longrightarrow 0$, we obtain

$$-\int f\varphi\, dx = \int DuD\varphi\, dx,$$

so that $u \in H^{1,1}(\mathbb{R}^n)$ is a weak solution of $\triangle u = f$. Since u also has compact support in $\mathbb{R}^n$, the regularization $u_h := (u * \xi_h) \in C_0^\infty(\mathbb{R}^n)$ ((ξ_h) being a Dirac-sequence) and satisfies $\triangle u_h = f_h$ in $\mathbb{R}^n$.
We assert that $u_h \longrightarrow u$ as $h \longrightarrow 0$ in $H^{2,p}(\mathbb{R}^n)$ and

$$\|D^2 u\|_p \leq C\|\triangle u\|_p. \tag{2.43}$$

Let us prove this: By Proposition 2.16,

$$\begin{aligned}\|D^2(u_h - u_{h'})\|_p &= \sum_{i,j} \|D_{ij}(u_h - u_{h'})\|_p \\ &\leq C\|\triangle(u_h - u_{h'})\|_p = C\|f_h - f_{h'}\|_p \longrightarrow 0,\end{aligned}$$

as $h, h' \longrightarrow 0$. Thus there exists

$$D_{ij}u := \lim_{h\downarrow 0} D_{ij}u_h \quad \text{in} \quad L^p(\mathbb{R}^n).$$

By A.7,

$$\begin{aligned}\|u_h - u_{h'}\| &\leq C\|D(u_h - u_{h'})\|_p \\ &= C\sum_i \|D_i(u_h - u_{h'})\|_p \\ &\leq \tilde{C}\sum_i \|D(D_i(u_h - u_{h'}))\|_p \\ &= \tilde{C}\sum_{i,j} \|D_{ij}(u_h - u_{h'})\|_p \longrightarrow 0\end{aligned}$$

Thus there exists $\tilde{u} \in L^p(\mathbb{R}^n)$, such that $u_h \longrightarrow \tilde{u}$ in $L^p(\mathbb{R}^n)$, in particular $u_{h_k} \longrightarrow \tilde{u}$ pointwisely for some subsequence (h_k) of (h). Since $u_{h_k} \longrightarrow u$ in $L^1(\mathbb{R}^n)$ and therefore $u_{h_{k_l}} \longrightarrow u$ pointwisely, it follows that $u = \tilde{u}$ a. e. and therefore $u \in L^p(\mathbb{R}^n)$. Now

$$\begin{aligned}u \in H^{1,1}(\mathbb{R}^n) &\Rightarrow \exists D_i u \in L^1(\mathbb{R}^n) \\ \|D_i u_h - D_i u_{h'}\|_p &\leq C\|D(D_i u_h - D_i u_{h'})\|_p \leq C\sum_{i,j} \|D_{ij}(u_h - u_{h'})\|_p \longrightarrow 0 \\ &\Rightarrow \exists \widetilde{D_i u} := \lim_{h\downarrow 0} D_i u_h \quad \text{in} \quad L^p(\mathbb{R}^n).\end{aligned}$$

But

$$\begin{aligned}D_i u_h(\cdot) &= \int u(y) D_i^x \xi_h(\cdot - y)\, dy \\ &= -\int u(y) D_i^y \xi_h(\cdot - y)\, dy\end{aligned}$$

$$= \int D_i^y(y)\xi_h(\cdot - y)\,dy = (D_i u)_h(\cdot) \longrightarrow D_i u \quad \text{in} \quad L^1(\mathbb{R}^n).$$

$$\Rightarrow \widetilde{D_i u} = D_i u \quad \text{a. e. and} \quad D_i u \in L^p(\mathbb{R}^n).$$

Let $\varphi \in C_0^\infty(\mathbb{R}^n)$, then

1. $u \in L^p(\mathbb{R}^n)$
2. $\int u D_i\varphi = -\int (D_i u)\varphi, D_i u \in L^p(\mathbb{R}^n)$
3. $\int u D_{ij}\varphi = \lim_{h\downarrow 0} \int u_h D_{ij}\varphi = \lim_{h\downarrow 0} \int \varphi D_{ij} u_h = \int \varphi D_{ij} u.$

Thus, $u \in H^{2,p}(\mathbb{R}^n)$ and $u_h \longrightarrow u$ in $H^{2,p}(\mathbb{R}^n)$. Since supp u compact, we have $u \in H_0^{2,p}(B_R)$ and hence,

$$\|D^2 u\|_p \le C\|\triangle u\|_p$$

by Proposition 2.16. Thus, the estimate (2.42) follows with constant C twice of that in (2.43). Since $u_h(x', 0) = 0$, we also obtain $u \in H_0^{1,p}(\Omega^+)$. □

For the global estimate, we require that boundary values are taken on in the sense of $H^{1,p}(\Omega)$. If T is a subset of $\partial\Omega$ and $u \in H^{1,p}(\Omega)$, we say that $u = 0$ on T in the sense of $H^{1,p}(\Omega)$, if u is the limit in $H^{1,p}(\Omega)$ of a sequence of functions in $C^1(\Omega)$ vanishing near T. With the aid of Lemma 2.22 we now extend Proposition 2.18 to derive a local boundary estimate. Our later application in mind, we only consider C^∞-domains for simplicity.

Proposition 2.23 (cf. [19, Theorem 9.13]) *Let Ω be a domain in $\mathbb{R}^n$ with a C^∞-boundary portion $T \subset \partial\Omega$. Let $u \in H^{2,p}(\Omega), 1 < p < \infty$, be a strong solution of $Lu = f$ in Ω with $u = 0$ on T in the sense of $H^{1,p}(\Omega)$, where L satisfies (2.32)–(2.34) with $a^{ij} \in C(\Omega \cup T)$. Then, for any domain $\Omega' \subset\subset \Omega \cup T$,*

$$\|u\|_{2,p;\Omega'} \le C(\|u\|_{p;\Omega} + \|f\|_{p;\Omega}), \tag{2.44}$$

where $C = C(n, p, \lambda, \Lambda, T, \Omega, \Omega', (a^{ij}))$.

*Proof.*Since T is of class C^∞, for each point $x_0 \in T$ there is a neighbourhood $\mathcal{N} = \mathcal{N}_{x_0}$ and a diffeomorphism $\psi = \psi_{x_0}$ from $\mathcal{N}$ onto the unit ball B in $\mathbb{R}^n$, such that $\psi(\mathcal{N} \cap \Omega) \subset \mathbb{R}^n_+, \psi(\mathcal{N} \cap \partial\Omega) \subset \partial\mathbb{R}^n_+, \psi \in C^\infty(\mathcal{N}), \psi^{-1} \in C^\infty(B)$. Writing $y = \psi(x) = (\psi_1(x), \ldots, \psi_n(x)), \tilde{u}(y) = u(x), x \in \mathcal{N}, y \in B$, we have

$$\tilde{L}\tilde{u} = \tilde{a}^{rs} D^y_{rs}\tilde{u} + \tilde{b}^r D^y_r \tilde{u} + \tilde{c}\tilde{u} = \tilde{f} \quad \text{in} \quad B^+,$$

where

$$\tilde{a}^{rs}(y) = D_{x_i}\psi_r D_{x_j}\psi_s a^{ij}(x), \tilde{b}^r(y) = D^2_{x_i x_j}\psi_r a^{ij}(x) + D_{x_i}\psi_r b^i(x),$$

$$\tilde{c}(y) = c(x), \tilde{f}(y) = f(x).$$

In fact, by the chain and product rule

$$a^{ij}(x) D^x_{ij} u(x) = a^{ij}(x) D^x_{ij} \tilde{u}(\psi(x))$$

$$
\begin{aligned}
&= a^{ij}(x) D_i^x \left(\sum_r D_r^y \tilde{u}(\psi(x)) D_j^x \psi_r(x) \right) \\
&= \sum_r a^{ij}(x) [D_i^x (D_r^y \tilde{u}(\psi(x))) D_j^x \psi_r(x) + D_r^y \psi(x) D_{ij}^x \psi_r(x)] \\
&= \sum_{r,s} a^{ij}(x) D_{rs}^y \tilde{u}(\psi(x)) D_i^x \psi_r(x) D_j^x \psi_s(x) + \\
&\quad \sum_r a^{ij}(x) D_r^y \tilde{u}(\psi(x)) D_{ij}^x \psi_r(x) \\
&= \tilde{a}^{rs} D_{rs}^y \tilde{u} + \sum_r a^{ij}(x) D_r^y \tilde{u}(\psi(x)) D_{ij}^x \psi_r(x)
\end{aligned}
$$

Moreover, $b^i D_i^x u(x) = D_i^x \psi_r(x) b^i(x) D_r^y \tilde{u}(y)$, so that $\tilde{f}(\psi(x)) = f(x) = L^x u(x) = \tilde{L}^y \tilde{u}(\psi(x))$ in Ω. $\tilde{L}$ satisfies conditions similar to (2.32)-(2.34) with constants $\tilde{\lambda}, \tilde{\Lambda}$ also depending on λ, Λ and ψ. Furthermore $\tilde{u} \in H^{2,p}(B^+)$ and $\tilde{u} = 0$ on $B \cap \partial\mathbb{R}_+^n$ in the sense of $H^{1,p}(B^+)$. We now proceed as in the proofs of Lemma 2.17 and Proposition 2.18 with the ball $B_R(x_0)$ replaced by the half ball $B_R^+(0) \subset B$ and with Lemma 2.22 used in place of Proposition 2.16 in order to obtain

$$
\|D^2 \tilde{u}\|_{p;B_\rho^+} \leq C \left[\|f\|_{p;B_R^+} + \frac{1}{(R-\rho)^2} \|u\|_{p;B_R^+} \right],
$$

for $\frac{\bar{R}_0}{2} \leq \rho < R \leq \bar{R}_0$. Taking $R = \bar{R}_0, \rho = \frac{\bar{R}_0}{2}$ and $\tilde{\mathcal{N}} = \tilde{\mathcal{N}}_{x_0} = \psi^{-1}(B_\rho)$, returning to our original coordinates, we therefore have

$$
\|D^2 u\|_{p;\tilde{\mathcal{N}} \cap \Omega} \leq C(\|u\|_{p;\mathcal{N} \cap \Omega} + \|f\|_{p;\mathcal{N} \cap \Omega}), \tag{2.45}
$$

where $C = C(n, p, \lambda, \Lambda, \delta, \psi)$. Finally, by covering $\Omega' \cap T$ with a finite number of such neighbourhoods $\tilde{\mathcal{N}}_1, \ldots, \tilde{\mathcal{N}}_m$, we obtain

$$
\begin{aligned}
&\|u\|_{2,p;\Omega'} \\
&\quad \leq \|u\|_{2,p;\Omega' \setminus \bigcup_i \tilde{\mathcal{N}}_i} + \sum_i (\|u\|_{p;\tilde{\mathcal{N}}_i} + \|Du\|_{p;\tilde{\mathcal{N}}_i} + \|D^2 u\|_{p;\tilde{\mathcal{N}}_i}) \\
&\qquad \text{(by A.2)} \\
&\quad \leq C'(\|u\|_{p;\Omega} + \|f\|_{p;\Omega}) + n\|u\|_{p;\Omega} + \sum_i (\|Du\|_{p;\Omega} + C''(\|u\|_{p;\Omega} + \|f\|_{p;\Omega})) \\
&\qquad \text{(by 2.18, (2.45))} \\
&\quad \leq C(\|u\|_{p;\Omega} + \|f\|_{p;\Omega})
\end{aligned}
$$

by A.6. □

When $T = \partial\Omega$ in Proposition 2.23, we may take $\Omega' = \Omega$ to obtain a global $H^{2,p}(\Omega)$ estimate. This estimate can, in fact, be refined as follows.

Proposition 2.24 (cf. [19, Theorem 9.14]) *Let Ω be a C^∞-domain in $\mathbb{R}^n$ and suppose the operator L satisfies the conditions (2.32)–(2.34). Then, if $u \in H^{2,p}(\Omega) \cap H_0^{1,p}(\Omega), 1 < p < \infty$, we have*

$$
\|u\|_{2,p;\Omega} \leq C\|Lu - \sigma u\|_{p,\Omega}, \tag{2.46}
$$

for any $\sigma \geq \sigma_0$, where C and σ_0 are positive constants only depending on $n, p, \lambda, \Lambda, \Omega, (a^{ij})$.

*Proof.*We define a domain Ω_0 in $\mathbb{R}^{n+1}(x,t)$ by $\Omega_0 := \Omega \times (-1,1)$ together with the operator L_0, given by $L_0 v := Lv + D_{tt}v$ for $v \in H^{2,p}(\Omega_0)$. Then, if $u \in H^{2,p}(\Omega) \cap H_0^{1,p}(\Omega)$ the function v, given by $v(x,t) := u(x)\cos\sigma^{\frac{1}{2}}t$ belongs to $H^{2,p}(\Omega_0)$ and vanishes on $\partial\Omega \times (-1,1)$ in the sense of $H^{1,p}(\Omega_0)$. Indeed, let $u_m \in C_0^\infty(\Omega), m \in \mathbb{N}$, such that $u_m \longrightarrow u$ in $H^{1,p}(\Omega)$. Define $v_m(x,t) := u_m(x)\cos\sigma^{\frac{1}{2}}t$,
then $v_m \in C^1(\Omega_0)$ and vanishes near $T := \partial\Omega \times (-1,1)$. Moreover,

$$\int |u_m(x)\cos\sigma^{\frac{1}{2}}t - u(x)\cos\sigma^{\frac{1}{2}}t|^p \, d(x,t) \leq \|u_m - u\|^p_{L^p(dx)} \longrightarrow 0.$$

Similarly $\|D_i v_m - D_i v\|_{L^p(d(x,t))} \longrightarrow 0$ for $i \in \{1, \ldots, n\}$.
Case $i = n+1$:

$$\begin{aligned} &\int |D_t(u_m(x)\cos\sigma^{\frac{1}{2}}t) - D_t(u(x)\cos\sigma^{\frac{1}{2}}t)|^p \, d(x,t) \\ &= \int |-u_m(x)\sigma^{\frac{1}{2}}\sin\sigma^{\frac{1}{2}}t + u(x)\sigma^{\frac{1}{2}}\sin\sigma^{\frac{1}{2}}t|^p \, d(x,t) \\ &\leq \sigma^{\frac{1}{2}}\|u_m - u\|^p_{L^p(dx)} \longrightarrow 0. \end{aligned}$$

Furthermore,

$$\begin{aligned} L_0 v &= Lv + D_{tt}v = \cos(\sigma^{\frac{1}{2}}t)Lu - \sigma u\cos\sigma^{\frac{1}{2}}t \\ &= (Lu - \sigma u)\cos\sigma^{\frac{1}{2}}t, \end{aligned}$$

so that by Proposition 2.23 with $\Omega' := \Omega \times (-\epsilon, \epsilon), 0 < \epsilon \leq \frac{1}{2}$, we get

$$\|D_{tt}v\|_{p;\Omega'} \leq C(\|Lu - \sigma u\|_{p;\Omega} + \|u\|_{p;\Omega}).$$

Indeed, L_0 fulfills (2.32)–(2.34) on Ω'. Therefore, by 2.23

$$\begin{aligned} \|D_{tt}v\|_{p;\Omega'} &\leq \|v\|_{2,p;\Omega'} \leq C(\|v\|_{p;\Omega} + \|L_0 v\|_{p;\Omega}) \\ &\leq C(\|u\|_{p;\Omega} + \|Lu - \sigma u\|_{p;\Omega}). \end{aligned}$$

But now, taking $\epsilon := \frac{\pi}{3\sigma^{\frac{1}{2}}}$, we have

$$\begin{aligned} \|D_{tt}v\|_{p;\Omega'} &= \sigma\|v\|_{p;\Omega'} \\ &= \sigma\left(\int_\Omega \int_{-\epsilon}^{\epsilon} |u(x)\cos\sigma^{\frac{1}{2}}t|^p \, dt \, dx\right)^{\frac{1}{p}} \\ &\geq \sigma(2\epsilon)^{\frac{1}{p}}(\cos\sigma^{\frac{1}{2}}\epsilon)\|u\|_{p;\Omega} \\ &\geq \frac{1}{2}\left(\frac{2\pi}{3}\right)^{\frac{1}{p}}\sigma^{1-\frac{1}{2p}}\|u\|_{p;\Omega}. \end{aligned}$$

Thus,

$$\|u\|_{p;\Omega} \leq \frac{C''}{\sigma^{1-\frac{1}{2p}}}\|D_{tt}v\|_{p;\Omega'} \leq \frac{C}{\sigma^{1-\frac{1}{2p}}}(\|Lu - \sigma u\|_{p;\Omega} + \|u\|_{p;\Omega}),$$

so that

$$\left(1 - \frac{C}{\sigma^{1-\frac{1}{2p}}}\right) \|u\|_{p;\Omega} \le \frac{C}{\sigma^{1-\frac{1}{2p}}} \|Lu - \sigma u\|_{p;\Omega}.$$

Hence, for σ large enough,

$$\|u\|_{p;\Omega} \le \|Lu - \sigma u\|, \tag{2.47}$$

and therefore by Proposition 2.23,

$$\begin{aligned} \|u\|_{2,p;\Omega} &\le C(\|u\|_{p;\Omega} + \|Lu\|_{p;\Omega}) \\ &\le C(\|Lu - \sigma u\|_{p;\Omega} + \|Lu\|_{p;\Omega}) \\ &\le C(2\|Lu - \sigma u\|_{p;\Omega} + \sigma\|u\|_{p;\Omega}) \\ &\le C'\|Lu - \sigma u\|_{p;\Omega}. \end{aligned}$$

□

Now we can prove a preliminary stage of Proposition 2.6.

Lemma 2.25 *Fix $x_0 \in \mathbb{R}^n$ and a symmetric, nondegenerate real matrix (a_{ij}). For $Lu := a^{ij}D_{ij}u(x)$ (constant coefficients!) and $2 \le p < \infty$ we have an estimate of the form*

$$\|u\|_{2,p;B_1(x_0)} \le C_1\|Lu\|_{p;B_1(x_0)}, \tag{2.48}$$

$\forall u \in (H^{2,p} \cap H_0^{1,p})(B_1(x_0))$.

Proof. Assume, (2.48) is wrong, then

$$\forall n \in \mathbb{N} \quad \exists u_n \in (H^{2,p} \cap H_0^{1,p})(B_1(x_0)) : \|u_n\|_{2,p;B_1(x_0)} > n\|Lu_n\|_{p;B_1(x_0)}.$$

Define $v_n := \frac{u_n}{\|u_n\|_{2,p;B_1(x_0)}}$, then

$$\|v_n\|_{2,p;B_1(x_0)} \equiv 1 \quad \text{and} \quad \|Lv_n\|_{p;B_1(x_0)} < \frac{1}{n}\|v_n\|_{2,p;B_1(x_0)} \longrightarrow 0.$$

In particular, (v_n) is bounded in $H^{2,p}(B_1(x_0))$ and since $(H^{2,p} \cap H_0^{1,p})(B_1(x_0))$ is weakly sequentially compact, we obtain for some subsequence,

$$v_m \longrightarrow v \in (H^{2,p} \cap H_0^{1,p})(B_1(x_0)) \quad \text{weakly.}$$

We now show that v cannot be 0. By Proposition 2.24, $1 = \|v_m\|_{2,p;B_1(x_0)} \le C(\sigma\|v_m\|_{p;B_1(x_0)} + \underbrace{\|Lv_m\|_{p;B_1(x_0)}}_{\longrightarrow 0})$.

Hence, $\liminf_m \|v_m\|_{p;B_1(x_0)} \ge (C\sigma)^{-1} > 0$. By Rellich-embedding, Proposition C.1, we have

$$v_m \longrightarrow v \quad \text{in} \quad L^p(B_1(x_0)) \quad \text{strongly.}$$

Thus, $\|v\|_{p;B_1(x_0)} = \lim_{m\longrightarrow\infty} \|v_m\|_{p;B_1(x_0)} > (c\sigma)^{-1} > 0$, which implies $v \neq 0$.
Since $\int_{B_1(x_0)} gD^\alpha v_m\, dx \longrightarrow \int_{B_1(x_0)} gD^\alpha v\, dx$ for any $|\alpha| \leq 2, g \in L^{p'}(B_1(x_0))$, we have

$$\int_{B_1(x_0)} gLv\, dx = \int_{B_1(x_0)} ga^{ij} D_{ij} v\, dx = \lim_{m\longrightarrow\infty} \int_{B_1(x_0)} gLv_m\, dx = 0$$

$\forall g \in L^{p'}(B_1(x_0))$, which implies $Lv = 0$.
If we can show that $v = 0$, we will arrive at a contradiction. In fact, since we have constant coefficients, L is of divergence form, too, and because of $p \geq 2$, $(H^{2,p} \cap H_0^{1,p})(B_1(x_0)) \subset H_0^{1,2}(B_1(x_0))$. Thus we can integrate by parts and obtain

$$0 = -\int_{B_1(x_0)} va^{ij} D_{ij} v\, dx = \int_{B_1(x_0)} a^{ij} D_i v D_j v\, dx.$$

Therefore, $Dv = 0$ and by Poincaré's lemma, $v = 0$. □

Second step:

Lemma 2.26 *Let $(a_{ij}), L, C_1, x_0, p$ be as in Lemma 2.25. Then we have an estimate of the form*

$$\|u\|_{2,p;B_r(x_0)} \leq C_1 \|Lu\|_{p;B_r(x_0)}$$

for any $r \leq 1, u \in (H^{2,p} \cap H_0^{1,p})(B_r(x_0))$.

Proof. Define $T_r : B_1(x_0) \longrightarrow B_r(x_0), x \longmapsto x_0 + r(x - x_0)$, then $|\det T_r| = r^n$. For $u \in (H^{2,p} \cap H_0^{1,p})(B_r(x_0))$, we have that

$$u_r := u \circ T_r \in (H^{2,p} \cap H_0^{1,p})(B_1(x_0))$$

and $D_i^x u_r(x) = rD_i^y u(y)$, $D_{ij}^x u_r(x) = r^2 D_{ij}^y u(y)$, where $y = T_r(x)$. In particular,

$$L^x u_r(x) = r^2 (L^y u)(y).$$

Hence, by integral transformation and Lemma 2.25,

$$\begin{aligned}
r^2 \|u\|_{2,p;B_r(x_0)} &= r^2 \sum_{|\alpha|\leq 2} \left(\int_{B_r(x_0)} |D^\alpha u(x)|^p\, dx \right)^{\frac{1}{p}} \\
&= r^2 \sum_{|\alpha|\leq 2} \left(\int_{B_1(x_0)} |D^\alpha u(T_r x)|^p r^n\, dx \right)^{\frac{1}{p}} \\
&= \sum_{|\alpha|\leq 2} r^{\frac{n}{p}+2} \left(\int_{B_1(x_0)} |r^{-|\alpha|} D^\alpha u_r(x)|^p\, dx \right)^{\frac{1}{p}} \\
&= \sum_{|\alpha|\leq 2} r^{\frac{n}{p}-|\alpha|+2} \|D^\alpha u_r\|_{p;B_1(x_0)} \\
&\leq r^{\frac{n}{p}} \|u_r\|_{2,p;B_1(x_0)} \qquad (r \leq 1)
\end{aligned}$$

$$
\begin{aligned}
&\le C_1 r^{\frac{n}{p}} \|Lu_r\|_{p;B_1(x_0)} \\
&= C_1 r^{\frac{n}{p}} r^{2-\frac{n}{p}} \|Lu\|_{p;B_r(x_0)} \\
&= C_1 r^2 \|Lu\|_{p;B_r(x_0)}.
\end{aligned}
$$

Divide both sides by r^2 to obtain the lemma. □

We now give the *proof of Proposition 2.6*:
For $x_0 \in \Omega$ choose $r_0 = r(x_0)$ small enough, so that

$$\max_{i,j} \|a_{ij}(x_0) - a_{ij}(\cdot)\|_{\infty,\overline{B_{r_0}(x_0)}} < \frac{1}{C_1},$$

where C_1 is the constant from Lemma 2.26. For $r \le r_0$ and $u \in (H^{2,p} \cap H_0^{1,p})(B_r(x_0))$ it follows from the same lemma that

$$
\begin{aligned}
&\|u\|_{2,p;B_r(x_0)} \\
&\le C_1 \|a^{ij}(x_0) D_{ij} u\|_{p;B_r(x_0)} \\
&\le C_1 \|Lu\|_{p;B_r(x_0)} + C_1 \|(a^{ij}(x_0) - a^{ij}(\cdot)) D_{ij} u\|_{p;B_r(x_0)} \\
&\le C_1 \|Lu\|_{p;B_r(x_0)} + C_1 \max_{i,j} \|a_{ij}(x_0) - a_{ij}(\cdot)\|_{\infty,\overline{B_r(x_0)}} \|u\|_{2,p;B_r(x_0)},
\end{aligned}
$$

and consequently,

$$\|u\|_{2,p;B_r(x_0)} \le \frac{C_1}{1 - C_1 \max_{i,j} \|a_{ij}(x_0) - a_{ij}(\cdot)\|_{\infty,\overline{B_{r_0}(x_0)}}} \|Lu\|_{p;B_r(x_0)}.$$

□

Combining Proposition 2.6 and (2.13), we are able to prove an existence result in L^p of the Dirichlet problem of type (2.13) with a weaker assumption on a^{ij}, which shall be used in Theorem 3.52.

Proposition 2.27 *Let L be strictly elliptic in $B := B_R$ with $a^{ij} \in C(\bar{B}), L = a^{ij}(x) D_i D_j$. Then the Dirichlet problem* (2.13) *has a solution $u \in H^{2,p}(B), p \in [1, \infty)$, if $f \in C_0^\infty(B)$.*

Proof. Approximate a^{ij} by $(a_m^{ij})_m \subset C^1(\bar{B})$ with respect to $\| \;\; \|_\infty$ (e. g., polynoms over $\bar{B}$ are dense in $C(\bar{B})$). Let $f \in C_0^\infty(B)$ and $m_0 \in \mathbb{N}$ such that a_m^{ij} satisfy the ellipticity condition for any $m \ge m_0$. We know that for any $m \ge m_0$ there exists $u_m \in C^{2,\alpha}(\bar{B}) \subset H^{2,p}(B)$ with

$$
\begin{aligned}
L_m u_m &= f \quad \text{in} \quad B, \\
u_m &= 0 \quad \text{on} \quad \partial B,
\end{aligned}
$$

in particular, $u_m \in (H^{2,p} \cap H_0^{1,p})(B)$ by C.11. By 2.6, we have

$$\|u_m\|_{2,p} \le C \|L_m u_m\|_p = C \|f\|_p$$

for $m \ge m_0$. Thus, (u_m) is bounded in $H^{2,p}$, which implies that there exists a subsequence (u_{m_l}) converging weakly in $(H^{2,p} \cap H_0^{1,p})(B)$ to some $u \in (H^{2,p} \cap$

$H_0^{1,p})(B)$. Let us denote this subsequence again by (u_m). We assert that $Lu = f$: in fact, for $g \in L^{p'}, |\alpha| \leq 2$ we have

$$\int gD^\alpha u_m\,dx \longrightarrow \int gD^\alpha u\,dx.$$

Moreover, for $\xi \in C_0^\infty(B), m, n \in \mathbb{N}$,

$$\begin{aligned}\int \xi f\,dx &= \int \xi a_m^{ij} D_{ij} u_m\,dx \\ &= \underbrace{\int \xi(a_m^{ij} - a_n^{ij}) D_{ij} u_m\,dx}_{I_{m,n}} + \underbrace{\int \xi a_n^{ij} D_{ij} u_m\,dx}_{J_{m,n}}.\end{aligned}$$

Now, $\lim_m \lim_n I_{m,n} = 0$ and since $a_n^{ij}\xi, a^{ij} \in L^{p'}$,

$$\lim_{m,n} J_{m,n} = \int \xi a^{ij} D_{ij} u\,dx.$$

This implies $Lu = f$ a. e.. □

2.3 The Existence Theorem

It is time to present the main theorem of this chapter, although we can only prove one half of it at the moment. The missing point in the proof of part (i) is worked out in the next section.

Suppose that A is a Borel measurable mapping on a domain $\Omega \subset \mathbb{R}^n$ with values in the space of nonnegative symmetric matrices in $\mathbb{R}^n$.

Theorem 2.28 *Let μ be a locally finite signed Borel measure on Ω, such that $a^{ij} \in L^1_{loc}(\mu)$, and for some $C > 0$, one has*

$$\int_\Omega a^{ij} D_i D_j \varphi\,d\mu \leq C(\sup_\Omega |\varphi| + \sup_\Omega |\nabla\varphi|) \tag{2.49}$$

for any nonnegative $\varphi \in C_0^\infty(\Omega)$. Then

(i) *If μ is nonnegative, then $(\det A)^{\frac{1}{n}}\mu$ has a density, which belongs to $L^{n'}_{loc}(\Omega, dx)$;*

(ii) *If A is locally Hölder continuous and nondegenerate, then μ has a density, which belongs to $L^r_{loc}(\Omega, dx)$ for any $r \in [1, n')$.*

Proof.(ii):Let $B_0 \subset\subset \Omega$ be a ball and $\zeta \in C_0^\infty(\Omega)$, such that $0 \leq \zeta \leq 1, \zeta \equiv 1$ in B_0 with support in another ball $B \subset \Omega$. Consider the signed measure $\nu := \zeta\mu$. We want to generalize (2.49) to $C^2(\bar{B})$-functions. Replacing φ in (2.49) by $\zeta\psi$ for some nonnegative function $\psi \in C^\infty(\Omega)$, we obtain

$$\int_B a^{ij} D_i D_j \psi\,d\nu \leq C_1(\sup_B |\psi| + \sup_B |\nabla\psi|) \tag{2.50}$$

by the computation below, where

$$C_1 = C + (C + 2n^2 \sup_{i,j} \|a^{ij}\|_{L^1(B,\mu)}) \sup_B |\nabla\zeta| + |a^{ij}\|_{L^1(B,\mu)} \sup_B |D_iD_j\zeta| \quad (2.51)$$

is independent from ψ:

$$\begin{aligned}
\int_B a^{ij} D_iD_j\psi \, d\nu &= \int_B a^{ij}(D_iD_j\psi)\zeta \, d\mu \\
&= \int_B a^{ij}(D_iD_j(\zeta\psi) - D_j\psi D_i\zeta - D_j\zeta D_i\psi - \psi D_iD_j\zeta) \, d\mu \\
&\leq C(\sup_B |\zeta\psi| + \sup_B |\nabla(\zeta\psi)|) + \sup_B |D_j\psi| \sup_B |D_i\zeta| \|a^{ij}\|_{L^1(B,\mu)} + \\
&\quad \sup_B |D_j\zeta| \sup_B |D_i\psi| \|a^{ij}\|_{L^1(B,\mu)} + \sup_B |\psi| \sup_B |D_iD_j\zeta| \|a^{ij}\|_{L^1(B,\mu)} \\
&\quad (\text{product rule,supp } \zeta \subset B) \\
&\leq C(\sup_B |\zeta\psi| + \sup_B |\zeta\nabla\psi + \psi\nabla\zeta|) + \\
&\quad 2n^2 \sup_B |\nabla\psi| \sup_B |\nabla\zeta| \sup_{i,j} \|a^{ij}\|_{L^1(B,\mu)} + \\
&\quad \sup_B |D_iD_j\zeta| \|a^{ij}\|_{L^1(B,\mu)} (\sup_B |\psi| + \sup_B |\nabla\psi|) \\
&\leq C(\sup_B |\psi| + \sup_B |\nabla\psi|) + C(\sup_B |\psi| + \sup_B |\nabla\psi|) \sup_B |\nabla\zeta| + \\
&\quad 2n^2 \sup_{ij} \|a^{ij}\|_{L^1(B,\mu)} \sup_B |\nabla\zeta| (\sup_B |\psi| + \sup_B |\nabla\psi|) + \\
&\quad \sup_B |D_iD_j\zeta| \|a^{ij}\|_{L^1(B,\mu)} (\sup_B |\psi| + \sup_B |\nabla\psi|) \quad (0 \leq \zeta \leq 1) \\
&= C_1(\sup_B |\psi| + \sup_B |\nabla\psi|)
\end{aligned}$$

We note that (2.50) remains valid for any nonnegative $\psi \in C^2(\bar{B})$, since for any mollifier ξ, $\|\xi_\epsilon * h - h\|_{\infty,\bar{B}} \longrightarrow 0$, as ϵ tends to zero for any $h \in C(\mathbb{R}^n)$ and $D^\alpha(\psi * \xi_\epsilon) = (D^\alpha\psi) * \xi_\epsilon$.
Let $\psi \in C^2(\bar{B})$, then $0 \leq \psi + \sup_B |\psi| \in C^2(\bar{B})$. Thus,

$$\begin{aligned}
\left| \int_B a^{ij} D_iD_j\psi \, d\nu \right| &= \left| \int_B a^{ij} D_iD_j(\psi + \sup_B |\psi|) \, d\nu \right| \\
&\leq C_1(\sup_B |\psi + \sup_B |\psi|| + \sup_B |\nabla\psi|) \\
&\leq 2C_1(\sup_B |\psi| + \sup_B |\nabla\psi|) \quad (2.52)
\end{aligned}$$

Now let $x_0 \in \Omega, p > n$. By Proposition 2.6 there exists $r(x_0) > 0$, such that

$$\|u\|_{2,p;B_r(x_0)} \leq C_2 \|Lu\|_{p;B_r(x_0)},$$

for any $u \in (W^{2,p} \cap W_0^{1,p})(B_r(x_0)), r \leq r(x_0)$, where $L := a^{ij}(x)D_iD_j$ is strictly elliptic in $B_r(x_0)$, because A is assumed to be nondegenerate. Consider $B^{x_0} := B_r(x_0)$, where $r \leq r(x_0)$ such that $\bar{B}^{x_0} \subset \Omega$. Fix $f \in C_0^\infty(B^{x_0})$ and $\zeta^{x_0} \in C_0^\infty(\Omega), B_0^{x_0}$, such that $(B_0^{x_0}, \zeta^{x_0}, B^{x_0})$ corresponds to (B_0, ζ, B) from above.

By Section 2.1, there exists $u \in C^{2,\alpha}(\bar{B}^{x_0})$, such that $Lu = f$ in B^{x_0} and $u = 0$ on ∂B^{x_0}. Moreover, $u \in (W^{2,p} \cap W_0^{1,p})(B^{x_0})$ (cf. [3, A 5.11]). By Sobolev embedding we obtain

$$\sup_{B^{x_0}} |\nabla u| + \sup_{B^{x_0}} |u| \leq C_2' \|u\|_{2,p;B^{x_0}} \leq C_3 \|f\|_{p;B^{x_0}}.$$

Together with (2.52) this yields

$$\int_{B^{x_0}} f \, d\nu^{x_0} \leq 2C_1C_3 \|f\|_{p;B^{x_0}} \quad \forall f \in C_0^\infty(B^{x_0}). \tag{2.53}$$

Hence ν^{x_0} is absolutely continuous with $\nu = g^{x_0} \, dx, g^{x_0} \in L^{p'}(B^{x_0})$.
In fact, define

$$l : C_0^\infty(B^{x_0}, dx) \longrightarrow \mathbb{R}, l(\cdot) := \int_{B^{x_0}} \cdot \, d\nu^{x_0}.$$

Because of (2.53), l can be extended uniquely to a continuous linear functional $\bar{l}$ on $L^p(B^{x_0}, dx)$. Consequently, by duality, there exists $g^{x_0} \in L^{p'}(B^{x_0})$, such that $\bar{l}(\cdot) = \int_{B^{x_0}} g^{x_0} \cdot dx$. In particular,

$$\int_{B^{x_0}} f \, d\nu^{x_0} = \int_{B^{x_0}} f g^{x_0} \, dx \quad \forall f \in C_0^\infty(B^{x_0}).$$

It follows that $\nu^{x_0} = g^{x_0} \, dx$ on $\mathcal{B}(B^{x_0})$, since for any $f \in C_0^\infty(B^{x_0})$

$$\begin{aligned} \int_{B^{x_0}} f \, d\nu^{x_0+} - \int_{B^{x_0}} f \, d\nu^{x_0-} &= \int_{B^{x_0}} f g^{x_0+} \, dx - \int_{B^{x_0}} f g^{x_0-} \, dx \\ \Rightarrow \int_{B^{x_0}} f \, d\nu^{x_0+} + \int_{B^{x_0}} f g^{x_0-} \, dx &= \int_{B^{x_0}} f g^{x_0+} \, dx + \int_{B^{x_0}} f \, d\nu^{x_0-} \\ \Leftrightarrow \int_{B^{x_0}} f(x)(\nu^{x_0+} + g^{x_0-}\lambda)(dx) &= \int_{B^{x_0}} f(x)(g^{x_0+}\lambda + \nu^{x_0-})(dx) (\lambda := \text{Leb.}), \end{aligned}$$

which implies that the positive measures $\nu^{x_0+} + g^{x_0-} \, dx, g^{x_0+} \, dx + \nu^{x_0-}$ coincide on $\mathcal{B}(B^{x_0})$, since $\sigma(C_0^\infty(B^{x_0})) = \mathcal{B}(B^{x_0})$ (for let $U \in \mathcal{B}(B^{x_0})$ be open, $x_0 \in B^{x_0}, r > 0$, such that $B_r(x_0) \subset U$. Choose $f \in C_0^\infty(B^{x_0})$ such that $B_r(x_0) = \{f > 0\} \in \sigma(C_0^\infty(B^{x_0}))$. Now, $U = \bigcup_{x_0 \in \mathbb{Q}^n \cap U} B_{r(x_0)}(x_0) \in \sigma(C_0^\infty(B^{x_0})))$.

Because of $\nu^{x_0} = \zeta^{x_0}\mu$, we have $\mu = g^{x_0} \, dx$ on $\mathcal{B}(B_0^{x_0})$, where $g^{x_0} \in L^{p'}(B_0^{x_0})$. Now let $N \in \mathcal{B}(\Omega), dx(N) = 0$. Moreover, let $K_m \subset \Omega, m \in \mathbb{N}$, be compact sets, such that $\Omega = \bigcup_{m \in \mathbb{N}} K_m, K_m \subset K_{m+1}$ and such that for any $K \subset \Omega$, K compact, we have that $K \subset K_{m_0}$ for some m_0. Then for any m: $K_m \subset \bigcup_{x_0 \in K_m} B_0^{x_0}$ Thus $\exists x_0^1, \dots, x_0^n \in K_m : K_m \subset \bigcup_{i=1}^n B_0^{x_0^i}$. Consequently,

$$|\mu(K_m \cap N)| \leq \sum_{i=1}^n |\mu_{|B_0^{x_0^i}}|(N) = \sum_{i=1}^n |g^{x_0^i} \, dx(N \cap B_0^{x_0^i})| = 0.$$

Therefore,

$$0 \leq |\mu(N)| = |\mu(\bigcup_{m \in \mathbb{N}} K_m \cap N)| \leq \sum_{m \in \mathbb{N}} |\mu(K_m \cap N)| = 0.$$

By uniqueness of the density, we conclude

$$\mu << dx \quad \text{with density in} \quad L^{p'}_{loc}(\Omega, dx) \quad \forall p > n.$$

(i): We start with a remark; the above reasoning does not work in this case even for bounded uniformly nondegenerate A, since the equation $Lu = f$ does not have to be solvable. For continuous A, the solution of this equation only is in $W^{2,p}$ and not in C^2. Therefore, one cannot pass from C_0^∞- functions to u in (2.52).

As above, by considering a suitable function ζ, we arrive at estimate (2.52) for the measure $\nu = \zeta\mu$ on the ball $B_{R_0}(x_0)$. Note that the support of the measure ν is contained in a ball $B_R(x_0)$ with radius $R = R_0 - r$, where $r > 0$ (for orientation: $B \cong B_{R_0}(x_0), B_0 \subset \operatorname{supp} \zeta \subset B_R(x_0)$, such that $\zeta \equiv 1$ in B_0). In this case, instead of solving the elliptic equation, we shall employ a result from [22] according to which, for any nonnegative continuous function f on $\mathbb{R}^n$ vanishing outside $\overline{B_R(x_0)}$ there exists a nonnegative continuous concave function z on $B_{R_0}(x_0)$ with the following properties:

$$-\alpha^{ij} D_i D_j z \geq |\det(\alpha^{ij})|^{\frac{1}{n}} f \tag{2.54}$$

in the sense of distributions on $B_{R_0}(x_0)$ for any nonnegative symmetric matrix (α^{ij}) and

$$\sup_{B_{R_0}(x_0)} z \leq N\|f\|_{n;B_{R_0}(x_0)}, \tag{2.55}$$

where N does not depend on f.
The next section is dedicated to this result. But first let us go on. Let g be a mollifier, g_ϵ its Dirac sequence and $v_\epsilon := v * g_\epsilon$ for suitable functions v. Then for any nonnegative, symmetric matrix (α^{ij}) and $\epsilon \in (0, r)$, one has the estimates

$$\begin{aligned} -\alpha^{ij} D_i D_j z_\epsilon(x) &\geq |\det(\alpha^{ij})|^{\frac{1}{n}} f_\epsilon(x) \quad \text{on} \quad B_R(x_0), &(2.56)\\ \sup_{B_R(x_0)} |z_\epsilon| &\leq N\|f\|_{n;B_{R_0}(x_0)}, &(2.57) \end{aligned}$$

where N does not depend on $f, (\alpha^{ij})$ and ϵ:

$$\begin{aligned} -\alpha^{ij} D_i D_j z_\epsilon(x) &= -\alpha^{ij} D_i D_j \int g_\epsilon(x-y) z(y)\, dy \\ &\geq |\det(\alpha^{ij})|^{\frac{1}{n}} \int g_\epsilon(x-y) f(y)\, dy \\ &= |\det(\alpha^{ij})|^{\frac{1}{n}} f_\epsilon(x). \end{aligned}$$

Clearly, the functions z_ϵ are smooth and nonnegative on $B_{R+\frac{r}{2}}(x_0)$, if $\epsilon < \frac{r}{2}$. For these ϵ, z_ϵ is concave on $B_{R+\frac{r}{2}}(x_0)$, too:

$$\begin{aligned} &z_\epsilon(\lambda x + (1-\lambda)x') \\ &\quad = \int g_\epsilon(\lambda x + (1-\lambda)x' - y) z(y)\, dy \end{aligned}$$

$$\begin{aligned}
&= \int g_\epsilon(y) z(\lambda(x-y) + (1-\lambda)(x'-y))\,dy \\
&\geq \int g_\epsilon(y)\lambda z(x-y)\,dy + \int g_\epsilon(y)(1-\lambda) z(x'-y)\,dy \\
&= \lambda z_\epsilon(x) + (1-\lambda) z_\epsilon(x').
\end{aligned}$$

We observe, that for any nonnegative continuously differentiable concave function w on $B_{R+\frac{r}{2}}(x_0)$, one has

$$|\nabla w(x)| \leq \frac{2}{r}\sqrt{n} \sup_{y\in B_R(x_0)} w(y) \quad \forall x \in B_R(x_0).$$

Indeed, consider the case $n = 1, x_0 = 0$:
Since $w \geq 0$ and concave, we have $\forall x \in B_R$

$$\begin{aligned}
\frac{\sup_{B_R} w(y)}{\frac{r}{2}} \geq \frac{w(-R)}{\frac{r}{2}} \geq w'(-R) \quad &\geq \quad w'(x) \\
\geq w'(R) \quad &\geq \quad -\frac{w(R)}{\frac{r}{2}} \geq -\frac{\sup_{B_R} w(y)}{\frac{r}{2}}.
\end{aligned}$$

Case $n \in \mathbb{N}$:

$$|\nabla w(x)| = \sqrt{(D_1 w(x))^2 + \cdots + (D_n w(x))^2} \leq \sqrt{n}\frac{2}{r} \sup_{B_R(x_0)} w(y). \tag{2.58}$$

Thus, using that ν is nonnegative, we obtain

$$\begin{aligned}
&\int |\det(a^{ij}(x))|^{\frac{1}{n}} f_\epsilon(x)\,\nu(dx) \\
&\leq \left|\int a^{ij}(x) D_i D_j z_\epsilon(x)\,\nu(dx)\right| && \text{(by (2.56))} \\
&\leq C_1 \sup_{B_R(x_0)} (|\nabla z_\epsilon| + |z_\epsilon|) && \text{(by (2.52))} \\
&\leq C_1 N (1 + \frac{2\sqrt{n}}{r}) \|f\|_{n;B_{R_0}(x_0)} && \text{(by (2.57),(2.58))}
\end{aligned}$$

for any $\epsilon \in (0, \frac{r}{2})$. By Lebesgue, for $\epsilon \downarrow 0$ we obtain that

$$\int |\det A|^{\frac{1}{n}} f\,d\nu \leq C_1 N (1 + \frac{2\sqrt{n}}{r}) \|f\|_{n;B_{R_0}(x_0)}.$$

As in case (ii), we complete the proof. □

Corollary 2.29 *Let μ be a locally finite signed Borel measure on Ω and let $a^{ij}, b^i, c \in L^1_{loc}(\Omega, \mu)$. Assume that*

$$\int_\Omega (L_{A,b}\varphi + c\varphi)\,d\mu \leq 0 \tag{2.59}$$

for any nonnegative $\varphi \in C_0^\infty(\Omega)$. Then

(i) *If μ is nonnegative, then $(\det A)^{\frac{1}{n}}\mu$ has a density, which belongs to $L^{n'}_{loc}(\Omega, dx)$;*

(ii) *If A is locally Hölder continuous and nondegenerate, then μ has a density, which belongs to $L^r_{loc}(\Omega, dx)$ for any $r \in [1, n')$.*

In particular, the above statements are true, if

$$\int_\Omega L_{A,b}\varphi \, d\mu = 0 \quad \forall \varphi \in C_0^\infty(\Omega).$$

Proof. Note that for any bounded open $\Omega_0 \subset \bar{\Omega}_0 \subset \Omega$, one has

$$\int_{\Omega_0} (b^i D_i \varphi) \, d\mu \le \sup_{\Omega_0} |\nabla \varphi| \int_{\Omega_0} |b| d|\mu| + \sup_{\Omega_0} |\varphi| \int_{\Omega_0} |c| d|\mu|,$$

for any smooth function φ with support in Ω_0. Now starting with B_0, B, ζ, ψ as in the proof of the Theorem, we obtain (2.50) with $C := \int_B |b| \, d|\mu| + \int_B |c| \, d|\mu|$, since

$$\int_d a^{ij} D_i D_j (\zeta\psi) \, d\mu \le C(\sup_B |\zeta\psi| + \sup_B |\nabla(\zeta\psi)|)$$

by (2.59). From here on, we can copy the rest of the proof of the Theorem. The in particular assertion follows by choosing $c = 0$. □

Corollary 2.30 *Let μ and ν be two locally finite signed Borel measures on Ω and let $a^{ij}, b^i, c \in L^1_{loc}(\Omega, \mu)$. Assume that*

$$\int_\Omega L_{A,b}\varphi + c\varphi \, d\mu = \int_\Omega \varphi \, d\nu \tag{2.60}$$

for any nonnegative $\varphi \in C_0^\infty(\Omega)$. Then

(i) *If μ is nonnegative, then $(\det A)^{\frac{1}{n}}\mu$ has a density, which belongs to $L^{n'}_{loc}(\Omega, dx)$;*

(ii) *If A is locally Hölder continuous and nondegenerate, then μ has a density, which belongs to $L^r_{loc}(\Omega, dx)$ for any $r \in [1, n')$.*

Proof. Starting with B_0, B, ζ, ψ as in the proof of the Theorem, we obtain (2.50) with $C := 2\max(\int_B |b| \, d\mu|, \int_B |c| \, d|\mu|, |\nu|(B))$, since

$$\begin{aligned}
\int_B a^{ij} D_i D_j(\zeta\psi) \, d\mu &= -\int_B b^i D_i(\zeta\psi) \, d\mu - \int_B c\zeta\psi \, d\mu + \int_B \zeta\psi \, d\nu \\
&\le \sup_B |\nabla(\zeta\psi)| \int_B |b| \, d|\mu| + \sup_B |\zeta\psi| \int_B |c| \, d|\mu| + \sup_B |\zeta\psi||\nu|(B) \\
&\le C(\sup_B |\zeta\psi| + \sup_B |\nabla(\zeta\psi)|)
\end{aligned}$$

by (2.60). From here on, we can copy the rest of the proof of the Theorem . □

2.4 Convex Polyhedra

Now we want to work out the missing point in the proof of Theorem 2.28, which can be found in [22], i. e. we want to prove the following

Proposition 2.31 *Let $\Omega \subset \mathbb{R}^n$ be a domain, $x_0 \in \Omega$ and $R_0, r > 0$, such that $\overline{B_{R_0}(x_0)} \subset\subset \Omega$ and $R := R_0 - r > 0$. Let $f \in C(\mathbb{R}^n)$ be nonnegative and $f \equiv 0$ outside $\overline{B_R(x_0)}$. Then there exists a nonnegative continuous concave function z on $\overline{B_R(x_0)}$ with the following properties:*

(i) $-\alpha^{ij} D_i D_j z \geq |\det(\alpha^{ij})|^{\frac{1}{n}} f$ *in the sense of distributions on $B_{R_0}(x_0)$ for any nonnegative symmetric matrix (α^{ij}),*

(ii) $\sup\limits_{B_{R_0}(x_0)} z \leq N \|f\|_{n, B_{R_0}(x_0)}$,

where N does not depend on f.

At first, we have to get familiar with convex polyhedras and polytopes. Properties of those, that are not explicitely proven here, can be found and well understood in e. g. [25].

Definition 2.32 *A convex polyhedron is the intersection of a finite number of closed half-spaces.*

An important role in the following will be played by functions $z : \mathbb{R}^n \longrightarrow \mathbb{R}$, such that $\{(x, z) \in \mathbb{R}^{n+1} | x \in \mathbb{R}^n, z \in (-\infty, z(x)]\}$ defines a convex polyhedron.

Remark 2.33 *All polyhedra in the following will be convex and this fact will not be specifically mentioned. Also note that the function z will always be concave, although it describes a convex polyhedron! We shall identify these functions with their corresponding polyhedra by saying the "polyhedron $z(x)$" instead of the "polyhedron $\{(x, z) \in \mathbb{R}^{n+1} | x \in \mathbb{R}^n, z \in (-\infty, z(x)]\}$".*

The plane $z = < p, x - x_0 > + z(x_0)$ in $\mathbb{R}^{n+1}$ will be termed a *plane of support* of the polyhedron $z(x)$ at the point $(x_0, z(x_0))$, if $< p, (x - x_0) > + z(x_0) \geq z(x)$ for any $x \in \mathbb{R}^n$.

Definition 2.34 *Let H be a supporting plane of the polyhedron $z(x)$. Then $H \cap z(x)$ is called a face of $z(x)$. A 0-dimensional face is called vertex, an $(n-1)$-dimensional face is called facet of $z(x)$.*

Suppose $P_z(x_0) = \{p \in \mathbb{R}^n | < p, (x - x_0) > + z(x_0) \geq z(x) \forall x\}$. Then $P_z(x)$ is a closed convex set in $\mathbb{R}^n$ for each x:
convexity:

$$\begin{aligned} &< \alpha p_1 + (1 - \alpha) p_1, x - x_0 > + z(x_0) \\ &\quad = < \alpha p_1, x - x_0 > + < (1 - \alpha) p_1, x - x_0 > + z(x_0) \\ &\quad \geq \alpha (z(x) - z(x_0)) + (1 - \alpha)(z(x) - z(x_0)) + z(x_0) \end{aligned}$$

$$= z(x)$$

closedness: Let $(p_n) \subset P_z(x_0), p_n \longrightarrow p$ in $\mathbb{R}^n$, then

$$< p, x - x_0 >=< \lim_{n \longrightarrow \infty} p_n, x - x_0 >\geq z(x) - z(x_0)$$

$P_z(x)$ will be called normal representation of the point x with help of the function z. Proceeding from the case $n = 1$ and noting that $P_u(x) = \{Du(x)\}$, if e. g. $u \in C^1$ is concave (since any supporting hyperplane must then be a tangent hyperplane to the graph of u), we heuristically may speak of $P_z(x)$ being the set of slopes of supporting hyperplanes at x lying above the graph of z. $P_z(x)$ is bounded for each x, since otherwise $P_z(x_0)$ would contain the ray $tp_0 + p_1, t \geq t_0$, entirely, for some x_0. We would conclude that

$$< tp_0, x - x_0 > + < p_1, x - x_0 > + z(x_0) \geq z(x)$$

for all x. However, if we take x so that $< p, x - x_0 >< 0$, then $z(x) = -\infty$ as $t \longrightarrow \infty$, which is impossible. Hence, the n-dimensional Lebesgue-measure $N_z(x)$ of the set $P_z(x)$ is finite for any x.

Let us prove that if $N_z(x_0) \neq 0$, then $(x_0, z(x_0))$ is an extreme point, i. e. a vertex of the polyhedron $z(x)$: the equations $x_0 = \frac{1}{2}(x_1 + x_2)$ and $z(x_0) = \frac{1}{2}(z(x_1) + z(x_2))$ imply that $x_0 = x_1 = x_2$. Indeed, $x_2 - x_0 = -(x_1 - x_0)$ and

$$\begin{aligned} z(x_0) & \\ &= \frac{1}{2}(z(x_1) + z(x_2)) \\ &\leq \frac{1}{2}\{ \min_{p \in P_z(x_0)} [< p, x_1 - x_0) > + z(x_0)] + \min_{p \in P_z(x_0)} [< p, x_2 - x_0 > + z(x_0)]\}. \end{aligned}$$

Hence, it follows that

$$\begin{aligned} 0 &\leq \min_{p \in P_z(x_0)} [< p, x_1 - x_0 >] + \min_{p \in P_z(x_0)} [- < p, x_1 - x_0 >] \\ &= \min_{p \in P_z(x_0)} [< p, x_1 - x_0 >] - \max_{p \in P_z(x_0)} [< p, x_1 - x_0 >] \\ &\leq \min_{p \in P_z(x_0)} [< p, x_1 - x_0 >] - \min_{p \in P_z(x_0)} [< p, x_1 - x_0 >] \leq 0. \end{aligned}$$

Assume, $\exists p_1, p_2 \in P_z(x_0)$ such that $< p_1, x_1 - x_0 > \quad > \quad < p_2, x_1 - x_0 >$. Then

$$\begin{aligned} 0 &> < p_2, x_1 - x_0 > + < -p_1, x_1 - x_0 > \\ &\geq \min_{p_2 \in P_z(x_0)} [< p_2, x_1 - x_0 >] + \min_{p_1 \in P_z(x_0)} [< -p_1, x_1 - x_0 >] \geq 0, \end{aligned}$$

which forms a contradiction. Therefore, $< p_1 - p_2, x_1 - x_0 >= 0$ for any $p_1, p_2 \in P_z(x_0)$. But since $N_z(x_0) \neq 0, P_z(x_0)$ has interior points and thus the last relation is possible only if $x_1 = x_0$.

The vertices of the polyhedron constitute a finite set (for a vertex is point of intersection of at least $n + 1$ halfspaces. Let $z(x) = \bigcap_{i=1}^m H_i$, then we have

$$\#\text{vert}(z(x)) \leq \binom{m}{n+1} + \cdots + \binom{m}{m} < \infty).$$

Therefore, $N_z(x)$ can be nonzero only at a finite number of points, say, $x_1, \ldots, x_n$. Set now, for any set Γ, $N_z(\Gamma) := \sum_{x_i \in \Gamma} N_z(x_i)$. This set function plays a basic role in our investigation. In the theory of convex surfaces, it is called the volume of the normal representation of the set Γ with the help of the function z.

Let $z(x)$ be a convex polyhedron, $\frac{d}{dn_r}$ the inward normal derivative on the boundary ∂B_r and dS_r the area element of ∂B_r (cf. C.7). For some nonsingular matrix σ and $t > 0$ define

$$\begin{aligned} T_t^\sigma f(x) &:= (2\pi t)^{-\frac{n}{2}} [\det \sigma]^{-1} \int_{\mathbb{R}^n} f(y) \exp\left[-\frac{|\sigma^{-1}(x-y)|^2}{2t}\right] dy, \\ e(s) &:= s^{-n+1} \int_s^\infty r^{n-1} \exp\left[-r^2\right] dr, \quad \text{for} \quad s > 0. \end{aligned}$$

Note that $(T_t^{E_n})_{t>0}$ coincides with the Brownian semigroup. To be able to construct the ordered concave function, which will complete the proof of Theorem 2.28, we need a handful lemmas.

Lemma 2.35 *There exists a positive constant ϵ just depending on the dimension n such that, for any positive r,*

$$\int_{\partial B_r} \frac{d}{dn_r} z(y)\, dS_r \geq \epsilon \sqrt[n]{N_z(B_{\frac{r}{2}})} r^{n-1}.$$

*Proof.*Let us first treat the case $r = 4$. We therefore have to show that

$$\int_{\partial B_4} \frac{d}{dn_4} z(y)\, dS_4 \geq \epsilon \sqrt[n]{N_z(B_2)}. \tag{2.61}$$

Further, if instead of $z(x)$ we consider the function $z(x) + < p', x > +b$, then neither side of inequality (2.61) will be effected:

$$|\{p| < p, x - x_i > +z(x_i) \geq z(x) \forall x\}| = |\{< p - p', x - x_i > +z(x_i) \geq z(x) \forall x\}|,$$

because of translation invariance of the Lebesgue measure; use the divergence theorem for the left-hand side.
Therefore to prove (2.61), we may assume that

$$z(0) = 0 \quad \text{and} \quad z(x) \leq 0 \quad \forall x$$

(choose a plane of support $s(x) =< p, x > +z(0)$ of $z(x)$ at the point $(0, z(0))$ and consider $z'(x) := z(x) - s(x)$). In addition, if $z =< p, x - x_0 > +z(x_0)$ is a plane of support and $x_0 \in B_2$, then $z(x_0) - z(x) \geq< p, x_0 - x >$ and

$$[z] := \sup_{x_1, x_2 \in B_3} \frac{|z(x_1) - z(x_2)|}{x_1 - x_2} \geq \sup_{|x - x_0| = 1} | < p, x - x_0 > | = |p|.$$

Hence, $P_z(x) \subset C_{[z]}$ for $x \in B_2$ and

$$N_z(B_2) = \sum_{\substack{x \in B_2 \\ x \text{ vertex}}} |P_z(x)| = \left| \bigcup_{\substack{x \in B_2 \\ x \text{ vertex}}} P_z(x) \right| \leq \omega_n [z]^n,$$

where we used that $P_z(x_0)$ and $P_z(y_0)$ have not got any interior points in common for $x_0 \neq y_0$. Indeed, let $p \in P_z(x_0)$, such that $< p, x - x_0 > +z(x_0) > z(x) \quad \forall x \in \mathbb{R}^n \backslash \{x_0\}$, in particular $< p, y_0 - x_0 > +z(x_0) > z(y_0)$. It follows that $< p, x_0 - y_0 > +z(y_0) < z(x_0)$, i. e. $p \notin P_z(y_0)$.
Therefore, (2.61) will be proved, if we can show that

$$\epsilon[z] \leq \int_{\partial B_4} \frac{d}{dn_4} z(x)\, dS_4 \tag{2.62}$$

for some positive constant ϵ only depending on n, and for any polyhedra $z(x)$, such that $z(0) = 0, z(x) \leq 0$ for any x.
Suppose that such a constant does not exist. Then one can find a sequence of polyhedra $z_k(x)$, such that

$$[z_k] = 1, \int_{\partial B_4} \frac{d}{dn_4} z_k(x)\, dS_4 \leq \frac{1}{k}, z_k(x) \leq 0, z_k(0) = 0.$$

If for fixed $x \in \partial B_4$ we regard $z_k(tx)$ as a function of the single variable $t \in [0, 1]$, we establish that

$$0 \leq z_k(tx) - z_k(x) \leq 4(1-t) \frac{dz_k}{dn_4}(x),$$

since

$$\begin{aligned} \frac{dz_k}{dn_4}(x) &= \lim_{h \longrightarrow 0} \frac{z_k(x + hn_4) - z_k(x)}{h} = \frac{1}{4} \lim_{t \uparrow 1} \frac{z_k(tx) - z_k(x)}{1-t} \\ &= -\frac{1}{4} \lim_{t \uparrow 1} \frac{\tilde{z}_k(1) - \tilde{z}_k(t)}{1-t} \geq \frac{1}{4} \frac{\tilde{z}_k(t) - \tilde{z}_k(1)}{1-t} \end{aligned}$$

by concavity. Hence, it follows that

$$0 \leq \int_{\partial B_4} z_k(tx)\, dS_4 - \int_{\partial B_4} z_k(x)\, dS_4 \leq 4 \frac{1-t}{k}. \tag{2.63}$$

The relations $[z_k] = 1$ and $z_k(0) = 0$ allow us to apply the Arzela-Ascoli theorem and therefore we obtain uniformly convergence towards some concave function z on $\bar{B}_3$ for a subsequence of (z_k), again denoted by (z_k). Taking $t = \frac{1}{2}$ in (2.63), we see that $\lim\limits_{k \longrightarrow \infty} \int_{\partial B_4} z_k(x)\, dS_4$ exists:
$t < \frac{3}{4} \Rightarrow \int_{\partial B_4} z_k(\underbrace{tx}_{\in B_3})\, dS_4$ converges to $\int_{\partial B_4} z(tx)\, dS_4$, because of uniform convergence. Therefore, by (2.63), we see that $\lim\limits_{k \longrightarrow \infty} \int_{\partial B_4} z_k(x)\, dS_4$ exists and

$$\lim_{k \longrightarrow \infty} \int_{\partial B_4} z_k(tx)\, dS_4 \left(= \lim_{k \longrightarrow \infty} \int_{\partial B_4} z_k(x)\, dS_4 \forall t \in \left(0, \frac{3}{4}\right)\right)$$

is independent from t for $t \in (0, \frac{1}{2})$. Now

$$\lim_{k \longrightarrow \infty} \int_{\partial B_4} z_k(tx)\, dS_4 = \int_{\partial B_4} z(tx)\, dS_4$$

$$= \lim_{t\downarrow 0}\int_{\partial B_4} z(tx)\,dS_4 = n\omega_n 4^{n-1}z(0) = 0,$$

thus

$$\lim_{k\longrightarrow\infty}\int_{\partial B_4} z_k(x)\,dS_4 = 0. \tag{2.64}$$

Again consider the condition $[z_k] = 1$. Let

$$z_k(x) = \min\{< p_j^k, x > +b_j^k | j = 1,\ldots,r_k\}.$$

We shall assume that any plane $z =< p_j^k, x > +b_j^k$ has at least one point in common with $z = z_k(x)$ for $x \in B_3$,if $j = 1,\ldots,s_k$ and none, if $s_k < j \le r_k$. Then $z_k(x) = \min\{< p_j^k, x > +b_j^k | j = 1,\ldots,s_k\}$ for $x \in B_3$ and so, for $x, y \in B_3$,

$$\begin{aligned} |z_k(x) - z_k(y)| &\le \max\{| < p_j^k, x - y > || j = 1,\ldots,s_k\} \\ &\le |x - y|\max\{|p_j^k|| j = 1,\ldots,s_k\} \end{aligned}$$

(suppose

$$\begin{aligned} |z_k(x) - z_k(y)| &= z_k(x) - z_k(y) =< p_{j_x}^k, x > +b_{j_x}^k - < p_{j_y}^k, y > -b_{j_y}^k \\ &\le < p_{j_y}^k, x > +b_{j_y}^k - < p_{j_y}^k, y > -b_{j_y}^k =< p_{j_y}^k, x - y > \\ &\le \max\{| < p_j^k, x - y > || j = 1,\ldots,s_k\}; \end{aligned}$$

the other case works analogously).
Consequently, $\max\{|p_j^k|| j = 1,\ldots,s_k\} \ge 1$. Suppose that, say $|p_1^k| \ge 1$ and $< p_1^k, x_1^k > +b_1^k = z_k(x_1^k)$, where $x_1^k \in B_3$. Then, for any x,

$$z_k(x) \le < p_1^k, x > +b_1^k =< p_1^k, x - x_1^k > + \underbrace{z_k(x_1^k)}_{\le 0} \le < p_1^k, x - x_1^k > .$$

Hence, we obtain

$$\int_{\partial B_4} z_k(x)\,dS_4 \le \int_{\partial B_4} 0\wedge < p_1^k, x - x_1^k > \,dS_4. \tag{2.65}$$

Finally, the function $0 \wedge [< p_1^k, x > - < p_1^k, x_1^k >]$ is nonincreasing with respect to $< p_1^k, x_1^k >$ and thus its largest value for $x_1^k \in B_3$ is

$$0 \wedge [< p_1^k, x > +3|p_1^k|] = 0 \wedge |p_1^k|[< \frac{p_1^k}{|p_1^k|}, x > +3].$$

This in conjunction with (2.65) yields

$$\int_{\partial B_4} z_k(x)\,dS_4 \le \int_{\partial B_4} 0 \wedge [< \frac{p_1^k}{|p_1^k|}, x > +3]\,dS_4 < 0,$$

the middle term in this last inequality being independent of k and p_1^k, contradicting (2.64).

Arbitrary case: by integral transformation (see, e. g. [17, Section 14, Satz 7])

$$\int_{\partial B_r} \frac{dz}{dn_r}(y)\, dS_r = \int_{\partial B_4} \frac{dz}{dn_4}\left(\frac{ry}{4}\right)\left(\frac{r}{4}\right)^{n-1} d\sigma_4 \tag{2.66}$$

Consider $\tilde{z}(y) := z\left(\frac{ry}{4}\right)$. We compute

$$\begin{aligned} \frac{d\tilde{z}}{dn_4}(y) &= < D\tilde{z}(y), -\frac{y}{|y|} >= \frac{r}{4} < Dz\left(\frac{4r}{y}\right), -\frac{y}{|y|} > \\ &= \frac{r}{4}\left(\frac{dz}{dn_r}\right)\left(\frac{4r}{y}\right) \end{aligned}$$

Hence,

$$(2.66) = \int_{\partial B_r} \frac{d\tilde{z}}{dn_4}(y)\, dS_4 \left(\frac{r}{4}\right)^{n-2} \geq \left(\frac{r}{4}\right)^{n-2} \epsilon \sqrt[n]{N_{\tilde{z}}(B_2)}.$$

By definition, $N_{\tilde{z}}(B_2) = \sum_{x_0 \in B_2} |P_{\tilde{z}}(x_0)|$ and

$$\begin{aligned} P_{\tilde{z}}(x_0) &= \{p| < p, x - x_0 >\geq \tilde{z}(x) - \tilde{z}(x_0) \quad \forall x\} \\ &= \{p| < \frac{4}{r}p, \frac{r}{4}x - \frac{r}{4}x_0 >\geq z\left(\frac{rx}{4}\right) - z\left(\frac{rx_0}{4}\right) \quad \forall x\} \\ &= T(P_z(Tx_0)), \end{aligned}$$

when $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n, x \longmapsto \frac{r}{4}x$.

$$\Rightarrow |P_{\tilde{z}}(x_0)| = |T(P_z(Tx_0))| = |\det T||P_z(Tx_0)| = \left(\frac{r}{4}\right)^n |P_z(Tx_0)|$$

Now $T(B_2) = B_{\frac{r}{2}}$, hence,

$$\begin{aligned} N_{\tilde{z}}(B_2)) &= \sum_{x_0 \in B_2} |P_{\tilde{z}}(x_0)| = \sum_{x_0 \in B_2} \left(\frac{r}{4}\right)^n |P_z(Tx_0)| \\ &= \sum_{x_1 \in B_{\frac{r}{2}}} \left(\frac{r}{4}\right)^n |P_z(x_1)| = \left(\frac{r}{4}\right)^n N_z(B_{\frac{r}{2}}). \end{aligned}$$

Consequently,

$$\int_{\partial B_r} \frac{dz}{dn_r}(y)\, dS_r \geq \left(\frac{r}{4}\right)^{n-2} \epsilon \sqrt[n]{\left(\frac{r}{4}\right)^n N_z(B_{\frac{r}{2}})} = r^{n-1}\tilde{\epsilon} \sqrt[n]{N_z(B_{\frac{r}{2}})}.$$

□

Definition 2.36 *The convex hull of a finite set of points is called a* convex polytope.

Remark 2.37 *Let us collect some facts about convex polytopes. A bounded convex polyhedron is a convex polytope. It only has got a finite number of distinct faces, and each face is a convex polytope itself. Moreover, a convex polytope P is the convex hull of its set of vertices, that is $P = conv(vert\ P)$. If $\{F_1, \ldots, F_r\}$ is a family of faces of a convex polytope, then $\bigcap_{i=1}^r F_i$ is also a face of it. (see e. g.*[25]*)*

Lemma 2.38 *Let $z(x) \leq 0$ outside B_r and let $z(x_0) > 0$ for some point $x_0 \in B_r$. Then any plane $z =< p, x > +b$ with $< p, x_0 > +b = z(x_0)$ and $< p, x > +b \geq z(x)$ for $x \in B_r$ is a support plane at one of the vertices lying on B_r.*

*Proof.*Let $z = \bigcap_{i=1}^k H_i, \{H_i | i = 1, \dots, k\}$ irredundant (i. e. one cannot resign on any of the H_i to describe z), $z(x) \leq 0$ on $(B_r)^c, z(x_0) > 0$ for some $x_0 \in B_r$. Clearly, z is bounded from above. Do not let $(x_0, z(x_0))$ be a vertex, because the assertion would be obvious, otherwise. Consider the convex polytope

$$z' := (z \cap \underbrace{\{(x, y) \in \mathbb{R}^{n+1} | x \in \mathbb{R}^n, y \geq \frac{1}{2} z(x_0)\}}_{=:H_{k+1}}) \subset B_r \times \mathbb{R}_+.$$

$E : \tilde{z} =< p, x > +b$ with $< p, x_0 > +b = z(x_0)$ and $< p, x > +b \geq z(x)$ for $x \in B_r$ is plane of support of z and z'. Because of Remark 2.37, $E \cap z' = \text{conv}(\text{vert}\ E \cap z')$, since $E \cap z'$ is a convex polytope. Thus $(x_0, z(x_0)) = \sum_{i=1}^m \mu_i (x^i, z(x_0^i))$, where $\mu_i \geq 0, \sum_{i=1}^m \mu_i = 1$

$$\Rightarrow z(x_0^{i_1}) \geq z(x_0) \quad \text{for some} \quad i_1 \in \{1, \dots, m\}$$

$$\Rightarrow (x_0^{i_1}, z(x_0^{i_1})) \in \text{vert}(E \cap z) \subset \text{vert}(z) \cap E$$

□

Lemma 2.39 *Let $z(x) \leq 0$ for $x \notin B_R$. Then for any x*

$$z(x) \leq N_0 \sqrt[n]{N_z(B_R)},$$

where $N_0 = 2R\omega_n^{-\frac{1}{n}}$ and ω_n is the volume of B_1.

*Proof.*Without loss of generality, $\bar{z} := \max_{x \in B_R} z(x) > 0$. Let $\{(x_i, z(x_i)), i = 1, \dots, r\}$ be the set of all vertices of $z(x)$, for which $x \in B_R$. This set is not empty, since by Lemma 2.38, $\bar{z} = z(x_{i_0})$ for some $i_0 \in \{1, \dots, r\}$. Further, consider the plane $z =< p, x > +b$, where $|p| < \frac{\bar{z}}{2R}$. For $b > \frac{3\bar{z}}{2}$, we have

$$< p, x > +b > -R|p| + \frac{3\bar{z}}{2} > \bar{z} \quad \forall x \in B_R.$$

If $b = 0$, $< p, x >< R|p| < \frac{\bar{z}}{2}$ for $x \in B_R$. Since the distance function is continuous, for some $x_0 \in \bar{B}_R$ we obtain

$$\inf_{x \in \bar{B}_R} \{\text{dist}((x, < p, x > +b), z(\bar{B}_R))\} = \text{dist}((x_0, < p, x_0 > +b), z(\bar{B}_R)) =: f(b).$$

Since f is continuous in $b, f(0) = 0, f(b) > \bar{z}$ for any $b > \frac{3}{2}\bar{z}$, there exists a least $b = b(p)$ among those, for which $< p, x > +b \geq z(x)$ for any $x \in B_R$. In addition,

$$z(x_0) =< p, x_{i_0} > +b(p)+ < p, x_0 - x_{i_0} >\geq \bar{z} - \frac{\bar{z}}{2R} |x_0 - x_{i_0}| > 0.$$

Therefore, $x_0 \in B_R$ and by Lemma 2.38, there exists an i, such that $p \in P_z(x_i)$, proving that the ball $\{p||p| < \frac{\bar{z}}{2R}\}$ is contained in $\bigcup_{i=1}^r P_z(x_i)$. From this follows the inequality for volumes satisfying

$$\left(\frac{\bar{z}}{2R}\right)^n \omega_n \leq N_z(B_R),$$

as required. □

Lemma 2.40 *Let f be a nonnegative continuous function vanishing outside of B_r. Then there exists a sequence of polyhedra $z_k(x)$, such that*

1. *$N_{z_k}(U) \longrightarrow \int_U f^n(x)\,dx$ for any open region $U \subset B_R$, where $0 < r < R$;*
2. *$\forall x, y$:*

$$|z_k(x) - z_k(y)| \leq 2N_0|x-y| \sqrt[n]{N_{z_k}(B_r)},$$
$$z_k(x) \geq -2N_0|x| \sqrt[n]{N_{z_k}(B_r)};$$

3. *the sequence $z_k(x)$ converges uniformly on any compactum to some concave function $z(x)$ vanishing on ∂B_R.*

*Proof.*Let U be a convex polyhedron in $\mathbb{R}^n$ with vertices on ∂B_R and such that $B_\rho \subset U \subset \bar{B}_R$ for some $r < \rho < R$. Let there be given points $x_1, \ldots, x_m \in B_r$ and positive numbers $\mu_1, \ldots, \mu_m$. After this proof we will show that there exists a polyhedron $z(x)$ such that $z_{|\partial U} = 0, N_z(x_i) = \mu_i, i = 1, \ldots, m$, and the set of vertices of $z(x)$ lying over U can be projected into $\{x_1, \ldots, x_m\}$.
Let this be $z(x) = \min\{< p_j, x > + b_j | j = 1, \ldots, r\}$ for $x \in U$, the collection of planes being chosen, such that for each j there is a point $x(j)$, lying strictly inside U, for which $z(x(j)) =< p_j, x(j) > + b_j$. If this condition is satisfied, then we can redefine $z(x)$ outside U and set

$$z(x) = \min\{< p_j, x > + b_j | j = 1, \ldots, r\} \quad \forall x.$$

By virtue of Lemma 2.38, for each j there exists, together with the point $x(j)$, a vertex, say $(x_{i(j)}, z(x_{i(j)}))$, such that $< p_j, x_{i(j)} > + b_j = z(x_{i(j)})$. Therefore, taking Lemma 2.39 into consideration, we obtain

$$0 \leq z(x) \leq < p_j, x - x_{i(j)} > + z(x_{i(j)}) \leq < p_j, x - x_{i(j)} > + N_0 \sqrt[n]{N_z(B_r)}$$

for $x \in B_\rho$. For $x = -\frac{(\rho - r)p_j}{|p_j|} + x_{i(j)}$ we have $|p_j| \leq \frac{N_0}{(\rho - r)} \sqrt[n]{N_z(B_r)}$ and hence, for any x, y,

$$\begin{aligned} |z(x) - z(y)| &\leq \max\{| < p_j, x - y > || j = 1, \ldots, r\} \quad \text{(see Lemma 2.35)} \\ &\leq \frac{N_0}{(\rho - r)}|x - y| \sqrt[n]{N_z(B_r)}. \end{aligned} \tag{2.67}$$

If we take $y = 0$, then since $z(0) \geq 0$,

$$z(x) \geq z(0) - \frac{N_0}{(\rho - r)}|x| \sqrt[n]{N_z(B_r)} \geq -\frac{N_0}{(\rho - r)}|x| \sqrt[n]{N_z(B_r)}. \tag{2.68}$$

Assertion 2 will follow from (2.67) and (2.68), if we can construct a sequence of polyhedra $z_k(x)$ of type $z(x)$:
Choose a sequence of polyhedra U_k in $\mathbb{R}^n$, such that the vertices of U_k lie on $\partial B_R, B_\rho \subset U_k \subset \bar{B}_R, U_k \subset U_{k+1}$ for any k, and $B_R \subset \bigcup_{k=1}^{\infty} U_k$. By definition of the Lebesgue-integral and the assumptions on f, we have

$$\int_{B_r} f^n \, dx = \lim_{k \longrightarrow \infty} \sum_{i=1}^{k2^k} \frac{i}{2^k} dx(A_{ik}),$$

where

$$A_{ik} := \begin{cases} \{f^n \geq i2^{-k}\} \cap \{f^n < (i+1)2^{-k}\} & , \text{for} \quad i = 0, \dots, k2^k - 1 \\ \{f^n \geq k\} & , \text{for} \quad i = k2^k \end{cases}$$

For $k \in \mathbb{N}, i = 1, \dots, k2^k =: r_k$ select $x_i^k \in B_r$ (if possible), such that $f(x_i^k) \in A_{ik}$ and define $\mu_i^k := \mu(x_i^k) := \frac{i}{2^k} dx(A_{ik})$. Then

$$\int_U f^n \, dx = \lim_{k \longrightarrow \infty} \sum_{x_i^k \in U} \mu_i^k \quad \forall U \subset B_R, U \text{ open.}$$

We construct $z_k(x)$ from $U_k, \{x_1^k, \dots, x_{r_k}^k\}$ and $\{\mu_1^k, \dots, \mu_{r_k}^k\}$ in the same way as $z(x)$ is constructed from $U, \{x_1, \dots, x_m\}$ and $\{\mu_1, \dots, \mu_m\}$. As above, by Lemma 2.39, (2.67) and (2.68), we have for any x, y that

$$-\frac{N_0}{(\rho - r)}|x| \sqrt[n]{N_{z_k}(B_r)} \leq z_k(x) \leq N_0 \sqrt[n]{N_{z_k}(B_r)}$$

and

$$|z_k(x) - z_k(y)| \leq \frac{N_0}{(\rho - r)}|x - y| \sqrt[n]{N_{z_k}(B_r)}.$$

Let $S \subset \mathbb{R}^n$ be compact, $S \subset B_\xi$, then

$$\sup_{k \in \mathbb{N}} \sup_{x \in S} |z_k(x)| \leq \sup_{k \in \mathbb{N}} \xi \frac{N_0}{(\rho - r)}|x| \sqrt[n]{N_{z_k}(B_r)} < \infty,$$

since

$$N_{z_k}(B_r) = \sum_{x_i^k \in B_r} N_{z_k}(x_i^k) = \sum_{x_i^k \in B_r} \mu_i^k < \int_{B_r} f^n \, dx < \infty \forall k.$$

Moreover,

$$\sup_{k \in \mathbb{N}} |z_k(x) - z_k(y)| \leq \sup_{k \in \mathbb{N}} \frac{N_0}{(\rho - r)}|x - y| \sqrt[n]{N_{z_k}(B_r)} \longrightarrow 0,$$

as $|x-y| \longrightarrow 0$. With the help of the Arzela-Ascoli theorem, we can pick out a subsequence of the (z_k), which fulfills the lemma. □

Let us now construct the missing polyhedron in the situation of the previous lemma (cf. [4, Sections 11.2, 11.3]). If you already believe Lemma 2.39, then you can skip off the next lemma and use the upper bound of Lemma 2.39 instead of that of the following lemma in the construction. Nevertheless, the proof of the following lemma is interesting and therefore presented.

Define $g(\zeta) := dx(B_\zeta(0))$, $T := g^{-1}$ and note that both functions are strictly increasing and continuous.

Lemma 2.41 *Let $z(x)$ be a convex polyhedron on U such that*

1. $z_{|\partial U} = 0$,
2. $N_z(U) < \infty$,

then $T(N_z(U))diam(U) \geq z(x) \geq 0$.

Proof. Let $x_0 \in U$ such that $z(x_0) = \sup_U z(x)$, K_0 a cone with peak $(x_0, z(x_0))$ and base ∂S, where $S := B_{\text{diam } U}(x_0) \subset \mathbb{R}^n$, and K_1 a cone with peak $(x_0, z(x_0))$ and base ∂U.
Clearly, $P_{K_0}(x_0) \subset P_{K_1}(x_0)$. Moreover, if z_1, z_2 are concave functions on U such that $z_{1|\partial U} = z_{2|\partial U}$ and $z_1 \leq z_2$ on U, then $P_{z_1}(U) \subset P_{z_2}(U)$ by shifting planes of support. Therefore,

$$N_{K_0}(S) \leq N_{K_1}(U) \leq N_z(U).$$

We have

$$g(\zeta_0) = N_{K_0}(S) \quad \text{, where} \quad \zeta_0 = \frac{z(x_0)}{\text{diam } U}.$$

Indeed, consider the one-dimensional case:

$$P_{K_0}(x_0) = \left\{ p \in \mathbb{R} | p \geq \frac{z(x) - z(x_0)}{x - x_0} \forall x \right\} = \left[-\frac{z(x_0)}{\text{diam } U}, \frac{z(x_0)}{\text{diam } U} \right].$$

In n dimensions, we have $P_{K_0}(x_0) = \bigcup_{x \in \mathbb{R}^n} P_{\Pi_x K_0}(x_0)$, where $\Pi_x K_0$ shall be the projection of K_0 onto the line $(x_0 + \lambda x)_{\lambda \in \mathbb{R}}$; the assertion follows.
Thus $\zeta_0 = T(N_{K_0}(S)) \leq T(N_z(U))$, since T is isotone. Consequently,

$$0 \leq z(x) \leq (\text{diam } U) T(N_z(U)) \quad \text{on} \quad U.$$

□

Let $B_1, \ldots, B_m$ be the vertices of U and H the set of all convex polyhedra $z(x)$, such that $z(B_j) = 0$ for $j = 1, \ldots, k, N_z(x_i) \leq \mu_i$ for $i = 1, \ldots, m$ and vert $z_{|U \setminus \partial U} \subset \{x_1, \ldots, x_m\}$. H is not empty, for $0 \in H$. Let now $z(x) \in H$ and $\mu_0 := \sum_{i=1}^m \mu_i$, then

$$N_z(U \setminus \partial U) = \sum_{i=1}^{m} N_z(x_i) \leq \sum_{i=1}^{m} \mu_i = \mu_0 < \infty,$$

and from the lemma it follows that

$$0 \leq z(x) \leq T(\mu_0) \text{ diam } U \quad \forall x \in U.$$

If $\delta := \min\limits_{i=1,\dots,m} \text{dist}(x_i, \partial U)$, then all functions $z(x) \in H$ fulfill the lipschitz-condition with common constant $M = T(\mu_0) \text{ diam}(U)\delta^{-1}$. Therefore, we can apply Arzela-Ascoli and obtain that $H \subset C(U)$ is compact.
Let $V(z)$ be the volume of $z_{|U} \cap \mathbb{R}^{n+1}_+ =: S_z$, then $V : H \longrightarrow \mathbb{R}$ continuous w.r.t. $\|\cdot\|_{\infty;U}$. Consequently, $V_0 := \sup\limits_H V(z) < \infty$ and $\exists \tilde{z} \in H$, such that $V_0 = V(\tilde{z})$.
We want to show by contradiction, that $\tilde{z}$ is the polyhedron in demand:
Let us assume the opposite. Then $N_{\tilde{z}}(x_i) \leq \mu_i$ for $i = 1, \dots, m$ and there exists $i_0 \in \{1, \dots, m\}$, such that $N_{\tilde{z}}(x_{i_0}) < \mu_{i_0}$. Consider the point $(x_{i_0}, \tilde{z}(x_{i_0}) + \epsilon)$ and let S_v denote the polytope belonging to $\partial[\text{conv}(\text{vert}(S_{\tilde{z}}) \cup \{(x_{i_0}, \tilde{z}(x_{i_0}) + \epsilon)\})]$.
For sufficient small ϵ, we have

$$\text{vert } v_{|U \setminus \partial U} \subset \{x_1, \dots, m\}, N_v(x_{i_0}) < \mu_{i_0}, N_v(x_i) \leq \mu_i \forall i \neq i_0,$$

which implies that $v \in H$. But, by our construction, $V(v) > V(\tilde{z}) = \sup\limits_{u \in H} V(u)$, which is a contradiction. □

Lemma 2.42 *Let $z(x)$ be a polyhedron, such that the integrability condition $K \geq z(x) \geq -K(1 + |x|)$ holds for some constant K and all x. Then, for any x and $t > 0$,*

$$z(x) \geq T_t^\sigma z(x) + c\epsilon \sqrt[n]{|\det \sigma|} \sqrt{2t} \int_0^\infty r^{n-1} e(r) \sqrt{N_z[A(\sigma, t, x, r)]}\, dr,$$

where $c = \pi^{-\frac{n}{2}}$, ϵ is the constant of Lemma 2.35 and

$$A(\sigma, t, x, r) = \left\{ y \in \mathbb{R}^n \,|\, \left| \frac{\sigma^{-1}}{\sqrt{2t}}(y - x) \right| < \frac{r}{2} \right\}.$$

Proof. Case $x = 0$ and $\sqrt{2t}\sigma = I$: with the help of integral transformation (see [17, Section 14, Satz 8]) we compute:

$$\begin{aligned}
&z(0) - T_t^\sigma z(0) \\
&\quad = c \int_{\mathbb{R}^n} [z(0) - z(y)] \exp[-|y|^2]\, dy \\
&\quad = c \int_0^\infty \exp[-r^2] r^{n-1} \int_{\partial B_1} [z(0) - z(ry)]\, dS_1\, dr \quad \text{(polar coord.)} \\
&\quad = c \int_0^\infty \exp\left[-r^2\right] r^{n-1} \int_{\partial B_1} [z(0) - z(r_1 y)] + \dots + [z(r_m) - z(ry)]\, dS_1\, dr,
\end{aligned}$$

where $0 = r_0 < r_1 < \dots < r_m = r$ and $r_i y$ are the non-smooth points of z

$$\quad = -c \int_0^\infty \exp[-r^2] r^{n-1} \int_{\partial B_1} \sum_{i=1}^m \int_{r_{i-1}}^{r_i} \frac{dz}{ds}(sy)\, ds\, dS_1\, dr \quad \text{(fund.-theorem)}$$

$$= c\int_0^\infty \exp[-r^2]r^{n-1}\int_{\partial B_1}\int_0^r \frac{dz}{dn_1}(sy)\,ds\,dS_1\,dr \quad \text{(def. of normal deriv.)}$$
$$= c\int_0^\infty \exp[-r^2]r^{n-1}\int_0^r\int_{\partial B_1} \frac{dz}{dn_1}(sy)\,dS_1\,ds\,dr \quad \text{(Fubini)}$$
$$= c\int_0^\infty \exp[-r^2]r^{n-1}\int_0^r s^{1-n}\int_{\partial B_s} \frac{dz}{dn_s}(y)\,dS_s\,ds\,dr \quad \text{(int. transform.)}$$
$$= c\int_0^\infty e(s)\int_{\partial B_s} \frac{dz}{dn_s}(y)\,dS_s\,ds \quad \text{(def. of } e(s))$$
$$\geq c\int_0^\infty e(s)\epsilon\sqrt[n]{N_z(B_{\frac{s}{2}})}s^{n-1}\,ds \quad \text{(Lemma 2.35)}$$

Arbitrary case: fix $x \in \mathbb{R}^n$, a nonsingular matrix σ and consider the bijection

$$\bar{y} := T(y) := \frac{\sigma^{-1}(y-x)}{\sqrt{2t}} \quad \text{and} \quad \tilde{z} := z \circ T^{-1}.$$

We have $\tilde{z}(0) = z(x)$ and

$$\begin{aligned} T_t^\sigma z(x) &= (2\pi t)^{-\frac{n}{2}}|\det\sigma|^{-1}\int_{\mathbb{R}^n} z(y)\exp\left[-\frac{|\sigma^{-1}(y-x)|^2}{2t}\right]dy \\ &= \pi^{-\frac{n}{2}}\int_{\mathbb{R}^n} z(x+\sqrt{2t}\sigma y)\exp[-|y|^2]\,dy \\ &= T_t^{\frac{I}{\sqrt{2t}}}\tilde{z}(0). \end{aligned}$$

Now, if $\sqrt[n]{N_{\tilde{z}}(B_{\frac{r}{2}})} = \sqrt{2t}\sqrt[n]{|\det\sigma|N_z[A(\sigma,t,x,r)]}$, the lemma follows:

$$A := A(\sigma,t,x,r) = \left\{y \,\middle|\, |T(y)| < \frac{r}{2}\right\} = T^{-1}(B_{\frac{r}{2}})$$

and

$$N_z(A) = \sum_{x_i \in A}|P_z(x_i)|$$

Thus, it remains to show: $|P_z(x_0)| = (2t)^{-\frac{n}{2}}|\det\sigma|^{-1}|P_{\tilde{z}}(Tx_0)|$. But

$$\begin{aligned} P_z(x_0) &= \{p| <p, x-x_0> \geq z(x)-z(x_0) \quad \forall x\} \\ &= \{p| <p, T^{-1}(Tx-Tx_0)> \geq \tilde{z}(Tx)-\tilde{z}(Tx_0) \quad \forall x\} \\ &= \{p| <(T^{-1})^t p, Tx-Tx_0> \geq \tilde{z}(Tx)-\tilde{z}(Tx_0) \quad \forall x\} \\ &= T^t(P_{\tilde{z}}(Tx_0)). \end{aligned}$$

Together with $|\det T^t| = |\det T| = |\det\sigma|^{-1}(2t)^{-\frac{n}{2}}$, we obtain

$$\begin{aligned} |P_z(x_0)| &= |T^t(P_{\tilde{z}}(Tx_0))| = |\det T^t||P_{\tilde{z}}(Tx_0)| \\ &= |\det\sigma|^{-1}(2t)^{-\frac{n}{2}}|P_{\tilde{z}}(Tx_0)|. \end{aligned}$$

□

After all the preparations, we now start to fill the hole in the proof of the main theorem.
Let $z_k(x)$ be the sequence, whose existence is asserted in Lemma 2.40. By Lemma 2.42,

$$z_k(x) \geq T_t^\sigma z_k(x) + c\epsilon \sqrt[n]{|\det\sigma|}\sqrt{2t}\int_0^\infty r^{n-1}e(r)\sqrt[n]{N_{z_k}[A(\sigma,t,x,r)]}\,dr$$

and by Fatou's lemma,

$$\begin{aligned} z(x) &= \lim_{k\longrightarrow\infty} z_k(x) \\ &\geq T_t^\sigma z(x) + c\epsilon \sqrt[n]{|\det\sigma|}\sqrt{2t}\int_0^\infty r^{n-1}e(r)\sqrt[n]{\int_{A(\sigma,t,x,r)} f^n(y)\,dy dr} \end{aligned}$$

for any x and $t > 0$ and nonsingular σ. Writing

$$r^{n-1}e(r) = \left(\int_{A(\sigma,t,x,r)}\left(r^{n-1}e(r)\frac{1}{\sqrt[n']{dy(A(\sigma,t,x,r))}}\right)^{n'} dy\right)^{\frac{1}{n'}},$$

we can apply Hölder's inequality and obtain

$$\begin{aligned} &z(x) \geq T_t^\sigma z(x) + \\ &\quad c\epsilon\sqrt[n]{|\det\sigma|}\sqrt{2t}\int_0^\infty\int_{A(\sigma,t,x,r)} f(y)r^{n-1}e(r)(dy(A(\sigma,t,x,r)))^{\frac{1-n}{n}}\,dy\,dr. \end{aligned}$$

Now,

$$\begin{aligned} (dy(A(\sigma,t,x,r)))^{\frac{1-n}{n}} &= (dy(T^{-1}(B_{\frac{r}{2}})))^{\frac{1-n}{n}} = (|\det T^{-1}|dy(B_{\frac{r}{2}}))^{\frac{1-n}{n}} \\ &= |\det T^{-1}|^{\frac{1-n}{n}}\left(\frac{r}{2}\right)^{1-n}\omega_n^{\frac{1-n}{n}} \end{aligned}$$

and

$$\int_{A(\sigma,t,x,r)} f(y)\,dy = \int_{B_{\frac{r}{2}}} f(\sqrt{2t}\sigma y + x)|\det T^{-1}|\,dy.$$

Consequently,

$$\begin{aligned} &z(x) \geq T_t^\sigma z(x) + \\ &\quad c\epsilon|\det\sigma|^{\frac{1}{n}}\sqrt{2t}\,\omega_n^{\frac{1-n}{n}}|\det T^{-1}|^{\frac{1}{n}}2^{n-1}\int_0^\infty e(r)\int_{B_{\frac{r}{2}}} f(\sqrt{2t}\sigma y + x)\,dy\,dr \\ &\quad = T_t^\sigma z(x) + \sqrt[n]{\det\sigma^2}tN_1\int_0^\infty e(r)\int_{B_{\frac{r}{2}}} f(x+\sqrt{2t}\sigma y)\,dy\,dr, \end{aligned} \tag{2.69}$$

where $N_1 := 2^n\omega_n^{\frac{1-n}{n}}c\epsilon$. Now let the matrix σ be fixed and set for $\delta > 0$,

$$z_\delta(x) = T_\delta^\sigma z(x) \quad\text{and}\quad f_\delta(x) = T_\delta^\sigma f(x).$$

Then, from (2.69), we obtain

$$
\begin{aligned}
&\frac{1}{t}[T_t^\sigma z_\delta(x) - z_\delta(x)]\\
&\quad = \frac{1}{t}[T_t^\sigma T_\delta^\sigma z(x) - T_\delta^\sigma z(x)]\\
&\quad = \frac{1}{t}[T_\delta^\sigma (T_t^\sigma z(x) - z(x))]\\
&\quad \leq -T_\delta^\sigma \left(N_1 \sqrt[n]{\det \sigma^2} \int_0^\infty e(r) \int_{B_{\frac{r}{2}}} f(\cdot + \sqrt{2t}\sigma\bar{y})\, d\bar{y}\, dr \right)(x)\\
&\quad = -(2\pi\delta)^{-\frac{n}{2}} |\det \sigma|^{-1} \int_{\mathbb{R}^n} N_1 \sqrt[n]{|\det \sigma|^2} \int_0^\infty e(r)\\
&\qquad \int_{B_{\frac{r}{2}}} f(y + \sqrt{2t}\sigma\bar{y})\, d\bar{y}\, dr \exp\left[-\frac{|\sigma^{-1}(x-y)|^2}{2\sigma}\right] dy \qquad (2.70)\\
&\quad = -N_1 \sqrt[n]{|\det \sigma|^2} \int_0^\infty e(r) \int_{B_{\frac{r}{2}}} (2\pi\delta)^{-\frac{n}{2}} |\det \sigma|^{-1}\\
&\qquad \underbrace{\int_{\mathbb{R}^n} f(y + \sqrt{2t}\sigma\bar{y}) \exp\left[-\frac{|\sigma^{-1}(x-y)|^2}{2\sigma}\right] dy}_{\int_{\mathbb{R}^n} f(y)\exp\left[-\frac{|\sigma^{-1}(x-y+\sqrt{2t}\sigma\bar{y})|^2}{2\delta}\right] dy}\, d\bar{y}\, dr\\
&\quad = -N_1 |\det \sigma^2|^{\frac{1}{n}} \int_0^\infty e(r) \int_{B_{\frac{r}{2}}} f_\delta(x + \sqrt{2t}\sigma y) dy\, dr.
\end{aligned}
$$

For $\delta \leq \frac{1}{2}$, the left-hand side of (2.70) tends to

$$L^\sigma z_\delta(x) := \sum_{i,j=1}^n a_{ij} \frac{\partial^2 z_\delta}{\partial x_i \partial x_j}(x),$$

as $t \longrightarrow 0$, where $a := (a_{ij}) = \frac{1}{2}\sigma\sigma^t$. Therefore, (2.70) implies that

$$L^\sigma z_\delta(x) \leq -\sqrt[n]{|\det \sigma^2|} N_2 f_\delta(x),$$

where $N_2 = c\epsilon \sqrt[n]{\omega_n} \int_0^\infty r^n e(r)\, dr$, since $f_\delta(x + \sqrt{2t}\sigma\bar{y}) \longrightarrow f_\delta(x)$ for any x by Lebesgue, as $t \downarrow 0$.
The last two convergence assertions walk along with wild computations and are put into appendix B for keeping orientation. We get near to (2.54), (2.55).

Let $u \in C_0^\infty(B_{R_0}(x_0))$ be nonnegative. Then for $\delta \leq \frac{1}{2}$,

$$
\begin{aligned}
-\int a^{ij} D_{ij} u(x) (T_\delta^\sigma z)(x)\, dx &= -\int \underbrace{a^{ij} D_{ij} (T_\delta^\sigma z)}_{=L^\sigma z_\delta}(x) u(x)\, dx\\
&\geq |\det \sigma^2|^{\frac{1}{n}} N_2 \int u f_\delta\, dx.
\end{aligned}
$$

By appendix B we obtain

$$-\int (a^{ij} D_{ij} u(x)) z(x)\, dx \geq |\det \sigma^2|^{\frac{1}{n}} N_2 \int uf\, dx,$$

as $\delta \downarrow 0$. This is (2.54) for nonnegative, symmetric, nonsingular matrices (α^{ij}) because of

Lemma 2.43 *Let (α^{ij}) be strictly positive and symmetric, then there exists a nonsingular matrix σ, such that $(\alpha^{ij}) = N_2^{\frac{1}{n}} \sigma\sigma^t$.*

*Proof.*By linear algebra results, there exists an orthogonal matrix T, such that $T(\alpha^{ij})T^t = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$, where $\lambda_1, \dots, \lambda_n \geq 0$. Define $(\tilde{\alpha}^{ij}) := \mathrm{diag}(\lambda_1^{\frac{1}{2}}, \dots, \lambda_n^{\frac{1}{2}})$, then $(T^t(\tilde{\alpha}^{ij})T)(T^t(\tilde{\alpha}^{ij})T) = T^t \mathrm{diag}(\lambda_1, \dots, \lambda_n)T = (\alpha^{ij})$. Thus, define

$$\sigma := N_2^{-\frac{1}{2n}} (T^t(\tilde{\alpha}^{ij})T) (= \sigma^t)$$

σ is nonsingular: Let $\sigma v = 0$. Then $0 = \|\sigma v\|^2 =< \sigma v, \sigma v >=< \sigma^t \sigma v, v >= N_2^{-\frac{1}{n}} < (\alpha^{ij})v, v > \geq \epsilon \|v\|^2$. Consequently, $v = 0$. □

Hence, we have (2.54), when (α^{ij}) is strictly positive and symmetric. Now define $(\alpha_\epsilon^{ij}) := (\alpha^{ij}) + \epsilon(\delta^{ij})$, where (α^{ij}) is as in the situation of (2.54), and let L^a, L^{a_ϵ} be the corresponding operators. Note that (α_ϵ^{ij}) is symmetric and

$$< (\alpha_\epsilon^{ij})v, v >=< (\alpha^{ij})v, v > +\epsilon < v, v > \geq \epsilon \|v\|^2 > 0.$$

Thus,

$$\begin{aligned} -\int L^a uz\, dx &= -\int L^{a_\epsilon} uz\, dx + \epsilon \int \delta^{ij} D_{ij} uz\, dx \\ &\geq N_2 |\det a^\epsilon|^{\frac{1}{n}} \int uf\, dx + \epsilon \int \delta^{ij} D_{ij} uz\, dx \\ &\longrightarrow N_2 |\det a|^{\frac{1}{n}} \int uf\, dx, \end{aligned}$$

for any $u \in C_0^\infty(B_{R_0}(x_0))$, since $\det(\cdot)$ is a polynome and therefore continuous. This is (2.54).

For (2.55) let $z_k, z, k \in \mathbb{N}$ as in Lemma 2.40. Then by Lemma 2.39

$$z_k(x) \leq N_0 \sqrt[n]{N_{z_k}(B_{R_0}(x_0))} \forall k \forall x.$$

Therefore, by Lemma 2.40,

$$\sup_{B_{R_0}(x_0)} z \leq N \|f\|_{n, B_{R_0}(x_0)},$$

which is (2.55).

Chapter 3

REGULARITY RESULTS

3.1 Weak Solutions of Elliptic Equations

3.1.1 Existence in $H_0^{1,2}$

In this section we prove the existence of weak solutions of elliptic partial differential equations of second order in divergence form on balls. We therefore need the following result of functional analysis.

Proposition 3.1 (Fredholm alternative, cf. [19, Theorem 5.3]) *Let T be a compact linear mapping of a normed linear space V into itself. Then either*

(i) the homogeneous equation $x - Tx = 0$ has a non-trivial solution $x \in V$ or

(ii) for any $y \in V$ the equation $x - Tx = y$ has a uniquely determined solution $x \in V$.

Furthermore, in case (ii), the operator $(I - T)^{-1}$, whose existence is asserted there, is also bounded.

Proof.

- Step 1: Let $S = I - T$ and let $N = S^{-1}(0) = \{x \in V | Sx = 0\}$ be the nullspace of S. Then there exists a constant K, such that

$$\operatorname{dist}(x, N) \leq K\|Sx\| \forall x \in V. \tag{3.1}$$

*Proof.*Suppose the result is not true. Then there exists a sequence $(x_n) \subset V$ satisfying $\|Sx_n\| = 1$ and $d_n = \operatorname{dist}(x_n, N) \longrightarrow \infty$. Choose a sequence $(y_n) \subset N$, such that $d_n \leq \|x_n - y_n\| \leq 2d_n$. Then, if

$$z_n := \frac{x_n - y_n}{\|x_n - y_n\|},$$

we have $\|z_n\| = 1$ and $\|Sz_n\| \leq d_n^{-1} \longrightarrow 0$, so that the sequence (Sz_n) converges to 0. But since T is compact, by passing to a subsequence if necessary, we may assume that the sequence (Tz_n) converges to an element $y_0 \in V$. Since $z_n = (S+T)z_n$, we also have (z_n) converging to y_0 and consequently $y_0 \in N$. However this leads to a contradiction as

$$\begin{aligned}\operatorname{dist}(z_n, N) &= \inf_{y\in N}\|z_n - y\| = \|x_n - y_n\|^{-1}\inf_{y\in N}\|x_n - y_n - \|x_n - y_n\|y\| \\ &= \|x_n - y_n\|^{-1}\operatorname{dist}(x_n, N) \geq \frac{1}{2}.\end{aligned}$$

□

- Step 2: Let $R = S(V)$ be the range of S. Then R is a closed subspace of V.

  *Proof.*Let (x_n) be a sequence in V, whose image (Sx_n) converges to an element $y \in V$. To show that R is closed, we must show that $y = Sx$ for some $x \in V$. By our previous result, the sequence (d_n), where $d_n = \operatorname{dist}(x_n, N)$ is bounded. Choosing $y_n \in N$ as before and writing $w_n = x_n - y_n$, we consequently have that the sequence (w_n) is bounded while the sequence (Sw_n) converges to y. Since T is compact, by passing to a subsequence if necessary, we may assume that (Tw_n) converges to an element $w_0 \in V$. Hence, the sequence (w_n) itself converges to $y + w_0$ and by the continuity of S, we have $S(y + w_0) = y$. Consequently, R is closed. □

- Step 3: If $N = \{0\}$, then $R = V$. That is, if case (i) of Proposition 3.1 does not hold, then case (ii) is true.

  *Proof.*By our previous result, the sets R_j defined by $R_j = S^j(V), j = 1, 2, \ldots$ form a non-increasing sequence of closed subspaces of V. Suppose that no two of them coincide. Then each is a proper subspace of its predecessor. Hence there exists a sequence $(y_n) \subset V$ of "nearly orthogonal" elements (see [3, 2.18]), such that $y_n \in R_n, \|y_n\| = 1$ and $\operatorname{dist}(y_n, R_{n+1}) \geq \frac{1}{2}$. Thus, if $n > m$,

$$Ty_m - Ty_n = y_m + (-y_n - Sy_m + Sy_n) = y_m - y$$

  for some $y \in R_{n+1}$. Hence, $\|Ty_m - Ty_n\| \geq \frac{1}{2}$ contrary to the compactness of T. Consequently, there exists an integer k such that $R_j = R_k$ for any $j \geq k$. Up to this point we have not used the condition $N = 0$. Now let y be an arbitrary element of V. Then $S^k y \in R_k = R_{k+1}$, so $S^k y = S^{k+1}x$ for some $x \in V$. Therefore, $S^k(y - Sx) = 0$ and therefore $y = Sx$, since $S^{-k}(0) = S^{-1}(0) = 0$. Consequently $R = R_j = V$ for each j. □

- Step 4: If $R = V$, then $N = \{0\}$. Consequently, either case (i) or case (ii) holds.

  *Proof.*This time we define a non-decreasing sequence of closed subspaces (N_j) by setting $N_j = S^{-j}(0)$. The closedness of N_j follows from the continuity of S. Suppose that no two of these spaces coincide. Then each one is a proper subspace of the following. Hence, as above, there exists a sequence $(y_n) \subset V$, such that $y_n \in N_n, \|y_n\| = 1$ and $\mathrm{dist}(y_n, N_{n-1}) \geq \frac{1}{2}$. Thus, if $n < m$,

$$Ty_m - Ty_n = y_m + (-y_n - Sy_m + Sy_n) = y_m - y$$

for some $y \in N_{m-1}$. Hence, $\|Ty_m - Ty_n\| \geq \frac{1}{2}$, contrary to the compactness of T. Therefore, $N_j = N_l$ for any $j \geq$ some $l \in \mathbb{N}$. Thus, if $R = V$, any element $y \in N_l$ satisfies $y = S^l x$ for some $x \in V$. Consequently, $S^{2l}x = 0$, so that $x \in N_{2l} = N_l$, whence $y = S^l x = 0$. □

The boundedness of $S^{-1} = (I - T)^{-1}$ in case (ii) follows from step 1 with $N = \{0\}$. □

Let $B \subset \mathbb{R}^n$ be a ball. We are going to consider equations of the form

$$\begin{aligned} Lu = D_i(a^{ij}D_ju + D_ja^{ij}u - b^iu) + cu &= D_if^i \quad \text{in} \quad B \\ u &= 0 \quad \text{on} \quad \partial B. \end{aligned} \tag{3.2}$$

Let $p > n, a^{ij} \in H^{1,p}(B), b^i, f^i, c \in L^p(B), A \geq \lambda I$. For $u, v \in H^{1,2}(B)$ we set

$$a(u,v) = \int_B (a^{ij}D_ju + d^iu)D_iv + cuv\,dx,$$

where $d^i := D_ja^{ij} - b^i$. We mix considerations in [13, Section I.1.2] and [19, Section 8.2] until we obtain the existence result needed in Corollary 3.56.

Definition 3.2 *For $T \in H^{-1,2}(B)$ and $g \in H^{1,2}(B)$ we say that $u \in H^{1,2}(B)$ is a weak solution of the Dirichlet problem*

$$\begin{aligned} Lu &= T \quad \text{in} \quad B \\ u &= g \quad \text{on} \quad \partial B, \end{aligned} \tag{3.3}$$

if u satisfies

$$\begin{aligned} a(u,v) &= \langle T, v \rangle \; \forall v \in H_0^{1,2}(B) \\ u - g &\in H_0^{1,2}(B). \end{aligned} \tag{3.4}$$

Lemma 3.3 *$a(\cdot,\cdot)$ is a bounded bilinear form on $H_0^{1,2}(B)$.*

Proof. Using Hölder,

$$\left| \int_B a^{ij}D_iuD_jv\,dx \right| \leq N \sup_{i,j} \|a^{ij}\|_\infty \|u\|_{1,2}\|v\|_{1,2}.$$

Case $n \geq 3$**:** Using generalized Hölder, $p > n$ and Proposition D.1, we obtain

$$\begin{aligned} \left| \int_B d^iuD_iv\,dx \right| &\leq \sum_i \|d^i\|_n \|u\|_{\frac{2n}{n-2}} \|D_iv\|_2 \leq C\|u\|_{1,2}\|v\|_{1,2} \\ \left| \int_B cuv\,dx \right| &\leq \|c\|_{\frac{n}{2}} \|u\|_{\frac{2n}{n-2}} \|v\|_{\frac{2n}{n-2}} \leq C\|u\|_{1,2}\|v\|_{1,2}. \end{aligned}$$

Case $n = 2$: $u \in H_0^{1,2}(B) \Rightarrow u \in H_0^{1,2-\epsilon}(B) \forall \epsilon \in [0,1]$. By Sobolev,

$$\|u\|_{\frac{4-2\epsilon}{\epsilon}} = \|u\|_{\frac{(2-\epsilon)2}{2-(2-\epsilon)}} \leq C\|u\|_{1,2-\epsilon} \leq C'\|u\|_{1,2}.$$

Choose $0 < \epsilon < \frac{p-2}{p-1}$, then $\frac{2p}{p-2} < \frac{4-2\epsilon}{\epsilon}$. Hence, with generalized Hölder,

$$\begin{aligned} \left|\int_B d^i u D_i v \, dx\right| &\leq \sum_i \|d^i\|_p \|u\|_{\frac{2p}{p-2}} \|D_i v\|_2 \\ &\leq C\|d\|_p \|u\|_{\frac{2p}{p-2}} \|v\|_{1,2} \\ &\leq C'\|d\|_p \|u\|_{1,2} \|v\|_{1,2}, \\ \left|\int_B cuv \, dx\right| &\leq C\|u\|_{1,2}\|v\|_{1,2} \quad \text{analogously} \quad . \end{aligned}$$

Thus we obtain

$$|a(u,v)| \leq C\|u\|_{1,2}\|v\|_{1,2}. \tag{3.5}$$

□

Lemma 3.4 *There exists $\bar{\mu} > 0$, such that $a(u,v) + \mu(u,v)$ is coercive on $H_0^{1,2}(B)$ for $\mu \geq \bar{\mu}$.*

In order to prove the lemma, we need the following fact: for $f \in L^p(B)$ and $\epsilon > 0$ there exists a decomposition $f = f_1 + f_2$, such that

$$\|f_2\|_p < \epsilon, \sup_B |f_1(x)| < K(\epsilon). \tag{3.6}$$

Choose

$$f_1(x) = \begin{cases} f(x) & , \text{if} |f(x)| < K \\ 0 & , \text{if} |f(x)| \geq K, \end{cases} \quad \text{i. e.} \quad f_2(x) = \begin{cases} 0 & , \text{if} |f(x)| < K \\ f(x) & , \text{if} |f(x)| \geq K, \end{cases}$$

where K is sufficiently large.

*Proof.*For $\epsilon > 0$ there are decompositions $d^i = d_1^i + d_2^i, c = c_1 + c_2$, such that

$$\begin{aligned} \|d_2\|_p + \|c_2\|_p &\leq \epsilon \\ \|d_1\|_\infty + \|c_1\|_\infty &\leq K(\epsilon). \end{aligned}$$

Set

$$\begin{aligned} a_2(u,v) &= \int_B (a^{ij} D_i u - d_2^j u) D_j v + c_2 uv \, dx \\ a_1(u,v) &= a(u,v) - a_2(u,v), \end{aligned}$$

then

$$\begin{aligned} a_2(u,u) = \int_B a^{ij} D_i u D_j u \, dx + \int_B d_2^j u D_j u + c_2 u^2 \, dx \\ \geq \lambda \|Du\|_2^2 - c'\epsilon \|u\|_{1,2}^2 - \epsilon \|u\|_{1,2}^2 \\ \geq c\lambda \|u\|_{1,2}^2 - (c'+1)\epsilon \|u\|_{1,2}^2, \end{aligned}$$

where we computed as in Lemma 3.3 and used the equivalence of norms on $H_0^{1,2}(B)$ (cf. Poincaré inequality A.7). For $a_1(u,u)$ we have

$$\begin{aligned}
|a_1(u,u)| &\le K(\epsilon)\left(\int_B \sum_j |D_j u u|\,dx + \int_B |u|^2\,dx\right)\\
&\le \underbrace{K(\epsilon)\|Du\|_2\|u\|_2}_{\le \epsilon\|Du\|_2^2 + \frac{K(\epsilon)}{4\epsilon}\|u\|_2^2} + \|u\|_2^2 K(\epsilon)\\
&\le \epsilon\|Du\|_2^2 + \left(\frac{K(\epsilon)}{4\epsilon} + K(\epsilon)\right)\|u\|_2^2\\
&\le c'''\epsilon\|u\|_{1,2}^2 + \left(\frac{K(\epsilon)}{4\epsilon} + K(\epsilon)\right)\|u\|_2^2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
a(u,u) &= a_1(u,u) + a_2(u,u)\\
&\ge -\epsilon c'''\|u\|_{1,2} - \left(\frac{K(\epsilon)}{4\epsilon} + K(\epsilon)\right)\|u\|_2^2 + c\lambda\|u\|_{1,2}^2 - (c'+1)\epsilon\|u\|_{1,2}^2\\
&= \left[c\lambda - \epsilon(c'''+c'+1)\right]\|u\|_{1,2}^2 - \left(\frac{K(\epsilon)}{4\epsilon} + K(\epsilon)\right)\|u\|_2^2.
\end{aligned}$$

Set $\epsilon := \frac{c\lambda - \frac{\lambda}{k_c}}{c'''+c'+1}$, where $k_c \in \mathbb{N}$ such that $c - \frac{1}{k_c} \in (0,1) (c \in (0,1)!)$, then

$$a(u,u) \ge \frac{\lambda}{k_c}\|u\|_{1,2}^2 - \bar{\mu}\|u\|_2^2 \ge \left(\frac{\lambda}{k_c} - \bar{\mu}\right)\|u\|_{1,2}^2. \tag{3.7}$$

□

Proposition 3.5 *Let $T \in H^{-1,2}(B)$. Then there exists $\bar{\mu} > 0$, such that for $\mu > \bar{\mu}$ the Dirichlet problem*

$$Lu + \mu u = T, u \in H_0^{1,2}(B), \tag{3.8}$$

has a unique weak solution.

*Proof.*By definition of weak solutions, the bilinear form corresponding to (3.8) is given by $a(u,v) + \mu(u,v)$. A weak solution satisfies

$$a(u,v) + \mu(u,v) = <T, v> \forall v \in H_0^{1,2}(B), u \in H_0^{1,2}(B). \tag{3.9}$$

From lemmas 3.3 and 3.4 we derive that $a(u,v) + \mu(u,v)$ is a bounded coercive bilinear form on $H_0^{1,2}(B)$ for $\mu \ge \bar{\mu}$. It follows from Lax-Milgram that (3.9) has a unique solution $u \in H_0^{1,2}(B)$. □

We define $I : H_0^{1,2}(B) \longrightarrow H^{-1,2}(B)$ by

$$(Iu)(v) := \int_B uv\,dx, v \in H_0^{1,2}. \tag{3.10}$$

Lemma 3.6 *I is compact.*

*Proof.*Let $I_2 : H_0^{1,2} \longrightarrow L^2$ be the natural embedding. I_2 is compact by C.1. Let $I_1 : L^2 \longrightarrow H^{-1,2}$ be given by (3.10). I_1 is continuous and therefore $I = I_1 \circ I_2$ compact. □

The equation $Lu = F$ for $u \in H_0^{1,2}, F \in H^{-1,2}$ is equivalent to the equation

$$L_\mu u + \mu I u := Lu - \mu I u + \mu I u = F, \mu > \bar{\mu} \quad \text{fixed.}$$

By Lax-Milgram, L_μ^{-1} is a one-to-one and continuous mapping of $H^{-1,2}$ onto $H_0^{1,2}$: in fact, $L_\mu u = G_u \Leftrightarrow a_\mu(u,v) = G_u v \quad \forall v \in H_0^{1,2}$. Now, by Lax-Milgram, there exists a unique $A \in L(H_0^{1,2})$, bijective with $A^{-1} \in L(H_0^{1,2})$, too, such that

$$(Au, v)_{1,2} = a_\mu(u,v) = G_u v \quad \forall v \in H_0^{1,2}.$$

Therefore $L_\mu = A$ by duality. Applying this to the above equation, we obtain the equivalent equation

$$u + \mu L_\mu^{-1} I u = L_\mu^{-1} F. \tag{3.11}$$

The mapping $T = -\mu L_\mu^{-1} I$ is compact by Lemma 3.6 and hence by the Fredholm alternative, Proposition 3.1, the existence of a function $u \in H_0^{1,2}$ satisfying (3.11) is a consequence of the uniqueness in $H_0^{1,2}$ of the trivial solution of the equation $Lu = 0$ (see proposition below):

$$u + \mu L_\mu^{-1} I u = 0 \Leftrightarrow L_\mu u + \mu u = 0 \Leftrightarrow Lu = 0 \Leftrightarrow u = 0.$$

Applying the Fredholm alternative, there exists a unique u, such that $u + \mu L_\mu^{-1} I u = L_\mu^{-1} F$, which is equivalent to $Lu = F$.

Proposition 3.7 *Problem (3.3) with $T = D_i f^i$ has at most one solution on $B = B_r$, if r is so small that $\frac{k_c \mu}{\lambda} c_1 r^2 < 1$, where $\mu > \bar{\mu}$ fixed, k_c, λ as above.*

The underlying idea of the following proof can be found in [23, Section 4.3].

*Proof.*Let u', u'' be solutions of (3.3), then $u = u' - u''$ is a solution of $Lu = 0$ in $B, u = 0$ on ∂B. By (3.7), we have

$$\int_{B_r} |Du|^2 + u^2 \, dx \le \frac{k_c}{\lambda} a(u,u) + \frac{k_c \bar{\mu}}{\lambda} \|u\|_2^2 = \frac{k_c \bar{\mu}}{\lambda} \|u\|_2^2.$$

Now by Poincaré, Corollary A.8,

$$\|u\|_{2;B_r}^2 \le c_1 r^2 \int_{B_r} |Du|^2 \, dx = c_1 r^2 \|Du\|_{2;B_r}^2.$$

Consequently, for sufficient small r,

$$0 \le \left(1 - \frac{k_c \bar{\mu}}{\lambda} c_1 r^2\right) \|Du\|_2^2 + \|u\|_2^2 \le 0$$

implies $u = 0$. □

3.1.2 L^p-subspaces

In this section we introduce Morrey-, Campanato- and BMO-spaces, which are subspaces of L^p and which are useful for regularity considerations of weak solutions of equations as in the last section. The original work of Bogachev, Krylov and Röckner in mind, we will only present the necessary facts, which are mostly taken from [13, II.9.1 and II.10.1].

Let Ω be a bounded domain in $\mathbb{R}^n$. We notate:

$$\begin{aligned} \Omega(x_0, R) &:= \Omega \cap B(x_0, R) \\ u_M &:= \fint_M u(x)\,dx = \frac{1}{|M|}\int_M u(x)\,dx \end{aligned}$$

for any measurable u and Borel set $M \neq \emptyset$ lying in the domain of u.

Definition 3.8 *If there exists a positive constant A, such that for any $x \in \Omega$ and any ρ with $0 < \rho < diam\,\Omega$ the estimate $|\Omega(x,\rho)| \geq A\rho^n$ is valid, we say that Ω is a domain of type (A).*

Definition 3.9 *Let $p \geq 1, \mu \geq 0$. The collection of functions $u \in L^p(\Omega)$ satisfying*

$$\sup_{\substack{x\in\Omega \\ 0<\rho<diam\,\Omega}} \rho^{-\mu} \int_{\Omega(x,\rho)} |u(z)|^p\,dz < \infty$$

with the norm

$$\|u\|_{L^{p,\mu}(\Omega)} := \left(\sup_{\substack{x\in\Omega \\ 0<\rho<diam\ \Omega}} \rho^{-\mu} \int_{\Omega(x,\rho)} |u(z)|^p\,dz \right)^{\frac{1}{p}}$$

constitutes a normed linear space and is called the Morrey space $L^{p,\mu}(\Omega)$.

$L^{p,\mu}(\Omega)$ is a Banach space: let (u_m) be $L^{p,\mu}$-cauchy, then $(u_m)L^p$-cauchy and thus there exists $u = L^p - \lim_{m\longrightarrow\infty} u_m$. Now, for any ρ, x:

$$\begin{aligned} \rho^{-\mu}\int_{\Omega(x,\rho)} |u - u_m|^p\,dz &= \lim_{n\longrightarrow\infty} \rho^{-\mu}\int_{\Omega(x,\rho)} |u_n - u_m|^p\,dz \\ &\leq \limsup_{n\longrightarrow\infty} \|u_n - u_m\|^p_{L^{p,\mu}}. \end{aligned}$$

Therefore,

$$\|u - u_m\|^p_{L^{p,\mu}} \leq \limsup_{n\longrightarrow\infty} \|u_m - u_n\|^p_{L^{p,\mu}} < \epsilon \quad \text{for sufficient large } m.$$

Lemma 3.10 *(i)* $L^{p,0}(\Omega) \cong L^p(\Omega)$

(ii) $L^{p,n}(\Omega) \cong L^\infty(\Omega)$

(iii) If $\mu > n$, then $L^{p,\mu}(\Omega) = \{0\}$.

(iv) If $p \le q$ and $\frac{n-\mu}{p} \ge \frac{n-\nu}{q}$, then $L^{q,\nu}(\Omega) \subset L^{p,\mu}(\Omega)$.

Proof.

(i) obvious

(ii) If $u \in L^\infty(\Omega)$, then

$$\|u\|_{L^{p,n}(\Omega)} \le \left(\sup_{\substack{x\in\Omega \\ 0<\rho<\text{diam }\Omega}} \rho^{-n}\|u\|_\infty^p |\Omega(x,\rho)| \right)^{\frac{1}{p}} \le \omega_n^{\frac{1}{p}}\|u\|_\infty.$$

Conversely, if $u \in L^{p,n}(\Omega)$, then for a. e. $x \in \Omega$, by 2.13, we have

$$u(x) = \lim_{\rho\longrightarrow 0} \fint_{\Omega(x,\rho)} u\, dz = \lim_{\rho\longrightarrow 0} \frac{1}{|B(x,\rho)|}\int_{\Omega(x,\rho)} u\, dz.$$

Therefore

$$\begin{aligned} |u(x)| &\le \sup_\rho \left(\frac{1}{\omega_n \rho^n} \int_{\Omega(x,\rho)} |u|\, dz \right) \\ &\le \sup_\rho \left[\frac{1}{\omega_n\rho^n} |\Omega(x,\rho)|^{1-\frac{1}{p}} \left(\int_{\Omega(x,\rho)} |u|^p\, dz \right)^{\frac{1}{p}} \right] \\ &\le \omega_n^{-\frac{1}{p}} \|u\|_{L^{p,n}(\Omega)}. \end{aligned}$$

(iii) If $u \in L^{p,\mu}(\Omega)$ for some $\mu > n$, then

$$\begin{aligned} \left| \frac{1}{|B(x,\rho)|} \int_{\Omega(x,\rho)} u\, dz \right| &\le \frac{1}{\omega_n\rho^n} (\omega_n\rho^n)^{1-\frac{1}{p}} \rho^{\frac{\mu}{p}} \left(\rho^{-\mu} \int_{\Omega(x,\rho)} |u|\, dz \right)^{\frac{1}{p}} \\ &\le C\rho^{\frac{\mu-n}{p}} \|u\|_{L^{p,\mu}(\Omega)} \xrightarrow[\rho\to 0]{} 0. \end{aligned}$$

Therefore $u(x) = 0$ for a. e. $x \in \Omega$ by 2.13.

(iv) If $u \in L^{q,\nu}(\Omega)$, then for $0 < \rho < \text{diam } \Omega$,

$$\begin{aligned} \rho^{-\mu} \int_{\Omega(x,\rho)} |u(z)|^p\, dz &\le \rho^{-\mu} |\Omega(x,\rho)|^{1-\frac{p}{q}} \left(\int_{\Omega(x,\rho)} |u|^q\, dz \right)^{\frac{p}{q}} \\ &= \rho^{-\mu} \omega_n^{1-\frac{p}{q}} \rho^{n-\frac{np}{q}} \rho^{\frac{\nu p}{q}} \left(\rho^{-\nu} \int_{\Omega(x,\rho)} |u|^q\, dz \right)^{\frac{p}{q}} \\ &= \omega_n^{1-\frac{p}{q}} \rho^{p\left[\frac{n-\mu}{p} - \frac{n-\nu}{q}\right]} \left(\rho^{-\nu} \int_{\Omega(x,\rho)} |u|^q\, dz \right)^{\frac{p}{q}} \\ &\le \omega_n^{1-\frac{p}{q}} (\text{diam}\Omega)^{p\left[\frac{n-\mu}{p} - \frac{n-\nu}{q}\right]} \|u\|_{L^{q,\nu}(\Omega)}^p. \end{aligned}$$

Thus $u \in L^{p,\mu}(\Omega)$. □

Next, we introduce the Campanato spaces. We will often use the following notations:

$$u_\rho := \fint_{B_\rho} u(z)\,dz \quad \text{and} \quad u_{x,\rho} := \fint_{B(x,\rho)} u(z)\,dz \quad \text{for} \quad x \in \Omega, 0 < \rho < d,$$

where we only integrate over $B_\rho^+, B(x,\rho)^+$, respectively, when reasonable by the context.

Definition 3.11 *Let $p \geq 1, \mu \geq 0$. The collection of functions $u \in L^p(\Omega)$ satisfying*

$$[u]_{p,\mu;\Omega} := \left(\sup_{\substack{x\in\Omega \\ 0<\rho<diam\ \Omega}} \rho^{-\mu} \int_{\Omega(x,\rho)} |u(z) - u_{x,\rho}|^p\,dz \right)^{\frac{1}{p}} < \infty$$

with the norm

$$\|u\|_{\mathcal{L}^{p,\mu}(\Omega)} := \|u\|_{p;\Omega} + [u]_{p,\mu;\Omega}$$

constitutes a normed linear space and is called the Campanato space $\mathcal{L}^{p,\mu}(\Omega)$.

$\mathcal{L}^{p,\mu}(\Omega)$ is a Banach space: in fact, let (u_m) be $\mathcal{L}^{p,\mu}$-cauchy, then there exists $u := L^p - \lim_{m\longrightarrow\infty} u_m$. For any x, ρ,

$$\begin{aligned} \rho^{-\mu} & \int_{\Omega(x,\rho)} |u - u_m - \fint_{\Omega(x,\rho)} u - u_m\,dz|^p\,dx \\ & = \rho^{-\mu} \int_{\Omega(x,\rho)\setminus N} |\lim_n u_{m_n} - u_m - \lim_n \fint_{\Omega(x,\rho)} u_{m_N} - u_m\,dz|^p\,dx, \end{aligned}$$

where N shall be the Lebesgue nullset, on which the for reason of L^p-convergence existing subsequence (u_{m_n}) does not converge pointwisely.

$$\begin{aligned} & \leq \liminf_n \rho^{-\mu} \int_{\Omega(x,\rho)} |u_{m_n} - u_m - \fint_{\Omega(x,\rho)} u_{m_n} - u_m\,dz|^p\,dx \quad \text{(Fatou)} \\ & \leq \liminf_n [u_{m_n} - u_m]^p_{p,\mu;\Omega}. \end{aligned}$$

It follows that

$$[u - u_m]^p_{p,\mu;\Omega} \leq \liminf_n [u_{m_n} - u_m]^p_{p,\mu;\Omega} < \epsilon$$

for sufficient large m.
Just like in $\mathcal{L}^{p,\mu}(\Omega)$, if $p \leq q$ and $\frac{n-\mu}{p} \geq \frac{n-\nu}{q}$, then $\mathcal{L}^{q,\nu}(\Omega) \subset \mathcal{L}^{p,\mu}(\Omega)$. In order to prove that $\mathcal{L}^{p,\mu}(\Omega) \cong L^{p,\mu}(\Omega)$ for $0 \leq \mu < n$ and suitable domains Ω, we need the following lemma.

Lemma 3.12 *Let Ω be a domain of type (A). If $u \in \mathcal{L}^{p,\mu}(\Omega)$, where $p \geq 1, \mu \geq 0$, then for any $x \in \bar{\Omega}$ and any $\tilde{r}, \tilde{R}$ with $0 < \tilde{r} < \tilde{R}$, we have*

$$|u_{x,\tilde{R}} - u_{x,\tilde{r}}| \leq C(p,A)[u]_{p,\mu;\Omega} \tilde{r}^{-\frac{n}{p}} \tilde{R}^{\frac{\mu}{p}}. \tag{3.12}$$

Proof. We have

$$|u_{x,\tilde{r}} - u_{x,\tilde{R}}|^p \le 2^{p-1}(|u(z) - u_{x,\tilde{R}}|^p + |u(z) - u_{x,\tilde{r}}|^p).$$

Integrating this inequality over $\Omega(x,\tilde{r})$, we obtain

$$|u_{x,\tilde{r}}-u_{x,\tilde{R}}|^p|\Omega(x,\tilde{r})| \le 2^{p-1}\left[\int_{\Omega(x,\tilde{R})}|u(z) - u_{x,\tilde{R}}|^p\,dz + \int_{\Omega(x,\tilde{r})}|u(z) - u_{x,\tilde{r}}|^p\,dz\right].$$

Since Ω is a domain of type (A), $|\Omega(x,\tilde{r})| \ge A\tilde{r}^n$, it follows that

$$\begin{aligned}|u_{x,\tilde{r}} - u_{x,\tilde{R}}|^p &\le 2^{p-1}A^{-1}\tilde{r}^{-n}(\tilde{R}^\mu[u]^p_{p,\mu;\Omega} + \tilde{r}^\mu[u]^p_{p,\mu;\Omega})\\ &\le C(p,A)[u]^p_{p,\mu;\Omega}\tilde{r}^{-n}\tilde{R}^\mu.\end{aligned}$$

Taking the p-th root on both sides, we get (3.12). □

Proposition 3.13 *Let Ω be a domain of type (A). Then*

$$\mathcal{L}^{p,\mu}(\Omega) \cong L^{p,\mu}(\Omega) \quad \textit{for} \quad 0 \le \mu < n.$$

Proof. First, we show that $u \in \mathcal{L}^{p,\mu}(\Omega)$ and $\|u\|_{\mathcal{L}^{p,\mu}(\Omega)} \le C\|u\|_{L^{p,\mu}(\Omega)}$, if $u \in L^{p,\mu}(\Omega), \mu \ge 0$: for any $x \in \Omega, \rho > 0$ we have

$$\begin{aligned}\int_{\Omega(x,\rho)}|u(z) - u_{x,\rho}|^p\,dz &\le 2^{p-1}\int_{\Omega(x,\rho)}(|u|^p + |u_{x,\rho}|^p)\,dz\\ &= 2^{p-1}\left[\int_{\Omega(x,\rho)}|u|^p\,dz + |\Omega(x,\rho)||u_{x,\rho}|^p\right].\end{aligned} \tag{3.13}$$

Moreover,

$$\begin{aligned}|u_{x,\rho}|^p &= \left|\frac{1}{|\Omega(x,\rho)|}\int_{\Omega(x,\rho)}u\,dz\right|^p\\ &\le \frac{1}{|\Omega(x,\rho)|^p}\left(\int_{\Omega(x,\rho)}|u|\,dz\right)^p\\ &\le \frac{1}{|\Omega(x,\rho)|^p}\left(|\Omega(x,\rho)|^{1-\frac{1}{p}}\left(\int_{\Omega(x,\rho)}|u|^p\,dz\right)^{\frac{1}{p}}\right)^p \quad \text{(Hölder)}\\ &= |\Omega(x,\rho)|^{-1}\int_{\Omega(x,\rho)}|u|^p\,dz.\end{aligned}$$

Substituting this inequality into (3.13), we get

$$\int_{\Omega(x,\rho)}|u(z) - u_{x,\rho}|^p\,dz \le 2^p\int_{\Omega(x,\rho)}|u|^p\,dz.$$

It follows that $[u]_{p,\mu;\Omega} \le 2\|u\|_{L^{p,\mu}(\Omega)}$ and $u \in \mathcal{L}^{p,\mu}(\Omega)$; furthermore, $\|u\|_{\mathcal{L}^{p,\mu}(\Omega)} \le C\|u\|_{L^{p,\mu}(\Omega)}$, where C depends on diam Ω (cf. Lemma 3.10(iv)).

Next, we show that $u \in L^{p,\mu}(\Omega)$ and $\|u\|_{L^{p,\mu}(\Omega)} \le C\|u\|_{\mathcal{L}^{p,\mu}(\Omega)}$, if $u \in \mathcal{L}^{p,\mu}(\Omega)$ and $0 \le \mu < n$: for any $x \in \Omega, \rho > 0$ we have

$$\rho^{-\mu}\int_{\Omega(x,\rho)}|u|^p\,dz \le 2^{p-1}\left(\rho^{-\mu}\int_{\Omega(x,\rho)}|u(z)-u_{x,\rho}|^p\,dz + \omega_n\rho^{n-\mu}|u_{x,\rho}|^p\right). \tag{3.14}$$

For $R > \rho > 0$, we have

$$|u_{x,\rho}|^p \le 2^{p-1}[|u_{x,R}|^p + |u_{x,R}-u_{x,\rho}|^p]. \tag{3.15}$$

In order to estimate $|u_{x,R}-u_{x,\rho}|$, in Lemma 3.12 we choose

$$\tilde{R} = R_i = \frac{R}{2^i}, \tilde{r} = R_{i+1} = \frac{R}{2^{i+1}}, (i = 0, 1, 2, \dots).$$

Then

$$|u_{x,R_i} - u_{x,R_{i+1}}| \le C(p,A)[u]_{p,\mu}\left(\frac{R}{2^i}\right)^{\frac{\mu}{p}}\left(\frac{R}{2^{i+1}}\right)^{-\frac{n}{p}}.$$

For any integer $h > 0$ it follows that

$$\begin{aligned}
|u_{x,R} - u_{x,R_{h+1}}| &\le C(p,A)[u]_{p,\mu}R^{\frac{\mu-n}{p}}\sum_{i=0}^{h}\underbrace{2^{-i\frac{\mu}{p}+(i+1)\frac{n}{p}}}_{2^{\frac{n}{p}}2^{\frac{i(n-\mu)}{p}}} \\
&\le C(p,A,n)[u]_{p,\mu}R^{\frac{\mu-n}{p}}\frac{2^{\frac{(n-\mu)(h+1)}{p}}-1}{2^{\frac{n-\mu}{p}}-1} \quad \text{(geom. series)} \\
&\le C(p,A,n,\mu)[u]_{p,\mu}R^{\frac{\mu-n}{p}}2^{\frac{(n-\mu)(h+1)}{p}} \\
&= C(p,A,n,\mu)[u]_{p,\mu}R_{h+1}^{\frac{\mu-n}{p}}.
\end{aligned}$$

For any fixed $\rho \in (0, \operatorname{diam}\Omega)$, we choose h, R such that

$$\operatorname{diam}\Omega \le 2^{h+1}\rho < 2\operatorname{diam}\Omega, R = 2^{h+1}\rho.$$

Then

$$|u_{x,R} - u_{x,\rho}| \le C(p,A,n,\mu)[u]_{p,\mu}\rho^{\frac{\mu-n}{p}} \tag{3.16}$$

and

$$|u_{x,R}| = \frac{1}{|\Omega|}\left|\int_\Omega u(z)\,dz\right| \le |\Omega|^{-\frac{1}{p}}\|u\|_{p;\Omega}. \tag{3.17}$$

Substituting (3.16) and (3.17) into (3.15), we get

$$|u_{x,\rho}|^p \le 2^{p-1}\left[\frac{1}{|\Omega|}\|u\|_p^p + C^p[u]_{p,\mu}^p\,\rho^{\mu-n}\right].$$

Substituting this inequality into (3.14) and using $n > \mu$, we derive

$$\begin{aligned}
\rho^{-\mu}\int_{\Omega(x,\rho)}|u|^p\,dz &\le 2^{p-1}[u]_{p,\mu}^p + \omega_n 2^{2p-2}\left[\frac{1}{|\Omega|}(\operatorname{diam}\Omega)^{n-\mu}\|u\|_p^p + C^p[u]_{p,\mu}^p\right] \\
&\le C[\|u\|_p^p + [u]_{p,\mu}^p],
\end{aligned}$$

where C depends on $p, A, n, \mu, \operatorname{diam}\Omega$. Therefore, $u \in L^{p,\mu}(\Omega)$ and $\|u\|_{p,\mu} \leq C\|u\|_{\mathcal{L}^{p,\mu}(\Omega)}$. □

We shall not need it, but for completeness, we state without proof the following (cf. [13, Proposition 9.1.5])

Proposition 3.14 (Integral characterization of Hölder continuity) *Let Ω be a domain of type (A). If $n < \mu \leq n+p$, then $\mathcal{L}^{p,\mu}(\Omega) \cong C^{0,\delta}(\bar{\Omega}), \delta = \frac{\mu-n}{p}$; if $\mu > n+p$, then $\mathcal{L}^{p,\mu}(\Omega) = \{constant\}$.*

Next, we introduce the space of bounded mean oscillation BMO.

Definition 3.15 *Suppose that Q^0 is a cube in $\mathbb{R}^n$. If $u \in L^1(Q^0)$ satisfies*

$$|u|_{*,Q^0} := \sup_{Q \subset Q^0} \fint_Q |u - u_Q|\, dx < \infty,$$

where the supremum is taken over all subcubes parallel to Q^0, then we say that $u \in BMO(Q^0)$.

The norm of an element in $\mathrm{BMO}(Q^0)$ is defined to be

$$\|u\|_{\mathrm{BMO}(Q^0)} := \|u\|_{1;Q^0} + |u|_{*,Q^0}.$$

With this norm, $\mathrm{BMO}(Q^0)$ is a Banach space; this follows analogously to the proof of completeness of the Campanato spaces $\mathcal{L}^{p,\mu}$.
We shall use Q_x to denote a cube with center at x and $Q_{x,r}$ to denote the cube in $\mathbb{R}^n$ centered at x with side length $2r$ and sides parallel to the coordinate axes. We often drop the x and r and simply denote cubes by $Q, Q^0, \ldots$, if there is not any confusion.

Lemma 3.16 *Let $u \in L^1(Q^0)$ such that*

$$|u|_{*,p;Q^0} := \left(\sup_{Q \subset Q^0} \frac{1}{|Q|} \int_Q |u - u_Q|^p\, dx \right)^{\frac{1}{p}} < \infty$$

(i. e. $u \in BMO(Q^0)$, if $p = 1$), then

$$\sup_{Q_x, x \in Q^0} \frac{1}{|Q \cap Q^0|} \int_{Q \cap Q^0} |u - u_{Q \cap Q^0}|^p\, dx < \infty.$$

*Proof.*Let $x \in Q^0 = Q_{x_0,r_0}$ and $Q = Q_{x,r}$. Without loss of generality $Q \neq Q \cap Q^0 \neq Q^0$.

- Case 1: Let $r \geq r_0$. Since $x \in Q^0$, we have $|Q \cap Q^0| \geq 2^{-n}|Q^0| = r_0^n$. Therefore, using

$$|u_M - u_N| \leq \frac{1}{|M|} \int_M |u - u_N|\, dx \tag{3.18}$$

in the last but one step,

$$
\begin{aligned}
&\frac{1}{|Q\cap Q^0|}\int_{Q\cap Q^0}|u-u_{Q\cap Q^0}|^p\,dx\\
&\le c(p)\frac{1}{|Q\cap Q^0|}\left(\int_{Q\cap Q^0}|u-u_{Q^0}|^p\,dx+\int_{Q\cap Q^0}|u_{Q^0}-u_{Q\cap Q^0}|^p\,dx\right)\\
&\le c(p)\left(\frac{|Q^0|}{|Q\cap Q^0|}|u|^p_{*,p;Q^0}+|u_{Q^0}-u_{Q\cap Q^0}|^p\right)\\
&\le c(p)\left(\frac{|Q^0|}{|Q\cap Q^0|}|u|^p_{*,p;Q^0}+\left(\frac{|Q^0|}{|Q\cap Q^0|}\frac{1}{|Q^0|}\int_{Q^0}|u-u_{Q^0}|\,dx\right)^p\right),\\
&\le 2^n c(p)|u|^p_{*,p;Q^0}+c(p)2^{pn}|u|^p_{*;Q^0}\quad\text{(independent of } Q\text{ !)}
\end{aligned}
$$

- Case 2: Let $r < r_0$, then $r^n \le |Q\cap Q^0|$ $(x\in Q^0!)$ and there exists $\tilde{Q}=\tilde{Q}_{\tilde{x},r}$ such that $Q\cap Q^0\subset\tilde{Q}=\tilde{Q}\cap Q^0$ (by parallel shifting in direction Q^0). Therefore,

$$\frac{\tilde{Q}}{|Q\cap Q^0|}\le\frac{(2r)^n}{r^n}=2^n.$$

Consequently,

$$
\begin{aligned}
&\frac{1}{Q\cap Q^0}\int_{Q\cap Q^0}|u-u_{Q\cap Q^0}|^p\,dx\\
&\le c(p)\left(\frac{|\tilde{Q}|}{|Q\cap Q^0|}\int_{\tilde{Q}}|u-u_{\tilde{Q}}|^p\,dx+|u_{\tilde{Q}}-u_{Q\cap Q^0}|^p\right)\\
&\qquad\text{(similarly as above)}\\
&\le 2^n c(p)|u|^p_{*,p;Q^0}+c(p)2^{pn}|u|^p_{*;Q^0}\quad\text{(independent of } Q\text{ !)}
\end{aligned}
$$

□

We need to get to know more about the BMO-space and present a result of F. John and L. Nirenberg [20].

Proposition 3.17 *There exist two constants C_1, C_2 only depending on n, such that for any $Q\subset Q^0$*

$$dx(\{x\in Q||u(x)-u_Q|>t\})\le C_1|Q|\exp\left[-\frac{C_2 t}{|u|_{*,Q^0}}\right]\;\forall u\in BMO(Q^0).$$

*Proof.*Since the asserted inequality is homogeneous in t, we may assume $|u|_{*,Q^0}=1$. Moreover, $u\in \mathrm{BMO}(Q^0)\Rightarrow u\in\mathrm{BMO}(Q)\forall Q\subset Q^0$, so we may also assume $Q=Q^0$. For $\alpha>1\ge \fint_{Q^0}|u-u_{Q^0}|\,dx$ we apply the Calderón-Zygmund decomposition, Lemma 2.12, to the function $|u(\cdot)-u_{Q^0}|$. There exist nonoverlapping cubes $(Q_j^{(1)})$, such that

$$\alpha<\fint_{Q_j^{(1)}}|u-u_{Q^0}|\,dx\le 2^n\alpha, \tag{3.19}$$

$$|u(x) - u_{Q^0}| \le \alpha \quad \text{a. e.} \quad x \in Q^0 \setminus \bigcup_j Q_j^{(1)}. \tag{3.20}$$

It follows that (see remark after Lemma 2.12)

$$\sum_j |Q_j^{(1)}| \le \frac{1}{\alpha} \int_{Q^0} |u - u_{Q^0}|\, dx \le \frac{1}{\alpha}|Q^0|, \tag{3.21}$$

$$|u_{Q_j^{(1)}} - u_{Q^0}| \le \fint_{Q_j^{(1)}} |u - u_{Q^0}|\, dx \le 2^n \alpha \tag{3.22}$$

Since $|u|_{*,Q^0} = 1$, we still have $\alpha > 1 \ge \fint_{Q_j^{(1)}} |u - u_{Q_j^{(1)}}|\, dx$ for any $Q_j^{(1)}$. Applying Lemma 2.12 again to the function $|u(\cdot) - u_{Q_j^{(1)}}|$, we obtain a sequence of nonoverlapping cubes $(u_{Q_j^{(2)}})$ (collecting all cubes obtained for each $u_{Q_j^{(1)}}$), such that

$$\sum_j |u_{Q_j^{(2)}}| \le \frac{1}{\alpha} \sum_j \int_{Q_j^{(1)}} |u - u_{Q_j^{(1)}}|\, dx, \tag{3.23}$$

$$|u(x) - u_{Q_j^{(1)}}| \le \alpha \quad \text{a. e.} \quad x \in Q_j^{(1)} \setminus \bigcup_i Q_i^{(2)}. \tag{3.24}$$

Again notice that we have $|u|_{*,Q^0} = 1$. (3.23) and (3.21) imply that

$$\sum_j |Q_j^{(2)}| \le \frac{1}{\alpha} \sum_j |Q_j^{(1)}| \le \frac{1}{\alpha^2}|Q^0|. \tag{3.25}$$

We shall show that

$$|u(x) - u_{Q^0}| \le 2 \cdot 2^n \alpha \quad \text{a. e.} \quad x \in Q^0 \setminus \bigcup_i Q_i^{(2)} : \tag{3.26}$$

in fact, if $x \in Q^0 \setminus \bigcup_j Q_j^{(1)}$, then (3.20) implies (3.26). Now suppose that $x \in \bigcup_j Q_j^{(1)} \setminus \bigcup_k Q_k^{(2)}$, then $x \in Q_j^{(1)}$ for some j and therefore, by (3.24) and (3.22),

$$|u(x) - u_{Q^0}| \le |u(x) - u_{Q_j^{(1)}}| + |u_{Q_j^{(1)}} - u_{Q^0}| \le 2 \cdot 2^n \alpha.$$

We inductively repeat the above decomposition. For any integer $k \ge 1$, there exist nonoverlapping cubes $Q_j^{(k)}$, such that

$$\sum_j |Q_j^{(k)}| \le \frac{1}{\alpha^k}|Q^0|, |u(x) - u_{Q^0}| \le k 2^n \alpha \quad \text{a. e.} \quad x \in Q^0 \setminus \bigcup_j Q_j^{(k)}.$$

It follows that

$$\mathrm{dx}(\{x \in Q^0 | |u(x) - u_{Q^0}| > 2^n k \alpha\}) \le \sum_j |Q_j^{(k)}|.$$

The above inequality is obviously valid for $k = 0$. For any $t \in (0, \infty)$, we choose $k \ge 0$, such that

$$2^n \alpha k < t \le 2^n \alpha (k+1).$$

Then

$$\begin{aligned}
&\mathrm{dx}(\{x \in Q^0 \| u(x) - u_{Q^0}| > t\}) \\
&\quad \le \mathrm{dx}(\{x \in Q^0 \| u(x) - u_{Q^0}| > 2^n k\alpha\}) \\
&\quad \le \frac{1}{\alpha^k}|Q^0| \\
&\quad \le \alpha \exp[-At]|Q^0|,
\end{aligned}$$

where $A = \frac{\log[\alpha]}{2^n\alpha}$, since $\alpha > 1$ and $\frac{t}{2^n\alpha} > k$. □

Corollary 3.18 *$BMO(Q^0) \cong \mathcal{L}^{p,n}(Q^0)$ for any $p \ge 1$.*

Proof.

- "$\Leftarrow$": Let $u \in \mathcal{L}^{1,n}(Q^0)$ and $Q = Q_{x,r} \subset Q^0$, then

  $$|Q| = (2r)^n, Q \subset B(x, r\sqrt{n}) \cap Q^0 =: B.$$

  Using (3.18) and $|B| \ge (2r)^n$, we obtain

  $$\begin{aligned}
  \frac{1}{|Q|}\int_Q |u - u_Q|\, dy &\le r^{-n}\int_B |u - u_B|\, dy + r^{-n}\int_Q |u_B - u_Q|\, dy \\
  &\le c(n)[u]_{1,n;Q^0} + |u_B - u_Q| \\
  &\le c(n)[u]_{1,n;Q^0} + \frac{1}{|Q|}\int_Q |u - u_B|\, dy \\
  &\le c(n)[u]_{1,n;Q^0} + \frac{|B|}{|Q|}\frac{1}{|B|}\int_B |u - u_B|\, dy \\
  &\le c(n)[u]_{1,n;Q^0},
  \end{aligned}$$

  since $\frac{|B|}{|Q|} \le \frac{\omega_n\sqrt{n}^n r^n}{r^n} = \omega_n\sqrt{n}^n$. Thus, $u \in \mathrm{BMO}(Q^0)$ and

  $$\|u\|_{\mathrm{BMO}} \le c(n)\|u\|_{\mathcal{L}^{1,n}(Q^0)}.$$

  If $p > 1$, then $\mathcal{L}^{p,n}(Q^0) \subset \mathcal{L}^{1,n}(Q^0)$ by Hölder. Hence,

  $$|u|_{*,Q^0} \le c(n)\|u\|_{\mathcal{L}^{1,n}(Q^0)} \le c(n, Q^0)\|u\|_{\mathcal{L}^{p,n}(Q^0)}$$

  and again $u \in \mathrm{BMO}(Q^0)$ with $\|u\|_{\mathrm{BMO}} \le c(n)\|u\|_{\mathcal{L}^{p,n}(Q^0)}$.

- "$\Rightarrow$": If $u \in \mathrm{BMO}(Q^0)$, then for $p \ge 1$ and $Q \subset Q^0$ we have

  $$\begin{aligned}
  \int_Q |u - u_Q|^p\, dx &= p\int_0^\infty t^{p-1}|\{x \in Q \| u(x) - u_Q| > t\}|\, dt \\
  &\le pC_1\int_0^\infty t^{p-1}|Q|\exp\left[-\frac{C_2 t}{|u|_{*,Q^0}}\right] dt \\
  &= pC_1\left(\frac{|u|_{*,Q^0}}{C_2}\right)^p |Q|\int_0^\infty e^{-t}t^{p-1}\, dt \\
  &\le c(p,n)|u|^p_{*,Q^0}|Q|.
  \end{aligned}$$

It follows that

$$\fint_Q |u - u_Q|^p \, dx \leq \underbrace{c(p,n)|u|^p_{*,Q^0}}_{\text{(independent of } Q!)} \quad . \tag{3.27}$$

We assert that $u \in \mathcal{L}^{p,n}(Q^0)$:
Let $x \in Q^0 = Q_{x_0,r_0}, \rho > 0, B := Q^0 \cap B(x,\rho)$, then

$$B \subset Q_{x,\rho} \cap Q^0 =: Q, |Q| \leq (2\rho)^n$$

and we compute

$$\begin{aligned}
\frac{1}{\rho^n} \int_B |u - u_B|^p \, dx &\leq \frac{c(p)}{\rho^n} \left(\int_B |u - u_Q|^p \, dx + \int_B |u_Q - u_B|^p \, dx \right) \\
&\leq c(p) \left(\frac{1}{|Q|} \int_Q |u - u_Q|^p \, dx + \omega_n |u_Q - u_B|^p \right) \\
&\leq c(p) \left(|u|^p_{*,p;Q^0} + \left(\frac{1}{|B|} \int_B |u - u_Q| \, dx \right)^p \right) \\
&\leq c(p) \left(|u|^p_{*,p;Q^0} + \left(\frac{|Q|}{|B|} \frac{1}{|Q|} \int_Q |u - u_Q| \, dx \right)^p \right) \\
&\leq c(p)|u|^p_{*,p;Q^0} + c(p,n)|u|^p_{*;Q^0} \\
&\leq c(n,p)|u|^p_{*;Q^0} \quad \text{(by (3.27))} \quad ,
\end{aligned}$$

which is finite and independent of ρ, if we can show that $f : \rho \longmapsto \frac{|Q|}{|B|}(q)$ is bounded, but

$$\frac{|Q|}{|B|}(\rho) \begin{cases} \leq \frac{2^n \rho^n}{2^{-n}\rho^n \omega_n} = \frac{2^{2n}}{\omega_n} & , \text{ if } \rho < r_0, \\ = 1 & , \text{ if } \rho > \operatorname{diam} Q^0 \end{cases}$$

and f is continuous. Thus $u \in \mathcal{L}^{p,n}(Q^0)$ and $[u]_{p,n;Q^0} \leq c(p,n)|u|_{*,Q^0}$.
Moreover,

$$\begin{aligned}
\|u\|^p_{p;Q^0} &\leq c \left(\int_{Q^0} |u - u_{Q^0}|^p \, dx + |u_{Q^0}|^p |Q^0| \right) \\
&\leq c(p,n,|Q^0|) \|u\|^p_{\mathrm{BMO}(Q^0)}.
\end{aligned}$$

Hence, $\|u\|_{\mathcal{L}^{p,n}(Q^0)} \leq C\|u\|_{\mathrm{BMO}(Q^0)}$.

□

We close this subsection with stating an interpolation theorem between L^p and BMO spaces, whose proof can be found in Appendix A.

Proposition 3.19 (Stampacchia interpolation) *Let Q^0 be a cube in $\mathbb{R}^n$. Suppose that for some $p \geq 1, T$ is both a bounded linear operator: $L^p(Q^0) \longrightarrow L^p(Q^0)$ and a bounded linear operator: $L^\infty(Q^0) \longrightarrow BMO(Q^0)$, i. e.*

$$\|Tu\|_{p;Q^0} \quad \leq \quad B_p \|u\|_{p;Q^0} \tag{3.28}$$

$$\|Tu\|_{BMO(Q^0)} \leq B_\infty \|u\|_{\infty;Q^0}. \tag{3.29}$$

Then for any $q \in [p, \infty), T$ *is a bounded linear operator:* $L^q(Q^0) \longrightarrow L^q(Q^0)$ *and*

$$\|Tu\|_{q;Q^0} \leq C\|u\|_{q;Q^0} \quad \forall u \in L^q(Q^0), \tag{3.30}$$

where C *only depends on* n, p, q, B_p, B_∞.

3.1.3 L^2-Theory of Elliptic PDEs in Divergence Form

For simplicity, let Ω denote a C^∞-domain in $\mathbb{R}^n$. We are going to consider equations of the form

$$Lu = D_i(a^{ij}D_ju) = D_if^i \quad \text{in} \quad \Omega \tag{3.31}$$

$$u = 0 \quad \text{on} \quad \partial\Omega. \tag{3.32}$$

Let $p > n, a^{ij} \in H^{1,p}(\Omega), A \geq \lambda I$. By the Sobolev embedding, there exists $\Lambda := \sup_{ij}\|a_{ij}\|_{\infty;\Omega} \in \mathbb{R}_+$. Of course, we can find $\nu \in \mathbb{R}_+$, such that

$$\nu^{-1}|\xi|^2 \leq a^{ij}(x)\xi_i\xi_j \leq \nu|\xi|^2 \quad \forall x \in \Omega, \xi \in \mathbb{R}^n. \tag{3.33}$$

For $u, v \in H^{1,2}(\Omega)$ we set

$$a(u,v) = \int_\Omega a^{ij}(x)D_juD_iv\,dx$$

$$< T, v > = \int_\Omega f^iD_iv\,dx$$

and give a reformulation of definition 3.2 corresponding to our setting, which is more specific than that around (3.2). We are going to follow considerations of [13, II.8].

Definition 3.20 *If* $u \in H^{1,2}_{loc}(\Omega)$ *satisfies*

$$\int a^{ij}D_juD_i\varphi\,dx = \int f^iD_i\varphi\,dx \quad \forall \varphi \in H^{1,2}_0(\Omega), \tag{3.34}$$

then u is said to be a weak solution of the elliptic equation (3.31). *If additionally* $u \in H^{1,2}_0(\Omega)$, *then u is said to be a weak solution of the Dirichlet problem* (3.31), (3.32).

Remark 3.21 *Let* $u \in H^{1,2}_0(\Omega)$ *be a weak solution of the Dirichlet problem* (3.31), (3.32) *with* $f_i \in L^2(\Omega)$. *Replacing* φ *by* u *in* (3.34), *by ellipticity and Cauchy-Schwarz, we obtain*

$$\frac{1}{\nu}\|Du\|^2_{2;\Omega} = \frac{1}{\nu}\int_\Omega |Du|^2\,dx \leq \int_\Omega f^iD_iu\,dx \leq \|f^i\|_{2;\Omega}\|D_iu\|_{2;\Omega}$$
$$\leq \|f\|_{2;\Omega}\|Du\|_{2;\Omega}.$$

Consequently, by the equivalence of norms on $H^{1,2}_0$ *(cf. Poincaré),*

$$\|u\|_{1,2;\Omega} \leq C\|Du\|_{2;\Omega} \leq C(\nu)\|f\|_{2;\Omega}.$$

Proposition 3.22 *Let* (3.33) *be in force. Suppose that* $f^i \in L^2(\Omega)$. *Then the Dirichlet problem* (3.31), (3.32) *admits an unique weak solution.*

Proof. $a(\cdot,\cdot)$ is a bounded bilinear form on $H_0^{1,2}(\Omega) \times H_0^{1,2}(\Omega)$. By ellipticity,

$$a(u,u) \geq \nu^{-1} \int |Du|^2\,dx \quad \forall u \in H_0^{1,2}(\Omega). \tag{3.35}$$

Thus, $a(\cdot,\cdot)$ is coercive. Since $< T, \cdot >$ is a bounded linear functional on $H_0^{1,2}(\Omega)$, there exists an unique $u \in H_0^{1,2}(\Omega)$ such that $a(u,v) =< T, v >$ $\forall v \in H_0^{1,2}(\Omega)$ by Lax-Milgram. □

Under higher regularity assumptions on the coefficients and the right-hand side of (3.31), we shall establish $H^{k,2}$-regularity for weak solutions.

Proposition 3.23 (Caccioppoli's inequality) *Under the assumptions of Proposition 3.22, if* $u \in H_{loc}^{1,2}(\Omega)$ *is a weak solution of* (3.31), *then for any* $x_0 \in \Omega, 0 < \rho < R < dist(x_0, \partial\Omega)$, *we have*

$$\int_{B(x_0,\rho)} |Du|^2\,dx \leq C\left[\frac{1}{(R-\rho)^2}\int_{B(x_0,\rho)} |u-a|^2\,dx + \int_{B(x_0,\rho)} |f-\alpha|^2\,dx\right], \tag{3.36}$$

where $a \in \mathbb{R}, \alpha \in \mathbb{R}^n$ *are arbitrary,* $C = C(n,\nu,a^{ij})$.

Proof. The integral identity (3.34) remains valid, if we replace f^i by $(f^i - \alpha^i)$; moreover, we choose $\varphi = \eta^2(u-a)$, where $\eta \in C_0^\infty(B(x_0,R))$ is a cutoff function:

$$0 \leq \eta \leq 1 \quad \text{on} \quad B(x_0,R), \eta \equiv 1 \quad \text{on} \quad B(x_0,\rho), |D\eta| \leq \frac{C}{R-\rho}. \tag{3.37}$$

Then

$$\begin{aligned} &\int_\Omega \eta^2 a^{ij}(x) D_j u D_i u\,dx \\ &= -\int_\Omega 2\eta(u-a)a^{ij}(x)D_j u D_i \eta\,dx + \int_\Omega \eta^2(f^j - \alpha^j)D_j u\,dx \\ &\quad + \int_\Omega 2\eta(f^j-\alpha^j)(u-a)D_j\eta\,dx. \end{aligned}$$

Therefore, by ellipticity and since $ab \le \epsilon a^2 + \frac{1}{4\epsilon} b^2$,

$$\begin{aligned}
&\nu^{-1} \int_{B(x_0,R)} |\eta Du|^2\, dx \\
&\le 2\nu \int_{B(x_0,R)} \eta |Du||u-a||D\eta|\, dx + \int_{B(x_0,R)} \eta^2 |f-\alpha||Du|\, dx + \\
&\quad 2 \int_{B(x_0,R)} \eta |f-\alpha||u-a||D\eta|\, dx \\
&\le C \int_{B(x_0,R)} \underbrace{\eta|Du|(|u-a||D\eta| + |f-\alpha|\eta)}_{\le \epsilon\eta^2|Du|^2 + \frac{1}{2\epsilon}|u-a|^2|D\eta|^2 + \frac{1}{2\epsilon}|f-\alpha|^2\eta^2} \, dx \\
&\quad + 2 \int_{B(x_0,R)} \underbrace{\eta|f-\alpha||u-a||D\eta|}_{\eta^2|f-\alpha|^2 + \frac{1}{4}|u-a|^2|D\eta|^2} \, dx \\
&\le C\epsilon \int_{B(x_0,R)} \eta^2 |Du|^2\, dx + \left(\frac{C}{2\epsilon} + \frac{1}{2}\right) \int_{B(x_0,R)} |u-a|^2 |D\eta|^2\, dx \\
&\quad + \left(\frac{C}{2\epsilon} + 1\right) \int_{B(x_0,R)} \eta^2 |f-\alpha|^2\, dx \\
&\le C\epsilon \int_{B(x_0,R)} \eta^2 |Du|^2\, dx + \frac{C'}{(R-\rho)^2} \int_{B(x_0,R)} |u-a|^2\, dx \\
&\quad + C' \int_{B(x_0,R)} |f-\alpha|^2\, dx,
\end{aligned}$$

where $C = \max(2\nu, 1), C' = C'(\epsilon, n)$. We take $\epsilon = \frac{\nu^{-1}}{2C}$. Then

$$\begin{aligned}
\int_{B(x_0,\rho)} |Du|^2\, dx &\le \int_{B(x_0,R)} |\eta Du|^2\, dx \\
&\le C' \left[\frac{1}{(R-\rho)}^2 \int_{B(x_0,R)} |u-a|^2\, dx + \int_{B(x_0,R)} |f-\alpha|^2\, dx \right]
\end{aligned}$$

□

Proposition 3.24 (Friedrichs' Theorem, [3, Proposition 10.17]) *Let $A \ge \nu^{-1} I$. Let $u \in H^{1,2}(\Omega)$ be a weak solution of (3.31) and $m \in \{0, 1, 2, \dots\}$. If $f_i \in H^{m+1,2}(\Omega), a_{ij} \in C^{m,1}(\Omega)$, then $u \in H^{m+2,2}_{loc}(\Omega)$.*

*Proof.*Let $m = 0$ and $D \subset\subset \Omega$. For $0 < h < \operatorname{dist}(D, \partial\Omega), k \in \{1, \dots, n\}$ and the canonical onb $\{e_1, \dots, e_n\}$ of $\mathbb{R}^n$ we set

$$D_k^h v(x) := \frac{1}{h}(v(x + he_k) - v(x)).$$

Hence, for any $\zeta \in C_0^\infty(D)$

$$
\begin{aligned}
&\int \sum_i D_i\zeta \left(\sum_j a_{ij} D_j D_k^h u + \sum_j (D_k^h a_{ij}) D_j u(x+he_k) - D_k^h f_i \right) dx \\
&= \frac{1}{h} \int \sum_i D_i\zeta \left(\sum_j a_{ij} D_j (u(x+he_k) - u) + \right. \\
&\qquad \left. \sum_j (a_{ij}(x+he_k) - a_{ij}) D_j u(x+he_k) - f_i(x+he_k) + f_i \right) dx \\
&= \frac{1}{h} \int \sum_i D_i\zeta \left(-\sum_j a_{ij} D_j u + \right. \\
&\qquad \left. \sum_j a_{ij}(x+he_k) D_j u(x+he_k) - f_i(x+he_k) + f_i \right) dx \\
&= 0,
\end{aligned}
\tag{3.38}
$$

i. e., setting

$$
f_{hi} := \sum_j (D_k^h a_{ij}) D_j u(x+he_k) - D_k^h f_i,
$$

$$
\int_D \sum_i D_i\zeta \left(\sum_j a_{ij} D_j (D_k^h u) + f_{hi} \right) dx = 0. \tag{3.39}
$$

Hence, $D_k^h u$ solves a PDE of the same type and we can apply Proposition 3.23 to obtain

$$
\|DD_k^h u\|_{2;B(x_0,\rho)} \le C(\|f_h\|_{2;D} + \|D_k^h u\|_{2;D}),
$$

for any ball $B(x_0,\rho) \subset\subset \Omega$. Therefore,

$$
\begin{aligned}
\|D_k^h u\|_{1,2;B(x_0,\rho)} &= \|D_k^h u\|_{2;B(x_0,\rho)} + \|DD_k^h u\|_{2;B(x_0,\rho)} \\
&\le C'(\|f_h\|_{2;D} + \|D_k^h u\|_{2;D}).
\end{aligned}
$$

Now, for any $u \in (H^{1,2} \cap C^\infty)(D)$,

$$
\begin{aligned}
\int_D |D_k^h u(x)|^2\, dx &= \int_D \left|\frac{1}{h}(u(x+he_k) - u(x))\right|^2 dx \\
&= \int_D \left|\frac{1}{h} \int_0^h D_k u(x+se_k)\, ds\right|^2 dx \\
&\le \int_D \frac{1}{h} \int_0^h |D_k u(x+se_k)|^2\, ds\, dx \\
&\le \int_\Omega |D_k u|^2\, dx.
\end{aligned}
\tag{3.40}
$$

Since $(H^{1,2} \cap C^\infty)(D) \subset H^{1,2}(D)$ dense, we can extend

$$T : (H^{1,2} \cap C^\infty)(D) \longrightarrow L^2(D), u \longmapsto D_k^h u$$

to all of $H^{1,2}(D)$, (3.40) remaining valid. Correspondingly,

$$\begin{aligned}
\|f_h\|_{2;D}^2 &\leq \sum_i \int_D (\max_{i,j} \operatorname{lip}(a_{ij}) \vee 1)^2 \left(\sum_j |D_j u(x + he_k)| + |D_k^h f_i| \right)^2 dx \\
&\leq C \left(\int_\Omega |Du|^2 \, dx + \sum_i \int_D |D_k^h f_i|^2 \, dx \right) \\
&\leq C'(\|Du\|_{2;\Omega} + \|D_k f\|_{2;\Omega})^2,
\end{aligned}$$

which shows that $(D_k^h u)_h$ is bounded in $H^{1,2}(B(x_0, \rho))$. Since $H^{1,2}(B(x_0, \rho))$ is reflexive, there exists $v_k \in H^{1,2}(B(x_0, \rho))$ such that for some subsequence $h \longrightarrow 0$,

$$D_k^h u \longrightarrow v_k \text{weakly in } H^{1,2}(B(x_0, \rho)).$$

It follows for $\zeta \in C_0^\infty(B(x_0, \rho))$ as $h \longrightarrow 0$, that

$$\begin{aligned}
\int_{B(x_0,\rho)} \zeta D_l v_k \, dx \longleftarrow \int_{B(x_0,\rho)} \zeta D_l D_k^h u \, dx
&= \frac{1}{h} \int_{B(x_0,\rho)} \zeta(x) D_l (u(x + he_k) - u(x)) \, dx \\
&= \int_{B(x_0,\rho)} \frac{1}{h} (\zeta(x - he_k) - \zeta(x)) D_l u(x) \, dx \\
&= - \int_{B(x_0,\rho)} \left(D_l \frac{1}{h} (\zeta(x - he_k) - \zeta(x)) \right) u(x) \, dx \\
&\longrightarrow \int_{B(x_0,\rho)} (D_{lk} \zeta) u \, dx,
\end{aligned}$$

i. e. $u \in H^{2,2}(B(x_0, \rho))$ with $D_{kl} u = D_l v_k = D_k v_l$ by definition of Sobolev spaces. Moreover, $D_k^h a_{ij} \longrightarrow D_k a_{ij}$ weakly in $L^2(B(x_0, \rho))$, since

$$\begin{aligned}
\int_{B(x_0,\rho)} \zeta D_k^h a_{ij} \, dx &= \int_{B(x_0,\rho)} a_{ij} D_k^{-h} \zeta \, dx \\
&\xrightarrow[h \to 0]{} - \int_{B(x_0,\rho)} a_{ij} D_k \zeta \, dx = \int_{B(x_0,\rho)} \zeta D_k a_{ij} \, dx.
\end{aligned}$$

Analogously, $D_k^h f_i \longrightarrow D_k f_i$ weakly in $L^2(B(x_0, \rho))$. We have even strong convergence of $D_j u(\cdot + he_k)$ to $D_j u$ in $L^2(D)$ as h tends to zero. Thus, we can pass to the limits in (3.39) and obtain for $\zeta \in C_0^\infty(B(x_0, \rho))$,

$$\int_{B(x_0,\rho)} \sum_i D_i \zeta \left(\sum_j a_{ij} D_j (D_k u) + \sum_j D_k a_{ij} D_j u - D_k f_i \right) dx = 0.$$

In the case $m \geq 1$ we can therefore take over the proof for u on Ω so far to the above PDE for $D_k u$ on $B(x_0, \rho)$ to obtain the full result iteratively. □

Corollary 3.25 *Suppose that* $a^{ij}, f^i \in C^\infty(\bar{\Omega})$ *with* (3.33). *If* $u \in H^{1,2}(\Omega)$ *is a weak solution of* (3.31), *then* $u \in C^\infty(\Omega)$.

*Proof.*By C.6, the embedding $J : C^{k,1}(\bar{\Omega}) \longrightarrow H^{k+1,\infty}(\Omega)$ is an isomorphism for each $k \in \mathbb{N}$. Now apply the Sobolev embedding D.10 to obtain the corollary. □

Corollary 3.26 *Suppose that* a^{ij} *are constants satisfying* (3.33), $f^i \equiv 0$. *If* $u \in H^{1,2}(\Omega)$ *is a weak solution of* (3.31), *then for any* $B(x_0, R) \subset\subset \Omega$ *and any positive integer* k,

$$\|u\|_{k,2;B(x_0,R-\epsilon)} \le C\|u\|_{2;B(x_0,R)}, C = C(n,\nu,k,R).$$

*Proof.*Under our assumptions, Corollary 3.25 implies that $u \in C^\infty(\Omega)$. It satisfies

$$\int_\Omega a^{ij} D_j u D_i \varphi \, dx = 0 \quad \forall \varphi \in H_0^{1,2}(\Omega). \tag{3.41}$$

We choose a sequence of R_k as follows:

$$R_0 = R, R_k = \frac{1}{2}(R - \epsilon + R_{k-1}), k \in \mathbb{N}.$$

Applying Lemma 3.23 to u, we obtain

$$\int_{B(x_0,R_1)} |D_s u|^2 \, dx \le \frac{C}{(R-R_1)^2} \int_{B(x_0,R)} |u|^2 \, dx, C = C(n,\nu), s = 1, \dots, n.$$

Note that $D_t u$ $(t = 1, \dots, n)$ satisfies

$$\int_\Omega a^{ij} D_j(D_t u) D_i \varphi \, dx = 0 \quad \forall \varphi \in H_0^{1,2}(\Omega), \text{supp } \varphi \subset \Omega :$$

in fact, employing (3.41), $u \in C^\infty(\Omega)$ and integration by parts, it follows that $a^{ij} D_i D_j u = 0$. Hence, $D_t(a^{ij} D_i D_j)u = a^{ij} D_i (D_j D_t) u = 0$.
Applying Lemma 3.23 to $D_t u$, we get

$$\begin{aligned}\int_{B(x_0,R_2)} |D_s D_t u|^2 \, dx &\le \frac{C}{(R_1-R_2)^2} \int_{B(x_0,R_1)} |D_t u|^2 \, dx \\ &\le \frac{C}{(R-R_1)^2 (R_1 - R_2)^2} \int_{B(x_0,R)} |u|^2 \, dx.\end{aligned}$$

Applying Lemma 3.23 k-times and Lebesgue, we are done, since $R_k \downarrow R - \epsilon$. □

Corollary 3.27 *Let* a^{ij} *be constants satisfying* (3.33), $f^i \equiv 0$. *If* $u \in H^{1,2}(B_R^+)$ *is a weak solution of* (3.31), $u = 0$ *on* $\Gamma_R := B_R \cap \{x_n = 0\}$, *then*

$$\|u\|_{k,2;B_{R-\epsilon}^+} \le C\|u\|_{2;B_R^+},$$

where $C = C(n, \nu, k, R)$.

Proof. Set $u(x', x_n) = -u(x', -x_n)$ for $x_n < 0$, then the extended function u satisfies $Lu = 0$ weakly:
Let $\varphi \in C_0^\infty(B_R)$ and for $\delta > 0$ let η be an even function in $C^\infty(\mathbb{R})$, such that $\eta(t) = 0$ for $|t| \le \delta, \eta(t) = 1$ for $|t| \ge 2\delta$ and $|\eta'| \le \frac{2}{\delta}$. Then

$$\begin{aligned} 0 &= \int_{B_R} a^{ij} D_i u D_j(\eta(x_n)\varphi)\, dx \\ &= \int_{B_R} a^{ij} \eta(x_n) D_i u D_j \varphi\, dx + \int_{B_R} \varphi \eta'(x_n) D_n u a^{ij}\, dx. \end{aligned}$$

Now

$$\left| \int_{B_R} \varphi \eta' D_n u\, dx \right| \le 8 \max |D\varphi| \int_{\{0 < x_n < 2\epsilon\}} |D_n u|\, dx \underset{\delta \downarrow 0}{\longrightarrow} 0.$$

(see proof of 2.22). Consequently, letting $\delta \downarrow 0$, we obtain

$$0 = \int_{B_R} a^{ij} D_i u D_j \varphi\, dx,$$

so that $u \in H^{1,2}(B_R)$ is a weak solution of (3.31). By Corollary 3.26,

$$\|u\|_{k,2;B^+_{R-\epsilon}} \le \|u\|_{k,2;B_{R-\epsilon}} \le C\|u\|_{2;B_R} \le 2C\|u\|_{2;B^+_R}.$$

□

The following considerations mainly base on [12] and shall lead to the following regularity result of weak solutions:

Proposition 3.28 (cf. [13, II Theorem 9.2.4]) *Suppose that a^{ij} are constants satisfying (3.33), $f^i \in \mathcal{L}^{2,\mu}(\Omega), 0 \le \mu \le n$ and $\partial\Omega$ is smooth. If $u \in H_0^{1,2}(\Omega)$ is a weak solution of (3.31), then $Du \in \mathcal{L}^{2,\mu}(\Omega; \mathbb{R}^n)$; furthermore,*

$$\|Du\|_{\mathcal{L}^{2,\mu}(\Omega)} \le C\|f\|_{\mathcal{L}^{2,\mu}(\Omega)},$$

where $C = C(n, \nu, \mu, diam\,\Omega)$.

We start with three fundamental lemmas.

Lemma 3.29 ([12, Lemma 6.1]) *Let $\varphi : (0,\infty) \longrightarrow \mathbb{R}$ and $B : (1,\infty) \longrightarrow \mathbb{R}$ be nonnegative, $A > 1, \alpha > 0$. Suppose that for any $p > 1$ there exists $t(p) > 0$, such that for $\rho, r \in (0, t(p)]$ with $1 < \frac{r}{\rho} \le p$ the following inequality holds:*

$$\varphi(\rho) \le A \left(\frac{\rho}{r}\right)^\alpha \varphi(r) + B(p)\rho^\alpha. \tag{3.42}$$

Then for any $\epsilon > 0$ and ρ, r, such that $0 < \rho < r < t(A^{\frac{1}{\epsilon}})$,

$$\varphi(\rho) \le A \left(\frac{\rho}{r}\right)^{\alpha-\epsilon} \varphi(r) + B(A^{\frac{1}{\epsilon}}) \frac{Ar^\epsilon}{A-1} \rho^{\alpha-\epsilon}. \tag{3.43}$$

*Proof.*Fix $\epsilon > 0$, set $p_\epsilon := A^{\frac{1}{\epsilon}}$. For $\rho, r \in (0, t(p_\epsilon)]$, such that $1 < \frac{r}{\rho} \leq p_\epsilon$, (3.43) follows immediately from (3.42), for

$$\begin{aligned}\varphi(\rho) &\leq A\left(\frac{\rho}{r}\right)^{\alpha}\varphi(r) + B(A^{\frac{1}{\epsilon}})\rho^{\alpha} \leq A\left(\frac{\rho}{r}\right)^{\alpha-\epsilon}\varphi(r) + B(A^{\frac{1}{\epsilon}})r^{\epsilon}\rho^{\alpha-\epsilon}\\ &\leq A\left(\frac{\rho}{r}\right)^{\alpha-\epsilon}\varphi(r) + B(A^{\frac{1}{\epsilon}})\frac{A}{A-1}r^{\epsilon}\rho^{\alpha-\epsilon}.\end{aligned}$$

Now suppose that $\rho, r \in (0, t(p_\epsilon)]$, $\frac{r}{\rho} > p_\epsilon$. There exists $h \in \mathbb{N}$, such that

$$\left(A^{\frac{h}{\epsilon}} =\right) p_\epsilon^h < \frac{r}{\rho} < p_\epsilon^{h+1}. \tag{3.44}$$

Therefore,

$$1 < \frac{r}{\rho p_\epsilon^h} < p_\epsilon, r < \rho p_\epsilon^{h+1},$$

which implies

$$1 < \left(\frac{\rho p_\epsilon^{h+1}}{\rho p_\epsilon^h} =\right)\frac{\rho p_\epsilon^h}{\rho p_\epsilon^{h-1}} = \cdots = \frac{\rho p_\epsilon}{\rho} = p_\epsilon.$$

We apply (3.42) to the list of pairs $(\rho p_\epsilon^h, r), (\rho p_\epsilon^{h-1}, \rho p_\epsilon^h), \ldots, (\rho, \rho p_\epsilon)$ and $p = p_\epsilon$ to obtain

i_0)

$$\varphi(\rho p_\epsilon^h) \leq A\left(\frac{\rho}{r}\right)^{\alpha} p_\epsilon^{\alpha h}\varphi(r) + B(p_\epsilon)\rho^{\alpha}p_\epsilon^{\alpha h}$$

i_1)

$$\begin{aligned}\varphi(\rho p_\epsilon^{h-1}) &\leq A\left(\frac{\rho p_\epsilon^{h-1}}{\rho p_\epsilon^h}\right)^{\alpha}\varphi(\rho p_\epsilon^h) + B(p_\epsilon)(\rho p_\epsilon^{h-1})^{\alpha}\\ &= \frac{A}{p_\epsilon^{\alpha}}\varphi(\rho p_\epsilon^h) + B(p_\epsilon)\rho^{\alpha}p_\epsilon^{\alpha(h-1)}\end{aligned}$$

$\ldots i_h$)

$$\varphi(\rho) \leq \frac{A}{p_\epsilon^{\alpha}}\varphi(\rho p_\epsilon) + B(p_\epsilon)\rho^{\alpha}.$$

Successive usage of the predecessor-estimate yields

$$\begin{aligned}\varphi(\rho) &\leq \frac{A}{p_\epsilon^{\alpha}}\varphi(\rho p_\epsilon) + B(p_\epsilon)\rho^{\alpha}\\ &\leq \frac{A}{p_\epsilon^{\alpha}}\left(\frac{A}{p_\epsilon^{\alpha}}\varphi(\rho p_\epsilon^2) + B(p_\epsilon)\rho^{\alpha}p_\epsilon^{\alpha}\right) + B(p_\epsilon)\rho^{\alpha}\\ &\leq \cdots \leq A^{h+1}\left(\frac{\rho}{r}\right)^{\alpha}\varphi(r) + B(p_\epsilon)\rho^{\alpha}\sum_{j=0}^{h}A^j.\end{aligned} \tag{3.45}$$

Therefore,

$$\begin{aligned}\varphi(\rho) &\leq A^{h+1}\left(\frac{\rho}{r}\right)^{\epsilon}\left(\frac{\rho}{r}\right)^{\alpha-\epsilon}\varphi(r)+B(p_\epsilon)\frac{A^{h+1}-1}{A-1}r^{\epsilon}\rho^{\alpha-\epsilon}\left(\frac{\rho}{r}\right)^{\epsilon}\\ &\leq A\left(\frac{\rho}{r}\right)^{\alpha-\epsilon}\varphi(r)+B(p_\epsilon)\frac{A}{A-1}r^{\epsilon}\rho^{\alpha-\epsilon}\quad (\text{since } \left(\tfrac{\rho}{r}\right)^{\epsilon}<A^{-h})\end{aligned}$$

□

Lemma 3.30 *Let φ, B, A be as in Lemma 3.29, $\alpha, \beta \in \mathbb{R}$ such that $0<\beta<\alpha$. Suppose that for any $p>1$ there exists $t(p)>0$ such that for $\rho, r \in (0, t(p)]$ with $1<\frac{r}{\rho}\leq p$ we have*

$$\varphi(\rho) \leq A\left(\frac{\rho}{r}\right)^{\alpha}\varphi(r)+B(p)\rho^{\beta}. \tag{3.46}$$

Then $\forall 0<\epsilon<\alpha-\beta, \forall r\in(0,t(A^{\frac{1}{\epsilon}})]$ and $\forall 0<\rho<r$

$$\varphi(\rho) \leq A\left(\frac{\rho}{r}\right)^{\alpha-2}\varphi(r)+B(A^{\frac{1}{\epsilon}})\frac{A^{\frac{\alpha-\beta}{\epsilon}}}{A^{\frac{\alpha-\beta}{\epsilon}}-A}\rho^{\beta}. \tag{3.47}$$

*Proof.*Fix $0<\epsilon<\alpha-\beta$ and set $p_\epsilon := A^{\frac{1}{\epsilon}}$. For $\rho, r \in (0, t(p_\epsilon)]$ with $1<\frac{r}{\rho}\leq p_\epsilon$ (3.47) follows from (3.46), since $\frac{r}{\rho}<1$ and

$$\frac{A^{\frac{\alpha-\beta}{\epsilon}}}{A^{\frac{\alpha-\beta}{\epsilon}}-A}>1.$$

Now, suppose $\rho, r \in (0, t(p_\epsilon)]$ and $\frac{r}{\rho}>p_\epsilon$. Analogously to Lemma 3.29, for some $h\in\mathbb{N}$ with $p_\epsilon^h<\frac{r}{\rho}\leq p_\epsilon^{h+1}$, we obtain

i_0)
$$\varphi(\rho p_\epsilon^h)\leq A\left(\frac{\rho}{r}\right)^{\alpha}p_\epsilon^{h\alpha}\varphi(r)+B(p_\epsilon)\rho^{\beta}p_\epsilon^{h\beta}$$

i_1)
$$\varphi(\rho p_\epsilon^{h-1})\leq \frac{A}{p_\epsilon^{\alpha}}\varphi(\rho p_\epsilon^h)+B(p_\epsilon)\rho^{\beta}p_\epsilon^{h-1\beta}$$

$\ldots i_h$)
$$\varphi(\rho)\leq \frac{A}{p_\epsilon^{\alpha}}\varphi(\rho p_\epsilon)+B(p_\epsilon)\rho^{\beta}$$

and instead of (3.45),

$$\begin{aligned}\varphi(\rho) &\leq \frac{A}{p_\epsilon^{\alpha}}\varphi(\rho p_\epsilon)+B(p_\epsilon)\rho^{\beta}\\ &\leq \frac{A}{p_\epsilon^{\alpha}}\left(\frac{A}{p_\epsilon^{\alpha}}\varphi(\rho p_\epsilon^2)+B(p_\epsilon\rho^{\beta}p_\epsilon^{\beta}\right)+B(p_\epsilon)\rho^{\beta}\\ &\leq \frac{A}{p_\epsilon^{\alpha}}\left(\frac{A}{p_\epsilon^{\alpha}}\left(\frac{A}{p_\epsilon^{\alpha}}\varphi(\rho p_\epsilon^3)+B(p_\epsilon)\rho^{\beta}p_\epsilon^{2\beta}\right)+B(p_\epsilon)\rho^{\beta}p_\epsilon^{\beta}\right)+B(p_\epsilon)\rho^{\beta}\\ &\leq \cdots \leq A^{h+1}\left(\frac{\rho}{r}\right)^{\alpha}\varphi(r)+B(p_\epsilon)\rho^{\beta}\sum_{j=0}^{h}\left(\frac{A}{p_\epsilon^{\alpha-\beta}}\right)^{j}\end{aligned} \tag{3.48}$$

We have $\left(\frac{\rho}{r}\right)^\epsilon < A^{-h}$ and

$$\sum_{j=0}^{h}\left(\frac{A}{p_\epsilon^{\alpha-\beta}}\right)^j = \sum_{j=0}^{h}\left(\frac{A}{A^{\frac{\alpha-\beta}{\epsilon}}}\right)^j = \sum_{j=0}^{h}(A^{1-\frac{\alpha-\beta}{\epsilon}})^j < \frac{1}{1-A^{1-\frac{\alpha-\beta}{\epsilon}}} = \frac{A^{\frac{\alpha-\beta}{\epsilon}}}{A^{\frac{\alpha-\beta}{\epsilon}}-A}.$$

Therefore,

$$\varphi(\rho) \le A\left(\frac{\rho}{r}\right)^{\alpha-\epsilon}\varphi(r) + B(A^{\frac{1}{\epsilon}})\rho^\beta \frac{A^{\frac{\alpha-\beta}{\epsilon}}}{A^{\frac{\alpha-\beta}{\epsilon}}-A}$$

as in Lemma 3.29. □

Lemma 3.31 (cf. [13, II.9.2.1]) *Let φ be a nonnegative and nondecreasing function. Suppose that*

$$\varphi(\rho) \le A\left[\left(\frac{\rho}{R}\right)^\alpha + \epsilon\right]\varphi(R) + BR^\beta \quad \forall 0 < \rho \le R \le R_0,$$

where A, α, β, R_0 are nonnegative constants, $\beta < \alpha$. Then there exist positive constants $\epsilon_0 = \epsilon_0(A, \alpha, \beta)$ and $C = C(A, \alpha, \beta)$, such that for $\epsilon < \epsilon_0$,

$$\varphi(\rho) \le C\left[\left(\frac{\rho}{R}\right)^\beta \varphi(R) + B\rho^\beta\right] \quad \forall 0 < \rho \le R \le R_0.$$

*Proof.*Under our assumption, for any $\tau \in (0,1)$ we have

$$\varphi(\tau R) \le A\tau^\alpha[1+\epsilon\tau^{-\alpha}]\varphi(R) + BR^\beta, R \le R_0,$$

where we assume without loss of generality that $A \ge 1$. At first, we take γ, such that $\beta < \gamma < \alpha$. Then we choose τ, such that $2A\tau^\alpha = \tau^\gamma$, i. e. $\tau = \exp[-\frac{\log(2A)}{\alpha-\gamma}]$. With these choices of γ, τ and ϵ_0, for $\epsilon < \epsilon_0$ we have

$$\begin{aligned}\varphi(\tau R) &\le 2A\tau^\alpha\varphi(R) + BR^\beta \\ &\le \tau^\gamma\varphi(R) + BR^\beta, R \le R_0.\end{aligned}$$

Now, we iterate. For any positive integer k, we have

$$\begin{aligned}\varphi(\tau^{k+1}R) &\le \tau^\gamma\varphi(t^kR) + B\tau^{k\beta}R^\beta \\ &\le \tau^\gamma(\tau^\gamma\varphi(t^{k-1}R) + B\tau^{(k-1)\beta}R^\beta) + B\tau^{k\beta}R^\beta \\ &\le \cdots \le \tau^{(k+1)\gamma}\varphi(R) + B\tau^{k\beta}R^\beta\sum_{j=0}^{k}\tau^{j(\gamma-\beta)} \\ &\le \tau^{(k+1)\gamma}\varphi(R) + B\tau^{k\beta}R^\beta\frac{1-\tau^{(\gamma-\beta)(k+1)}}{1-\tau^{\gamma-\beta}} \\ &\le \tau^{(k+1)\gamma}\left[\varphi(R) + BR^\beta\frac{1}{\tau^\beta-\tau^\gamma}\right],\end{aligned}$$

i. e.

$$\varphi(\tau^{k+1}R) \le C\tau^{(k+1)\beta}[\varphi(R) + BR^\beta], C = C(A,\alpha,\beta). \tag{3.49}$$

For any $\rho \le R$, we choose k such that $\tau^{k+1}R < \rho \le \tau^k R$. Inequality (3.49), isotony of φ and our choice of k imply that

$$\begin{aligned}\varphi(\rho) \le \varphi(\tau^k R) &\le C\tau^{k\beta}[\varphi(R) + BR^\beta] \\ &\le C\tau^{-\beta}\left(\frac{\rho}{R}\right)^\beta [\varphi(R) + BR^\beta] \\ &\le C\tau^{-\beta}\left[\left(\frac{\rho}{R}\right)^\beta \varphi(R) + B\rho^\beta\right] \\ &\le C_1\left[\left(\frac{\rho}{R}\right)^\beta \varphi(R) + B\rho^\beta\right],\end{aligned}$$

where $C_1 = C_1(A, \alpha, \beta)$. □

In the next lemmas, we establish some L^2-estimates for weak solutions.

Lemma 3.32 (cf. [12, 5.1]) *Let $u \in H_0^{1,2}(\Omega)$ be a weak solution of $Lu = D_j f^j$. Then there exists $c = c(\nu)$, such that for any $\alpha \in \mathbb{R}^n$*

$$\int_\Omega |Du|^2\,dx \le c(\nu)\int_\Omega |f - \alpha|^2\,dx \tag{3.50}$$

*Proof.*Since u is a weak solution, we have

$$\int_\Omega a^{ij} D_j u D_i \varphi\,dx = \int_\Omega (f_j - \alpha_j) D^j \varphi\,dx \quad \forall \varphi \in H_0^{1,2}(\Omega).$$

For $\varphi = u$ by ellipticity and Cauchy-Schwarz,

$$\begin{aligned}\nu^{-1}\int_\Omega |Du|^2\,dx &\le \sum_j \|f_j - \alpha_j\|_{2;\Omega}\|D_j u\|_{2;\Omega} \\ &\le \|f - \alpha\|_{2;\Omega}\|Du\|_{2;\Omega}.\end{aligned}$$

Dividing by $\|Du\|_{2;\Omega}$ yields the lemma. □

Lemma 3.33 *Let $u \in H^{1,2}(B_r^+)$ be solution of $Lu = D_j f^j$ in B_r^+ and $u = 0$ on $\Gamma_r = \partial B_r^+ \cap \{x_n = 0\}$ (i. e., $\exists u_n \in C^1(\bar{B}_r^+), u_n = 0 \quad on \quad \Gamma_r, n \in \mathbb{N} : u_n \longrightarrow u \quad in \quad H^{1,2}(B_r^+)$). Then there exists $c = c(\nu)$, such that for any $0 < \rho < r, \alpha \in \mathbb{R}^n$,*

$$\int_{B_\rho^+} |Du|^2\,dx \le c(\nu)\left(\frac{1}{(r-\rho)^2}\int_{B_r^+} |u|^2\,dx + \int_{B_r^+} |f - \alpha|^2\,dx\right). \tag{3.51}$$

*Proof.*Since u is a weak solution, for any $\varphi \in H_0^{1,2}(B_r^+), \alpha \in \mathbb{R}^n$ we have

$$\int_{B_r^+} a^{ij} D_j u D_i \varphi\,dx = \int_{B_r^+} (f_j - \alpha_j) D^j \varphi\,dx. \tag{3.52}$$

Let $\theta \in C_0^\infty(B_r)$ with $0 \le \theta \le 1, \theta \equiv 1 \quad$ on $\quad B_\rho, |D_j\theta| \le \frac{K}{r-\rho}$ and $\theta_+ = \theta_{|B_r^+}$. Since $u = 0$ on Γ_r, $\theta_+^2 u \in H_0^{1,2}(B_r^+)$. Take $\varphi = \theta_+^2 u$ and go on as in the proof of 3.23. □

Lemma 3.34 *For any $u \in H^{1,2}(B_r^+)$ with $u = 0$ on Γ_r one has*

$$\int_{B_r^+} |u|^2\,dx \le \frac{r^2}{2} \int_{B_r^+} |D_n u|^2\,dx. \tag{3.53}$$

*Proof.*By A.4, we may assume without loss of generality that $u \in C^1(\bar{B}_r^+), u = 0$ on Γ_r. Let $\bar{x} = (x_1, \dots, x_{n-1}, 0) \in \Gamma_r$ and $x \in B_r^+$. Since $u(\bar{x}) = 0$, we have

$$u(x) = \int_0^{x_n} D_n u(x_1, \dots, x_{n-1}, t)\,dt$$

and thus

$$\begin{aligned} |u(x)|^2 &\le \left(\int_0^{\sqrt{r^2 - |\bar{x}|^2}} |1_{[0,x_n]}| |D_n u(x_1, \dots, x_{n-1}, t)|\,dt \right)^2 \\ &\le x_n \int_0^{\sqrt{r^2 - |\bar{x}|^2}} |D_n u(x_1, \dots, x_{n-1}, t)|^2\,dt \end{aligned}$$

by Hölder. Integrating over B_r^+ yields

$$\begin{aligned} \int_{B_r^+} |u(x)|^2\,dx &= \int_{\Gamma_r} \int_0^{\sqrt{r^2 - |\bar{x}|^2}} |u(x)|^2\,dx_n\,d\bar{x} \\ &\le \int_{\Gamma_r} \int_0^{\sqrt{r^2 - |\bar{x}|^2}} x_n \int_0^{\sqrt{r^2 - |\bar{x}|^2}} |D_n u(x_1, \dots, x_{n-1}, t)|^2\,dt\,dx_n\,d\bar{x} \\ &= \int_{\Gamma_r} \int_0^{\sqrt{r^2 - |\bar{x}|^2}} |D_n u(\bar{x}, t)|^2\,dt \int_0^{\sqrt{r^2 - |\bar{x}|^2}} x_n\,dx_n\,d\bar{x} \\ &\le \frac{r^2}{2} \int_{B_r^+} |D_n u|^2\,dx \end{aligned}$$

by the Fundamental Theorem of analysis. □

Properties of homogeneous solutions in the constant case

Lemma 3.35 (cf. [13, II.9.2.2.(i)]) *Let $u \in H^{1,2}(\Omega)$ be a weak solution of $Lu = 0$, where we assume that a^{ij} are constants satisfying* (3.33)*. Then there exists a constant $C > 0$, such that for any $x_0 \in \Omega$ and ρ, R with $0 < \rho \le R <$ dist $(x_0, \partial\Omega) =: d$,*

$$\int_{B(x_0,\rho)} |u|^2\,dx \le C \left(\frac{\rho}{R}\right)^n \int_{B(x_0,R)} |u|^2\,dx. \tag{3.54}$$

*Proof.*Corollary 3.26 implies that $u \in H^{k,2}(B(x_0, \frac{R}{2}))$ and that

$$\|u\|_{k,2;B(x_0,\frac{R}{2})} \le C \|u\|_{2;B(x_0,\frac{R}{2})} \quad (k \in \mathbb{N}), \tag{3.55}$$

where $C = C(n, \nu, k, R)$. We choose $k > n$. By the Sobolev embedding, D.10,

$$\sup_{B(x_0, \frac{R}{2})} |u| \le C\|u\|_{k,2;B(x_0,\frac{R}{2})}. \tag{3.56}$$

It follows from (3.56) that for $0 < \rho < \frac{R}{2}$,

$$\int_{B(x_0,\rho)} |u|^2\,dx \le \omega_n \rho^n \sup_{B(x_0,\frac{R}{2})} |u|^2 \le C\rho^n \|u\|^2_{k,2;B(x_0,\frac{R}{2})}.$$

By (3.55),

$$\int_{B(x_0,\rho)} |u|^2\,dx \le C(R)\rho^n \int_{B(x_0,R)} |u|^2\,dx, \tag{3.57}$$

where $C(R)$ also depends on R in addition to the dependencies on n, ν, d. Using a rescaling technique, we next show that $C(R) = C_1 R^{-n}$, where $C_1 = C_1(n, \nu, d)$. In fact, at first we fix $a < d$. Then (3.57) implies

$$\int_{B(x_0,r)} |u|^2\,dx \le C(a) r^n \int_{B(x_0,a)} |u|^2\,dx$$

for any $r < \frac{a}{2}$. Set $y = a\frac{x-x_0}{R} = T(x), r = a\frac{\rho}{R}$. Then $|x - x_0| < \rho$ implies $|y| < r < \frac{a}{2}$; therefore,

$$\int_{B(x_0,\rho)} |u(x)|^2\,dx = \left(\frac{R}{a}\right)^n \int_{B_r} |u(x_0 + Ra^{-1}y)|^2\,dy$$

(since $T : B(x_0, \rho) \longrightarrow B_r, \det T = \left(\frac{a}{R}\right)^n$)

$$\begin{aligned}
&\le \left(\frac{R}{a}\right)^n C(a) r^n \int_{B_a} \left| u\left(x_0 + \frac{R}{a} y\right)\right|^2 dy \\
&= \left(\frac{R}{a}\right)^n C(a) r^n \left(\frac{a}{R}\right)^n \int_{B(x_0,R)} |u(x)|^2\,dx \\
&= \rho^n C(a) \left(\frac{a}{R}\right)^n \int_{B(x_0,R)} |u(x)|^2\,dx \\
&= \rho^n R^{-n} C_1 \int_{B(x_0,R)} |u(x)|^2\,dx,
\end{aligned}$$

where $C_1 = C_1(n, \nu, d)$. This proves that (3.54) is valid for $0 < \rho < \frac{R}{2}$. For $\frac{R}{2} < \rho \le R$ one has

$$\int_{B_\rho} u^2\,dx \le \int_{B_R} u^2\,dx = \left(\frac{R}{\rho}\right)^n \left(\frac{\rho}{R}\right)^n \int_{B_R} u^2\,dx \le 2^n \left(\frac{\rho}{R}\right)^n \int_{B_R} u^2\,dx.$$

□

Corollary 3.36 (cf. [12, Corollary 7.1]) *Let u be as in Lemma 3.35. For any ρ with $0 < \rho \leq R < d, m \in \mathbb{N}^n$*

$$\int_{B_\rho} |D^m u|^2\, dx \leq c(\nu, d) \left(\frac{\rho}{R}\right)^n \int_{B_R} |D^m u|^2\, dx. \tag{3.58}$$

Proof. $D^m u$ satisfies the assumptions of Lemma 3.35 on B_R, since $u \in C^\infty(B_R)$ by 3.24, and $L(D^m u) = D^m L u = 0$, because of constant coefficients. □

Lemma 3.37 *Let $u \in C^\infty(B_R)$ be a solution of $Lu = 0$ on B_R. Then there exists a positive constant $C_1(\nu)$, such that for any $0 < \rho \leq R, \alpha \in \mathbb{R}$,*

$$\int_{B_\rho} |u - u_\rho|^2\, dx \leq C_1(\nu) \left(\frac{\rho}{R}\right)^{n+2} \int_{B_R} |u - \alpha|^2\, dx. \tag{3.59}$$

Proof. For any $\alpha \in \mathbb{R}$, $u - \alpha$ satisfies the assumptions of Lemma 3.35 on B_R. Analogously to Lemma 3.35,

$$\|u - \alpha\|^2_{k,2;B_{\frac{R}{2}}} \leq C(\nu, R, k) \int_{B_R} |u - \alpha|^2\, dx. \tag{3.60}$$

For sufficient large k, by D.10 we obtain

$$\sum_j \sup_{B\left(0,\frac{R}{2}\right)} |D_j u|^2 \leq C(n, R) \|u - \alpha\|^2_{k,2;B_{\frac{R}{2}}} \leq C(\nu, R) \int_{B_R} |u - \alpha|^2\, dx. \tag{3.61}$$

Let $0 < \rho \leq \frac{R}{2}$. For $x \in \bar{B}_\rho$, by the mean-value theorem we have

$$|u(x) - u(0)|^2 \leq C(n) \rho^2 \sum_j \sup_{B\left(0,\frac{R}{2}\right)} |D_j u|^2. \tag{3.62}$$

From (3.61) and (3.62), for $0 < \rho \leq \frac{R}{2}$ it follows that

$$\int_{B_\rho} |u - u_\rho|^2\, dx \leq \int_{B_\rho} |u - u(0)|^2\, dx \leq C(\nu, R) \rho^{n+2} \int_{B_R} |u - \alpha|^2\, dx, \tag{3.63}$$

where the first inequality follows from 3.39 and the rest of the proof is done as in 3.35. □

Corollary 3.38 *Let u be as in Lemma 3.37. Then for any $\rho \in (0, R], m \in \mathbb{N}^n, \alpha \in \mathbb{R}^n$,*

$$\int_{B_\rho} |D^m u - (D^m u)_\rho|^2\, dx \leq C_1(\nu) \left(\frac{\rho}{R}\right)^{n+2} \int_{B_R} |D^m u - \alpha|^2\, dx. \tag{3.64}$$

Proof. $D^m u$ verifies the assumptions of Lemma 3.37. □

Lemma 3.39 $\rho \longmapsto \Phi(\rho) = \int_{B(x_0,\rho)} |u(x) - u_{x_0,\rho}|^2\, dx$ *is nondecreasing.*

*Proof.*Setting

$$\Psi(t) = \int_{B(x_0,\rho)} |u(x) - t|^2\, dx, t \in \mathbb{R},$$

we derive that $\Phi(\rho) = \min_{t\in\mathbb{R}} \Psi(t)$. In fact,

$$\begin{aligned}
&\int_{B(x_0,\rho)} |u(x) - u_{x_0,\rho}|^2 - |u(x) - t|^2\, dx \\
&= \int_{B(x_0,\rho)} (u(x) - u_{x_0,\rho} + u(x) - t)(u(x) - u_{x_0,\rho} - u(x) + t)\, dx \\
&= \int_{B(x_0,\rho)} (2u(x) - (u_{x_0,\rho} + t))(t - u_{x_0,\rho})\, dx \\
&= 2(t - u_{x_0,\rho})u_{x_0,\rho} - t^2 + u^2_{x_0,\rho} \\
&= 2tu_{x_0,\rho} - t^2 - u^2_{x_0,\rho} \\
&= -(u^2_{x_0,\rho} - 2tu_{x_0,\rho} + t^2) \\
&= -(u_{x_0,\rho} - t)^2 < 0.
\end{aligned}$$

Thus for $\rho \leq R$ we have

$$\int_{B(x_0,\rho)} |u(x) - u_{x_0,\rho}|^2\, dx \leq \int_{B(x_0,\rho)} |u(x) - u_{x_0,R}|^2\, dx \leq \int_{B(x_0,R)} |u(x) - u_{x_0,R}|^2\, dx.$$

□

Properties of solutions of $Lu = D_i f^i$ in the continuous case

In the following we consider the equation

$$Lu = D_j f^j \quad \text{with} \quad f_j \in L^2(B_r), a_{ij} \in C(\bar{B}_r). \tag{3.65}$$

We set $\omega^2(r) := \sup_{ij} \sup_{\bar{B}_r} |a_{ij}(x) - a_{ij}(0)|^2$ and also write

$$\omega^2(r) = \sup_{ij} \sup_{\bar{B}^{(+)}(x_0,r)} |a_{ij}(x) - a_{ij}(x_0)|^2$$

later on without redefining; furthermore, let L_0 denote the constant coefficient operator

$$L_0 = a^{ij}(0) D_i D_j. \tag{3.66}$$

Lemma 3.40 *Let* $u \in H^{1,2}(B_r)$ *be a solution of* (3.65) *in* B_r. *Then there exists* $C(\nu)$, *such that for any* $\rho \in (0, r]$

$$\int_{B_\rho} |Du|^2\, dx \leq C(\nu) \left(\left[\left(\frac{\rho}{R} \right)^n + \omega^2(r) \right] \int_{B_r} |Du|^2\, dx + \int_{B_r} |f|^2\, dx \right). \tag{3.67}$$

*Proof.*From

$$\int_{B_r} a^{ij}(x) D_j u D_i \varphi \, dx = \int_{B_r} f^j D_j \varphi \, dx \quad \forall \varphi \in H_0^{1,2}(B_r)$$

it follows that

$$\int_{B_r} a^{ij}(0) D_j u D_i \varphi \, dx = \int_{B_r} [f_i + (a^{ij}(0) - a^{ij}(x)) D_j u] D_i \varphi \, dx. \tag{3.68}$$

Now decompose u in $v + w$, where v solves

$$\int_{B_r} a^{ij}(0) D_j v D_i \varphi \, dx = 0 \quad \forall \varphi \in H_0^{1,2}(B_r), v - u \in H_0^{1,2}(B_r) \tag{3.69}$$

and w solves

$$\int_{B_r} a^{ij}(0) D_j w D_i \varphi \, dx = \int_{B_r} [f_i + (a_{ij}(0) - a_{ij}(x)) D^j u] D_i \varphi \, dx \quad \forall \varphi \in H_0^{1,2}(B_r). \tag{3.70}$$

Since v solves $L_0 v = 0$ in B_r, we may assume that $v \in C^\infty(B_R)$ by Proposition 3.24 for any $B_R \subset\subset B_r$. Fix $\frac{r}{2} \leq R < r$. By Corollary 3.36, we obtain

$$\begin{aligned} \int_{B_\rho} |Dv|^2 \, dx &\leq C(\nu) \left(\frac{\rho}{R}\right)^n \int_{B_R} |Dv|^2 \, dx \\ &\leq 2^n C(\nu) \left(\frac{\rho}{r}\right)^n \int_{B_r} |Dv|^2 \, dx, \end{aligned} \tag{3.71}$$

for $\rho \in (0, R]$. Since R is arbitrary, (3.71) is valid for $0 < \rho \leq r$.
$w \in H_0^{1,2}(B_r)$ solves $L_0 w = D_i(f_i + [a^{ij}(0) - a^{ij}(x)] D_j u)$, where $f_i + [a_{ij}(0) - a_{ij}(x)] D^j u \in L^2(B_r)$ by assumption. We apply Lemma 3.32 with $\alpha = 0$ and obtain

$$\int_{B_\rho} |Dw|^2 \, dx \leq C(\nu) \int_{B_r} |f|^2 + \omega^2(r) |Du|^2 \, dx. \tag{3.72}$$

For $0 < \rho \leq r$, from (3.71), (3.72) and $u = v + w$ it follows that

$$\begin{aligned} &\int_{B_\rho} |Du|^2 \, dx \\ &\leq C(\nu) \left\{ \left[\left(\frac{\rho}{r}\right)^n + \omega^2(r) \right] \int_{B_r} |Du|^2 \, dx + \left(\frac{\rho}{r}\right)^n \int_{B_r} |Dw|^2 \, dx + \int_{B_r} |f|^2 \, dx \right\} \\ &\leq C(\nu) \left\{ \left[\left(\frac{\rho}{r}\right)^n (1 + \omega^2(r)) + \omega^2(r) \right] \int_{B_r} |Du|^2 \, dx + \left[\left(\frac{\rho}{r}\right)^n + 1 \right] \int_{B_r} |f|^2 \, dx \right\}. \end{aligned} \tag{3.73}$$

□

Lemma 3.41 *Let $u \in H^{1,2}(B_r)$ be a solution of $Lu = D_j f^j$ in B_r. Then there exists $C(\nu) > 0$ such that for any $\rho \in (0, r)$ and $\alpha, \beta \in \mathbb{R}^n$,*

$$\begin{aligned} &\int_{B_\rho} |Du - (Du)_\rho|^2 \, dx \leq \\ &C(\nu) \left(\left(\frac{\rho}{r}\right)^{n+2} \int_{B_r} |Du - \beta|^2 \, dx + w^2(r) \int_{B_r} |Du|^2 \, dx + \int_{B_r} |f - \alpha|^2 \, dx \right). \end{aligned} \tag{3.74}$$

Proof. Decompose u in $v+w$ as in Lemma 3.40. Then v satisfies the assumptions of Lemma 3.37 with B_r instead of B_R. Applying Corollary 3.38, for $0<\rho<r, \beta\in\mathbb{R}^n$ we obtain

$$\int_{B_\rho} |Dv-(Dv)_\rho|^2\,dx \le 2^{n+2}C(\nu)\left(\frac{\rho}{r}\right)^{n+2}\int_{B_r}|Dv-\beta|^2\,dx. \tag{3.75}$$

Applying Lemma 3.32 to w, for $0<\rho\le r, \alpha\in\mathbb{R}^n$, we obtain

$$\int_{B_\rho}|Dw|^2\,dx \le C(\nu)\int_{B_r}|f-\alpha|^2+\omega^2(r)|Du|^2\,dx. \tag{3.76}$$

For $0<\rho<r$, from (3.75), (3.76) it follows that

$$\begin{aligned}
&\int_{B_\rho}|Du-(Du)_\rho|^2\,dx\\
&\le 2\int_{B_\rho}|Dv-(Dv)_\rho|^2\,dx + 2\underbrace{\int_{B_\rho}|Dw-(Dw)_\rho|^2\,dx}_{\le 2\int_{B_\rho}|Dw|^2\,dx}\\
&\le C(\nu)\left[\left(\frac{\rho}{r}\right)^{n+2}\int_{B_r}|Dv-\beta|^2\,dx+\int_{B_r}|Dw|^2\,dx\right]\\
&\le C(\nu)\left[\left(\frac{\rho}{r}\right)^{n+2}\int_{B_r}|Du-\beta|^2\,dx+\left(1+\left(\frac{\rho}{r}\right)^{n+2}\right)\int_{B_r}|Dw|^2\,dx\right]\\
&\le C(\nu)\left[\left(\frac{\rho}{r}\right)^{n+2}\int_{B_r}|Du-\beta|^2\,dx+\left(1+\left(\frac{\rho}{r}\right)^{n+2}\right)\omega^2(r)\int_{B_r}|Du|^2\,dx+\right.\\
&\qquad\left.\int_{B_r}|f-\alpha|^2\,dx\right].
\end{aligned} \tag{3.77}$$

□

Regularity of Du in the constant coefficient case

Proposition 3.42 *Let $u\in H^{1,2}(\Omega)$ be a weak solution of $Lu=D_if^i$, where we assume that a_{ij} are constants satisfying* (3.33). *If $f_i\in\mathcal{L}^{2,\mu}(\Omega), 0\le\mu<n+2$, then $Du\in\mathcal{L}^{2,\mu}_{loc}(\Omega;\mathbb{R}^n)$.*

Proof. For $x_0\in\tilde\Omega\subset\subset\Omega, 0<R<\frac{d}{2}:=\frac{1}{2}\mathrm{dist}(\tilde\Omega,\partial\Omega)$, similarly to Lemma 3.40, we let $v\in H^{1,2}(B(x_0,R))$ be the solution of the following elliptic equation

$$\int_{B(x_0,R)} a^{ij}D_jvD_i\varphi\,dx = 0 \quad \forall\varphi\in H^{1,2}_0(B(x_0,R)), v-u\in H^{1,2}_0(B(x_0,R)). \tag{3.78}$$

Such a solution exists, for we can solve $Lw=D_if^i, w\in H^{1,2}_0(B(x_0,R))$ uniquely by Proposition 3.22, then define $v:=u-w$ to obtain

$$Lv=Lu-Lw=0, u-v=w\in H^{1,2}_0(B(x_0,R)).$$

By Corollary 3.38,

$$\int_{B(x_0,\rho)} |Dv-(Dv)_{x_0,\rho}|^2\,dx \le C\left(\frac{\rho}{R}\right)^{n+2}\int_{B(x_0,R)}|Dv-(Dv)_{x_0,R}|^2\,dx, \quad (3.79)$$

where $0<\rho\le R, C=C(n,\nu,d)$. For simplicity, we denote $B(x_0,\rho)$ by B_ρ and $(Du)_{x_0,\rho}$ by $(Du)_\rho$. Since for any suitable set M

$$\int_M u_M^2\,dx = \frac{1}{|M|}\|u\|_{1;M}^2 \le \|u\|_{2;M}^2, \quad (3.80)$$

we compute

$$\begin{aligned}\int_{B_\rho}|Du-(Du)_\rho|^2\,dx &\le 2\int_{B_\rho}|Dv-(Dv)_\rho|^2\,dx + 2\int_{B_\rho}|Dw-(Dw)_\rho|^2\,dx\\ &\le C\left(\frac{\rho}{R}\right)^{n+2}\int_{B_R}|Dv-(Dv)_R|^2\,dx + 8\int_{B_\rho}|Dw|^2\,dx\\ &\le C\left(\frac{\rho}{R}\right)^{n+2}\int_{B_R}|Du-(Du)_R|^2\,dx + C\int_{B_R}|Dw|^2\,dx.\end{aligned} \quad (3.81)$$

Now, we estimate $\int_{B_R}|Dw|^2\,dx$. Note that $w=u-v\in H_0^{1,2}(B_R)$ satisfies

$$\int_{B_R} a^{ij}D_jwD_i\varphi\,dx = \int_{B_R}[f^i-(f^i)_R]D_i\varphi\,dx \quad \forall\varphi\in H_0^{1,2}(B_R)$$

(this follows from $\int cD_i\xi\,dx=0 \quad \forall\xi\in C_0^\infty, c\in\mathbb{R}$ and $C_0^\infty\subset H_0^{1,2}$ dense). By choosing $\varphi=w$ in the above equation and using the ellipticity condition and Hölder, we derive

$$\nu^{-1}\|Dw\|_{2;B_R}^2 = \nu^{-1}\int_{B_R}|Dw|^2\,dx \le \|f-f_R\|_{2;B_R}\|Dw\|_{2;B_R}.$$

Therefore,

$$\int_{B_R}|Dw|^2\,dx \le C\int_{B_R}|f-f_R|^2\,dx. \quad (3.82)$$

Since $f^i\in\mathcal{L}^{2,\mu}(\Omega)$, we have

$$\int_{B_R}|f-f_R|^2\,dx \le R^\mu[f]_{2,\mu;\Omega}^2.$$

Substituting this inequality into (3.82), we get

$$\int_{B_R}|Dw|^2\,dx \le C[f]_{2,\mu;\Omega}^2R^\mu. \quad (3.83)$$

Substituting (3.83) into (3.81), we conclude that

$$\int_{B_\rho}|Du-(Du)_\rho|^2\,dx \le C\left(\frac{\rho}{R}\right)^{n+2}\int_{B_R}|Du-(Du)_R|^2\,dx + C[f]_{2,\mu;\Omega}^2R^\mu,$$

where $C = C(n, \nu, d)$. In Lemma 3.31 we choose

$$\varphi(\rho) := \int_{B_\rho} |Du - (Du)_\rho|^2\, dx, A = C, B = C[f]^2_{2,\mu;\Omega}, \alpha = n+2, \beta = \mu, \epsilon = 0.$$

Then

$$\int_{B_\rho} |Du - (Du)_\rho|^2\, dx \le C\left(R^{-\mu}\int_{B_R} |Du - (Du)_R|^2\, dx + [f]^2_{2,\mu;\Omega}\right)\rho^\mu.$$

It follows that

$$\rho^{-\mu}\int_{B_\rho} |Du - (Du)_\rho|^2\, dx \le C\left(\|Du\|^2_{2;\Omega} + [f]^2_{2,\mu;\Omega}\right),$$

where $C = C(n, \nu, \mu, d), 0 < \rho \le R$. This implies $Du \in \mathcal{L}^{2,\mu}_{\text{loc}}(\Omega; \mathbb{R}^n)$. □

Regularity at the boundary – constant coefficients

The following considerations prepare us for generalizing Proposition 3.42 to Ω.

Lemma 3.43 *Let $u \in H^{1,2}(B_r^+)$ solve $Lu = 0$ in B_r^+, $u = 0$ on Γ_r, where the coefficients of L are assumed to be constants satisfying 3.33. Then there exists $C = C(\nu) > 0$ such that for $0 < \rho \le r$*

$$\int_{B_\rho^+} u^2\, dx \le C(\nu)\left(\frac{\rho}{R}\right)^{n+2}\int_{B_r^+} u^2\, dx. \tag{3.84}$$

*Proof.*By Corollary 3.27,

$$\|u\|^2_{k,2;B^+(0,\frac{r}{2})} \le C(\nu, k, r)\int_{B_r^+} u^2\, dx. \tag{3.85}$$

At this point we remind ourselves at the proofs of Lemma 3.35 and Lemma 3.37. Let k be sufficiently large, then with Sobolev,

$$\sum_j \sup_{B^+(0,\frac{r}{2})} |D_j u|^2 \le C(n, r)\|u\|^2_{k,2;B^+(0,\frac{r}{2})}. \tag{3.86}$$

By (3.85) and (3.86), we obtain for $\rho \in \left(0, \frac{r}{2}\right]$

$$\begin{aligned}\int_{B_\rho^+} u^2\, dx &= \int_{B_\rho^+} |u(x) - u(0)|^2\, dx \\ &\le C(n)\rho^{n+2}\sum_j \sup_{B^+(0,\frac{r}{2})} |D_j u|^2 \\ &\le C(\nu, r)\rho^{n+2}\int_{B_r^+} u^2\, dx.\end{aligned} \tag{3.87}$$

For defining the dependence of $C(\nu, r)$ with respect to r, we at first consider $v := u(r\cdot)(\in H^{1,2}(B_1^+), v = 0$ on $\Gamma_1, Lv = 0$ in $B_1^+)$. (3.87) for v and $B^+\left(0, \frac{\rho}{r}\right)$, B_1^+ respectively leads to

$$\int_{B^+\left(0,\frac{\rho}{r}\right)} |v(x)|^2\, dx \le C(\nu, 1)\left(\frac{\rho}{r}\right)^{n+2} \int_{B_1^+} v^2\, dx.$$

Now consider $x \longmapsto \frac{x}{r}$; then

$$\int_{B_\rho^+} u^2\, dx \le C(\nu)\left(\frac{\rho}{r}\right)^{n+2} \int_{B_r^+} u^2\, dx. \tag{3.88}$$

Modifying $C(\nu)$, (3.88) remains valid for any $0 < \rho \le r$. □

Corollary 3.44 *Let u be as in Lemma 3.43. Then for $0 < \rho \le r, h \in \{1, \dots, n-1\}$,*

$$\int_{B_\rho^+} |D_h u|^2\, dx \le C(\nu)\left(\frac{\rho}{r}\right)^{n+2} \int_{B_r^+} |D_h u|^2\, dx. \tag{3.89}$$

Proof. $D_h u$ satisfies the assumptions of Lemma 3.43. □

Lemma 3.45 *Let $u \in H^{1,2}(B_r^+)$ be solution of $Lu = 0$ in B_r^+, $u = 0$ on Γ_r. Then there exists $C = C(\nu) > 0$, such that for any $0 < \rho \le r$,*

$$\int_{B_\rho^+} |D_n u|^2\, dx \le C(\nu)\left(\frac{\rho}{r}\right)^{n} \int_{B_r^+} |D_n u|^2\, dx \tag{3.90}$$

and

$$\int_{B_\rho^+} |D_n u - (D_n u)_\rho|^2\, dx \le C(\nu)\left(\frac{\rho}{R}\right)^{n+2} \int_{B_r} |D_n u - (D_n u)_r|^2\, dx. \tag{3.91}$$

Proof. After odd reflection and 3.24, we may assume $u \in C^\infty(\overline{B\left(0, \frac{r}{2}\right)})$. By (3.85) and Sobolev, for $0 < \rho \le \frac{r}{2}$, we obtain

$$\begin{aligned}\int_{B_\rho^+} |D_n u|^2\, dx &\le C(n)\rho^n \sup_{\overline{B^+\left(0,\frac{r}{2}\right)}} |D_n u|^2 \\ &\le C(n, r)\rho^n \|u\|^2_{k,2;B^+\left(0,\frac{r}{2}\right)} \\ &\le C(\nu, r)\rho^n \int_{B_r^+} u^2\, dx.\end{aligned} \tag{3.92}$$

Since $u = 0$ on Γ_r, we may apply Lemma 3.34 and obtain

$$\int_{B_\rho^+} |D_n u|^2\, dx \le C(\nu, r) r^2 \rho^n \int_{B_r^+} |D_n u|^2\, dx. \tag{3.93}$$

(3.90) follows with the same homothety argument as between (3.87) and (3.88). Note that $u - cx_n$ satisfies the assumptions of the lemma for any $c \in \mathbb{R}$ and that (3.91) is not affected, if we replace u by $u - cx_n$. By (3.85),

$$\|u - cx_n\|^2_{k,2;B^+(0,\frac{r}{2})} \le C(n,k,r) \int_{B_r^+} |u - cx_n|^2 \, dx \tag{3.94}$$

and together with Sobolev and the mean-value theorem

$$\begin{aligned} \int_{B_\rho^+} |D_n u - D_n u(0)|^2 \, dx &\le C\rho^{n+2} \sup_{B^+(0,\frac{r}{2})} \sum_{|m|=2} |D^m(u - cx_n)|^2 \\ &\le C(\nu, r)\rho^{n+2} \int_{B_r^+} |u - cx_n|^2 \, dx. \end{aligned} \tag{3.95}$$

On the other hand, by Lemma 3.34,

$$\int_{B_r^+} |u - cx_n|^2 \, dx \le \frac{r^2}{2} \int_{B_r^+} |D_n u - c|^2 \, dx. \tag{3.96}$$

Thus, for $0 < \rho \le \frac{r}{2}$,

$$\int_{B_\rho^+} |D_n u - D_n u(0)|^2 \, dx \le C(\nu, r) r^2 \rho^{n+2} \int_{B_r^+} |D_n u - c|^2 \, dx. \tag{3.97}$$

Applying the homothety argument, which lead to (3.88), we obtain

$$\int_{B_\rho^+} |D_n u - D_n u(0)|^2 \, dx \le C(\nu) \left(\frac{\rho}{r}\right)^{n+2} \int_{B_r^+} |D_n u - c|^2 \, dx.$$

Setting $c = (D_n u)_r$ and using Lemma 3.39, we obtain

$$\begin{aligned} \int_{B_\rho^+} |D_n u - (D_n u)_\rho|^2 \, dx &\le \int_{B_\rho^+} |D_n u - D_n u(0)|^2 \, dx \\ &\le C(\nu) \left(\frac{\rho}{r}\right)^{n+2} \int_{B_r^+} |D_n u - (D_n u)_r|^2 \, dx. \end{aligned}$$

The case $\frac{r}{2} < \rho \le r$ follows as usual. □

Regularity at the boundary – continuous coefficients

Lemma 3.46 *Let $u \in H^{1,2}(B_r^+)$ be solution of $Lu = D_i f^i$ with $a_{ij} \in C(\bar{B}_r^+)$, $f_i \in L^2(B_r^+)$ and $u = 0$ on Γ_r. Then there exists a positive constant $C(\nu)$ such that for any $\rho \in (0, r)$,*

$$\int_{B_\rho^+} |Du|^2 \, dx \le C(\nu) \left(\left[\left(\frac{\rho}{r}\right)^n + \omega^2(r) \right] \int_{B_r^+} |Du|^2 \, dx + \int_{B_r^+} |f|^2 \, dx \right). \tag{3.98}$$

Proof. Decompose u in $v + w$ as around (3.69), (3.70) with B_r replaced by B_r^+. v satisfies the assumptions of Lemma 3.43 and Lemma 3.45. Hence, for $0 < \rho < r$ by (3.89) and (3.90), it follows that

$$\int_{B_\rho^+} |Dv|^2\,dx \le C(\nu)\left(\frac{\rho}{r}\right)^n \int_{B_r^+} |Dv|^2\,dx. \tag{3.99}$$

Applying Lemma 3.32 on w with $\Omega = B_r^+$ and $\alpha = 0$, for $0 < \rho \le r$ we obtain

$$\int_{B_\rho^+} |Dw|^2\,dx \le C(\nu)\int_{B_r^+} |f|^2 + \omega^2(r)|Du|^2\,dx. \tag{3.100}$$

The assertion follows from (3.99) and (3.100). □

Lemma 3.47 *Under the same assumptions as in Lemma 3.46 there exists a positive constant $C(\nu)$, such that for any $\rho \in (0,r), \alpha \in \mathbb{R}^n$*

$$\sum_{j=1}^{n-1}\int_{B_\rho^+} |D_j u|^2\,dx \le$$
$$C(\nu)\left[\left(\frac{\rho}{r}\right)^{n+2}\sum_{j=1}^{n-1}\int_{B_r^+} |D_j u|^2\,dx + \omega^2(r)\int_{B_r^+}|Du|^2\,dx + \int_{B_r^+}|f-\alpha|^2\,dx\right] \tag{3.101}$$

and

$$\int_{B_\rho^+} |D_n u - (D_n u)_\rho|^2\,dx \le \tag{3.102}$$
$$C(\nu)\left[\left(\frac{\rho}{r}\right)^{n+2}\int_{B_r^+} |D_n u - (D_n u)_r|^2\,dx + \omega^2(r)\int_{B_r^+}|Du|^2\,dx + \int_{B_r^+}|f-\alpha|^2\,dx\right].$$

Proof. We argue as in the previous lemma. Let $u = v + w$ as in (3.69), (3.70) with B_r replaced by B_r^+. Apply Lemma 3.32 on w, (3.89) on v for $j = 1, \dots, n-1$ and (3.91), for $j = n$ respectively. □

The following proposition is fundamental for passing from Proposition 3.42 to its global version 3.28. The proof is kind of hair-raising, because of its case distinctions. This might be the reason, why it has not been done in detail in the literature we took into consideration.

Proposition 3.48 *Let $u \in H^{1,2}(B_1^+)$ be a solution of $Lu = D_i f^i$ in B_1^+, $u = 0$ on Γ_1, where $f_i \in \mathcal{L}^{2,\mu}(B_1^+), 0 < \mu \le n, a_{ij} \in C^{0,\gamma}(\bar{B}_1^+), 0 < \gamma \le 1$. Let $R \in (0,1)$. Then $Du \in \mathcal{L}^{2,\mu}(B_R^+;\mathbb{R}^n)$ and we have the estimate*

$$\|Du\|^2_{\mathcal{L}^{2,\mu}(B_R^+)} \le C(\nu,\mu,R)(\|Du\|^2_{2;B_1^+} + \|f\|^2_{\mathcal{L}^{2,\mu}(B_1^+)}). \tag{3.103}$$

Proof. Fix $R \in (0,1)$ and set $\delta_0 := \frac{1-R}{8}$. Let $x_0 \in B_R^+$ and $r \in (0, \delta_0)$.
Case $\mu < n$.

1. Case $B(x_0, r) \subset B_1^+$:
   By Lemma 3.40,

$$\int_{B(x_0,\rho)} |Du|^2\, dx \leq$$
$$C(\nu)\left[\left(\frac{\rho}{r}\right)^n + \omega^2(r)\right]\int_{B(x_0,r)} |Du|^2\, dx + C(\nu)\underbrace{\int_{B(x_0,r)} |f|^2\, dx}_{\leq \|f\|^2_{L^{2,\mu}(B_1^+)} r^\mu}.$$

   Remember that $\mathcal{L}^{2,\mu} \cong L^{2,\mu}$, if $0 \leq \mu < n$. Since $a_{ij} \in C(\bar{B}_1^+)$, there exists $r_0 \in (0, \delta_0)$, such that for $0 < \rho < \tilde{r} < r_0$,

$$\int_{B(x_0,\rho)} |Du|^2\, dx \leq C\left[\left(\frac{\rho}{\tilde{r}}\right)^n + \epsilon\right]\int_{B(x_0,\tilde{r})} |Du|^2\, dx + C\|f\|^2_{\mathcal{L}^{2,\mu}(B_1^+)}\tilde{r}^\mu,$$

   where $C = C(\nu, R)$. Now apply Lemma 3.31 with

$$\varphi(\rho) = \int_{B(x_0,\rho)} |Du|^2\, dx, A = C, B = C\|f\|^2_{\mathcal{L}^{2,\mu}(B_1^+)}, \alpha = n, \beta = \mu.$$

   Then there exists $\tilde{r}_0 \in (0, r_0)$, such that for $0 < \rho \leq \tilde{r} \leq \tilde{r}_0$,

$$\int_{B(x_0,\rho)} |Du|^2\, dx \leq C\left(\tilde{r}^{-\mu}\int_{B_1^+} |Du|^2\, dx + \|f\|^2_{\mathcal{L}^{2,\mu}(B_1^+)}\right)\rho^\mu.$$

   Therefore,

$$\rho^{-\mu}\int_{B(x_0,\rho)} |Du - (Du)_{x_0,\rho}|^2\, dx \leq C(\|Du\|^2_{2;B_1^+} + \|f\|^2_{\mathcal{L}^{2,\mu}(B_1^+)}),$$

   where $C = C(\nu, \mu, R)$.

2. Case $B(x_0, r) \cap B_1^+ = B^+(x_0, r)$, i. e. $x_0 \in \Gamma_R$:
   Decompose u in $v + w$ as in Lemma 3.46 with B_r^+ replaced by $B^+(x_0, r)$ and $a_{ij}(0)$ replaced by $a_{ij}(x_0)$. From this lemma, it follows for $\rho \in (0, r)$ that

$$\int_{B^+(x_0,\rho)} |Du|^2\, dx \leq C(\nu)\left[\left(\frac{\rho}{r}\right)^n + \omega^2(r)\right]\int_{B_r^+} |Du|^2\, dx + C(\nu)\int_{B_r^+} |f|^2\, dx.$$

   Analogously to 1., we obtain for $0 < \rho \leq \tilde{r} \leq \tilde{r}_0$,

$$\rho^{-\mu}\int_{B(x_0,\rho)} |Du|^2\, dx \leq C(\|Du\|^2_{2;B_1^+} + \|f\|^2_{\mathcal{L}^{2,\mu}(B_1^+)}).$$

   Remember $\mathcal{L}^{p,\mu} \cong L^{p,\mu} \quad \forall 0 \leq \mu < n$.

3. Case $0 < \operatorname{dist}(x_0, \Gamma_R) =: d < r, B(x_0, \rho) \subset B_1^+$ (in particular $\rho < d$):
Let y_0 be the projection of x_0 onto Γ_R. Decompose u in $v + w$ as in Lemma 3.46 with $B^+(y_0, 4r)$ instead of B_r^+. Then

$$\int_{B(x_0,\rho)} |Dv|^2\,dx \le C\left(\frac{\rho}{d}\right)^n \int_{B(x_0,d)} |Dv|^2\,dx$$

by Lemma 3.35 applied to $D_i v$, since after odd reflection and in view of Friedrich's theorem we may assume $v \in C^\infty(\overline{B^+(y_0, 2r)})$.

$$\begin{aligned}
&\le C\left(\frac{\rho}{d}\right)^n \int_{B^+(y_0,2d)} |Dv|^2\,dx \\
&\le C\left(\frac{\rho}{d}\right)^n \tilde{C}\left(\frac{2d}{2r}\right)^{n+2} \int_{B^+(y_0,2r)} |Dv|^2\,dx
\end{aligned}$$

by Lemma 3.43 applied to $D_i v$.

$$\le C\left(\frac{\rho}{r}\right)^n \int_{B^+(y_0,2r)} |Dv|^2\,dx,$$

where $C = C(\nu, R)$. Clearly,

$$\int_{B(x_0,\rho)} |Du|^2\,dx \le 2\int_{B(x_0,\rho)} |Dv|^2\,dx + 2\int_{B(x_0,\rho)} |Dw|^2\,dx.$$

Thus for $\rho < d$,

$$\int_{B(x_0,\rho)} |Du|^2\,dx \le C\left(\frac{\rho}{r}\right)^n \int_{B^+(y_0,2r)} |Dv|^2\,dx + C\int_{B^+(y_0,2r)} |Dw|^2\,dx. \tag{3.104}$$

Since

$$\int_{B^+(y_0,4r)} a^{ij}(x_0) D_i w D_j \varphi\,dx = \int_{B^+(y_0,4r)} (f_i + [a^{ij}(x_0) - a^{ij}(x)] D_i u) D_j \varphi\,dx$$

for any $\varphi \in H_0^{1,2}(B^+(y_0, 4r))$, it follows for $\varphi = w$ that

$$\int_{B^+(y_0,4r)} |Dw|^2\,dx \le C\int_{B^+(y_0,4r)} |f|^2\,dx + C\omega^2(4r)\int_{B^+(y_0,4r)} |Du|^2\,dx. \tag{3.105}$$

Substituting (3.105) into (3.104) yields

$$\int_{B(x_0,\rho)} |Du|^2\,dx \le C\left[\left(\frac{\rho}{r}\right)^n + \omega^2(4r)\right]\int_{B^+(y_0,4r)} |Du|^2\,dx + C\|f\|_{L^{2,\mu}(B_1^+)} r^\mu.$$

Note that $\omega^2(4r) < \epsilon$ for $r < \frac{r_0}{4}$. Similarly to case 1., apply Lemma 3.31 with

$$\varphi(\rho) := \begin{cases} \int_{B(x_0,\rho)} |Du|^2\,dx, & \rho < d \\ \int_{B^+(y_0,4\rho)} |Du|^2\,dx, & \rho \ge d, \end{cases}$$

which is nonnegative and isotone. Consequently, for $0 < \rho \le \tilde{r}_1 \wedge d$,

$$\rho^{-\mu} \int_{B(x_0,\rho)} |Du|^2 \, dx \le C(\|Du\|^2_{2;B_1^+} + \|f\|^2_{\mathcal{L}^{2,\mu}(B_1^+)}), \tag{3.106}$$

where $C = C(\nu, \mu, R)$.

4. Case $0 < d < r, B(x_0, \rho) \not\subset B_1^+$:

$$\int_{B^+(x_0,\rho)} |Du|^2 \, dx \le \int_{B^+(y_0,2\rho)} |Du|^2 \, dx$$
$$\le C(\nu) \left[\left(\frac{2\rho}{2r} \right)^n + \omega^2(2r) \right] \int_{B^+(y_0,2r)} |Du|^2 \, dx + C(\nu) \int_{B^+(y_0,2r)} |f|^2 \, dx.$$

Note that $\omega^2(2r) < \epsilon$ for $r < \frac{r_0}{2}$. Applying Lemma 3.31 as in case 3., we obtain (3.106) for $0 < \rho \le \tilde{r}_1$.

Up to now we have shown that for any $\rho < (\tilde{r}_1 \wedge \tilde{r}_0), x_0 \in B_R^+$,

$$\rho^{-\mu} \int_{B(x_0,\rho) \cap B_1^+} |Du|^2 \, dx \le C(\|Du\|^2_{2;B_1^+} + \|f\|^2_{\mathcal{L}^{2,\mu}(B_1^+)}),$$

where $C = C(\nu, \mu, R)$. But for $\rho \ge (\tilde{r}_1 \wedge \tilde{r}_0)$ we have

$$\rho^{-\mu} \int_{B(x_0,\rho) \cap B_R^+} |Du|^2 \, dx \le (\tilde{r}_1 \wedge \tilde{r}_0)^{-\mu} \|Du\|^2_{2,B_1^+} \le C(\|Du\|^2_{2;B_1^+} + \|f\|^2_{\mathcal{L}^{2,\mu}(B_1^+)}).$$

Consequently,

$$\sup_{\substack{x_0 \in B_R^+ \\ \rho > 0}} \rho^{-\mu} \int_{B(x_0,\rho) \cap B_1^+} |Du|^2 \, dx \le C(\nu, \mu, R)(\|Du\|^2_{2;B_1^+} + \|f\|^2_{\mathcal{L}^{2,\mu}(B_1^+)}).$$

Case $\mu = n$.

1. Case $B(x_0, r) \subset B_1^+$:
By Lemma 3.41 for $\rho \in (0, r]$,

$$\int_{B(x_0,\rho)} |Du - (Du)_{x_0,\rho}|^2 \, dx \le$$
$$C(\nu) \left[\left(\frac{\rho}{r} \right)^{n+2} \int_{B(x_0,r)} |Du - (Du)_{x_0,r}|^2 \, dx + \omega^2(r) \int_{B(x_0,r)} |Du|^2 \, dx + \int_{B(x_0,r)} |f - f_{x_0,r}|^2 \, dx \right].$$

Note that $\omega^2(r) < \epsilon$ for $r < r_0$ and $\int_{B(x_0,r)} |f - f_{x_0,r}|^2 \, dx \le \|f\|^2_{\mathcal{L}^{2,n}(B_1^+)} r^n$.
Now apply Lemma 3.31 with $\alpha = n + 2, \beta = n$,

$$\varphi(\rho) = \int_{B(x_0,\rho)} |Du - (Du)_{x_0,\rho}|^2 \, dx, A = C(\nu), B = C(\nu) \|f\|^2_{\mathcal{L}^{2,n}(B_1^+)}.$$

Then for $0 < \rho < r_0$

$$\rho^{-n} \int_{B(x_0,\rho)} |Du - (Du)_{x_0,\rho}|^2 \, dx \le C(\|Du\|^2_{2;B_1^+} + \|f\|^2_{\mathcal{L}^{2,n}(B_1^+)}).$$

2. Case $B(x_0, r) \cap B_1^+ = B^+(x_0, r)$, i. e. $x_0 \in \Gamma_R$:
Since $\mathcal{L}^{2,n} \subset \mathcal{L}^{2,n-2\gamma} \cong L^{2,n-2\gamma}$, we may apply case $\mu < n$ on $\tilde{\mu} = n - 2\gamma$ to obtain $Du \in L^{2,n-2\gamma}(B^+(\frac{1+R}{2}))$ and

$$\|Du\|^2_{L^{2,n-2\gamma}(B^+(\frac{1+R}{2}))} \leq C(\nu, \gamma, R)(\|Du\|^2_{2;B_1^+} + \|f\|^2_{L^{2,n-2\gamma}(B_1^+)}). \quad (3.107)$$

At this point we apply Lemma 3.47 and obtain the estimates (3.101) for $1 \leq j \leq n-1$ and (3.102) for $j = n$. From these and (3.107) it follows for $0 < \rho < \tilde{r} \leq \delta_0$,

$$\sum_{j=1}^{n-1} \int_{B^+(x_0,\rho)} |D_j u|^2\, dx$$
$$\leq C \left[\left(\frac{\rho}{\tilde{r}}\right)^{n+2} \sum_{j=1}^{n-1} \int_{B^+(x_0,\tilde{r})} |D_j u|^2\, dx \right] +$$
$$C\tilde{r}^n [\|Du\|^2_{L^{2,n-2\gamma}(B^+(\frac{1+R}{2}))} + \|f\|^2_{\mathcal{L}^{2,n}(B_1^+)}]$$

(since $\omega^2(r) \leq \text{höl}^2_\gamma(a_{ij}) r^{2\gamma}$)

$$\leq C \left[\left(\frac{\rho}{\tilde{r}}\right)^{n+2} \sum_{j=1}^{n-1} \int_{B^+(x_0,\tilde{r})} |D_j u|^2\, dx + \tilde{r}^n [\|Du\|^2_{2;B_1^+} + \|f\|^2_{\mathcal{L}^{2,n}(B_1^+)}] \right] \quad (3.108)$$

by case $\mu < n$, where $C = C(\nu, \gamma, R)$. Analogously for $j = n$,

$$\int_{B^+(x_0,\rho)} |D_n u - (D_n u)_{x_0,\rho}|^2\, dx$$
$$\leq C(\nu, \gamma, R) \left[\left(\frac{\rho}{\tilde{r}}\right)^{n+2} \int_{B^+(x_0,\tilde{r})} |D_n u - (D_n u)_{x_0,\tilde{r}}|^2\, dx \right] + \quad (3.109)$$
$$C(\nu, \gamma, R)\tilde{r}^n [\|Du\|^2_{2;B_1^+} + \|f\|^2_{\mathcal{L}^{2,n}(B_1^+)}].$$

We set

$$\varphi(t) := \sum_{j=1}^{n-1} \int_{B^+(x_0,t)} |D_j u|^2\, dx \quad (3.110)$$
$$B(t) := C(\nu, \gamma, R)[\|Du\|^2_{2;B_1^+} + \|f\|^2_{\mathcal{L}^{2,n}(B_1^+)}] t^n$$
$$\psi(t) := \int_{B^+(x_0,t)} |D_n u - (D_n u)_{x_0,t}|^2\, dx \quad (3.111)$$

Summarizing, from (3.108) and (3.109) one obtains for arbitrary fixed $p > 1$ that for any pair $(\rho, \tilde{r})$, such that $0 < \rho < \tilde{r} \leq \delta_0, 1 < \frac{\tilde{r}}{\rho} \leq p$

$$\varphi(\rho) \leq C(\nu, \gamma, R) \left(\frac{\rho}{\tilde{r}}\right)^{n+2} \varphi(\tilde{r}) + B(p)\rho^n$$

and

$$\psi(\rho) \le C(\nu,\gamma,R)\left(\frac{\rho}{\tilde r}\right)^{n+2}\psi(\tilde r) + B(p)\rho^n.$$

At this point we apply Lemma 3.30 and obtain for $x_0 \in \Gamma_R$ and $0<\rho<\tilde r$ that

$$\begin{aligned}
&\sum_{j=1}^{n-1}\int_{B^+(x_0,\rho)}|D_ju|^2\,dx \\
&\le C(\nu,\gamma,R,n)\{\left(\frac{\rho}{\tilde r}\right)^{n+2-\epsilon}\left[\sum_{j=1}^{n-1}\int_{B^+(x_0,\tilde r)}|D_ju|^2\,dx\right] + \rho^n\|Du\|^2_{2;B_1^+} \\
&\qquad + \rho^n\|f\|^2_{\mathcal{L}^{2,n}(B_1^+)}\}
\end{aligned}$$

for any $\epsilon \in (0,2)$. Therefore, with $\epsilon = 1$

$$\begin{aligned}
&\sum_{j=1}^{n-1}\rho^{-n}\int_{B^+(x_0,\rho)\cap B_R^+}|D_ju-(D_ju)_{x_0,\rho}|^2\,dx \\
&\le C'\sum_{j=1}^{n-1}\rho^{-n}\int_{B^+(x_0,\rho)}|D_ju|^2\,dx \\
&\le C\rho^{-n}\left[\left(\frac{\rho}{\tilde r}\right)^{n+1}\int\limits_{B^+(x_0,\tilde r)}|Du|^2\,dx + \rho^n[\|Du\|^2_{2;B_1^+} + \|f\|^2_{\mathcal{L}^{2,n}(B_1^+)}]\right] \\
&\le C[\|Du\|^2_{2;B_1^+} + \|f\|^2_{\mathcal{L}^{2,n}(B_1^+)}]
\end{aligned}$$

and for $j=n$

$$\rho^{-n}\int\limits_{B^+(x_0,\rho)\cap B_R^+}|D_nu-(D_nu)_{x_0,\rho}|^2\,dx \le C[\|Du\|_{2;B_1^+} + \|f\|^2_{\mathcal{L}^{2,n}(B_1^+)}],$$

where $C = C(\nu,\gamma,R,n)$. Consequently, for $x_0 \in \Gamma_R$ and $0<\rho<\tilde r$,

$$\rho^{-n}\int\limits_{B^+(x_0,\rho)\cap B_R^+}|Du-(Du)_{x_0,\rho}|^2\,dx \le C(\nu,\gamma,R,n)[\|Du\|_{2;B_1^+}+\|f\|^2_{\mathcal{L}^{2,n}(B_1^+)}].$$

3. Case $0 < d < r, B(x_0,\rho) \subset B_1^+$ (in particular $\rho < d$):

$$\begin{aligned}
&\int_{B(x_0,\rho)} |Du - (Du)_{x_0,\rho}|^2\,dx \\
&\leq C(\nu)\left[\left(\frac{\rho}{d}\right)^{n+2}\int_{B(x_0,d)} |Du-(Du)_{x_0,d}|^2\,dx + \omega^2(d)\int_{B(x_0,d)}|Du|^2\,dx + \right.\\
&\qquad \left.\int_{B(x_0,d)} |f - f_{x_0,d}|^2\,dx\right]
\end{aligned}$$

(by Lemma 3.41)

$$\begin{aligned}
&\leq C(\nu)\left[\left(\frac{\rho}{d}\right)^{n+2}\int_{B^+(y_0,2d)} |Du-(Du)_{y_0,2d}|^2\,dx + \right.\\
&\qquad \left.\omega^2(2d)\int_{B^+(y_0,2d)}|Du|^2\,dx + \int_{B^+(y_0,2d)} |f - f_{y_0,2d}|^2\,dx\right]
\end{aligned}$$

(isotony)

$$\begin{aligned}
&\leq C(\nu,\gamma,R)\{\left(\frac{\rho}{d}\right)^{n+2}\int_{B^+(y_0,2d)} |Du-(Du)_{y_0,2d}|^2\,dx + \\
&\qquad (2d)^n[\|Du\|^2_{L^{2,n-2\gamma}(B^+(\frac{1+R}{2}))} + \|f\|^2_{\mathcal{L}^{2,n}(B_1^+)}]\} \\
&=: (*),
\end{aligned} \tag{3.112}$$

since again $\omega^2(r) \leq \text{höl}^2_\gamma(a_{ij})r^{2\gamma}$. Decompose u in $v + w$, where v solves the problem

$$\begin{cases} L_0 v = 0 & \text{in} \quad B^+(y_0,4r) \\ v - u \in H_0^{1,2}(B^+(y_0,4r)) \end{cases}$$

and w solves the problem

$$\begin{cases} L_0 w = \sum_i D_i\{f_i - (f_i)_{y_0,4r} + \sum_j [a_{ij}(x_0) - a_{ij}(x)]D_j u\} & \text{in} \quad B^+(y_0,4r) \\ w \in H_0^{1,2}(B^+(y_0,4r)). \end{cases}$$

Now

$$
\begin{aligned}
&\int_{B^+(y_0,2d)} |Dv - (Dv)_{y_0,2d}|^2\,dx \\
&\le (2d)^2 C \sum_{i=1}^{n} \int_{B^+(y_0,2d)} |D(D_i v)|^2\,dx \quad \text{(by Corollary A.8)} \\
&\le (2d)^{n+2} C \sum_{ij} \sup_{B^+(y_0,2d)} |D_i D_j v|^2 \\
&\le C(k)(2d)^{n+2} \|v\|^2_{k,2;B^+(y_0,2r)} \quad \text{(by Sobolev, if } k \text{ large enough)} \\
&\le C(k,r)(2d)^{n+2} \int_{B^+(y_0,4r)} |v|^2\,dx \quad \text{(by Corollary 3.27)} \\
&\le C(k,r)(2d)^{n+2} \int_{B^+(y_0,4r)} |D_n v|^2\,dx \quad \text{(by Lemma 3.34)} \quad .
\end{aligned}
$$

Again note (cf. proof of Lemma 3.45) that the last inequality is not affected, if we replace v by $v - (D_n v)_{y_0,4r} x_n$. We obtain thus

$$
\begin{aligned}
&\int_{B^+(y_0,2d)} |Dv - (Dv)_{y_0,2d}|^2\,dx \\
&\le C(k,r)(2d)^{n+2} \int_{B^+(y_0,4r)} |D_n v - (D_n v)_{y_0,4r}|^2\,dx \\
&\le C(k,r)(2d)^{n+2} \int_{B^+(y_0,4r)} |Dv - (Dv)_{y_0,4r}|^2\,dx.
\end{aligned}
$$

Therefore, similar to (3.81), since $u = v + w$

$$
\begin{aligned}
(*) \le C\{&\left(\frac{\rho}{d}\right)^{n+2} (2d)^{n+2} \int_{B^+(y_0,4r)} |Du - (Du)_{y_0,4r}|^2\,dx + \\
&C' \int_{B^+(y_0,4r)} |Dw|^2\,dx + \\
&(2d)^n [\|Du\|^2_{L^{2,n-2\gamma}(B^+(\frac{1+R}{2}))} + \|f\|^2_{\mathcal{L}^{2,n}(B_1^+)}]\}.
\end{aligned}
\tag{3.113}
$$

Again similarly to the proof of Proposition 3.42 and to (3.108),

$$
\begin{aligned}
&\int_{B^+(y_0,4r)} |Dw|^2\,dx \le \\
&\int_{B^+(y_0,4r)} |f - f_{y_0,4r}|^2\,dx + C\omega^2(4r) \int_{B^+(y_0,4r)} |Du|^2\,dx \\
&\le C(4r)^n \|f\|^2_{\mathcal{L}^{2,n}(B_1^+)} + C(\gamma)(4r)^{2\gamma}(4r)^{n-2\gamma} \|Du\|^2_{L^{2,n-2\gamma}(B^+(\frac{1+R}{2}))} \\
&\le C(\alpha,R)(4r)^n [\|Du\|^2_{L^{2,n-2\gamma}(B^+(\frac{1+R}{2}))} + \|f\|^2_{\mathcal{L}^{2,n}(B_1^+)}].
\end{aligned}
\tag{3.114}
$$

Thus with help of (3.113), (3.114), case $\mu < n$ and since $d < r < 1$,

$$\begin{aligned}\int_{B(x_0,\rho)} &|Du - (Du)_{x_0,\rho}|^2\,dx \\ &\le C\{\left(\frac{\rho}{r}\right)^{n+2}\int_{B^+(y_0,4r)}|Du-(Du)_{y_0,4r}|^2\,dx + \\ &\quad r^n[\|Du\|^2_{2;B_1^+}+\|f\|^2_{\mathcal{L}^{2,n}(B_1^+)}]\},\end{aligned} \tag{3.115}$$

where $C = C(\nu, \alpha, R)$. We set

$$B(t) = C(\nu,\alpha,R)[\|Du\|^2_{2;B_1^+}+\|f\|^2_{\mathcal{L}^{2,n}(B_1^+)}]t^n, t(p) = \delta_0,$$

$$\varphi(t) := \begin{cases} \int\limits_{B(x_0,t)} |Du-(Du)_{x_0,t}|^2\,dx, & \text{if } \quad t<d \\ 2\sum\limits_{j=1}^{n-1}\int\limits_{B^+(y_0,4t)}|D_ju|^2\,dx + \int\limits_{B^+(y_0,4t)}|D_nu-(D_nu)_{y_0,4t}|^2\,dx, \end{cases}$$

if $t \ge d$. Then for fixed $p > 1$ and any pair $(\rho, \tilde{r})$ with $0 < \rho < \tilde{r} \le \delta_0, 1 < \frac{\tilde{r}}{\rho} \le p$,

$$\varphi(\rho) \le C(\nu,\gamma,R)\left(\frac{\rho}{\tilde{r}}\right)^{n+2}\varphi(\tilde{r}) + B(p)\rho^n.$$

In fact, for $t < \tilde{r} < d$ use the first step of (3.112) and the Hölder continuity of a_{ij} as in (3.114), for $t < d < \tilde{r}$ use (3.115) and for $d < t < \tilde{r}$ use (3.108), (3.109) and (3.111). Now apply Lemma 3.30 to obtain

$$\frac{1}{\rho^n}\int_{B(x_0,\rho)}|Du-(Du)_{x_0,\rho}|^2\,dx \le C(\nu,\gamma,R,n)\{\|Du\|^2_{2;B_1^+}+\|f\|^2_{\mathcal{L}^{2,n}(B_1^+)}\}$$

for any $x_0 \in B_R^+, 0 < \rho < d$.

4. Case $0 < d < r, B(x_0,\rho) \not\subset B_1^+, \rho < \frac{r}{2}$:

$$\int_{B(x_0,\rho)}|Du-(Du)_{x_0,\rho}|^2\,dx \le \int_{B^+(y_0,2\rho)}|Du-(Du)_{y_0,2\rho}|^2\,dx.$$

As in case 2.), for the projection $y_0 \in \Gamma_R$ and $0 < 2\rho \le \delta_0$

$$\rho^{-n}\int\limits_{B^+(y_0,2\rho)\cap B_R^+}|Du-(Du)_{y_0,2\rho}|^2\,dx \le C[\|Du\|_{2;B_1^+}+\|f\|^2_{\mathcal{L}^{2,n}(B_1^+)}],$$

where $C = C(\nu,\gamma,R,n)$, and by monotony,

$$\int_{B(x_0,\rho)\cap B_R^+}|Du-(Du)_{x_0,\rho}|^2\,dx \le \int\limits_{B^+(y_0,2\rho)\cap B_R^+}|Du-(Du)_{y_0,2\rho}|^2\,dx.$$

Hence, 4.) is done.

Up to now, we have shown

$$\rho^{-n} \int\limits_{B^+(x_0,\rho)\cap B_R^+} |Du - (Du)_{x_0,\rho}|^2\, dx \leq C[\|Du\|_{2;B_1^+} + \|f\|^2_{\mathcal{L}^{2,n}(B_1^+)}]$$

for $\rho < \frac{r}{2}, x_0 \in B_R^+$, where $C = C(\nu, \gamma, R)$. For $\rho \geq \frac{r}{2}$ we proceed as at the end of case $\mu < n$. □

We are now prepared for the proof of 3.28: Since Ω is of class C^∞, there exist finitely many domains $\Omega_0, \Omega_1, \dots, \Omega_m$, such that $\Omega_0 \subset\subset \Omega = \bigcup_{j=0}^m \Omega_j$ and for each $\Omega_j, 1 \leq j \leq m$, there exists a diffeomorphism T_j of class C^∞, such that $T_j(\Omega_j \cap \bar{\Omega}) \subset B_1^+ \cup \Gamma_1$ and $T_j(\Omega_j \cap \partial\Omega) \subset \Gamma_1$. By Theorem 3.42, we know that $Du \in \mathcal{L}^{2,\mu}(\Omega_0; \mathbb{R}^n)$ and we have the estimate

$$\|Du\|^2_{\mathcal{L}^{2,\mu}(\Omega_0)} \leq C\{\|Du\|^2_{2;\Omega} + \|f\|^2_{\mathcal{L}^{2,\mu}(\Omega)}\}. \tag{3.116}$$

We fix $j \in \{1, \dots, m\}$ and denote again by u the restriction of u on $\Omega_j \cap \Omega$. Let $y \in B_1^+$ arbitrary. Consider $v := u \circ T_j^{-1}$ on B_1^+. In B_1^+, v solves an equation of type

$$L_j v = D^i F_i(y) \quad \text{weakly,}$$

where $(y \longmapsto F_j(y)) \in \mathcal{L}^{2,\mu}(B_1^+)$ and L_j is elliptic in B_1^+ with some ellipticity constant $K\nu$ (see B.2). Applying Proposition 3.48, for $0 < R < 1$ we have that $(y \longmapsto Dv(y)) \in \mathcal{L}^{2,\mu}(B_R^+; \mathbb{R}^n)$ and

$$\|Dv\|^2_{\mathcal{L}^{2,\mu}(B_R^+)} \leq C\{\|Dv\|^2_{2;B_1^+} + \|F\|^2_{\mathcal{L}^{2,\mu}(B_R^+)}\}. \tag{3.117}$$

Integral transformations (see B.2) yield

$$\begin{aligned} \|Du\|^2_{\mathcal{L}^{2,\mu}(\Omega_{j,R})} &\leq C\{\|Du\|^2_{2;\Omega_j\cap\Omega} + \|f\|^2_{\mathcal{L}^{2,\mu}(\Omega_j\cap\Omega)}\} \\ &\leq C\{\|Du\|^2_{2;\Omega} + \|f\|^2_{\mathcal{L}^{2,\mu}(\Omega)}\}, \end{aligned} \tag{3.118}$$

where $\Omega_{j,R} := T_j^{-1}(B_R^+)$. Since $R \in (0,1)$ was arbitrary, we may choose R sufficient close to 1, such that $\Omega_0, \Omega_{1,R}, \dots, \Omega_{m,R}$ still cover Ω. From (3.116) and (3.118) it follows that

$$\|Du\|^2_{\mathcal{L}^{2,\mu}(\Omega)} \leq C\{\|Du\|^2_{2;\Omega} + \|f\|^2_{\mathcal{L}^{2,\mu}(\Omega)}\}. \tag{3.119}$$

Using 3.21, i. e.

$$\|Du\|^2_{2;\Omega} \leq C\|f\|^2_{2;\Omega} \leq C\|f\|^2_{\mathcal{L}^{2,\mu}(\Omega)},$$

we arrive at the assertion

$$\|Du\|^2_{\mathcal{L}^{2,\mu}(\Omega)} \leq C\|f\|^2_{\mathcal{L}^{2,\mu}(\Omega)}.$$

□

Remark 3.49 (cf. [13, II.9.2.4]) *Since* $L^{2,\mu}(\Omega) \subset \mathcal{L}^{2,\mu}(\Omega)$ *for* $0 \leq \mu < n+2$ *and*

$$\|\cdot\|_{\mathcal{L}^{2,\mu}(\Omega)} \leq C\|\cdot\|_{L^{2,\mu}(\Omega)},$$

the assumption on f_i *in Proposition 3.28 can be changed to* $f_i \in L^{2,\mu}(\Omega), 0 \leq \mu \leq n$*. In this case, the weak solution* u *of the Dirichlet problem* (3.31), (3.32) *satisfies* $Du \in \mathcal{L}^{2,\mu}(\Omega;\mathbb{R}^n)$ *and*

$$\|Du\|_{\mathcal{L}^{2,\mu}(\Omega)} \leq C\|f\|_{L^{2,\mu}(\Omega)},$$

$C = C(n, \nu, \mu, diam\,\Omega)$.

3.1.4 L^p-Theory of Elliptic PDEs in Divergence Form

Constant Coefficients

Proposition 3.50 (cf. [13, II.10.2.1]) *Suppose that* $u \in H_0^{1,2}(B_R)$ *satisfies* (3.34) *on* B_R*, where* a_{ij} *are constants satisfying* (3.33), $f_i \in L^p(B_R), p \geq 2$. *Then* $Du \in L^p(B_R;\mathbb{R}^n)$ *and*

$$\|Du\|_{p;B_R} \leq C\|f\|_{p;B_R},$$

where $C = C(n, \nu, p)$ *is independent of* R.

*Proof.*Let $u \in H_0^{1,2}(B_R)$ the unique solution of (3.34) corresponding to f. We define an operator T as follows: $Tf = Du$. By choosing $\varphi = u$ in (3.34), we obtain the estimate

$$\|Du\|_{2;B_R} \leq C\|f\|_{2;B_R}, C = C(n, \nu).$$

This shows that T is a bounded linear operator from $L^2(B_R;\mathbb{R}^n)$ to $L^2(B_R;\mathbb{R}^n)$. On the other hand, Remark 3.49 implies that if $f \in L^{2,n}(B_R;\mathbb{R}^n)$, then $Du \in \mathcal{L}^{2,n}(B_R;\mathbb{R}^n)$ and

$$\|Du\|_{\mathcal{L}^{2,n}(B_R)} \leq C\|f\|_{L^{2,n}(B_R)}, C = C(n, \nu).$$

Using Lemma 3.10 and Corollary 3.18, we conclude that T is a bounded linear operator from $L^\infty(B_R;\mathbb{R}^n)$ to $\mathrm{BMO}(B_R;\mathbb{R}^n)$. By 3.19, for $2 \leq p < \infty, T$ is a bounded linear operator from $L^p(B_R;\mathbb{R}^n)$ to $L^p(B_R;\mathbb{R}^n)$, i. e., if $f \in L^p(B_R;\mathbb{R}^n)$, then $Du \in L^p(B_R;\mathbb{R}^n)$ and

$$\|Du\|_{p;B_R} \leq C\|f\|_{p;B_R}, C = C(n, \nu, p).$$

□

Non-Constant Coefficients

We now generalize to the case of continuous coefficients using [18, section 4.3]. Look at a solution $u \in H_0^{1,p}(B_R)$ of

$$\int_{B_R} a^{ij}(x) D_i u D_j \varphi \, dx = \int_{B_R} f^j D_j \varphi \, dx \quad \forall \varphi \in H_0^{1,p'}(B_R).$$

We write

$$\begin{aligned}
&\int_{B_R} a^{ij}(x_0) D_i u D_j \varphi \, dx \\
&= \int_{B_R} [a^{ij}(x_0) - a^{ij}(x)] D_i u D_j \varphi \, dx + \int_{B_R} a^{ij}(x) D_i u D_j \varphi \, dx \qquad (3.120) \\
&= \int_{B_R} [a^{ij}(x_0) - a^{ij}(x)] D_i u D_j \varphi \, dx + \int_{B_R} f^j D_j \varphi \, dx.
\end{aligned}$$

We fix $V \in H^{1,p}(B_R)$ and take a solution $v \in H_0^{1,p}(B_R)$ of the equation

$$\begin{aligned}
&\int_{B_R} a^{ij}(x_0) D_i v D_j \varphi \, dx \\
&= \int_{B_R} [a^{ij}(x_0) - a^{ij}(x)] D_i V D_j \varphi \, dx + \int_{B_R} f^j D_j \varphi \, dx \\
&= \sum_j \int_{B_R} \underbrace{\left(\sum_i [a_{ij}(x_0) - a_{ij}(x)] D_i V + f_j \right)}_{\in L^p(B_R)} D_j \varphi \, dx,
\end{aligned}$$

then

$$\|Dv\|_{p;B_R} \leq C \|[A(x_0) - A] DV\|_{p;B_R} + C \|f\|_{p;B_R}$$

for any $f \in L^p(B_R; \mathbb{R}^n)$, in particular for $f = 0$. Consider the solution operator $\tilde{T} : H_0^{1,p}(B_R) \longrightarrow H_0^{1,p}(B_R), V \longmapsto v$:

$$\begin{aligned}
\|\tilde{T}V_1 - \tilde{T}V_2\|_{1,p;B_R} &= \|v_1 - v_2\|_{1,p;B_R} \\
&\leq C(R^p + 1) \|Dv_1 - Dv_2\|_{p;B_R} \quad \text{(by A.8)} \\
&\leq C(R^p + 1) \|[A(x_0) - A] D(V_1 - V_2)\|_{p;B_R} \\
&\leq C(R^p + 1) \sup_{B_R} |A(x_0) - A| \|V_1 - V_2\|_{1,p;B_R}.
\end{aligned}$$

Consequently, $\tilde{T}$ is a contraction for sufficient small R. Hence, there exists a unique fixpoint, which solves (3.120) and therefore equals u. Moreover,

$$\|Du\|_{p;B_R} \leq C \|f\|_{p;B_R}. \qquad (3.121)$$

3.2 The Regularity Theorem

After the huge amount of preparations, we are now able to understand the proofs given in the original work of Krylov, Bogachev and Röckner concerning the regularity of densities of type cf. Chapter 2.

Note that, if $pq \geq p+q, |b| \in L^p(B_R)$ and $\mu \in L^q(B_R)$, then by Hölder's inequality, $|b|\mu \in L^r(B_R)$ and

$$\|b\mu\|_{r;B_R} \leq \|b\|_{p;B_R}\|\mu\|_{q;B_R} \quad \text{with} \quad r = \frac{pq}{p+q}; \tag{3.122}$$

in fact,

$$\frac{1}{p} + \frac{1}{q} = \frac{p+q}{pq} = \frac{1}{r}, \text{ therefore } \frac{1}{\left(\frac{p}{r}\right)} + \frac{1}{\left(\frac{q}{r}\right)} = 1 = \frac{r}{p} + \frac{r}{q},$$

so that

$$\|b\mu\|_r = \|b^r\mu^r\|_1^{r^{-1}} \leq \|b^r\|_{\frac{p}{r}}^{r^{-1}}\|\mu^r\|_{\frac{q}{r}}^{r^{-1}} = \|b\|_p\|\mu\|_q.$$

Lemma 3.51 *Let p and q be two numbers satisfying $p \geq n, q \geq p'$ but not such that $p = n = q'$. Let $R_1 > 0$. Assume that the functions $a_{ij} \in H^{1,p}(B_{R_1})$ are continuous and $A \geq \lambda I$ for some $\lambda > 0$. Then there exists $N > 0$ and $R_0 > 0$ only depending on p, q, n, λ, R_1, the modulus of continuity of A, $\|a^{ij}\|_{1,p;B_{R_1}}$ and the rate of decreasing to zero of $\|Da_{ij}\|_{n;B_R}$ as $R \longrightarrow 0$, such that for any $R \leq R_0$ and $\varphi \in H_0^{1,q}(B_R)$ one has*

$$f := a^{ij}D_iD_j\varphi \in H^{-1,q}(B_R)$$

and

$$\|D\varphi\|_{q;B_R} \leq N\|f\|_{-1,q;B_R}. \tag{3.123}$$

Proof. We may assume that $R_1 = 1$.

- Step 1: $f \in H^{-1,q}(B_R) = (H_0^{1,q'}(B_R))'$
  If we assume, that for $\zeta \in (H^{1,p} \cap L^\infty)(B_R)$ the linear operator

$$T_\zeta : H_0^{1,q'} \longrightarrow H_0^{1,q'}, \qquad \psi \longmapsto \zeta\psi$$

  is continuous, then for $u \in H_0^{1,q'}$,

$$\begin{aligned} |fu| &= \left|\int_{B_R} a^{ij}(D_iD_j\varphi)u\,dx\right| \\ &= \left|\int T_{a^{ij}}(u)D_iD_j\varphi\,dx\right| \\ &= \left|-\int D_i(T_{a^{ij}}(u))D_j\varphi\,dx\right| \\ &\leq \|DT_{a^{ij}}(u)\|_{q'}\|D\varphi\|_q \quad \text{(by Hölder)} \\ &\leq C\|D\varphi\|_q\|Du\|_{q'} \end{aligned}$$

and step 1 would follow by equivalence of norms on $H_0^{1,q'}$. Let us justify our assumption; again by equivalence of norms, it suffices to show that $\|D(\zeta\psi)\|_{q'} \leq C\|D\psi\|_{q'}$. Let us first estimate $\|\psi D\zeta\|_{q'}$ in all cases for q' and then use the product rule (see (3.124) below).

- Case $q' < n$: By assumption and D.1, $D_i\zeta \in L^p(B_R), \psi \in L^{\frac{nq'}{n-q'}}(B_R)$. Since

$$p\frac{nq'}{n-q'} \geq p + \frac{nq'}{n-q'} \Leftrightarrow pnq' \geq pn - pq' + nq'$$
$$\Leftrightarrow pn(q'-1) + q'(p-n) \geq 0,$$

which is true, (3.122) is applicable and implies $|\psi D\zeta| \in L^s$ with

$$s = \frac{\frac{pq'n}{n-q'}}{p + \frac{nq'}{n-q'}} = q'\frac{pn}{pn - pq' + nq'} \geq q'$$

and again by D.1,

$$\|\psi D\zeta\|_s \leq \|\psi\|_{\frac{q'n}{n-q'}} \|D\zeta\|_p \leq C(n,q)\|D\psi\|_{q'}\|D\zeta\|_p.$$

By Hölder,

$$\|\psi D\zeta\|_{q'} \leq \mathrm{dx}(B_R)^{\frac{1}{q'}-\frac{1}{s}}\|\psi D\zeta\|_s$$

- Case $q' > n$: For $\psi \in H_0^{1,q'}(B_R)$, by D.1 we have

$$\sup_{B_R}|\psi| \leq C(n,q)\mathrm{dx}(B_R)^{\frac{1}{n}-\frac{1}{q'}}\|D\psi\|_{q'}$$

The general assumption $q \geq p'$ implies $p \geq q'$, thus again by Hölder

$$\begin{aligned}\|\psi D\zeta\|_{q'} &\leq \|\psi\|_\infty \|D\zeta\|_{q'}\\ &\leq C(n,q,R)\|D\psi\|_{q'}\|D\zeta\|_{q'}\\ &\leq C(n,q,p,R)\|D\psi\|_{q'}\|D\zeta\|_p.\end{aligned}$$

- Case $q' = n$: Since $\psi \in H_0^{1,n}(B_R)$, by D.1 we obtain $\psi \in L^r(B_R) \quad \forall r \in [1,\infty)$. From our assumptions $q' = n, p \geq n, \neg(p = n = q')$ it follows that $p > q'$. For $0 < \epsilon < 1$ we have by case $q' < n$ and Hölder,

$$\|\psi D\zeta\|_{q'-\epsilon} \leq C\|D\zeta\|_p\|D\psi\|_{q'-\epsilon} \leq \tilde{C}\|D\zeta\|_p\|D\psi\|_{q'}.$$

Now,

$$|\psi|^{q'-\epsilon}|D\zeta|^{q'-\epsilon} \xrightarrow[\epsilon\to 0]{} |\psi|^{q'}|D\zeta|^{q'} \quad \text{pointwisely}$$

and

$$\begin{aligned}|\psi D\zeta|^{q'-\epsilon} &= 1_{\{|\psi D\zeta|\geq 1\}}|\psi D\zeta|^{q'-\epsilon} + 1_{\{|\psi D\zeta|<1\}}|\psi D\zeta|^{q'-\epsilon}\\ &\leq 1_{\{|\psi D\zeta|\geq 1\}}|\psi D\zeta|^{q'} + 1_{\{|\psi D\zeta|<1\}} =: h \in L^1(B_R),\end{aligned}$$

since $1 \in L^1(B_R)$ and

$$\begin{aligned}\int |\psi D\zeta|^{q'}\,dx &\le \||D\zeta|^{q'}\|_{\frac{p}{q'}}\||\psi|^{q'}\|_{\left(\frac{p}{q'}\right)'}\\ &= \|D\zeta\|_p^{q'}\|\psi\|_{\frac{pq'}{p-q'}}^{q'} < \infty.\end{aligned}$$

Therefore, by Lebesgue with majorante h,

$$\int |\psi|^{q'-\epsilon}|D\zeta|^{q'-\epsilon}\,dx \xrightarrow[\epsilon\to 0]{} \int |\psi|^{q'}|D\zeta|^{q'}\,dx.$$

By a diagonal argument, we find a subsequence (ϵ_k), such that $\|\psi D\zeta\|_{q'-\epsilon_k} \xrightarrow[\epsilon_k\to 0]{} \|\psi D\zeta\|_{q'}$ and

$$\|\psi D\zeta\|_{q'} \le C\|D\zeta\|_p\|D\psi\|_{q'}.$$

By assumption and combination of the three cases, we obtain

$$\begin{aligned}\|D(\zeta\psi)\|_{q'} &= \|\psi D\zeta\|_{q'} + \|\zeta D\psi\|_{q'}\\ &\le C\|D\zeta\|_p\|D\psi\|_{q'} + \|\zeta\|_\infty\|D\psi\|_{q'}\\ &= (C\|D\zeta\|_p + \|\zeta\|_\infty)\|D\psi\|_{q'}.\end{aligned} \tag{3.124}$$

- Step 2: we assert (3.123).
Note that $D_i(a^{ij}D_j\varphi) = a^{ij}D_iD_j\varphi + D_ia^{ij}D_j\varphi =: f + g$, since integration by parts yields

$$\begin{aligned}\int_{B_R} a^{ij}D_iD_j\varphi u\,dx &= -\int D_i(a^{ij}u)D_j\varphi\,dx\\ &= -\int D_ia^{ij}D_j\varphi u + a^{ij}D_iuD_j\varphi\,dx\\ &= -\int D_ia^{ij}D_j\varphi u\,dx + \int uD_i(a^{ij}D_j\varphi)\,dx.\end{aligned}$$

Since $f+g \in H^{-1,q}(B_R)$, by Lemma C.13 there exist $v_1, \ldots, v_n \in L^q(B_R)$, such that

$$(f+g)u = \int_{B_R} v^iD_iu\,dx \quad \forall u \in H_0^{1,q'}(B_R),$$

or in other words,

$$-\int_{B_R} a^{ij}(x)D_j\varphi D_iu\,dx = \int_{B_R} v^iD_iu\,dx \quad \forall u \in H_0^{1,q'}(B_R). \tag{3.125}$$

We now interpret φ as a solution of the PDE (3.125), u playing the test-function's part. Now, we apply (3.121) with $\varphi, q, -v$ instead of u, p, f and then (C.4) to obtain

$$\begin{aligned}\|D\varphi\|_{q;B_R} &\le C\|v\|_{q;B_R}\\ &\le N_1\|f+g\|_{-1,q;B_R}\\ &\le N_1(\|f\|_{-1,q;B_R} + \|g\|_{-1,q;B_R}),\end{aligned} \tag{3.126}$$

where N_1 is independent of $R \in (0,1]$ and φ. Now, we consider three cases in order to get rid of the g-term on the right-hand side of the last inequality.

- Case $q > n'$: By Lemma D.3(i) and (3.122), we have

$$\|g\|_{-1,q;B_R} \le N_2 \|g\|_{\frac{qn}{q+n}} = N_2 \|D_i a^{ij} D_j \varphi\|_{\frac{qn}{q+n}}$$
$$\le N_2 \|D\varphi\|_q \|a\|_n \quad \underbrace{(qn > q+n \Leftrightarrow q > n')}_{(*)}$$

  where $a = (a^j), a^j = D_i a^{ij}$ and N_2 is independent of R. Using (3.126),

$$\|D\varphi\|_q \le N_1 N_2 \|a\|_n \|D\varphi\|_q + N_1 \|f\|_{-1,q}. \tag{3.127}$$

  N_1, N_2 are independent of R and f, and since $|Da^{ij}| \in L^p(B_R;\mathbb{R}^n), p \ge n$, we can choose R so small that

$$N_1 N_2 \|a\|_{n;B_R} \le \frac{1}{2}.$$

  For such an R, (3.127) implies (3.123).
- Case $p' < q \le n'$: $q > p' \Rightarrow pq \ge p + q$, hence by (3.122),

$$|a||D\varphi| \in L^{\tilde{r}}, \tilde{r} := \frac{pq}{p+q}. \tag{3.128}$$

  Also from $q > p'$ and (*) it follows that for r defined by

$$\frac{rn}{r+n} = \frac{pq}{p+q}, \quad \text{i. e.} \quad r = \frac{n\frac{pq}{p+q}}{n - \frac{pq}{p+q}}$$

  we have $r > n' \ge q$. By (*), (3.122) and Lemma D.3(i) we have

$$\|g\|_{-1,r} \le N \|g\|_{\frac{rn}{r+n}} \le N \|a\|_p \|D\varphi\|_q.$$

  By (C.3), Proposition C.13, norm equivalence and Hölder, we also have

$$\|g\|_{-1,q;B_R} \le \|\tilde{v}\|_q \le N_3 R^{n\left(\frac{1}{q}-\frac{1}{r}\right)} \|\tilde{v}\|_r$$
$$\le \tilde{N}_3 R^{n\frac{r-q}{rq}} \|\tilde{v}\|^*_{L^r_n} \le N_3 \|g\|_{-1,r},$$

  where $D_i \tilde{v}^i$ shall be the representation of $g \in H^{-1,r}$, for which $\|\tilde{v}\|^*_{L^r_n} = \|g\|_{-1,r}$. Note that $D_i \tilde{v}^i$ also is a representation of $g \in H^{-1,q}$, for which $\|g\|_{-1,q} \le \|\tilde{v}\|_q$, since $C_0^\infty(B_R) \subset (H_0^{1,r'} \cap H_0^{1,q'})(B_R)$. Altogether,

$$\|g\|_{-1,q} \le N \|a\|_p \|D\varphi\|_q$$

  and we continue as above.

– Case $q = p' < n'$: By Lemma D.3, for $R \in (0,1)$ we have

$$\|g\|_{-1,q} \leq N_4 R^{1-\frac{n}{p}} \|g\|_1 \leq N_4 R^{1-\frac{n}{p}} \|a\|_p \|D\varphi\|_q,$$

where we used Hölder and $p' = q$ for the last inequality. Now, we continue as above. □

Theorem 3.52 *Let $p \geq n, q \in (1,\infty), R_1 > 0$, let $a^{ij} \in H^{1,p}(B_{R_1})$ be continuous, $A \geq \lambda I$ for some $\lambda > 0$. Then there exist $R_0, N_0 > 0$ with the following properties. Let $R < R_0$ and let μ be a measure of finite total variation on B_R, such that for any $\varphi \in C_0^2(B_R) := C^2(\bar{B}_R) \cap \{u | u_{|\partial B_R} = 0\}$ we have*

$$\left| \int_{B_R} a^{ij} D_i D_j \varphi \, d\mu \right| \leq N \|D\varphi\|_q \tag{3.129}$$

with N independent of φ. Furthermore, assume one of the following:

1. *$p > n$ or*
2. *$p = n > q'$ and $\mu \in \bigcup_{r>1} L^r(B_R)$.*

Then $\mu \in H_0^{1,q' \wedge p}(B_R)$ (where we identify μ with its density) and

$$\|\mu\|_{1,q' \wedge p; B_R} \leq N_0.$$

In addition, R_0 only depends on $p, n, q, \lambda, R_1, \|a^{ij}\|_{1,p;B_{R_1}}$, the rate of decreasing of $\|Da^{ij}\|_{n;B_R}$ as $R \longrightarrow 0$, and N_0 depends on the same quantities and N.

Proof. We break the proof into two cases. Let $R < R_1$.

- Case $q \geq p'$: Take $f \in C_0^\infty(B_R)$ and solve the equation

$$\begin{aligned} a^{ij} D_i D_j \varphi &= f \quad \text{in} \quad B_R \\ \varphi &= 0 \quad \text{on} \quad \partial B_R. \end{aligned} \tag{3.130}$$

  If 1.) holds, then $p > n$ and A is Hölder continuous in $\bar{B}_R$ by Sobolev. By (2.13), there exists an unique solution $\varphi \in C_0^2(B_R)$, which we can substitute into (3.129). Note, that a density $\mu \in L^s_{loc}(B_R)$ exists for $s \in [1, n')$ by Theorem 2.28. Also note that 1.) holds, if $q = n'$.
  If 2.) holds, then by continuity of A and Proposition 2.27, $D_i D_j \varphi \in \bigcap_{p \in [1,\infty)} L^p(B_R)$. Owing to $\mu \in \bigcup_{r>1} L^r(B_R)$, we again can substitute φ into (3.129): in fact, let $\mu \in L^r(B_R)$, then $a^{ij}\mu \in L^r(B_R)$. Since $C_0^2(B_R) \subset (H^{2,q\vee r'} \cap H_0^{1,q \vee r'})(B_R)$ dense by Lemma 3.53, there exist $\varphi_m \in C_0^2(B_R)$, such that

$$\begin{aligned} \left| \int_{B_R} a^{ij} D_i D_j \varphi \mu \, dx \right| &= \lim_{m \longrightarrow \infty} \left| \int_{B_R} a^{ij} D_i D_j \varphi_m \mu \, dx \right| \\ &\leq N \lim_{m \longrightarrow \infty} \|D\varphi_m\|_{q;B_R} = N \|D\varphi\|_{q;B_R}. \end{aligned}$$

Now,

$$\left|\int_{B_R} f\mu\,dx\right| = \left|\int_{B_R} a^{ij}D_iD_j\varphi\mu\,dx\right| \le N\|D\varphi\|_{q;B_R} \quad \text{(by (3.129))} \le N\|f\|_{-1,q;B_R} \quad \text{(by Lemma 3.51)} \quad ,$$

which implies the first case by duality.

- Case $1 < q < p' < n'$: Observe that (3.129) is satisfied with $r = \frac{p'+n'}{2}$ in place of q: indeed, $n' > r > p' > q$, therefore

$$\|D\varphi\|_{q;B_R} \le \mathrm{dx}(B_R)^{\frac{1}{q}-\frac{1}{r}}\|D\varphi\|_{r;B_R}.$$

By the first case, we have $\mu \in H_0^{r',1}(B_R)$ for sufficient small R. Since $r' > n$, by the Sobolev embedding, μ is bounded in B_R. Furthermore, we note that (3.129) means that

$$L : \varphi \longmapsto \int_{B_R} a^{ij}D_iD_j\varphi\mu\,dx$$

is a linear functional defined on $C_0^2(B_R)(\subset H_0^{1,q}(B_R)$ dense) and bounded w. r. t. the $H_0^{q,1}(B_R)$-norm. By duality between $H_0^{1,q}(B_R)$ and $H^{-1,q'}(B_R)$, we have that

$$L\varphi = \int_{B_R} a^{ij}D_iD_j\varphi\mu\,dx = \int_{B_R} \varphi D_i f^i\,dx,$$

where $D_i f^i \in H^{-1,q'}(B_R) \subset H^{-1,p}(B_R)$ is the representation of L of type (C.3). Thus, μ is a generalized solution of the equation

$$D_j(a^{ij}D_i\mu) = D_j f^j - D_j(\mu D_i a^{ij}) =: g.$$

Here $\mu D_i a^{ij} \in L^p(B_R)$, since μ is bounded, so that $g \in H^{-1,p}(B_R)$. We assert that $\mu \in H_0^{1,p}(B_R)$:
$\mu \in L^p(B_R)$, since $\mu \in C(\bar{B}_R)$, moreover, $D\mu \in L^p(B_R;\mathbb{R}^n)$ by (3.121), hence $\mu \in H^{1,p}(B_R)$. Since $\mu \in H_0^{1,r'}(B_R) \cap C(\bar{B}_R), \mu \equiv 0$ on ∂B_R, which implies our claim.

□

Lemma 3.53 ([19, Exercise 9.6]) $C_0^2(B_R) \subset (H^{2,p} \cap H_0^{1,p})(B_R)$ *dense* $\forall p \in (1,\infty)$.

*Proof.*Let $u \in (H^{2,p} \cap H_0^{1,p})(B_R), B := B_1(0)$. Since B_R is of class C^∞, there exist $U^1, \dots, U^m \subset \mathbb{R}^n$ with $\partial B_R \subset \bigcup_{i=1}^m U^i$ and C^∞-diffeomorphisms $\psi_1, \dots, \psi_m$ with

$$\psi_i \in C^\infty(\bar{U}^i), \psi^{-1} \in C^\infty(\bar{B}), \psi_i(U^i \cap B_R) \subset \mathbb{R}^n_+, \psi_i(U^i \cap \partial B_R) \subset \partial\mathbb{R}^n_+.$$

Let $\zeta \in C_0^\infty(B), \zeta \equiv 1$ on $B_\rho(0)$, where $\rho < 1$ is chosen such that

$$\bigcup_{i=1}^m \psi^{-1}(B_\rho(0)) \supset \partial B_R.$$

Add $U^0 = B_\tau(0) \subset\subset B_R$, such that $B_R \subset \bigcup_{i=1}^m \psi^{-1}(B_\rho(0)) \cup U^0$. Since $u \in (H^{2,p} \cap H_0^{1,p})(B_R)$,

$$\tilde{u}_i := \zeta(u \circ \psi^{-1}) \in (H^{2,p} \cap H_0^{1,p})(B^+)$$

by the chain and product rule in Sobolev spaces, and moreover, $\tilde{u}_i = 0$ near $(\partial B)^+$. After odd reflection and extension by zero, as in Lemma 2.22, we obtain that $\tilde{u}_i \in H^{2,p}(\mathbb{R}^n)$. Therefore, using a usual mollifier, there exist

$$\tilde{u}_i^\epsilon \in C_0^\infty(\mathbb{R}^n), \epsilon > 0 \quad \text{, such that} \quad \tilde{u}_i^\epsilon \longrightarrow \tilde{u}_i \quad \text{in} \quad H^{2,p}(\mathbb{R}^n), \tilde{u}_i^\epsilon(x',0) = 0.$$

In particular, $\tilde{u}_i^\epsilon \longrightarrow \tilde{u}_i = u \circ \psi_i^{-1}$ in $H^{2,p}(B_\rho)$. Therefore,

$$C^\infty(\bar{U}^i) \ni u_i^\epsilon := \tilde{u}_i^\epsilon \circ \psi_i \longrightarrow u \quad \text{in} \quad H^{2,p}(\psi_i^{-1}(B_\rho)) \quad \forall 1 \le i \le m. \tag{3.131}$$

Since $C^\infty(\bar{U}^0) \subset H^{2,p}(U^0)$ dense by A.4, we also find (u_0^ϵ) with $u_0^\epsilon \longrightarrow u$ in $H^{2,p}(U^0)$. Let $(\eta_i)_{i=0}^m$ be a partition of unity, subordinated to

$$V_0 := U^0, V_1 := \psi_1^{-1}(B_\rho(0)), \dots, V_m := \psi_m^{-1}(B_\rho(0)).$$

Define $u_\epsilon := \sum_{i=0}^m \eta_i u_i^\epsilon \in C^\infty(\bar{B}_R)$ (after extension with zero even $C_0^\infty(\mathbb{R}^n)$), then

$$\begin{aligned}\|u - u_\epsilon\|_{2,p;B_R} &= \|\sum_{i=0}^m \eta_i(u - u_i^\epsilon)\|_{2,p;B_R} \\ &\le \sum_{i=0}^m \|\eta_i(u - u_i^\epsilon)\|_{2,p;V_i \cap B_R} \\ &\longrightarrow 0,\end{aligned}$$

because of (3.131), the product rule and $|D^\alpha \eta_i| \le C \quad \forall |\alpha| \le 2, i \in \{0, \dots, m\}$. Moreover, for $x \in \partial B_R$ with $x \in V_i, i \in I \subset \{1, \dots, m\}$,

$$u_\epsilon(x', 0) = \sum_{i \in I} (\eta_i u_i^\epsilon)(x', 0) = 0.$$

□

We now arrive at the main theorem of this chapter.

Theorem 3.54 *Let $p > n, r \in (p', \infty), \mu \in L^r_{loc}(\Omega, dx), a^{ij} \in H^{1,p}_{loc}(\Omega)$, and let either $\beta \in L^p_{loc}(\Omega, dx)$ or $\beta \in L^p_{loc}(\Omega, \mu)$. Suppose that A is locally uniformly nondegenerate and that for any $\varphi \in C_0^\infty(\Omega)$ we have*

$$\left| \int_\Omega a^{ij} D_i D_j \varphi \mu \, dx \right| \le \int_\Omega (|\varphi| + |D\varphi|) |\beta\mu| \, dx. \tag{3.132}$$

Then $\mu \in H^{1,p}_{loc}(\Omega)$.

Proof. We compute

$$r > p' = \frac{p}{p-1} \Rightarrow pr > p + r \Rightarrow q' := q'(r) := \frac{pr}{p+r} > 1$$

Hence,

$$1 < q = \frac{q'}{q'-1} = \frac{\frac{pr}{p+r}}{\frac{pr}{p+r} - 1} = \frac{pr}{pr - r - p}.$$

If $\beta \in L^p_{\text{loc}}(\Omega)$, then $\beta\mu \in L^{q'}_{\text{loc}}(\Omega)$ by (3.122).
Case $\beta \in L^p_{\text{loc}}(\Omega, \mu)$: We have

$$|\beta|^{q'}|\mu|^{q'} = |\beta|^{\frac{pr}{p+r}}|\mu|^{\frac{r}{p+r}}|\mu|^{\frac{pr-r}{p+r}},$$

where

$$|\beta|^{\frac{pr}{p+r}}|\mu|^{\frac{r}{p+r}} \in L^s_{\text{loc}}(\Omega), s = \frac{p+r}{r} \quad \left(\Rightarrow s' = \frac{p+r}{p}\right)$$

and

$$|\mu|^{\frac{pr-r}{p+r}} \in L^{s'}_{\text{loc}}(\Omega),$$

because $\frac{pr-r}{p} = r - \frac{r}{p} < r$ and $\mu \in L^r_{\text{loc}}(\Omega)$. For $R > 0$ such that $B_R := B_R(x_0) \subset\subset \Omega, \eta \in C_0^\infty(B_R)$ and $\varphi \in C_0^2(B_R)$ we have

$$\begin{aligned}
&\left|\int_{B_R} a^{ij} D_i D_j \varphi(\eta\mu)\, dx\right| \\
&= \left|\int_{B_R} a^{ij} D_i D_j(\varphi\eta)\mu - a^{ij}\varphi\mu D_i D_j \eta - 2a^{ij} D_i\varphi D_j\eta\mu \, dx\right| \\
&\le \left|\int_{B_R} a^{ij} D_i D_j(\varphi\eta)\mu\, dx\right| + \left|\int_{B_R} a^{ij}\varphi\mu D_i D_j \eta\, dx\right| + 2\int_{B_R} \|A\||D\eta||D\varphi||\mu|\, dx \\
&\le N'(\eta) \underbrace{\int_{B_R} (|\varphi| + |D\varphi|)|\beta\mu|\, dx}_{\le \tilde{N}\|\beta\mu\|_{q'}\|D\varphi\|_q} + N\|D^2\eta\|_\infty \|\|A\||\mu|\|_{q'}\|D\varphi\|_q + \\
&\qquad 2\|D\eta\|_\infty \|\|A\||\mu|\|_{q'}\|D\varphi\|_q
\end{aligned}$$

by (3.122), Poincaré and since $C_0^\infty(B_R) \subset \{u \in C^2(B_R) | \text{supp } u \text{ compact}\}$ dense, which can be shown by mollifying and is needed to apply (3.132)

$$\le N_1 \|D\varphi\|_{q;B_R}, \tag{3.133}$$

where N_1 does not depend on φ. Applying Theorem 3.52, 1.), we obtain $\eta\mu \in H_0^{1,q'\wedge p}(B_R)$, if R is small enough. Since we can take any point as x_0 and $q' - p = \frac{pr}{p+r} - p = -\frac{p^2}{p+r} < 0$, we have

$$\mu \in H^{1,q'}_{\text{loc}}(\Omega). \tag{3.134}$$

Note, $q' < n \Leftrightarrow r < \frac{pn}{p-n}$. In this case, by Sobolev embedding,

$$\mu \in L^{r_1}_{\text{loc}}(\Omega) \quad \text{with} \quad r_1 = \frac{q'n}{n-q'} = \frac{\frac{pr}{p+r}n}{n - \frac{pr}{p+r}} = \frac{prn}{(p+r)n - pr}.$$

Thus, on the interval $\left(\frac{p}{p-1}, \frac{pn}{p-n}\right)$ we have a mapping

$$T : r \longmapsto r_1$$

with the property that

$$(\mu \in L^r_{\text{loc}}(\Omega), q'(r) < n \Rightarrow \mu \in L^{r_1}_{\text{loc}}(\Omega) \quad \text{and therefore} \quad \mu \in H^{1,q'(r_1)}_{\text{loc}}(\Omega))$$

by the first part of this proof. Note that $(\cdot)'$ is antitone and

$$\frac{r_1}{r} = \frac{pn}{pn - r(p-n)} > \frac{pn}{pn - p'(p-n)} = \frac{n'}{p'} > 1,$$

where the first inequality sign dues to $r > p'$ and the second to $p > n$. Consequently, T is strictly increasing on $\left(p', \frac{pn}{p-n}\right)$ with $Tr > \frac{n'r}{p'}, T^k r > \left(\frac{n'}{p'}\right)^k r$. Hence,

$$\exists k_0 \in \mathbb{N}, \quad \text{such that} \quad T^{k_0} r < \frac{pn}{p-n}, T^{k_0+1} r \geq \frac{pn}{p-n}.$$

Therefore, after k_0 applications of T to the given r, we will arrive at a point

$$s \in \left(p', \frac{pn}{p-n}\right),$$

such that $t = T(s) \geq \frac{pn}{p-n}$ and $\mu \in L^t(B_R)$, i. e. we are leaving the domain of T and the (bootstrapping-) process stops. Without loss of generality we may assume that $t > \frac{pn}{p-n}$: if $t = \frac{pn}{p-n}$, then $\mu \in L^{t-\epsilon}_{\text{loc}}(\Omega)$ and we can apply the above considerations on $r := t - \epsilon_0$, where $\epsilon_0 > 0$ shall be so small that $T(t-\epsilon_0) > \frac{pn}{p-n}$. This shows that we could have assumed from the very beginning that $r > \frac{pn}{p-n}$, that is $q' > n$. In that case, (3.134) and Sobolev imply that the function μ is locally bounded, which shows that (3.133) remains true with $q' = p$. Now apply Theorem 3.52 again to finish the proof. □

It should be noted that the assertion of Theorem 3.54 is valid under the following alternative assumptions on a^{ij}, β, μ:

$$\text{a)} \begin{cases} \mu \quad \text{locally finite signed Borel measure on } \Omega \\ a^{ij} \in H^{1,p}_{\text{loc}}(\Omega) \quad \text{continuous} \\ A \quad \text{locally uniformly nondegenerate} \\ \beta \in L^p_{\text{loc}}(\Omega, \mu) \\ (3.132) \end{cases}$$

*Proof.*By (3.132), $\beta \in L^p_{\text{loc}}(\Omega,\mu)$ and Hölder, we have for any $\tilde{R} > 0$ such that $B_{\tilde{R}} := B_{\tilde{R}}(x_0) \subset\subset \Omega$,

$$\left| \int_{B_{\tilde{R}}} a^{ij} D_i D_j \varphi \, d\mu \right| \leq C \sup_{B_{\tilde{R}}} (|\varphi| + |D\varphi|),$$

where C does not depend on $\varphi \in C_0^\infty(B_{\tilde{R}})$. Applying Theorem 2.28, we obtain $\mu \in L^r_{\text{loc}}(B_{\tilde{R}}) \quad \forall r \in [1, n')$. Since $p' < n'$, we have $\mu \in L^r_{\text{loc}}(B_{\tilde{R}}, \text{dx})$ for some $r > p'$. Now, we can copy the proof of Theorem 3.54 with Ω replaced by $B_{\tilde{R}}$ and the case $\beta \in L^p_{\text{loc}}(\Omega,\mu)$. $\square$

or

$$\text{b)} \begin{cases} \mu \quad \text{locally finite signed Borel measure on } \Omega \\ a^{ij} \in H^{1,p}_{\text{loc}}(\Omega) \quad \text{continuous} \\ A \quad \text{locally uniformly nondegenerate} \\ \beta \in L^1_{\text{loc}}(\Omega,\mu) \cap L^p_{\text{loc}}(\Omega, \text{dx}) \\ (3.132) \end{cases}$$

*Proof.*By (3.132) and $\beta \in L^1_{\text{loc}}(\Omega,\mu)$, we have for any $\tilde{R} > 0$, such that $B_{\tilde{R}} := B_{\tilde{R}}(x_0) \subset\subset \Omega$,

$$\left| \int_{B_{\tilde{R}}} a^{ij} D_i D_j \varphi \, d\mu \right| \leq C \sup_{B_{\tilde{R}}} (|\varphi| + |D\varphi|),$$

where C does not depend on $\varphi \in C_0^\infty(B_{\tilde{R}})$. Now proceed as in a), case $\beta \in L^p_{\text{loc}}(\Omega, \text{dx})$. $\square$

Corollary 3.55 *Let μ be a locally finite Borel measure on Ω. Let A be locally uniformly nondegenerate in Ω with $a^{ij} \in H^{1,p}_{loc}(\Omega)$, where $p > n$, and let either $b^i, c \in L^p_{loc}(\Omega, dx)$ or $b^i, c \in L^p_{loc}(\Omega,\mu)$. Assume that for any $\varphi \in C_0^\infty(\Omega)$ one has*

$$\int_\Omega [a^{ij} D_i D_j \varphi + b^i D_i \varphi + c\varphi] \, d\mu = 0, \tag{3.135}$$

where we assume that b^i, c are locally μ-intgrable. Then μ has a density in $H^{1,p}_{loc}(\Omega)$, which is locally Hölder continuous.

*Proof.*By (3.135),

$$\int_\Omega a^{ij} D_i D_j \varphi \, d\mu = - \int_\Omega b^i D_i \varphi + c\varphi \, d\mu.$$

Therefore,

$$\left| \int_\Omega a^{ij} D_i D_j \varphi \, d\mu \right| \leq \int_\Omega (|\varphi| + |D\varphi|) |\beta| \, d|\mu|,$$

where $\beta = |b| + |c|$. Now proceed as in a), b) respectively. $\square$

Corollary 3.56 *Let $p > n, a^{ij} \in H^{1,p}_{loc}(\Omega), b^i, f^i, c \in L^p_{loc}, A$ locally uniformly nondegenerate in Ω. Assume that μ is a locally finite Borel measure on Ω, such that $b^i, c \in L^1_{loc}(\Omega, \mu)$ and for any $\varphi \in C_0^\infty(\Omega)$ one has*

$$\int_\Omega a^{ij} D_i D_j \varphi + b^i D_i \varphi + c\varphi \, d\mu = \int_\Omega f^i D_i \varphi \, dx, \tag{3.136}$$

then $\mu \in H^{1,p}_{loc}(\Omega)$.

*Proof.*Let $B := B_{\tilde{R}}(x_0) \subset\subset \Omega$. By Subsection 3.1.1 we obtain a solution $u \in H_0^{1,2}(B)$ of the problem

$$D_i(a^{ij} D_j u + D_j a^{ij} u - b^i u) + cu = D_i f^i, u_{|\partial B} = 0,$$

i. e.,

$$\int_B (a^{ij} D_j u + D_j a^{ij} u - b^i u) D_i \varphi + \varphi c u \, dx = \int_B f^i D_i \varphi \, dx$$

for any testfunction φ. Therefore,

$$\begin{aligned} \int_B a^{ij} D_i D_j \varphi \, du &= - \int_B (b^i u + f^i) D_i \varphi - cu\varphi \, dx \\ &\le \sup_B (|\varphi| + |D\varphi|) \int_B |bu| + |f| + |cu| \, dx \\ &\le C \sup_B (|\varphi| + |D\varphi|), \end{aligned} \tag{3.137}$$

where C does not depend on φ. Applying Theorem 2.28, we obtain $u \in L^r_{\text{loc}}(\Omega, \text{dx})$ for some $r > p'$.

Define $\beta := |b| + |c| \in L^p_{\text{loc}}$, then as in Theorem 3.54, $\beta u \in L^{q'}_{\text{loc}}(B)$. For $R > 0$, such that $B_R := B_R(x_0) \subset\subset B$ and $\varphi \in C_0^2(B_R)$, we have

$$\begin{aligned} &\left| \int_{B_R} a^{ij} D_i D_j \varphi u \, dx \right| \\ &\le \int_{B_R} (|\varphi| + |D\varphi|) |\beta u| \, dx + \int_{B_R} (|\varphi| + |D\varphi|) |f| \, dx \quad \text{(by (3.137))} \\ &\le N_3 \|D\varphi\|_{q;B_R} + \|f\|_{p;B_R} \||\varphi| + |D\varphi|\|_{p';B_R} \quad \text{(by Hölder)} \\ &\le N_4 \|D\varphi\|_{q;B_R} \quad \text{(Poincaré and } p' < q) \quad . \end{aligned}$$

As in Theorem 3.54, it follows that $u \in H^{1,q'}(B_R)$, if R is small enough. Following the proof of Theorem 3.54, we see that u is locally bounded. Therefore,

$$\begin{aligned} \left| \int_{B_R} a^{ij} D_i D_j \varphi \, du \right| &\le \sup_{B_R} |u| \int_{B_R} (|\varphi| + |D\varphi|) |\beta| \, dx + \int_{B_R} (|\varphi| + |D\varphi|) |f| \, dx \\ &\le N_5 \|\beta\|_{p;B_R} \|D\varphi\|_{p';B_R} + \|f\|_{p;B_R} \|D\varphi\|_{p';B_R} \\ &\le N_6 \|D\varphi\|_{p';B_R}. \end{aligned}$$

Application of Theorem 3.52 yields $u \in H_0^{1,p}(B_R)$ for some suitable small R. Consider the measure $\mu - u\,dx$ on B_R. It is

$$\int_{B_R} a^{ij} D_i D_j \varphi + b^i D_i \varphi + c\varphi \, d(\mu - u\,dx) = 0 \quad \forall \varphi \in C_0^\infty(B_R).$$

Now, it remains to apply Corollary 3.55. □

Appendix A

INTERPOLATION INEQUALITIES

The following interpolation inequalities were needed in chapter 2 and shall be proven here.

A.1 Marcinkiewicz interpolation

Proposition A.1 (cf. [13, I Theorem 1.2]) *Let $1 \leq p < q \leq \infty$. Suppose that a linear map T is of both weak type (p,p) and weak type (q,q), i. e.,*

$$\|Tf\|_{L^p_w} \leq B_p\|f\|_p \quad \forall f \in L^p(\Omega) \tag{A.1}$$
$$\|Tf\|_{L^q_w} \leq B_q\|f\|_q \quad \forall f \in L^q(\Omega). \tag{A.2}$$

Then for any $r \in (p,q)$, T is of strong type (r,r) and

$$\|T\|_{(r,r)} \leq CB_p^\theta B_q^{1-\theta},$$

where C only depends on r,p,q and $\theta = \frac{p(q-r)}{r(q-p)}$.

*Proof.*Let $f \in L^r(\Omega)$. Decompose f as $f = f_1^s + f_2^s$, where

$$f_2^s(x) = \begin{cases} 0 & \text{, for} \quad |f(x)| > \gamma s, \\ f(x) & \text{, for} \quad |f(x)| \leq \gamma s, \end{cases}$$

and γ is a constant to be determined and $s \geq 0$. We observe that $f_2^s \in L^q(\Omega)$, since

$$\|f_2^s\|_q^q = \int_{\{|f|\leq\gamma s\}} |f|^q\,dx = \int_{\{|f|\leq\gamma s\}} |f|^{q-r}|f|^r\,dx \leq |\gamma s|^{q-r}\int_\Omega |f|^r\,dx < \infty$$

and $f_1^s \in L^p(\Omega)$, since

$$\|f_1^s\|_p = \|f - f_2^s\|_p = \|f\|_{p;\{|f|>\gamma s\}} \leq \mathrm{dx}(\{|f| > \gamma s\})^{\frac{1}{p}-\frac{1}{r}}\|f\|_r < \infty.$$

Moreover,

$$dx(\{|Tf| > s\}) \leq dx\left(\left\{|Tf_1^t| > \frac{s}{2}\right\}\right) + dx\left(\left\{|Tf_2^t| > \frac{s}{2}\right\}\right) \quad \forall s,t \geq 0. \tag{A.3}$$

If $q < \infty$, then from (A.1), (A.2) and (A.3) we derive

$$\lambda_{Tf}(s) \le s^{-p}(2B_p)^p \|f_1^t\|_p^p + s^{-q}(2B_q)^q \|f_2^t\|_q^q \quad \forall t \ge 0.$$

By Lemma 2.8,

$$\begin{aligned}
\int_\Omega |Tf|^r\,dx &= \int_0^\infty rs^{r-1}\lambda_{Tf}(s)\,ds \\
&\le (2B_p)^p \int_0^\infty rs^{r-p-1} \int_{\{|f|>\gamma s\}} |f|^p\,dx\,ds \\
&\quad +(2B_q)^q \int_0^\infty rs^{r-q-1} \int_{\{|f|\le\gamma s\}} |f|^q\,dx\,ds \\
&= (2B_p)^p r \int_\Omega |f|^p \int_0^{\frac{|f|}{\gamma}} s^{r-p-1}\,ds\,dx \\
&\quad +(2B_q)^q r \int_\Omega |f|^q \int_{\frac{|f|}{\gamma}}^\infty s^{r-q-1}\,ds\,dx \\
&= \left[\frac{(2B_p)^p r}{r-p}\gamma^{p-r} + \frac{(2B_q)^q r}{q-r}\gamma^{q-r}\right] \int_\Omega |f|^r\,dx,
\end{aligned}$$

where we used

$$\begin{aligned}
\int_0^\infty s^{r-p-1} \int_{\{|f|>\gamma s\}} |f|^p\,dx\,ds &= \int_\Omega |f|^p \int_0^\infty 1_{\left\{\frac{|f|}{\gamma}>s\right\}} s^{r-p-1}\,ds\,dx \\
&= \int_\Omega |f|^p \int_0^{\frac{|f|}{\gamma}} s^{r-p-1}\,ds\,dx.
\end{aligned}$$

Choosing $\gamma = (B_p^{-p}B_q^q)^{\frac{1}{p-q}}$ (choice as in [13] does not lead to the required θ), we obtain

$$\begin{aligned}
\|T\|_{(r,r)}^r &\le \frac{(2B_p)^p r}{r-p}(B_p^{-p}B_q^q)^{\frac{p-r}{p-q}} + \frac{(2B_q)^q r}{q-r}(B_p^{-p}B_q^q)^{\frac{q-r}{p-q}} \\
&= \frac{2^p r}{r-p} B_p^{p\frac{r-q}{p-q}} B_q^{q\frac{p-r}{p-q}} + \frac{2^q r}{q-r} B_p^{p\frac{r-q}{p-q}} B_q^{q\frac{p-r}{p-q}} \\
&= \left(\frac{2^p r}{r-p} + \frac{2^q r}{q-r}\right) B_p^{p\frac{r-q}{p-q}} B_q^{q\frac{p-r}{p-q}},
\end{aligned}$$

where we used that $p - p\frac{p-r}{p-q} = p\frac{r-q}{p-q}$ and $q + q\frac{p-r}{p-q} = q\frac{p-r}{p-q}$. In other words,

$$\|T\|_{(r,r)} \le CB_p^\theta B_q^{1-\theta}.$$

In the case $q = \infty$, we can take γ so large that the second term on the right-hand side of (A.3) vanishes. In fact, (A.2) implies that $\|Tf_2\|_\infty \le B_\infty \|f_2\|_\infty \le B_\infty \gamma s$; thus we can choose $\gamma = (2B_\infty)^{-1}$. It follows similarly to above that

$$\begin{aligned}
\int_\Omega |Tf|^r\,dx &= \int_0^\infty rs^{r-1}\lambda_{Tf}(s)\,ds \\
&\le \int_0^\infty rs^{r-1}\lambda_{Tf_1}\left(\frac{s}{2}\right)\,ds
\end{aligned}$$

$$
\begin{aligned}
&\leq \int_0^\infty r s^{r-1-p} 2^p B_p^p \|f_1\|_p^p \, ds \\
&= (2B_p)^p r \int_\Omega |f|^p \int_0^{|f|2B_\infty} s^{r-p-1} \, ds \, dx \\
&= (2B_p)^p r \int_\Omega |f|^p \frac{1}{r-p} |f|^{r-p} (2B_\infty)^{r-p} \, dx \\
&= C B_p^p B_\infty^{r-p} \int_\Omega |f|^r \, dx.
\end{aligned}
$$

Hence,

$$\|T\|_{(r,r)} \leq C B_p^{\frac{p}{r}} B_\infty^{\frac{(r-p)}{r}}.$$

□

A.2 Interpolation in Sobolev spaces

Lemma A.2 *1. $\forall p, a, b \in [0, \infty)$,*

$$(a+b)^p \leq 2^p (a^p + b^p).$$

2. $\forall p, a, b \in [0, \infty)$,

$$(a^p + b^p) \leq 2^{p^2} (a+b)^p.$$

Proof.

1. Without loss of generality, $a > b$.

$$(a+b)^p = a^p \left(1 + \frac{b}{a}\right)^p \leq 2^p a^p = 2^p (\max(a,b))^p \leq 2^p (a+b)^p.$$

2. By the first part of this lemma,

$$(a^p + b^p)^{\frac{1}{p}} \leq 2^p ((a^p)^{\frac{1}{p}} + (b^p)^{\frac{1}{p}}) = 2^p (a+b).$$

Since $x \longmapsto x^s$ is isotone for $s \in \mathbb{R}_+$, the lemma follows by taking the p-th power on both sides. □

Proposition A.3 (cf. [19, 7.27]) *Let $1 < p < \infty, u \in H_0^{2,p}(\Omega)$. Then for any $\epsilon > 0$,*

$$\|D_i u\|_{p;\Omega} \leq \epsilon \|D^2 u\|_{p;\Omega} + \frac{C}{\epsilon} \|u\|_{p;\Omega} \leq \epsilon \|u\|_{2,p;\Omega} + \frac{C}{\epsilon} \|u\|_{p;\Omega}. \tag{A.4}$$

*Proof.*Let us first suppose $u \in C_0^2(\mathbb{R})$ and consider an interval (a, b) of length $b - a = \epsilon$. For $x' \in \left(a, a + \frac{\epsilon}{3}\right), x'' \in \left(b - \frac{\epsilon}{3}, b\right)$, by the mean-value theorem we have

$$|u'(\bar{x})| = \frac{|u(x') - u(x'')|}{|x' - x''|} \leq \frac{3}{\epsilon} (|u(x')| + |u(x'')|)$$

for some $\bar{x} \in (a,b)$. Consequently, for any $x \in (a,b)$, we can compute

$$
\begin{aligned}
|u'(x)| &\leq |u'(x) - u'(\bar{x})| + |u'(\bar{x})| \\
&\leq \int_a^b |u''|\,dx + \frac{3}{\epsilon}(|u(x')| + |u(x'')|) \\
\Rightarrow \frac{\epsilon^2}{9}|u'(x)| &= \int_a^{a+\frac{\epsilon}{3}} \int_{b-\frac{\epsilon}{3}}^b |u'(x)|\,dx''\,dx' \\
&\leq \frac{\epsilon^2}{9}\int_a^b |u''|\,dx + \frac{3}{\epsilon}\int_a^{a+\frac{\epsilon}{3}} \int_{b-\frac{\epsilon}{3}}^b |u(x')|\,dx''\,dx' \\
&\quad + \frac{3}{\epsilon}\int_a^{a+\frac{\epsilon}{3}} \int_{b-\frac{\epsilon}{3}}^b |u(x'')|\,dx''\,dx' \\
&\leq \frac{\epsilon^2}{9}\int_a^b |u''|\,dx + \int_a^{a+\frac{\epsilon}{3}} |u(x')|\,dx' + \int_{b-\frac{\epsilon}{3}}^b |u(x'')|\,dx'' \\
&\leq \frac{\epsilon^2}{9}\int_a^b |u''|\,dx + \int_a^b |u|\,dx \\
\Rightarrow \quad |u'(x)| &\leq \int_a^b |u''|\,dx + \frac{9}{\epsilon^2}\int_a^b |u|\,dx,
\end{aligned}
$$

so that by Hölder's inequality

$$
\begin{aligned}
|u'(x)|^p &\leq \left(\int_a^b |u''|\,dx + \frac{9}{\epsilon^2}\int_a^b |u|\,dx,\right)^p \\
&\leq \left(\|1\|_{p'}\||u''| + \frac{9}{\epsilon^2}|u|\|_p\right)^p \\
&= \epsilon^{p-1}\int_a^b \left(|u''| + \frac{9}{\epsilon^2}|u|\right)^p\,dx \\
&\leq \epsilon^{p-1}\int_a^b 2^{p-1}\left(|u''|^p + \frac{9^p}{\epsilon^{2p}}|u|^p\right)\,dx \\
&\leq 2^{p-1}\left(\epsilon^{p-1}\int_a^b |u''|^p\,dx + \frac{9^p}{\epsilon^{p+1}}\int_a^b |u|^p\,dx\right).
\end{aligned}
$$

Hence, integrating with respect to x over (a,b) we have

$$
\int_a^b |u'(x)|^p\,dx \leq 2^{p-1}\left(\epsilon^p\int_a^b |u''|^p\,dy + \left(\frac{9}{\epsilon}\right)^p\int_a^b |u|^p\,dy\right).
$$

Consequently, if we subdivide $\mathbb{R}$ into intervals of length ϵ, and sum up all those inequalities, we obtain

$$
\int |u'(x)|^p\,dx \leq 2^{p-1}\left(\epsilon^p\int |u''|^p\,dy + \left(\frac{9}{\epsilon}\right)^p\int |u|^p\,dy\right), \tag{A.5}
$$

which is the desired result in the one-dimensional case.
To extend to higher dimensions we fix $i \in \{1,\dots,n\}$ and apply (A.5) to $u \in$

$C_0^2(\Omega)$ regarded as a function of x_i only. By succesive integration over the remaining variables we thus obtain

$$\int_\Omega |D_i u|^p \, dx \leq 2^{p-1} \left(\epsilon^p \int_\Omega |D_{ii} u|^p \, dx + \left(\frac{9}{\epsilon}\right)^p \int_\Omega |u|^p \right),$$

so that

$$\begin{aligned} \|D_i u\|_{p;\Omega} &\leq 2^{\frac{p-1}{p}} \left(\epsilon \|D_{ii} u\|_{p;\Omega} + \frac{9}{\epsilon} \|u\|_{p;\Omega} \right) \\ &\leq \tilde{\epsilon} \|D^2 u\|_{p;\Omega} + \frac{36}{\tilde{\epsilon}} \|u\|_{p;\Omega}. \end{aligned} \tag{A.6}$$

□

Before we can show the same inequality for $u \in H^{2,p}(\Omega)$, we need some preparation. Since our estimates in section 2.2 take place on balls, we may assume Ω to be of class C^∞ in the following propositions for simplicity. Originally, it would be enough to assume Ω to be a $C^{k-1,1}$-domain for $k \geq 1$.

Proposition A.4 (cf. [19, 7.25]) (i) *$C^\infty(\bar{\Omega}) \subset H^{k,p}(\Omega)$ dense for $1 \leq p < \infty$,*

(ii) *for any open set $\Omega' \supset\supset \Omega$ there exists a bounded linear extension operator E from $H^{k,p}(\Omega)$ into $H_0^{k,p}(\Omega')$, such that $Eu = u$ in Ω and*

$$\|Eu\|_{k,p;\Omega'} \leq C \|u\|_{k,p;\Omega} \tag{A.7}$$

$$\|Eu\|_{p;\Omega'} \leq C \|u\|_{p;\Omega} \tag{A.8}$$

for any $u \in H^{k,p}(\Omega)$, where $C = C(\Omega, \Omega')$.

*Proof.*Let us first consider the density result (i) for the half-space $\mathbb{R}^n_+$. In this case the translated mollifications of u, given by

$$\begin{aligned} v_h(x) &= u_h(x + 2he_n) \\ &= \int_{\{y_n > 0\}} u(y) \varphi_h(x + 2he_n - y) \, dy, \quad h > 0, \end{aligned} \tag{A.9}$$

converge to u in $H^{k,p}(\mathbb{R}^n_+)$ as $h \longrightarrow 0$ (cf. [3, 2.5,2.12]) , (φ_h) being a Dirac-sequence. Setting $x = (x', x_n)$, we may define an extension $E_0 u$ of u to all of $\mathbb{R}^n$ by

$$E_0 u(x) := \begin{cases} u(x) & , \text{for} \quad x_n > 0 \\ \sum_{i=1}^k c_i u\left(x', -\frac{x_n}{i}\right) & , \text{for} \quad x_n < 0, \end{cases} \tag{A.10}$$

where c_i are constants determined by the system of equations

$$\sum_{i=1}^k c_i \left(-\frac{1}{i}\right)^m = 1, \quad m = 0, 1, \ldots, k-1,$$

i. e. by (Vandermondematrix)$c = 1$. If $u \in C^k(\bar{\mathbb{R}}^n_+) \cap H^{k,p}(\mathbb{R}^n_+)$, it follows that $E_0 u \in C^{k-1}(\mathbb{R}^n) \cap H^{k,p}(\mathbb{R}^n)$ and, moreover,

$$\|E_0 u\|_{k,p;\mathbb{R}^n} \leq C\|u\|_{k,p;\mathbb{R}^n_+}. \tag{A.11}$$

Therefore, by approximation we obtain that E_0 maps $H^{k,p}(\mathbb{R}^n_+)$ into $H^{k,p}(\mathbb{R}^n)$ and satisfies (A.11) for any $u \in H^{k,p}(\mathbb{R}^n_+)$.
Having treated the half-space case, let us now suppose that Ω is a C^∞-domain in $\mathbb{R}^n$. According to definition 2.20, there exist a finite number of open sets $\Omega_j \subset \Omega', j = 1, \ldots, N$, which cover $\partial\Omega$, and corresponding mappings ψ_j of Ω_j onto the unit ball $B = B_1(0)$ in $\mathbb{R}^n$, such that

(i) $\psi_j(\Omega_j \cap \Omega) = B^+ = B \cap \mathbb{R}^n_+$,

(ii) $\psi_j(\Omega_j \cap \partial\Omega) = B \cap \partial\mathbb{R}^n_+$,

(iii) $\psi_j \in C^\infty(\Omega_j), \quad \psi_j^{-1} \in C^\infty(B)$.

We let $\Omega_0 \subset\subset \Omega$ be a subdomain of Ω, such that $\{\Omega_j\}, j = 0, \ldots, N$, is a finite covering of $\bar{\Omega}$, and let $\eta_j, j = 0, \ldots, N$ be a partition of unity subordinated to this covering. Then $(\eta_j u) \circ \psi_j^{-1} \in H^{k,p}(\mathbb{R}^n_+)$ and hence $E_0[(\eta_j u) \circ \psi_j^{-1}] \in H^{k,p}(\mathbb{R}^n)$, whence $E_0[(\eta_j u) \circ \psi_j^{-1}] \circ \psi_j \in H_0^{k,p}(\Omega_j), j = 1, \ldots, N$, since $supp\, \eta_j \subset \Omega_j$. Thus the mapping E defined for $u \in H^{k,p}(\Omega)$ by

$$Eu = u\eta_0 + \sum_{j=1}^N E_0[(\eta_j u) \circ \psi_j^{-1}] \circ \psi_j \tag{A.12}$$

satisfies $Eu \in H_0^{k,p}(\Omega'), Eu = u$ in Ω and (A.7), where $C = C(N, \psi_j, \eta_j) = C(\Omega, \Omega')$. Furthermore, $(Eu)_h \longrightarrow u$ in $H^{k,p}(\Omega)$ as $h \longrightarrow 0$.

It remains to show (A.8). Since the mappings ψ_j are only defined on Ω_j, we set $E_0[(\eta_j u) \circ \psi_j^{-1}] \circ \psi_j(x) := 0$ for $x \notin \Omega_j$. Hence, since $\det D\psi_j \neq 0$ on Ω_j,

$$\begin{aligned}
&\|E_0[(\eta_j u) \circ \psi_j^{-1}] \circ \psi_j\|^p_{p;\Omega'} = \|E_0[(\eta_j u) \circ \psi_j^{-1}] \circ \psi_j\|^p_{p;\Omega_j} \\
&\quad = \int_{\Omega_j} |E_0[(\eta_j u) \circ \psi_j^{-1}]|^p \circ \psi_j(x) \frac{|\det D\psi_j|(x)}{|\det D\psi_j| \circ \psi_j^{-1} \circ \psi_j(x)}\, dx \\
&\quad = \int_B |E_0[(\eta_j u) \circ \psi_j^{-1}]|^p |\det D\psi_j|^{-1} \circ \psi_j^{-1}\, dx \\
&\quad \leq \int_{B\cap H^+} |(\eta_j u) \circ \psi_j^{-1}|^p |\det D\psi_j^{-1}|\, dx \quad (\det D(Id) = 1) \\
&\qquad + \int_{B\cap H^-} |E_0[(\eta_j u) \circ \psi_j^{-1}]|^p |\det D\psi_j^{-1}|\, dx \\
&\quad \leq \int_{\Omega\cap\Omega_j} |\eta_j u|^p\, dx \\
&\qquad + \sum_{i=1}^k c_i \int_{B\cap H^-} |(\eta_j u) \circ \psi_j^{-1}(x', -\frac{x_n}{i})|^p |\det D\psi_j^{-1}|(x', x_n)\, dx.
\end{aligned}$$

Let us estimate:

$$
\begin{aligned}
& c_1 \int_{B\cap H^-} |(\eta_j u)\circ\psi_j^{-1}(x',-x_n)|^p |\det D\psi_j^{-1}|(x',x_n)\,dx \\
= {} & c_1 \int_{B\cap H^+} |(\eta_j u)\circ\psi_j^{-1}|^p(x)|\det D\psi_j^{-1}|(x',-x_n)\frac{|\det D\psi_j^{-1}|(x)}{|\det D\psi_j^{-1}|(x)}\,dx \\
= {} & c_1 \int_{\Omega\cap\Omega_j} |\eta_j u|^p \underbrace{\frac{|\det D\psi_j^{-1}|(\psi(x'),-\psi_n(x))}{|\det D\psi_j^{-1}|(\psi(x))}}_{=:g(x)}\,dx \\
\leq {} & C_{\psi,\eta}\int_{\Omega\cap\Omega_j}|\eta_j u|^p\,dx,
\end{aligned}
$$

where we used integral transformation several times and boundedness of g in $\Omega\cap\Omega_j$. In fact, since $\psi:\Omega_j\longrightarrow B$ is a diffeomorphism, $|\det D\psi_j^{-1}|$ is bounded from below and above on $\psi(\Omega\cap\Omega_j)$ and $(\psi',-\psi_n)(\Omega\cap\Omega_j)$, respectively. Therefore g is bounded on $\Omega\cap\Omega_j$ and the inequality follows. We estimate the other summands analogously and obtain

$$\|E_0[(\eta_j u)\circ\psi_j^{-1}]\circ\psi_j\|_{p;\Omega'}\leq C\|u\|_{p;\Omega}.$$

Summing up our estimates, we arrive at (A.8). □

We now prove the analogous result of Proposition A.3 for $u\in H^{2,p}(\Omega)$ with the help of E.

Proposition A.5 (cf. [19, 7.28]) *Let $u\in H^{2,p}(\Omega)$. Then for any $\epsilon>0, i\in 1,\dots,n$,*

$$\|D_i u\|_{p;\Omega}\leq\epsilon\|u\|_{2,p;\Omega}+\frac{C}{\epsilon}\|u\|_{p;\Omega}, \tag{A.13}$$

where $C=C(\Omega)$.

*Proof.*Let $\Omega'\supset\supset\Omega$. Consider $Eu\in H_0^{2,p}(\Omega')$. A.3 gives

$$\|D_i u\|_{p;\Omega}\leq\|D_i(Eu)\|_{p;\Omega'}\leq\epsilon\|Eu\|_{2,p;\Omega'}+\frac{C'}{\epsilon}\|Eu\|_{p;\Omega'}.$$

Using (A.7) and (A.8), we obtain (A.13). □

Remark A.6 *We can go round the first derivatives on the right hand side, since*

$$
\begin{aligned}
\|Du\|_{p;\Omega} &= \sum_i \|D^i u\|_{p;\Omega} \\
&\leq \left(n\epsilon+\frac{C}{\epsilon}\right)\|u\|_{p;\Omega}+n\epsilon\sum_i\|D^i u\|_{p;\Omega}+n\epsilon\sum_{i,j}\|D^{ij}u\|p;\Omega
\end{aligned}
$$

Therefore,

$$(1-n\epsilon)\|Du\|_{p;\Omega} \leq \left(n\epsilon + \frac{C}{\epsilon}\right)\|u\|_{p;\Omega} + n\epsilon\sum_{i,j}\|D^{ij}u\|_{p;\Omega}$$

and consequently

$$\begin{aligned}\|Du\|_{p;\Omega} &\leq \frac{n\epsilon+\frac{C}{\epsilon}}{1-n\epsilon}\|u\|_{p;\Omega} + \frac{n\epsilon}{1-n\epsilon}\sum_{i,j}\|D^{ij}u\|_{p;\Omega} \\ &=: \tilde{\epsilon}\|D^2u\|_{p;\Omega} + C_{\tilde{\epsilon}}\|u\|_{p;\Omega}.\end{aligned}$$

Proposition A.7 (Poincaré, cf. [13, Appendix 1, Theorem 3.1])

(i) *Let* Ω *be a bounded domain in* $\mathbb{R}^n$. *If* $u \in H_0^{1,p}(\Omega), 1 \leq p < \infty$, *then*

$$\int_\Omega |u|^p\,dx \leq C(n,p,\Omega)\int_\Omega |Du|^p\,dx.$$

(ii) *Let* Ω *be of class* C^∞, *then for* $u \in H^{1,p}(\Omega), 1 < p < \infty$,

$$\int_\Omega |u-u_\Omega|^p\,dx \leq C(n,p,\Omega)\int_\Omega |Du|^p\,dx. \tag{A.14}$$

Proof.

(i) We first consider the case $u \in C_0^1(\Omega)$. Assume without loss of generality that $\Omega \subset\subset Q := \{x \in \mathbb{R}^n \,|\, |x_i| < a, i = 1,\ldots,n\}$. Set

$$\tilde{u}(x) := \begin{cases} u(x), & x \in \Omega \\ 0, & x \in Q\backslash\Omega \end{cases}$$

For any $x \in Q$,

$$\begin{aligned}|\tilde{u}(x)|^p &= \left|\int_{-a}^{x_1} D_1\tilde{u}(t,x_2,\ldots,x_n)\,dt\right|^p \\ &\leq \left(\int_{-a}^{x_1} 1|D_1\tilde{u}(t,x_2,\ldots,x_n)|\,dt\right)^p \\ &\leq \|1\|^p_{p';(-a,a)}\|D_1\tilde{u}\|^p_{p;(-a,a)} \\ &= (2a)^{p-1}\int_{-a}^{a}|D_1\tilde{u}|^p\,dx_1 \\ &\leq (2a)^{p-1}\int_{-a}^{a}|D\tilde{u}|^p\,dx_1.\end{aligned}$$

It follows that

$$\begin{aligned}\int_\Omega |u(x)|^p\,dx &= \int_Q |\tilde{u}(x)|^p\,dx \leq (2a)^{p-1}\int_Q\int_{-a}^{a}|D\tilde{u}|^p\,dx_1\,dx \\ &\leq (2a)^p\int_Q |D\tilde{u}|^p\,dx = (2a)^p\int_\Omega |Du|^p\,dx.\end{aligned}$$

Since $C_0^1(\Omega) \subset H_0^{1,p}(\Omega)$ dense, the assertion follows.

(ii) Since (A.14) is invariant under adding a constant to u, we can assume without loss of generality that $u_\Omega = 0$. If (A.14) is not valid, then for any positive integer k, there exists $u_k \in H^{1,p}(\Omega)$ satisfying $\fint_\Omega u_k\,dx = 0$ and

$$\int_\Omega |u_k|^p\,dx > k \fint_\Omega |Du_k|^p\,dx.$$

Set $w_k = \frac{u_k}{\|u_k\|_{p;\Omega}}$, $k \in \mathbb{N}$, then $w_k \in H^{1,p}(\Omega)$ satisfies

(1) $\fint_\Omega w_k\,dx = 0$

(2) $\|w_k\|_{p;\Omega} = 1$

(3) $\int_\Omega |Dw_k|^p\,dx \le \frac{1}{k}$.

From (2) and (3) we deduce that $\|w_k\|_{1,p;\Omega}$ is bounded. By using C.3, we can find a subsequence $w_{k_j} \in H^{1,p}(\Omega)$ and $w \in H^{1,p}(\Omega)$, such that

$$w_{k_j} \longrightarrow w \quad \text{strongly in} \quad L^p(\Omega) \tag{A.15}$$

$$Dw_{k_j} \longrightarrow Dw \quad \text{weakly in} \quad L^p(\Omega;\mathbb{R}^n). \tag{A.16}$$

By (3) and (A.16), $Dw(x) = 0$ a. e. $x \in \Omega$. It follows that

$$w \equiv \text{const a. e. } x \in \Omega.$$

By (1) and (A.15), $\fint_\Omega w\,dx = 0$; therefore,

$$w \equiv 0 \quad \text{a. e.} \quad x \in \Omega. \tag{A.17}$$

However, (2) and (A.15) imply that $\|w\|_{p;\Omega} = 1$, which contradicts (A.17).

□

Corollary A.8 *(i) If $u \in H_0^{1,p}(B_R), 1 \le p < \infty$, then*

$$\int_{B_R} |u|^p\,dx \le C(n,p)R^p \int_{B_R} |Du|^p\,dx.$$

(ii) If $u \in H^{1,p}(B_R), 1 < p < \infty$, then

$$\int_{B_R} |u - u_R|^p\,dx \le C(n,p)R^p \int_{B_R} |Du|^p\,dx.$$

Proof. If $R = 1$, Proposition A.7 implies the assertion. Otherwise, use integral transformation. □

A.3 Stampacchia Interpolation

We are now going to prove the Stampacchia interpolation theorem, Proposition 3.19, as is done in [13, Appendix 4]. At first, some preparations are necessary.

For $f \in L^1(Q^0)$, the function

$$M_0 f(x) := \sup_{r>0} \fint_{Q_{x,r} \cap Q^0} |f(y)|\, dy, x \in Q^0 \tag{A.18}$$

is called the centered Hardy-Littlewood maximal function. Sometimes, it is more convenient to define

$$Mf(x) := \sup_{Q: x \in Q} \fint_{Q \cap Q^0} |f(y)|\, dy, x \in Q^0, \tag{A.19}$$

where Q is a cube with center inside Q^0. Mf is called the Hardy-Littlewood maximal function on Q^0.

Lemma A.9 $M_0 f(x) \le Mf(x) \le 2^n M_0 f(x), x \in Q^0$.

Proof. The first inequality is obvious, because more cubes are admitted. Let $x \in Q^0$ and $Q = Q_{\tilde{x},\tilde{r}}$ be a cube with $x \in Q, \tilde{x} \in Q^0$, then

$$\begin{aligned}
\fint_{Q \cap Q^0} |f(y)|\, dy &\le \frac{1}{|Q \cap Q^0|} \int_{Q_{x,2\tilde{r}} \cap Q^0} |f(y)|\, dy \\
&= 2^n \frac{1}{2^n |Q \cap Q^0|} \int_{Q_{x,2\tilde{r}} \cap Q^0} |f(y)|\, dy \\
&\le 2^n \fint_{Q_{x,2\tilde{r}} \cap Q^0} |f(y)|\, dy \quad (\text{since } 2^n |Q \cap Q^0| \ge |Q_{x,2\tilde{r}} \cap Q^0|) \quad .
\end{aligned}$$

Consequently, $Mf(x) \le 2^n M_0 f(x)$. □

Lemma A.10 *(1) Mf is a measurable function on Q^0.*

(2) M is sublinear, i. e. $|M(f+g)| \le Mf + Mg$.

(3) If $f \in L^\infty(Q^0)$, then $Mf \in L^\infty(Q^0)$ and $\|Mf\|_{\infty;Q^0} \le \|f\|_{\infty;Q^0}$.

Proof.

- (2),(3): obvious
- (1): Consider $G := \{x \in Q^0 | Mf(x) > t\}$. We assert that G is open. Let therefore be $x \in G$, such that $Mf(x) > t' > t$, then there exists $Q = Q_{\tilde{x},\tilde{r}}$ and $\epsilon < t' - t$ such that $x \in Q$ and $\fint_{Q \cap Q^0} |f(y)|\, dy > t' - \epsilon$. Let $\xi \in \mathbb{R}^n, |\xi|$ small. If $x + \xi \in Q$, then $Mf(x+\xi) > t$. If $x + \xi \notin Q$, then define

$$Q' := Q_{\tilde{x}, \tilde{r} + \operatorname{dist}(x+\xi, Q)} \supset Q$$

and obtain

$$\begin{aligned}
&\frac{1}{|Q' \cap Q^0|}\int_{Q'\cap Q^0}|f(y)|\,dy\\
&\geq \frac{1}{|Q' \cap Q^0|}\int_{Q\cap Q^0}|f(y)|\,dy\\
&\geq \frac{1}{|Q\cap Q^0| + \underbrace{2^n(\mathrm{dist}(x+\xi,Q)\tilde{r} + \mathrm{dist}(x+\xi,Q)^2)}_{=:g(\xi)}}\int_{Q\cap Q^0}|f(y)|\,dy.
\end{aligned}$$

Note that g is continuous and tends to 0 as $\xi \longrightarrow 0$. Choose $\xi_0 \in \mathbb{R}^n$ with $|\xi_0|$ so small that

$$\frac{1}{|Q\cap Q^0| + g(\xi_0)}\int_{Q\cap Q^0}|f(y)|\,dy = t' - \epsilon > t.$$

Thus $B(x, |\xi_0|) \subset G$. □

After the next covering lemma, we give a deeper property of the maximal function.

Lemma A.11 *Let $G \subset \mathbb{R}^n$ be a bounded set. If $r : x \longmapsto r(x)$ is a function defined on G, such that $0 < r < 1$, then there exists a sequence of points $x_i \in G, i \in \mathbb{N}$, such that*

$$Q_{x_i,r(x_i)} \cap Q_{x_j,r(x_j)} = \emptyset, \text{ if } i \neq j \tag{A.20}$$

$$\bigcup_i Q_{x_i,3r(x_i)} \supset G. \tag{A.21}$$

*Proof.*Consider the following family of cubes:

$$Q_{2^{-k},2^{-(k+1)}} := \{Q_{x,r(x)} | x \in G, 2^{-(k+1)} \leq r(x) < 2^{-k}\}, k \in \mathbb{N}.$$

Since G is bounded, we can choose finitely many (say n_1) cubes given by

$$\tilde{Q}_{1,\frac{1}{2}} := \left\{Q_{x_i,r(x_i)} | x_i \in G, \frac{1}{2} \leq r(x_i) < 1, i = 1, \dots, n_1\right\},$$

such that

(1) $Q_{x_i,r(x_i)} \cap Q_{x_j,r(x_j)} = \emptyset$, if $i, j = 1, \dots, n_1, i \neq j$,

(2) each cube in $Q_{1,\frac{1}{2}}$ must intersect with at least one cube in $\tilde{Q}_{1,\frac{1}{2}}$.

Next, we choose a subfamily of $Q_{\frac{1}{2},\frac{1}{4}}$ given by

$$\tilde{Q}_{\frac{1}{2},\frac{1}{4}} := \left\{Q_{x_i,r(x_i)} | x_i \in G, \frac{1}{4} \leq r(x_i) < \frac{1}{2}, i = n_1 + 1, \dots, n_2\right\}, n_2 \geq n_1,$$

such that

(1) $Q_{x_i,r(x_i)} \cap Q_{x_j,r(x_j)} = \emptyset$, if $i, j = 1, \dots, n_2, i \neq j$,

(2) each cube in $Q_{\frac{1}{2},\frac{1}{4}}$ must intersect with at least one cube in

$$\tilde{Q}_{1,\frac{1}{2}} \cup \tilde{Q}_{\frac{1}{2},\frac{1}{4}} = \{Q_{x_j,r(x_j)} | x_i \in G, i = 1, \dots, n_2\}.$$

(It might happen that $n_2 = n_1$; in this case $\tilde{Q}_{\frac{1}{2},\frac{1}{4}} = \emptyset$.) Continuing this process, if $x_1, \dots, x_{n_k}$ are chosen, then we choose a subfamily of $Q_{2^{-k},2^{-(k+1)}}$ given by

$$\tilde{Q}_{2^{-k},2^{-(k+1)}} := \{Q_{x_i,r(x_i)} | x_i \in G, 2^{-(k+1)} \leq r(x_i) < 2^{-k}, i = n_k + 1, \dots, n_{k+1}\},$$

$n_{k+1} \geq n_k$ with the following properties:

(1) $Q_{x_i,r(x_i)} \cap Q_{x_j,r(x_j)} = \emptyset$, if $i, j = 1, \dots, n_{k+1}, i \neq j$,

(2) each cube in $Q_{2^{-k},2^{-(k+1)}}$ must intersect with at least one cube in

$$\bigcup_{j=0}^{k} \tilde{Q}_{2^{-j},2^{-(j+1)}} = \{Q_{x_i,r(x_i)} | x_i \in G, i = 1, \dots, n_{k+1}\}.$$

(It might happen that $n_{k+1} = n_k$; in this case $\tilde{Q}_{2^{-k},2^{-(k+1)}} = \emptyset$.) By our selection, (A.20) is obviously satisfied. We next prove (A.21). For $x \in G$ let $k_0 \in \mathbb{N}$ be such that $2^{-(k_0+1)} \leq r(x) < 2^{-k_0}$. Then $x \in Q_{x,r(x)} \in Q_{2^{-k_0},2^{-(k_0+1)}}$. Therefore, there exists x_i, such that

$$Q_{x_i,r(x_i)} \in \bigcup_{j=0}^{k_0} \tilde{Q}_{2^{-j},2^{-(j+1)}}, Q_{x_i,r(x_i)} \cap Q_{x_j,r(x_j)} = \emptyset, 2r(x_i) \geq r(x).$$

Thus $|x - x_i| \leq r(x) + r(x_i) \leq 3r(x_i)$, i. e., $x \in B(x_i, 3r(x_i)) \subset Q_{x_i,3r(x_i)}$. □

Proposition A.12 (Hardy-Littlewood) *The operator M is of weak type* $(1,1)$ *and*

$$\|Mf\|_{L^1_w(Q^0)} \leq c(n)\|f\|_{1;Q^0} \forall f \in L^1(Q^0). \tag{A.22}$$

*Proof.*Let $x \in G := \{x \in Q^0 | M_0 f(x) > s\}$. From the definition of M_0 we deduce that there exists $Q_{x,r(x)}$, such that

$$-\!\!\!\!\!\!\int_{Q_{x,r(x)} \cap Q^0} |f(y)|\, dy > s, \text{ i. e.} |Q_{x,r(x)} \cap Q^0| < \frac{1}{s} \int_{Q_{x,r(x)} \cap Q^0} |f(y)|\, dy. \tag{A.23}$$

By the covering lemma, there exist countably many cubes $Q_{x_i,r(x_i)}, i \in \mathbb{N}$, such that

(1) $Q_{x_i,r(x_i)} \cap Q_{x_j,r(x_j)} = \emptyset$ for $i \neq j$,

(2) $G \subset \bigcup_i Q_{x_i,3r(x_i)} \cap Q^0$.

It follows that

$$|G| \leq \sum_i |Q_{x_i,3r(x_i)} \cap Q^0| \leq 3^n \sum_i |Q_{x_i,r(x_i)} \cap Q^0|.$$

By (A.23),

$$|G| \leq \frac{3^n}{s} \sum_i \int_{Q_{x_i,r(x_i)} \cap Q^0} |f(y)|\, dy \leq \frac{3^n}{s} \|f\|_{1;Q^0}.$$

This proves that M_0 is of weak type $(1,1)$. Using (A.9), we see that M is also of weak type $(1,1)$:

$$|\{x \in Q^0 | Mf(x) > s\}| \leq |\{2^n M_0 f > s\} \cap Q^0| \leq \frac{6^n}{s} \|f\|_{1;Q^0}.$$

□

Corollary A.13 *For $1 < p < \infty$, the maximal operator M is of strong type (p,p).*

Proof. M is of weak type $(1,1), (\infty, \infty)$ (see (3.30)). Applying Marcinkiewicz interpolation, we obtain the corollary. □

For $f \in L^1(Q^0)$, the function

$$f^{\#}(x) := \sup_{x \in Q} \fint_{Q \cap Q^0} |f - f_{Q \cap Q^0}|\, dy, x \in Q^0 \tag{A.24}$$

is called the maximal mean oscillation function of f on Q^0, where Q has center inside Q^0.

Lemma A.14 *(i) $f \in BMO(Q^0) \Leftrightarrow f^{\#} \in L^\infty(Q^0)$*

(ii) If $1 < p < \infty$, then

$$\|f^{\#}\|_{p;Q^0} \leq c(n,p) \|f\|_{p;Q^0}. \tag{A.25}$$

(iii) If $f \in L^1(Q^0)$, then $f^{\#} \in L^1_w(Q^0)$.

Proof.

(i) follows from Lemma 3.16 with $p = 1$

(ii) We have $f^{\#} \leq 2Mf$. By Corollary A.13, M is of strong type (p,p). It follows that the sharp operator $(\cdot)^{\#}$ must also be of strong type (p,p).

(iii) $\mathrm{dx}(\{x \in Q^0 | f^{\#}(x) > s\}) \leq \mathrm{dx}(\{x \in Q^0 | Mf(x) > \frac{s}{2}\}) \leq \frac{6^n 2}{s} \|f\|_{1;Q^0}$

□

Another important property of the mean oscillation function was discovered by Fefferman and Stein (cf. [16]).

Proposition A.15 *Let $f \in L^1(Q^0)$. If $f^\# \in L^p(Q^0)$ for some $1 < p < \infty$, then $f \in L^p(Q^0)$ and*

$$\|f\|_{p;Q^0} \le c(n,p)(\|f^\#\|_{p;Q^0} + |f|_{Q^0}|Q^0|^{\frac{1}{p}}). \tag{A.26}$$

*Proof.*For any fixed α such that

$$\alpha > \fint_{Q^0} |f|\, dx \tag{A.27}$$

we apply Lemma 2.12 to the function $|f|$. There exists a sequence of nonoverlapping cubes (Q_j^α) such that

$$\alpha < \fint_{Q_j^\alpha} |f|\, dx \le 2^n \alpha, j = 1, 2, \dots \tag{A.28}$$

$$|f(x)| \le \alpha \quad \text{a. e.} \quad x \in Q^0 \setminus \bigcup_j Q_j^\alpha. \tag{A.29}$$

If we do the decomposition simultaneously for all values of α satisfying (A.27), then $(Q_j^{\alpha_1})$ are subcubes of $Q_j^{\alpha_2}$, when $\alpha_1 > \alpha_2$ (see proof of Lemma 2.12). Set

$$\mu(\alpha) = \sum_j |Q_j^\alpha| \quad \text{for} \quad \alpha > \fint_{Q^0} |f|\, dx.$$

Then $a \longmapsto \mu(\alpha)$ is a nonincreasing function. From (A.29) we deduce that

$$\lambda(\alpha) := \mathrm{dx}(\{x \in Q^0 \| f| > \alpha\}) \le \mu(\alpha) \tag{A.30}$$

We claim that if

$$\frac{\alpha}{2^{n+1}} > \fint_{Q^0} |f|\, dx = |f|_{Q^0}, \tag{A.31}$$

then

$$\mu(\alpha) \le \mathrm{dx}\left(\left\{x \in Q^0 | f^\#(x) > \frac{\alpha}{A}\right\}\right) + \frac{2}{\alpha}\mu\left(\frac{\alpha}{2^{n+1}}\right), \tag{A.32}$$

where A is an arbitrary positive number.
Denote any cube $Q_{j_0}^{\alpha 2^{-(n+1)}}$ by $\tilde{Q}^0$. For each cube Q_j^α, there are two cases:

- Case 1: $\tilde{Q}^0 \subset \{x \in Q^0 | f^\# > \frac{\alpha}{A}\}$. Then

$$\sum_{Q_j^\alpha \subset \tilde{Q}^0} |Q_j^\alpha| \le \mathrm{dx}\left(\left\{x \in Q^0 | f^\#(x) > \frac{\alpha}{A}\right\} \cap \tilde{Q}^0\right). \tag{A.33}$$

- Case 2: $\tilde{Q}^0 \not\subset \{x \in Q^0 | f^\# > \frac{\alpha}{A}\}$. Then there exists $x_0 \in \tilde{Q}^0$, such that $f^\#(x_0) \le \frac{\alpha}{A}$. By definition of $f^\#$, we have

$$\fint_{\tilde{Q}^0} |f(x) - f_{\tilde{Q}^0}|\, dx \le \frac{\alpha}{A}. \tag{A.34}$$

On the other hand, (A.28) implies that

$$|f|_{\tilde{Q}^0} \leq 2^n \left(\frac{\alpha}{2^{n+1}}\right) \leq \frac{\alpha}{2}; \quad |f|_{Q_j^\alpha} > \alpha \quad \text{anyway.}$$

It follows that

$$\begin{aligned}
\int_{Q_j^\alpha} |f(x) - f_{\tilde{Q}^0}|\, dx &\geq \int_{Q_j^\alpha} ||f| - |f_{\tilde{Q}^0}||\, dx \\
&\geq \int_{Q_j^\alpha} |f| - |f_{\tilde{Q}^0}|\, dx \\
&\geq \int_{Q_j^\alpha} |f|\, dx - \int_{Q_j^\alpha} |f|_{\tilde{Q}^0}\, dx \\
&= (|f|_{Q_j^\alpha} - |f|_{\tilde{Q}^0})|Q_j^\alpha| \\
&> \frac{\alpha}{2}|Q_j^\alpha|,
\end{aligned}$$

where Q_j^α is an arbitrary element in (Q_j^α), which is contained in $\tilde{Q}^0$. We sum up over all such cubes. Then

$$\sum_{Q_j^\alpha \subset \tilde{Q}^0} |Q_j^\alpha| < \frac{2}{\alpha} \sum_{Q_j^\alpha \subset \tilde{Q}^0} \int_{Q_j^\alpha} |f(x) - f_{\tilde{Q}^0}|\, dx \leq \frac{2}{A}|\tilde{Q}^0| \quad \text{(by (A.34))} \quad . \tag{A.35}$$

Combining the two cases, we deduce from (A.33) and (A.35) that

$$\sum_{Q_j^\alpha \subset \tilde{Q}^0} |Q_j^\alpha| \leq \mathrm{dx}\left(\left\{x \in Q^0 | f^\#(x) > \frac{\alpha}{A}\right\} \cap \tilde{Q}^0\right) + \frac{2}{A}|\tilde{Q}^0|,$$

where $\tilde{Q}^0$ is an arbitrary cube from the set $\left\{Q_j^\beta | \beta = \frac{\alpha}{2^{n+1}}\right\}$. By summing over all such cubes, we obtain (A.32):

$$\begin{aligned}
\mu(\alpha) = \sum_j |Q_j^\alpha| &= \sum_j \sum_{Q_i^\alpha \subset Q_j^\beta} |Q_i^\alpha| \\
&\leq \mathrm{dx}\left(\left\{x \in Q^0 | f^\#(x) > \frac{\alpha}{A}\right\}\right) + \frac{2}{A}\mu\left(\frac{\alpha}{2^{n+1}}\right).
\end{aligned}$$

For any $s > 2^{n+1}|f|_{Q^0}$, we set

$$I_s = p \int_0^s \alpha^{p-1} \mu(\alpha)\, d\alpha.$$

Since $\mu(\alpha) \leq |Q^0|$, the above integral is well defined. Using (A.32), we get

$$\begin{aligned}
I_s &= p \int_0^{2^{n+1}|f|_{Q^0}} \alpha^{p-1}\mu(\alpha)\, d\alpha + p \int_{2^{n+1}|f|_{Q^0}}^s \alpha^{p-1}\mu(\alpha)\, d\alpha \\
&\leq |Q^0|(2^{n+1}|f|_{Q^0})^p + \frac{2p}{A}\int_0^s \alpha^{p-1}\mu\left(\frac{\alpha}{2^{n+1}}\right) d\alpha \\
&\quad + p \int_0^\infty \mathrm{dx}\left(\left\{x \in Q^0 | f^\#(x) > \frac{\alpha}{A}\right\}\right) \alpha^{p-1}\, d\alpha \\
&\leq 2^{(n+1)p}|Q^0||f|_{Q^0}^p + \frac{2}{A}2^{(n+1)p} I_s + A^p \|f^\#\|_{p;Q^0}^p,
\end{aligned}$$

where we applied Lemma 2.7 and the transformation $\alpha \longmapsto 2^{n+1}\alpha$. We choose $A = 4 \cdot 2^{(n+1)p}$ in the above inequality. Then

$$I_s \leq 2^{(n+1)(p+1)}|Q^0||f|^p_{Q^0} + 2A^p\|f^\#\|^p_{p;Q^0}.$$

By (A.30),

$$p\int_0^s \lambda(\alpha)\alpha^{p-1}\,d\alpha \leq 2A^p\|f^\#\|^p_{p;Q^0} + 2^{(n+1)(p+1)}|Q^0||f|^p_{Q^0}.$$

Letting $s \longrightarrow \infty$, we conclude (A.26) by Lemma 2.7. □

Proof of the Stampacchia interpolation theorem. Define $T^\# u := (Tu)^\#$. By Lemma A.14 and the assumptions, we deduce that

$$\|T^\# u\|_{q;Q^0} \leq C(n,p)\|Tu\|_{q;Q^0} \leq C(n,p)B_q\|u\|_{q;Q^0}.$$

By the definition of $f^\#$,

$$\|T^\# u\|_{\infty;Q^0} \leq |Tu|_{*;Q^0} \leq 2B_\infty\|u\|_{\infty;Q^0}.$$

Thus, the Marcinkiewicz interpolation theorem implies that $T^\#$ is of strong type (p,p) for $p \in [q,\infty)$ and

$$\|T^\# u\|_{p;Q^0} \leq CB_q^{\frac{q}{p}}B_\infty^{\frac{p-q}{p}}\|u\|_{p;Q^0}\forall u \in L^p(Q^0),$$

where C only depends on n,p,q. Now we use the Fefferman-Stein theorem to conclude that for $u \in L^p(Q^0)$

$$\begin{aligned}
\|Tu\|_{p;Q^0} &\leq C(\|T^\# u\|_{p;Q^0} + |Q^0|^{\frac{1}{p}}|Tu|_{Q^0})\\
&\leq C(\|u\|_{p;Q^0} + |Q^0|^{\frac{1}{p}-1}\|Tu\|_{1;Q^0})\\
&\leq C(\|u\|_{p;Q^0} + |Q^0|^{\frac{1}{p}-\frac{1}{q}}\|Tu\|_{q;Q^0})\\
&\leq C(\|u\|_{p;Q^0} + |Q^0|^{\frac{1}{p}-\frac{1}{q}}B_q\|u\|_{q;Q^0})\\
&\leq C(\|u\|_{p;Q^0} + |Q^0|^{\frac{1}{p}-\frac{1}{q}}B_q|Q^0|^{\frac{1}{q}-\frac{1}{p}}\|u\|_{p;Q^0})\\
&\leq C\|u\|_{p;Q^0},
\end{aligned}$$

where $C = C(n,p,q,B_q,B_\infty)$. □

Appendix B

SOME COMPUTATIONS

B.1 Refering to Section 2.4

In the situation at the end of Section 2.4 let us show that

$$\frac{1}{t}[T_t^\sigma z_\delta(x) - z_\delta(x)] \longrightarrow \sum_{i,j=1}^n a^{ij} \frac{\partial^2 z_\delta(x)}{\partial x_i \partial x_j} =: L^\sigma z_\delta(x), \tag{B.1}$$

as $t \longrightarrow 0$, where $a = (a^{ij}) = \frac{1}{2}\sigma\sigma^t$: At first,

$$|\sigma^{-1}(x-y)|^2 =< \underbrace{(\sigma\sigma^t)^{-1}}_{=:A^{-1}=:(A_{ij}^{-1})} (x-y), x-y >= \left(\sum_{k,l=1}^n A_{kl}^{-1}(x_k - y_k)(x_l - y_l) \right).$$

Define

$$p_t^\sigma(x,y) := (2\pi t)^{-\frac{n}{2}} |\det\sigma|^{-1} \exp\left[-\frac{|\sigma^{-1}(x-y)|^2}{2t} \right],$$

then

$$\begin{aligned}
\frac{d}{dt} p_t^\sigma(x,y) &= -(2\pi)^{-\frac{n}{2}} \frac{n}{2} t^{-(\frac{n}{2}+1)} |\det\sigma|^{-1} \exp\left[-\frac{|\sigma^{-1}(x-y)|^2}{2t} \right] + \\
&\qquad \frac{|\sigma^{-1}(x-y)|^2}{2t} t^{-1} p_t^\sigma(x,y) \\
&= -\frac{1}{2t} p_t^\sigma(x,y) \left(n - \frac{|\sigma^{-1}(x-y)|^2}{2t} \right), \\
\frac{\partial p_t^\sigma(x,y)}{\partial x_i} &= -p_t^\sigma(x,y) \frac{1}{t} \left(\sum_{k=1}^n A_{ik}^{-1}(x_k - y_k) \right), \\
\frac{\partial^2 p_t^\sigma(x,y)}{\partial x_i \partial x_j} &= p_t^\sigma(x,y) \frac{1}{t^2} \left(\sum_{k=1}^n A_{jk}^{-1}(x_k - y_k) \right) \left(\sum_{k=1}^n A_{ik}^{-1}(x_k - y_k) \right) - \\
&\qquad p_t^\sigma(x,y) \frac{1}{t} A_{ij}^{-1} \\
&= p_t^\sigma(x,y) \left[\frac{1}{t^2} \left(\sum_{k=1}^n A_{jk}^{-1}(x_k - y_k) \right) \left(\sum_{k=1}^n A_{ik}^{-1}(x_k - y_k) \right) - \frac{1}{t} A_{ij}^{-1} \right]
\end{aligned}$$

Now

$$\begin{aligned}
\sum_{ij} A_{ij} \frac{\partial^2 p_t^\sigma(x,y)}{\partial x_i \partial x_j} &= \sum_{k,l} p_t^\sigma(x,y) \frac{1}{t^2} \sum_{i,j} A_{ij} A_{jk}^{-1}(x_k - y_k) A_{ik}^{-1}(x_l - y_l) - \\
&\quad p_t^\sigma(x,y) \frac{1}{t} (\sum_{i,j} A_{ij} A_{ij}^{-1}) \\
&= \sum_{k,l} p_t^\sigma(x,y) \frac{1}{t^2} \sum_i \delta_{ik}(x_k - y_k) A_{il}^{-1}(x_l - y_l) - \frac{1}{2} p_t^\sigma(x,y) \frac{n}{t} \\
&\quad \text{(since } A_i^j A_{ji}^{-1} = \delta_{ii} = 1) \\
&= \sum_{k,l} p_t^\sigma(x,y) \frac{1}{t^2} (x_k - y_k) A_{kl}^{-1}(x_l - y_l) - \frac{n}{t} p_t^\sigma(x,y) \\
&= \frac{p_t^\sigma(x,y)}{t^2} |\sigma^{-1}(x-y)|^2 - \frac{p_t^\sigma(x,y)}{t} n \\
\Rightarrow L_x^\sigma p_t^\sigma(x,y) &:= \sum_{ij} a_{ij} \frac{\partial^2 p_t^\sigma(x,y)}{\partial x_i \partial x_j} = \frac{d}{dt} p_t^\sigma(x,y)
\end{aligned}$$

Let $f \in C^2(\mathbb{R}^n), |D^\alpha f(x)| \leq M(1+|x|)$ for $|\alpha| \leq 2$, where $M = M(\alpha)$. Then, for $\epsilon \in (0,t)$, with $p_t^\sigma := T_t^\sigma$:

$$\begin{aligned}
p_t^\sigma f(x) - p_\epsilon^\sigma f(x) &= \int_\epsilon^t \int f(y) \underbrace{\frac{d}{ds} p_s^\sigma(x,y)}_{= L_y^\sigma p_s^\sigma(x,y)} \, dy \, ds \\
&= \int_\epsilon^t \int (L_y^\sigma f)(y) p_s^\sigma(x,y) \, dy \, ds \\
&= \int_\epsilon^t p_s^\sigma (L^\sigma f)(x) \, ds \qquad \text{(B.2)}
\end{aligned}$$

Lemma B.1 *Letting $\epsilon \longrightarrow 0$ in (B.2), we obtain*

$$p_t^\sigma f(x) - f(x) = \int_0^t p_s^\sigma (L^\sigma f)(x) \, ds.$$

*Proof.*left-hand side: If $f \in C_b(\mathbb{R}^n)$, then $p_t^\sigma f(x) \longrightarrow f(x) \forall x$:

$$\begin{aligned}
|p_t^\sigma f(x) - f(x)| &\leq \int |f(y) - f(x)| p_t^\sigma(x,y) \, dy \\
&= \int_{B_\epsilon(x)} |f(y) - f(x)| p_t^\sigma(x,y) \, dy + \int_{B_\epsilon(x)^c} \underbrace{|f(x) - f(y)| p_t^\sigma(x,y)}_{\longrightarrow 0 \text{ pointwise, as } t \longrightarrow 0} \, dy \\
&\leq \sup_{y \in B_\epsilon(x)} |f(y) - f(x)| + 2\|f\|_\infty \int 1_{B_\epsilon(x)^c} \frac{p_t^\sigma(x,y)}{p_1^\sigma(x,y)} p_1^\sigma(x,y) \, dy.
\end{aligned}$$

Now

$$1_{B_\epsilon(x)^c}(y) \frac{p_t^\sigma(x,y)}{p_1^\sigma(x,y)} = 1_{B_\epsilon(x)^c}(y) t^{-\frac{n}{2}} e^{-\frac{|\sigma^{-1}(x-y)|^2}{2} \overbrace{\left(\frac{1}{t} - 1\right)}^{\geq \frac{1}{2t} \text{ for } t \leq \frac{1}{2}}}$$

$$\leq \quad t^{-\frac{n}{2}} \exp\left[-\frac{|\sigma^{-1}(x-y)|^2}{4t}\right] 1_{B_\epsilon(x)^c}(y)$$
,

and, since for $x \geq 0$ we have $x^n \leq n! \exp[x]$, it follows for $t \leq \frac{1}{2}$,

$$\begin{aligned} t^{-\frac{n}{2}} \exp\left[-\frac{|\sigma^{-1}(x-y)|^2}{4t}\right] &\leq t^{-\frac{n}{2}} \frac{(4t)^n n!}{|\sigma^{-1}(x-y)|^{2n}} \\ &\leq \frac{4^n n!}{|\sigma^{-1}(x-y)|^{2n}} \end{aligned} \tag{B.3}$$

Therefore,

$$1_{B_\epsilon(x)^c}(y) \frac{p_t^\sigma(x,y)}{p_1^\sigma(x,y)} \leq \frac{4^n n!}{|\sigma^{-1}(x-y)|^{2n}} 1_{B_\epsilon(x)^c}(y) \leq \frac{\|\sigma\|^{2n}}{\epsilon^{2n}} 4^n n!,$$

which is a majorante with respect to a probability measure.

$$\Rightarrow \limsup_{t \longrightarrow 0} |p_t^\sigma f(x) - f(x)| \leq \sup_{y \in B_\epsilon(x)} |f(y) - f(x)| \longrightarrow 0,$$

as $\epsilon \downarrow 0$, by Lebesgue.

For $f \in C(\mathbb{R}^n)$, such that $|f(x)| \leq M(1+|x|)$, we take $\chi_n \in C_0^\infty(\mathbb{R}^n)$ such that $1_{B_n(0)} \leq \chi_n \leq 1, \chi_n \uparrow$. Then

$$|p_t^\sigma f(x) - f(x)| \leq |p_t^\sigma(f(1-\chi_n))(x)| + |p_t^\sigma(f\chi_n)(x) - (f\chi_n)(x)| + |f(1-\chi_n)(x)|.$$

Now

$$\begin{aligned} |p_t^\sigma(f(1-\chi_n))(x)| &\leq \int |f|(1-\chi_n)(y) p_t^\sigma(x,y)\, dy \\ &\leq M \int_{B_n(0)^c} (1+|y|) p_t^\sigma(x,y)\, dy \\ &\leq M \int_{B_n(0)^c} (1+|x|+|y-x|) p_t^\sigma(x,y)\, dy \\ &\leq \underbrace{M(1+|x|) \int_{B_n(0)^c} p_t^\sigma(x,y)\, dy}_{\longrightarrow 0, \text{as } t\downarrow 0, \text{if } n \text{ is large enough}} + M\|\sigma\| \int_{B_n(0)^c} |\sigma^{-1}(x-y)| p_t^\sigma(x,y)\, dy, \end{aligned}$$

since

$$\frac{|x-y|}{|\sigma^{-1}(x-y)|} \leq \sup_{x \neq y} \frac{|x-y|}{|\sigma^{-1}(x-y)|} = \sup_{x \neq y} \frac{|\sigma(x-y)|}{|x-y|} = \|\sigma\|.$$

We assert that the second summand tends to zero by Lebesgue as $t \downarrow 0$:

$$\begin{aligned} &M\|\sigma\| \int_{B_n(0)^c} |\sigma^{-1}(x-y)| p_t^\sigma(x,y)\, dy \\ &\quad = M\|\sigma\| \int_{B_n(0)^c} |\sigma^{-1}(x-y)| \frac{p_t^\sigma(x,y)}{p_1^\sigma(x,y)} p_1^\sigma(x,y)\, dy \end{aligned}$$

$$\leq \frac{M4^n n! \|\sigma\|^{2n}}{\mathrm{dist}(x, B_n(0)^c)},$$

if n is large enough and $t \leq \frac{1}{2}$, since

$$\begin{aligned} 1_{B_n(0)^c}(y)|\sigma^{-1}(x-y)|\frac{p_t^\sigma(x,y)}{p_1^\sigma(x,y)} &\leq 1_{B_n(0)^c}(y)\frac{4^n n!}{|\sigma^{-1}(x-y)|^{2n-1}} \\ &\leq 1_{B_n(0)^c}(y)\frac{4^n n!\|\sigma\|^{2n-1}}{|x-y|^{2n-1}} \\ &\leq \frac{4^n n!\|\sigma\|^{2n-1}}{\mathrm{dist}(x, B_n(0)^c)} \forall x \in B_n(0). \end{aligned}$$

This is our majorante and because of pointwise convergence of the integrand, the assertion is proved. Consequently, $|p_t^\sigma f(x) - f(x)| \longrightarrow 0$ as $t \downarrow 0$.

right-hand side: We want to apply Lebesgue again. Since pointwise convergence is shown, it would be enough to prove that

$$\int_0^t |p_s^\sigma(L^\sigma f)(x)|\, ds < \infty \forall x.$$

Now, since $D_{ij} f(x) \leq M(1+|x|)$, we have by definition of L^σ, that $|\underbrace{L^\sigma f}_{=:g}|(x) \leq M'(1+|x|)$. With that,

$$\begin{aligned} |p_s^\sigma(g)(x)| &= \left|(2\pi s)^{-\frac{n}{2}}|\det\sigma|^{-1}\int g(y)\exp\left[-\frac{|\sigma^{-1}(x-y)|^2}{2s}\right] dy\right| \\ &\leq (2\pi s)^{-\frac{n}{2}}|\det\sigma|^{-1}\int M'(1+|y|)\exp\left[-\frac{|\sigma^{-1}(x-y)|^2}{2s}\right] dy \\ &\leq (2\pi s)^{-\frac{n}{2}}|\det\sigma|^{-1}\int M' \underbrace{|y-x|}_{\leq 1+|y-x|^2} \exp\left[-\frac{|\sigma^{-1}(x-y)|^2}{2s}\right] dy + M'(1+|x|) \\ &\quad \text{(since } y = y-x+x) \\ &\leq M' + M''(\sigma,n)s + M'(1+|x|). \end{aligned}$$

Thus,

$$\int_0^t |p_s^\sigma(g)(x)|\, ds \leq M(1+|x|) + tM'' < \infty.$$

□

Let us try to turn to (B.1). Since z is continuous and uniform limit of piecewise linear functions z_k, each of them fulfilling $|z_k(x)| \leq M(1+|x|)$, we know that

$$|z(x)| \leq M(1+|x|) \quad \text{and} \quad z_\delta = p_\delta^\sigma z \in C^2(\mathbb{R}^n).$$

Moreover,

$$|z_\delta|, |D_i z_\delta|, |D_{ij} z_\delta| \leq M(1+|\cdot|), \tag{B.4}$$

as we will see later. Therefore, we can apply the first part of Lemma B.1 to obtain

$$p_s^\sigma L^\sigma z_\delta(x) \longrightarrow L^\sigma z_\delta(x),$$

as $s \downarrow 0$ for any x. Let $\epsilon > 0$, then there exists t_0 such that

$$|p_s^\sigma L^\sigma z_\delta(x) - L^\sigma z_\delta(x)| < \epsilon \quad \forall s \in (0, t_0).$$

Hence, for $t \leq t_0$,

$$\frac{1}{t}\int_0^t p_s^\sigma(L^\sigma z_\delta)(x) - (L^\sigma z_\delta)(x)\, ds| \leq \epsilon.$$

Let us look at (B.4): Let $\epsilon > 0$.

$$\begin{aligned}
&|p_\delta^\sigma z(x)| \\
&\quad \leq (2\pi\delta)^{-\frac{n}{2}} |\det\sigma|^{-1} \int |z(y)| \exp\left[-\frac{|\sigma^{-1}(x-y)|^2}{2\delta}\right] dy \\
&\quad \leq \sup_{y \in B_\epsilon(x)} |z(y)| + \\
&\qquad (2\pi\delta)^{-\frac{n}{2}} |\det\sigma|^{-1} \int_{B_\epsilon(x)^c} M(1+|y|) \exp\left[-\frac{|\sigma^{-1}(x-y)|^2}{2\delta}\right] dy \\
&\quad \leq \sup_{x \neq y} \frac{|x-y|}{|z(y)|} + M + \\
&\qquad M(2\pi\delta)^{-\frac{n}{2}} |\det\sigma|^{-1} \int_{B_\epsilon(x)^c} |y| \exp\left[-\frac{|\sigma^{-1}(x-y)|^2}{2\delta}\right] dy \\
&\quad \leq \sup_{x \neq y} \frac{|x-y|}{|z(y)|} + M + M|x| + M \int_{B_\epsilon(x)^c} |x-y| \frac{p_t^\sigma(x,y)}{p_1^\sigma(x,y)} p_1^\sigma(x,y)\, dy
\end{aligned}$$

An above computation in mind, we obtain for $\delta \leq \frac{1}{2}$,

$$\begin{aligned}
1_{B_\epsilon(x)^c}(y)|x-y| \frac{p_t^\sigma(x,y)}{p_1^\sigma(x,y)} &\leq 1_{B_\epsilon(x)^c}(y)|x-y| \frac{4^n n!}{|\sigma^{-1}(x-y)|^{2n}} \\
&\leq \frac{\|\sigma\|^{2n}}{\epsilon^{2n-1}} 4^n n! \\
\Rightarrow |p_\delta^\sigma z(x)| &\leq \left(\sup_{y \in B_\epsilon(x)} |z(y)| + M + M\frac{\|\sigma\|^{2n}}{\epsilon^{2n-1}} 4^n n!\right)(1+|x|).
\end{aligned}$$

Similiar computations yield the estimates for the partial derivatives. □

Our last task in this section is to show that

$$f_\delta(x + \sqrt{2t}\sigma y) \longrightarrow f_\delta(x) \quad \text{as} \quad t \downarrow 0:$$

Remember, that $f \equiv 0$ in B_R^c. Therefore,

$$\begin{aligned}
&f_\delta(x + \sqrt{2t}\sigma\bar{y}) \\
&\quad = (2\pi\sigma)^{-\frac{n}{2}} |\det\sigma|^{-1} \int_{B_R} f(y) \exp\left[-\frac{|\sigma^{-1}(x + \sqrt{2t}\sigma\bar{y} - y)|^2}{2\delta}\right] dy \\
&\quad \longrightarrow f_\delta(x) \quad \text{as} \quad t \downarrow 0 \quad \text{by Lebesgue.}
\end{aligned}$$

B.2 Refering to Proposition 3.28

(compare with [12, Appendix I and III]) Let $T \in C^\infty(\bar\Omega_1;\bar\Omega_2)$ be a diffeomorphism and u satisfy

$$\int_{\Omega_1} a^{ij} D_j u D_i \varphi \, dx = \int_{\Omega_1} f^j D_j \varphi \, dx. \tag{B.5}$$

If $\psi \in H_0^{1,2}(\Omega_2)$, then

$$\begin{aligned}
&\int_{\Omega_1} a^{ij} D_i(u\circ T) D_j(\psi\circ T)\, dx \\
&= \sum_{i,j} \int_{\Omega_1} a_{ij} \left(\sum_k D_k u(Tx) D_i T_k(x) \right) \left(\sum_l D_l \psi(Tx) D_j T_l(x) \right) dx \\
&= \sum_{i,j,k,l} \int_{\Omega_1} a_{ij} D_i T_k(x) D_j T_l(x) D_k u(Tx) D_l \psi(Tx)\, dx \\
&= \sum_{i,j,k,l} \int_{\Omega_1} a_{ij} D_i T_k \circ T^{-1}(Tx) D_j T_l \circ T^{-1}(Tx) D_k u(Tx) D_l \psi(Tx)\, dx \\
&= \sum_{i,j,k,l} \int_{\Omega_2} a_{ij} (D_i T_k \circ T^{-1})(y) (D_j T_l \circ T^{-1})(y) D_k u(y) D_l \psi(y) |\det T|^{-1}(y)\, dy
\end{aligned}$$

by integral transformation and since a_{ij} are constants.

$$\begin{aligned}
&= \sum_{k,l} \int_{\Omega_2} \left(\sum_{i,j} a_{ij} (D_i T_k \circ T^{-1})(y) (D_j T_l \circ T^{-1})(y) |\det T^{-1}|(y) \right) \\
&\qquad D_k u(y) D_l \psi(y)\, dy \\
&=: \sum_{k,l} \int_{\Omega_2} b_{kl}(y) D_k u(y) D_l \psi(y)\, dy,
\end{aligned}$$

from what we see that the ellipticity constant of the b_{kl} is

$$\frac{\nu^{-1}}{\min_{y\in\Omega_2} |\det T^{-1}|(y)}.$$

On the other side,

$$\begin{aligned}
&\int_{\Omega_1} f^i D_i(\psi \circ T)\,dx \\
&\quad = \int_{\Omega_1} f^i(x) \left(\sum_j D_j\psi(Tx) D_i T_j(x) \right) dx \\
&\quad = \sum_{i,j} \int_{\Omega_1} f_i(x) D_i T_j(x) D_j \psi(Tx)\,dx \\
&\quad = \sum_{i,j} \int_{\Omega_1} f_i \circ T^{-1}(Tx)(D_iT_j) \circ T^{-1}(Tx) D_j\psi(Tx)\,dx \\
&\quad = \sum_{i,j} \int_{\Omega_2} (f_i \circ T^{-1})(y)((D_iT_j)\circ T^{-1})(y)|\det T|^{-1}(y) D_j\psi(y)\,dy \\
&\quad =: \int_{\Omega_2} F^j(y) D_j\psi(y)\,dy.
\end{aligned}$$

Hence, defining the equation

$$\int_{\Omega_2} b^{kl}(y) D_k v D_l \psi\,dy = \int_{\Omega_2} F^k(y) D_k\psi(y)\,dy, \tag{B.6}$$

we can say:
u is a weak solution of (B.6) $\Leftrightarrow$ $u \circ T$ is a weak solution of (B.5).
or in other words:
u is a weak solution of (B.5) $\Leftrightarrow$ $u \circ T^{-1}$ is a weak solution of (B.6).

Since T is a C^∞-diffeomorphism and a_{ij} are constants, $b_{kl} \in C^\infty(\bar{\Omega}_2)$.
We now assert that $F_h \in \mathcal{L}^{2,\mu}(\Omega_2)$:
Step 1: $f_j \circ T^{-1} \in \mathcal{L}^{2,\mu}(\Omega_2)$

*Proof.*Set $J(x) := |\det T|(x), J^{-1}(y) := |\det T^{-1}|(y)$. Since J_1, J_2 are bounded, there exist $c_1, c_2 > 0$, such that for $x_1, x_2 \in \bar{\Omega}_1$,

$$c_1|x_1 - x_2| \leq |Tx_1 - Tx_2| \leq C_2|x_1 - x_2|.$$

Therefore, for $x \in \Omega_1, \rho > 0$,

$$\Omega_2(T(x), c_1\rho) := B(Tx, c_1\rho) \cap \Omega_2 \subset T(\Omega_1(x,\rho)) \subset \Omega_2(Tx, c_2\rho); \tag{B.7}$$

in fact, let $x \in \Omega_1$ and $y \in \Omega_2 \cap B(Tx, c_1\rho)$, then

$$|T^{-1}(y) - x| \leq \frac{1}{c_1}|y - Tx| \leq \frac{1}{c_1} c_1\rho = \rho.$$

Let $y \in T(\Omega_1(x,\rho))$, then $|y - Tx| \leq c_2|T^{-1}(y) - x| \leq c_2\rho$.
Analogously,

$$\Omega_1\left(T^{-1}(y), \frac{\rho}{c_2}\right) \subset T^{-1}(\Omega_2(y,\rho)) \subset \Omega_1\left(T^{-1}(y), \frac{\rho}{c_2}\right). \tag{B.8}$$

By monotony and integral transformation,

$$\begin{aligned}
&\int_{\Omega_2(y_0,\rho)} |f_j(T^{-1}y) - (f_j \circ T^{-1})_{y_0,\rho}|^2\, dy \\
&\quad\le \int_{\Omega_2(y_0,\rho)} |f_j(T^{-1}y) - (f_j)_{\Omega_1(T^{-1}y_0,\rho c_2^{-1})}|^2\, dy \\
&\quad\le \int_{T^{-1}(\Omega_2(y_0,\rho))} |f_j(x) - (f_j)_{\Omega_1(T^{-1}y_0,\rho c_2^{-1})}|^2 |J(x)|\, dx \\
&\quad\le \|J\|_\infty \int_{\Omega_1(T^{-1}y_0,\rho c_2^{-1})} |f_j(x) - (f_j)_{\Omega_1(T^{-1}y_0,\rho c_2^{-1})}|^2\, dx \\
&\quad\le \|J\|_\infty \rho^\mu c_2^{-\mu} \|f_j\|^2_{\mathcal{L}^{2,\mu}(\Omega_1)}.
\end{aligned}$$

Consequently, $f_j \circ T^{-1} \in \mathcal{L}^{2,\mu}(\Omega_2)$ with

$$\|f_j \circ T^{-1}\|_{\mathcal{L}^{2,\mu}(\Omega_2)} \le (\|J\|_\infty c_2^\mu)^{\frac{1}{2}} \|f_j\|_{\mathcal{L}^{2,\mu}(\Omega_1)}.$$

With help of (B.7) we get the estimate

$$\|\cdot \circ T\|_{\mathcal{L}^{2,\mu}(\Omega_1)} \le C\|\cdot\|_{\mathcal{L}^{2,\mu}(\Omega_2)}$$

analogously. □

Step 2: Of course, $(D_iT_j) \circ T^{-1} J^{-1} \in C^\infty(\bar\Omega_2)$.

- Case $0 \le \mu < n$: By Proposition 3.13, $\mathcal{L}^{2,\mu}(\Omega_2) \cong L^{2,\mu}(\Omega_2)$.
  Let $\varphi \in C(\bar\Omega_2)$ and $u \in L^{2,\mu}(\Omega_2)$, then

  $$\int_{\Omega_2(y_0,r)} |\varphi u|^2\, dx \le \sup_{\bar\Omega_2} |\varphi|^2 \int_{\Omega_2(y_0,r)} |u|^2\, dx.$$

  Hence,

  $$\begin{aligned}
  \|u\varphi\|_{\mathcal{L}^{p,\mu}(\Omega_2)} &\le C\|u\varphi\|_{L^{2,\mu}(\Omega_2)} \le C\|\varphi\|_\infty \|u\|_{L^{2,\mu}(\Omega_2)} \\
  &\le \tilde C \|\varphi\|_\infty \|u\|_{\mathcal{L}^{2,\mu}(\Omega_2)}.
  \end{aligned}$$

- Case $\mu = n$: Let $\varphi \in C^{0,\alpha}(\bar\Omega_2)$ for some $0 < \alpha \le 1, u \in \mathcal{L}^{2,n}(\Omega_2)$.
  For $y_0 \in \Omega_2$ and $r > 0$ we have

  $$\begin{aligned}
  &\int_{\Omega_2(y_0,r)} |u\varphi - (u\varphi)_{y_0,r}|^2\, dy \\
  &= \int_{\Omega_2(y_0,r)} |u\varphi(y_0) - u[\varphi(y_0) - \varphi] - \varphi(y_0)u_{y_0,r} - (u[\varphi(y_0) - \varphi])_{y_0,r}|^2\, dy \\
  &\le C\{\sup_{\bar\Omega_2} |\varphi|^2 \int_{\Omega_2(y_0,r)} |u - u_{y_0,r}|^2\, dy + \int_{\Omega_2(y_0,r)} |u|^2 |\varphi(y_0) - \varphi|^2\, dy\},
  \end{aligned}$$

  where we used (3.80),

  $$\begin{aligned}
  &\le C\{\sup_{\bar\Omega_2} |\varphi|^2 \int_{\Omega_2(y_0,r)} |u - u_{y_0,r}|^2\, dy + r^{2\alpha} \|\varphi\|^2_{C^{0,\alpha}(\bar\Omega_2)} \int_{\Omega_2(y_0,r)} |u|^2\, dy\} \\
  &= (*).
  \end{aligned}$$

Now, $\mathcal{L}^{2,n}(\Omega_2) \subset \mathcal{L}^{2,n-\epsilon}(\Omega_2) \cong L^{2,n-\epsilon}(\Omega_2)$, hence for $\epsilon = 2\alpha$

$$\begin{aligned}\int_{\Omega_2(y_0,r)} |u|^2\,dx &\leq r^{n-2\alpha}\|u\|^2_{L^{2,n-2\alpha}(\Omega_2)} \\ &\leq Cr^{n-2\alpha}\|u\|^2_{\mathcal{L}^{2,n-2\alpha}(\Omega_2)} \\ &\leq C' r^{n-2\alpha}\|u\|^2_{\mathcal{L}^{2,n}(\Omega_2)}.\end{aligned}$$

Ergo,

$$\begin{aligned}(*) &\leq C\{\sup_{\bar{\Omega}_2}|\varphi|^2 r^n [u]^2_{\mathcal{L}^{2,n}(\Omega_2)} + r^n \|\varphi\|_{C^{0,\alpha}(\bar{\Omega}_2)}\|u\|^2_{\mathcal{L}^{2,n}(\Omega_2)}\} \\ &\leq Cr^n\|\varphi\|_{C^{0,\alpha}(\bar{\Omega}_2)}\|u\|^2_{\mathcal{L}^{2,n}(\Omega_2)}.\end{aligned}$$

Therefore

$$[u\varphi]^2_{\mathcal{L}^{2,n}(\Omega_2)} \leq C\|\varphi\|_{C^{0,\alpha}(\bar{\Omega}_2)}\|u\|^2_{\mathcal{L}^{2,n}(\Omega_2)},$$

which implies $u\varphi \in \mathcal{L}^{2,n}(\Omega_2)$ and

$$\begin{aligned}\|u\varphi\|_{\mathcal{L}^{2,n}(\Omega_2)} &= \|u\varphi\|_{2;\Omega} + [u\varphi]_{\mathcal{L}^{2,n}(\Omega_2)} \\ &\leq C\|\varphi\|_{C^{0,\alpha}(\bar{\Omega}_2)}\|u\|_{\mathcal{L}^{2,n}(\Omega_2)}.\end{aligned}$$

Altogether, $F_h \in \mathcal{L}^{2,\mu}(\Omega_2) \quad \forall 0 \leq \mu \leq n$ with

$$\begin{aligned}\|F_h\|_{\mathcal{L}^{2,\mu}(\Omega_2)} &\leq C\|F_h \circ T\|_{\mathcal{L}^{2,\mu}(\Omega_2)} \\ &= c\|\sum_j f_j D_j T_h \frac{1}{J(x)}\|_{\mathcal{L}^{2,n}(\Omega_1)} \\ &\leq C\|f\|_{\mathcal{L}^{2,n}(\Omega_1)}.\end{aligned}$$

Appendix C

PROPERTIES OF $H^{m,p}$

C.1 $m \in \mathbb{N}$

This section is based on [3, Chapter 5]. Let in the whole section $\Omega \subset \mathbb{R}^n$ be open and bounded, $m \in \mathbb{N}$.

Proposition C.1 (Rellich-Embedding) *Let $1 \le p < \infty$ and $m \ge 1$. If $u_k \longrightarrow u$ weakly in $H_0^{m,p}(\Omega)$, then $u_k \longrightarrow u$ strongly in $H^{m-1,p}(\Omega)$.*

Proof. Let $m = 1$ (otherwise apply the following argumentation to each $\partial^s u_k$ with $|s| \le m-1$). If we set $u_k \equiv 0$ in $\mathbb{R}^n \setminus \Omega$, then $u_k \in H^{1,p}(\mathbb{R}^n)$. For the Dirac-sequence (φ_ϵ) of some $\varphi \in C_0^\infty(B_1(0))$ we have that $\varphi_\epsilon * u_k \in C^\infty(\mathbb{R}^n)$ and $\varphi_\epsilon * u_k \longrightarrow \varphi_\epsilon * u$ in $L^p(\mathbb{R}^n)$ as $k \longrightarrow \infty$: in fact, outside $B_\epsilon(\Omega), \varphi_\epsilon * u_k$ and $\varphi_\epsilon * u$ are vanishing and for $x_k \in \bar{B}_\epsilon(\Omega), k \in \mathbb{N}$, such that $x_k \longrightarrow x$, we have $\varphi_\epsilon(x_k - \cdot) \longrightarrow \varphi_\epsilon(x - \cdot)$ uniformly, thus in $L^{p'}(\Omega)$, too. Since $u_k \longrightarrow u$ also weakly in $L^p(\Omega)$, it follows that (u_k) is bounded in L^p and

$$\begin{aligned} \varphi_\epsilon * u_k(x_k) &= \int_\Omega \varphi_\epsilon(x_k - y) u_k(y)\, dy \\ &\longrightarrow \int_\Omega \varphi_\epsilon(x-y) u(y)\, dy = \varphi_\epsilon * u(x) \quad \text{(cf. [3, 5.3.5])} \quad . \end{aligned}$$

From this, we derive $\varphi_\epsilon * u_k \longrightarrow \varphi_\epsilon * u$ pointwisely as $k \longrightarrow \infty$ and since

$$\varphi_\epsilon * u_k(\cdot) \le \|\varphi_\epsilon\|_\infty C \|u_k\|_{p;\Omega} \le \|\varphi_\epsilon\|_\infty C C_1 \le C'(\epsilon),$$

where C' does not depend on k, we conclude that $\varphi_\epsilon * u_k \longrightarrow \varphi_\epsilon * u$ in $L^p(\mathbb{R}^n)$ as k tends to infinity for any ϵ.
Moreover, for arbitrary $v \in H^{1,p}(\mathbb{R}^n)$ we have an estimate of the form

$$\|v - \varphi_\epsilon * v\|_{p;\mathbb{R}^n} \le \epsilon \|Dv\|_{p;\mathbb{R}^n} : \tag{C.1}$$

$$\begin{aligned} \|v - \varphi_\epsilon * v\|_{p;\mathbb{R}^n} &= \left(\int_{\mathbb{R}^n} \left| v(x) - \int_{\mathbb{R}^n} \varphi_\epsilon(x-y) v(y)\, dy \right|^p dx \right)^{\frac{1}{p}} \\ &= \left(\int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} \varphi_\epsilon(x-y)(v(x) - v(y))\, dy \right|^p dx \right)^{\frac{1}{p}} \end{aligned}$$

$$\leq \sup_{|h|\leq\epsilon}\left(\int_{\mathbb{R}^n}|v(x+h)-v(x)|^p\right)^{\frac{1}{p}} \quad \text{(cf. [3, 2.8])} \quad .$$

For $v \in H^{1,p}(\mathbb{R}^n)\cap C^\infty(\mathbb{R}^n)$ we have

$$\begin{aligned}\int_{\mathbb{R}^n}|v(x+h)-v(x)|^p\,dx &= \int_{\mathbb{R}^n}\left|\int_0^1 < Dv(x+sh),h > ds\right|^p dx \\ &\leq |h|^p\int_0^1\int_{\mathbb{R}^n}|Dv(x+sh)|^p\,dx\,ds \\ &\qquad \text{(Cauchy-Schw., Jensen, Fubini)} \\ &\leq |h|^p\int_{\mathbb{R}^n}|Dv|^p\,dx.\end{aligned}$$

Since $H^{1,p}(\mathbb{R}^n)\cap C^\infty(\mathbb{R}^n)$ dense in $H^{1,p}(\mathbb{R}^n)$, the above estimate remains valid for $v \in H^{1,p}(\mathbb{R}^n)$ and we obtain (C.1). Now we replace v by u_k and obtain

$$\|u-u_k\|_{p;\mathbb{R}^n} \leq \|u-\varphi_\epsilon * u\|_p + \underbrace{\|\varphi_\epsilon * u - \varphi_\epsilon * u_k\|_p}_{\longrightarrow 0\forall\epsilon} + \underbrace{\|\varphi_\epsilon * u_k - u_k\|_p}_{\leq\epsilon\|Du_k\|_p\leq\epsilon C_1}.$$

Since $\varphi_\epsilon * u \longrightarrow u$ in $L^p(\mathbb{R}^n)$ the proposition follows. □

Remark C.2 *We have not used that Du belongs to $L^p(\mathbb{R}^n)$. Actually, we have proven the following implication:*
u_k bounded in $H_0^{1,p}(\Omega), u \in L^p(\Omega)$ and $\int_\Omega \zeta u_k \longrightarrow \int_\Omega \zeta u \forall\zeta \in C_0^\infty(\Omega)$

$$\Rightarrow u_k \longrightarrow u \quad \textit{strongly in} \quad L^p(\Omega).$$

During the following considerations it would be enough to assume Ω to be of class $C^{0,1}$. Since we are only interested in the case Ω = ball and want to use the extension operator E from A.4, we assume Ω to be of class C^∞ for the rest of this section, if not otherwise stated, for simplicity (see Definition 2.20). According to that definition, $\mathcal{U} := \{B_{r(x_0)}(x_0)|x_0 \in \partial\Omega\}$ is an open covering of $\partial\Omega$. By compactness, there exist $U^1,\ldots,U^m \in \mathcal{U}$ such that $\bigcup_{i=1}^m U^i \supset \partial\Omega$. Add U^0 with $\bar{U}^0 \subset \Omega$ and obtain an open covering of $\bar{\Omega}$. Let

$$\eta^j \in C_0^\infty(U^j), 0 \leq \eta^j \leq 1, \sum_{j=0}^m \eta^j = 1 \quad \text{in} \quad \bar{\Omega}$$

be a partition of unity subordinated to $\{U^j|j=0,\ldots,m\}$. If then $u \in H^{m,p}(\Omega)$, we have $u = \sum_{j=0}^m \eta^j u$. In particular, $\eta^0 u \in H_0^{m,p}(\Omega)$ (for $p < \infty$) and if we define $Q^j := U^j \cap \Omega$, then $\eta^j u \in H^{m,p}(Q^j)$.

Proposition C.3 *Let $1 \leq p < \infty, m \in \mathbb{N}$. If $u_k \longrightarrow u$ weakly in $H^{m,p}(\Omega)$, then $u_k \longrightarrow u$ strongly in $H^{m-1,p}(\Omega)$.*

Proof. With our notation $\eta^j u_k \longrightarrow \eta^j u$ weakly in $H^{m,p}(\Omega)$ for each j: in fact, let $l \in H^{m,p}(\Omega)'$. Define $\tilde{l} : H^{m,p}(\Omega) \longrightarrow \mathbb{R}, v \longmapsto l(\eta^j v)$, then

$$|\tilde{l}v| = |l(\eta^j v)| \leq \|l\| \|\eta^j v\|_{m,p;\Omega} \leq \|l\| \|\eta^j\|_{C^m(\mathbb{R}^n)} \|v\|_{m,p;\Omega}.$$

Thus, $\tilde{l} \in H^{m,p}(\Omega)'$. Consequently, $|\tilde{l}(u_k) - \tilde{l}(u)| \longrightarrow 0$, in other words,

$$|l(\eta^j u_k) - l(\eta^j u)| \longrightarrow 0.$$

Since $\eta^0 u_k$ and $\eta^0 u \in H_0^{m,p}(\Omega)$, $\eta^0 u_k \longrightarrow \eta^0 u$ strongly in $H^{m-1,p}(\Omega)$ by C.1. Let $\Omega' \supset\supset \Omega$ and E be the extension operator from A.4. We assert that $E(\eta^j u_k) \longrightarrow E(\eta^j u)$ weakly in $H_0^{m,p}(\Omega')$: in fact, let $l \in H_0^{m,p}(\Omega)'$, then $l \circ E \in H^{m,p}(\Omega)'$, since for any $u \in H^{m,p}(\Omega)$,

$$\begin{aligned} |(l \circ E)(u)| &= |l(Eu)| \\ &\leq C\|Eu\|_{2,p;\Omega'} \leq C'\|u\|_{2,p;\Omega}. \end{aligned}$$

Therefore,

$$|l(E(\eta^j u_k)) - l(E(\eta^j u))| = |(l \circ E)(\eta^j u_k - \eta^j u)| \longrightarrow 0.$$

So we can apply C.1 to $E(\eta^j u_k), E(\eta^j u)$ and obtain the proposition, because $E_{|\Omega} = \mathrm{Id}$ in Ω. □

Lemma C.4 *For $m \in \mathbb{N}$ and $1 < p \leq \infty$ the following equivalence holds:*

$$f \in H^{m,p}(\Omega) \Leftrightarrow$$
$$f \in L^1_{loc}(\Omega) \quad and \quad \left| \int_\Omega f D^s \zeta \, dx \right| \leq C\|\zeta\|_{p';\Omega} \forall |s| \leq m, \zeta \in C_0^\infty(\Omega).$$

Proof. $\Rightarrow$: $\left|\int_\Omega f D^s \zeta \, dx\right| = \left|\int_\Omega D^s f \zeta \, dx\right| \leq \|D^s f\|_{p;\Omega} \|\zeta\|_{p';\Omega}$
$\Leftarrow$: For $0 \leq |s| \leq m$ and $\zeta \in C_0^\infty(\Omega)$ set $F_s(\zeta) := \int_\Omega f D^s \zeta \, dx$. Since $C_0^\infty(\Omega) \subset L^{p'}(\Omega)$ dense, we can extend F_s to $L^{p'}(\Omega)$ and obtain by duality the existence of functions $f^{(s)} \in L^p(\Omega)$ such that

$$F_s(g) = \int_\Omega g((-1)^{|s|} f^{(s)}) \, dx \quad \forall g \in L^{p'}(\Omega).$$

Thus,

$$\int_\Omega f D^s \zeta \, dx = (-1)^{|s|} \int_\Omega f^{(s)} \zeta \, dx \forall \zeta \in C_0^\infty(\Omega),$$

For $|s| = 0$ we have that $\int_\Omega f \zeta \, dx = \int_\Omega f^{(0)} \zeta \, dx \quad \forall \zeta \in C_0^\infty(\Omega)$. Therefore $f = f^{(0)}$ and $f \in H^{m,p}(\Omega)$. □

Lemma C.5 *Let $x_0, x_1 \in \Omega$. Then there exists $\gamma \in C^\infty([0,1]; \Omega)$ such that $\gamma(0) = x_0, \gamma(1) = x_1$ and for some constant C, which only depends on Ω, we have for its length*

$$L(\gamma) := \int_0^1 |\gamma'(t)| \, dt \leq \sup_{0 \leq t \leq 1} |\gamma'(t)| \leq C|x_1 - x_0|.$$

*Proof.*step 1: Construction of a $C^{0,1}$-path γ such that lip $(\gamma) \leq C|x_1 - x_0|$.
We start from our covering $(U^j)_{j=1,\dots,m}$ of $\partial\Omega$ and choose points $z^j \in Q^j$. Then we take an open set $D \subset\subset \Omega$, such that $z^1, \dots, z^m \in \bar{D}$ and $D, U^1, \dots, U^m$ cover $\bar{\Omega}$. Moreover, cover $\bar{D}$ with finitely many balls $U^j := B_\delta(z^j) \subset \Omega, j = m+1, \dots, l$.
We distinguish three cases. If $x_0, x_1 \in U^j$ for some $j > m$, then define

$$\gamma(t) := (1-t)x_0 + tx_1.$$

If $x_0, x_1 \in U^j$ for some $j \leq k$, then define

$$\gamma(t) := \tau((1-t)\tau^{-1}(x_0) + t\tau^{-1}(x_1)),$$

where $\tau(y) := \sum_{i=1}^{n-1} y_i e_i^j + (y_n + g^j(y'))e_n^j$. This defines a lipschitz continuous path γ in Ω from x_0 to x_1 with

$$\begin{aligned} \mathrm{lip}(\gamma) &= \sup_{s,t\in[0,1]} \frac{|\gamma(t) - \gamma(s)|}{|(1-t)\tau^{-1}(x_0) + t\tau^{-1}(x_1) - (1-s)\tau^{-1}(x_0) - s\tau^{-1}(x_1)|} \cdot \\ &\quad \frac{|(s-t)\tau^{-1}(x_0) + (t-s)\tau^{-1}(x_1)|}{|s-t|} \\ &\leq \mathrm{lip}(\tau)|\tau^{-1}(x_0) - \tau^{-1}(x_1)| \leq \mathrm{lip}(\tau)\mathrm{lip}(\tau^{-1})|x_1 - x_0|. \end{aligned}$$

In the third case x_0 and x_1 do not both lie in the same of our $\bar{\Omega}$-covering sets. We assert that there exists a constant $c > 0$, just depending on our covering, such that $|x_1 - x_0| \geq c$. This follows from the fact that for $x \in \bar{\Omega} \cup U^j$ lying close enough to $\partial U^j, x \in U^k$ for some $k \neq j$; let $I := \{1, \dots, l\}$ and define

$$\begin{aligned} c := \min\{\mathrm{dist}((\bigcap_{i\in I_0} U^i) \setminus (\bigcup_{i\in I\setminus I_0} U^i), ((\bigcap_{i\in I_1} U^i) \setminus (\bigcup_{i\in I\setminus I_1} U^i))| \\ \emptyset \neq I_0 \subset I, \emptyset \neq I_1 \subset I, I_0 \cap I_1 = \emptyset\}. \end{aligned}$$

Then $0 < c \leq \mathrm{diam}\Omega$ (set $\mathrm{dist}(\emptyset, A) := \infty$) by construction. Now let $I_{x_0}, I_{x_1} \subset I$, such that $x_0 \in \bigcap_{i\in I_{x_0}} U^i, x_1 \in \bigcap_{i\in I_{x_1}} U^i$, then $|x_0 - x_1| \geq c$. Since Ω is a domain and therefore connected, there exists a continuous path $\gamma_{j,k} : [0,1] \longrightarrow \Omega$ with $\gamma_{j,k}(0) = z^j, \gamma_{j,k}(1) = z^k$. If we work on $\gamma_{j,k}$ as we will do on the entire path in step 2, we may assume $\gamma_{j,k}$ to be smooth. Then $\mathrm{lip}(\gamma_{j,k}) < \infty$ and because of $|x_1 - x_0| \geq c$, there exists $C = C(\gamma_{j,k})$, such that

$$\mathrm{lip}(\gamma_{j,k}) < Cc \leq C|x_1 - x_0|.$$

Now let $x_0 \in U^{j_0}, x_1 \in U^{j_1}$. At first, according to case 1 and 2, we connect x_0 and z^{j_0} within U^{j_0} by some path, whose lipschitz constant is estimated by $C|z^{j_0} - x_0| \leq C\mathrm{diam}U^{j_0}$. Then we connect z^{j_0} and z^{j_1} by γ_{j_0,j_1} and z^{j_1} with x_1 within U^{j_1}. Reparametrizing the concatenation of these paths, step 1 follows.

step 2: Let γ be the result of step 1. Define $\gamma(t) := x_0$ for $t < 0, \gamma(t) := x_1$ for $t > 1$ and $\gamma_\epsilon := \varphi_\epsilon * \gamma$, where $\varphi \in C_0^\infty((-1,1))$ and (φ_ϵ) denotes its Dirac sequence. For any t,

$$\gamma'_\epsilon(t) \quad = \quad \lim_{t_0 \longrightarrow t} \frac{\gamma_\epsilon(t) - \gamma_\epsilon(t_0)}{t - t_0} \leq \mathrm{lip}(\gamma_\epsilon)$$

$$\begin{aligned}
&& \Rightarrow & \quad \|\gamma_\epsilon'\|_\infty \leq \mathrm{lip}(\gamma_\epsilon) \\
\sup_{s,t} \frac{|\gamma_\epsilon(t)-\gamma_\epsilon(s)|}{|t-s|} & = & \sup_{s,t} & \frac{\left|\int_{\mathbb{R}} \varphi_\epsilon(r)(\gamma(t-r)-\gamma(s-r))\,dr\right|}{|(t-r)-(s-r)|} \\
& \leq & & \mathrm{lip}(\gamma) \\
& \Rightarrow & & \|\gamma_\epsilon'\|_\infty \leq \mathrm{lip}(\gamma_\epsilon) \leq \mathrm{lip}(\gamma).
\end{aligned}$$

Moreover,

$$\gamma_\epsilon(-\epsilon) = \int_{B_\epsilon(0)} \varphi_\epsilon(s) \underbrace{\gamma(-\epsilon - s)}_{=x_0 \forall s \in B_\epsilon(0)} \, ds = x_0$$

$$\gamma_\epsilon(1+\epsilon) = \int_{B_\epsilon(0)} \varphi_\epsilon(s) \underbrace{\gamma(1+\epsilon - s)}_{=x_1 \forall s \in B_\epsilon(0)} \, ds = x_1.$$

Consider the transformation

$$T_\epsilon : [0,1] \longrightarrow [-\epsilon, 1+\epsilon], t \longmapsto (1+2\epsilon)t - \epsilon.$$

and define $\tilde{\gamma}_\epsilon := \gamma_\epsilon \circ T_\epsilon$, then $\tilde{\gamma}_\epsilon \in C^\infty([0,1];\mathbb{R}^n), \tilde{\gamma}_\epsilon(0) = x_0, \tilde{\gamma}_\epsilon(1) = x_1$. Set $\epsilon_0 := \mathrm{dist}(\gamma([0,1]), \partial\Omega)$, then for any $t \in [0,1]$,

$$\begin{aligned}
|\tilde{\gamma}_\epsilon(t) - \gamma(t)| & = |\gamma_\epsilon \circ T_\epsilon(t) - \gamma(t)| \\
& = |(\varphi_\epsilon * \gamma)(T_\epsilon(t)) - \gamma(t)| \\
& = |\int_{\mathbb{R}} \varphi_\epsilon(s)(\gamma(T_\epsilon(t) - s) - \gamma(t))\,ds| \\
& \leq \int_{\mathbb{R}} \varphi_\epsilon(s)|\gamma(T_\epsilon(t)-s) - \gamma(t)|\,ds < \epsilon_0,
\end{aligned}$$

if ϵ is sufficiently small, because γ is uniformly continuous and $|T_\epsilon(t) - s - t| \leq |T_\epsilon(t) - t| + |s| \leq 4\epsilon$. Thus for suitable $\epsilon, \tilde{\gamma}_\epsilon([0,1]) \subset \Omega$ and step 2 is proven. □

Proposition C.6 *$C^{m-1,1}(\bar{\Omega}) \subset H^{m,\infty}(\Omega)$ and the embedding $J : C^{m-1,1}(\bar{\Omega}) \longrightarrow H^{m,\infty}(\Omega)$ is an isomorphism with $\|J\| \leq 1$. In particular, any function in $H^{m,\infty}(\Omega)$ has got an unique representation in $C^{m-1,1}(\bar{\Omega})$.*

*Proof.*It suffices to consider the case $m = 1$. So let $u \in C^{0,1}(\bar{\Omega})$. Letting h tend to 0 we have for arbitrary $\zeta \in C_0^\infty(\Omega)$,

$$\begin{aligned}
\left|\int_\Omega u D_i \zeta \, dx\right| & \longleftarrow \left|\int_\Omega u(x) \frac{\zeta(x+he_i) - \zeta(x)}{h} \, dx\right| \\
& = \left|\int_\Omega \frac{u(x-he_i) - u(x)}{h} \zeta(x)\, dx\right| \\
& \leq \mathrm{lip}(u) \int_\Omega |\zeta| \, dx.
\end{aligned}$$

Hence, $u \in H^{1,\infty}(\Omega)$ by Lemma C.4. Now let $u \in H^{1,\infty}(\Omega), u_\epsilon := u * \varphi_\epsilon$, where (φ_ϵ) shall be the Dirac-sequence corresponding to some $\varphi \in C_0^\infty(B_1(0))$. We proved in Lemma C.5 that for $x_0, x_1 \in \Omega$ there exists a path $\gamma \in C^\infty([0,1],\Omega)$ with $\gamma(0) = x_0, \gamma(1) = x_1$ and length $\int_1^0 |\gamma'|\, dt \leq C|x_1 - x_0|$, where C only depends on Ω. We obtain

$$\begin{aligned} |u_\epsilon(x_1) - u_\epsilon(x_0)| &= \left| \int_0^1 (u_\epsilon \circ \gamma)'\, dx \right| \\ &\leq \int_0^1 |Du_\epsilon(\gamma(t))||\gamma'(t)|\, dt \\ &\leq \sup_{0\leq t\leq 1} |Du_\epsilon(\gamma(t))| C|x_1 - x_0| \end{aligned}$$

and for sufficient small ϵ we have for $x = \gamma(t), 0 \leq t \leq 1$,

$$\begin{aligned} |Du_\epsilon(x)| &= |D(u * \varphi_\epsilon)(x)| = |(Du * \varphi_\epsilon)(x)| \\ &= \sum_{i=1}^n \left| \int_\Omega \varphi_\epsilon(x-y) D_i u(y)\, dy \right| \leq \|Du\|_{\infty;\Omega}. \end{aligned}$$

Since $u_\epsilon \longrightarrow u$ in $L^p(\Omega)$ for $p < \infty$, there exists a subsequence $\epsilon \longrightarrow 0$ such that $u_\epsilon \longrightarrow u$ a. e. in Ω. Hence we have shown that for any $x_0, x_1 \in \Omega$,

$$\frac{|u(x_1) - u(x_0)|}{|x_1 - x_0|} \leq C\|Du\|_{\infty;\Omega},$$

i. e. u is lipschitz continuous outside a nullset. Modify u on this nullset to obtain $u \in C^{0,1}(\bar{\Omega})$. □

Definition C.7 *Recalling definition 2.20 and that $\partial\Omega$ is compact, we can choose finitely many $U^j, j = 1, \dots, m$ of type $B_{r^j}(x_j)$, that cover $\partial\Omega$. Let $(e_i^j)_{i=1}^n$ be the corresponding modified coordinate systems. We say that $f : \partial\Omega \longrightarrow \mathbb{R}$ is measurable (integrable), if with our notation for $j = 1, \dots, m$ the functions*

$$y \longmapsto f\left(\sum_{i=1}^{n-1} y_i e_i^j + g^j(y) e_n^j\right)$$

for $y \in \mathbb{R}^{n-1}, |y| < r^j$ are measurable (integrable). The boundary integral of f over $\partial\Omega$ is then defined by

$$\int_{\partial\Omega} f\, dS := \sum_{j=1}^m \int_{\partial\Omega} \eta^j f\, dS,$$

and if $supp(f) \subset U^j$,

$$\int_{\partial\Omega} f\, dS := \int_{\mathbb{R}^{n-1}} f\left(\sum_{i=1}^{n-1} y_i e_i^j + g^j(y) e_n^j\right) \sqrt{1 + |Dg^j(y)|^2}\, dx.$$

Note that Dg^j is a well defined measurable and bounded function by C.6.

For $1 \le p \le \infty$ we set

$$L^p(\partial\Omega) := \{f : \partial\Omega \longrightarrow \mathbb{R} | f \quad \text{is measurable and} \quad \|f\|_{p;\partial\Omega} < \infty\},$$

where $\|f\|_{p;\partial\Omega} := \left(\int_{\partial\Omega} |f|^p \, dS\right)^{\frac{1}{p}}$ and $\|f\|_{\infty;\partial\Omega} := \operatorname{ess\,sup}_{\partial\Omega}|f|$. Since Dg^j are measurable functions, we can define

$$\nu(x) := (1 + |Dg^j(y)|^2)^{-\frac{1}{2}} \left(\sum_{i=1}^{n-1} D_i g^j(y) e_i^j - e_n^j \right)$$

for $x = \sum_{i=1}^{n-1} y_i e_i^j + g^j(y) e_n^j \in U^j$ with $|y| < r^j$ in each local neighbourhood $U^j, j = 1, \dots, m$. We call $\nu(x)$ the outer normal of $\partial\Omega$ in x. ν is measurable in $\partial\Omega$ with $|\nu| = 1$, thus $\nu \in L^\infty(\partial\Omega)$. Because of the above representation of x, $\nu(x)$ is orthogonal to

$$\tau_k(x) := D_{y_k} \left(\sum_{i=1}^{n-1} y_i e_i^j + g^j(y) e_n^j \right) = e_k^j + \partial_k g^j(y) e_n^j$$

for $1 \le k \le n-1$. Moreover, $\nu(x)$ points outside, i. e. $x + \epsilon\nu(x) \notin \Omega$ for small $\epsilon > 0$: for this, we have to show that $(x + \epsilon\nu(x))_n^j < g^j(((x + \epsilon\nu(x))^j)')$:

$$\begin{aligned} (x + \epsilon\nu(x))_n^j &= g^j(y) - \frac{\epsilon}{\sqrt{1 + |Dg^j(y)|^2}} \\ g^j(((x + \epsilon\nu(x))^j)') &= g^j \left(y + \frac{\epsilon}{\sqrt{1 + |Dg^j(y)|^2}} Dg^j(y) \right). \end{aligned}$$

Thus

$$g^j(y) - g^j \left(y + \frac{\epsilon}{\sqrt{1 + |Dg^j(y)|^2}} Dg^j(y) \right) \le 0 < \frac{\epsilon}{\sqrt{1 + |Dg^j(y)|^2}},$$

since $Dg^j(y)$ points to the direction of strongest ascent.

Proposition C.8 *There exists exactly one continuous linear map*

$$B : H^{1,p}(\Omega) \longrightarrow L^p(\partial\Omega),$$

such that $Bu = u_{|\partial\Omega}$ for $u \in H^{1,p}(\Omega) \cap C^0(\bar{\Omega})$. (In this case we sometimes write u instead of Bu.)

Proof. The case $p = \infty$ follows from C.6. So let $p < \infty$ and $u \in H^{1,p}(\Omega)$. We know that $v := \eta^j u \in H^{1,p}(Q^j)$ and since $\eta^j \in C_0^\infty(U^j)$, for some $\delta > 0$ we have

$$v(x) = 0 \quad \forall |(x^j)'| \ge r^j - \delta \quad \text{and} \quad \forall x_n^j - g^j((x^j)') \ge h^j - \delta.$$

For $s > 0$ we define functions $v_s : \mathbb{R}^{n-1} \longrightarrow \mathbb{R}$ by

$$v_s(y) := v(y, g^j(y) + s),$$

where $(y,h) := \sum_{i=1}^{n-1} y_i e_i^j + h e_n^j$. We assert that for a. e. $s_1, s_2 > 0$ and then for a. e. $y \in \mathbb{R}^{n-1}$ we have

$$\begin{aligned} v_{s_2}(y) - v_{s_1}(y) &= v(y, g^j(y) + s_2) - v(y, g^j(y) + s_1) \\ &= \int_{g^j(y)+s_1}^{g^j(y)+s_2} D_{e_n^j} v(y,h)\, dh \end{aligned} \tag{C.2}$$

In fact, approximate v by $v_k \in H^{1,p}(Q^j) \cap C^\infty(Q^j)$, $k \in \mathbb{N}$. For v_k, (C.2) is true and moreover,

$$\begin{aligned} &\int_0^{h^j} \int_{B_{r^j}(0)} |v(y, g^j(y) + s) - v_k(y, g^j(y) + s)|\, dy\, ds \\ &\quad = \int_{Q^j} |v - v_k|\, dx \longrightarrow 0 \quad \text{as} \quad k \longrightarrow \infty \end{aligned}$$

and

$$\begin{aligned} &\int_0^{h^j} \int_{B_{r^j}(0)} \int_{g^j(y)}^{g^j(y)+s} |D_{e_n^j} v(y,h) - D_{e_n^j} v_k(y,h)|\, dh\, dy\, ds \\ &\quad \le h^j \int_{Q^j} |D_{e_n^j}(v - v_k)|\, dx \longrightarrow 0 \quad \text{as} \quad k \longrightarrow \infty. \end{aligned}$$

Now for $s_1 < s_2$ by Hölder

$$\begin{aligned} &\int_{B_{r^j}(0)} |v_{s_2} - v_{s_1}|^p\, dy \\ &= \int_{B_{r^j}(0)} \left| \int_{g^j(y)+s_1}^{g^j(y)+s_2} D_{e_n^j} v(y,h)\, dh \right|^p dy \\ &\le \int_{B_{r^j}(0)} \left(\int_{g^j(y)+s_1}^{g^j(y)+s_2} |D_{e_n^j} v(y,h)|\, dh \right)^p dy \\ &\le \int_{B_{r^j}(0)} \|1\|^p_{\frac{p}{p-1};[g^j(y)+s_1, g^j(y)+s_2]} \|D_{e_n^j} v(y,h)\|^p_{p;[g^j(y)+s_1, g^j(y)+s_2]}\, dy \\ &= \int_{B_{r^j}(0)} |s_2 - s_1|^{p-1} \int_{g^j(y)+s_1}^{g^j(y)+s_2} |D_{e_n^j} v(y,h)|^p\, dh\, dy \\ &\le |s_2 - s_1|^{p-1} \int_{\{x \in Q^j \,|\, s_1 < x_n^j - g^j((x^j)') < s_2\}} |Dv|^p\, dx, \end{aligned}$$

thus

$$\|v_{s_2} - v_{s_1}\|_{p;B_{r^j}(0)} \le |s_2 - s_1|^{1-\frac{1}{p}} \|Dv\|_{p;\{x \in Q^j \,|\, s_1 < x_n^j - g^j((x^j)') < s_2\}}.$$

Since the right hand side tends to 0 as $s_1, s_2 \longrightarrow 0$, the functions v_s build a Cauchy sequence in $L^p(\mathbb{R}^{n-1})$ for $s \longrightarrow 0$, hence $v_s \longrightarrow v_0$ in $L^p(\mathbb{R}^{n-1})$ for some $v_0 \in L^p(\mathbb{R}^{n-1})$. Set

$$B^j v(y, g^j(y)) := v_0(y).$$

By C.7, $B^j v \in L^p(\partial\Omega)$ with

$$\begin{aligned}\|B^j v\|_{p;\partial\Omega} &\leq C\|v_0\|_{p;B_{r^j}(0)} \\ &= C\|v_s - v_{s_0}\|_{p;B_{r^j}(0)} \\ &= C\lim_{s\downarrow 0}\|v_s - v_{s_0}\|_{p;B_{r^j}(0)} \\ &\leq C s_0^{1-\frac{1}{p}}\|Dv\|_{p;Q_j},\end{aligned}$$

if $h^j - \delta < s_0 < h^j$. For $u \in H^{1,p}(\Omega)$ we then define

$$Bu := \sum_{j=1}^{m} B^j(\eta^j u).$$

In particular, $Bu = u_{|\partial\Omega}$, if $u \in C(\bar{\Omega})$. Thus the existence of B is proven. Uniqueness follows by A.4 (i). □

Proposition C.9 (weak Gaussian theorem) *1. If $u \in H^{1,1}(\Omega)$, then for $i = 1, \ldots, n$,*

$$\int_\Omega D_i u = \int_{\partial\Omega} u\nu_i \, dS,$$

where ν shall be the outer normal to $\partial\Omega$ from C.7.

2. If $u \in H^{1,p}(\Omega)$ and $v \in H^{1,p'}(\Omega)$ such that $\frac{1}{p} + \frac{1}{p'} = 1$, then for $i = 1, \ldots, n$

$$\int_\Omega (uD_i v + vD_i u) = \int_{\partial\Omega} uv\nu_i \, dS.$$

*Proof.*2.) Approximate u and v by functions in $C^\infty(\bar{\Omega})$ according to A.4 and obtain that $uv \in H^{1,1}(\Omega)$ with $D_i(uv) = uD_i v + vD_i u$ and that $B(uv) = B(u)B(v)$ in $L^1(\partial\Omega)$. Thus 2.) is reduced to 1.).

1.) Approximate u by functions in $C^\infty(\bar{\Omega})$ according to A.4. Then use the classical Gaussian theorem and the continuity of B. □

Lemma C.10 *Let $u : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that $u_{|\Omega_+} \in H^{1,1}(\Omega_+)$ and $u_{|\Omega_-} \in H^{1,1}(\Omega_-)$, where*

$$\Omega_\pm := \{(y,h) \in \mathbb{R}^n \,|\, \pm(h - g(y)) > 0\}.$$

If $B_\pm$ are the boundary value operators w. r. t. $\Omega_\pm$ from C.8, then

$$u \in H^{1,1}(\mathbb{R}^n) \Leftrightarrow B_+u = B_-u.$$

Proof.$\Rightarrow$: For $s \in \mathbb{R}$ define $u_s(y) := u(y, g(y) + s)$. By (C.2),

$$\int_{\mathbb{R}^{n-1}} |u_\epsilon - u_{-\epsilon}| \, dx \quad \leq \quad \int_{\mathbb{R}^{n-1}} \int_{g(y)-\epsilon}^{g(y)+\epsilon} |D_{e_n} u(y,h)| \, dh \, dy$$

$$\leq \int_{\mathbb{R}^{n-1}} \int_{g(y)-\epsilon}^{g(y)+\epsilon} |Du(y,h)|\, dh\, dy \longrightarrow 0 \quad \text{for} \quad \epsilon \downarrow 0.$$

Thus by definition of B (see also the proof of C.8), $B_+u = B_-u$.

$\Leftarrow$: Let $\nu_\pm$ denote the outer normal to $\Omega_\pm$, then $\nu_- = -\nu_+$. Thus, by C.9 for $\zeta \in C_0^\infty(\mathbb{R}^n)$,

$$\begin{aligned}\int_{\mathbb{R}^n} (uD_i\zeta + \zeta D_i u)\, dx &= \int_{\Omega_+} (uD_i\zeta + \zeta D_i u)\, dx + \int_{\Omega_-} (uD_i\zeta + \zeta D_i u)\, dx \\ &= \int_{\partial\Omega_+} \zeta(B_+u)\nu_+\, dS + \int_{\partial\Omega_-} \zeta(B_-u)\nu_-\, dS \\ &= \int_{\text{graph}(g)} \zeta \underbrace{\left((B_+u)\nu_+ + (B_-u)\nu_-\right)}_{=0} dS.\end{aligned}$$

□

Let us now show that functions in $H_0^{1,p}(\Omega)$ have zero boundary values.

Lemma C.11 *For* $1 \leq p < \infty$,

$$H_0^{1,p}(\Omega) = \{u \in H^{1,p}(\Omega) | Bu = 0\}.$$

Proof. By definition, any function $u \in H_0^{1,p}(\Omega)$ can be approximated by $u_k \in C_0^\infty(\Omega)$. Therefore $0 = Bu_k \longrightarrow Bu$ in $L^p(\partial\Omega)$. Now let $u \in H^{1,p}(\Omega)$ such that $Bu = 0$. We also have $B(\eta^j u) = \eta^j B(u) = 0$ on $\partial\Omega$ (see proof of C.9). If we define

$$v_j(x) := \begin{cases} (\eta^j u)(x) & \text{, if } x \in Q^j \\ 0 & \text{, if } x \notin Q^j \end{cases}$$

for $j = 1, \ldots, m$, then $v_j \in H^{1,p}(\mathbb{R}^n)$, because of C.10, just like

$$v_{j\delta}(x) := v_j(x - \delta e_n^j),$$

for $\delta > 0$, and as δ tends to zero, $v_{j\delta} \longrightarrow v_j$ in $H^{1,p}(\mathbb{R}^n)$. Hence

$$u_\delta := \eta^0 u + \sum_{j=1}^m v_{j\delta} \longrightarrow u$$

in $H^{1,p}(\Omega)$ as $\delta \longrightarrow 0$. But u_δ has compact support in Ω and can therefore be approximated by functions in $C_0^\infty(\Omega)$ through convolution. A diagonal argument yields the lemma. □

C.2 $m = -1$

(compare with [1, Lemma 3.7ff.]) Consider $L_n^p := \underbrace{L^p \times \cdots \times L^p}_{n}$ with norm

$$\|f\|_{L_n^p}^* := \left(\sum_{j=1}^n \|f_j\|_p^p\right)^{\frac{1}{p}},$$

where $L^p = L^p(\Omega)$ for some suitable domain Ω (e. g. $\Omega = B(0,R)$). We want to show that any $L \in H^{-1,p}$ can be written as

$$L = D_i f^i, f^i \in L^p, i = 1\dots,n, \tag{C.3}$$

such that

$$\|L\|_{H^{-1,p}} \geq C\|f\|_p. \tag{C.4}$$

Lemma C.12 *Let $1 < p < \infty$. To any $L \in (L_n^{p'})'$ there corresponds an unique $v \in L_n^p$, such that for any $u \in L_n^{p'}$,*

$$Lu =< u_j, v^j > .$$

Moreover, $\|L\|_{(L_n^{p'})'} = \|v\|^*_{L_n^p}$. Thus $(L_n^{p'})' \cong L_n^p$.

*Proof.*For $1 \leq j \leq n$ and $w \in L^{p'}$, let $w_{(j)} = (0,\dots,0,w,0,\dots,0)$. Setting $L_j w := Lw_{(j)}$, we see that $L_j \in (L^{p'})'$. Hence, by Riesz, there exists $v_j \in L^p$, such that

$$Lw_j = L_j w =< w, v_j > \quad \forall w \in L^{p'}.$$

If $u \in L_n^{p'}$, then

$$Lu = L\left(\sum_{j=1}^n u_{j\,(j)}\right) = \sum_{j=1}^n Lu_{j\,(j)} =< u^j, v_j > .$$

By Hölder,

$$|Lu| \leq \|u^j\|_{p'}\|v_j\|_p \leq \|u\|^*_{L_n^{p'}}\|v\|^*_{L_n^p}$$

so that $\|L\|_{(L_n^{p'})'} \leq \|v\|^*_{L_n^p}$. We show that these norms are in fact equal as follows. Define

$$u_j(x) := \begin{cases} |v_j(x)|^{p-2}v_j(x) & \text{, if } v_j(x) \neq 0, \\ 0 & \text{, otherwise,} \end{cases}$$

then

$$|L(u_1,\dots,u_n)| = \left|\sum_{j=1}^n \int |v_j|^p\,dx\right| = \|v^p\|^*_{L_n^1} = \|u\|^*_{L_n^{p'}}\|v\|^*_{L_n^p}.$$

□

Proposition C.13 *Let $1 < p < \infty$. For any $L \in (H_0^{1,p'}, \|\cdot\|^*_{H_0^{1,p'}} := \|Du\|^*_{L_n^{p'}})'$ there exists an element $f \in L_n^p$, such that*

$$Lu =< D_i u, f^i > \quad \forall u \in H_0^{1,p'}$$

and

$$\|L\|_{-1,p} = \min_{\substack{f \in L_n^p \\ with (C.3)}} \|f\|^*_{L_n^p}. \tag{C.5}$$

Proof. Consider $P : H_0^{1,p'} \longrightarrow L_n^{p'}, u \longmapsto Du$. A linear functional L^* on $P(H_0^{1,p'})$ is defined as follows:

$$L^*(Pu) := Lu, u \in H_0^{1,p'}.$$

Since $\|Pu\|^*_{L_n^{p'}} = \|u\|^*_{H_0^{1,p'}}$, we have

$$L^* \in (P(H_0^{1,p'}))' \quad \text{and} \quad \|L^*\|_{(P(H_0^{1,p'}))'} = \|L\|_{H^{-1,p}}.$$

By the Hahn-Banach theorem there exists a norm preserving extension $\tilde{L}$ of L^* to all of $L_n^{p'}$ and by Lemma C.12 there exists $f \in L_n^p$, such that

$$\tilde{L}u =< f^i, u_i > .$$

Thus, for $u \in H_0^{1,p'}$, we obtain

$$Lu = L^*(Pu) = \tilde{L}(Pu) =< D_i u, f^i > .$$

Moreover,

$$\|L\|_{-1,p} = \|L^*\|_{P(H_0^{1,p'})'} = \|\tilde{L}\|_{(L_n^{p'})'} = \|f\|^*_{L_n^p}. \tag{C.6}$$

Now, any element $f \in L_n^p$, for which (C.3) holds for any $u \in H_0^{1,p'}$ corresponds to an extension L of L^* and will therefore have norm $\|f\|_{L_n^p}$ not less than $\|L\|_{-1,p}$. Combining this with (C.6), we obtain (C.5). □

Let now $L \in (H_0^{1,p'}, \|\cdot\|_{1,p'})'$, then by norm equivalence (cf. A.7, A.2),

$$L \in (H_0^{1,p'}, \|\cdot\|^*_{H_0^{1,p'}})'.$$

Therefore, there exists $f_0 \in L_n^p$, such that $L = D_i f_0^i$ and

$$\|L\|_{(H_0^{1,p'}, \|\cdot\|^*_{H_0^{1,p'}})'} = \|f_0\|^*_{L_n^p}.$$

Now, by norm equivalence,

$$\begin{aligned} \|L\|_{-1,p} &= \sup_{u \in H_0^{1,p'}} \frac{|Lu|}{\|u\|_{1,p'}} \geq \sup_{u \in H_0^{1,p'}} \frac{|Lu|}{C_1 \|u\|^*_{H_0^{1,p'}}} \\ &= C_1^{-1} \|L\|_{(H_0^{1,p'}, \|\cdot\|^*_{H_0^{1,p'}})'} = C_1^{-1} \|f_0\|^*_{L_n^p} \\ &\geq C_1^{-1} C_2 \|f_0\|_p, \end{aligned}$$

which shows (C.4).

Appendix D

SOBOLEV EMBEDDINGS

We now present one of the most important tools of the regularity examinations of this work. We often refered to the results of this section, which are, e. g., that functions in $H^{m,p}$ automatically have $C^{k,\alpha}$-versions, if $m - \frac{n}{p} \geq k + \alpha$.
Let $\Omega \subset \mathbb{R}^n$ be a domain. To spare time, the reader should only concentrate on the case $p < n$ in the following proposition, because we will develop even better results for the case $p > n$ in the rest of this section – in a total different manner. Perhaps the reader is interested in both proofs.

Proposition D.1 (cf. [19, Theorem 7.10], [3, Chapter8]) *Let $1 \leq p, q < \infty$ such that $-\frac{n}{q} = 1 - \frac{n}{p}$. If $u \in H^{1,p}(\mathbb{R}^n)$ then $u \in L^q(\mathbb{R}^n)$ and*

$$\|u\|_q \leq q\frac{n-1}{n}\|Du\|_p. \tag{D.1}$$

In particular, $H_0^{1,p}(\Omega) \subset L^{\frac{np}{n-p}}(\Omega)$, if $p < n$. For $p > n$ and bounded Ω, we have $H_0^{1,p}(\Omega) \subset C^0(\bar{\Omega})$ and

$$\sup_\Omega |u| \leq C|\Omega|^{\frac{1}{n}-\frac{1}{p}}\|Du\|_p.$$

*Proof.*By density it is enough to show the assertion for $C_0^\infty(\mathbb{R}^n)$-functions. For if $u \in H^{1,p}(\mathbb{R}^n)$ is the limit of $u_k \in C_0^\infty(\mathbb{R}^n), k \in \mathbb{N}$ w. r. t. $\|\cdot\|_{1,p}$, then $\|u_k\|_q \leq q\frac{n-1}{n}\|Du_k\|_p$ and $\|u_k - u_l\|_q \leq q\frac{n-1}{n}\|D(u_k - u_l)\|_p$, thus (u_k) is Cauchy in L^q, therefore $u_k \longrightarrow \tilde{u}$ in L^q for some $\tilde{u} \in L^q$ and $\|\tilde{u}\|_q \leq q\frac{n-1}{n}\|Du\|_p$. Since for bounded sets $D \subset \mathbb{R}^n, u_k \longrightarrow u$ and $u_k \longrightarrow \tilde{u}$ in L^1, it follows that $\tilde{u} = u$ a. e. in $\mathbb{R}^n$.
So let $u \in C_0^\infty(\mathbb{R}^n)$. Note in the following argumentation that $u \in C_0^1(\mathbb{R}^n)$ would be enough. At first, let us consider the case $p = 1$, i. e. $q = \frac{n}{n-1}$.
Since $u(x_1, \ldots, x_{i-1}, \xi, x_{i+1}, \ldots, x_n) = 0$ for large ξ, we have for $x \in \mathbb{R}^n$ and $i = 1, \ldots, n$ that

$$\begin{aligned} |u(x)| &= \left|\int_{x_i}^\infty D_i u(x_1, \ldots, x_{i-1}, \xi, x_{i+1}, \ldots, x_n)\, d\xi\right| \\ &\leq \int_{\mathbb{R}} |D_i u(x_1, \ldots, x_{i-1}, \xi, x_{i+1}, \ldots, x_n)|\, d\xi \\ &=: \int_{\mathbb{R}} |D_i u|\, d\xi_i \end{aligned}$$

Multiplication of these n inequalities yields

$$|u(x)|^{\frac{n}{n-1}} \le \prod_{i=1}^{n} \left(\int_{\mathbb{R}^n} |D_i u| \, d\xi_i \right)^{\frac{1}{n-1}} .$$

Integration w. r. t. x_1 leads to

$$\int_{\mathbb{R}} |u|^{\frac{n}{n-1}} \, d\xi_1 \le \left(\int_{\mathbb{R}} |D_1 u| \, d\xi_1 \right)^{\frac{1}{n-1}} \int_{\mathbb{R}} \prod_{i=2}^{n} \left(\int_{\mathbb{R}} |D_i u| \, d\xi_i \right)^{\frac{1}{n-1}} \, d\xi_1$$

and by Hölder,

$$\le \left(\int_{\mathbb{R}} |D_1 u| \, d\xi_1 \right)^{\frac{1}{n-1}} \prod_{i=2}^{n} \left(\int_{\mathbb{R}} \int_{\mathbb{R}} |D_i u| \, d\xi_1 \, d\xi_i \right)^{\frac{1}{n-1}} .$$

Now we integrate w. r. t. x_2 and obtain the desired estimate in the case $n = 2$, because for $n = 2$ we have

$$\int_{\mathbb{R}} |u|^2 \, d\xi_1 \le \left(\int_{\mathbb{R}} |D_1 u| \, d\xi_1 \right) \left(\int_{\mathbb{R}} \int_{\mathbb{R}} |D_2 u| \, d\xi_1 \, d\xi_2 \right)$$

and therefore

$$\|u\|_2^2 \le \left(\int_{\mathbb{R}} \int_{\mathbb{R}} |D_1 u| \, d\xi_1 \, d\xi_2 \right) \left(\int_{\mathbb{R}} \int_{\mathbb{R}} |D_2 u| \, d\xi_1 \, d\xi_2 \right) \le \|Du\|_1^2.$$

In the case $n \ge 3$, again by Hölder, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}} \int_{\mathbb{R}} |u|^{\frac{n}{n-1}} \, d\xi_1 \, d\xi_2 \\
&\quad \le \left(\int_{\mathbb{R}} \int_{\mathbb{R}} |D_2 u| \, d\xi_1 \, d\xi_2 \right)^{\frac{1}{n-1}} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |D_1 u| \, d\xi_1 \right)^{\frac{1}{n-1}} \\
&\qquad \prod_{i=3}^{n} \left(\int_{\mathbb{R}^2} |D_i u| \, d\xi_1 \, d\xi_i \right)^{\frac{1}{n-1}} \, d\xi_2 \\
&\quad \le \left(\int_{\mathbb{R}} \int_{\mathbb{R}} |D_2 u| \, d\xi_1 \, d\xi_2 \right)^{\frac{1}{n-1}} \left(\int_{\mathbb{R}} \int_{\mathbb{R}} |D_1 u| \, d\xi_1 \, d\xi_2 \right)^{\frac{1}{n-1}} \\
&\qquad \prod_{i=3}^{n} \left(\int_{\mathbb{R}^3} |D_i u| \, d\xi_1 \, d\xi_2 \, d\xi_i \right)^{\frac{1}{n-1}} .
\end{aligned}$$

Proceeding in this manner, we inductively obtain

$$\begin{aligned}
&\int_{\mathbb{R}^j} |u|^{\frac{n}{n-1}} \, d\xi_1 \dots d\xi_j \\
&\quad \le \prod_{i=1}^{j} \left(\int_{\mathbb{R}^j} |D_i u| \, d\xi_1 \dots d\xi_j \right)^{\frac{1}{n-1}} ,
\end{aligned}$$

thus for $j = n$,

$$\int_{\mathbb{R}^n} |u|^{\frac{n}{n-1}}\, d\xi \le \prod_{i=1}^{n} \left(\int_{\mathbb{R}^n} |D_i u|\, d\xi \right)^{\frac{1}{n-1}} \le \left(\int_{\mathbb{R}^n} |Du|\, dx \right)^{\frac{n}{n-1}},$$

i. e. the assertion $\|u\|_{\frac{n}{n-1}} \le \|Du\|_1$.

For $p > 1$ we want to apply the above case to $v := u^{\frac{q(n-1)}{n}}$. Since $u \in C_0^\infty(\mathbb{R}^n)$ and $\frac{q(n-1)}{n} > p > 1$, we have $v \in C_0^1(\mathbb{R}^n)$ and

$$|Dv| = \frac{q(n-1)}{n} |u|^{\frac{q(n-1)}{n}-1} |Du|.$$

Therefore by what we have shown above,

$$\begin{aligned}
\left(\int_{\mathbb{R}^n} |u|^q\, dx \right)^{\frac{n-1}{n}} &= \left(\int_{\mathbb{R}^n} |v|^{\frac{n}{n-1}}\, dx \right)^{\frac{n-1}{n}} \le \int_{\mathbb{R}^n} |Dv|\, dx \\
&= \frac{q(n-1)}{n} \int_{\mathbb{R}^n} |u|^{\frac{q(n-1)}{n}-1} |Du|\, dx \\
&\le \frac{q(n-1)}{n} \left(\int_{\mathbb{R}^n} |u|^{(\frac{q(n-1)}{n}-1)p'}\, dx \right)^{\frac{1}{p'}} \|Du\|_p \quad \text{(by Hölder)}
\end{aligned}$$

Now $\frac{n-1}{n} - \frac{1}{p'} = -\frac{1}{n} + \frac{1}{p} = \frac{1}{q}$ and thus $\left(\frac{q(n-1)}{n} - 1 \right) p' = q$. Hence,

$$\left(\int_{\mathbb{R}^n} |u|^q\, dx \right)^{\frac{1}{q}} \le \frac{q(n-1)}{n} \|Du\|_p.$$

For $p > n$, let us write $\tilde{u} = \frac{|u|}{\|Du\|_p}$ and assume that $|\Omega| = 1$. We then obtain for $\gamma > 1$,

$$\begin{aligned}
\|\tilde{u}^\gamma\|_{n'} &= \frac{1}{\|Du\|_p^\gamma} \||u|^\gamma\|_{n'} \\
&\le \gamma \frac{1}{\|Du\|_p^\gamma} \||u|^{\gamma-1}\|_{p'} \|Du\|_p \quad \text{(see (??))} \\
&= \gamma \frac{1}{\|Du\|_p^{\gamma-1}} \||u|^{\gamma-1}\|_{p'} \\
&= \gamma \|\tilde{u}^{\gamma-1}\|_{p'},
\end{aligned}$$

so that

$$\begin{aligned}
\|\tilde{u}\|_{\gamma n'} &= \|\tilde{u}^\gamma\|_{n'}^{\frac{1}{\gamma}} \le \gamma^{\frac{1}{\gamma}} \|\tilde{u}^{\gamma-1}\|_{p'}^{\frac{1}{\gamma}} \quad \text{(by (??))} \\
&= \gamma^{\frac{1}{\gamma}} \|\tilde{u}\|_{p'(\gamma-1)}^{\frac{\gamma-1}{\gamma}} \le \gamma^{\frac{1}{\gamma}} \|\tilde{u}\|_{p'\gamma}^{1-\frac{1}{\gamma}},
\end{aligned}$$

since $L^p(\Omega) \downarrow_p$.

Let us substitute for γ the values $\delta^\nu, \nu \in \mathbb{N}$, where $\delta = \frac{n'}{p'} > 1$. Then we obtain for any ν,

$$\|\tilde{u}\|_{n'\delta^\nu} \le \delta^{\nu\delta^{-\nu}} \|\tilde{u}\|_{p'\left(\frac{n'}{p'}\right)^\nu}^{1-\delta^{-\nu}} = \delta^{\nu\delta^{-\nu}} \|\tilde{u}\|_{n'\delta^{\nu-1}}^{1-\delta^{-\nu}}.$$

We assert that for any $k \in \mathbb{N} : \|\tilde{u}\|_{\delta^k} \leq \delta^{\sum_{\nu=1}^k \nu\delta^{-\nu}}$. In fact, for $k = 1$:

$$\|\tilde{u}\|_{\delta} \leq \|\tilde{u}\|_{\delta n'} \leq \delta^{\delta^{-1}} \|\tilde{u}\|_{n'}^{1-\delta^{-1}} \leq \delta^{\delta^{-1}},$$

since $\|\tilde{u}\|_{n'} = \frac{1}{\|Du\|_p} \||u|\|_{n'} \leq \frac{\|Du\|_1}{\|Du\|_p^{\gamma}} \leq 1$.
$k \longmapsto k+1$:

$$\begin{aligned}
\|\tilde{u}\|_{\delta^{k+1}} &\leq \|\tilde{u}\|_{n'\delta^{k+1}} \\
&\leq \delta^{(k+1)\delta^{-(k+1)}} \|\tilde{u}\|_{n'\delta^k}^{1-\delta^{-(k+1)}} \\
&\leq \delta^{(k+1)\delta^{-(k+1)}} \left(\delta^{\sum_{\nu=1}^k \nu\delta^{-\nu}}\right)^{1-\delta^{-(k+1)}} \\
&= \delta^{\sum_{\nu=1}^{k+1} \nu\delta^{-\nu}} \underbrace{\delta^{-\left(\delta^{\sum_{\nu=1}^k \nu\delta^{-\nu}}\right)\delta^{-(k+1)}}}_{\leq 1, \text{ since } -\left(\delta^{\sum_{\nu=1}^k \nu\delta^{-\nu}}\right)\delta^{-(k+1)} \leq 0} .
\end{aligned}$$

Moreover, $\delta^{\sum_{\nu=1}^k \nu\delta^{-\nu}} \uparrow_k$, since $\delta > 1$, and $\exists \sum_{\nu=1}^{\infty} \nu\delta^{-\nu} =: \chi$ by quotient criterium. Consequently, as $k \longrightarrow \infty$,

$$\sup_{\Omega} \tilde{u} \leq \chi$$

and hence

$$\sup_{\Omega} |u| \leq \chi \|Du\|_p.$$

To get rid of the restriction $|\Omega| = 1$, we consider the transformation $T : \Omega \longrightarrow \mathbb{R}^n, x \longmapsto \frac{x}{|\Omega|^{\frac{1}{n}}}$ for arbitrary bounded domain Ω. With integral transformation we obtain $|T(\Omega)| = |\det DT||\Omega| = 1$ and

$$\begin{aligned}
\sup_{\Omega} |u(x)| &= \sup_{y \in T(\Omega)} |u \circ T^{-1}(y)| \\
&\leq \chi \|D(u \circ T^{-1})\|_{p;T(\Omega)} \\
&= \chi |\Omega|^{\frac{1}{n} - \frac{1}{p}} \|Du\|_{p;\Omega}.
\end{aligned}$$

In order to transfer the estimates (D.1) to arbitrary $u \in H_0^{1,p}$, we let (u_m) be a sequence of $C_0^1(\Omega)$-functions tending to u in $H^{1,p}(\Omega)$. Applying the estimates (D.1) on the differences $u_{m_1} - u_{m_2}$, we see that (u_m) will be a Cauchy sequence in $L^{\frac{np}{n-p}}(\Omega)$ for $p < n$, and in $C^0(\bar{\Omega})$ for $p > n$. Consequently, the limit function u will lie in the desired spaces and satisfy the stated estimates. □

Remark D.2 *The first estimate in D.1 remains valid for any $u \in L^r(\mathbb{R}^n)$ such that $Du \in L^p(\mathbb{R}^n)$, where $1 \leq r \leq q$.*

*Proof.*Just like in the proof of D.1 it suffices to approximate u in the L^p-norm of the gradient and locally in L^1 by functions in $H^{1,p}$ with compact support. At first let us approximate $u \in L^r(\mathbb{R}^n)$ with $Du \in L^p(\mathbb{R}^n)$ by bounded functions. Therefore let

$$\psi_\delta(z) := \frac{z}{1+\delta\sqrt{1+z^2}} (\Rightarrow |\psi_\delta| \le \delta^{-1})$$

for $\delta > 0$. Then $u_\delta := \psi_\delta(u) \in L^\infty(\mathbb{R}^n) \cap L^r(\mathbb{R}^n)$, because $|u_\delta| \le |u|$, and since $u_\delta \longrightarrow u$ a. e. as $\delta \longrightarrow 0$, it follows that $u_\delta \longrightarrow u$ in $L^r(\mathbb{R}^n)$ by Lebesgue. Since $Du_\delta = \psi'_\delta(u)Du$ and

$$|\psi'_\delta(z)| = \left| \frac{1 + \frac{\delta}{\sqrt{1+z^2}}}{1 + 2\delta\sqrt{1+z^2} + \delta^2 + \delta^2 z^2} \right| \le 1 + \delta$$

and $\psi'_\delta(z) \longrightarrow 1$ as $\delta \longrightarrow 0$ for any z, it follows that $Du_\delta \longrightarrow Du$ in $L^p(\mathbb{R}^n)$.
If now $u \in L^\infty(\mathbb{R}^n) \cap L^r(\mathbb{R}^n)$, then $|u|^q \le C|u|^r$, where C depends on $\|u\|_\infty$. Thus, $u \in L^q(\mathbb{R}^n)$ and we can therefore assume that $r = q$. If $r = q$, then we approximate $u \in L^q(\mathbb{R}^n)$ by cutting off. Therefore define for $1 < R$

$$\varphi_R(x) := \varphi_R(|x|) := \begin{cases} 1 & , for \quad |x| \le R \\ \frac{\log R^2 - \log|x|}{\log R} & , for \ R \le |x| \le R^2 \\ 0 & , for R^2 \le |x| \ . \end{cases}$$

Then φ_R has compact support and $|\varphi'_R(t)| < 1$. Thus $\varphi_R \in C^{0,1}(\bar{B}_{2R^2}) \subset H^{1,\infty}(\bar{B}_{2R^2})$ by C.6, moreover $\varphi_R \in H^{1,s}(\bar{B}_{2R^2}) \forall s \ge 1$, because of compact support. Now define $u_R := u\varphi_R$ for $u \in L^q(\mathbb{R}^n)$. Since $q\frac{n-1}{n} > p$, we have $q > p$ and hence $u_R, Du_R = uD\varphi_R + \varphi_R Du \in L^p(\mathbb{R}^n)$, because of compact support, i. e. $u_R \in H^{1,p}(\mathbb{R}^n)$ and (D.1) is fulfilled for any $R > 1$.
By Lebesgue: $u_R \longrightarrow u$ in $L^q(\mathbb{R}^n)$ and $\varphi_R Du \longrightarrow Du$ in $L^p(\mathbb{R}^n)$. It remains to show that $uD\varphi_R \longrightarrow 0$ in $L^p(\mathbb{R}^n)$. Now, since $s := \frac{q}{p} > 1$, by Hölder

$$\int_{\mathbb{R}^n} |uD\varphi_R|^p \, dx \le \left(\int_{\mathbb{R}^n} |u|^q \, dx \right)^{\frac{1}{s}} \left(\int_{\mathbb{R}^n} |D\varphi_R|^{ps'} \, dx \right)^{\frac{1}{s'}} .$$

Moreover, $ps' = \frac{ps}{s-1} = \frac{q}{\frac{q}{p}-1} = n$ by assumption and

$$\begin{aligned} \int_{\mathbb{R}^n} |D\varphi_R|^n \, dx &= \int_{B_{R^2} \setminus B_R} \frac{1}{(|x| \log R)^n} \, dx = (\log R)^{-n} \int_{B_{R^2} \setminus B_R} |x|^{-n} \, dx \\ &= (\log R)^{-n} C(n) \log R = C(n)(\log R)^{1-n} \longrightarrow 0 \end{aligned}$$

as $R \longrightarrow \infty$ (remember that $n \ge 2$). □

By turning to dual spaces, we arrive at the following lemma. (My calculations returned another exponent for R in (iii) than in [9, Lemma 2.5], but we will not need it.)

Lemma D.3 **(i)** *Let $n' < r < \infty$ and $R > 0$. Then $L^{\frac{rn}{r+n}}(B_R) \subset H^{-1,r}(B_R)$, and the embedding operator is bounded. In addition, there exists N independent of R, such that*

$$\|u\|_{-1,r;B_R} \le N \|u\|_{\frac{rn}{r+n};B_R} \tag{D.2}$$

for any $u \in L^{\frac{rn}{r+n}}(B_R)$ and $R > 0$.

(ii) *Let $1 < r < n'$ and $R > 0$. Then $L^1(B_R) \subset H^{-1,r}(B_R)$ and the embedding operator is bounded. In addition, there exists N independent of R, such that*

$$\|u\|_{-1,r;B_R} \le NR^{1-\frac{n}{r'}}\|u\|_{1;B_R} \tag{D.3}$$

for any $u \in L^1(B_R)$ and $R > 0$.

(iii) *Let $r = n', s > 1, R > 0$. Then $L^s(B_R) \subset H^{-1,r}(B_R)$. In addition, there exists N independent of R, such that*

$$\|u\|_{-1,r;B_R} \le \begin{cases} N\|u\|_{s;B_R}R^{\frac{n(s+1)}{s}} & \text{for } s \le n, \\ N\|u\|_{s;B_R}R^{\frac{n-2s}{s}} & \text{for } s > n. \end{cases} \tag{D.4}$$

for any $u \in L^s(B_R)$ and $R > 0$.

Proof.(i) By D.1 we know that $H_0^{1,r} \subset L^{\frac{nr}{n-r}}$, if $1 \le r < n$. Thus, fractions and duality considerations yield

$$L^{\frac{nr'}{n+r'}} = L^{\frac{nr}{nr-n+r}} = \left(L^{\frac{nr}{n-r}}\right)' \subset \left(H_0^{1,r}\right)' = H^{-1,r'} \quad \forall 1 \le r < n.$$

Since $r \ge 1$, we have $r' \ge 1$; moreover $\frac{r'}{r'-1} < n \Leftrightarrow r' > n'$. Therefore, $L^{\frac{nr}{n+r}} \subset H^{-1,r} \quad \forall n' < r < \infty$. By D.1, we know for the embedding operator $I : H_0^{1,r} \longrightarrow L^{\frac{nr}{n-r}}$ that

$$\|Iu\|_{\frac{nr}{n-r}} \le N(n,r)\|u\|_{1,r}.$$

Since $\|I\| = \|I'\|$, if I' denotes the dual operator of I, (D.2) follows.
(ii) Let $u \in L^1(B_R)$. Define

$$T_u g := \int_{B_R} gu\,dx, g \in C_0^\infty(B_R),$$

then T_u can be extended to a bounded linear functional on $H_0^{1,r'}(B_R)$ with $\|T_u\| \le \|u\|_1$. Relabeling the extension by u, by norm equivalence and D.1 ($r' > n$!) we obtain

$$\begin{aligned} \|u\|_{-1,r;B_R} &= \sup_{g \in H_0^{1,r'}(B_R)} \frac{|T_u g|}{\|g\|_{1,r';B_R}} \le C_1 \sup_{g \in H_0^{1,r'}(B_R)} \frac{|T_u g|}{\|Dg\|_{r';B_R}} \\ &\le C_1C_2|B_R|^{\frac{1}{n}-\frac{1}{r'}} \sup_{g \in H_0^{1,r'}(B_R)} \frac{|T_u g|}{\|g\|_\infty} \le C_1C_2\omega_n R^{1-\frac{n}{r'}}\|u\|_{1;B_R}. \end{aligned}$$

(iii) By D.1, $H_0^{1,n}(B_R) \subset L^{s'}(B_R)$ for any $s' \in [1,\infty)$, thus $L^s(B_R) \subset H^{-1,n'}$ for any $s > 1$. By Hölder and again by D.1, the following embedding operators are continuous, if $1 \le t < n$ and $s' \le \frac{nt}{n-t}$:

- $I_1 : H_0^{1,n}(B_R) \longrightarrow H_0^{1,t}(B_R)$,
- $I_2 : H_0^{1,t}(B_R) \longrightarrow L^{\frac{nt}{n-t}}(B_R)$,

- $I_3 : L^{\frac{nt}{n-t}}(B_R) \longrightarrow L^{s'}(B_R)$.

Let $s > 1$ be fixed.Now $\frac{nt}{n-t} \geq s' \Leftrightarrow t \geq \frac{sn}{sn-n+s}$ and $\frac{sn}{sn-n+s} \geq 1 \Leftrightarrow s \leq n$. Choose

$$t_s := \begin{cases} \frac{sn}{sn-n+s}, & \text{for } s \leq n, \\ 2, & \text{for } s > n, \end{cases}$$

then

$$L^s = \left(L^{s'}\right)' \subset \left(L^{\frac{nt_s}{n-t_s}}\right)' = L^{\frac{nt_s'}{n+t_s'}} \subset H^{-1,t_s'} \subset H^{-1,n'}.$$

Again by Hölder and D.1, we have

- $\|I_1 v\|_{1,t_s} \leq |B_R|^{\frac{1}{t_s}-\frac{1}{n}} \|v\|_{1,n} = \omega_n^{\frac{1}{t_s}-\frac{1}{n}} R^{\frac{n}{t_s}-1} \|v\|_{1,n}$,
- $\|I_2 v\|_{\frac{nt_s}{n-t_s}} \leq C(n,t_s)\|v\|_{1,t_s}$,
- $\|I_3 v\|_{s'} \leq |B_R|^{\frac{n-t_s}{nt_s}-\frac{1}{s'}} \|v\|_{\frac{nt_s}{n-t_s}} = \omega_n^{\frac{n-t_s}{nt_s}-\frac{1}{s'}} R^{\frac{n-t_s}{t_s}-\frac{n}{s'}} \|v\|_{\frac{nt_s}{n-t_s}}$.

Consequently, for $I := I_3 \circ I_2 \circ I_1$,

$$\begin{aligned} \|Iu\|_{s'} &\leq \omega_n^{\frac{n-t_s}{nt_s}-\frac{1}{s'}} R^{\frac{n-t_s}{t_s}-\frac{n}{s'}} C(n,t_s) \omega_n^{\frac{1}{t_s}-\frac{1}{n}} R^{\frac{n}{t_s}-1} \|u\|_{1,n} \\ &= \omega_n^{\frac{n-t_s}{nt_s}-\frac{1}{s'}+\frac{1}{t_s}-\frac{1}{n}} C(n,t_s) R^{\frac{2(n-t_s)}{t_s}-\frac{n}{s'}} \|u\|_{1,n}. \end{aligned}$$

Since $\|I'\| = \|I\|$, it follows that

$$\|u\|_{-1,n'} \leq NR^{\frac{2(n-t_s)}{t_s}-\frac{n}{s'}} \|u\|_s \leq \begin{cases} NR^{\frac{n(s+1)}{s}} \|u\|_s & \text{for } s \leq n, \\ NR^{\frac{n-2s}{s}} \|u\|_s & \text{for } s > n. \end{cases}$$

□

The rest of this section is based on [3, Chapter 8].

Proposition D.4 (Embeddings within Hölder-spaces) *Let $\Omega \subset \mathbb{R}^n$ be open and bounded an let $k_1, k_2 \geq 0, 0 \leq \alpha_1, \alpha_2 \leq 1$ such that $k_1 + \alpha_1 > k_2 + \alpha_2$. Then the embedding $J : C^{k_1,\alpha_1}(\bar{\Omega}) \longrightarrow C^{k_2,\alpha_2}(\bar{\Omega})$ is compact. In the case $k_1 > 0$ we assume that Ω is a C^∞-domain.*

*Proof.*Let (f_i) be a bounded sequence in $C^{k_1,\alpha_1}(\bar{\Omega})$. We have to show that some subsequence converges in $C^{k_2,\alpha_2}(\bar{\Omega})$. At first, let $k_2 = k_1 = 0$, thus $0 \leq \alpha_2 < \alpha_1 \leq 1$. By the Arzela-Ascoli theorem there exists $f \in C(\bar{\Omega})$ such that for some subsequence $f_i \longrightarrow f$ uniformly in $\bar{\Omega}$.
For $|y - x| \leq \delta$ we have

$$\begin{aligned} \frac{|(f-f_i)(y) - (f-f_i)(x)|}{|y-x|^{\alpha_2}} &= \lim_{j \longrightarrow \infty} \frac{|(f_j-f_i)(y) - (f_j-f_i)(x)|}{|y-x|^{\alpha_2}} \\ &\leq \delta^{\alpha_1-\alpha_2} \sup_j \|f_j - f_i\|_{C^{0,\alpha_1}(\bar{\Omega})}, \end{aligned}$$

and for $|y-x| \geq \delta$

$$\frac{|(f-f_i)(y)-(f-f_i)(x)|}{|y-x|^{\alpha_2}} \leq 2\delta^{-\alpha_2}\|f-f_i\|_{C(\bar{\Omega})},$$

thus altogether with some constant c

$$\sup_{x,y\in\bar{\Omega}} \frac{|(f-f_i)(y)-(f-f_i)(x)|}{|y-x|^{\alpha_2}} \leq \underbrace{c\delta^{\alpha_1-\alpha_2}}_{\longrightarrow 0,\text{ as }\delta\longrightarrow 0} + 2\delta^{-\alpha_2} \underbrace{\|f-f_i\|_{C(\bar{\Omega})}}_{\longrightarrow 0 \text{ as } i\longrightarrow\infty},$$

i. e. the Hölder constant of $f-f_i$ tends to zero as i grows to infinity.
From now on we assume Ω to be a C^∞-domain. We now show that the embedding $C^1(\bar{\Omega}) \longrightarrow C^{0,1}(\bar{\Omega})$ is well-defined and continuous. As is shown in Lemma C.5, for fixed $x_0, x_1 \in \Omega$ there exists $\gamma \in C^\infty([0,1];\Omega)$ such that $\gamma(0)=x_0, \gamma(1)=x_1$ and $\int_0^1 |\gamma'(t)|\,dt \leq C|x_1-x_0|$, where $C=C(\Omega)$.
For $f \in C^1(\bar{\Omega})$ we obtain

$$\begin{aligned} |f(x_1)-f(x_0)| &= |\int_0^1 (f\circ\gamma)'(t)\,dt| \leq \|Df\|_\infty \int_0^1 |\gamma'(t)|\,dt \\ &\leq C\|f\|_{C^1(\bar{\Omega})}|x_1-x_0|, \end{aligned}$$

i. e. the lipschitz constant is estimated by the C^1-norm.
We now consider the case $k_2 = k_1 \geq 1$, thus again $0 \leq \alpha_2 < \alpha_1 \leq 1$. Then $(D^s f_i)$ are bounded sequences in $C^1(\bar{\Omega})$ for $|s| < k_1$, thus in $C^{0,1}(\bar{\Omega})$ and for $|s| = k_1$ they are bounded in $C^{0,\alpha_1}(\bar{\Omega})$. By what has been shown above we can successively (in s) choose subsequences such that finally, for a subsequence (renamed as (f_i)),

$$D^s f_i \longrightarrow g_s \quad \text{in} \quad C^{0,\alpha_2}(\bar{\Omega}) \quad \forall |s| \leq k_1$$

for some $g_s \in C^{0,\alpha_2}(\bar{\Omega})$. In particular, (f_i) is Cauchy in $C^{k_1}(\bar{\Omega})$, which implies that $f := g_0 \in C^{k_1}(\bar{\Omega})$ with $D^s f = g_s$, i. e. $f_i \longrightarrow f$ in $C^{k_1,\alpha_2}(\bar{\Omega})$.
Let now $k_1 > k_2$. By the above considerations, we have that in the case $\alpha_2 < 1$ the embedding $C^{k_2,1}(\bar{\Omega}) \longrightarrow C^{k_2,\alpha_2}(\bar{\Omega})$ is compact and in the case $\alpha_1 > 0$ so is the embedding $C^{k_1,\alpha_1}(\bar{\Omega}) \longrightarrow C^{k_1}(\bar{\Omega})$. Moreover the embedding $C^{k_1}(\bar{\Omega}) \longrightarrow C^{k_1-1,1}(\bar{\Omega})$ is continuous, which follows from the continuity of the embedding $C^1(\bar{\Omega}) \longrightarrow C^{0,1}(\bar{\Omega})$. So it remains to consider $C^{k_1-1,1}(\bar{\Omega}) \longrightarrow C^{k_2,1}(\bar{\Omega})$, which is the identity in the case $k_2 = k_1 - 1$. In the case $k_2 < k_1 - 1$ (e. g. $\alpha_1 = 0, \alpha_2 = 1$) $C^{k_1-1,1}(\bar{\Omega}) \longrightarrow C^{k_2+1,1}(\bar{\Omega})$ and $C^{k_2+1}(\bar{\Omega}) \longrightarrow C^{k_2,1}(\bar{\Omega})$ are continuous and $C^{k_2+1,1}(\bar{\Omega}) \longrightarrow C^{k_2+1}(\bar{\Omega})$ is compact. Hence the concatenation is compact. □

Proposition D.5 (Embeddings within Sobolev-spaces) *Let $\Omega \subset \mathbb{R}^n$ be open, $m_1 > m_2 \geq 0, 1 \leq p_1 < \infty$ and $1 \leq p_2 < \infty$. Then*

1. *If $m_1 - \frac{n}{p_1} = m_2 - \frac{n}{p_2}$, the embedding $J : H_0^{m_1,p_1}(\Omega) \longrightarrow H_0^{m_2,p_2}(\Omega)$ exists and is continuous, i. e.*

$$\|u\|_{m_2,p_2;\Omega} \leq C(n,m_1,p_1)\|u\|_{m_1,p_1;\Omega}.$$

2. *If Ω is bounded and $m_1 - \frac{n}{p_1} > m_2 - \frac{n}{p_2}$, the embedding $J : H_0^{m_1,p_1}(\Omega) \longrightarrow H_0^{m_2,p_2}(\Omega)$ exists and is compact.*

3. *If Ω is a C^∞-domain, each of both assertions above is valid for the spaces $H^{m_i,p_i}(\Omega)$. C will additionally depend on Ω.*

Remark D.6 *In 3.) a $C^{0,1}$-domain would be enough, but we will use Proposition A.4 later in the proof and may have Ω =ball in mind anyway.*

*Proof.*1.)Let $m_2 = m_1 - 1$. By iteration we then obtain assertion 1.) also for smaller m_2. If $u \in H_0^{m_1,p_1}(\Omega)$, then $u \in H_{(0)}^{m_1,p_1}(\mathbb{R}^n)$ by extending u by zero outside Ω. Thus, $D^s u \in H_0^{1,p_1}(\mathbb{R}^n)$ for each $|s| \leq m_1 - 1 = m_2$. Hence, by D.1 we obtain $D^s u \in L^{p_2}(\mathbb{R}^n)$ with

$$\|D^s u\|_{p_2;\Omega} \leq C(p_2, n)\|DD^s u\|_{p_1;\Omega} \leq C(p_2, n)\|u\|_{m_1,p_1;\Omega}.$$

2.)We restrict ourselves to the case $m_1 = 1, m_2 = 0, p := p_1, q := p_2$. The general version follows analoguously to 1.) and by using that for continuous linear operators $T_1 \circ T_2$ is compact, if T_1 or T_2 is compact. So we have to show, that any bounded sequence $(u_k) \subset H_0^{1,p}(\Omega)$ contains a subsequence, which converges in $L^q(\Omega)$. We will prove this with help of D.1 and C.1.
Since $q < \infty$, we can choose $1 \leq p_0 \leq p$ and $q < q_0 < \infty$ such that

$$1 - \frac{n}{p} \geq 1 - \frac{n}{p_0} = -\frac{n}{q_0} > -\frac{n}{q}.$$

Since Ω is bounded, (u_k) is also bounded in $H_0^{1,p_0}(\Omega)$ and therefore in $L^{q_0}(\Omega)$ by D.1, thus bounded in $L^{q_1}(\Omega)$ for any $q_1 \leq q_0$. Choose $q_1 = q_0$. Since $L^{q_1}(\Omega)$ is reflexive, (u_k) contains an L^{q_1}-weakly converging subsequence, i. e.

$$\int g u_{k_i}\, dx \longrightarrow \int g u\, dx \quad \forall g \in (L^{q_1})'$$

for some $u \in L^{q_1}(\Omega)$; in particular,

$$\int g u_{k_i}\, dx \longrightarrow \int g u\, dx \quad \forall g \in (L^1)'.$$

Thus (u_k) contains an L^1-weakly converging subsequence, renamed as (u_k). Since $(u_k) \subset H_0^{1,1}(\Omega)$ bounded, $u_k \longrightarrow u$ strongly in $L^1(\Omega)$ by C.2. Now $1 \leq q < q_1$, therefore for any $\epsilon > 0$ we have an elementary inequality of the form

$$a^q \leq \epsilon a^{q_1} + C_\epsilon a \quad \forall a \geq 0, C_\epsilon = C_\epsilon(\epsilon, q, q_1)$$

(in fact, define $f : [0, \infty) \longrightarrow \mathbb{R}, f(a) := a^{q-1} - \epsilon a^{q_1 - 1}$, then $f(0) = 0$ and

$$\left(f'(a) < 0 \Leftrightarrow a > \left(\frac{q-1}{\epsilon(q_1 - 1)} \right)^{\frac{1}{q_1 - q}} \right)$$

and thus f is bounded from above).
Hence,

$$\int_\Omega |u_k - u_l|^q \le \epsilon \underbrace{\int_\Omega |u_k - u_l|^{q_1}}_{\text{bounded in } k,l} + C_\epsilon \underbrace{\int_\Omega |u_k - u_l|}_{\longrightarrow 0 \text{ as } k,l\longrightarrow\infty},$$

which implies that (u_k) is Cauchy and therefore converging in $L^q(\Omega)$.

3.)Let $E : H^{m_1,p_1}(\Omega) \longrightarrow H_0^{m_1,p_1}(B_1(\Omega))$ be as in Proposition A.4. Then in the situation of 1.) we have for $u \in H^{m_1,p_1}(\Omega)$ that $Eu \in H_0^{m_2,p_2}(\Omega)$ by 1.) and therefore

$$\begin{aligned} \|u\|_{m_2,p_2;\Omega} &\le \|Eu\|_{m_2,p_2;B_1(\Omega)} \le C\|Eu\|_{m_1,p_1;B_1(\Omega)} \quad \text{(again by 1.))} \\ &\le C'\|u\|_{m_1,p_1;\Omega} \quad \text{(by Proposition A.4)} \quad . \end{aligned}$$

In the situation of 2.) (Eu_k) is bounded in $H_0^{m_1,p_1}(B_1(\Omega))$, whenever (u_k) is bounded in $H^{m_1,p_1}(\Omega)$. Thus, by 2.) a subsequence of (Eu_k) is converging strongly in $H_0^{m_2,p_2}(B_1(\Omega))$, thus the corresponding subsequence of (u_k) is converging strongly in $H^{m_2,p_2}(\Omega)$. □

Before we come to the embeddings from Sobolev-spaces into Hölder-spaces, we need some preparations.

Proposition D.7 *Let $\Omega \subset \mathbb{R}^n$ be open and bounded, $1 < p < \infty$ such that $1 - \frac{n}{p} > 0$. Then for any $u \in H_0^{1,p}(\Omega)$, we have $u \in L^\infty(\Omega)$ and*

$$\|u\|_{L^\infty(\Omega)} \le C(n, p, \textit{diam }\Omega)\|Du\|_{p;\Omega}.$$

*Proof.*It suffices to consider $u \in C_0^\infty(\Omega)$. Let R :=diam Ω, i. e. $\Omega \subset B_R(x_0) \forall x_0 \in \Omega$. Then for any $\xi \in \partial B_1(0)$

$$\begin{aligned} |u(x_0)| &= \left| \int_0^R \frac{d}{dr}(u(x_0 + r\xi))\, dr \right| \\ &\le \int_0^R |D_\xi u(x_0 + r\xi)|\, dr \\ &= \int_0^R | < \xi, Du(x_0 + r\xi) > |\, dr \\ &\le \int_0^R |Du(x_0 + r\xi)|\, dr. \end{aligned}$$

Integrating this inequality w. r. t. ξ, dS, we obtain

$$\begin{aligned} dS(\partial B_1(0))|u(x_0)| &\leq \int_0^R \int_{\partial B_1(0)} |Du(x_0 + r\xi)| \, dS(\xi) \, dr \\ &= \int_{B_R(x_0)} \frac{|Du(x)|}{|x - x_0|^{n-1}} \, dx \end{aligned}$$

(by [17, Section 14, Satz 8] applied to $|Du(\cdot)||\cdot|^{1-n}$ and translation invariance)

$$\leq \left(\int_{B_R(x_0)} \frac{1}{|x - x_0|^{p'(n-1)}} \, dx \right)^{\frac{1}{p'}} \|Du\|_{p;\Omega}$$

by Hölder. The first factor is independent of x_0 and finite, if $p'(n-1) < n$, i. e. $p' < n'$, which is equivalent to $p > n$. □

Proposition D.8 (Morrey) *Let $\Omega \subset \mathbb{R}^n$ open, $0 < \alpha \leq 1$ and $u \in H_0^{1,1}(\Omega)$ such that*

$$\int_{B_r(x_0) \cap \Omega} |Du| \, dx \leq M r^{n-1+\alpha}$$

for any $x_0 \in \Omega$ and $r > 0$. Then for a. e. $x_1 \neq x_2 \in \bar{\Omega}$,

$$\frac{|u(x_1) - u(x_2)|}{|x_1 - x_2|^\alpha} \leq C(n, \alpha) M.$$

Proof. We can assume $\Omega = \mathbb{R}^n$, since u can be extended to all of $\mathbb{R}^n$ by 0. For any ball $B_r(x_0), x_0 \in \mathbb{R}^n$, we have

$$\int_{B_r(x_0)} |Du| \, dx \leq \int_{B_{2r}(x_1) \cap \Omega} |Du| \, dx \leq M(2r)^{n-1+\alpha},$$

if $x_1 \in B_r(x_0) \cap \Omega$, and

$$\int_{B_r(x_0)} |Du| \, dx = 0,$$

if $B_r(x_0) \cap \Omega = \emptyset$. At first, we prove the estimate of the Hölder-constant for $u \in C^\infty(\mathbb{R}^n)$. For $x_1, x_2 \in \mathbb{R}^n$ let $x_0 := \frac{x_1 + x_2}{2}$ and $\rho := \frac{|x_1 - x_2|}{2}$. We have

$$\begin{aligned} \omega_n \rho^n |u(x_1) - u(x_2)| &= \int_{B_\rho(x_0)} |u(x_1) - u(x_2)| \, dx \\ &\leq \int_{B_\rho(x_0)} |u(x_1) - u(x)| \, dx + \int_{B_\rho(x_0)} |u(x_2) - u(x)| \, dx. \end{aligned}$$

By symmetry, it remains to estimate the first integral. Now for $x \in B_\rho(x_0)$

$$\begin{aligned} |u(x) - u(x_1)| &= \left| \int_0^1 \frac{d}{dt}(u(x_1 + t(x - x_1))) \, dt \right| \\ &= \left| \int_0^1 D_{x - x_1} u(x_1 + t(x - x_1)) \, dt \right| \end{aligned}$$

$$
\begin{aligned}
&= \left| \int_0^1 < x - x_1, Du(x_1 + t(x - x_1)) > \, dt \right| \\
&\leq |x - x_1| \int_0^1 |Du(x_1 + t(x - x_1))| \, dt.
\end{aligned}
$$

Since $2\rho = |x_2 - x_1| = 2|x_0 - x_1|$ and $|x - x_0| \leq \rho$, it follows that $|x - x_1| = |x - x_0 + x_0 - x_1| \leq 2\rho$. Hence, integrating w. r. t. x,

$$
\begin{aligned}
\int_{B_\rho(x_0)} |u(x) - u(x_1)| \, dx &\leq 2\rho \int_0^1 \int_{B_\rho(x_0)} |Du(x_1 + t(x - x_1))| \, dx \, dt \\
&= 2\rho \int_0^1 t^{-n} \int_{B_{t\rho}(x_1 + t(x_0 - x_1))} |Du(y)| \, dy \, dt \\
&\quad \text{(integral transformation } y(x) := x_1 + t(x - x_1)) \\
&\leq 2\rho \int_0^1 t^{-n} M (t\rho)^{n-1+\alpha} \, dt \\
&= 2\rho \int_0^1 t^{\alpha - 1} M \rho^{n-1+\alpha} \, dt \\
&= \frac{2M}{\alpha} \rho^{n+\alpha}.
\end{aligned}
$$

Now let $\varphi \in C_0^\infty(B_1(0))$ be a mollifier with corresponding Dirac-sequence (φ_ϵ) and $u \in H^{1,1}(\mathbb{R}^n)$. The functions $u_\epsilon := u * \varphi_\epsilon \in C^\infty(\mathbb{R}^n)$ fulfill the assumptions of this proposition, because

$$
\begin{aligned}
\int_{B_r(x_0)} |Du_\epsilon(x)| \, dx &= \int_{B_r(x_0)} \sum_{i=1}^n \left| \int_{\mathbb{R}^n} D_i u(x - y) \varphi_\epsilon(y) \, dy \right| \, dx \\
&\leq \int_{B_r(x_0)} \sum_{i=1}^n \int_{\mathbb{R}^n} |D_i u(x - y)| |\varphi_\epsilon(y)| \, dy \, dx \\
&\leq n \int_{\mathbb{R}^n} \int_{B_r(x_0)} |Du(x - y)| \, dx \varphi_\epsilon(y) \, dy \\
&\leq n M r^{n-1+\alpha} \int_{\mathbb{R}^n} \varphi_\epsilon(y) \, dy \\
&= n M r^{n-1+\alpha}.
\end{aligned}
$$

Moreover, we have a. e. convergence of some subsequence of (φ_ϵ) towards u as ϵ tends to 0. Altogether: for $u \in C^\infty(\mathbb{R}^n)$ we have

$$
|u(x_1) - u(x_2)| \leq 2 \frac{2M}{\alpha \omega_n} \rho^\alpha \Rightarrow \frac{|u(x_1) - u(x_2)|}{|x_1 - x_2|^\alpha} \leq \frac{4M}{2^\alpha \alpha \omega_n}.
$$

For $u \in H^{1,1}(\mathbb{R}^n)$ we have

$$
\frac{|u_\epsilon(x_1) - u_\epsilon(x_2)|}{|x_1 - x_2|^\alpha} \leq \frac{4M}{2^\alpha \alpha \omega_n}
$$

and because of $u_\epsilon \longrightarrow u$ a. e. for a subsequence, the assertion follows. $\square$

Remark D.9 *Let Ω be bounded and $u \in H_0^{1,p}(\Omega)$ such that $1 - \frac{n}{p} > 0$. Then with $\alpha := 1 - \frac{n}{p}$*

$$\begin{aligned}\int_{B_r(x_0)\cap\Omega} |Du| &\leq \omega_n^{\frac{1}{p'}} r^{\frac{n}{p'}} \|Du\|_{p;\Omega} \\ &=: Mr^{n-1+\alpha}\end{aligned}$$

and D.8 can be applied.

Proposition D.10 (Embeddings of Sobolev-spaces into Hölder-spaces) *Let $\Omega \subset \mathbb{R}^n$ be open and bounded, $m \geq 1, 1 \leq p < \infty, k \geq 0, 0 \leq \alpha \leq 1$. Then*

1. *If $m - \frac{n}{p} = k + \alpha, 0 < \alpha < 1$, then the embedding $J : H_0^{m,p}(\Omega) \longrightarrow C^{k,\alpha}(\bar{\Omega})$ exists and is continuous. I. e. for $u \in H_0^{m,p}(\Omega)$ there exists a unique continuous function, which is equal to u a. e. (and again denoted by u), such that*
$$|u|_{k,\alpha;\Omega} \leq C(\Omega, n, m, p, k, \alpha)\|u\|_{m,p;\Omega}.$$
2. *If $m - \frac{n}{p} > k + \alpha$, the embedding $J : H_0^{m,p}(\Omega) \longrightarrow C^{k,\alpha}(\bar{\Omega})$ exists and is compact.*
3. *If Ω is a C^∞-domain, then assertions 1.) and 2.) remain valid for $H^{m,p}(\Omega)$ instead of $H_0^{m,p}(\Omega)$.*

Remark D.11 *In 3.) a $C^{0,1}$-domain would be enough, but we will use Proposition A.4 later in the proof and may have $\Omega =$ball in mind anyway.*

*Proof.*For the proof of D.10, part 1.) we can assume $k = 0$. Otherwise we apply the following consideration for $|s| \leq k$ to each function $D^s u \in H_0^{m-k,p}(\Omega)$. Next, we are going to reduce the proof to the case $m = 1$:
If $m > 1$, then $D_i u \in H_0^{m-1,p}(\Omega)$ for $i = 1, \ldots, n$. Define $q := \frac{n}{1-\alpha}(> n)$, then

$$(m-1) - \frac{n}{p} = \alpha - 1 = -\frac{n}{q}.$$

By D.5, $D_i u \in L^q(\Omega)$ and

$$\|D_i u\|_{q;\Omega} \leq C(n, m, p)\|D_i u\|_{m-1,p;\Omega}. \tag{D.5}$$

Let us now show that $u \in H_0^{1,q}(\Omega)$: for that, we first show that $u \in L^q(\Omega)$. Define

$$r := \frac{n}{1 + \frac{n}{p} - (m-1)} \left(= \begin{cases} \frac{n}{1+\frac{n}{q}} & < \; n \\ \frac{n}{2-\alpha} & > \; 1 \end{cases} \right),$$

then $-\frac{n}{q} = 1 - \frac{n}{r} = (m-1) - \frac{n}{p}$ and $1 \leq r < n < q$. We already know that $D_i u \in L^q(\Omega)$ and $u \in L^p(\Omega)$. Since $|\Omega| < \infty$ and $r < q$, we have $D_i u \in L^r(\Omega)$. Analogously, $u \in L^1(\Omega)$. Now apply D.2 on D.1 with r instead of p, and 1 instead of r, respectively, to obtain $u \in L^q(\Omega)$ and (D.6). To show that $u \in H_0^{1,q}(\Omega)$, we must be able to approximate u by C_0^∞-functions. Since

$u \in H_0^{m,p}(\Omega)$, there exist $u_k \in C_0^\infty(\Omega), k \in \mathbb{N}$ such that $\|u_k - u\|_{m,p;\Omega} \longrightarrow 0$. Now,

$$\begin{aligned}\|u\|_{q;\Omega} &\leq q\frac{n-1}{n}\|Du\|_{r;\Omega} \\ &\leq q\frac{n-1}{n}|\Omega|^{\frac{1}{r}-\frac{1}{q}}\|Du\|_{q;\Omega} \quad (r < q).\end{aligned} \tag{D.6}$$

Therefore, by (D.5),

$$\|u_k - u\|_{q;\Omega} \leq C'\|D(u_k - u)\|_{q;\Omega} \longrightarrow 0.$$

Since $1 - \frac{n}{q} = \alpha$, we have reduced the proof to the case $m = 1$. For $m = 1$ Proposition D.10.1 follows from D.7 and D.8 (see D.9).

2.) Choose $\tilde{m} \leq m$ and $1 < \tilde{p} < \infty, \tilde{k} \geq 0, 0 < \tilde{\alpha} < 1$, such that

$$m - \frac{n}{p} \geq \tilde{m} - \frac{n}{\tilde{p}} = \tilde{k} + \tilde{\alpha} > k + \alpha$$

($\tilde{m} = m, \tilde{p} = p$ possible, if $\frac{n}{p} \notin \mathbb{N}$). If we choose R such that $\Omega \subset B_R(0)$, we can extend functions in $H_0^{m,p}(\Omega)$ to $H_0^{m,p}(B_R(0))$ by setting them 0 in $B_R(0)\backslash\Omega$. The embedding $H_0^{m,p}(B_R(0)) \longrightarrow H_0^{\tilde{m},\tilde{p}}(B_R(0))$ is continuous in the case $\tilde{m} = m$ by Hölder and in the case $\tilde{m} < m$ by D.5. Then the embedding into $C^{\tilde{k},\tilde{\alpha}}(\bar{B}_R(0))$ is continuous by 1.) and $C^{\tilde{k},\tilde{\alpha}}(\bar{B}_R(0)) \longrightarrow C^{k,\alpha}(\bar{B}_R(0))$ is compact by D.4.

3.) Let $u \in H^{m,p}(\Omega)$, then $Eu \in H_0^{m,p}(\Omega')$, where $\Omega' \supset\supset \Omega$ and E is the extension operator from A.4. Thus we apply 1.) and 2.) to Eu and obtain the desired result by the properties of E. □

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